# ATAR Notes Specialist Maths Term 1 Lecture

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April 7, 2018

# Plan for today

- ▶ **Session 1 Vectors** and **Complex Numbers Part 1**
- ▶ **Session 2 Complex Numbers Part II** and **Calculus intro**
- ▶ **Session 3 More calculus** ... than you'll ever want to know

# What's up?

- Hope year 12 is going well for you guys =) it's been a while
- Take a chill pill and relax cos it's gonna be fun guys
- Having trouble with Specialist at this point of year? Make use of your holidays!
- Don't underestimate the power of **notetaking**

# What's up?

**Notetaking**

Writing up **notes** for maths (and also other courses) is super important.

- Forces you to **work through** problems and derivations
- Don't write down a formula without understanding

**No point reading or listening about math ... need to work through with paper and pencil**

# I present you this...

**What one fool can do, another can.**

*(Ancient Simian Proverb)*

From *S. Thompson, Calculus Made Easy (1914)*

# So strap yourselves in...

You should recognise these formulas and know how to use them...

**Geometric progression and series**

- $\sum_{k=0}^{n} ar^k = a + ar + ... + ar^n = \dfrac{a(1 - r^n)}{1 - r}$

- $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ... = \dfrac{a}{1 - r}$

  $\Rightarrow$ ... under what conditions?

**Ellipses and hyperbolae**

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow$ ellipse

- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow$ hyperbola... what kind?

- $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1 \Rightarrow$ hyperbola... what kind?

Of course we can have the necessary translations: $x \to x - h$ ; $y \to y - k$.

**Example 1:** Plot the ellipse with equation: $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$ Hence, draw the hyperbola with equation: $\dfrac{x^2}{3} - \dfrac{y^2}{4} = 1$

# Geometry

- Circle theorems (refer to Year 11)
- Sine rule:
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$
- Cosine rule:
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$
- Pythagoras theorem (and its converse)
$$a^2 + b^2 = c^2 \Leftrightarrow \text{ABC is a right angled triangle}$$

# Vectors

- **Vectors** are a general class of mathematical objects that satisfy some expected properties. **For our purposes**, vectors can be treated as just 'arrows'
- **Vectors** are special because they satisfy a very specific set of properties - they come from **vector spaces**, e.g. $\mathbb{R}^2, \mathbb{R}^3$
- **Geometric interpretation:** vectors are directed line segments with a **head** and **tail**

# Vector toolbox

- ▶ **Scalar multiplication:**
  $\vec{v} \to \lambda \vec{v}$ stretches $\vec{v}$ by $|\lambda|$ and possibly reverses direction.
- ▶ **Addition:**
  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, add head $\to$ tail
- ▶ **Subtraction:**
  $\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$
- ▶ **Magnitude:**
  $|\vec{a}| = $ length of $\vec{a}$, $|\vec{a}| \geq 0$
- ▶ **Scalar product:**
  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$
- ▶ **Unit vector:**
  $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

# Vector toolbox
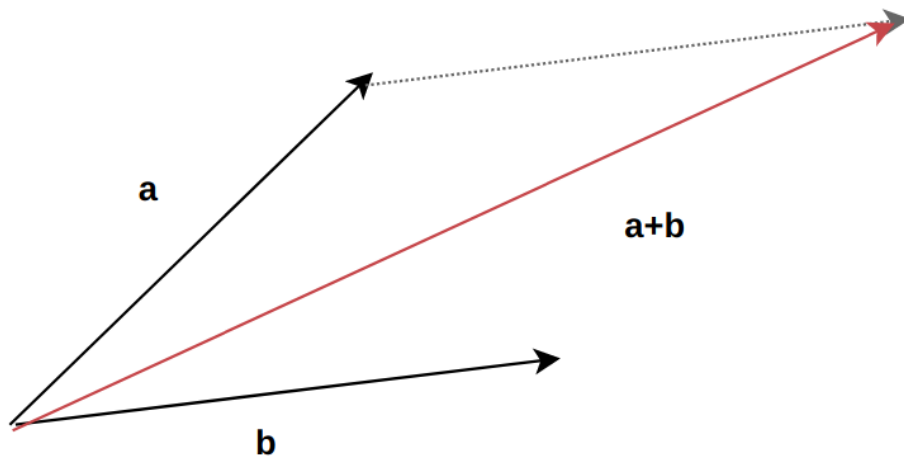
# Vector toolbox

**Scalar product**

The scalar ($\equiv$ dot) product between two vectors is super useful.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$$

In fact, the **scalar product** is what defines the concept of **distance** and **orientation** in our vector space.

