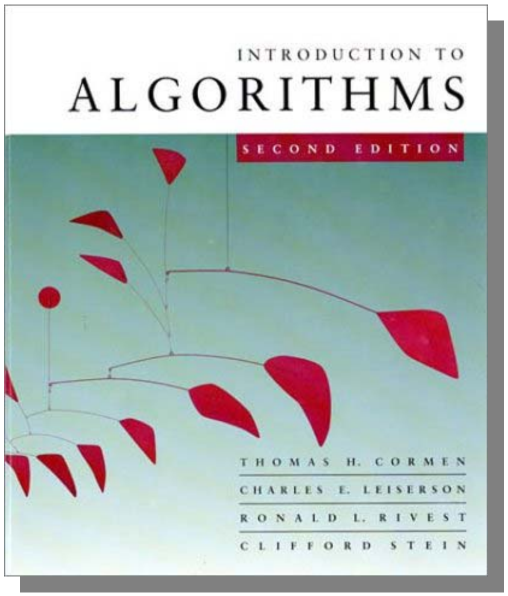
Introduction to Algorithms

6.046J/18.401J



LECTURE 20

Network Flow I

- Flow networks
- Maximum-flow problem
- Flow notation
- Properties of flow
- Cuts
- Residual networks
- Augmenting paths

Prof. Charles E. Leiserson

Flow networks

Definition. A ***flow network*** is a directed graph $G = (V, E)$ with two distinguished vertices: a ***source*** s and a ***sink*** t. Each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

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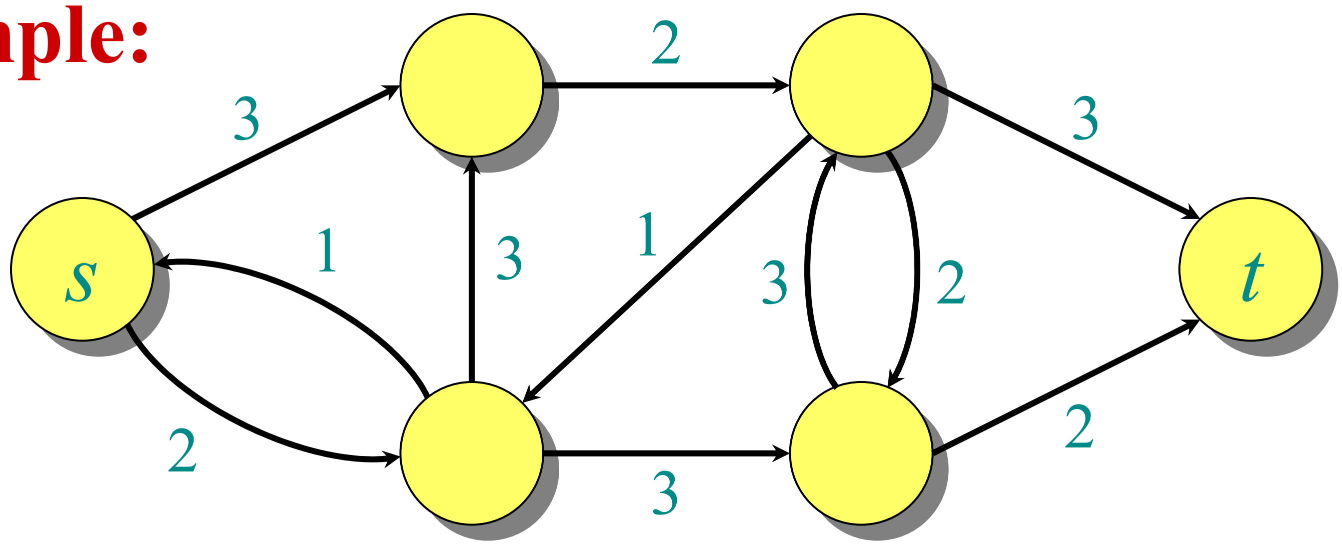
Example:



