University of Primorska
UP FAMNIT
Academic year 2021/2022

Algebra I

Exam

– February 10, 2023 –

Time: 135 *minutes. Maximum number of points:* 100. *You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!*

1. (a) Write the definition of the scalar (dot) product and state at least 3 of its properties. Then, prove the following statement: For any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$, it holds that $\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}| \cos \varphi$, where φ is the angle between vectors $\vec{u}$ and $\vec{v}$. *(7 points)*

   (b) In $\mathbb{R}^3$, derive the equation of a line in vectorial, parametric and canonical form. *(6 points)*

   (c) Write down and prove Cramer's rule for solving systems of linear equations. *(7 points)*

2. (a) Let the vectors $\vec{v}$ and $\vec{u}$ be given as $\vec{v} = t\vec{a} + 17\vec{b}$ and $\vec{u} = 3\vec{a} - \vec{b}$. Find all values of the parameter $t \in \mathbb{R}$ for which the vectors $\vec{v}$ in $\vec{u}$ are orthogonal, using the facts that $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\measuredangle(a, b) = \frac{2\pi}{3}$. *(10 točk)*

   (b) Are the vectors $\vec{a} = (-1, 3, 2)$, $\vec{b} = (2, -3, -4)$ and $\vec{c} = (-3, 12, 6)$ co-planar? If yes, write $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$. *(10 točk)*

3. We are given lines $\ell = (7, 0, 1) + \lambda(2, 1, -2)$ and $q : x + 3 = 4 - 4y = 20 - 4z$.

   (a) Find the intersection of lines ℓ and q. *(7 points)*

   (b) Find the equation of the plane containing lines ℓ and q. *(7 points)*

   (c) Compute the angle between lines ℓ and q. *(6 points)*

4. Find the values of a and b for which the system of linear equations

$$
\begin{aligned}
x + ay + z &= 3 \\
x + 2y + 2z &= b \\
x + 5y + 3z &= 9
\end{aligned}
$$

   is consistent. Under which condition will this system have a unique solution (write the solution)? *(20 points)*

5. Show that the determinant of the $n \times n$ matrix

$$
\begin{bmatrix}
3 & 2 & 0 & \cdots & 0 & 0 & 0 \\
1 & 3 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 3 & 2 \\
0 & 0 & 0 & \cdots & 0 & 1 & 3
\end{bmatrix}
$$

   is equal to $2^{n+1} - 1$. *(20 points)*