



Algebra I
EXAM
– JANUARY 26, 2022 –

Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) Write the equation of a plane in vectorial, general and parametric vectorial form. Then, derive the equation of a plane through 3 non-collinear points in terms of determinants. (6 points)

(b) Write the characterization of:

- Two parallel lines in terms of the cross product. (2 points)
- Two parallel planes (given in general forms) in terms of the cross product. (2 points)

Prove at least one of these characterizations. (3 points)

- Write the definition of rank of a matrix in terms of linear independency. (3 points)
- Prove the statement: A non-homogeneous system $AX = B$ has a solution if and only if $\text{rank}(A) = \text{rank}(A|B)$. (4 points)

2. Let A, B, C and D be consecutive points of a parallelogram. Point E divides the diagonal AC so that $|AE| : |EC| = 1 : 3$. Point F divides the diagonal BD so that $|BD| : |BF| = 4 : 3$. Let S be the point of intersection of line segments AF and ED .

- (a) Write the vector \overrightarrow{AS} as a linear combination of vectors $\vec{e} = \overrightarrow{AC}$ and $\vec{f} = \overrightarrow{BD}$. (8 points)

Hint: $|AS| = \frac{2}{3}|AF|$.

- (b) If $A(1, -2, 2)$, $B(3, -1, 4)$ and $C(2, 3, -3)$, find the angle between line segments AB and BC and determine the area of the parallelogram. (12 points)

Hint: The angle between two vectors can be obtained from the equation $\cos \varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

3. Let $\Pi : x - 4y + 2z - 7 = 0$ be a plane in space and let ℓ be the line given by the intersection of the planes $x - 2y - 4z = -3$ and $2x + y - 3z + 1 = 0$. Find the line p on the plane Π that is orthogonal to ℓ and contains the point given by the intersection between ℓ and Π . (20 points)

4. Discuss the solutions of the following system of linear equations with respect to parameters $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

(20 points)

5. Compute the determinant

$$\det \left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \cdots \begin{bmatrix} n & n-1 \\ n+1 & n \end{bmatrix} \right)$$

for every $n \in \mathbb{N}$.

(20 points)