Algebra I - Matrični račun: 10. vaje

Determinante. Osnovne lastnosti determinant.

1. Pokažite, da je determinanta $n \times n$ matrike $\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$ enaka $n+1$.

2. Naj bosta $a, b \in \mathbb{R}$. Izračunajte determinanto $n \times n$ matrike

$$\begin{bmatrix} a & b & b & \cdots & b & b \\ b & a & 0 & \cdots & 0 & 0 \\ b & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & 0 & \cdots & a & 0 \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}.$$

3. Izračunajte determinanto naslednjih matrik:

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ -1 & 0 & 3 & -4 & 1 \\ -2 & 3 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

4. Z uporabo osnovnih operacij na vrsticah pokažite, da je naslednja determinanta enaka 0.

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix}$$

5. Naj bo $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Pokažite, da za matriko $A \in \mathbb{R}^{2\times 2}$ velja:

$$\det(A) = 1 \Longleftrightarrow A^T E A = E.$$

6. Naj bodo $A, B, X \in \mathbb{R}^{n\times n}$ in naj velja $\det(B) = \det(A) - 1$ ter $3A^2X = XB$. Poiščite $\det(X)$.

LEARN TO STUDY USING…

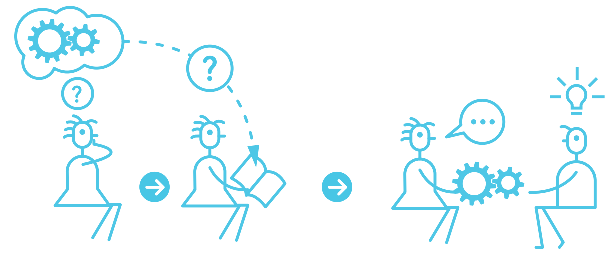
Elaboration

EXPLAIN AND DESCRIBE IDEAS WITH MANY DETAILS

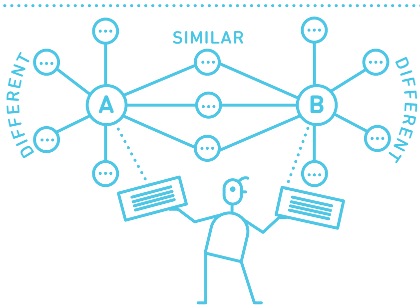
LEARNINGSCIENTISTS.ORG

HOW TO DO IT

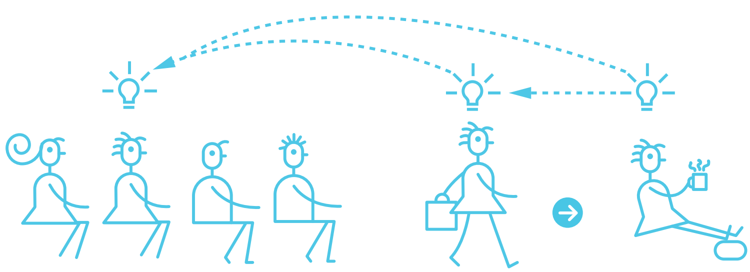
Ask yourself questions while you are studying about how things work and why, and then find the answers in your class materials and discuss them with your classmates.



As you elaborate, make connections between different ideas to explain how they work together. Take two ideas and think of ways they are similar and different.



Describe how the ideas you are studying apply to your own experiences or memories. As you go through your day, make connections to the ideas you are learning in class.



HOLD ON!

Make sure the way you are explaining and describing an idea is accurate. Don't overextend the elaborations, and always check your class materials or ask your teacher.

Work your way up so that you can describe and explain without looking at your class materials.

RESEARCH

Read more about elaboration as a study strategy

McDaniel, M. A., & Donnelly, C. M. (1996). Learning with analogy and elaborative interrogation. *Journal of Educational Psychology, 88*, 508-519.

Wong, B. Y. L. (1985). Self-questioning instructional research: A review. *Review of Educational Research*, *55*, 227-268.

Content by Yana Weinstein (University of Massachusetts Lowell) & Megan Smith (Rhode Island College) | Illustrations by Oliver Caviglioli (teachinghow2s.com/cogsci)
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Navodila.

