13 Properties of matrix multiplication. Matrix Inversion. Computing an Inverse.

Identity Matrix

The $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is called the ***identity matrix*** of order n. For every $m \times n$ matrix $\mathbf{A}$,

$$\mathbf{A}\mathbf{I}_n = \mathbf{A} \quad \text{and} \quad \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

The subscript on $\mathbf{I}_n$ is neglected whenever the size is obvious from the context.

Reverse Order Law for Transposition

For conformable matrices $\mathbf{A}$ and $\mathbf{B}$,

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$$

The case of conjugate transposition is similar. That is,

$$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*.$$

Matrix Inversion

For a given square matrix $\mathbf{A}_{n\times n}$, the matrix $\mathbf{B}_{n\times n}$ that satisfies the conditions

$$\mathbf{AB} = \mathbf{I}_n \quad \text{and} \quad \mathbf{BA} = \mathbf{I}_n$$

is called the ***inverse*** of $\mathbf{A}$ and is denoted by $\mathbf{B} = \mathbf{A}^{-1}$. Not all square matrices are invertible—the zero matrix is a trivial example, but there are also many nonzero matrices that are not invertible. An invertible matrix is said to be ***nonsingular,*** and a square matrix with no inverse is called a ***singular matrix.***

Matrix Equations

- If $\mathbf{A}$ is a nonsingular matrix, then there is a unique solution for $\mathbf{X}$ in the matrix equation $\mathbf{A}_{n\times n}\mathbf{X}_{n\times p} = \mathbf{B}_{n\times p}$, and the solution is

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}.$$

- A system of n linear equations in n unknowns can be written as a single matrix equation $\mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{n\times 1}$, so it follows that when $\mathbf{A}$ is nonsingular, the system has a unique solution given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

1. A Pitfall. For two $n \times n$ matrices, what is $(A+B)^2$?

2. Show that, for every matrix A, the products $A^\top A$ and $AA^\top$ are symmetric matrices.

3. **Trace of a Product**. The trace of a square matrix is the sum of its main diagonal entries. Although matrix multiplication is not commutative, the trace function is one of the few cases where the order of the matrices can be changed without affecting the results.
For matrices $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $B \in \mathrm{Mat}_{n\times m}(\mathbb{R})$, prove that

$$\mathrm{trace}(AB) = \mathrm{trace}(BA).$$

4. Show that, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\delta = ad - bc \neq 0$ then

$$A^{-1} = \frac{1}{\delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

5. Let A, B and C denote three given matrices

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}.$$

Solve (for X) the following matrix equations:

(a) $3X + A = B$.

(b) $AX = B$.

(c) $(2 - X)B = XA^\top + 3I$.

(d) $XA = B$.

(e) $AXB = C$.

(f) $X^{-1}A = B^{-1}$.

(g) $AX + I = X - 2I$.

(h) $A^{-1}X = X - I$.

(i) $(A + 3I)(X - I) = B$.

(j) $B^{-1}XA = (3B + 2I)^{-1}$.

(k) $(AXB)^{-1} = B^{-1}(X^{-1} + B)$.

6. For all matrices $A \in \mathrm{Mat}_{n\times k}(\mathbb{R})$ and $B \in \mathrm{Mat}_{k\times n}(\mathbb{R})$, show that the block matrix

$$L = \begin{bmatrix} I - BA & B \\ 2A - ABA & AB - I \end{bmatrix},$$

has the property $L^2 = I$. Matrices with this property are said to be **involutory**, and they occur in the science of cryptography.

Existence of an Inverse

For an $n \times n$ matrix $\mathbf{A}$, the following statements are equivalent.

- $\mathbf{A}^{-1}$ exists $\quad(\mathbf{A}$ is nonsingular).
- $rank\,(\mathbf{A}) = n$.
- $\mathbf{A} \xrightarrow{\text{Gauss–Jordan}} \mathbf{I}$.
- $\mathbf{Ax} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$.