Flow networks

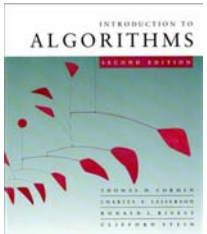
Definition. A ***positive flow*** on G is a function $p : V \times V \rightarrow \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,

$$0 \le p(u, v) \le c(u, v).$$

- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = 0\,.$$



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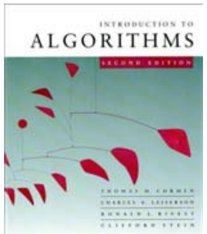
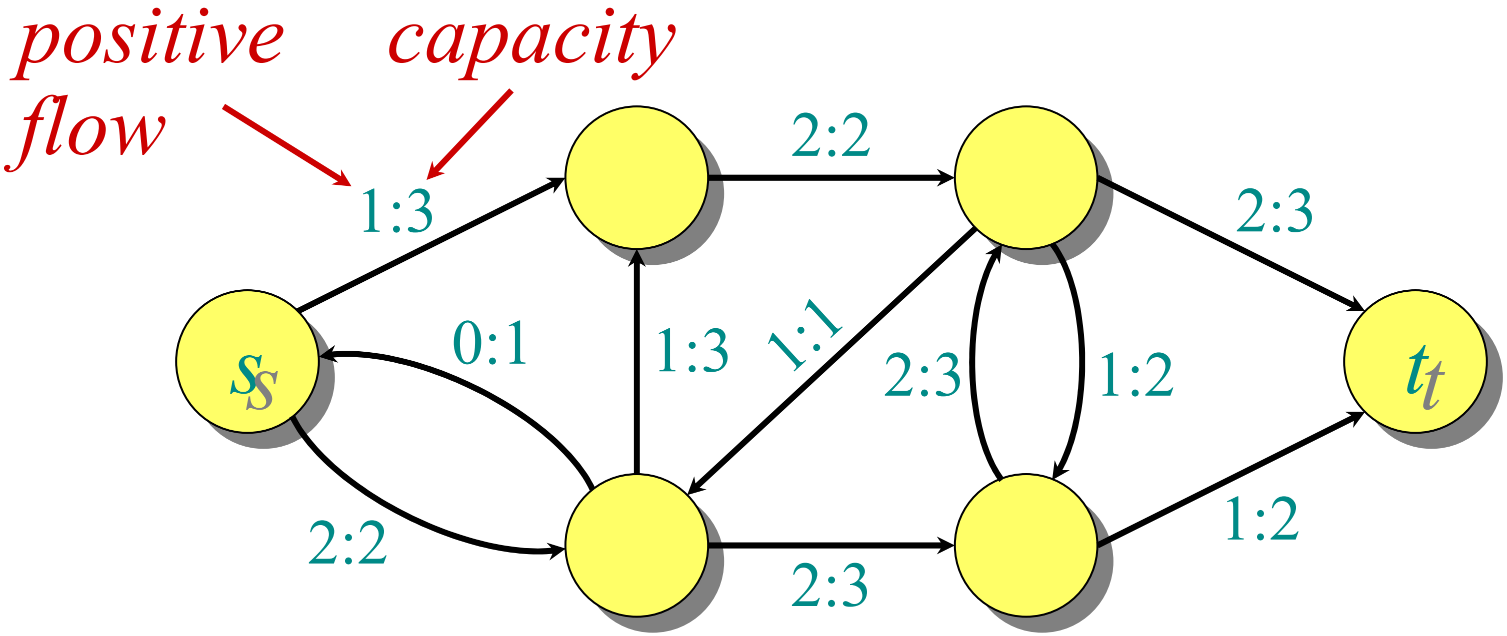
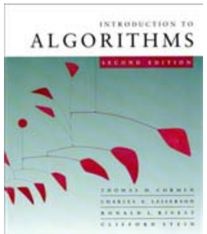
The ***value*** of a flow is the net flow out of the source:

