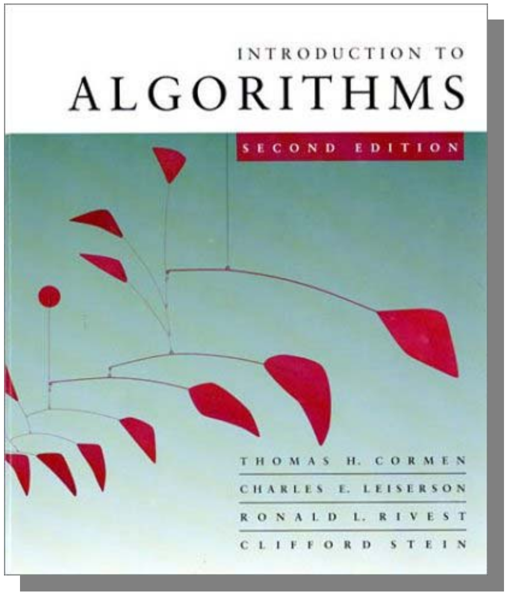
Introduction to Algorithms

6.046J/18.401J

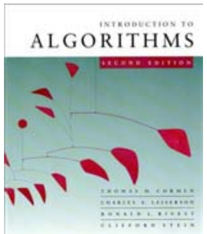


LECTURE 16

Shortest Paths III

- All-pairs shortest paths
- Matrix-multiplication algorithm
- Floyd-Warshall algorithm
- Johnson's algorithm

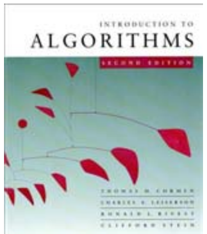
Prof. Charles E. Leiserson



Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(E + V \lg V)$
- General
  - Bellman-Ford algorithm: $O(VE)$
- DAG
  - One pass of Bellman-Ford: $O(V + E)$



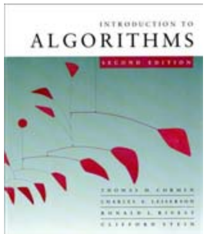
Shortest paths

Single-source shortest paths

- Nonnegative edge weights
  - Dijkstra's algorithm: $O(E + V \lg V)$
- General
  - Bellman-Ford: $O(VE)$
- DAG
  - One pass of Bellman-Ford: $O(V + E)$

All-pairs shortest paths

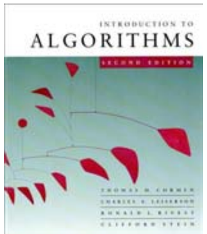
- Nonnegative edge weights
  - Dijkstra's algorithm $|V|$ times: $O(VE + V^2 \lg V)$
- General
  - Three algorithms today.



All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, with edge-weight function $w : E \to \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.



All-pairs shortest paths

Input: Digraph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, with edge-weight function $w : E \to \mathbb{R}$.

Output: $n \times n$ matrix of shortest-path lengths $\delta(i, j)$ for all $i, j \in V$.

IDEA:
- Run Bellman-Ford once from each vertex.
- Time $= O(V^2E)$.
- Dense graph (n^2 edges) $\Rightarrow \Theta(n^4)$ time in the worst case.

Good first try!

Dynamic programming

Consider the $n \times n$ adjacency matrix $A = (a_{ij})$ of the digraph, and define

$d_{ij}^{(m)}$ = weight of a shortest path from i to j that uses at most m edges.