Properties of Matrix Inversion

For nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$, the following properties hold.

- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.
- The product $\mathbf{AB}$ is also nonsingular.
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ (the reverse order law for inversion).
- $\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^T\right)^{-1}$ and $\left(\mathbf{A}^{-1}\right)^* = \left(\mathbf{A}^*\right)^{-1}$.

Computing an Inverse

Gauss–Jordan elimination can be used to invert $\mathbf{A}$ by the reduction

$$[\mathbf{A}\,|\,\mathbf{I}] \xrightarrow{\text{Gauss–Jordan}} [\mathbf{I}\,|\,\mathbf{A}^{-1}].$$

The only way for this reduction to fail is for a row of zeros to emerge in the left-hand side of the augmented array, and this occurs if and only if $\mathbf{A}$ is a singular matrix.

7. If possible, find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

A ***minor determinant*** (or simply a ***minor***) of $\mathbf{A}_{m\times n}$ is defined to be the determinant of any $k \times k$ submatrix of $\mathbf{A}$. For example,

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \quad\text{and}\quad \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6 \quad\text{are } 2\times 2 \text{ minors of } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

An individual entry of $\mathbf{A}$ can be regarded as a 1×1 minor, and $\det(\mathbf{A})$ itself is considered to be a 3×3 minor of $\mathbf{A}$.

Invertibility and Determinants

- $\mathbf{A}_{n\times n}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$ or, equivalently,
- $\mathbf{A}_{n\times n}$ is singular if and only if $\det(\mathbf{A}) = 0$.

Rank and Determinants

- $rank\,(\mathbf{A}) =$ the size of the largest nonzero minor of $\mathbf{A}$.

Cofactors

The ***cofactor*** of $\mathbf{A}_{n\times n}$ associated with the (i,j)-position is defined as

$$\mathring{\mathrm{A}}_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the $n-1 \times n-1$ minor obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$. The matrix of cofactors is denoted by $\mathring{\mathbf{A}}$.

Cofactor Expansions

- $\det(\mathbf{A}) = a_{i1}\mathring{\mathrm{A}}_{i1} + a_{i2}\mathring{\mathrm{A}}_{i2} + \cdots + a_{in}\mathring{\mathrm{A}}_{in}$ (about row i).
- $\det(\mathbf{A}) = a_{1j}\mathring{\mathrm{A}}_{1j} + a_{2j}\mathring{\mathrm{A}}_{2j} + \cdots + a_{nj}\mathring{\mathrm{A}}_{nj}$ (about column j).

Determinant Formula for $\mathbf{A}^{-1}$

The ***adjugate*** of $\mathbf{A}_{n\times n}$ is defined to be $\mathrm{adj}\,(\mathbf{A}) = \mathring{\mathbf{A}}^T$, the transpose of the matrix of cofactors—some older texts call this the ***adjoint*** matrix. If $\mathbf{A}$ is nonsingular, then

$$\mathbf{A}^{-1} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{\mathrm{adj}\,(\mathbf{A})}{\det(\mathbf{A})}.$$

8. For $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}$ determine the cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$. Compute the entire matrix of cofactors $\mathring{A}$. Use determinants to compute entries $(A^{-1})_{12}$ and $(A^{-1})_{31}$. Using determinant formula compute A^{-1}.

Problems from exam

1. Find the inverse of the matrix A, $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by Gauss-Jordan Method.

2. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1} and show that $A^3 = A^{-1}$.

3. Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{bmatrix}$, find $\operatorname{adj} A$.

5. Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by elementary row operations.

6. Express $\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix}$ as the sum of a lower triangular matrix and an upper triangular matrix with zero leading diagonal.

7. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular?

8. Factories the matrix A, $\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular with each diagonal element one and U is upper triangular matrix.

9. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)^\top = B^\top A^\top$.