$$\sum_{v \in V} p(s,v) - \sum_{v \in V} p(v,s) .$$



A flow on a network





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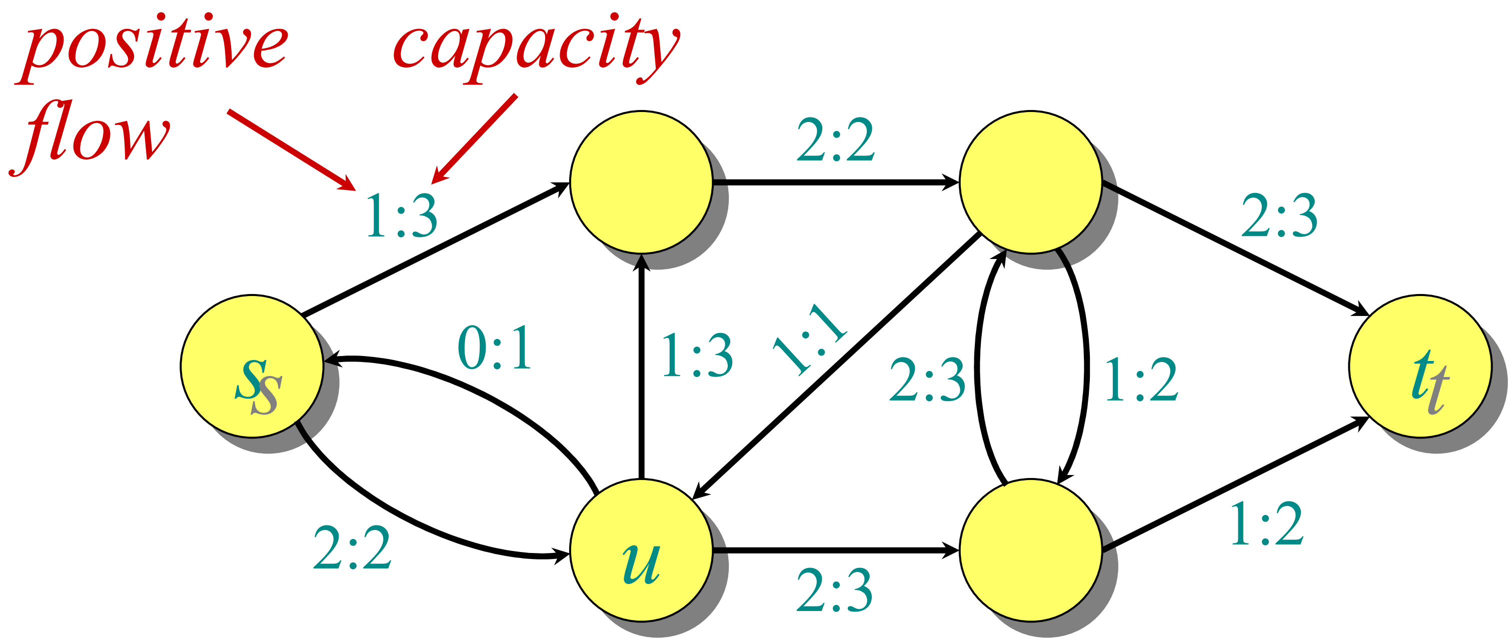


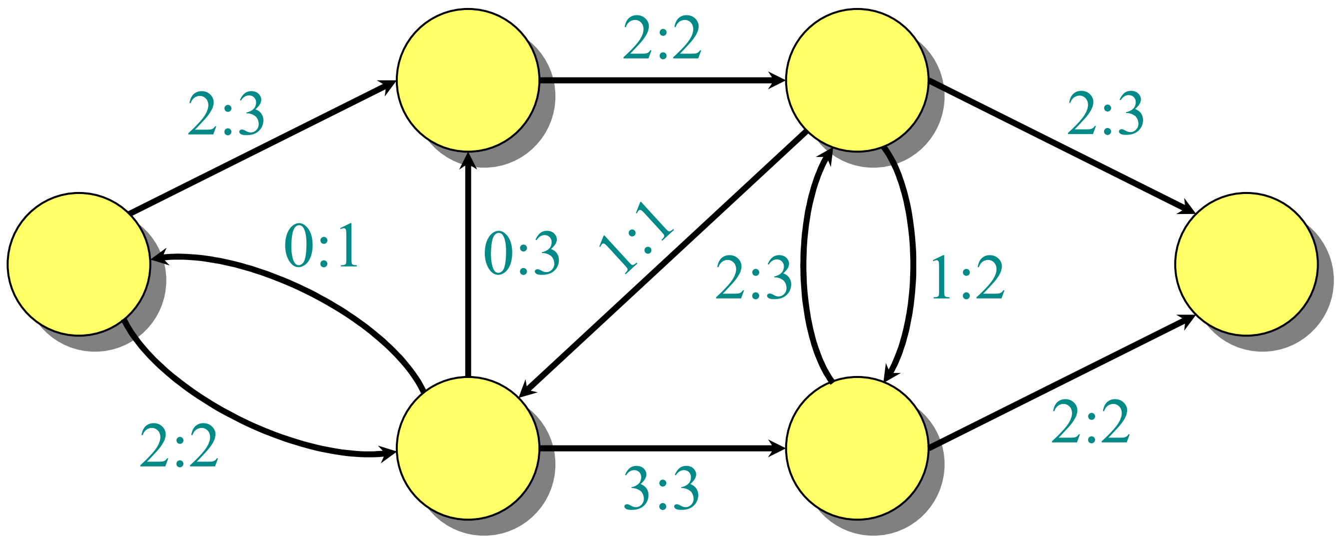
Flow conservation (like Kirchoff's current law):