For instance, the magnitude of a vector is formally defined as:
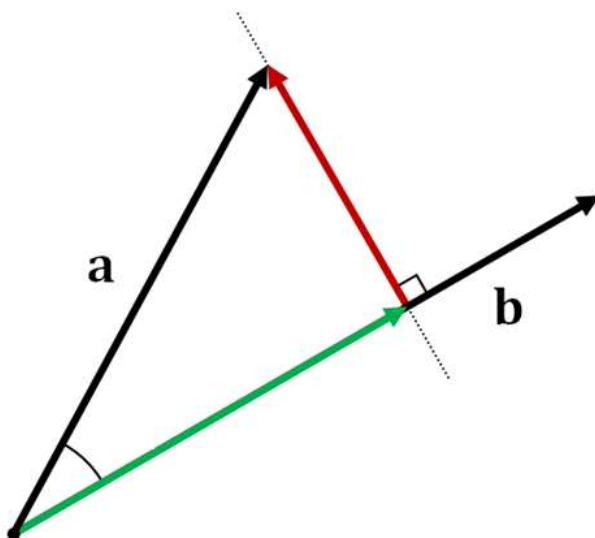
$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

The scalar product we are using is the **Euclidean** inner product. There are other strange beasts out there, though, that give us *distorted* concepts of length.

# Vector toolbox

**Vector projections** Given two vectors $\vec{a}, \vec{b}$, we can consider $\vec{a}_{\parallel}$, the projection of $\vec{a}$ onto $\vec{b}$
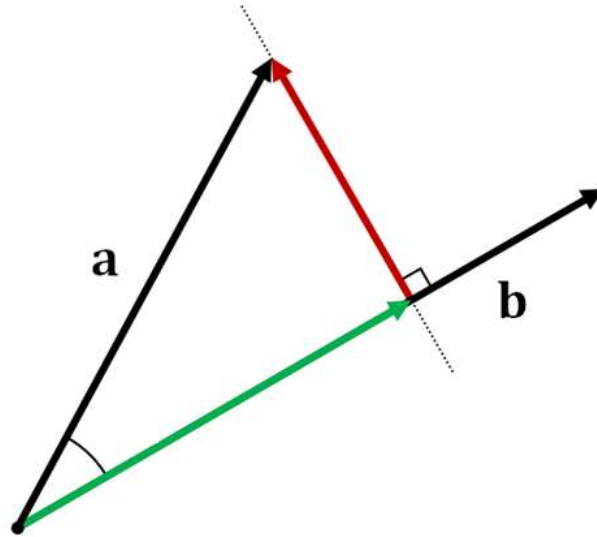
$$\vec{a}_{\parallel} = \left( \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \right) \vec{b} = (\vec{a} \cdot \hat{b})\hat{b}$$

# Vector toolbox

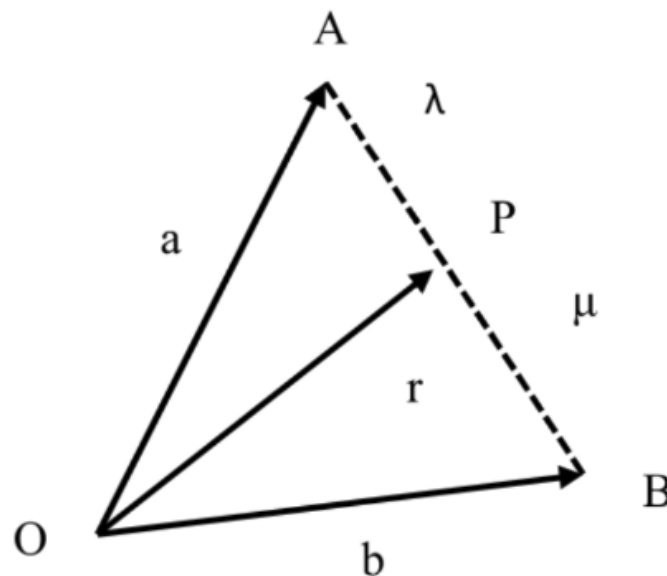**Vector projection** Once we have $\vec{a}_\parallel$, $\vec{a}_\perp$ can simply be found by subtracting:

$$\vec{a}_\perp = \vec{a} - \vec{a}_\parallel$$

# Ratio theorem

This is an important theorem...

