Week 14

Consistency

Each of the following is equivalent to saying $[A|b]$ is consistent:

- In row reducing, a row of the form $(0\ 0 \cdots 0 \mid \alpha)$, $\alpha \neq 0$ never appears
- b is a nonbasic column in $[A|b]$
- $\text{rank}[A|b] = \text{rank}(A)$
- b is a combination of the basic columns in A

Homogenous Systems

A system of m linear equations with n unknowns in which the right-hand side consists entirely of 0's is said to be a <u>homogenous system</u>.

* Consistency is never an issue with homogenous systems

Gauss-Jordan method

- At each step, the pivot element is forced to be 1
- A each step, all terms above and below the pivot are eliminated

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_n \end{array}\right]$$

here, $x_i = s_i$

Summary

Let $[A|b]$ be the augmented matrix for a consistent $m \times n$ non-homogenous system in which $\text{rank}(A) = r$.

- $[A|b]$ has a general solution of the form

$$\vec{x} = \vec{p} + x_{f_1}\vec{h}_1 + x_{f_2}\vec{h}_2 + \cdots + x_{f_{n-r}}\vec{h}_{n-r}$$

where the free variables x_{f_i} range over all possible values.

- Column $\vec{p}$ is a particular solution of the non-homogenous system
- The expression $x_{f_1}\vec{h}_1 + \cdots + x_{f_{n-r}}\vec{h}_{n-r}$ is the general solution of the associated homogenous system $[A|\vec{0}]$.
- Column $\vec{p}$ as well as the columns $\vec{h}_i$ are independent of the row-echelon form to which $[A|b]$ is reduced.
- The system contains a unique solution if and only if any of the following is true:
  * $\text{rank}(A) = n$ = # of unknowns
  * there are no free variables
  * the associated homogenous system possesses only the trivial solution $(x_1 = x_2 = \cdots = x_n = 0)$

Effects of row operations on determinants

Let B be the matrix obtained from $A_{n \times n}$ by one of the three elementary row operations:

Type I: interchange rows i and j
Type II: multiply row i by $\alpha \neq 0$
Type III: add α times row i to row j

The value of $\det(B)$ is as follows:

- $\det(B) = -\det(A)$ for Type I operations
- $\det(B) = \alpha \det(A)$ for Type II operations
- $\det(B) = \det(A)$ for Type III operations

Note: $\det(A^T) = \det(A)$

This implies that column operations can be applied to a matrix while calculating its determinant with the same consequences as row operations.

Invertibility and Determinants

- $A_{n\times n}$ is invertible if and only if $\det(A) \neq 0$
- $A_{n\times n}$ does not have an inverse if and only if $\det(A) = 0$

Product Rules

- $\det(AB) = \det(A)\det(B)$

  * a priori, $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D)$ if A and D are square

Block Determinants

If A and D are square matrices, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{cases} \det(A)\det(D - CA^{-1}B) & \text{when } A^{-1} \text{ exists} \\ \det(D)\det(A - BD^{-1}C) & \text{when } D^{-1} \text{ exists} \end{cases}$$

The matrices $D - CA^{-1}B$ and $A - BD^{-1}C$ are called the Schur complements of A and D, respectively.

Rank-One Update

For u, v both $n \times 1$ matrices (i.e. column vectors)

- $\det(I + uv^T) = 1 + v^T u$
- $\det(A + uv^T) = \det(A)(1 + v^T A^{-1} u)$

Cramer's Rule

In a nonsingular system $A_{n\times n}x = b$, the i^{th} unknown is

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where $A_i = [A_{*1}] \cdots [A_{*i-1}]\ b\ [A_{*i+1}] \cdots [A_{*n}]$.

That is, A_i is identical to A except in the i^{th} column, which has been replaced with b.

① Line $p: (1,0,0) + \lambda(1,1,1)$.
Find points on p which are equidistant from planes
$\Sigma: x+y-z=-1$ and $\pi: x-y+z=5$.

<u>Solution:</u> A point on line p (call the point t) is of the form $t(1+\lambda, 1, 1)$.

<u>Want:</u> $d(t,\Sigma) = d(t,\pi)$. <u>Know:</u> $\vec{n}_\Sigma = (1,1,-1)$, $d_\Sigma = -1$
$\vec{n}_\pi = (1,-1,1)$, $d_\pi = -5$

Using formula of distance between a point and a plane:

$$d(t,\Sigma) = \frac{|\vec{n}_\Sigma \cdot t - d_\Sigma|}{|\vec{n}_\Sigma|} = \frac{|\vec{n}_\pi \cdot t - d_\pi|}{|\vec{n}_\pi|} = d(t,\pi)$$

$$\cancel{\sqrt{3}}\cdot\frac{|1\cdot(1+\lambda) + 1\cdot\lambda + (-1)\cdot\lambda - (-1)|}{\sqrt{1^2 + \cancel{1^2} + (-1)^2}} = \cancel{\sqrt{3}}\cdot\frac{|1\cdot(1+\lambda) + (-1)\cdot\lambda + \lambda\cdot 1 - 5|}{\sqrt{1^2 + \cancel{(-1)^2} + 1^2}}$$

$$|1+\lambda+\lambda-\lambda+1| = |1+\lambda-\lambda+\lambda-5|$$

$$|\lambda+2| = |\lambda-4|$$

Because of absolute value, we must consider two cases:

① $\lambda+2 = \lambda-4$

② $\lambda+2 = -(\lambda-4)$

But in case ①, subtracting λ from each side yields $2=-4$, a contradiction ↯

So, the only possibility is case ②

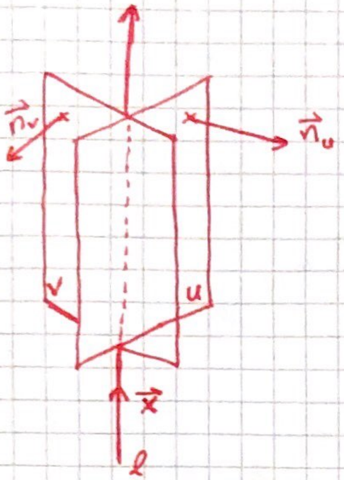
$$\lambda+2 = -(\lambda-4) = -\lambda+4$$

$$2\lambda = 2 \Rightarrow \boxed{\lambda = 1}$$

Then, point we are looking for is $\boxed{t(2,1,1)}$ ☺

② Intersection of two planes. Find the line given by

$U: x + 2y - z = 4$ and $V: 2x + y - 5z = 2$.



