Algebra I - Matrični račun: 4. vaje

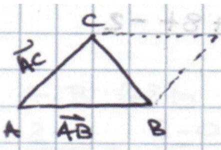
Geometrijski pomen vektorskega produkta. Mešani produkt. Premice.

1. Izračunajte ploščino trikotnika, z oglišči v točkah $A(0,0,0)$, $B(2,-1,5)$ in $C(1,4,-2)$.
2. Za poljubno sodo naravno število n poenostavite izraz
$$((\ldots((((\vec{a}\times\vec{b})\times \underbrace{\vec{a})\times\vec{a})\times\vec{a})\ldots)\times\vec{a})\times\vec{a}}_{n}.$$
3. Imejmo vektorje $\vec{a}=(1,2,-3)$, $\vec{b}=(0,3,1)$ in $\vec{c}=(-1,0,-1)$. Izračunajte: $(\vec{a},\vec{b},\vec{c})$, $(\vec{b},\vec{a},\vec{c})$, $(\vec{a},\vec{b},\vec{b})$.
4. Izračunajte prostornino paralelepipeda, določenega z vektorji $\vec{a}=(1,2,-1)$, $\vec{b}=(0,7,-4)$ in $\vec{c}=(-2,0,3)$.
5. Za katere vrednosti $t\in\mathbb{R}$ je prostornina paralelepipeda, določenega z vektorji $\vec{a}=(t,-1,4)$, $\vec{b}=(2,0,-2)$ in $\vec{c}=(-2,t,1)$ enaka 8?
6. V vseh treh oblikah zapišite enačbo premice, ki gre skozi točki $A(3,1,-2)$ in $B(0,2,-1)$.
7. Dani sta točki $A(2,3,-1)$ in $B(4,-1,1)\in\mathbb{R}^3$. Za vsako od točk $P(2,4,-1)$, $Q(3,1,0)$, $R(-2,11,-5)$ raziščite, ali leži na daljici AB in ali leži na premici skozi A in B.
8. Ali sta premici $p: \ 2-x=\frac{y}{2}=\frac{z+1}{3}$ in $q: \ \frac{x-1}{2}=\frac{2-y}{4}=\frac{2-z}{6}$ enaki?

Navodila.

1. Izračunajte ploščino trikotnika, z oglišči v točkah $A(0,0,0)$, $B(2,-1,5)$ in $C(1,4,-2)$.

① $A(0,0,0)$
$B(2,-1,5)$
$C(1,4,-2)$



$\vec{AB} = (2,-1,5)$
$\vec{AC} = (1,4,-2)$

$$S_\triangle = \frac{1}{2} S_{\text{▱}} = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|(2,-1,5)\times(1,4,-2)| =$$
$$= \frac{1}{2}|(2-20, -(-4-5), 8+1)| =$$
$$= \frac{1}{2}|(-18, 9, 9)| = \frac{1}{2}\sqrt{18^2+9^2+9^2} =$$
$$= \frac{1}{2}\sqrt{324+81+81} = \frac{1}{2}\sqrt{486} \approx \underline{11{,}02}$$

2. Za poljubno sodo naravno število n poenostavite izraz

$$((\ldots((((\vec{a}\times\vec{b})\times\underbrace{\vec{a})\times\vec{a})\times\vec{a})\ldots)\times\vec{a})\times\vec{a}}_{n}.$$

② $((\ldots((((\vec{a}\times\vec{b})\times\underbrace{\vec{a})\times\vec{a})\times\vec{a})\ldots)\times\vec{a})\times\vec{a}}_{n} = \vec{v}_n$ n je poljubno sodo naravno število

ker bo rešitev vektor

$\vec{v}_1 = (\vec{a}\times\vec{b})\times\vec{a} = (\vec{a}\cdot\vec{a})\cdot\vec{b} - (\vec{b}\cdot\vec{a})\vec{a} =$
$= |\vec{a}|^2\vec{b} - (\vec{b}\cdot\vec{a})\vec{a}$

$(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\cdot\vec{w})\cdot\vec{v} - (\vec{v}\cdot\vec{w})\cdot\vec{u}$

$\vec{v}_2 = \vec{v}_1\times\vec{a} = (|\vec{a}|^2\vec{b} - (\vec{b}\cdot\vec{a})\vec{a})\times\vec{a} =$
$= |\vec{a}|^2\vec{b}\times\vec{a} - \underbrace{(\vec{b}\vec{a})\vec{a}\times\vec{a}}_{\vec{0}} = |\vec{a}|^2\,\vec{b}\times\vec{a} = -|\vec{a}|^2\vec{a}\times\vec{b}$

$(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$
$(\alpha\vec{u})\times\vec{v} = \alpha(\vec{u}\times\vec{v})$
$\vec{a}\times\vec{a} = \vec{0}$

$\vec{v}_3 = \vec{v}_2\times\vec{a} = -|\vec{a}|^2\underbrace{(\vec{a}\times\vec{b})\times\vec{a}}_{\vec{v}_1} = -|\vec{a}|^2\cdot\vec{v}_1$