1. Pokažite, da je determinanta $n \times n$ matrike $\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}$ enaka $n+1$.

5. naloga na prejšnjih vajah + 1. naloga na teh vajah

(1) $A_n = \begin{bmatrix} 2 & 1 & & \emptyset \\ 1 & 2 & 1 & \\ & \ddots & \ddots & \\ \emptyset & & 2 & 1 \\ & & 1 & 2 \end{bmatrix}_{n \times n}$ $\det(A_n) = n+1$

$A_1 = [2]$ $\det(A_1) = 2 = n+1$

$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\det(A_2) = 4 - 1 = 3 = n+1$

$A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{matrix}$ $\det(A_3) = 8 + 0 + 0 - 0 - 2 - 2 = 4$

$A_4 = \begin{bmatrix} +2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $\det(A_4) = 2 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} +1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} =$

$= 2 \cdot \det(A_3) - 1 \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot \det(A_3) - \det(A_2) =$

$= 2 \cdot 4 - 3 = 5 = n+1$

Predpostavimo, da trditev velja za $t \leq n$, ($\det(A_t) = t+1$)
pokažimo, da velja za $t+1$.

$A_n = \begin{bmatrix} +2 & 1 & & \emptyset \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & \\ \emptyset & & 2 & 1 \\ & & 1 & 2 \end{bmatrix}_{n \times n}$

$\det(A_n) = 2 \cdot \begin{vmatrix} 2 & 1 & & \\ 1 & 2 & \ddots & \emptyset \\ & \ddots & \ddots & \\ \emptyset & & 2 & 1 \\ & & 1 & 2 \end{vmatrix}_{(n-1)\times(n-1)} - 1 \cdot \begin{vmatrix} 1 & 0 & \cdots & & 0 \\ 1 & 2 & 1 & 0 \cdots & 0 \\ 0 & 1 & 2 & 1 \; 0 \cdots & 0 \\ \vdots & & \ddots & 2 & 1 \\ 0 & & & 1 & 2 \end{vmatrix}_{(n-1)\times(n-1)}$

$\stackrel{\text{i.p.}}{=} 2 \cdot \det(A_{n-1}) - 1 \cdot 1 \cdot \begin{vmatrix} 2 & 1 & & \emptyset \\ 1 & 2 & \ddots & \\ \emptyset & \ddots & 2 & 1 \\ & & 1 & 2 \end{vmatrix}_{(n-2)\times(n-2)} = 2 \cdot \det(A_{n-1}) - \det(A_{n-2}) =$

$\stackrel{\text{po i.p.}}{=} 2 \cdot ((n-1)+1) - 1 \cdot ((n-2)+1) =$

$= 2n - n + 1 = \underline{\underline{n+1}}$

2. Naj bosta $a, b \in \mathbb{R}$. Izračunajte determinanto $n \times n$ matrike

$$\begin{bmatrix} a & b & b & \cdots & b & b \\ b & a & 0 & \cdots & 0 & 0 \\ b & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & 0 & \cdots & a & 0 \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}.$$

(2) $\begin{bmatrix} a & b & b & \cdots & b \\ b & a & 0 & & \\ b & 0 & a & & \emptyset \\ \vdots & & & \ddots & \\ b & & \emptyset & & a \end{bmatrix}$

$A_1 = [a] = \quad \det(A_1) = a$

$A_2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \det(A_2) = a \cdot a - b \cdot b = a^2 - b^2$

$A_3 = \begin{bmatrix} a & b & b \\ b & a & 0 \\ b & 0 & a \end{bmatrix} \begin{matrix} a & b \\ b & a \\ b & 0 \end{matrix} \quad \det(A_3) = a^3 + 0 + 0 - ab^2 - ab^2 = a^3 - 2ab^2$

