Introduction to Algorithms

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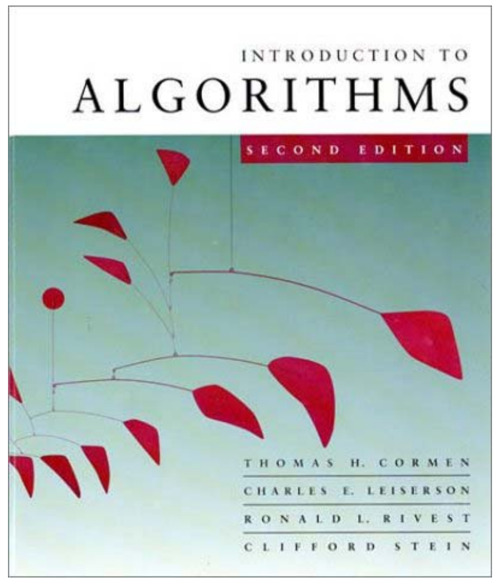
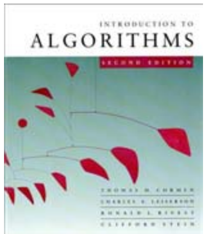


Lecture 24

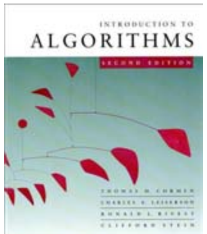
Prof. Piotr Indyk



Dealing with Hard Problems

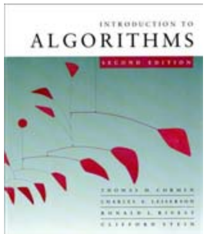
- What to do if:
  - Divide and conquer
  - Dynamic programming
  - Greedy
  - Linear Programming/Network Flows
  - …

  does not give a polynomial time algorithm?



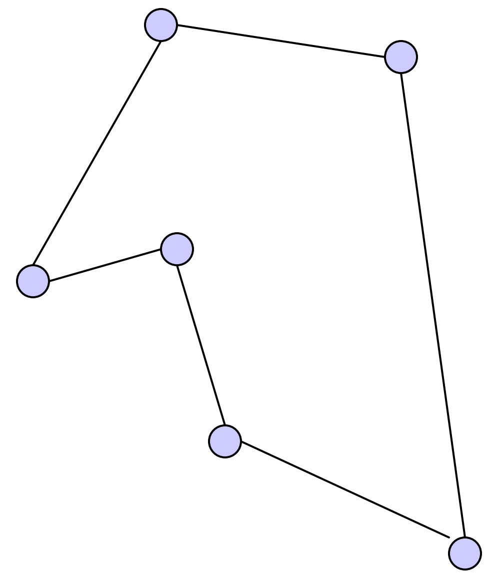
Dealing with Hard Problems

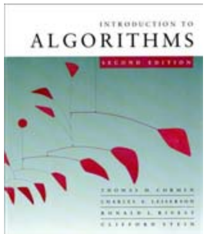
- Solution I: Ignore the problem
  - Can’t do it ! There are thousands of problems for which we do not know polynomial time algorithms
  - For example:
    - Traveling Salesman Problem (TSP)
    - Set Cover



Traveling Salesman Problem

- Traveling Salesman Problem (TSP)
  - Input: undirected graph with lengths on edges
  - Output: shortest cycle that visits each vertex exactly once
- Best known algorithm: $O(n\ 2^n)$ time.



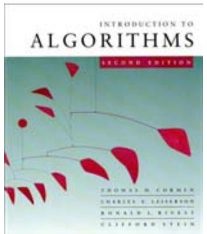


Set Covering

- Set Cover:
  – Input: subsets $S_1 \ldots S_n$ of X, $\cup_i S_i = X$, $|X|=m$
  – Output: $C \subseteq \{1 \ldots n\}$, such that $\cup_{i \in C} S_i = X$, and $|C|$ minimal
- Best known algorithm: $O(2^n\ m)$ time(?)

Bank robbery problem:

- X={plan, shoot, safe, drive, scary}
- Sets:
  – S_{Joe}={plan, safe}
  – S_{Jim}={shoot, scary, drive}
  – ….



Dealing with Hard Problems

- Exponential time algorithms for small inputs. E.g., $(100/99)^n$ time is not bad for $n < 1000$.
- Polynomial time algorithms for some (e.g., average-case) inputs
- Polynomial time algorithms for all inputs, but which return approximate solutions

Approximation Algorithms

- An algorithm A is ρ-approximate, if, on any input of size n:
  - The cost C_A of the solution produced by the algorithm, and
  - The cost C_{OPT} of the optimal solution