$$\vec{r} = \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda + \mu}$$

# Cartesian representation

We've been speaking about vectors in an abstract manner. In $\mathbb{R}^2, \mathbb{R}^3$, we can **project** a vector $\vec{a}$ onto a **basis** $\{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{a} = (\vec{a} \cdot \vec{i})\vec{i} + (\vec{a} \cdot \vec{j})\vec{j} + (\vec{a} \cdot \vec{k})\vec{k}$$
$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$
$$= (a_1, a_2, a_3)$$

We say that $a_1, a_2, a_3$ are **projections** of our vector $\vec{a}$ in the $x, y, z$ directions respectively. Can we derive this from our previous definitions of **vector projections**?

# Cartesian representation

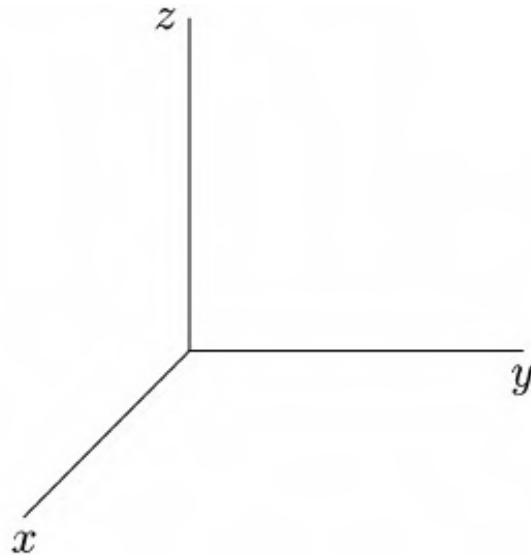Using the **Cartesian representation** of vectors, we find that:

- **Scalar multiplication:** $\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$
- **Addition:**
  $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- **Magnitude:** $|(a_1, a_2, a_3)| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- **Scalar product:**
  $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$
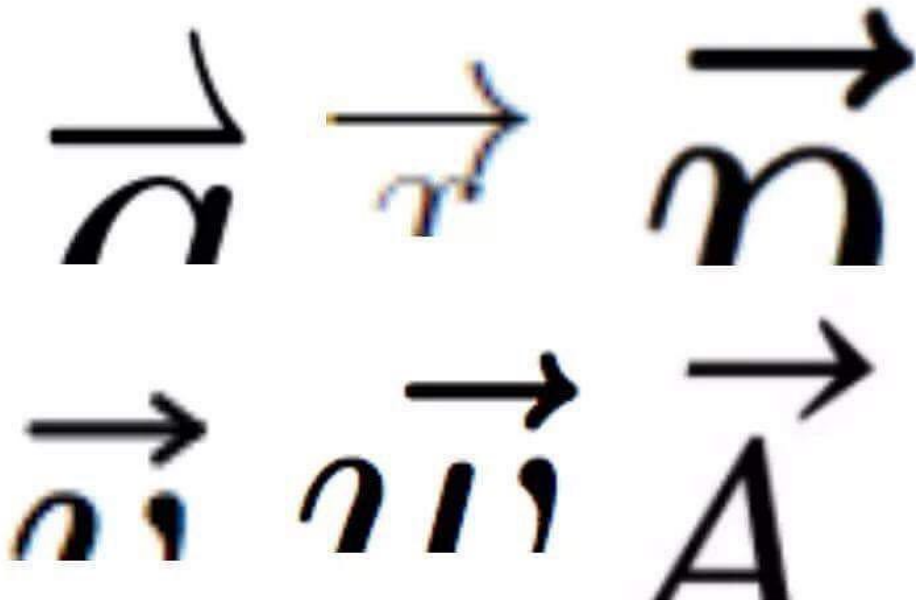
# Pythagoras in 3D

We said previously that

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

This is **Pythagoras' theorem** for 3 dimensions. Proof?