$A_4 = \begin{bmatrix} a^+ & b^- & b^+ & b^- \\ b & a & 0 & 0^+ \\ b & 0 & a & 0^- \\ b & 0 & 0 & a^+ \end{bmatrix} \quad \det(A_4) = -b \begin{vmatrix} b & a & 0 \\ b & 0 & a \\ b & 0 & 0 \end{vmatrix} + a \begin{vmatrix} a & b & b \\ b & a & 0 \\ b & 0 & a \end{vmatrix}$

$= -b(0 + a^2b + 0 - 0 - 0 - 0) + a(a^3 + 0 + 0 - ab^2 - ab^2)$

$= -a^2b^2 + a^4 - 2a^2b^2$

$= a^4 - 3a^2b^2$

Predpostavka: $\det(A_n) = a^n - (n-1)a^{n-2}b^2$

Pokažimo, da predpostavka velja za $\forall n \in \mathbb{N}$.

$A_n = \begin{bmatrix} a & b & \cdots & b \\ b & a & & \emptyset \\ \vdots & & \ddots & a \\ b & \emptyset & & a \end{bmatrix}$

$\det(A_n) = (-1)^{n+1} \cdot b \begin{vmatrix} b & a & 0 & \cdots & 0 \\ b & 0 & a & & \emptyset \\ \vdots & & & \ddots & a \\ b & 0 & \emptyset & & 0 \end{vmatrix}_{(n-1)\times(n-1)} + \overset{+1}{(-1)^{2n}} \cdot a \begin{vmatrix} a & b & \cdots & b \\ b & a & & \emptyset \\ \vdots & & \ddots & \\ b & \emptyset & & a \end{vmatrix}_{(n-1)\times(n-1)}$

$= (-1)^{n+1} \cdot b (-1)^{n-1+1} \cdot b \begin{vmatrix} a & 0 & \cdots & 0 \\ 0 & a & & \emptyset \\ \vdots & & \ddots & \\ 0 & 0 & \emptyset & a \end{vmatrix}_{(n-2)\cdot(n-2)} + a \cdot \det(A_{n-1}) =$

$= (-1)^{2n+1} \cdot b \cdot b \cdot (-1)^n \cdot a^{n-2} + a \cdot \det(A_{n-1})$

$= \overset{-1}{(-1)^{2n+1}} b^2 a^{n-2} + a \det(A_{n-1})$

$\overset{\text{po i.p.}}{=} -a^{n-1} b^2 + a \cdot (a^{n-1} - ((n-1)-1) a^{(n-1)-2} \cdot b^2)$

$= -a^{n-2}b^2 + a(a^{n-1} - (n-2)a^{n-3}b^2)$

$= -a^{n-2}b^2 + a^n - (n-2)a^{n-2}b^2$

$= \underline{a^n - (n-1)a^{n-2}b^2}$

3. Izračunajte determinanto naslednjih matrik:

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 2 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 3 & -1 \\ 0 & 1 & 0 & 2 & 0 \\ -1 & 0 & 3 & -4 & 1 \\ -2 & 3 & 4 & 0 & -3 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

(3) $A = \begin{bmatrix} ^{+}1 & ^{-}-2 & 3 & 4 \\ 2 & ^{+}0 & 1 & 3 \\ 1 & ^{-}0 & -1 & 2 \\ 0 & ^{+}1 & 1 & 3 \end{bmatrix}$

$$\det A = \begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 0 & 1 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{vmatrix} = -(-2)\cdot \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} + 1\cdot \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 1 & -1 & 2 \end{vmatrix} =$$

$$= 2\cdot(-6+0\cancel{+3}+0-4\cancel{-3}) + 1\,(2\cancel{+9}-8-4\cancel{+3}-12) = -20-10 = \underline{-30}$$

$$\det B = \begin{vmatrix} ^{+}1 & 2 & 0 & 3 & -1 \\ ^{-}0 & 1 & 0 & 2 & 0 \\ ^{+}-1 & 0 & 3 & -4 & 1 \\ ^{-}-2 & 3 & 4 & 0 & -3 \\ ^{+}0 & ^{-}0 & ^{+}1 & ^{-}2 & ^{+}0 \end{vmatrix} = 1\cdot \begin{vmatrix} ^{+}1 & 2 & 3 & -1 \\ ^{-}0 & ^{+}1 & ^{-}-2 & 0 \\ -1 & 0 & -4 & 1 \\ -2 & 3 & 0 & -3 \end{vmatrix} - 2\cdot \begin{vmatrix} ^{+}1 & ^{-}2 & 0 & -1 \\ 0 & ^{+}1 & 0 & 0 \\ -1 & 0 & 3 & 1 \\ -2 & 3 & 4 & -3 \end{vmatrix} =$$

