Week 10

The problem is to calculate, if possible, a common solution for a system of m linear algebraic equations with n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad &\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where the x_i's are the unknowns and the a_{ij}'s and b_i's are known constants. There are exactly 3 possibilities:

1. Unique solution - there is one and only one set of values for the x_i's that satisfy all equations simultaneously

2. No solution - there is no set of values for the x_i's that satisfy all equations simultaneously

3. Infinitely many solutions

Standard Exercises

1. Let E_k denote the k^{th} equation $E_k : a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$ and write the system as

$$S = \left\{ \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{matrix} \right\}$$

For a linear system S, describe each of the three elementary operations.

1. Interchange the i^{th} and j^{th} equations

$$S = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_j \\ \vdots \\ E_m \end{matrix} \right\} \longrightarrow S' = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_j \\ \vdots \\ E_i \\ \vdots \\ E_m \end{matrix} \right\}$$

2. Replace the i^{th} equation with a non zero multiple of itself

$$S = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_m \end{matrix} \right\} \longrightarrow S' = \left\{ \begin{matrix} E_1 \\ \vdots \\ \alpha E_i \\ \vdots \\ E_m \end{matrix} \right\}$$

3. Replace the j^{th} equation by a combination of itself plus a multiple of the i^{th} equation

$$S = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_j \\ \vdots \\ E_m \end{matrix} \right\} \longrightarrow S' = \left\{ \begin{matrix} E_1 \\ \vdots \\ E_j + \alpha E_i \\ \vdots \\ E_m \end{matrix} \right\}$$

② Consider the following three systems where the coefficients are the same for each system, but the right-hand sides are different

$$\begin{array}{rcl|c|c} 4x - 8y + 5z & = & 1 & 0 & 0 \\ 4x - 7y + 4z & = & 0 & 1 & 0 \\ 3x - 4y + 2z & = & 0 & 0 & 1 \end{array}$$

Solve all three systems at one time by performing Gaussian elimination on an augmented matrix of the form $[A \mid b_1 \mid b_2 \mid b_3]$

$$\left[\begin{array}{ccc|c|c|c} 4 & -8 & 5 & 1 & 0 & 0 \\ 4 & -7 & 4 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c|c|c} 1 & -4 & 3 & 1 & 0 & -1 \\ 4 & -7 & 4 & 0 & 1 & 0 \\ 3 & -4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c|c|c} 1 & -4 & 3 & 1 & 0 & -1 \\ 0 & 9 & -8 & -4 & 1 & 4 \\ 0 & 8 & -7 & -3 & 0 & 4 \end{array}\right]$$

$$\xrightarrow{R_2 - R_3} \left[\begin{array}{ccc|c|c|c} 1 & -4 & 3 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 8 & -7 & -3 & 0 & 4 \end{array}\right] \xrightarrow[R_3 - 8R_2]{R_1 + 4R_2} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -1 & -3 & 4 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 5 & -8 & 4 \end{array}\right] \xrightarrow[R_2 + R_3]{R_1 + R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & -4 & 3 \\ 0 & 1 & 0 & 4 & -7 & 4 \\ 0 & 0 & 1 & 5 & -8 & 4 \end{array}\right]$$

Solutions:
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -4 & 3 \\ 4 & -7 & 4 \\ 5 & -8 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \text{ or } \begin{bmatrix} -4 \\ -7 \\ -8 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

this matrix is A^{-1}!

③ Suppose that matrix B is obtained by performing a sequence of row operations on matrix A. Explain why A can be obtained by performing row operations on B.

We can apply the inverse operations!

That is, if it takes n row operations to transform A into B and:

- Step k is to swap rows i and j, then when transforming B into A, step (n-k) is to swap rows j and i
- step k is to replace row i, E_i, with multiple $\alpha \cdot E_i$, then when transforming B into A, step (n-k) is to replace row i with multiple $\frac{1}{\alpha} E_i$
- Step k is to combine row j, E_j with $E_j + \alpha E_i$, step (n-k) when transforming B into A is to replace row j, E_j, with $E_j - \alpha E_i$.

④ Find angles α, β, γ such that

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

where $0 \le \alpha < 2\pi$, $0 \le \beta < 2\pi$, and $0 \le \gamma < \pi$.

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 3 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{array}\right] \Rightarrow -8\tan\gamma = 0 \Rightarrow \boxed{\gamma = 0}$$

$$R_2 \Rightarrow 4\cos\beta = -4 \Rightarrow \cos\beta = -1 \Rightarrow \boxed{\beta = \pi}$$

$$R_1 \Rightarrow 2\sin\alpha + 1 = 3$$

$$\Rightarrow 2\sin\alpha = 2 \Rightarrow \sin\alpha = 1 \Rightarrow \boxed{\alpha = \frac{\pi}{2}}$$

① Let $A \in \mathbb{C}^{3\times3}$. Prove that if $A^T = -A$ then $\det(A) = 0$.

from notes ∇ $\begin{cases} \star \ \det(cA) = c^m \det(A) \quad \text{if } A \text{ is } m\times m \\ \star \ \det(AB) = \det(A)(\det(B)) \quad \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \\ \star \ \det(A^T) = \det(A) \end{cases}$

$A^T = -A \quad \Rightarrow \quad \det(A^T) = \det(-A)$

$\Rightarrow \det(A) = (-1)^3 \det(A) \Rightarrow \det(A) = -\det(A)$

$\Rightarrow \boxed{\det(A) = 0}$

② Suppose $A, B, C \in \mathbb{R}^{4\times4}$ such that $\det(A) = 1$, $\det(B) = 4$, $\det(C) = 2$.

Compute $\det(AB^{-1}C^T(2B)^2)$

$\det(AB^{-1}C^T(2B)^2) = \det(A)\det(B^{-1})\det(C^T)\det((2B)^2)$

$= \dfrac{\det(A)\ \det(C)\ \det(2B)\det(2B)}{\det(B)}$

$= \det(A)\det(C)\, 2^4 \cdot 2^4 \det(B)$

$= (1)(2)(16)(16)(4) = \boxed{2048}$

③ Let $A \in \mathbb{C}^{2\times2}$ such that

$$A\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}1&1\\1&1\end{bmatrix}A.$$

Prove that A is symmetric.

$A\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}a_{11}+a_{12} & a_{11}+a_{12}\\ a_{21}+a_{22} & a_{21}+a_{22}\end{bmatrix}$

$\begin{bmatrix}1&1\\1&1\end{bmatrix}A = \begin{bmatrix}a_{11}+a_{21} & a_{12}+a_{22}\\ a_{11}+a_{21} & a_{12}+a_{22}\end{bmatrix}$

$\Rightarrow a_{11}+a_{12} = a_{11}+a_{21}$

$\Rightarrow a_{12} = a_{21}$

$\Rightarrow \boxed{A \text{ Symmetric } ∇}$

④ Find $A, B \in \mathbb{C}^{m\times m}$, nonzero, such that $AB = 0$.

$A = \begin{bmatrix}1&-1\\-1&1\end{bmatrix} \qquad B = \begin{bmatrix}1&1\\1&1\end{bmatrix}$

$AB = \begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

⑤ Is it always true that if $AC = BC$ then $A = B$?
When does $AC = BC$ imply $A = B$?

$$AC = BC \iff AC - BC = 0 \iff (A-B)C = 0$$

From last exercise, If $A - B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

then $A \neq B$ and $(A-B)C = 0$?

★ If C has an inverse $\Rightarrow ACC^{-1} = BCC^{-1}$

$$\Rightarrow A = B \quad \square$$