Claim: We have

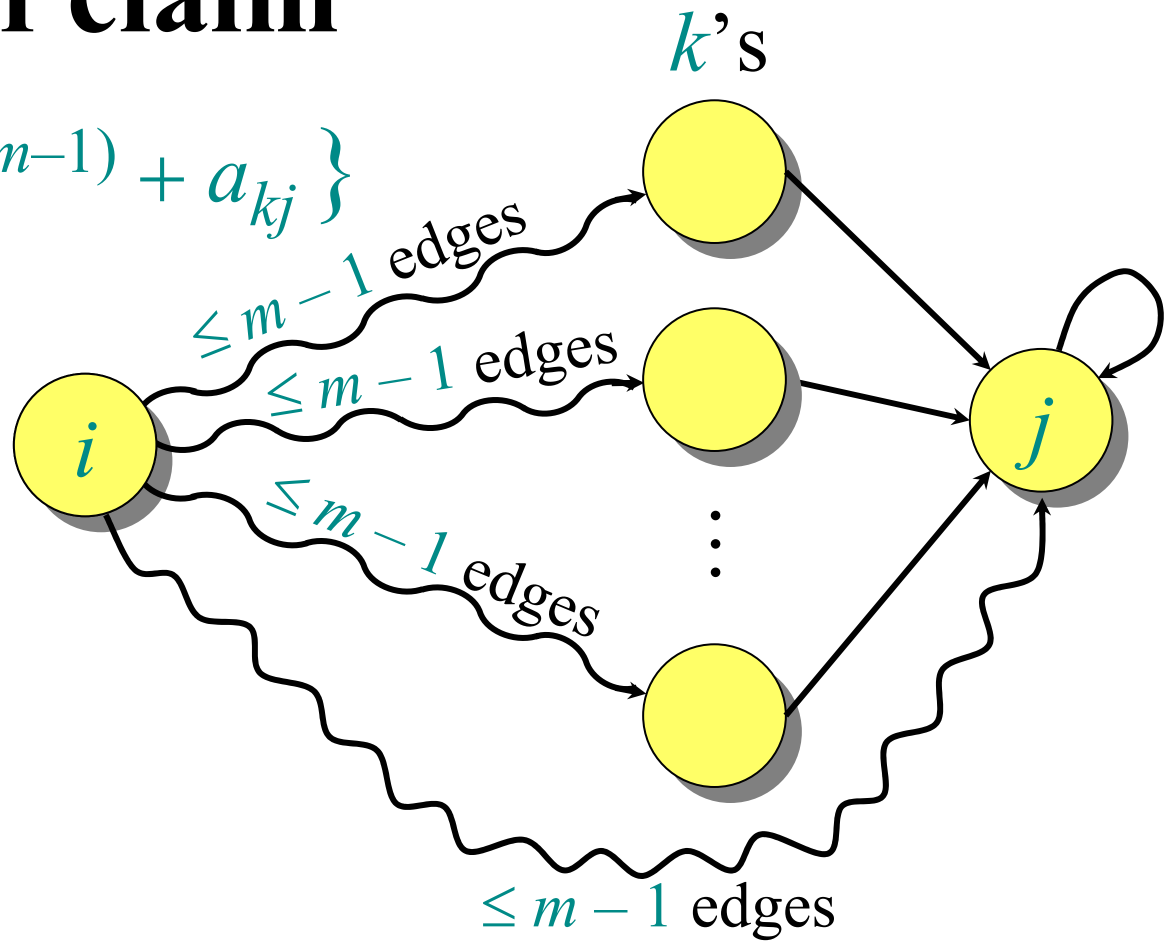
$$d_{ij}^{(0)} = \begin{cases} 0 & \text{if } i = j, \\ \infty & \text{if } i \neq j; \end{cases}$$

