ATAR Notes Specialist Maths Term 1 Lecture

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What's up?

- Hope year 12 is going well for you guys =) it's been a while
- Take a chill pill and relax cos it's gonna be fun guys
- Having trouble with Specialist at this point of year? Make use of your holidays!
- Don't underestimate the power of **notetaking**

What's up?

Notetaking

Writing up **notes** for maths (and also other courses) is super important.

- Forces you to **work through** problems and derivations
- Don't write down a formula without understanding

No point reading or listening about math ... need to work through with paper and pencil

I present you this...

What one fool can do, another can.

(Ancient Simian Proverb)

From *S. Thompson, Calculus Made Easy (1914)*

So strap yourselves in...

You should recognise these formulas and know how to use them...

Geometric progression and series

- $\sum_{k=0}^{n} ar^k = a + ar + ... + ar^n = \dfrac{a(1-r^n)}{1-r}$
- $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ... = \dfrac{a}{1-r}$
  $\Rightarrow$ ... under what conditions?

Ellipses and hyperbolae

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow$ ellipse
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow$ hyperbola... what kind?
- $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1 \Rightarrow$ hyperbola... what kind?

Of course we can have the necessary translations: $x \to x - h$; $y \to y - k$.

Example: Plot the ellipse with equation: $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$ Hence, draw the hyperbola with equation: $\dfrac{x^2}{3} - \dfrac{y^2}{4} = 1$

Geometry

- Circle theorems (refer to Year 11)
- Sine rule:
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$
- Cosine rule:
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$
- Pythagoras theorem (and its converse)

$$a^2 + b^2 = c^2 \Leftrightarrow \text{ABC is a right angled triangle}$$

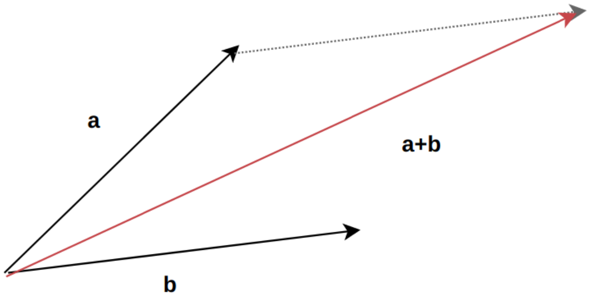
Vectors

- Recall that **vectors** are mathematical objects that are **groups** of numbers.
- **Vectors** are special because they satisfy a very specific set of properties - they come from **vector spaces**, e.g. $\mathbb{R}^2, \mathbb{R}^3$
- **Geometric interpretation:** vectors are directed line segments with a **head** and **tail**

Vector toolbox

- **Scalar multiplication:**
  $\vec{v} \to \lambda\vec{v}$ stretches $\vec{v}$ by $|\lambda|$ and possibly reverses direction.
- **Addition:**
  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, add head $\to$ tail
- **Subtraction:**
  $\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$
- **Magnitude:**
  $|\vec{a}| =$ length of $\vec{a}$, $|\vec{a}| \geq 0$
- **Scalar product:**
  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$
- **Unit vector:**
  $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Vector toolbox



Vector toolbox

Scalar product

The scalar ($\equiv$ dot) product between two vectors is super useful.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$$

In fact, the **scalar product** is what defines the concept of **distance** and **orientation** in our vector space.

For instance, the magnitude of a vector is formally defined as:

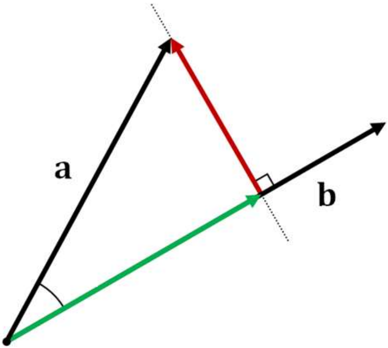
$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

The scalar product we are using is the **Euclidean** inner product. There are other strange beasts out there, though.

