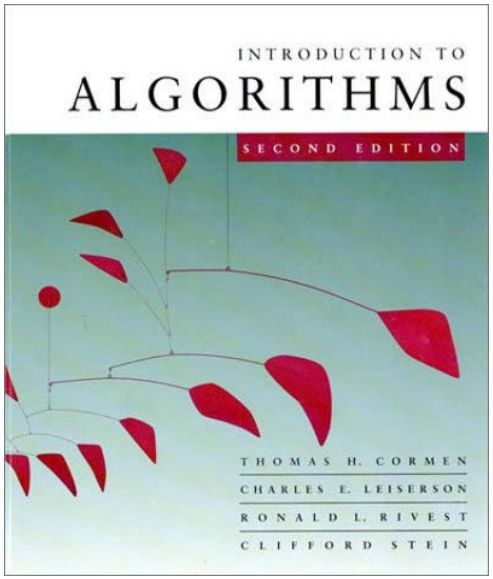
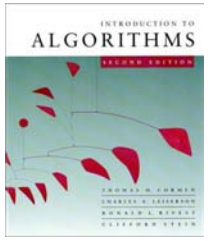
Introduction to Algorithms

6.046J/18.401J



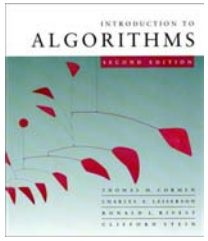
Lecture 22

Prof. Piotr Indyk



Today

- String matching problems
- HKN Evaluations (last 15 minutes)
- Graded Quiz 2 (outside)



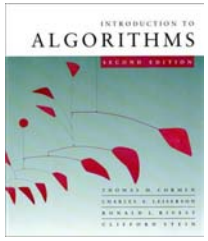
String Matching

- Input: Two strings $T[1\ldots n]$ and $P[1\ldots m]$, containing symbols from alphabet Σ.

  E.g. :
  - $\Sigma=\{a,b,\ldots,z\}$
  - $T[1\ldots 18]=$"to be or not to be"
  - $P[1..2]=$"be"

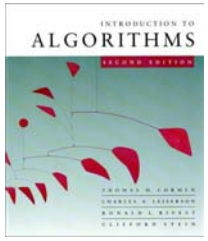
- Goal: find all "shifts" $0\leq s \leq n-m$ such that $T[s+1\ldots s+m]=P$

  E.g. 3, 16



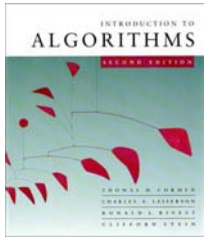
Simple Algorithm

```
for s ← 0 to n-m
    Match ← 1
    for  j ← 1 to m
        if T[s+j] ≠P[j] then
            Match ← 0
            exit loop
    if Match=1 then output s
```



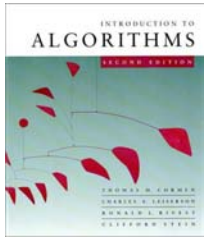
Results

- Running time of the simple algorithm:
  - Worst-case: $O(nm)$
  - Average-case (random text): $O(n)$
    - T_s = time spent on checking shift s
    - $E[T_s] \leq 2$
    - $E[\sum_s T_s] = \sum_s E[T_s] = O(n)$



Worst-case

- Is it possible to achieve O(n) for any input ?
  - Knuth-Morris-Pratt'77: deterministic
  - Karp-Rabin'81: randomized



Karp-Rabin Algorithm

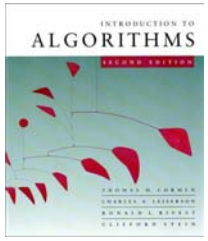
- A very elegant use of an idea that we have encountered before, namely…

  HASHING !

- Idea:
  - Hash all substrings

    $T[1\ldots m], T[2\ldots m+1], \ldots, T[m-n+1\ldots n]$
  - Hash the pattern $P[1\ldots m]$
  - Report the substrings that hash to the same value as P

- Problem: how to hash $n-m$ substrings, each of length m, in $O(n)$ time ?