- Flow into u is $2 + 1 = 3$.
- Flow out of u is $0 + 1 + 2 = 3$.

The value of this flow is $1 - 0 + 2 = 3$.

The maximum-flow problem

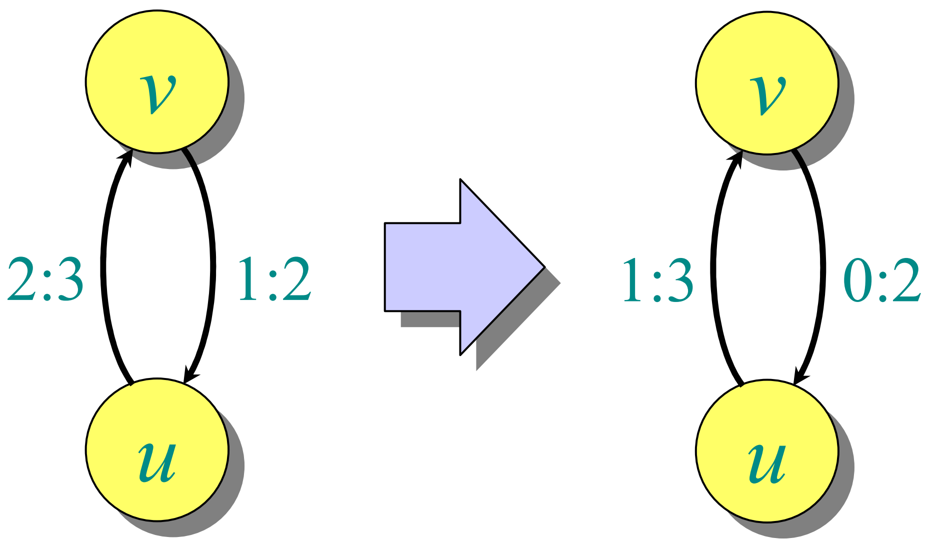
Maximum-flow problem: Given a flow network *G*, find a flow of maximum value on *G*.



The value of the maximum flow is 4.

Flow cancellation

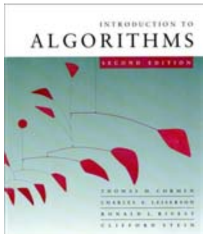
Without loss of generality, positive flow goes either from *u* to *v*, or from *v* to *u*, but not both.



Net flow from *u* to *v* in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

INTUITION: View flow as a *rate*, not a *quantity*.



A notational simplification

IDEA: Work with the net flow between two vertices, rather than with the positive flow.

Definition. A ***(net) flow*** on G is a function $f : V \times V \to \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,

$$f(u, v) \le c(u, v).$$

- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0.$$

- ***Skew symmetry:*** For all $u, v \in V$,

$$f(u, v) = -f(v, u).$$

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One summation instead of two.

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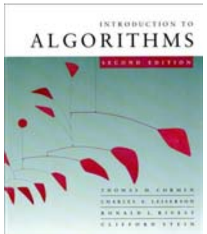

Equivalence of definitions

Theorem. The two definitions are equivalent.

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Proof. ($\Rightarrow$) Let $f(u, v) = p(u, v) - p(v, u)$.