Vector toolbox

Vector projections Given two vectors $\vec{a}, \vec{b}$, we can consider $\vec{a}_{\|}$, the projection of $\vec{a}$ onto $\vec{b}$

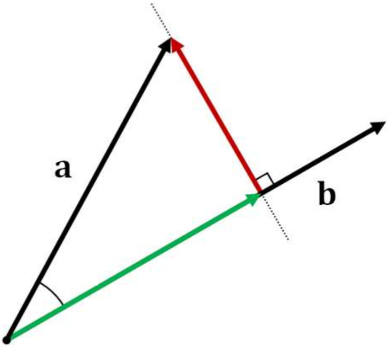
$$\vec{a}_{\|} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\right) \vec{b} = (\vec{a} \cdot \hat{b})\hat{b}$$



Vector toolbox

Vector projection Once we have $\vec{a}_{\parallel}$, $\vec{a}_{\perp}$ can simply be found by subtracting:

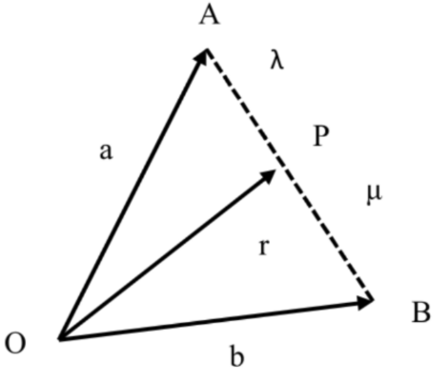
$$\vec{a}_{\perp} = \vec{a} - \vec{a}_{\parallel}$$



Ratio theorem

This is an important theorem...

$$\vec{r} = \frac{\mu\vec{a} + \lambda\vec{b}}{\lambda + \mu}$$



Cartesian representation

We've been speaking about vectors in an abstract manner. In $\mathbb{R}^2, \mathbb{R}^3$, we can **project** a vector $\vec{a}$ onto a **basis** $\{\vec{i}, \vec{j}, \vec{k}\}$

$$
\begin{aligned}
\vec{a} &= (\vec{a} \cdot \vec{i})\vec{i} + (\vec{a} \cdot \vec{j})\vec{j} + (\vec{a} \cdot \vec{k})\vec{k} \\
&= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \\
&= (a_1, a_2, a_3)
\end{aligned}
$$

We say that a_1, a_2, a_3 are **projections** of our vector $\vec{a}$ in the x, y, z directions respectively. Can we derive this from our previous definitions of **vector projections**?

Cartesian representation

Using the **Cartesian representation** of vectors, we find that:

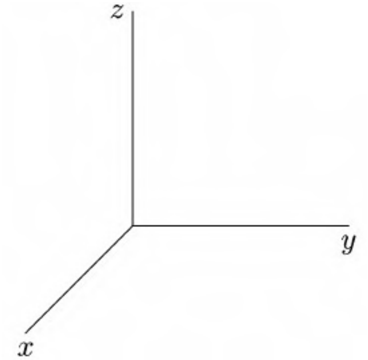
- **Scalar multiplication:** $\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$
- **Addition:**
  $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- **Magnitude:** $|(a_1, a_2, a_3)| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- **Scalar product:**
  $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$

Pythagoras in 3D

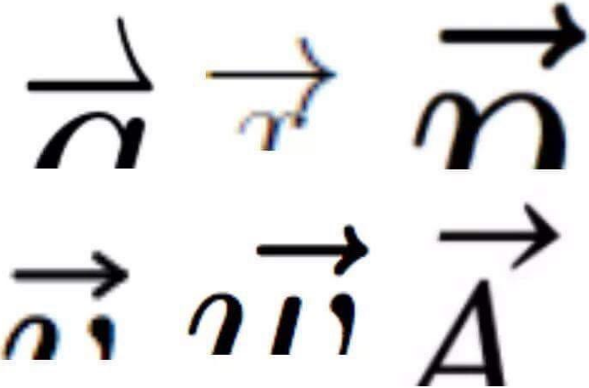
We said previously that

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

This is **Pythagoras' theorem** for 3 dimensions. Proof?



Guys with hair like this have a 125% chance of having **both a magnitude and a direction in space.**



Linear independence

Any set of vectors will either be linearly **dependent** or otherwise **independent**.

Formal definition

$\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent** if there exists k_1, k_2, k_3 **not all zero** such that:

$$k_1\vec{a} + k_2\vec{b} + k_3\vec{c} = \vec{0}$$

Practical definition

$\vec{a}, \vec{b}, \vec{c}$ are **linearly dependent** if we can write **one vector** as a **linear combination** of the others.