Guys with hair like this have a 125% chance of having  both a **magnitude** and a direction in space.

# Linear independence

**Any** set of vectors will either be linearly **dependent** or otherwise **independent**.

**Formal definition**

$\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent** if there exists $k_1, k_2, k_3$ **not all zero** such that:

$$k_1\vec{a} + k_2\vec{b} + k_3\vec{c} = \vec{0}$$

**Practical definition**

$\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent** if we can write **one vector** as a **linear combination** of the others.
E.g. $\vec{c} = m\vec{a} + n\vec{b}$, and so on.

# Linear independence

$$\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$$
$$\vec{b} = -2\vec{j} + 5\vec{k}$$
$$\vec{c} = 2\vec{i} + 17\vec{k}$$

$\{\vec{a}, \vec{b}, \vec{c}\}$ are **linearly dependent** since $\vec{c} = 2\vec{a} + 3\vec{b}$

# Linear independence

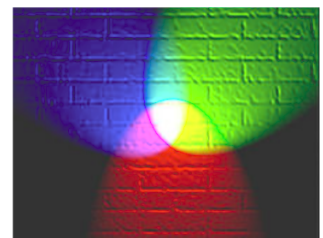If a set of vectors is **not** linearly dependent, then it is **linearly independent**.
**Useful special cases:**

- 2 vectors are dependent only if they are **parallel**
- In 2 dimensions, any set of $>2$ vectors will be dependent
- In 3 dimensions, any set of $>3$ vectors will be dependent
- Any 3 **coplanar** vectors will be dependent

- Here's an analogy:
- Think of a collection of colours: red, green, blue, purple, white
- Realise that we can narrow this down to just three colours, from which we can reconstruct the rest

Purple = red + blue      White = red + green + blue

- The set {red, green, blue, purple, white} is 'linearly dependent' – purple and white are **linear combinations** of red, green, blue
- Thus, we can reduce our set to just bare basics: {red, green, blue}
- This set is now linearly independent!

- In a similar way... let's look at the set of vectors containing
  $$\vec{a} = 2\vec{i} + 3\vec{j}, \qquad \vec{b} = -\vec{i} + \vec{j} + 2\vec{k}, \qquad \vec{c} = 5\vec{j} + 4\vec{k}, \qquad \vec{d} = \vec{i} + 9\vec{j} + 6\vec{k}$$
- Are we at bare basics?
- $\vec{c} = 2\vec{b} + \vec{a}$ and $\vec{d} = \vec{a} + \vec{b} + \vec{c} = 2\vec{a} + 3\vec{b}$
- So $\vec{c}$ and $\vec{d}$ can be expressed in terms of $\vec{a}$ and $\vec{b}$ alone!
- $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is thus linearly dependent
- We can throw out unnecessary baggage... $\{\vec{a}, \vec{b}\}$ is linearly dependent!

## Example: VCAA 2009 Exam 2

**Question 14**

The vectors $\underset{\sim}{u} = m\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = \underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j} + m\underset{\sim}{k}$, where $m$ is a real constant, are **linearly dependent** for

A. $m = 0$

B. $m = 1$

C. $m = 2$

D. $m = 3$

E. $m = 4$

# A trick for nerds

Consider the vectors
$$\vec{a} = (a_1, a_2, a_3), \quad \vec{b} = (b_1, b_2, b_3), \quad \vec{c} = (c_1, c_2, c_3).$$

$\vec{a}, \vec{b}, \vec{c}$ are **linearly independent** *if and only if*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

This determinant is easy to check using a CAS.

## Example: VCAA 2009 Exam 1

**Question 3**

Resolve the vector $5\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ into two vector components, one which is parallel to the vector $-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and one which is perpendicular to it.

3 marks

**Challenge problem**

Using vector projections, find the point on the line $y = 3x$ that is closest to the point $(4, 2)$. What is the minimal distance between the point $(4, 2)$ and the line?

