hw9 by Zhivko Stoimchev 89221056

A. Calculate each of the following limits, or explain why it does not converge. You may use any theorems that were stated in lecture.

i) $\lim\limits_{x \to 1+} |x-2|/(x-2)$

(using math logic, we calculate for 1):

$\lim |1-2|/(1-2) = |-1|/-1 = \frac{1}{-1} = -1$, so $L = -1$, and the limit ~~conv~~ converges

ii) $\lim\limits_{x \to 1+} |x-1|$

(using math logic, we calculate for 1):

$\lim\limits_{x \to 1+} |x-1| = |1-1| = 0$, so $L = 0$, and the limit converges.

iii) $\lim\limits_{x \to 1+} |x-1|/(x-1)^2$

(using math logic, we calculate for 1):

$\lim\limits_{x \to 1+} |x-1|/(x-1)^2 = |1-1|/(1-1)^2 = 0/0$ $\Rightarrow$ this means we need another approach!

Because x approaches 1 from the right side, we can lose the absolute value, and we have:

~~$\lim\limits_{x \to 1+} |x-1|/(x-1)^2 = \lim\limits_{x \to 1+} (x-1)/(x-1)^2 = \lim\limits_{x \to 1+} 1/(x-1)$~~

$\lim\limits_{x \to 1+} |x-1|/(x-1)^2 = \lim\limits_{x \to 1+} (\cancel{x-1})/(x-1)^{\cancel{2}} = \lim\limits_{x \to 1+} 1/(x-1)$

from the above we have that $x > 1$, and $x - 1 > 0$.

since $x-1$ is positive, it also approaches 0 from the right side, we conclude that the function

$$\lim_{x \to 1+} 1/x-1 = \infty,$$

increases without bound.

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B. i) Prove directly from the definitions that if $g: \mathbb{R} \to [1, \infty)$ is a function so that $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} 1/g(x) = 1/L$

First, we need to assume that $g: \mathbb{R} \to [1, \infty)$ is a function with $\lim_{x \to c} g(x) = L$. Now we get that:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } [0 < |x-c| < \delta] \Rightarrow [|g(x) - L| < \varepsilon]$$

Now we calculate:

$$0 < |g(x)| - |L| \leq |g(x) - L| < \varepsilon$$

$$|g(x)| - |L| < \varepsilon \quad \text{, from which we get}$$

$$-\varepsilon < |g(x)| - |L| < \varepsilon \quad \left(\text{means } \varepsilon = \frac{L}{2}\right)$$

$$-\frac{L}{2} < |g(x)| - |L| < \frac{L}{2}$$

$$L - \frac{L}{2} < |g(x)| < \frac{L}{2} + L$$

$$\frac{L}{2} < |g(x)| < \frac{3L}{2}$$

$$\frac{L}{2} < |g(x)|$$

$$\frac{1}{g(x)} < \frac{2}{L}$$

From the above we have:

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| < \frac{2}{L} - \frac{1}{L} = \frac{1}{L} \qquad \left(\frac{1}{L} < \frac{L}{2}, \text{ where } \frac{L}{2} = \varepsilon\right)$$

and for $L > \sqrt{2}$, it means that $\left|\frac{1}{g(x)} - \frac{1}{L}\right| < \varepsilon$, so $2 < L^2$, for $\forall L > \sqrt{2}$

ii) Prove the same result of the previous part, using Relating Sequences to Functions.

$\lim_{x \to c} f(x) = L$, where $\lim_{x \to c} 1/f(x) = 1/L$, which we want to prove

RSTF implies that if $\lim_{x \to c} f(x) = L$, then

$\forall a_n$ in the domain of $f \wedge c \notin a_n$, we have

$\lim_{n \to \infty} a_n = c$, which implies that $\lim_{n \to \infty} f(a_n) = L$

From the definition, we have

$\forall \varepsilon > 0, \exists N > 0, \text{ s.t. } [n > N] \Rightarrow [|a_n - c| < \varepsilon]$

now we calculate:

$$|a_n - c| < \varepsilon$$

$$|a_n| - |c| \leq |a_n - c| < \varepsilon \quad \text{, where } \varepsilon = \frac{c}{2}$$

$$|a_n| - |c| < \varepsilon$$

$$-\varepsilon < |a_n| - |c| < \varepsilon$$

$$-\frac{c}{2} < |a_n| - |c| < \frac{c}{2}$$

$$\frac{c}{2} < |a_n| < \frac{3c}{2}$$

$$\underline{\frac{1}{a_n} < \frac{2}{c}}$$

From the above we have:

$$\left|\frac{1}{f(a_n)} - \frac{1}{f(c)}\right| = \left|\frac{1}{f(a_n)} - \frac{1}{c}\right| < \frac{2}{f(c)} - \frac{1}{L} = \frac{2}{L} - \frac{1}{L} = \frac{1}{L}$$

so, $\frac{1}{L} < \frac{L}{2}$, which means $L^2 < 2 \Rightarrow$ for $\forall L > \sqrt{2}$.

and for $L > \sqrt{2}$, we have $\left|\frac{1}{f(a_n)} - \frac{1}{L}\right| < \varepsilon$

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C. Prove that $\lim_{x\to c} f(x)=\infty$ if and only if for every sequence a_n that is in the domain of f and converges to c (but never equal to c), we have $\lim_{n\to\infty} f(a_n)=\infty$. (That is, prove a version of Relating sequences to functions for infinite limits. Make sure you handle both directions of the if and only if).