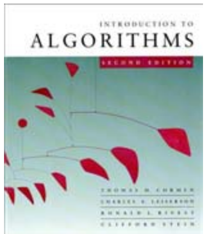
and for $m = 1, 2, \ldots, n-1$,

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \} .$$

Proof of claim

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$



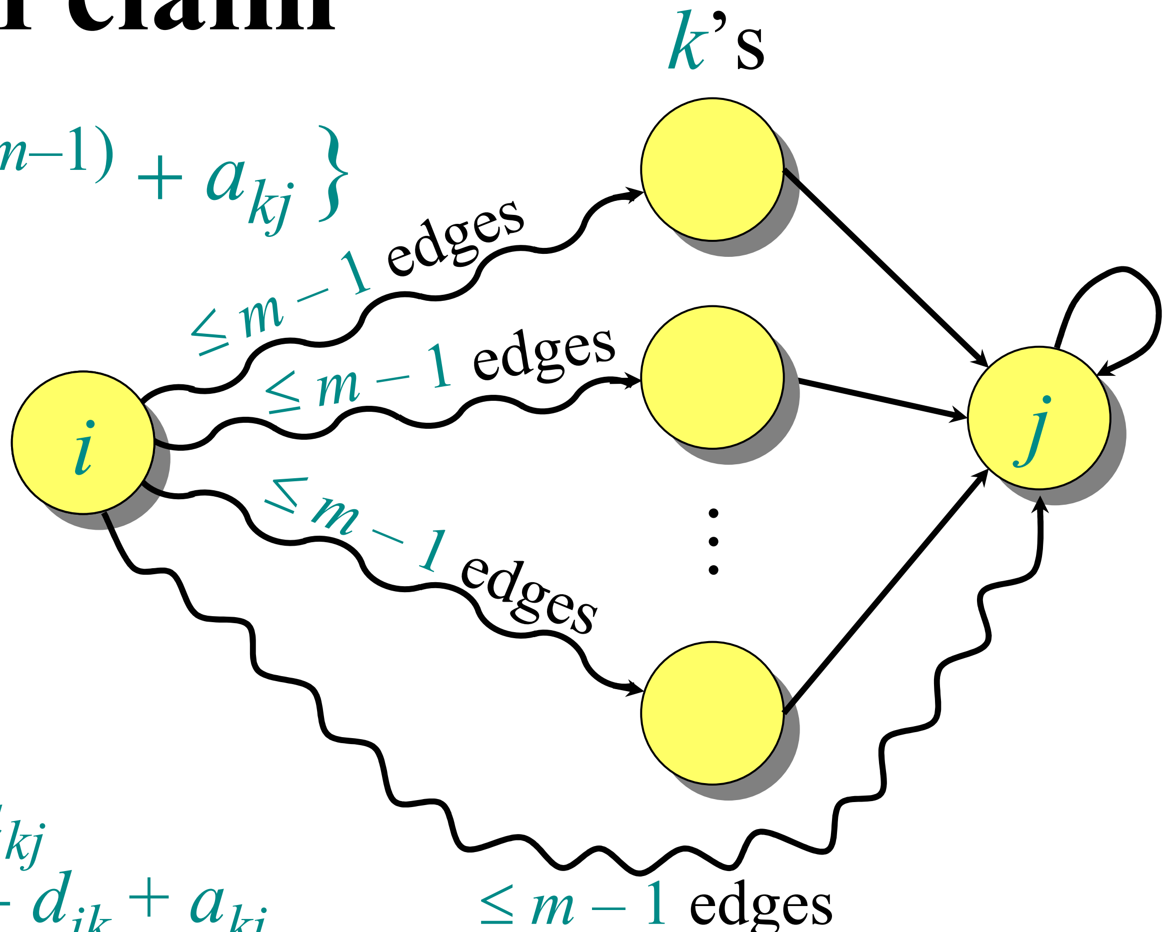


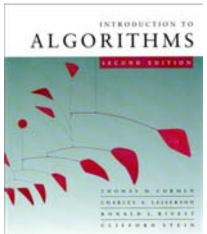
Proof of claim

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$

Relaxation!

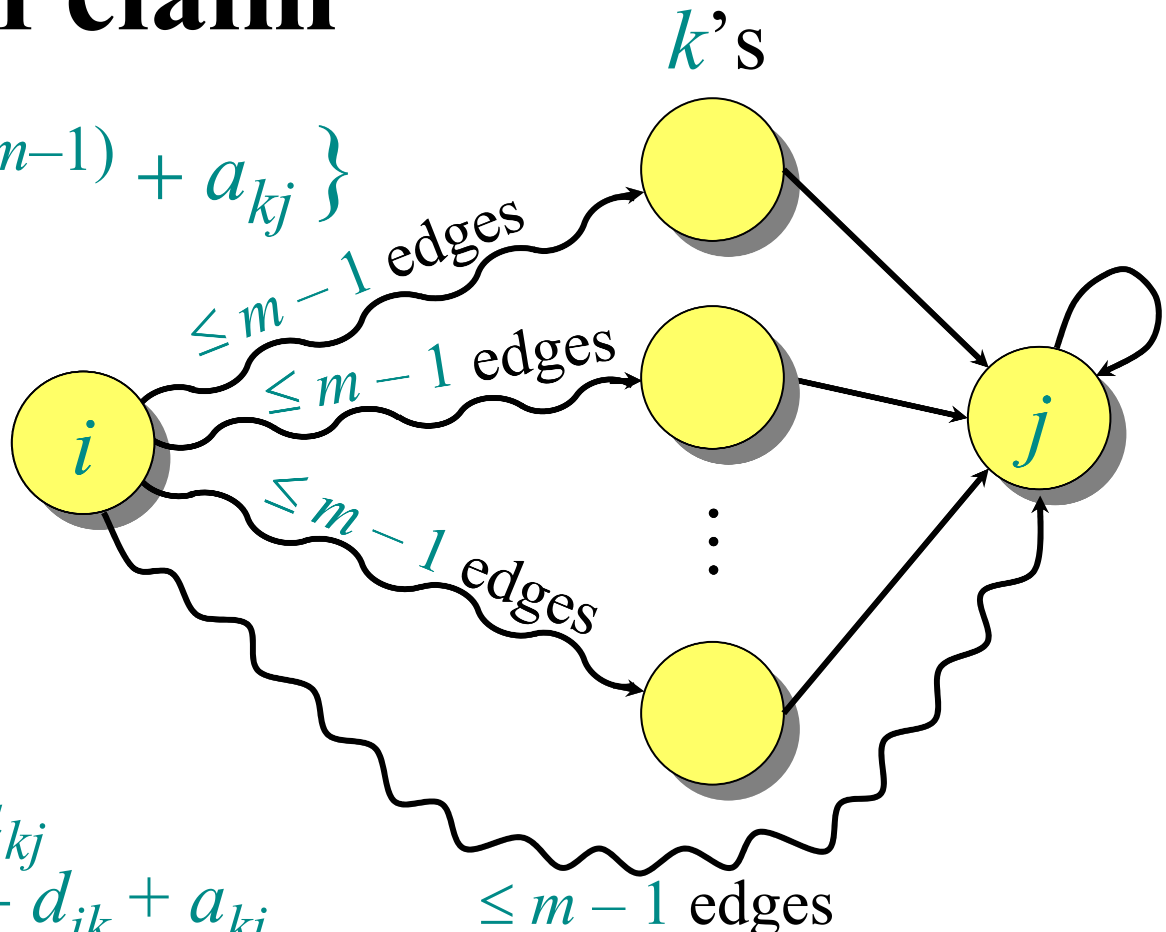
for $k \leftarrow 1$ **to** n
  do if $d_{ij} > d_{ik} + a_{kj}$
    then $d_{ij} \leftarrow d_{ik} + a_{kj}$