# Vector proofs

**Vector proofs** involve a wide range of techniques, but in general...

- ▸ We represent vectors using a **general notation** e.g. $\vec{a}$, **not** not in component $(\vec{i}, \vec{j}, \vec{k})$ form.
- ▸ **Dot product** is one of the most valuable tools you have!

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

- ▸ Start with what you want to prove - **draw a diagram**

**Problem:** Prove the cosine rule.

# Complex numbers basics

**Definition of a complex number...**

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$$

**Complex numbers** follow the basic arithmetic and algebraic properties we are used to applying to **reals**.

**However**, we do have some extras...

# Complex conjugate

The **conjugate** of a complex number $z$ is denoted $\bar{z}$, and:

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

The conjugate satisfies some **very important** properties...

- $z\bar{z} = (x + iy)(x - iy) = |z|^2$
- $z + \bar{z} = 2\mathrm{Re}(z)$
- $z - \bar{z} = 2i\mathrm{Im}(z)$

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z/w} = \bar{z}/\bar{w}$

# Polar form

Complex numbers can be expressed in **polar form** - this makes multiplication, division and exponentiation a lot easier.

$$z = r\mathrm{cis}(\theta) = r(\cos(\theta) + i\sin(\theta))$$

We write $r = |z|$ and $\theta = \arg(z)$

# Polar form

**Polar form** makes some things much easier...

- $z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$
- $z_1 / z_2 = \dfrac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$
- $z^n = r^n \text{cis}(n\theta)$ $\quad \leftarrow \quad$ De Moivre's Theorem

Importantly for the **conjugate**, we have:

$$z = r\text{cis}(\theta) \Rightarrow \bar{z} = r\text{cis}(-\theta)$$

**Problem.** Find in **Cartesian form**: $u = \text{cis}(\pi/4), v = \text{cis}(\pi/3)$.

Hence, find the values of $\sin(\pi/12)$ and $\cos(\pi/12)$

# Factorisation over $\mathbb{C}$

**Fundamental Theorem of Algebra (formal statement)**
Every polynomial in $P$ in $z$:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_2 z^2 + a_1 z + a_0$$

where $a_n \neq 0$ and $a_0, ..., a_n \in \mathbb{C}$ has at least **one** linear factor in $\mathbb{C}$.

**Version for p l e b s:**
Any polynomial of degree $n$ will have $n$ linear factors over $\mathbb{C}$.

**Quadratics:** we typically **complete the square**... best shown by example.

Factorise $z^2 + 2z + 5$

For higher order polynomials typically you'll need to use the **remainder theorem** to reduce to a quadratic

## Conjugate factor theorem

If the polynomial $P(z)$ has **real coefficients** and $P(\alpha) = 0$ where $\alpha \in \mathbb{C}$, then $P(\overline{\alpha}) = 0$ also.

I.e. if $(z - \alpha)$ is a factor, then so is $(z - \overline{\alpha})$

## Problem: VCAA 2014 Exam 1

**Question 3** (5 marks)

Let $f$ be a function of a complex variable, defined by the rule $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

**a.** Given that $z = i$ is a solution of $f(z) = 0$, write down a quadratic factor of $f(z)$. 2 marks

# Problem: VCAA 2014 Exam 1

**b.** Given that the other quadratic factor of $f(z)$ has the form $z^2 + bz + c$, find all solutions of $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$ in cartesian form.

3 marks

## plz don't kill me for this

who would win??!!!1111

An irreducible quadratic

$z^2 + 2x + 2$

one dreamy boi

$z = x + iy$

# *n*th roots

We can find the *n*th roots of any complex number using de Moivre's Theorem in the following procedure...

$$z^n = a, n \in \mathbb{N}$$

- ▶ There will be *n* solutions $w_1, ..., w_n$ (c.f. Fundamental Theorem of Algebra)
- ▶ Solutions will lie on the circle with radius $|a|^{1/n}$ at even intervals of $2\pi/n$.