E.g. $\vec{c} = m\vec{a} + n\vec{b}$, and so on.

Linear independence

$$\vec{a} = \vec{i} + 3\vec{j} + \vec{k}$$
$$\vec{b} = -2\vec{j} + 5\vec{k}$$
$$\vec{c} = 2\vec{i} + 17\vec{k}$$

$\{\vec{a}, \vec{b}, \vec{c}\}$ are **linearly dependent** since $\vec{c} = 2\vec{a} + 3\vec{b}$

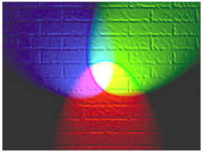
Linear independence

If a set of vectors is **not** linearly dependent, then it is **linearly independent**.

Useful special cases:

- 2 vectors are dependent only if they are **parallel**
- In 2 dimensions, any set of >2 vectors will be dependent
- In 3 dimensions, any set of >3 vectors will be dependent
- Any 3 **coplanar** vectors will be dependent

- Here's an analogy:
- Think of a collection of colours: red, green, blue, purple, white
- Realise that we can narrow this down to just three colours, from which we can reconstruct the rest



Purple = red + blue   White = red + green + blue

- The set {red, green, blue, purple, white} is 'linearly dependent' – purple and white are **linear combinations** of red, green, blue
- Thus, we can reduce our set to just bare basics: {red, green, blue}
- This set is now linearly independent!

- In a similar way... let's look at the set of vectors containing
  $\vec{a} = 2\vec{\imath} + 3\vec{\jmath}, \qquad \vec{b} = -\vec{\imath} + \vec{\jmath} + 2\vec{k}, \qquad \vec{c} = 5\vec{\jmath} + 4\vec{k}, \qquad \vec{d} = \vec{\imath} + 9\vec{\jmath} + 6\vec{k}$
- Are we at bare basics?
- $\vec{c} = 2\vec{b} + \vec{a}$ and $\vec{d} = \vec{a} + \vec{b} + \vec{c} = 2\vec{a} + 3\vec{b}$
- So $\vec{c}$ and $\vec{d}$ can be expressed in terms of $\vec{a}$ and $\vec{b}$ alone!
- $\{\vec{a}, \vec{b}, \vec{c}, \vec{d}\}$ is thus linearly dependent
- We can throw out unnecessary baggage... $\{\vec{a}, \vec{b}\}$ is linearly dependent!

Example: VCAA 2009 Exam 2

Question 14

The vectors $\underset{\sim}{u} = m\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = \underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{w} = \underset{\sim}{i} + \underset{\sim}{j} + m\underset{\sim}{k}$, where m is a real constant, are **linearly dependent** for

A. $m = 0$

B. $m = 1$

C. $m = 2$

D. $m = 3$

E. $m = 4$

Example: VCAA 2009 Exam 1

Question 3

Resolve the vector $5\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ into two vector components, one which is parallel to the vector $-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and one which is perpendicular to it.

3 marks

Challenge problem

Using vector projections, find the point on the line $y = 3x - 2$ that is closest to the point $(4, 2)$. What is the minimal distance between the point $(4, 2)$ and the line?

Complex numbers basics

Definition of a complex number...

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$$

Complex numbers follow the basic arithmetic and algebraic properties we are used to applying to **reals**.

However, we do have some extras...

Complex conjugate

The **conjugate** of a complex number z is denoted $\bar{z}$, and:

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

The conjugate satisfies some **very important** properties...

- $z\bar{z} = (x + iy)(x - iy) = |z|^2$
- $z + \bar{z} = 2\mathrm{Re}(z)$
- $z - \bar{z} = 2i\mathrm{Im}(z)$

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z/w} = \bar{z}/\bar{w}$

Polar form

Complex numbers can be expressed in **polar form** - this makes multiplication, division and exponentiation a lot easier.

$$z = r\mathrm{cis}(\theta) = r(\cos(\theta) + i\sin(\theta))$$

We write $r = |z|$ and $\theta = \arg(z)$

Polar form

Polar form makes some things much easier...

- $z_1 z_2 = r_1 r_2 \mathrm{cis}(\theta_1 + \theta_2)$
- $z_1 / z_2 = \dfrac{r_1}{r_2} \mathrm{cis}(\theta_1 - \theta_2)$
- $z^n = r^n \mathrm{cis}(n\theta) \quad \leftarrow$ De Moivre's Theorem

Problem. Find in **Cartesian form**: $u = \mathrm{cis}(\pi/4), v = \mathrm{cis}(\pi/3)$.