- ***Capacity constraint:*** Since $p(u, v) \le c(u, v)$ and $p(v, u) \ge 0$, we have $f(u, v) \le c(u, v)$.
- ***Flow conservation:***

$$\begin{aligned} \sum_{v \in V} f(u,v) &= \sum_{v \in V} \big(p(u,v) - p(v,u)\big) \\ &= \sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) \end{aligned}$$

- ***Skew symmetry:***

$$\begin{aligned} f(u, v) &= p(u, v) - p(v, u) \\ &= -\big(p(v, u) - p(u, v)\big) \\ &= -f(v, u). \end{aligned}$$

Proof (continued)

($\Leftarrow$) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \leq 0. \end{cases}$$

- ***Capacity constraint:*** By definition, $p(u, v) \geq 0$. Since $f(u, v) \leq c(u, v)$, it follows that $p(u, v) \leq c(u, v)$.
- ***Flow conservation:*** If $f(u, v) > 0$, then $p(u, v) - p(v, u) = f(u, v)$. If $f(u, v) \leq 0$, then $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$ by skew symmetry. Therefore,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = \sum_{v \in V} f(u,v). \quad \square$$

Notation

Definition. The ***value*** of a flow f, denoted by $|f|$, is given by

$$|f| = \sum_{v \in V} f(s, v) = f(s, V).$$

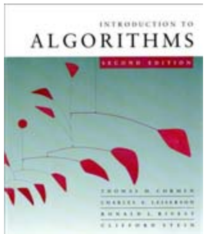
Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

- **Example** — flow conservation:
  $f(u, V) = 0$ for all $u \in V - \{s, t\}$.

Simple properties of flow

Lemma.

- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. $\square$



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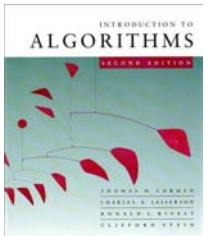
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. $\square$

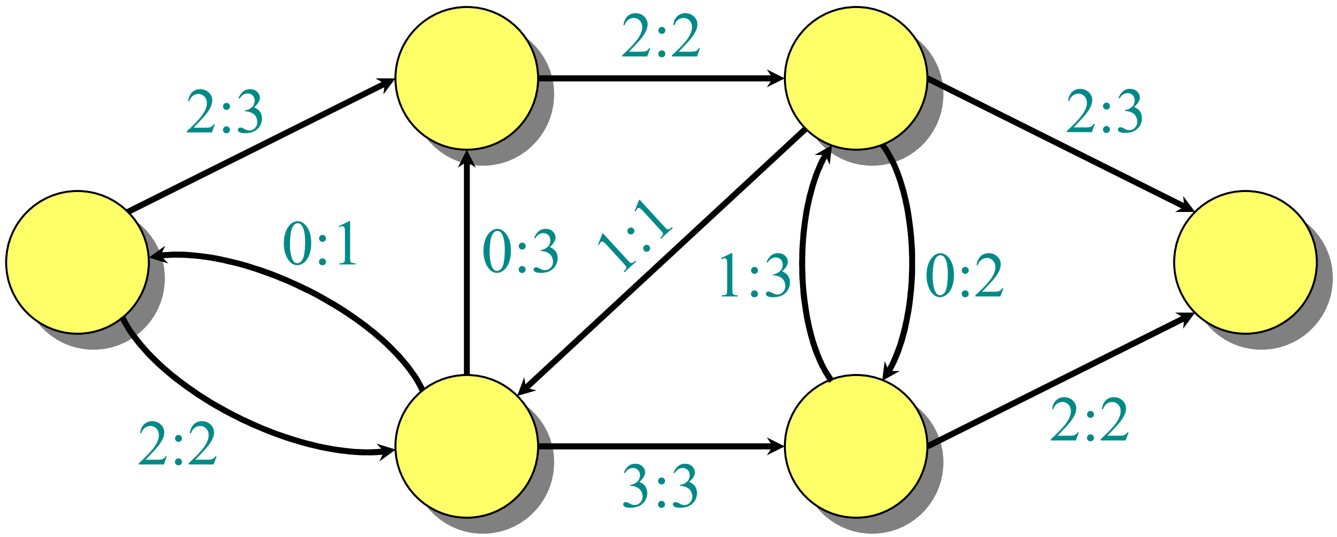
Theorem. $|f| = f(V, t)$.

Proof.