13.1 Solutions

A Pitfall. For two $n \times n$ matrices, what is $(\mathbf{A}+\mathbf{B})^2$? **Be careful!** Because matrix multiplication is not commutative, the familiar formula from scalar algebra is not valid for matrices. The distributive properties must be used to write

$$\begin{aligned}(\mathbf{A}+\mathbf{B})^2 &= \underbrace{(\mathbf{A}+\mathbf{B})}(\mathbf{A}+\mathbf{B}) = \underbrace{(\mathbf{A}+\mathbf{B})}\mathbf{A} + \underbrace{(\mathbf{A}+\mathbf{B})}\mathbf{B} \\ &= \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2,\end{aligned}$$

and this is as far as you can go. The familiar form $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$ is obtained only in those rare cases where $\mathbf{AB} = \mathbf{BA}$. To evaluate $(\mathbf{A}+\mathbf{B})^k$, the distributive rules must be applied repeatedly, and the results are a bit more complicated—try it for $k = 3$.

For every matrix $\mathbf{A}_{m\times n}$, the products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric matrices because

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T\mathbf{A}^{T^T} = \mathbf{A}^T\mathbf{A} \quad \text{and} \quad \left(\mathbf{A}\mathbf{A}^T\right)^T = \mathbf{A}^{T^T}\mathbf{A}^T = \mathbf{A}\mathbf{A}^T.$$

Problem: For matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times m}$, prove that

$$\mathit{trace}\,(\mathbf{AB}) = \mathit{trace}\,(\mathbf{BA}).$$

Solution:

$$\begin{aligned}\mathit{trace}\,(\mathbf{AB}) &= \sum_i [\mathbf{AB}]_{ii} = \sum_i \mathbf{A}_{i*}\mathbf{B}_{*i} = \sum_i\sum_k a_{ik}b_{ki} = \sum_i\sum_k b_{ki}a_{ik} \\ &= \sum_k\sum_i b_{ki}a_{ik} = \sum_k \mathbf{B}_{k*}\mathbf{A}_{*k} = \sum_k [\mathbf{BA}]_{kk} = \mathit{trace}\,(\mathbf{BA}).\end{aligned}$$

Note: This is true in spite of the fact that $\mathbf{AB}$ is $m \times m$ while $\mathbf{BA}$ is $n \times n$. Furthermore, this result can be extended to say that any product of conformable matrices can be permuted *cyclically* without altering the trace of the product. For example,

$$\mathit{trace}\,(\mathbf{ABC}) = \mathit{trace}\,(\mathbf{BCA}) = \mathit{trace}\,(\mathbf{CAB}).$$

However, a noncyclical permutation may not preserve the trace. For example,

$$\mathit{trace}\,(\mathbf{ABC}) \neq \mathit{trace}\,(\mathbf{BAC}).$$

If

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where} \quad \delta = ad - bc \neq 0,$$

then

$$\mathbf{A}^{-1} = \frac{1}{\delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

because it can be verified that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_2$.

$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}$, $B^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{3}{5} \end{pmatrix}$, $C^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

a. $X = \frac{1}{3}(B - A) = \begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

b. $X = A^{-1}B = \begin{pmatrix} \frac{4}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

c. $X = (-2B + 3I)(-B - A^\top)^{-1} = \begin{pmatrix} \frac{11}{23} & -\frac{7}{23} \\ -\frac{8}{23} & \frac{3}{23} \end{pmatrix}$.

d. $X = BA^{-1} = \begin{pmatrix} \frac{3}{2} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

e. $X = A^{-1}CB^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{15} & \frac{7}{15} \end{pmatrix}$.

f. $X = AB = \begin{pmatrix} 7 & -4 \\ -3 & 6 \end{pmatrix}$.

g. $X = -3(A-I)^{-1} = \begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & -\frac{3}{2} \end{pmatrix}$.

h. $X = -(A^{-1}-I)^{-1} = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & \frac{3}{2} \end{pmatrix}$.