Proof of claim

$$d_{ij}^{(m)} = \min_k \{ d_{ik}^{(m-1)} + a_{kj} \}$$

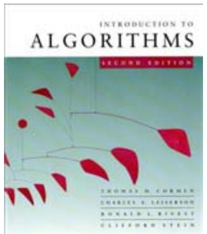


Relaxation!

for $k \leftarrow 1$ **to** n
  do if $d_{ij} > d_{ik} + a_{kj}$
    then $d_{ij} \leftarrow d_{ik} + a_{kj}$

Note: No negative-weight cycles implies

$$\delta(i, j) = d_{ij}^{(n-1)} = d_{ij}^{(n)} = d_{ij}^{(n+1)} = \ldots$$

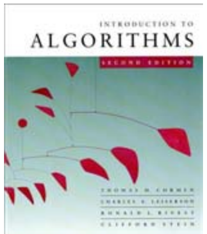


Matrix multiplication

Compute $C = A \cdot B$, where C, A, and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Time $= \Theta(n^3)$ using the standard algorithm.



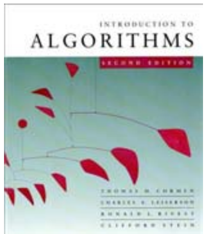
Matrix multiplication

Compute $C = A \cdot B$, where C, A, and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

Time $= \Theta(n^3)$ using the standard algorithm.

What if we map "+" $\to$ "min" and "·" $\to$ "+"?



Matrix multiplication

Compute $C = A \cdot B$, where C, A, and B are $n \times n$ matrices:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \, .$$

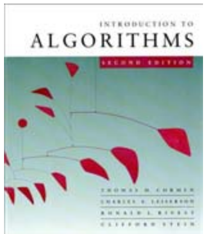
Time $= \Theta(n^3)$ using the standard algorithm.

What if we map "+" $\rightarrow$ "min" and "$\cdot$" $\rightarrow$ "+"?

$$c_{ij} = \min_k \{a_{ik} + b_{kj}\} .$$

Thus, $D^{(m)} = D^{(m-1)}$ "$\times$" A.

Identity matrix $= \mathrm{I} = \begin{pmatrix} 0 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty \\ \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 \end{pmatrix} = D^0 = (d_{ij}{}^{(0)})$.



Matrix multiplication (continued)

The (min, +) multiplication is ***associative***, and with the real numbers, it forms an algebraic structure called a ***closed semiring***.