**Example:** find the sixth roots of unity.

From *S. Thompson, Calculus Made Easy (1914)*

# Differentiation

**Assumed knowledge:** most of Methods calculus. For those of
you who haven't seen...

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}e^x = e^x$$

Also the **chain rule**:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

# A few new functions...

You will have encountered already...

$$\arcsin(x) = \sin^{-1}(x)$$
$$\arccos(x) = \cos^{-1}(x)$$
$$\arctan(x) = \tan^{-1}(x)$$

We seek their **derivatives...**

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}$$
$$\arccos'(x) = \frac{-1}{\sqrt{1 - x^2}}$$
$$\arctan'(x) = \frac{1}{1 + x^2}$$
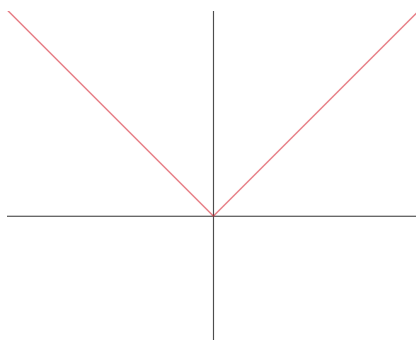
Let's derive these ...

$$\arctan'(x) = \frac{1}{1 + x^2}$$

Let $y = \tan(x)$, $\quad x \in (-\pi/2, \pi/2)$, $\quad y \in \mathbb{R}$. Thus $x = \arctan(y)$.

## The modulus function

You've probably seen already the modulus function...

$$|x| = x\mathrm{sgn}(x) = \text{'positive value of x'}$$

From the graph of $|x|$ we see that our function is **continuous everywhere**, and **differentiable** away from the origin.

$$\frac{d}{dx}|x| = \begin{cases} +1, x > 0 \\ -1, x < 0 \end{cases}$$

## Implicit differentiation

So far, we have been able to differentiate expressions where $y$ is **explicitly** related to $x$.

Implicit relations allow us to find derivatives where $y$ and $x$ are **implicitly** tangled together...

for example in the expression $x\cos(y) + y\cos(x) = 1$

**Quick rule:** to find $dy/dx$, differentiate as per usual w.r.t $x$, using chain/product/quotient rules as required, but write

$$\frac{d}{dx}(y) = \frac{dy}{dx} \quad \text{(obviously!!!)}$$

**Example:** given that $x\cos(y) + y\cos(x) = 1$, find an expression for $dy/dx$.

Here's a cheap way to do the same thing...

To differentiate some relation, rearrange into the form $F(x, y) = 0$ and then find **partial derivatives** $F_x, F_y$. From this, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

- ▶ To find $F_x$, differentiate $F(x, y)$ w.r.t $x$, treating $y$ like a **constant**
- ▶ To find $F_y$, differentiate $F(x, y)$ w.r.t $y$, treating $x$ like a **constant**

# Concavity and the second derivative test

- ▸ We know that **stationary points** can be a local **maximum**, **minimum**, or **point of inflection**
- ▸ **Without** sketching a graph, how would we be able to classify stationary points of a graph?
- ▸ We first introduce the idea of **concavity**

# Concavity

**Concavity** of a curve at a given point tells us whether the curve is 'curving upwards' or 'curving downwards'

- ▸ Consider the second derivative $f''(x) = \dfrac{d^2 f}{dx^2}$
- ▸ **Case:** $f''(x) > 0 \Rightarrow$ concave up (slope is increasing)
- ▸ **Case:** $f''(x) < 0 \Rightarrow$ concave down (slope is decreasing)
- ▸ **Case:** $f''(x) = 0 \Rightarrow$ zero concavity, i.e. locally our curve has no 'curvature'

# Second derivative test

Thus, given a **stationary point** at $x = x_0$, we can check the value of $f''(x_0)$ to determine the type of stationary point.

- $f''(x_0) > 0 \Rightarrow$ concave up $\Rightarrow$ local minimum
- $f''(x_0) < 0 \Rightarrow$ concave down $\Rightarrow$ local maximum
- $f''(x_0) = 0 \Rightarrow$ no concavity $\Rightarrow$ inconclusive

This is the **second derivative test.**

# Second derivative test

**What if $f''(x_0) = 0$?**

If $f''$ **switches sign** around $x_0$...

