Algebra I - Matričniračun: 7. vaje

Matrike (1. del) - Računske operacije z matrikami

1. Naj bodo dane matrike

$A = \begin{bmatrix} 2 & -1 & -2 \\ 4 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 \\ 4 & -3 \\ 6 & -2 \end{bmatrix}$ in $D = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

   (a) Določite velikost matrik A, B, C in D?

   (b) Poiščite naslednje elemente: $a_{21}, a_{32}, b_{12}, b_{22}, c_{13}, c_{31}, d_{33}$ in d_{32}.

   (c) Izračunajte naslednje matrike:

(i)	$A+B$	(vi)	$C-D$	(xi)	AC
(ii)	$A-C$	(vii)	$-2A$	(xii)	C^TD
(iii)	$A+D$	(viii)	$-B^T$	(xiii)	DC^T
(iv)	$B-2C$	(ix)	C^T	(xiv)	$B\cdot(-B^T)$
(v)	$B+3D$	(x)	BC	(xv)	B^TBC^T

V katerih primerih operacija ni izvedljiva?

2. Poiščite x, y, z, w, tako, da velja:

$$\begin{bmatrix} x-2 & y \\ 2 & 5-z \end{bmatrix} = \begin{bmatrix} 2 & x-y \\ w-1 & w+z \end{bmatrix}.$$

3. Dana je matrika

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Izračunajte A^2, A^3 ter A^4. Koliko je A^n za $n \in \mathbb{N}$?

4. Za naslednje pare matrik A in B ugotovite, ali sta matriki $C = A^2 + 2AB + B^2$ in $D = (A+B)^2$ enaki. Odgovor utemeljite.

   (a) $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$ in $B = \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$ in $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$.

5. Dani sta realni matriki

$A = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$ in $B = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}$.

Če obstaja, določite $y \in \mathbb{R}$ tako, da bo matrika $D = B^TAB$ diagonalna.

6. Pokažite, da sta za množenje matriki

$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ in $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

komutativni pri poljubnih $a, b, c, d \in \mathbb{R}$.

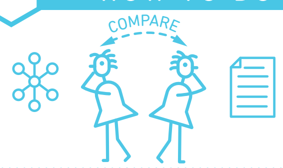
LEARN TO STUDY USING...

Dual Coding

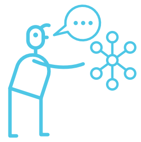
COMBINE WORDS AND VISUALS

LEARNINGSCIENTISTS.ORG

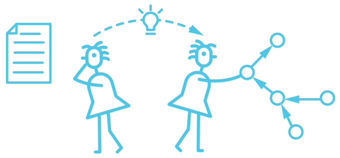
HOW TO DO IT



Look at your class materials and find visuals. Look over the visuals and compare to the words.



Look at visuals, and explain in your own words what they mean.

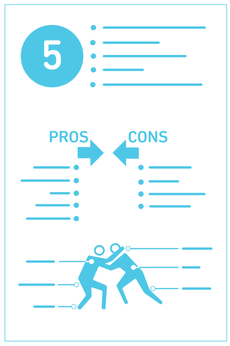


Take information that you are trying to learn, and draw visuals to go along with it.

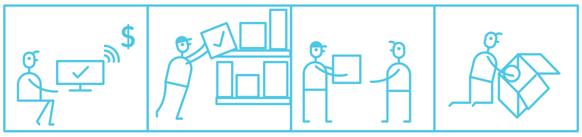
HOLD ON!

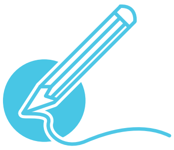
Try to come up with different ways to represent the information visually, for example an infographic, a timeline, a cartoon strip, or a diagram of parts that work together.

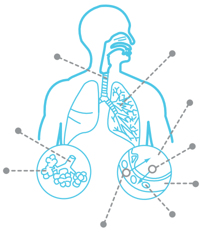
INFOGRAPHIC



CARTOON STRIP

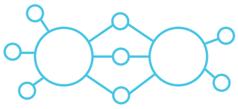






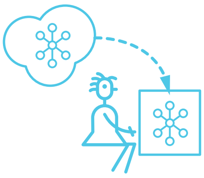
DIAGRAM

TIMELINE



GRAPHIC ORGANIZER

Work your way up to drawing what you know from memory.



RESEARCH

Read more about dual coding as a study strategy

Mayer, R. E., & Anderson, R. B. (1992). The instructive animation: Helping students build connections between words and pictures in multimedia learning. *Journal of Educational Psychology*, *4*, 444-452.

Content by Yana Weinstein (University of Massachusetts Lowell) & Megan Smith (Rhode Island College) | Illustrations by Oliver Caviglioli (teachinghow2s.com/cogsci)
Funding provided by the APS Fund for Teaching and Public Understanding of Psychological Science

Navodila.