i. $X = (A+3I)^{-1}B + I = \begin{pmatrix} \frac{47}{30} & -\frac{2}{15} \\ -\frac{1}{6} & \frac{4}{3} \end{pmatrix}$.

j. $X = B(3B+2I)^{-1}A^{-1} = \begin{pmatrix} \frac{21}{158} & \frac{17}{474} \\ -\frac{1}{79} & \frac{6}{79} \end{pmatrix}$.

k. $X = A^{-1}(I-A)B^{-1} = \begin{pmatrix} -\frac{1}{6} & 0 \\ -\frac{2}{15} & -\frac{2}{5} \end{pmatrix}$.

Recall that matrices A and B are said to be conformable for multiplication in the order AB whenever A has exactly as many columns as B has rows – i.e., A is $m \times p$ and B is $p \times n$.

Block Matrix Multiplication

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are partitioned into submatrices—often referred to as ***blocks***—as indicated below.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1r} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{s1} & \mathbf{A}_{s2} & \cdots & \mathbf{A}_{sr} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \cdots & \mathbf{B}_{1t} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \cdots & \mathbf{B}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{r1} & \mathbf{B}_{r2} & \cdots & \mathbf{B}_{rt} \end{pmatrix}.$$

If the pairs $(\mathbf{A}_{ik}, \mathbf{B}_{kj})$ are conformable, then $\mathbf{A}$ and $\mathbf{B}$ are said to be ***conformably partitioned***. For such matrices, the product $\mathbf{AB}$ is formed by combining the blocks exactly the same way as the scalars are combined in ordinary matrix multiplication. That is, the (i,j)-block in $\mathbf{AB}$ is

$$\mathbf{A}_{i1}\mathbf{B}_{1j} + \mathbf{A}_{i2}\mathbf{B}_{2j} + \cdots + \mathbf{A}_{ir}\mathbf{B}_{rj}.$$

Use block multiplication to verify $L^2 = I$ - be careful not to commute any of the terms when forming the various products.

Problem: For $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}$, determine the cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$.

Solution:

$\mathring{A}_{21} = (-1)^{2+1}M_{21} = (-1)(-19) = 19$ and $\mathring{A}_{13} = (-1)^{1+3}M_{13} = (+1)(18) = 18$.

The entire matrix of cofactors is $\mathring{\mathbf{A}} = \begin{pmatrix} -54 & -20 & 18 \\ 19 & 7 & -6 \\ -6 & -2 & 2 \end{pmatrix}$.

Problem: Use determinants to compute $\left[\mathbf{A}^{-1}\right]_{12}$ and $\left[\mathbf{A}^{-1}\right]_{31}$ for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}.$$

Solution: The cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$ were determined in Example 6.2.3 to be $\mathring{A}_{21} = 19$ and $\mathring{A}_{13} = 18$, and it's straightforward to compute $\det(\mathbf{A}) = 2$, so

$$\left[\mathbf{A}^{-1}\right]_{12} = \frac{\mathring{A}_{21}}{\det(\mathbf{A})} = \frac{19}{2} \quad \text{and} \quad \left[\mathbf{A}^{-1}\right]_{31} = \frac{\mathring{A}_{13}}{\det(\mathbf{A})} = \frac{18}{2} = 9.$$

Using the matrix of cofactors $\mathring{\mathbf{A}}$ computed in Example 6.2.3, we have that

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{1}{2}\begin{pmatrix} -54 & 19 & -6 \\ -20 & 7 & -2 \\ 18 & -6 & 2 \end{pmatrix}.$$

Problems from exam

Find the inverse of the matrix A, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ *by Gauss-Jordan Method.*

Sol. Writing the given matrix side by side with unit matrix I_3, we get

$$[A : I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Operating } R_{12}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Operating } R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right] \text{ Operating } R_1 - 2R_2,\ R_3 + 5R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \text{ Operating } \frac{1}{2} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \text{ Operating } R_1 + R_3,\ R_2 - 2R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \equiv [I_3 : A^{-1}]$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$,

(*i*) *Find* A^{-1} (*ii*) *Show that* $A^3 = A^{-1}$.