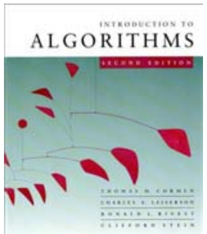
$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V-s, V) \qquad \textit{Omit braces.} \\
&= f(V, V-s) \\
&= f(V, t) + f(V, V-s-t) \\
&= f(V, t). \quad \square
\end{aligned}
$$



Flow into the sink

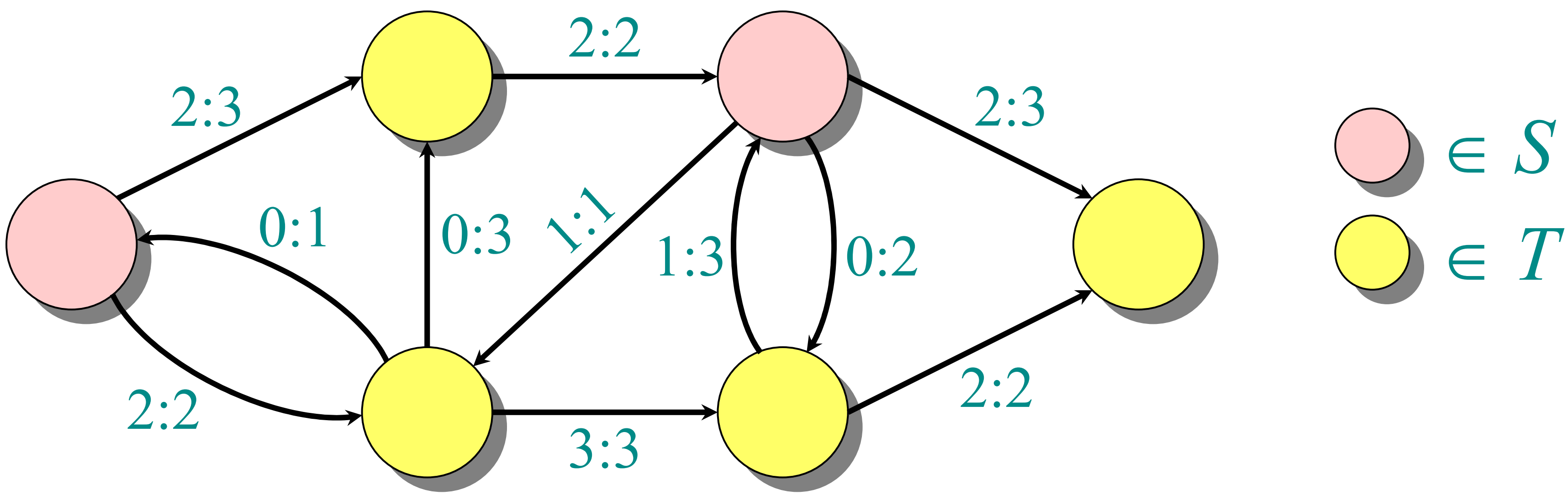


$|f| = f(s, V) = 4$ $\qquad$ $f(V, t) = 4$



Cuts

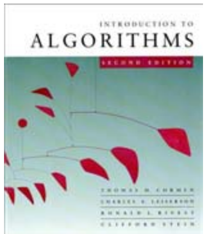
Definition. A ***cut*** (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G, then the ***flow across the cut*** is $f(S, T)$.



