Homework 2

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(A) a) $A|B = \{x \in \mathbb{Q}^{\geq 0} \mid x^2 < 7\} \mid \{x \in \mathbb{Q}^{\geq 0} \mid x^2 \geq 7\}$ $\sqrt{7}$

b) $A|B = \{x \in \mathbb{Q}^{\geq 0} \mid x^3 < 9\} \mid \{x \in \mathbb{Q}^{\geq 0} \mid x^3 \geq 9\}$ $\sqrt[3]{9}$

c) $A = \{x \in \mathbb{Q}^{\geq 0} \mid x > 2 \wedge x^2 > 3\}$ $\sqrt{3}+2$

$B = \mathbb{Q}^{\geq 0} \setminus A$

(B) $A|B = \left([0, \frac{x_1}{y_1}) \mid [\frac{x_1}{y_1}, \infty)\right)$ $\quad E|F = \left([0, \frac{x_3}{y_3}) \mid [\frac{x_3}{y_3}, \infty)\right)$

$C|D = \left([0, \frac{x_2}{y_2}) \mid [\frac{x_2}{y_2}, \infty)\right)$ $\quad \frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3} \in \mathbb{Q}^{\geq 0}$

want to show $\underbrace{A|B \cdot C|D}$ $= C|D \cdot A|B$

LS $\{a \cdot b \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2}\} = \left([0, a \cdot b) \mid [a \cdot b, \infty)\right)$

RS $\{b \cdot a \mid 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq a < \frac{x_1}{y_1}\} = \left([0, b \cdot a) \mid [b \cdot a, \infty)\right)$

commutativity in $\mathbb{R}^{\geq 0}$ $\quad a \cdot b = b \cdot a \rightarrow$ property of $\mathbb{Q}^{\geq 0}$

$\Rightarrow \left([0, a \cdot b) \mid [a \cdot b, \infty)\right) = \left([0, b \cdot a) \mid [b \cdot a, \infty)\right)$

want to show $\underbrace{(A|B \cdot C|D) \cdot E|F}$ $= A|B \cdot (C|D \cdot E|F)$

LS $\{(a \cdot b) \cdot c \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= \left([0, (a \cdot b) \cdot c) \mid [(a \cdot b) \cdot c, \infty)\right)$

RS $\{a \cdot (b \cdot c) \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= \left([0, a \cdot (b \cdot c)) \mid [a \cdot (b \cdot c), \infty)\right)$ $\quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \rightarrow$ property of $\mathbb{Q}^{\geq 0}$

associativity in $\mathbb{R}^{\geq 0}$

$\Rightarrow \left([0, a \cdot (b \cdot c)) \mid [a \cdot (b \cdot c), \infty)\right) = \left([0, (a \cdot b) \cdot c) \mid [(a \cdot b) \cdot c, \infty)\right)$

(C) $x_1 = A|B = \left([0, \frac{x_1}{y_1}) \mid [\frac{x_1}{y_1}, \infty)\right)$ $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3} \in \mathbb{Q}^{\geq 0}$

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$x_2 = C|D = \left([0, \frac{x_2}{y_2}) \mid [\frac{x_2}{y_2}, \infty)\right)$

$x_3 = E|F = \left([0, \frac{x_3}{y_3}) \mid [\frac{x_3}{y_3}, \infty)\right)$

want to show $((A|B) + (C|D)) \cdot (E|F) = (A|B)\cdot(E|F) + (B|C)\cdot(E|F)$

LS: $\left\{ (a+b)\cdot c \mid 0 \leq a < \frac{x_1}{y_1} \land 0 \leq b < \frac{x_2}{y_2} \land 0 \leq c < \frac{x_3}{y_3} \right\}$

$= \left([0, (a+b)\cdot c) \mid [(a+b) c, \infty)\right)$

RS $\left\{ ac + bc \mid 0 \leq a < \frac{x_1}{y_1} \land 0 \leq b < \frac{x_2}{y_2} \land 0 \leq c < \frac{x_3}{y_3} \right\}$

$= \left([0, ac + bc) \mid [ac + bc, \infty)\right)$

distributivity in $\mathbb{Q}^{\geq 0}$ $(a+b)\cdot c = ac + bc$ - Property of $\mathbb{Q}^{\geq 0}$

$\Rightarrow \left([0, (a+b)\cdot c) \mid [(a+b)\cdot c, \infty)\right) = \left([0, ac+bc) \mid [ac+bc, \infty)\right)$

(D) S, T - bounded sets $S\cdot T = \{x \cdot y : x \in S \land y \in T\}$

Show that if S and T are bounded so is $S\cdot T$

$s \in S$ $t \in T$

$\inf s = A_* | B_*$ $\inf t = C_* | D_*$

$s = A_s | B_s$ $t = C_t | D_t$

$\sup s = A | B$ $\sup t = C | D$

$\inf s \cdot \inf t = \inf$ (infimum of $S \cdot T$)

$\sup s \cdot \sup t = \sup$ (supremum of $S\cdot T$)

$\inf s < s < \sup s$ $\inf t < t < \sup t$

$A_* \subseteq A_s \subseteq A$ $C_* \subseteq C_t \subseteq C$

$\inf s \cdot \inf t \leq s \cdot t < \sup s \cdot \sup t$

E. Prove that $\mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$ is a field (Use properties of $\mathbb{R}$)

Properties of a field:

1 → $(\mathbb{Q}\sqrt{2}, +)$ → abelian group (Closed, Commutative, associative, inverse and identity)

2 → $(\mathbb{Q}\sqrt{2}, \cdot)$ → abelian group

3 → $(\mathbb{Q}\sqrt{2}, +, \cdot)$ → distributative law

1. $(a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}) = \underbrace{(a_1+a_2)}_{\mathbb{Q}}+\underbrace{(b_1+b_2)}_{\mathbb{Q}}\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ → closure

- Commutativity and associativity apply because $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and those are properties of $\mathbb{Q}$ (and $\mathbb{R}$)

additive identity: $0+0\cdot\sqrt{2}=0$

inverse: $a_1+b_1\sqrt{2} - a_1 - b_1\sqrt{2} = 0$

2. $(a_1+b_1\sqrt{2})\cdot(a_2+b_2\sqrt{2}) = (a_1a_2)+(a_1b_2\sqrt{2})+(a_2b_1\sqrt{2})+2b_1b_2 =$

$= \underbrace{(a_1a_2)}_{\mathbb{Q}}+\underbrace{(a_1b_2+a_2b_1)}_{\mathbb{Q}}\sqrt{2}+\underbrace{2b_1b_2}_{\mathbb{Q}} \in \mathbb{Q}(\sqrt{2})$ - Closure

- Commutativity and associativity apply because $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and these are properties of $\mathbb{Q}$ (and $\mathbb{R}$)

multiplicative identity: 1

inverse: $a_1+b_1\sqrt{2} \cdot \frac{1}{a_1+b_1\sqrt{2}} = 1$

3. $((a_1+b_1\sqrt{2})+(a_2+b_2\sqrt{2}))\cdot(a_3+b_3\sqrt{2}) =$

$= (a_1+b_1\sqrt{2})\cdot(a_3+b_3\sqrt{2})+(a_2+b_2\sqrt{2})\cdot(a_3+b_3\sqrt{2}) \in \mathbb{R}$

→ distributativity holds because it holds in $\mathbb{R}$