Sol. (*i*)
$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} \text{ Operating } C_2 + C_3$$

$$= \begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} \text{ Expanding by 3rd row}$$

$$= 1.$$

$\because$ $|A| = 1 \neq 0$, $\therefore$ A^{-1} exists.

$$A_1 = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1,\ A_2 = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1,$$

$$A_3 = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0,\ B_1 = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$$

$$B_2 = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3,\ B_3 = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$C_1 = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2,\ C_2 = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3,\ C_3 = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$$

$\therefore$
$$\text{adj } A = \text{transpose of } \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \qquad (\because\ |A| = 1)$$

(*ii*)
$$A^2 = A \times A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $A^4 = I$

Now, $A^4 = I \Rightarrow A \,.\, A^3 = I = A \,.\, A^3$

$\Rightarrow$ A^3 is inverse of A *i.e.*, $A^3 = A^{-1}$. Hence shown.

Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Sol. The determinant of the given matrix is

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (say)}$$

If $A_1, A_2, ...$ be the co-factors of $a_1, a_2, ...$ in Δ, then $A_1 = -24$, $A_2 = -8$, $A_3 = -12$, $B_1 = 10$, $B_2 = 2$, $B_3 = 6$, $C_1 = 2$, $C_2 = 2$, $C_3 = 2$.

Thus, $$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8$$

and $$\text{adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence the inverse of the given matrix A

$$= \frac{\text{adj } A}{\Delta} = -\frac{1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

If $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{bmatrix}$, *find adj A.*

Sol. Here $$|\,A\,| = \begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ say}$$

$\therefore$ $$A_1 = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = -2,\ B_1 = -\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} = -2$$

$$C_2 = -\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = -9,\ C_1 = \begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = 1,\ C_3 = 5$$

$$A_2 = -\begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} = -12,\ B_2 = \begin{vmatrix} 1 & 0 \\ -2 & 6 \end{vmatrix} = 6,\ A_3 = 8,\ B_3 = -4$$

Thus the matrix of co-factors is $\begin{bmatrix} -2 & -2 & 1 \\ -12 & 6 & -9 \\ 8 & -4 & 5 \end{bmatrix}$

Hence $$\text{adj } A = \begin{bmatrix} -2 & -12 & 8 \\ -2 & 6 & -4 \\ 1 & -9 & 5 \end{bmatrix}.$$

Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ *by elementary row operations.*

Sol. Writing the given matrix side by side with unit matrix of order 3, we get

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

Operate R_{13},

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \end{array}\right]$$

Operating $R_2 - 4R_1$, $R_3 - 2R_1$,

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & -5 & -15 & 0 & 1 & -4 \\ 0 & -1 & -4 & 1 & 0 & -2 \end{array}\right]$$

Operating $\dfrac{-R_2}{5}$, $-R_3$, we get

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & -\dfrac{1}{5} & \dfrac{4}{5} \\ 0 & 1 & 4 & -1 & 0 & 2 \end{array}\right]$$

Operating $R_1 - 2R_2$, $R_3 - R_2$,

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & \dfrac{2}{5} & \dfrac{-3}{5} \\ 0 & 1 & 3 & 0 & -\dfrac{1}{5} & \dfrac{4}{5} \\ 0 & 0 & 1 & -1 & \dfrac{1}{5} & \dfrac{6}{5} \end{array}\right]$$

Operating $R_1 + 2R_3$, $R_2 - 3R_3$,

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \dfrac{4}{5} & \dfrac{9}{5} \\ 0 & 1 & 0 & 3 & -\dfrac{4}{5} & -\dfrac{14}{5} \\ 0 & 0 & 1 & -1 & \dfrac{1}{5} & \dfrac{6}{5} \end{array}\right]$$