1. Naj bodo dane matrike

$$A=\begin{bmatrix}2&-1&-2\\4&0&-1\\1&2&3\end{bmatrix},\ B=\begin{bmatrix}1&-2\\-1&3\end{bmatrix},\ C=\begin{bmatrix}2&-1\\4&-3\\6&-2\end{bmatrix}\text{ in } D=\begin{bmatrix}0&0&1\\-1&0&1\\0&-1&0\end{bmatrix}.$$

   (a) Določite velikost matrik A, B, C in D?

   (b) Poiščite naslednje elemente: $a_{21}, a_{32}, b_{12}, b_{22}, c_{13}, c_{31}, d_{33}$ in d_{32}.

   (c) Izračunajte naslednje matrike:

(i)	$A+B$	(vi)	$C-D$	(xi)	AC
(ii)	$A-C$	(vii)	$-2A$	(xii)	C^TD
(iii)	$A+D$	(viii)	$-B^T$	(xiii)	DC^T
(iv)	$B-2C$	(ix)	C^T	(xiv)	$B\cdot(-B^T)$
(v)	$B+3D$	(x)	BC	(xv)	B^TBC^T

   V katerih primerih operacija ni izvedljiva?

① $A=\begin{bmatrix}2&-1&-2\\4&0&-1\\1&2&3\end{bmatrix}_{3\times3}$ $B=\begin{bmatrix}1&-2\\-1&3\end{bmatrix}_{2\times2}$ $C=\begin{bmatrix}2&-1\\4&-3\\6&-2\end{bmatrix}_{3\times2}$ $D=\begin{bmatrix}0&0&1\\-1&0&1\\0&-1&0\end{bmatrix}_{3\times3}$

$a_{21}=4$ $a_{32}=2$ $b_{12}=-2$ $b_{22}=3$ $c_{13}=/$ $c_{31}=-2$ $d_{33}=0$ $d_{32}=-1$

i) $A+B=//$

ii) $A-C=//$

iii) $A+D=\begin{bmatrix}2&-1&-1\\3&0&0\\1&1&3\end{bmatrix}$

iv) $B-2C=//$

v) $B+3D=//$

vi) $C-D=//$

vii) $-2A=\begin{bmatrix}-4&2&4\\-8&0&2\\-2&-4&-6\end{bmatrix}$

viii) $-B^T=\begin{bmatrix}-1&1\\2&-3\end{bmatrix}$

ix) $C^T=\begin{bmatrix}2&4&6\\-1&-3&-2\end{bmatrix}$

x) $BC=//$

xi) $AC=\begin{bmatrix}-12&5\\2&-2\\28&-13\end{bmatrix}$ $3\times3\ \ 3\times2$

xii) $C^TD=\begin{bmatrix}-4&-6&6\\3&2&-4\end{bmatrix}$ $2\times3\ \ 3\times3$

xiii) $DC^T=//$ $3\times3\ \ 2\times3$

xiv) $B\cdot(-B^T)=\begin{bmatrix}-5&7\\7&-10\end{bmatrix}$ $2\times2\ \ 2\times2$

xv) $B^TBC^T=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}\begin{bmatrix}1&-2\\-1&3\end{bmatrix}\begin{bmatrix}2&4&6\\-1&-3&-2\end{bmatrix}=\begin{bmatrix}2&-5\\-5&13\end{bmatrix}\begin{bmatrix}2&4&6\\-1&-3&-2\end{bmatrix}=\begin{bmatrix}9&23&22\\-23&-59&-56\end{bmatrix}$

$2\times2\ \ 2\times2\ \ 2\times3$, 2×2

2. Poiščite x, y, z, w, tako, da velja:

$$\begin{bmatrix}x-2&y\\2&5-z\end{bmatrix}=\begin{bmatrix}2&x-y\\w-1&w+z\end{bmatrix}.$$

② $\begin{bmatrix}x-2&y\\2&5-z\end{bmatrix}=\begin{bmatrix}2&x-y\\w-1&w+z\end{bmatrix}$

$x-2=2 \to x=4$

$y=x-y \to 2y=4 \to y=2$

$2=w-1 \to w=3$

$5-z=w+z \to 2z=5-w$

$2z=2 \to z=1$

$\begin{bmatrix}4-2&2\\2&5-1\end{bmatrix}=\begin{bmatrix}2&4-2\\3-1&3+1\end{bmatrix}$

3. Dana je matrika

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Izračunajte A^2, A^3 ter A^4. Koliko je A^n za $n \in \mathbb{N}$?

③ $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^4 = A^2 \cdot A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^5 = A^4 \cdot A = I \cdot A = A$
$A^6 = A^5 \cdot A = A \cdot A = A^2$
$A^7 = A^6 \cdot A = A^2 \cdot A = A^3$
$A^8 = A^7 \cdot A = A^3 \cdot A = A^4$ ~~.......~~ $= I$
$A^9 = A$

$$A^n = \begin{cases} A; & n = 1, 5, 9, 13, \ldots \\ A^2; & n = 2, 6, 10, 14, \ldots \\ A^3; & n = 3, 7, 11, 15, \ldots \\ A^4 = I; & n = 4, 8, 12, 16, \ldots \end{cases}$$

$A^{2020} = A^4 = I$

$2020 : 4 = 505$

4. Za naslednje pare matrik A in B ugotovite, ali sta matriki $C = A^2 + 2AB + B^2$ in $D = (A+B)^2$ enaki. Odgovor utemeljite.