$\vec{v}_4 = \vec{v}_3\times\vec{a} = -|\vec{a}|^2\cdot\underbrace{\vec{v}_1\times\vec{a}}_{\vec{v}_2} = -|\vec{a}|^2\cdot\vec{v}_2 = -|\vec{a}|^2(-|\vec{a}|^2\vec{a}\times\vec{b}) = |\vec{a}|^4\vec{a}\times\vec{b}$

Domneva: Za $n = 2k$ $k\in\mathbb{N}$ bo:
$\vec{v}_n = \vec{v}_{2k} = (-1)^k|\vec{a}|^{2k}\vec{a}\times\vec{b}$

Dokaz z indukcijo: (po k)
$k=1$: $\vec{v}_2 = (-1)^1|\vec{a}|^{2\cdot 1}\vec{a}\times\vec{b} = -|\vec{a}|^2\vec{a}\times\vec{b}$
$k\to k+1$: $\vec{v}_{2(k+1)} = (-1)^{k+1}|\vec{a}|^{2(k+1)}\vec{a}\times\vec{b}$

$\vec{v}_{2(k+1)} = \vec{v}_{2k+2} = (\vec{v}_{2k}\times\vec{a})\times\vec{a}$
$\overset{\text{i.p.}}{=} (((-1)^k|\vec{a}|^{2k}\underbrace{\vec{a}\times\vec{b})\times\vec{a}}_{\vec{v}_1})\times\vec{a}$
$= (-1)^k|\vec{a}|^{2k}\cdot\vec{v}_2 =$
$= (-1)^k|\vec{a}|^{2k}\cdot(-|\vec{a}|^2\vec{a}\times\vec{b}) =$
$= (-1)^{k+1}|\vec{a}|^{2k+2}\vec{a}\times\vec{b} =$
$= (-1)^{k+1}|\vec{a}|^{2(k+1)}\vec{a}\times\vec{b}$

domneva je dokazana!

3. Imejmo vektorje $\vec{a} = (1,2,-3)$, $\vec{b} = (0,3,1)$ in $\vec{c} = (-1,0,-1)$. Izračunajte: $(\vec{a},\vec{b},\vec{c})$, $(\vec{b},\vec{a},\vec{c})$, $(\vec{a},\vec{b},\vec{b})$.

③ $\vec{a} = (1,2,-3)$
$\vec{b} = (0,3,1)$
$\vec{c} = (-1,0,-1)$

$(\vec{a},\vec{b},\vec{c}) = (\vec{a}\times\vec{b})\vec{c} = (2+9, -(1), 3)\cdot(-1,0,-1) =$
$= -11+0-3 = \underline{-14}$

$(\vec{b},\vec{a},\vec{c}) = ((\vec{b}\times\vec{a})\cdot\vec{c}) = (-\vec{a}\times\vec{b})\cdot\vec{c} = -(\vec{a}\times\vec{b})\vec{c} = \underline{14}$
$(-11,1,-3)(-1,0,-1) = 11+0+3 = 14$

$(\vec{a},\vec{b},\vec{b}) = (\vec{a}\times\vec{b})\cdot\vec{b} = (11,-1,3)\cdot(0,3,1) = 0-3+3 = \underline{\underline{0}}$

4. Izračunajte prostornino paralelepipeda, določenega z vektorji $\vec{a} = (1, 2, -1)$, $\vec{b} = (0, 7, -4)$ in $\vec{c} = (-2, 0, 3)$.

(4) $\vec{a}=(1,2,-1)$
$\vec{b}=(0,7,-4)$
$\vec{c}=(-2,0,3)$

$V=|(\vec{a},\vec{b},\vec{c})|=\underline{\underline{23}}$

$(\vec{a},\vec{b},\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$
$=(-1,4,7)\cdot(-2,0,3)=2+0+21=\underline{\underline{23}}$

5. Za katere vrednosti $t \in \mathbb{R}$ je prostornina paralelepipeda, določenega z vektorji $\vec{a} = (t, -1, 4)$, $\vec{b} = (2, 0, -2)$ in $\vec{c} = (-2, t, 1)$ enaka 8?

(5) $V=8$
$\vec{a}=(t,-1,4)$
$\vec{b}=(2,0,-2)$
$\vec{c}=(-2,t,1)$
$t\in\mathbb{R}$?