Solution: We are looking for line $\ell: (x_0, y_0, z_0) + \lambda \vec{x}$

$\vec{x} \perp \vec{n_v}$, $x \perp \vec{n_u}$, $\vec{n_u} \perp \vec{n_v}$

So $\vec{x} \parallel \vec{n_u} \times \vec{n_v}$, $\vec{n_u} = (1, 2, -1)$

$\vec{n_v} = (2, 1, -5)$

$$\vec{n_u} \times \vec{n_v} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 1 & -5 \end{vmatrix} = i\left((2)\cdot(-5) - (-1)\cdot 1\right) - j\left(1\cdot(-5) - (-1)\cdot 2\right) + k\left(1\cdot 1 - 2\cdot 2\right)$$

$$= i \cdot 11 + j \cdot 3 + k \cdot (-3)$$

$$= (11, 3, -3)$$

So $\boxed{\vec{x} = (11, 3, -3).}$ Now, $\ell: (x_0, y_0, z_0) + \lambda(11, 3, -3)$.

Only need (x_0, y_0, z_0), some point which lies on both plane U and plane V.

Set up a system of equations (but only two equations with three unknowns... ‼)

These are 2D planes which cross all 3 axes (else they would be $2x + y + 0 \cdot z$ or something)

So pick one coordinate to be zero ‼ (which one? Doesn't matter ‼ as long as it's the same for both)

I select $z = 0$. Now, the system of equations relies only on x & y.

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 2 \end{array}\right] \xrightarrow{R_2 - 2\cdot R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -6 \end{array}\right] \xrightarrow{R_2/(-3)} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 - 2\cdot R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right] \Rightarrow x = 0,\ y = 2, \text{ so point } \boxed{(x_0, y_0, z_0) = (0, 2, 0)}$$

$$\boxed{\ell: (0, 2, 0) + \lambda(11, 3, -3)}$$

4 Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

does not have a solution unless $a + c = 2b$.
In that case, write the solution of the system.

Solution:

I want 1 to go here...
I see an "easy" way to get it

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 5 & 6 & 7 & c \\ 4 & 5 & 6 & b \\ 3 & 4 & 5 & a \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} 2 & 2 & 2 & c-a \\ 1 & 1 & 1 & b-a \\ 3 & 4 & 5 & a \end{array}\right]$$

Not done yet, but observing rows R_1 and R_2, we see

$$\begin{aligned} 2x + 2y + 2z &= c - a \\ x + y + z &= b - a \end{aligned}$$

Clearly, $2x + 2y + 2z = 2\cdot(x + y + z)$
So it must be the case that $c - a = 2\cdot(b - a)$

$+2a$

$= 2b - 2a \quad +2a$

$\Rightarrow c + a = 2b$:)

gives 1 in top left

0 in bottom left

$$\xrightarrow[R_3 - 3\cdot R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & c-b \\ 1 & 1 & 1 & b-a \\ 0 & 1 & 2 & 4a-3b \end{array}\right]$$

Now, back to solving the system...

Want 1 in top-left position,
0's below it.

$$\xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & c-b \\ 0 & 0 & 0 & 2b-a-c \\ 0 & 1 & 2 & 4a-3b \end{array}\right]$$

Note: $2b - a - c = 0$

Now, $y + 2z = 4a - 3b$

$x + y + z = c - b$

3 unknowns, only two equations ?

Let $\boxed{z = t} \in \mathbb{R}$.

Then, $\boxed{y = 4a - 3b - 2t}$

$x + (4a - 3b - 2t) + t = c - b$

$x + 4a - 3b - t = c - b$

$\boxed{x = c + 2b - 4a + t}$

Solution:

$$\boxed{(c + 2b - 4a + t,\ 4 - 3b - 2t,\ t)}$$

⑤ Using elementary row operations, show that

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix} = 0.$$

Idea: When computing determinants, we can "expand" around either rows or columns, abiding by the pattern $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Observe the given matrix and look for a row or column that seems easy to reduce.

Each column has an appearance of x & a, b & y, or c & z ...

Probably won't want to reduce the columns. BUT R_3 has x & a in C_1, y & b in C_2, and z & c in C_3!

This seems like a good one to reduce :)

$$\xrightarrow{R_3 - 2R_2} \begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ -a-2 & -b-2 & -c-2 \end{vmatrix}$$

$$\xrightarrow{R_3 + R_1} \begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 0 & 0 & 0 \end{vmatrix}$$

expand around me :)

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} b+2 & c+2 \\ y+1 & z+1 \end{vmatrix} - 0 \cdot \begin{vmatrix} a+2 & c+2 \\ x+1 & z+1 \end{vmatrix} + 0 \cdot \begin{vmatrix} a+2 & b+2 \\ x+1 & y+1 \end{vmatrix}$$

$$= 0 - 0 + 0 = 0$$:)