- We know that $y = f(x)$ changes its concavity around $x_0$
- Thus $y = f(x)$ has a **point of inflection** at $x_0$
- Depending on whether $f'(x_0) = 0$
  this could be a **stationary point of inflection**
  or a **non-stationary point of inflection**

# Second derivative test

**What if** $f''(x_0) = 0$?

In other cases, we actually **can't make any conclusions**!

- ▶ Consider $y = x^4$ and $y = x^5$.
- ▶ **Both** of these satisfy $dy/dx = d^2y/dx^2 = 0$ at $x = 0$
  - ▶ $y = x^4$ has a local minimum
  - ▶ $y = x^5$ has an inflection.
- ▶ **Thus**, in general we will need **further investigation** in these cases

# Second derivative test

**Problem**: find and classify all the stationary points of $y = x^2 e^{-x}$

# Integration techniques

We will follow on from what we learned in Methods.

**Definition of antiderivative**

For a (sufficiently nice) function $f(x)$, we say $F(x)$ is an antiderivative of $f$ if

$$\frac{dF}{dx} = f$$

# Integration techniques

**Connection between definite integrals and the antiderivative**

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F$ is **any** antiderivative of $f$.

Our definite integral itself can be interpreted as the (signed) area bounded by $f$ and the $x$-axis on the interval $[a, b]$

# Substitution of variables

This is a very powerful technique that allows us to compute **indefinite integrals** ($\equiv$ antiderivatives) of a wide range of functions.

Consider an integral of the general form:

$$\int u'(t)f(u(t))dt$$

We can make a substitution $v = u(t) \Rightarrow dv = u'(t)dt$ This brings us to

$$\int u'(t)f(u(t))dt = \int u'(t)f(u(t))\frac{dv}{u'(t)} = \int f(v)dv$$

# Substitution of variables

**Problem:** Compute

$$\int xe^{x^2}dx$$

# Substitution of variables

**Problem:** Compute
$$\int \frac{3 \; dx}{x \ln^2(x)}$$

# Substitution of variables

We can also perform substitution of variables for **definite integrals** - we just need to also substitute the terminals.

# Substitution of variables

**Problem (VCAA 2017 Exam 1):** Find

$$\int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)}$$

# Integration of trig functions

**Important identities**

▶ $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$

▶ $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

▶ $\sin(2x) = 2\sin(x)\cos(x)$

**Example:** $\displaystyle\int \sin^2(x)\cos^2(x)\ dx$

# Other integration techniques

**Partial fractions..**

Typically used to integrate rational functions of the form $P(x)/Q(x)$ with (degree of P) < (degree of Q)

*In the denominator...*

▸ Linear term $(ax + b) \Rightarrow \dfrac{A}{ax + b}$

▸ Repeated linear term $(ax + b)^n \Rightarrow \dfrac{A_1}{ax + b} + ... + \dfrac{A_n}{(ax + b)^n}$

▸ Quadratic linear term $ax^2 + bx + c \Rightarrow \dfrac{Ax + B}{ax^2 + bx + c}$

# Other integration techniques

**Problem:**

$$\int \frac{1}{(x^2 + 3x + 2)} \, dx = \int \frac{1}{(x + 2)(x + 1)} \, dx$$

# Other integration techniques

**Problem:**

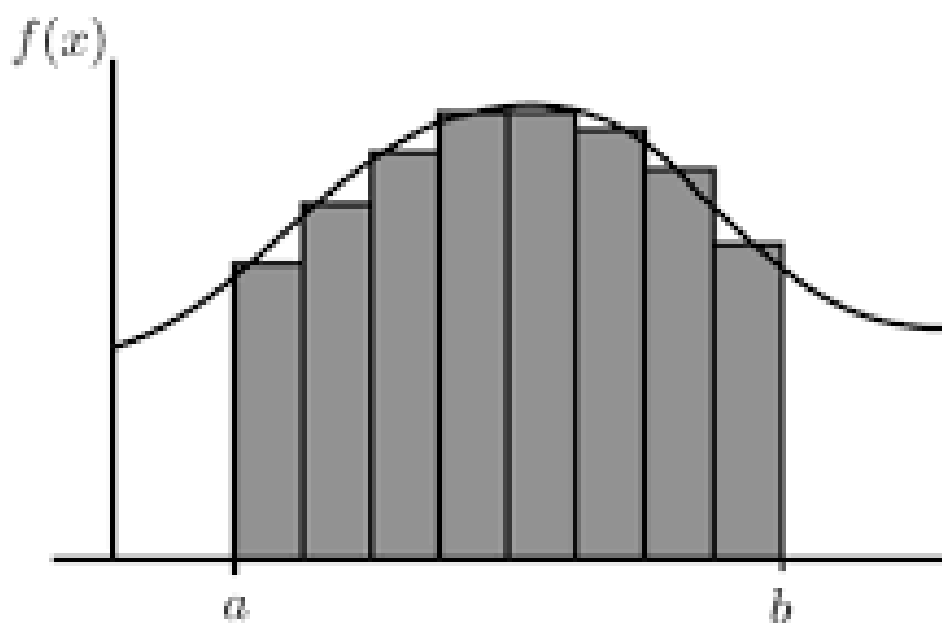Find $\dfrac{d}{dx}x\ln x$. Using the result, compute $\displaystyle\int \ln x \; dx$