$$= 1\cdot\left(1\cdot \begin{vmatrix} 1 & 3 & -1 \\ -1 & -4 & 1 \\ -2 & 0 & -3 \end{vmatrix} - 2\cdot \begin{vmatrix} 1 & 2 & -1 \\ -1 & 0 & 1 \\ -2 & 3 & -3 \end{vmatrix}\right) - 2\cdot 1\cdot \begin{vmatrix} 1 & 0 & -1 \\ -1 & 3 & 1 \\ -2 & 4 & -3 \end{vmatrix} =$$

$$= (12-6+0+8+0-9) - 2\cdot(+0-4\cancel{+3}+0\cancel{-3}-6) - 2\cdot(-9-0\cancel{+4}-6\cancel{-4}-0) =$$

$$= 5 - 2\cdot(-10) - 2\cdot(-15) = \underline{55}$$

4. Z uporabo osnovnih operacij na vrsticah pokažite, da je naslednja determinanta enaka 0.

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix}$$

(4) $\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix} v_3 - v_1 \sim \begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x+2 & 2y+2 & 2z+2 \end{vmatrix} /:2 \sim \begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ x+1 & y+1 & z+1 \end{vmatrix}$

Enaki vrstici $\Rightarrow$ det $= 0$

5. Naj bo $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Pokažite, da za matriko $A \in \mathbb{R}^{2\times 2}$ velja:

$$\det(A) = 1 \Longleftrightarrow A^T E A = E.$$

(5) $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A \in \mathbb{R}^{2\times 2}$

$\det(A) = 1 \Longleftrightarrow A^T \cdot E \cdot A = E$

Naj bo $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^T E A = E \Longleftrightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} c & -a \\ d & -b \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} ca - ac & cb - ad \\ da - bc & db - bd \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Longleftrightarrow \begin{bmatrix} 0 & cb - ad \\ da - bc & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Longleftrightarrow cb - ad = -1 \quad /\cdot(-1) \rightarrow \quad ad - cb = 1$

$da - bc = 1$

$\Longleftrightarrow da - bc = 1$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = da - bc$

6. Naj bodo $A, B, X \in \mathbb{R}^{n\times n}$ in naj velja $\det(B) = \det(A) - 1$ ter $3A^2X = XB$. Poiščite $\det(X)$.

(6) $A, B, X \in \mathbb{R}^{n\times n}$ $\det(B) = \det(A) - 1$ $3A^2 \cdot X = X \cdot B$

$\det(X) = ?$

$\det(3 \cdot I \cdot A^2 \cdot X) = \det(X \cdot B)$

$\det(3I) \cdot \det(A^2) \cdot \det(X) = \det(X) \cdot \det(B)$

$3^n \cdot (\det(A))^2 \cdot \det(X) = \det(X) \cdot \det B$

$3^n (\det(A))^2 \cdot \det(X) = \det(X) \cdot (\det(A) - 1)$

Označimo: $a = \det(A)$

$x = \det(X)$

$3^n \cdot a^2 \cdot x = x(a-1)$

$3^n a^2 \cdot x - ax + x = 0$

$(3^n a^2 - a + 1) X = 0$

$3^n a^2 - a + 1 = 0 \quad \vee \quad X = 0 \rightarrow \boxed{\det(X) = 0}$!

$D = b^2 - 4ac$

$= 1 - 4 \cdot 3^n \cdot 1$

$= 1 - 4 \cdot 3^n < 0$

ker $n \in \mathbb{N}$

$\Rightarrow$ ni realni ničel

$\Rightarrow 3^n a^2 - a + 1 \neq 0$ Vedno !!