$V=|(\vec{a},\vec{b},\vec{c})|=8$

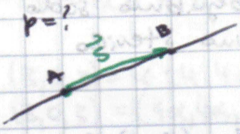
$(\vec{a},\vec{b},\vec{c})=(\vec{a}\times\vec{b})\,\vec{c}=$
$=(2,-(-2t-8),2)\cdot(-2,t,1)=$
$=(2,2t+8,2)\cdot(-2,t,1)=$
$=-4+2t^2+8t+2=$
$=2t^2+8t-2$

$2t^2+8t-2=8 \quad /:2$
$t^2+4t-5=0$
$(t-1)(t+5)=0$
$t_1=1$
$t_2=-5$

$2t^2+8t-2=-8 \quad /:2$
$t^2+4t+3=0$
$(t+3)(t+1)=0$
$t_1=-1$
$t_2=-3$

6. V vseh treh oblikah zapišite enačbo premice, ki gre skozi točki $A(3, 1, -2)$ in $B(0, 2, -1)$.

(6) $A(3,1,-2)$
$B(0,2,-1)$
$p=?$



$p: \frac{x-3}{-3}=\frac{y-1}{1}=\frac{z+2}{1}=\lambda$

kanonična oblika (iz parametrične izrazimo λ)

$\vec{s}=\vec{AB}=(0,2,-1)-(3,1,-2)=\underline{(-3,1,1)}$

vektorska oblika:
$p=\underbrace{(3,1,-2)}_{\text{točka}}+\lambda\underbrace{(-3,1,1)}_{\text{smer}}$

$\begin{cases} x=-3\lambda+3 \\ y=\lambda+1 \\ z=\lambda-2 \end{cases}$

parametrična oblika:
$(x,y,z)=(3-3\lambda,\ 1+1\lambda,\ -2+1\lambda)$

7. Dani sta točki $A(2, 3, -1)$ in $B(4, -1, 1) \in \mathbb{R}^3$. Za vsako od točk $P(2, 4, -1)$, $Q(3, 1, 0)$, $R(-2, 11, -5)$ raziščite, ali leži na daljici AB in ali leži na premici skozi A in B.

(7) $A(2,3,-1)$
$B(4,-1,1)$
$P(2,4,-1)$
$Q(3,1,0)$
$R(-2,11,-5)$

$\vec{s}=(4,-1,1)-(2,3,-1)=(2,-4,2)$

$p: \frac{x-2}{2}=\frac{y-3}{-4}=\frac{z+1}{2}=\lambda$

$p_P: \underbrace{\frac{2-2}{2}}_{=0} \neq \underbrace{\frac{4-3}{-4}}_{=-1/4} \neq \underbrace{\frac{-1+1}{2}}_{=0} \neq \lambda$

$p_Q: \underbrace{\frac{3-2}{2}}_{1/2} \stackrel{?}{=} \underbrace{\frac{1-3}{-4}}_{1/2} \stackrel{?}{=} \underbrace{\frac{0+1}{2}}_{1/2} \stackrel{?}{=} \lambda \Rightarrow \lambda=\frac{1}{2} \Rightarrow$ leži na daljici AB $\Rightarrow$ leži na premici p

$p_R: \underbrace{\frac{-2-2}{2}}_{-2} \stackrel{?}{=} \underbrace{\frac{11-3}{-4}}_{-2} \stackrel{?}{=} \underbrace{\frac{-5+1}{2}}_{-2} \stackrel{?}{=} \lambda \Rightarrow \lambda=2 \Rightarrow$ ne leži na daljici AB, leži pa na premici p

8. Ali sta premici $p: 2-x=\frac{y}{2}=\frac{z+1}{3}$ in $q: \frac{x-1}{2}=\frac{2-y}{4}=\frac{2-z}{6}$ enaki?

$\frac{2-x}{1} = -\frac{x-2}{1} = \frac{x-2}{-1}$

$p: 2-x=\frac{y}{2}=\frac{z+1}{3} \rightsquigarrow \vec{s}_p=(-1,2,3),\ A(2,0,-1)$

$q: \frac{x-1}{2}=\frac{2-y}{4}=\frac{2-z}{6} \rightsquigarrow \vec{s}_q=(2,-4,-6),\ B(1,2,2)$

$\frac{2-y}{4} = \frac{y-2}{-4}$, $\frac{2-z}{6} = \frac{z-2}{-6}$

$p: x=-\lambda+2,\ y=2\lambda,\ z=3\lambda-1$

$q: x=2\alpha+1,\ y=-4\alpha+2,\ z=-6\alpha+2$

$-\lambda+2=2\alpha+1 \rightarrow -\lambda=2\alpha+1-2=2\alpha-1$

$2\lambda=-4\alpha+2 \rightarrow \lambda=-2\alpha+1$

$3\lambda-1=-6\alpha+2$

$3(-2\alpha+1)-1=-6\alpha+2$

$-6\alpha+3-1=-6\alpha+2$

$2=2$ ✓

$\Rightarrow$ p in q sta enaki!!

Izberemo 2 točki, ki ležita na premici p:

$p=(2,0,-1)+\lambda(-1,2,3)$

$A(2,0,1)$

$B(1,2,2) \rightarrow$ pri $\lambda=1$

Preverimo ali A, B ležita na premici q:

za A: $\frac{2-1}{2}=\frac{2-0}{4}=\frac{2-1}{6}$

$\frac{1}{2}=\frac{2}{4}=\frac{3}{6} \Rightarrow$ A leži na premici q

za B: $\frac{1-1}{2}=\frac{2-2}{4}=\frac{2-2}{6}=0 \Rightarrow$ B leži na premici q

p in q imata 2 skupni točki $\Rightarrow$ p in q sta enaki