Consequently, we can compute

$$\begin{aligned} D^{(1)} &= D^{(0)} \cdot A = A^1 \\ D^{(2)} &= D^{(1)} \cdot A = A^2 \\ &\vdots \\ D^{(n-1)} &= D^{(n-2)} \cdot A = A^{n-1}, \end{aligned}$$

yielding $D^{(n-1)} = (\delta(i, j))$.

Time $= \Theta(n \cdot n^3) = \Theta(n^4)$. No better than $n \times$ B-F.

Improved matrix multiplication algorithm

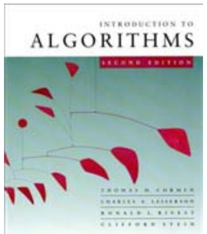
Repeated squaring: $A^{2k} = A^k \times A^k$.

Compute $\underbrace{A^2, A^4, \ldots, A^{2^{\lceil \lg(n-1) \rceil}}}_{O(\lg n) \text{ squarings}}$.

Note: $A^{n-1} = A^n = A^{n+1} = \cdots$.

Time $= \Theta(n^3 \lg n)$.

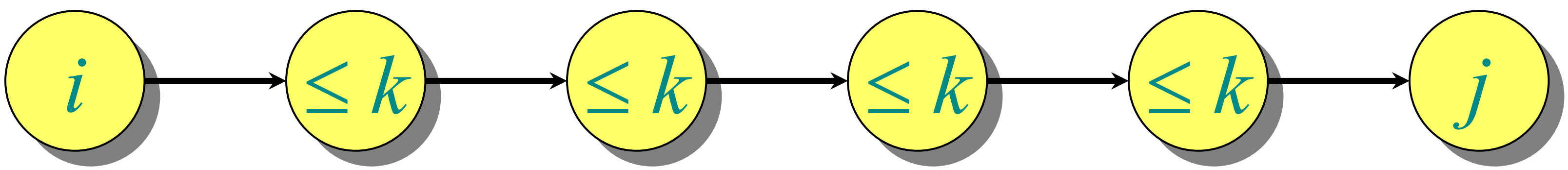
To detect negative-weight cycles, check the diagonal for negative values in $O(n)$ additional time.



Floyd-Warshall algorithm

Also dynamic programming, but faster!

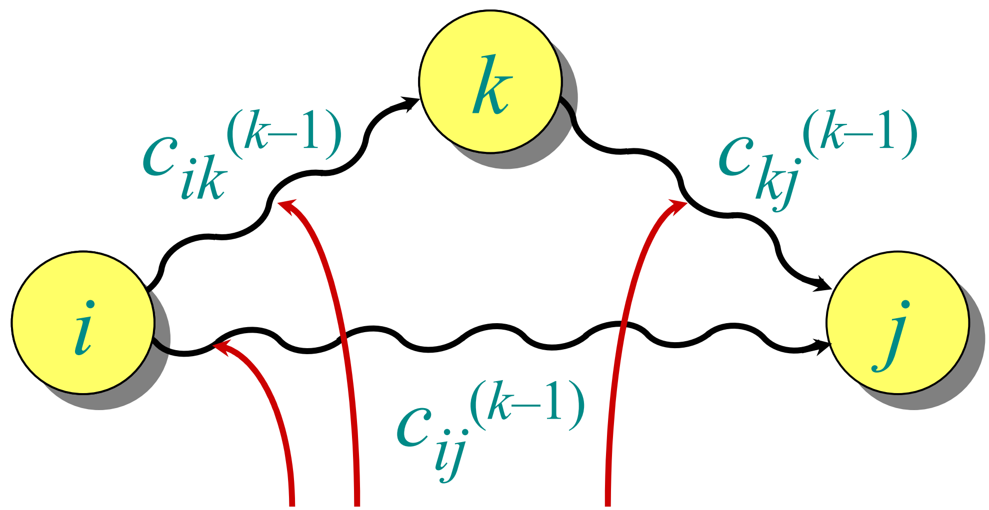
Define $c_{ij}^{(k)}$ = weight of a shortest path from i to j with intermediate vertices belonging to the set $\{1, 2, \ldots, k\}$.



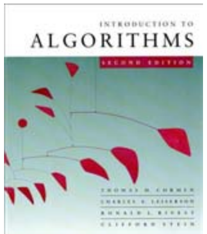
Thus, $\delta(i, j) = c_{ij}^{(n)}$. Also, $c_{ij}^{(0)} = a_{ij}$.