Hence, find the values of $\sin(\pi/12)$ and $\cos(\pi/12)$

Factorisation over $\mathbb{C}$

Fundamental Theorem of Algebra (formal statement)
Every polynomial in P in z:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_2 z^2 + a_1 z + a_0$$

where $a_n \neq 0$ and $a_0, ..., a_n \in \mathbb{C}$ has at least **one** linear factor in $\mathbb{C}$.

Version for p l e b s:
Any polynomial of degree n will have n linear factors over $\mathbb{C}$.

Quadratics: we typically **complete the square**... best shown by example.

Factorise $z^2 + 3z + 4$

For higher order polynomials typically you'll need to use the **remainder theorem** to reduce to a quadratic

Problem: VCAA 2014 Exam 1

Question 3 (5 marks)

Let f be a function of a complex variable, defined by the rule $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

a. Given that $z = i$ is a solution of $f(z) = 0$, write down a quadratic factor of $f(z)$. 2 marks

Problem: VCAA 2014 Exam 1

b. Given that the other quadratic factor of $f(z)$ has the form $z^2 + bz + c$, find all solutions of $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$ in cartesian form. 3 marks

plz don't kill me for this

who would win??!!!1111

An irreducible quadratic

$z^2 + 2x + 2$

one dreamy boi

$z = x + iy$

nth roots

We can find the nth roots of any complex number using de Moivre's Theorem in the following procedure...

$$z^n = a, n \in \mathbb{N}$$

- There will be n solutions $w_1, ..., w_n$ (c.f. Fundamental Theorem of Algebra)
- Solutions will lie on the circle with radius $|a|^{1/n}$ at even intervals of $2\pi/n$.

Example: find the fourth roots of unity.

CHAPTER I.

TO DELIVER YOU FROM THE PRELIMINARY TERRORS.

THE preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means "a little bit of."

Thus dx means a little bit of x; or du means a little bit of u. Ordinary mathematicians think it more polite to say "an element of," instead of "a little bit of." Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

(2) $\int$ which is merely a long S, and may be called (if you like) "the sum of."

Thus $\int dx$ means the sum of all the little bits of x; or $\int dt$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the

From *S. Thompson, Calculus Made Easy (1914)*

Differentiation

Assumed knowledge: most of Methods calculus. For those of you who haven't seen...

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$
$$\frac{d}{dx}e^x = e^x$$

Also the **chain rule**:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

A few new functions...

You will have encountered already...

$$\begin{aligned} \arcsin(x) &= \sin^{-1}(x) \\ \arccos(x) &= \cos^{-1}(x) \\ \arctan(x) &= \tan^{-1}(x) \end{aligned}$$

We seek their **derivatives...**

$$\begin{aligned} \arcsin'(x) &= \frac{1}{\sqrt{1-x^2}} \\ \arccos'(x) &= \frac{-1}{\sqrt{1-x^2}} \\ \arctan'(x) &= \frac{1}{1+x^2} \end{aligned}$$

Let's derive these ...

$$\arctan'(x) = \frac{1}{1+x^2}$$

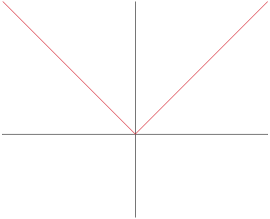
Let $y = \tan(x), \quad x \in (-\pi/2, \pi/2), \quad y \in \mathbb{R}$. Thus $x = \arctan(y)$.

The modulus function

You've probably seen already the modulus function...

$$|x| = x\mathrm{sgn}(\mathrm{x}) = \text{'positive value of x'}$$

From the graph of $|x|$ we see that our function is **continuous everywhere**, and **differentiable** away from the origin.



$$\frac{d}{dx}|x| = \begin{cases} +1, x > 0 \\ -1, x < 0 \end{cases}$$

Implicit differentiation

So far, we have been able to differentiate expressions where y is **explicitly** related to x.