Attempt 0

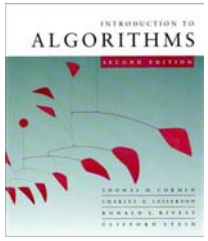
- In Lecture 7, we have seen

$$h_a(x)=\sum_i a_i x_i \bmod q$$

  where $a=(a_1,\ldots,a_r)$, $x=(x_1,\ldots,x_r)$
- To implement it, we would need to compute

$$h_a(\ T[s\ldots s+m-1]\ )=\sum_i a_i\ T[s+i] \bmod q$$

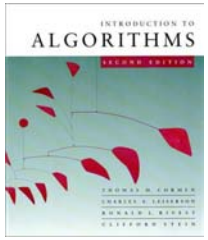
  for $s=0\ldots n-m$
- How to compute it in $O(n)$ time ?
- A big open problem!



Attempt 1

- Assume $\Sigma=\{0,1\}$
- Think about each $T^s=T[s+1\ldots s+m]$ as a number in binary representation, i.e.,

  $t_s=T[s+1]2^{m-1}+T[s+2]2^{m-2}+\ldots+T[s+m]2^0$
- Find a fast way of computing t_{s+1} given t_s
- Output all s such that t_s is equal to the number p represented by P



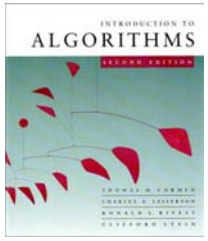
The great formula

- How to transform

  $t_s = T[s+1]2^{m-1} + T[s+2]2^{m-2} + \ldots + T[s+m]2^0$

  into

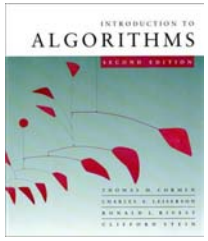
  $t_{s+1} = T[s+2]2^{m-1} + T[s+3]2^{m-2} + \ldots + T[s+m+1]2^0$?
- Three steps:
  - Subtract $T[s+1]2^{m-1}$
  - Multiply by 2 (i.e., shift the bits by one position)
  - Add $T[s+m+1]2^0$
- Therefore: $t_{s+1} = (t_s - T[s+1]2^{m-1})*2 + T[s+m+1]2^0$



Algorithm

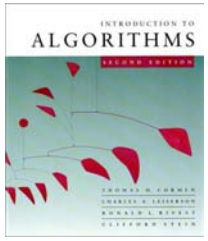
$$t_{s+1} = (t_s - T[s+1]2^{m-1})*2 + T[s+m+1]2^0$$

- Can compute t_{s+1} from t_s using 3 arithmetic operations
- Therefore, we can compute all $t_0, t_1, \ldots, t_{n-m}$ using $O(n)$ arithmetic operations
- We can compute a number corresponding to P using $O(m)$ arithmetic operations
- Are we done ?



Problem

- To get $O(n)$ time, we would need to perform each arithmetic operation in $O(1)$ time
- However, the arguments are m-bit long !
- If m large, it is unreasonable to assume that operations on such big numbers can be done in $O(1)$ time
- We need to reduce the number range to something more managable



Attempt 2: Hashing

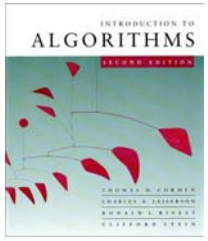
- We will instead compute

  $t'_s = T[s+1]2^{m-1} + T[s+2]2^{m-2} + \ldots + T[s+m]2^0 \bmod q$

  where q is an "appropriate" prime number
- One can still compute t'_{s+1} from t'_s :

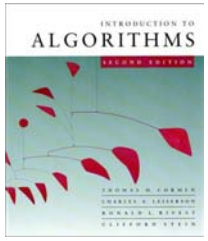
$$t'_{s+1} = (t'_s - T[s+1]2^{m-1})*2 + T[s+m+1]2^0 \bmod q$$

- If q is not large, i.e., has $O(\log n)$ bits, we can compute all t'_s (and p') in $O(n)$ time



Problem

- Unfortunately, we can have false positives, i.e., $T^s \neq P$ but $t_s \bmod q = p \bmod q$
- Need to use a random q
- We will show that the probability of a false positive is small → randomized algorithm