Thus the inverse of the given matrix is

$$\begin{bmatrix} -2 & \dfrac{4}{5} & \dfrac{9}{5} \\ 3 & -\dfrac{4}{5} & -\dfrac{14}{5} \\ -1 & \dfrac{1}{5} & \dfrac{6}{5} \end{bmatrix} \text{ i.e., } \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}.$$

Express $\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix}$ *as the sum of a lower triangular matrix and an upper triangular matrix with zero leading diagonal.*

Sol. Let $\quad L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ be the lower triangular matrix and

$$U = \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}$$

be the upper triangular matrix with zero leading diagonal such that

$$\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} + \begin{bmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix} = \begin{bmatrix} a & p & q \\ b & c & r \\ d & e & f \end{bmatrix}$$

Equating the corresponding elements on the two sides, we get

$2 = a, \quad 5 = p, \quad -7 = q, \quad -9 = b, \quad 12 = c, \quad 4 = r,$

$15 = d, \quad -13 = e, \quad 6 = f$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ -9 & 12 & 0 \\ 15 & -13 & 6 \end{bmatrix} \text{ and } U = \begin{bmatrix} 0 & 5 & -7 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ *is singular ?*

Sol. If the matrix is singular, the determinantal value is zero.

$$i.e., \quad \Rightarrow \quad \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$$\text{L.H.S.} = (3-x)\,[(4-x)(-1-x) + 4] - 2\,[2(-1-x) + 2] + 2\,[-8 + 2(4-x)]$$

$$= (3-x)(x^2 - 3x) + 4x - 4x = x(3-x)(x-3) = -x(x-3)^2$$

$$\Rightarrow \quad x = 0, 3.$$

Factorise the matrix A, $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ *into the form LU, where L is lower triangular with each diagonal element one and U is upper triangular matrix.*

Sol. Let $$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, \quad U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix}$$

$$\text{R.H.S.} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+p \end{bmatrix}$$

Equating the corresponding elements on the two sides, we get

$5 = d,\quad -2 = e,\quad 1 = f,\quad 7 = ad,\quad 1 = ae + g,$

$-5 = af + h,\quad 3 = bd,\quad 7 = be + cg,\quad 4 = bf + ch + p$

$$\Rightarrow \quad d = 5, \quad a = 7/5, \quad 1 = \frac{7}{5}(-2) + g \;\Rightarrow\; g = \frac{19}{5}$$

$$e = -2, \quad f = 1,$$

$$\frac{7}{5}(1) + h = -5 \quad \text{or} \quad h = -5 - \frac{7}{5} \quad \text{or} \quad h = \frac{-32}{5}$$

$$b = \frac{3}{d} \quad \text{or} \quad \frac{3}{5}$$

$$7 = \frac{3}{5}(-2) + c\left(\frac{19}{5}\right) \;\Rightarrow\; \frac{41}{5} = \frac{19c}{5} \;\Rightarrow\; c = \frac{41}{19}$$

$$4 = \frac{3}{5}(1) + \frac{41}{19}\left(-\frac{32}{5}\right) + p$$

$$\frac{17}{5} = -\frac{41 \times 32}{19 \times 5} + p$$

or $$p = \frac{41 \times 32}{19 \times 5} + \frac{17}{5} = \frac{41 \times 32 + 19 \times 17}{19 \times 5} = \frac{1635}{95} = \frac{327}{19}$$

Hence $$d = 5, \quad e = -2, \quad f = 1$$

$$a = \frac{7}{5}, \quad g = \frac{19}{5}, \quad h = -\frac{32}{5}$$

$$b = \frac{3}{5}, \quad c = \frac{41}{19}, \quad p = \frac{327}{19}$$

$$\Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}.$$

If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, *verify that* $(AB)' = B'A'$.

Sol.

$$AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \; A' = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 14 \\ 1 & 2 & 5 \\ -3 & 6 & 0 \end{bmatrix}$$

Hence $(AB)' = B'A'$ is verified.