hw8 Zhivko Stoimchev 89221056

A. Let H be the set of all points (x,y) in $\mathbb{R}^2$ s.t. $x^2+xy+3y^2=3$. Show that H is closed subset of $\mathbb{R}^2$ (considered with the Euclidean metric). Is H bounded?

$H=\{(x,y) \mid x^2+xy+3y^2=3\}$

Now we let (x_n,y_n) be $\to (x,y)$ and $(x_n,x_n)\in H$ s.t. $x_n^2+x_ny_n+3y_n^2=3$. Now we take it as a limit:

$\lim_{n\to\infty}(x_n^2+x_ny_n+3y_n^2)=3 \Rightarrow x^2+xy+3y^2=3 \Rightarrow (x,y)\in H$

To check if it's bounded: (using math logic):

$x^2+xy+3y^2=3$

$\Rightarrow \frac{1}{2}x^2+\frac{1}{2}(x+y)^2+\frac{5}{2}y^2=3/\cdot 2$

$x^2+(x+y)^2+5y^2=6$

$x^2\le 6$ and $5y^2\le 6$

$\Rightarrow |x|<\sqrt{}$ and $|y|\le\sqrt{\frac{6}{5}}$

which means $(x,y)\in [-\sqrt{6},\sqrt{6}]\times[-\sqrt{\frac{6}{5}},\sqrt{\frac{6}{5}}] \to$ this is M

from which we conclude ~~that $H\in M$.~~

$(x_n,y_n)\to(x,y) \Leftrightarrow x_n\to x \wedge y_n\to y$

Now we prove:

we prove in opposite direction $(\Leftarrow)$: Let $x_n\to x \wedge y_n\to y$

so we have: $\forall\varepsilon_1\ \exists n_1\ \forall n>n_1,\ |x_1-x|<\varepsilon_1=\frac{\varepsilon}{\sqrt{2}}$

$\forall\varepsilon_1\ \exists n_2\ \forall n>n_2\ |y_n-y|<\varepsilon_1=\frac{\varepsilon}{\sqrt{2}}$

Now we let $\varepsilon>0$ and $\exists n\ge\max\{n_1,n_2\}$.

so we have $\sqrt{(x_n-x)^2+(y_n-y)^2}\le\sqrt{\varepsilon_1^2+\varepsilon_1^2}=\sqrt{2}\cdot\varepsilon_1=\varepsilon$

$\Rightarrow$ Now we let: $\forall\varepsilon_1>0,\ \exists n_1,\ \forall n>n_1,\ \sqrt{(x_n-x)^2+(y_n-y)^2}<\varepsilon$.

Now let $\varepsilon>0$, $\exists n_0=n_1$ $\forall n>n_0$

$|x_n-x|\le\sqrt{(x_n-x)^2+(y_n-y)^2}<\varepsilon$ which means that

$\Rightarrow \lim_{n\to\infty}x_1=x$

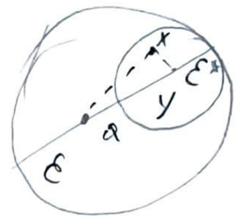
hw8 Zhivko Stoimchev 89221056

B. Given a metric space M with metric d, verify that any ε-ball is an open set.

- Let point $a \in M$.

From the ε-ball def, we have

$$B_\varepsilon(a) = \{x \in M \mid d(x, a) < \varepsilon\}, \text{ or } B_\varepsilon^{(d)}(a).$$



From taking ε^* as follows:

$\varepsilon^* = \varepsilon - d(y, a)$

and $x \in B_\varepsilon(y)$

↳ we get $d(x, y) < \varepsilon^*$

From Δ-inequality, we get.

$d(x, a) \le d(x, y) + d(y, a) < \varepsilon - d(y, a)$, which is equal to ε^*

$d(x, a) \le d(x, y) + d(y, a) < \varepsilon^* + d(y, a)$, —— // —— to ε

or $d(x, a) < \varepsilon$

From the above we get that $B_\varepsilon^{(d)} a$ is open ball.

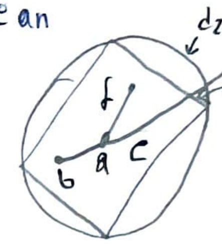
We also have that $B_\varepsilon^{(d)}(y) \subseteq B_\varepsilon^{(d)}(a)$.

With this, from the open set def, we have that $B_\varepsilon^{(d)}(a)$ is open set, which means that any ε-ball is an open set.

hw8 Zhivko Stoimchev 89221056

C. Show that a set in $\mathbb{R}^2$ is open in the Euclidean metric $\Leftrightarrow$ it is open in the max metric.

Hint: As usual, there are two directions to prove in an $\Leftrightarrow$. The picture on p73 of the notes may be helpful.



