

1.1 A: Prove that for all positive natural numbers n
 $1 \cdot 2 + 2 \cdot 3 + \dots + n \cdot (n+1) = n \cdot (n+1)(n+2)/3$

Proof: Base step: $n=1$

$$1 \cdot (1+1) = 1 \cdot (1+1)(1+2)/3$$

$$2 = 2$$

$$\text{LHS} = \text{RHS} \quad \checkmark$$

Induction step: $n=k$

$$1 \cdot 2 + 2 \cdot 3 + \dots + k(k+1) = k(k+1)(k+2)/3$$

1.2

~~now assume that $n=k+1$ is true~~

1.3

$$1 \cdot 2 + 2 \cdot 3 + \dots + k(k+1) + (k+1)(k+2) = k(k+1)(k+2)/3 + (k+1)(k+2)$$

$$\text{RHS} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= (k+1)(k+2) \cdot \left[\frac{k}{3} + 1 \right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$$= \text{? } n=k+1$$

$$= \frac{n(n+1)(n+2)}{3}$$

$$\Rightarrow \text{LHS} = \text{RHS}$$

B: For each of the following subsets of \mathbb{Z} , explain whether the subset is well-ordered or not (by the usual ordering on \mathbb{Z}).

i) even numbers:

Proof: For a set to be well-ordered, it has to have a smallest integer, which \mathbb{Z} doesn't have. So, we conclude that the subset of even numbers from \mathbb{Z} is not well-ordered.



ii) perfect squares:

$$0^2 = 0$$

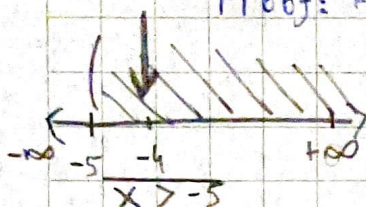
$$1^2 = 1 = (-1)^2$$

$$2^2 = 4 = (-2)^2$$

Proof: The subset of perfect squares of \mathbb{Z} , which has smallest perfect square, 0, or 1 is \times well-ordered, because of the condition that a subset needs to have a least integer to be well-ordered.

iii) the integers that are strictly greater than -5

Proof: For the subset of strictly greater integers than -5 , whose smallest integer is -4 , it holds that this subset is well-ordered subset.



2.1 Using the ordered pairs definition of the integers \mathbb{Z} , verify the associativity property for \mathbb{Z} . (You may use the properties of the naturals \mathbb{N} , including associativity, as stated in the notes.)

Task: prove that $a + (b + c) = (a + b) + c$

Proof: we'll be using ordered pairs, so that

$$x_1 = 2 \quad y_1 = 0$$

$$x_2 = 5 \quad y_2 = 1$$

$$x_3 = 7 \quad y_3 = 3$$

With the definition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

we replace for x_n : $(2, 0) + ((5, 1) + (7, 3)) = ((2, 0) + (5, 1)) + (7, 3)$

$$= (2, 0) + (5 + 7, 1 + 3) = (2 + 5, 0 + 1) + (7, 3)$$

$$= (2, 0) + (12, 4) = (7, 1) + (7, 3)$$

$$= (2 + 12, 0 + 4) = (7 + 7, 1 + 3)$$

$$= (14, 4) = (14, 4) \quad \checkmark \text{ it holds!}$$

we proved that $a + (b + c) = (a + b) + c$ is true (holds)

* same holds for multiplication

$$(2, 0) \cdot ((5, 1) \cdot (7, 3)) = ((2, 0) \cdot (5, 1)) \cdot (7, 3)$$

$$= (2, 0) \cdot (35, 3) = (10, 0) \cdot (7, 3)$$

$$= (70, 0) = (70, 0) \quad \checkmark$$

(it holds!)

Identity Proved Δ

22

3.1 D: Let G be the set $\{0, \infty\}$. Find a binary operation \oplus so that G is a group. Prove that your operation really yields a group!
(One efficient way to specify \oplus is to write out an addition table.)

Task: prove that our operation really yields a group
Proofs:

- i, ii) Let's assume 3.2 that axioms for closure and associativity under $(*)$ multiplication hold.
iii) Now, we give our operation an ~~element~~ identity, which means that it leaves unchanged every element of the set when the operation is applied.

The identity for multiplication is 1. Let's assume that our identity is described as h . So, this means that $0 \cdot e = 0$, as so $\infty \cdot e = \infty$.

- iv) Every element has an inverse element, so when multiplied with itself, it gives us 0. 3.3

$*$	0	∞
0	0	∞
∞	∞	0



- i) closure under multiplication \checkmark
ii) associativity under multiplication \checkmark
iii) existence of Identity \checkmark
iv) existence of Inverses \checkmark

My
It holds that our operation yields a group!

Index of comments

- 1.1 A
 3 / 5
- 1.2 Be careful here:
 You assumed that $n=k$ works
 ... but
 You want to prove that $n = k+1$ works by assuming the previous
- 1.3 Where did you apply the IH?
- 2.1 C
 1 / 5
- 2.2 One particular case doesn't prove a general statement
- 3.1 D
 2 / 5
- 3.2 You need to prove them. There's no point to assume their validity here.
- 3.3 The exercise is incomplete:
 * most of the properties remain unproven.