Analysis I
UP - FAMNIT
Fall 2022 October 26/27, 2022

Selected exercises 04

1. Let t_n be a bounded sequence and let s_n be a sequence such that $\lim s_n = 0$. Prove that $\lim s_n t_n = 0$.
2. Suppose s_n and t_n are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove that $\lim s_n = 0$.
3. Let $a, b, c \in \mathbb{R}$. Prove that $|a - b| < c$ if and only if $b - c < a < b + c$.
4. Prove that if a_n and b_n are sequences such that $a_n \leq b_n$ for all but finitely many values of n and $a_n \to a$, $bn \to b$, then $a \leq b$. ($a_n \to a$ means that $\lim_{n\to\infty} a_n = a$).
5. Give an example of each of the following or prove that such a request is impossible.
   (a) Unbounded convergent sequence.
   (b) Divergent sequences s_n and t_n, but whose sum $s_n + t_n$ converges.
   (c) A sequence with an infinite number of 1's that does not converge to 1.
   (d) An unbounded sequence a_n and a convergent sequence b_n with $a_n - b_n$ bounded.
6. Prove that every convergent sequence is bounded.
7. If $\lim b_n = B \neq 0$, prove that there exists a number N such that $|b_n| \geq \frac{1}{2}|B|$ for all $n > N$.
8. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B \neq 0$, prove
   (a) $\lim\limits_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{B}$ (b) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$
9. Show that if $a_n \to a$, then the sequence of absolute values $|a_n|$ converges to $|a|$. Is the opposite true? if we know that $|a_n| \to |a|$, can we deduce that $a_n \to a$?