Implicit relations allow us to find derivatives where y and x are **implicitly** tangled together...

for example in the expression $x\cos(y) + y\cos(x) = 1$

Quick rule: to find dy/dx, differentiate as per usual w.r.t x, using chain/product/quotient rules as required, but write

$$\frac{d}{dx}(y) = \frac{dy}{dx} \quad \text{(obviously!!!)}$$

Example: given that $x\cos(y) + y\cos(x) = 1$, find an expression for dy/dx.

Here's a cheap way to do the same thing...

To differentiate some relation, rearrange into the form $F(x, y) = 0$ and then find **partial derivatives** F_x, F_y. From this, we have:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

- To find F_x, differentiate $F(x, y)$ w.r.t x, treating y like a **constant**
- To find F_y, differentiate $F(x, y)$ w.r.t y, treating x like a **constant**

Concavity and the second derivative test

- We know that **stationary points** can be a local **maximum**, **minimum**, or **point of inflection**
- **Without** sketching a graph, how would we be able to classify stationary points of a graph?
- We first introduce the idea of **concavity**

Concavity

Concavity of a curve at a given point tells us whether the curve is 'curving upwards' or 'curving downwards'

- Consider the second derivative $f''(x) = \dfrac{d^2 f}{dx^2}$
- **Case:** $f''(x) > 0 \Rightarrow$ concave up (slope is increasing)
- **Case:** $f''(x) < 0 \Rightarrow$ concave down (slope is decreasing)
- **Case:** $f''(x) = 0 \Rightarrow$ zero concavity, i.e. locally our curve has no 'curvature'

Second derivative test

Thus, given a **stationary point** at $x = x_0$, we can check the value of $f''(x_0)$ to determine the type of stationary point.

- $f''(x_0) > 0 \Rightarrow$ concave up $\Rightarrow$ local minimum
- $f''(x_0) < 0 \Rightarrow$ concave down $\Rightarrow$ local maximum
- $f''(x_0) = 0 \Rightarrow$ no concavity $\Rightarrow$ inconclusive

This is the **second derivative test.**

Second derivative test

What if $f''(x_0) = 0$?

If f'' **switches sign** around x_0...

- We know that $y = f(x)$ changes its concavity around x_0
- Thus $y = f(x)$ has a **point of inflection** at x_0
- Depending on whether $f'(x_0) = 0$
  this could be a **stationary point of inflection**
  or a **non-stationary point of inflection**

Second derivative test

What if $f''(x_0) = 0$?

In other cases, we actually **can't make any conclusions**!

- Consider $y = x^4$ and $y = x^5$.
- **Both** of these satisfy $dy/dx = d^2y/dx^2 = 0$ at $x = 0$
  - $y = x^4$ has a local minimum
  - $y = x^5$ has an inflection.
- **Thus**, in general we will need **further investigation** in these cases

Second derivative test

Problem: find and classify all the stationary points of $y = x^2 e^{-x}$

Integration techniques

We will follow on from what we learned in Methods.

Definition of antiderivative

For a (sufficiently nice) function $f(x)$, we say $F(x)$ is an antiderivative of f if

$$\frac{dF}{dx} = f$$

Integration techniques

Connection between definite integrals and the antiderivative

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is **any** antiderivative of f.

Our definite integral itself can be interpreted as the (signed) area bounded by f and the x-axis on the interval $[a, b]$

Substitution of variables

This is a very powerful technique that allows us to compute **indefinite integrals** ($\equiv$ antiderivatives) of a wide range of functions.

Consider an integral of the general form:

$$\int u'(t)f(u(t))dt$$

We can make a substitution $v = u(t) \Rightarrow dv = u'(t)dt$ This brings us to

$$\int u'(t)f(u(t))dt = \int u'(t)f(u(t))\frac{dv}{u'(t)} = \int f(v)dv$$

Substitution of variables

Problem: Compute

$$\int xe^{x^2}\,dx$$

Substitution of variables

Problem: Compute
$$\int \frac{3 \ dx}{x \ln^2(x)}$$

Substitution of variables

We can also perform substitution of variables for **definite integrals** - we just need to also substitute the terminals.

Substitution of variables

Problem (VCAA 2017 Exam 1): Find

$$\int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)}$$

Integration of trig functions

Important identities

- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$
- $\sin(2x) = 2\sin(x)\cos(x)$

Example: $\displaystyle\int \sin^2(x)\cos^2(x)\;\mathrm{d}x$