Floyd-Warshall recurrence

$$c_{ij}^{(k)} = \min \{c_{ij}^{(k-1)}, c_{ik}^{(k-1)} + c_{kj}^{(k-1)}\}$$



intermediate vertices in $\{1, 2, \ldots, k\}$



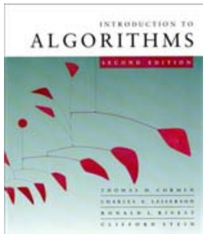
Pseudocode for Floyd-Warshall

for $k \leftarrow 1$ **to** n
    do for $i \leftarrow 1$ **to** n
        do for $j \leftarrow 1$ **to** n
            do if $c_{ij} > c_{ik} + c_{kj}$
                then $c_{ij} \leftarrow c_{ik} + c_{kj}$

relaxation

Notes:

- Okay to omit superscripts, since extra relaxations can't hurt.
- Runs in $\Theta(n^3)$ time.
- Simple to code.
- Efficient in practice.



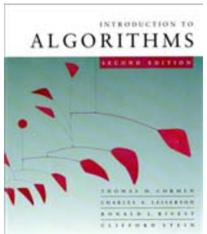
Transitive closure of a directed graph

Compute $t_{ij} = \begin{cases} 1 & \text{if there exists a path from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$

IDEA: Use Floyd-Warshall, but with $(\vee, \wedge)$ instead of $(\min, +)$:

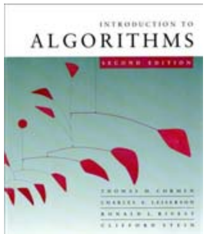
$$t_{ij}^{(k)} = t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}).$$

Time $= \Theta(n^3)$.



Graph reweighting

Theorem. Given a function $h : V \to \mathbb{R}$, ***reweight*** each edge $(u, v) \in E$ by $w_h(u, v) = w(u, v) + h(u) - h(v)$. Then, for any two vertices, all paths between them are reweighted by the same amount.



Graph reweighting

Theorem. Given a function $h : V \to \mathbb{R}$, ***reweight*** each edge $(u, v) \in E$ by $w_h(u, v) = w(u, v) + h(u) - h(v)$. Then, for any two vertices, all paths between them are reweighted by the same amount.

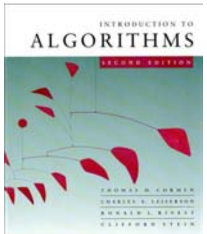
Proof. Let $p = v_1 \to v_2 \to \cdots \to v_k$ be a path in G. We have

$$
\begin{aligned}
w_h(p) &= \sum_{i=1}^{k-1} w_h(v_i, v_{i+1}) \\
&= \sum_{i=1}^{k-1} \big(w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\big) \\
&= \sum_{i=1}^{k-1} w(v_i, v_{i+1}) + h(v_1) - h(v_k) \\
&= w(p) + h(v_1) - h(v_k). \quad \square
\end{aligned}
$$

Same amount!

Shortest paths in reweighted graphs

Corollary. $\delta_h(u, v) = \delta(u, v) + h(u) - h(v)$. $\square$

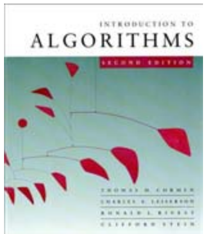


Shortest paths in reweighted graphs

Corollary. $\delta_h(u, v) = \delta(u, v) + h(u) - h(v)$. $\square$

IDEA: Find a function $h : V \to \mathbb{R}$ such that $w_h(u, v) \geq 0$ for all $(u, v) \in E$. Then, run Dijkstra's algorithm from each vertex on the reweighted graph.

NOTE: $w_h(u, v) \geq 0$ iff $h(v) - h(u) \leq w(u, v)$.



Johnson's algorithm

1. Find a function $h : V \to \mathbb{R}$ such that $w_h(u, v) \geq 0$ for all $(u, v) \in E$ by using Bellman-Ford to solve the difference constraints $h(v) - h(u) \leq w(u, v)$, or determine that a negative-weight cycle exists.
   - Time $= O(VE)$.
2. Run Dijkstra's algorithm using w_h from each vertex $u \in V$ to compute $\delta_h(u, v)$ for all $v \in V$.
   - Time $= O(VE + V^2 \lg V)$.
3. For each $(u, v) \in V \times V$, compute
$$\delta(u, v) = \delta_h(u, v) - h(u) + h(v) .$$
   - Time $= O(V^2)$.

Total time $= O(VE + V^2 \lg V)$.