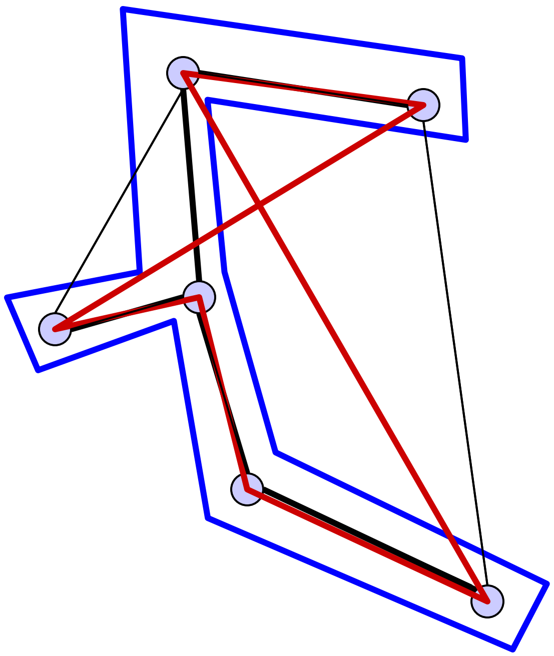
  are such that $C_A \le \rho\, C_{OPT}$
- We will see:
  - 2-approximation algorithm for TSP in the plane
  - ln(m)-approximation algorithm for Set Cover

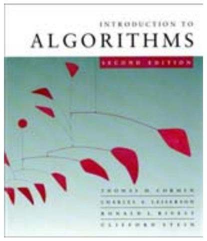
Comments on Approximation

- " $C_A \le \rho\ C_{OPT}$ " makes sense only for minimization problems
- For maximization problems, replace by " $C_A \ge 1/\rho\ C_{OPT}$ "
- Additive approximation " $C_A \le \rho + C_{OPT}$ " also makes sense, although difficult to achieve

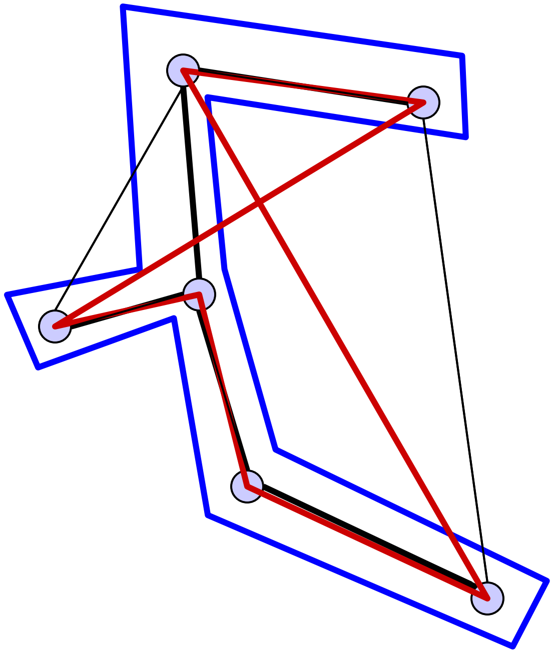
2-approximation for TSP

- Compute MST T
  - An edge between any pair of points
  - Weight = distance between endpoints
- Compute a tree-walk W of T
  - Each edge visited twice
- Convert W into a cycle C using shortcuts





2-approximation: Proof



- Let C_{OPT} be the optimal cycle
- $\text{Cost}(T) \le \text{Cost}(C_{OPT})$
  - Removing an edge from C gives a spanning tree, T is a spanning tree of minimum cost
- $\text{Cost}(W) = 2\ \text{Cost}(T)$
  - Each edge visited twice
- $\text{Cost}(C) \le \text{Cost}(W)$
  - Triangle inequality

$\Rightarrow \text{Cost}(C) \le 2\ \text{Cost}(C_{OPT})$

Approximation for Set Cover

Greedy algorithm:

- Initialize $C=\emptyset$
- Repeat until all elements are covered:
  - Choose S_i which contains largest number of yet-not-covered elements
  - Add i to C
  - Mark all elements in S_i as covered

Greedy Algorithm: Example

- X={1,2,3,4,5,6}
- Sets:
  - S_1={1,2}
  - S_2={3,4}
  - S_3={5,6}
  - S_4={1,3,5}
- Algorithm picks C={4,1,2,3}
- Not optimal!

ln(m)-approximation

- Notation:
  - C_{OPT} = optimal cover
  - $k=|C_{OPT}|$
- Fact: At any iteration of the algorithm, there exists S_j which contains at $\geq 1/k$ fraction of yet-not-covered elements
- Proof: by contradiction.
  - If all sets cover $<1/k$ fraction of yet-not-covered elements, there is no way to cover them using k sets
  - But C_{OPT} does that !
- Therefore, at each iteration greedy covers $\geq 1/k$ fraction of yet-not-covered elements

ln(m)-approximation

- Let u_i be the number of yet-not-covered elements at the end of step $i=0,1,2,\ldots$
- We have

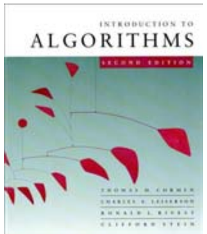
$$u_{i+1} \le u_i (1-1/k)$$

$$u_0 = m$$

- Therefore, after $t=k \ln m$ steps, we have

$$u_t \le u_0 (1-1/k)^t \le m (1-1/k)^{k \ln m} < m\ 1/e^{\ln m} = 1$$

- I.e., all elements are covered by the $k \ln m$ sets chosen by greedy algorithm
- Opt size is $k \Rightarrow$ greedy is $\ln(m)$-approximate



Approximation Algorithms

- Very rich area
  - Algorithms use greedy, linear programming, dynamic programming
    - E.g., 1.01-approximate TSP in the plane
  - Sometimes can show that approximating a problem is as hard as finding exact solution !
    - E.g., 0.99 ln(m)-approximate Set Cover