Let A be a set, and $\forall$ point $a \in A$, for $\varepsilon > 0$, there is δ such that $\delta > 0$. And, we apply the def. for open sets, where we choose $\delta < \varepsilon$, from which we get $d_1(a,b) < \delta \Rightarrow d_2(a,b) < \varepsilon$ (*)

and $d_1(a,b) < \delta \Leftarrow d_2(a,b) < \varepsilon$ (**)

prove that these metrices definned the same open set A.

(*) $d_1(a,b) < d_2(a,b)$. then $B_1(a,\delta) \subseteq B_2(a,\varepsilon)$

If A is an open set in d_1, $\forall a \in A$, $\delta > 0 \wedge \varepsilon > 0$,

$B_1(a,\delta) \subseteq A \Rightarrow B_2(a,\varepsilon) \subseteq A$.

which means A is open set.

(**) now, $d_2(a,b) \leq d_1(a,b)$, then $B_2(a,\varepsilon) \subseteq B_1(a,\delta)$.

If A is an open set in d_2, ~~for~~ $\forall a \in A$, $\delta > 0 \wedge \varepsilon > 0$,

$B_2(a,\varepsilon) \subseteq A \Rightarrow B_1(a,\delta) \subseteq A$.

which means A is open set

hw8 Zhivko Stoimchev 89221056.

D. By showing that any sequence in $A \cup L$ has the same limit as some sequence in A, prove that $\bar{A} \subseteq A \cup L$, where L is the set of acc. points of sequences in A.

By knowing that L is the set of acc. points of a seq. in A, we have that

$A \cup L$ contains all the points a_i,
and $A \cup L$ contains the limit of seq. in A.

Because $A \cup L$ contains all limit points of A, we conclude that $A \cup L$ is closed.

By letting $a \in \bar{A}$ (for our example), we have $a \in A$, which is the same as $a \in \bar{A} \setminus A$.

Since $a \in A$ is trivial, we work with $a \in \bar{A} \setminus A$, and we get that $a \notin A$ and $a \in \bar{A}$.

And because A doesn't contain limit points, we conclude that a is a limit of point of A, where $a \in L$.

So, from $a \in \bar{A}$ and $a \in L$, we conclude that $\bar{A} \subseteq A \cup L$.

hw3 Zhivko Stoimchev 89221056

ŠTUDENTSKA ORGANIZACIJA UNIVERZE NA PRIMORSKEM

E. Show that both $\mathbb{R}$ and $\mathbb{R}^2$ may be covered by countably many open balls

We have $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$

$I_n = (r_n - 1, r_n + 1)$, where $r_n \in \mathbb{Q}$

$\bigcup I_n = \mathbb{R}$ s.t. $\Rightarrow I_n \subseteq \mathbb{R} \wedge \bigcup I_n \subseteq \mathbb{R}$

(going backwards)

$(\Leftarrow)$ $x = \mathbb{R}$ s.t. $\Rightarrow (x - \frac{1}{3}, x + \frac{1}{3}) \cap \mathbb{Q} \neq 0$, where r_x is the cross-section!

which means that $\exists r_x \in \mathbb{Q}$ and $r_x \in (x - \frac{1}{3}, x + \frac{1}{3})$.

where $\Rightarrow x \in (r_x - 1, r_x + 1)$ for some r_x.

Because of $|r_x - x| < \frac{1}{3} < 1$, we have that $\mathbb{R} \subseteq \bigcup I_n$

$\mathbb{Q}^2 = \bigcup_{n=1}^{\infty} \{(x_n, y_n)\}$

$I_n = B(x_n, y_n)$

$\Rightarrow 1 = \{(x, y) \mid \sqrt{(x - x_n)^2 + (y - y_n)^2} < 1\}$

$\bigcup I_n = \mathbb{R}^2$

(going backwards)

$(\Leftarrow)$ We now let $(x, y) \in \mathbb{R}^2$, and get

$B(x, y)(\frac{1}{3}) = \{x', y') \mid d(x', y'), (x, y)) < \frac{1}{3}\}$

$B(x, y)(\frac{1}{3}) \cap \mathbb{Q}^2 \neq 0$

from which we can conclude (using math logic that):

$(z_1, z_2) \in \mathbb{Q}^2$ and $\sqrt{(x - z_1)^2 + (y - z_2)^2} < \frac{1}{3} < 1$

to have at last

$d((x, y), (z_1, z_2)) < 1$ which means that ~~(x, y) are close~~ $(x, y) \in I_n$