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2)$$
$$= 4$$

Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

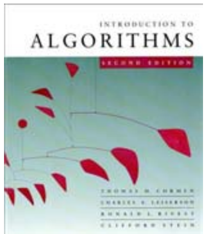


Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

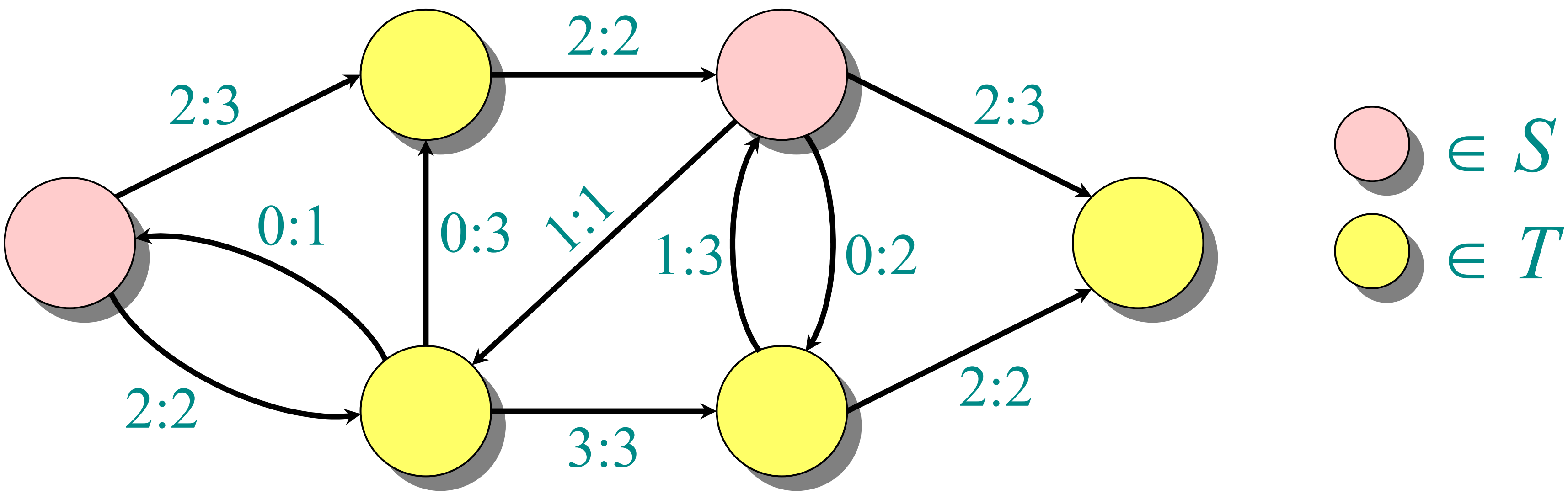
Proof.

$$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(s, V) + f(S-s, V) \\ &= f(s, V) \\ &= |f|. \quad \square \end{aligned}$$



Capacity of a cut

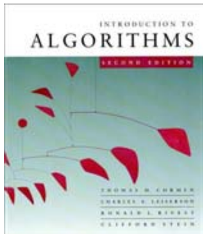
Definition. The ***capacity of a cut*** (S, T) is $c(S, T)$.



$$c(S, T) = (3 + 2) + (1 + 2 + 3)$$
$$= 11$$

Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.



Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

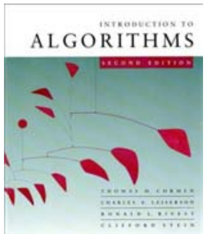
$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)\,. \quad \square
\end{aligned}
$$

Residual network

Definition. Let f be a flow on $G = (V, E)$. The ***residual network*** $G_f(V, E_f)$ is the graph with strictly positive ***residual capacities***

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.



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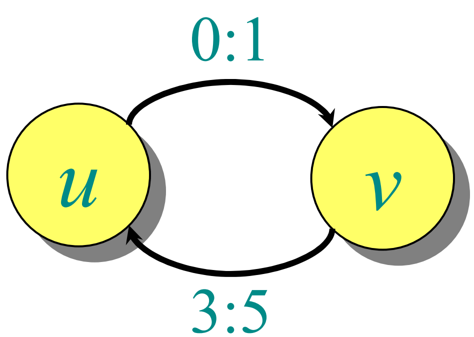
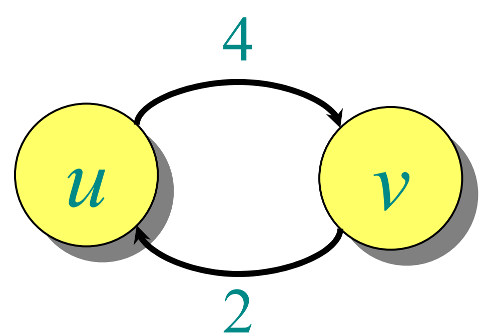
G: 

G_f: 