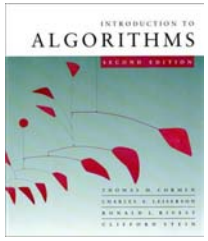
False positives

- Consider any $t_s \neq p$. We know that both numbers are in the range $\{0 \ldots 2^m-1\}$
- How many primes q are there such that

  $$t_s \bmod q = p \bmod q \equiv (t_s-p) = 0 \bmod q \ ?$$

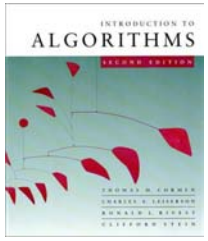
- Such prime has to divide $x=(t_s-p) \leq 2^m$
- Represent $x=p_1^{e1}p_2^{e2}\ldots p_k^{ek}$, p_i prime, $e_i \geq 1$

  What is the largest possible value of k ?
  - Since $2 \leq p_i$, we have $x \geq 2^k$
  - But $x \leq 2^m$
  - $k \leq m$
- There are $\leq m$ primes dividing x



Algorithm

- Algorithm:
  - Let $\prod$ be a set of $2nm$ primes, each having $O(\log n)$ bits
  - Choose q uniformly at random from $\prod$
  - Compute $t_0 \bmod q$, $t_1 \bmod q$, …., and $p \bmod q$
  - Report s such that $t_s \bmod q = p \bmod q$
- Analysis:
  - For each s, the probability that $T^s \neq P$ but
    $$t_s \bmod q = p \bmod q$$
    is at most $m/2nm = 1/2n$
  - The probability of *any* false positive is at most $(n-m)/2n \leq 1/2$

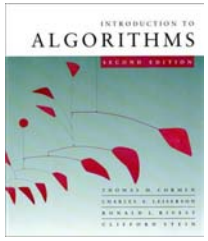


"Details"

- How do we know that such $\prod$ exists ?

  (That is, a set of $2nm$ primes, each having $O(\log n)$ bits)

- How do we choose a random prime from $\prod$ in $O(n)$ time ?



Prime density

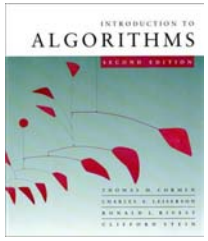
- Primes are "dense". I.e., if PRIMES(N) is the set of primes smaller than N, then asymptotically

$$|\text{PRIMES}(N)|/N \sim 1/\ln N$$

- If N large enough, then

$$|\text{PRIMES}(N)| \geq N/(2\ln N)$$

- Proof: Trust me.



Prime density continued

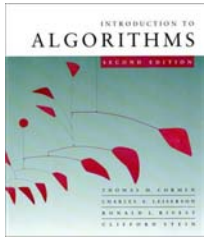
- Set $N=C\ mn\ \ln(mn)$
- There exists $C=O(1)$ such that

  $$N/(2\ln N) \geq 2mn$$

  (Note: for such N we have $PRIMES(N) \geq 2mn$)
- Proof:

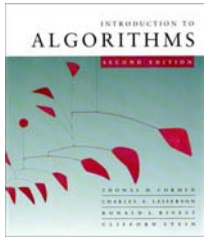
$$C\ mn\ \ln(mn) / [2\ \ln(C\ mn\ \ln(mn))\ ]$$
$$\geq C\ mn\ \ln(mn) / [2\ \ln(C\ (mn)^2)\ ]$$
$$= C\ mn\ \ln(mn) / 4[\ \ln(C) + \ln(mn)]$$

- All elements of $PRIMES(N)$ are $\log N = O(\log n)$ bits long



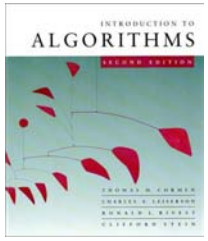
Prime selection

- Still need to find a random element of PRIMES(N)
- Solution:
  - Choose a random element from {1 … N}
  - Check if it is prime
  - If not, repeat



Prime selection analysis

- A random element q from $\{1\ldots N\}$ is prime with probability $\sim 1/\ln N$
- We can check if q is prime in time polynomial in $\log N$:
  - Randomized: Rabin, Solovay-Strassen in 1976
  - Deterministic: Agrawal et al in 2002
- Therefore, we can generate random prime q in $o(n)$ time



Final Algorithm

- Set $N = C\ mn\ \ln(mn)$
- Repeat
  - Choose q uniformly at random from $\{1 \ldots N\}$
- Until q is prime
- Compute $t_0 \bmod q$, $t_1 \bmod q$, …., and $p \bmod q$
- Report s such that $t_s \bmod q = p \bmod q$