(a) $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$ in $B = \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$ in $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$.

4) a) $A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix}$

$D = (A+B)^2 = \left(\begin{bmatrix} 9 & 1 \\ -1 & 11 \end{bmatrix}\right)^2 = \begin{bmatrix} 80 & 20 \\ -20 & 120 \end{bmatrix}$

$\begin{bmatrix} 9 & 1 \\ -1 & 11 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ -1 & 11 \end{bmatrix} = \begin{bmatrix} 80 & 20 \\ -20 & 120 \end{bmatrix}$

$C = A^2 + 2AB + B^2$

$A^2 = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 35 & -10 \\ 10 & 35 \end{bmatrix}$

$2AB = 2\begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix} = 2\begin{bmatrix} 20 & 5 \\ -5 & 30 \end{bmatrix} = \begin{bmatrix} 40 & 10 \\ -10 & 60 \end{bmatrix}$

$B^2 = \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ -20 & 45 \end{bmatrix}$

$C = \begin{bmatrix} 80 & 20 \\ -20 & 120 \end{bmatrix}$

$(A+B)(A+B) = A \cdot A + \underbrace{A \cdot B + B \cdot A}_{} + B \cdot B$

!!

$AB = AB$

komutativni matriki

b) $A=\begin{bmatrix}1&1&1\\1&1&-1\\0&2&1\end{bmatrix}$ $\quad D=(A+B)^2=\left(\begin{bmatrix}2&2&1\\2&2&1\\1&1&2\end{bmatrix}\right)^2=\begin{bmatrix}9&9&6\\9&9&6\\6&6&6\end{bmatrix}$

$B=\begin{bmatrix}1&1&0\\1&1&2\\1&-1&1\end{bmatrix}$

$$A^2=\begin{bmatrix}1&1&1\\1&1&-1\\0&2&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&-1\\0&2&1\end{bmatrix}=\begin{bmatrix}2&4&1\\2&0&-1\\2&4&-1\end{bmatrix}$$

$$2AB=2\begin{bmatrix}1&1&1\\1&1&-1\\0&2&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&2\\1&-1&1\end{bmatrix}=2\begin{bmatrix}3&1&3\\1&3&1\\3&1&5\end{bmatrix}=\begin{bmatrix}6&2&6\\2&6&2\\6&2&10\end{bmatrix}$$

$$B^2=\begin{bmatrix}1&1&0\\1&1&2\\1&-1&1\end{bmatrix}\begin{bmatrix}1&1&0\\1&1&2\\1&-1&1\end{bmatrix}=\begin{bmatrix}2&2&2\\4&0&4\\1&-1&-1\end{bmatrix}$$

$$D=A^2+2AB+B^2=\begin{bmatrix}10&8&9\\8&6&5\\9&5&8\end{bmatrix}$$

$C\neq D \Rightarrow$ ker $AB\neq BA$

5. Dani sta realni matriki
$A=\begin{bmatrix}2&-3\\-3&6\end{bmatrix}$ in $B=\begin{bmatrix}1&0\\y&1\end{bmatrix}$.
Če obstaja, določite $y\in\mathbb{R}$ tako, da bo matrika $D=B^TAB$ diagonalna.

5) $A=\begin{bmatrix}2&-3\\-3&6\end{bmatrix}$ $\quad B=\begin{bmatrix}1&0\\y&1\end{bmatrix}$ $\quad B^T=\begin{bmatrix}1&y\\0&1\end{bmatrix}$

$$D=B^TAB=\begin{bmatrix}1&y\\0&1\end{bmatrix}\begin{bmatrix}2&-3\\-3&6\end{bmatrix}B=\begin{bmatrix}-3y+2&6y-3\\-3&6\end{bmatrix}\begin{bmatrix}1&0\\y&1\end{bmatrix}$$

$$=\begin{bmatrix}-3y+2+y(6y-3)&6y-3\\-3+6y&6\end{bmatrix}$$

$6y-3=0$
$\underline{\underline{y=\frac{1}{2}}}$

$$\Rightarrow\begin{bmatrix}-\frac{3}{2}+2+\frac{1}{2}(3-3)&0\\0&6\end{bmatrix}\Rightarrow\begin{bmatrix}1/2&0\\0&6\end{bmatrix}$$

6. Pokažite, da sta za množenje matriki

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{ in } B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

komutativni pri poljubnih $a, b, c, d \in \mathbb{R}$.

6 $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

$$AB = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix}$$

$$BA = \begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} ac+bd & bc+ad \\ ad+bc & bd+ac \end{bmatrix}$$

$AB = BA$ za poljubne $a, b, c, d \in \mathbb{R}$