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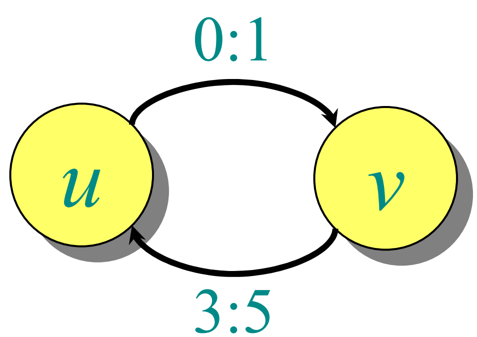
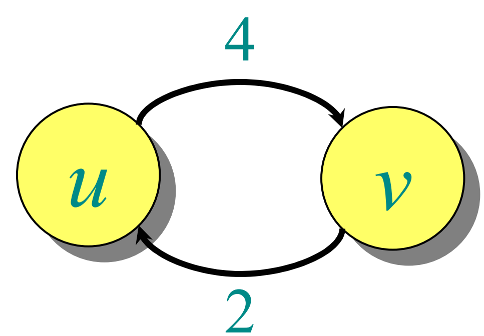
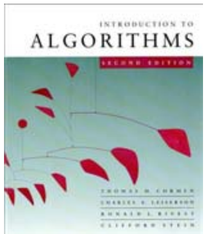


G_f:

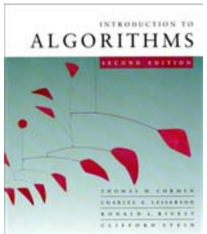


Lemma. $|E_f| \le 2|E|$. $\square$



Augmenting paths

Definition. Any path from s to t in G_f is an ***augmenting path*** in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v)\in p} \{c_f(u,v)\}$.



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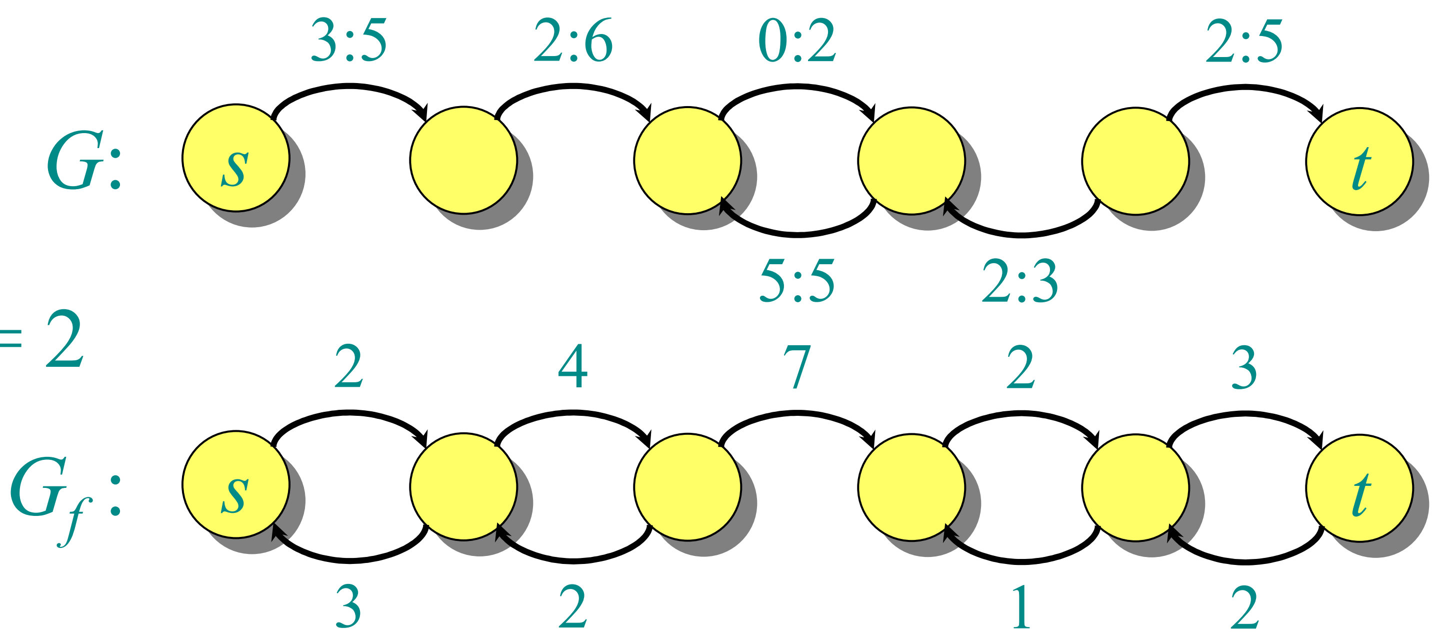
Ex.:



$c_f(p) = 2$

Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. f is a maximum flow.
2. G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

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1. f is a maximum flow.
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Proof (and algorithms). Next time. $\square$