



Algebra I, exam, 27th of January 2023.

Time: 120 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (Theoretical problem). Let \vec{u} , \vec{v} and \vec{w} denote arbitrary vectors in the space \mathbb{R}^3 . Prove the following property of vector product (10 points)

$$(\vec{u} \times \vec{v}) \times \vec{w} = \langle \vec{u}, \vec{w} \rangle \vec{v} - \langle \vec{v}, \vec{w} \rangle \vec{u}$$

2. Let $\vec{a} + 2\vec{b}$ and $\vec{a} - \vec{b}$ be orthogonal unit vectors. Find the angle between \vec{a} and \vec{b} . (20 points)

3. Let $A(2, 3, -1)$ and $B(4, -1, 1)$ be two points in the space \mathbb{R}^3 . For each of the following three points $P(2, 4, -1)$, $Q(3, 1, 0)$, $R(-2, 11, -5)$ find out if it lies at a segment \overline{AB} or if it lies at a line which pass through the points A and B (or neither of these two). (20 points)

4. (Theoretical problem). Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ denote arbitrary matrix in the space $\text{Mat}_{n \times n}(\mathbb{R})$. Use definition of determinant and prove that $\det(A^T) = \det(A)$. (10 points)

5. Let A denote a matrix bellow

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1+x \end{bmatrix}.$$

- (a) Find all values of x for which the given matrix A is a non-singular (invertible) matrix. (10 points)
(b) For $x = 2$ compute A^{-1} . (5 points)
(c) With respect to the part (b) above, solve the following matrix equation $XA - I = A^2$. (5 points)
6. Let $a, b, c \in \mathbb{R}$. Show that the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= b - c \\ 2x + 10y + 10z &= 2a - 2c \\ -3x - 15y - 15z &= 0 \end{aligned}$$

does not have a solution, except the case when $a = c$ (carefully explain your answer). Obtain the general solution for this case. (20 points)

Some intructions: Please write the exam exclusively with a fountain pen or ballpoint pen in blue or black colour. Attach and submit this sheet together with the solution sheets! Once when you finish with exam, please enumerate all solution sheets on the following way: current-page/total-number-of-pages.