We need to prove both $(\Rightarrow)$ and $(\Leftarrow)$.

$\Rightarrow$ For $\lim_{x\to c} f(x)=\infty$, we have def.

$$\forall M>0,\ \exists \delta>0 \text{ s.t. } [0<|x-c|<\delta] \Rightarrow [f(x)>M]$$

For $\lim_{n\to\infty} a_n=c$, we have the def.

$$\forall \varepsilon>0,\ \exists N \text{ s.t. } [n\geq N] \Rightarrow [|a_n-c|<\varepsilon]$$

Now we take arbitrary $\varepsilon=\delta$, and we get the following:

$$\forall M>0,\ \exists \varepsilon \text{ s.t. } \underbrace{[0<|x-c|<\varepsilon]}_{\text{this is } \lim_{n\to\infty} a_n=c} \Rightarrow |f(x)>M|$$

$$\forall M>0,\ \exists \varepsilon \text{ s.t. } [|a_n-c|<\varepsilon] \Rightarrow |f(x)>M|$$

because (a_n) is the domain of f, we have

$$\forall M>0,\ \exists \varepsilon \text{ s.t. } (|a_n-c|<\varepsilon] \Rightarrow \cancel{f(a_n)}\ |f(a_n)>M|$$

so $\lim_{n\to\infty} f(a_n)=\infty$

$\square$

$\Leftarrow$ First assume that every sequence of a_n provides $\lim_{x \to c} f(x) \neq \infty$, so there isn't δ to satisfy

$[0 < |x-c| < \delta] \Rightarrow [f(x) > M]$ for $M > 0$.

From the above we conclude that for each n there is:

$0 < |a_n - c| < n$, but $f(a_n) \geq M$. (from previous)

Since $\lim_{n \to \infty} a_n = c$ and $\lim_{n \to \infty} f(a_n) = \infty$, we have that

for large enough n, we have $f(a_n) > M$,

that proves our contradiction,

this proves that $\lim_{x \to c} f(x) = \infty$. ☒ Proof by contradiction.

D. Which of the following families of real functions are algebras of functions? (For each, either show it is an algebra of functions, or else identify a necessary property that is not satisfied).

i) Polynomials of degree at most 2. (That is, polynomials of the form ax^2+bx+c, where any or all of a, b, c may be 0.)

a) contains constant function 1.

b) $c \cdot f(x)$, where $c \in \mathbb{R}$, forms a constant function

c) $f(x)+f(y)$ forms a constant function

d) $f(x)-f(y)$ forms a constant function

c) and d) come from AoL.

ii) Functions f s.t. $f(x) \leq 6$.

We know that range can be changed, which implies that $c \cdot f(x)$, where $c \in \mathbb{R}$, can give ~~[illegible]~~ greater number than 6, which won't be in A

From the above, we conclude that A is not algebra of functions

iii) Functions f that are everywhere defined, and that are continuous at the point $c=2$.

For this, we need to show:

a) constant function $1 \in A$, because f is continuous at $c=2$.

b) $c \cdot f(x) = g(x)$. $g(x)$ is stretched version of $f(x)$, and will also be continuous at $c=2$

c) $f(x), g(x) \in A$, are functions that are continuous at $c=2$. then $f(x) + g(x)$ is also going to be continuous at $c=2$, which implies that $f(x)+g(x) \in A$.

d) $f(x), g(x) \in A$, functions continuous at $c=2$, because $f(x) \cdot g(x)$ is also continuous at $c=2$ (by AoL), from which we get that $f(x) \cdot g(x) \in A$.

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E. Prove that $f(x) = x \cdot |x|$ is continuous at all points c in $\mathbb{R}$

we need to show that $\lim_{x \to c} f(x) = f(c)$

Lets point $c \in \mathbb{R}$.

if $c < 0$, then

$\lim_{x \to c} x \cdot |x| = \lim_{x \to c} x \cdot (-x) = -\lim_{x \to c} x^2 = c^2 = c(-c) = c \cdot |c| = f(c).$

if $c > 0$, then

$\lim_{x \to c} x \cdot |x| = \lim_{x \to 0} x \cdot x = \lim_{x \to 0} x^2 = c^2 = c \cdot c = c \cdot |c| = f(c).$

if $c = 0$, then

for 0^+: $\lim_{x \to 0^+} x|x| = \lim_{x \to 0^+} x \cdot x = \lim_{x \to 0^+} x^2 = 0^2 = 0 = c \cdot |c| = f(c) = f(0)$

for 0^-: $\lim_{x \to 0^-} x|x| = \lim_{x \to 0^-} x(-x) = \lim_{x \to 0^-} x^2 = -0^2 = 0 = c \cdot |c| = f(0)$

With this we prove that $f(x) = x \cdot |x|$ is continuous at all points c in $\mathbb{R}$!

$\square$