Analysis I
UP - FAMNIT
Fall 2022 — October 20, 2022

Selected exercises 03

1. Decide if the following sets are bounded from above/below in $\mathbb{R}$:

   (a) $S = \{1, 2, 3\}$
   (b) $S = \{x \ : \ x \geq 5\}$
   (c) $S = \{0\} \cup \{x \ : \ x > 0\}$
   (d) $S = \{x^2 \ : \ x < -2\}$
   (e) $S = \{x^3 \ : \ x \in \mathbb{Z}\}$
   (f) $S = \{x^3 - x \ : \ x \geq 1\}$

2. Compute the supremum and infimum (if they exist) in the set S and prove by definition.

   (a) $S = [0, 2), \ S \subset \mathbb{R}$
   (b) $S = (-1, \sqrt{2}), \ S \subset \mathbb{Q}$

3. Find sets S and T with the following properties:

   (a) $\sup S = \inf T, \ S \cap T = \varnothing$
   (b) $\inf S = \sup T, \ S \subseteq T$
   (c) $\inf T = \min S, \ T$ has no minimal element.

4. Let $S \subseteq \mathbb{R}^{\geq 0}$ be a bounded set. Prove that the set T is bounded as well.

   (a) $T = \{5s \ : \ s \in S\}$
   (b) $T = \{-s^2 \ : \ s \in S\}$
   (c) $T = \{s^2 - s \ : \ s \in S\}$

5. Let A and B be non-empty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$. Analogously, define $A - B$. Prove the following equalities, or give a counterexample.

   (a) $\sup (A + B) = \sup A + \sup B$
   (b) $\sup (A - B) = \sup A - \sup B$
   (c) $\inf (A + B) = \inf A + \inf B$
   (d) $\inf (A - B) = \inf A - \sup B$

6. (a) Given the complex numbers $a = 2 + 3i$ and $b = -1 + i$, compute $|a - 5b|$.
   (b) Give a complex number with modulus (a.k.a. magnitude or absolute value) 2, and that is not a real number.

7. Give first few members of the following sequences. For each sequence determine if it is bounded, increasing/decreasing, convergent and calculate the limit (if it exists):

   (a) $a_n = \frac{3}{3n+7}$
   (b) $a_n = \sqrt{n^4 + n^2} - n^2$
   (c) $a_n = \frac{n+(-1)^n}{n-(-1)^n}$
   (d) $a_n = \frac{n^2+n}{n-1}$
   (e) $a_n = \cos(n\pi)$
   (f) $a_n = (-3)^n$
   (g) $a_n = \frac{2n^3+6n-3}{7n-3n^3+2}$