Homework 2

Ⓐ a) $A|B = \{x \in \mathbb{Q}^{\geq 0} \mid x^2 < 7\} \,|\, \{x \in \mathbb{Q}^{\geq 0} \mid x^2 \geq 7\}$ $\sqrt{7}$

b) $A|B = \{x \in \mathbb{Q}^{\geq 0} \mid x^3 < 9\} \,|\, \{x \in \mathbb{Q}^{\geq 0} \mid x^3 \geq 9\}$ $\sqrt[3]{9}$

c) $A = \{x \in \mathbb{Q}^{\geq 0} \mid x > 2 \wedge x^2 > 3\}$ $\sqrt{3}+2$

$B = \mathbb{Q}^{\geq 0} \setminus A$

Ⓑ $A|B = \left([0, \frac{x_1}{y_1}) \,|\, [\frac{x_1}{y_1}, \infty)\right)$ $E|F = \left([0, \frac{x_3}{y_3}) \,|\, [\frac{x_3}{y_3}, \infty)\right)$

$C|D = \left([0, \frac{x_2}{y_2}) \,|\, [\frac{x_2}{y_2}, \infty)\right)$ $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3} \in \mathbb{Q}^{\geq 0}$

want to show $\underbrace{A|B \cdot C|D} = C|D \cdot A|B$

LS $\{a \cdot b \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2}\} = \left([0, a \cdot b) \,|\, [a \cdot b, \infty)\right)$

RS $\{b \cdot a \mid 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq a < \frac{x_1}{y_1}\} = \left([0, b \cdot a) \,|\, [b \cdot a, \infty)\right)$

commutativity in $\mathbb{R}^{\geq 0}$ $a \cdot b = b \cdot a \rightarrow$ property of $\mathbb{Q}^{\geq 0}$

$\Rightarrow \left([0, a \cdot b) \,|\, [a \cdot b, \infty)\right) = \left([0, b \cdot a) \,|\, [b \cdot a, \infty)\right)$

want to show $\underbrace{(A|B \cdot C|D) \cdot E|F} = A|B \cdot (C|D \cdot E|F)$

LS $\{(a \cdot b) \cdot c \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= \left([0, (a \cdot b) \cdot c) \,|\, [(a \cdot b) \cdot c, \infty)\right)$

RS $\{a \cdot (b \cdot c) \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= \left([0, a \cdot (b \cdot c)) \,|\, [a \cdot (b \cdot c), \infty)\right)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c) \rightarrow$ property of $\mathbb{Q}^{\geq 0}$

associativity in $\mathbb{R}^{\geq 0}$

$\Rightarrow \left([0, a \cdot (b \cdot c)) \,|\, [a \cdot (b \cdot c), \infty)\right) = \left([0, (a \cdot b) \cdot c) \,|\, [(a \cdot b) \cdot c, \infty)\right)$

C $\alpha = A|B = ([0, \frac{x_1}{y_1}) \mid [\frac{x_1}{y_1}, \infty))$ $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3} \in \mathbb{Q}^{\geq 0}$

$\beta = C|D = ([0, \frac{x_2}{y_2}) \mid [\frac{x_2}{y_2}, \infty))$

$\gamma = E|F = ([0, \frac{x_3}{y_3}) \mid [\frac{x_3}{y_3}, \infty))$

want to show $((A|B) + (C|D)) \cdot (E|F) = (A|B) \cdot (E|F) + (B|C) \cdot (E|F)$

LS: $\{(a+b) \cdot c \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= ([0, (a+b) \cdot c) \mid [(a+b) c, \infty))$

RS $\{ac + bc \mid 0 \leq a < \frac{x_1}{y_1} \wedge 0 \leq b < \frac{x_2}{y_2} \wedge 0 \leq c < \frac{x_3}{y_3}\}$

$= ([0, ac + bc) \mid [ac + bc, \infty))$

distributivity in $\mathbb{Q}^{\geq 0}$ $(a+b) \cdot c = ac + bc$ - Property of $\mathbb{Q}^{\geq 0}$

$\Rightarrow ([0, (a+b) \cdot c) \mid [(a+b) \cdot c, \infty)) = ([0, ac + bc) \mid [ac + bc, \infty))$

D S, T - bounded sets $S \cdot T = \{x \cdot y : x \in S \wedge y \in T\}$

Show that if S and T are bounded so is $S \cdot T$

$s \in S$ $t \in T$

$\inf s = A_* | B_*$ $\inf t = C_* | D_*$

$s = A_s | B_s$ $t = C_t | D_t$

$\sup s = A | B$ $\sup t = C | D$

$\inf s \cdot \inf t = \inf$ (infinum of $S \cdot T$)

$\sup s \cdot \sup t = \sup$ (supremum of $S \cdot T$)

$\inf s < s < \sup s$ $\inf t < t < \sup t$

$A_* \subseteq A_s \subseteq A$ $C_* \subseteq C_t \subseteq C$

$\inf s \cdot \inf t \leq s \cdot t < \sup s \cdot \sup t$

E. Prove that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field. (Use properties of $\mathbb{R}$)

<u>Properties of a field:</u>

1 → $(\mathbb{Q}\sqrt{2}, +)$ → abelian group (Closed, Commutative, associative, inverse and identity)
2 → $(\mathbb{Q}\sqrt{2}, \cdot)$ → abelian group
3 → $(\mathbb{Q}\sqrt{2}, +, \cdot)$ → distributive law

1. $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = \underbrace{(a_1 + a_2)}_{} + \underbrace{(b_1 + b_2)}_{\mathbb{Q}}\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ → closure

- Commutativity and associativity apply because $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and those are properties of $\mathbb{Q}$ (and $\mathbb{R}$)

additive identity: $0 + 0 \cdot \sqrt{2} = 0$
inverse: $a_1 + b_1\sqrt{2} - a_1 - b_1\sqrt{2} = 0$

2. $(a_1 + b_1\sqrt{2}) \cdot (a_2 + b_2\sqrt{2}) = (a_1 a_2) + (a_1 b_2 \sqrt{2}) + (a_2 b_1 \sqrt{2}) + 2b_1 b_2 =$

$= \underbrace{(a_1 a_2)}_{\mathbb{Q}} + \underbrace{(a_1 b_2 + a_2 b_1)}_{\mathbb{Q}}\sqrt{2} + \underbrace{2b_1 b_2}_{\mathbb{Q}} \in \mathbb{Q}(\sqrt{2})$ - Closure

- Commutativity and associativity apply because $a_1, b_1, a_2, b_2 \in \mathbb{Q}$ and these are properties of $\mathbb{Q}$ (and $\mathbb{R}$)

multiplicative identity: 1
inverse: $a_1 + b_1\sqrt{2} \cdot \frac{1}{a_1 + b_1\sqrt{2}} = 1$

3. $((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) \cdot (a_3 + b_3\sqrt{2}) =$
$= (a_1 + b_1\sqrt{2}) \cdot (a_3 + b_3\sqrt{2}) + (a_2 + b_2\sqrt{2}) \cdot (a_3 + b_3\sqrt{2}) \in \mathbb{R}$

→ distributivity holds because it holds in $\mathbb{R}$