# Interesting problem

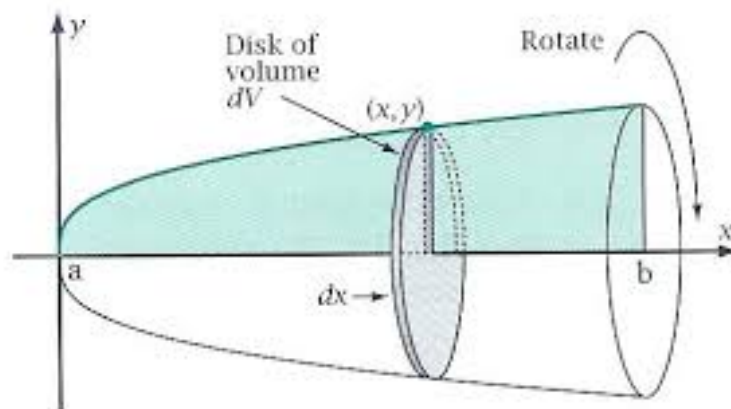Compute $\dfrac{d}{dx}(\sin(x))^x$

# Applications of integration

- A **very** cliche application of integration is to find areas
- $\int f(x) \, dx$ can be interpreted as the (signed) area bound by $f(x)$ and the $x$-axis.
- This is understood generally according the **Riemann** sense of the integral

$$\int_a^b f(x) \, dx := \lim_{n \to \infty} \left[ \sum_{k=1}^{n} f(x_k)\Delta_k \right]$$

# Volumes

Consider rotating a graph $f(x)$ about the $x$-axis by $2\pi$ rads.



Each 'disk' has a volume

$$\Delta V = \pi r^2 h = \pi (f(x))^2 \Delta x$$

# Volumes

And so

$$\text{volume of revolution} = \lim_{n \to \infty} \sum_{1 \le k \le n} \pi (f(x))^2 \Delta x_k$$

$$= \pi \int_a^b (f(x))^2 \; dx$$

Thus, we have the general formula for the volume generated by rotating $y = f(x)$ by $2\pi$ rad around the $x$-axis:

$$V = \pi \int_a^b y^2 dx$$

# Volumes

**Problem:** calculate the volume of the solid generated by rotating $y = x^2, 1 \leq x \leq 2$, about the $x$-axis by $2\pi$ rad.

# Arc length

**Ever wondered how long a curve is?**

Here's our chance.
Define a little piece of $x \Rightarrow \; \mathrm{d}x$
... and a little piece of corresponding $y$-ness $\Rightarrow \; \mathrm{d}y = f'(x) \; \mathrm{d}x$

Thus, by Pythagoras' theorem we have

$$\mathrm{d}\ell = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

# Arc length

And thus...

$$d\ell = \sqrt{dx^2 + dy^2}$$

$$\Rightarrow \frac{d\ell}{dx} = \frac{1}{dx}\sqrt{dx^2 + dy^2}$$

$$\Rightarrow \frac{d\ell}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow L = \int_{x=a}^{x=b} d\ell$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

# Arc length

Thus the **arc length** of a curve $y = f(x)$ between $x = a$ and $x = b$ is given by the formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$