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ŠTUDENTSKA ORGANIZACIJA UNIVERZE NA PRIMORSKEM

A. Show that any seq. of positive real numbers either has a subsequence that converges, or else a subseq. that diverges to ∞

- Let a_n be a seq. of positive real numbers.

<u>Need to prove:</u>

a) a_n has a subseq. that converge

b) b_n has a subseq. that diverge to ∞

<u>Proof of a)</u> Suppose that a_n is bounded. Then there exists $\cancel{\exists}$ real number $M > 0$ s.t. $a_n \leq M$ for all $n \in \mathbb{N}$. In that case, the seq. a_n is bounded of positive real numbers. ~~B~~ With this, by Bolzano-Weierstrass theorem, a_n has a convergent subseq. a_{n_k}. This concludes that the subseq. (a_{n_k}) converges (in this case).

BW Thm: every bounded seq. has a convergent subsequence -(from Tutorials)

<u>Proof of b)</u> Suppose that a_n is unbounded. Then $\exists$ subseq. a_{n_k} of a_n s.t. $a_{n_k} \to \infty$ as $k \to \infty$.
This concludes that the subseq. (a_{n_k}) diverges to ∞. (in this case).

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B. For each of the following series, determine whether the series converges or diverges. (Sum from 1 to ∞) For those that converge, find the value that they converge to.

i) $\sum (4^n+2^n)/3^n$ $\cancel{=\sum \frac{4^n}{3^n} + \sum \frac{2^n}{3^n}}$ $\quad a_n = \frac{4^n+2^n}{3^n}$

By applying ratio test, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{3^{n+1}}}{\frac{4^n}{3^n}} + \frac{\frac{2^{n+1}}{3^{n+1}}}{\frac{2^n}{3^n}} = \frac{1}{1} + \frac{1}{1} = 2 > 1 \Rightarrow \text{diverges}$$

ii) $\sum \sin^n(3)$

The seq. $\sin^n(3)$ converges to $\frac{1}{1-\sin(3)}$ because

$\sum r^n = \frac{1}{1+r}$ if $|r|<1$ and $\sin 3 = 0.14$

iii) $\sum 1/(n^2+2n)$, we conclude that $\frac{1}{n^2+2n} < \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ converges, by <u>direct comparison</u> test we have that $\sum \frac{1}{n^2+2n}$ also converges, to 1.

iv) $\sum(\sqrt{n+2}-\sqrt{n})$ $\Rightarrow$ first we rationalise:

$\Rightarrow \sqrt{n+2}-\sqrt{n} \cdot \frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}}$. From this we have

$$2\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 2\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = 2\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{\frac{n}{n}+\frac{2}{n}}+\frac{\sqrt{n}}{\sqrt{n}}} = 2\lim_{n\to\infty} \frac{1}{\sqrt{1+0}+1} = 2\lim \frac{1}{2} = \frac{2}{2} = 1$$

$\Rightarrow$ we have $0<1<\infty$, [redacted] $\sum(\sqrt{n+2}-\sqrt{n})$ is equiconvergent to $\sum \frac{1}{\sqrt{n}}$, so they converge both.

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C. For each of the following series, determine whether the series converges, diverges to $\pm\infty$, or diverges, not to $\pm\infty$. Don't try to find what the series converge to.

i) $\sum \sin^2(3n+2)/(2n^2-3)$, $a_n = \sin^2(3n+2)/(2n^2-3)$

we define $b_n = \frac{1}{n^2}$, and it converges, so we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim \frac{\sin^2\frac{3n+2}{2n^2-3}}{\frac{1}{n^2}\left(\frac{3n+2}{2n^2-3}\right)^2}\cdot\left(\frac{3n+2}{2n^2-3}\right)^2 = \lim \frac{n^2(9n^2+12n+4)}{4n^4-12n^2+9} = \frac{9}{4} \in \mathbb{R}^+$$

that means that $\sum a_n$ and $\sum b_n$ have the same nature, and $\sum a_n$ is convergent.

ii) $\sum 1/(n-2\pi)$, $\underbrace{a_n = 1/(n-2\pi) > 1/n = b_n}$, and $\sum \frac{1}{n}$ is divergent

$a_n = \frac{1}{n-2\pi}$

$b_n = \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = \lim \frac{n}{n-2\pi} = 1 \in \mathbb{R}^+$$

$\Rightarrow$ this means that $\sum a_n$ and $\sum b_n$ have same nature, and this shows that $\Rightarrow \sum a_n$ is divergent

iii) $\sum (n+1)/(n^3+2)$, $a_n = n+1/(n^3+2)$ and $b_n = n/n^3$, or $1/n^2$

$\Rightarrow \sum b_n$ is convergent (because $\alpha = 2 > 1$), and we have

$$\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n+1}{n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^3+n^2}{n^3+n^2} = 1 \in \mathbb{R}^+$$

$\Rightarrow$ this means that $\sum a_n$ and $\sum b_n$ have same nature, and this shows that

$\Rightarrow \sum a_n$ converges

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D. Determine whether each of the following series are absolutely convergent, conditionally convergent, or divergent.

i) $\cancel{\pm 2}$ $\sum (-2)^{n^2} / (n+2)!$ $\Rightarrow \sum \left| \frac{(-2)^{n^2}}{(n+2)!} \right| = \sum \frac{2^{n^2}}{(n+2)!} \Rightarrow a_n = \frac{2^{n^2}}{(n+2)!}$

by applying ratio test we get: because we can't go any further, we apply another ratio test!

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{(n+1)^2}}{(n+3)!}}{\frac{2^{n^2}}{(n+2)!}} = \frac{2^{n^2+2n+1}}{(n+3)2^{n^2}} = \frac{2^{2n+1}}{n+3}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{2n+1}}{n+3}}{\frac{2^{2n}}{n+2}} = \frac{2(n+2)}{n+3} = \frac{2n+4}{n+3}$$ Now we compute the limit:

$$\Rightarrow \lim \frac{2n+4}{n+3} = \lim \frac{\frac{2n}{n} + \frac{4}{n}^{\to 0}}{\frac{n}{n} + \frac{3}{n}^{\to 0}} = \frac{2}{1} = 2 > 0$$, which means $\sum a_n = +\infty$

The seq. is divergent

ii) $\sum (-3)^n / (3n^2+6^n)$: First we use ~~comparison test~~, we get:

$|a_n| = \left| \frac{(-3)^n}{3n^2+6^n} \right| = \frac{3^n}{3n^2+6^n}$ for $3n^2+6^n > 6^n /:$ $\Rightarrow \frac{1}{3n^2+6n} < \frac{1}{6^n} / \cdot 3^n \Rightarrow \underbrace{\frac{3^n}{3n^2+6^n}}_{a_n} < \underbrace{\frac{3^n}{6^n}}_{b_n}$

$b_n = \frac{3^n}{6^n} = \left(\frac{1}{2}\right)^n / \frac{1}{2^n}$

Now we use route test (from Tutorials), and get

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$$, which means the seq. is convergent

$\Rightarrow$ Since b_n is convergent, $|a_n|$ absolutely converges by comparison test.

iii) $\sum (-4)^n / (4^n \sqrt{2n+1}) \sim \sum \frac{(-1)^n \cancel{4^n}}{\cancel{4^n}\sqrt{2n+1}} = \sum \frac{(-1)^n}{\sqrt{2n+1}}$

with absolute value, we have $\sum \frac{1}{\sqrt{2n+1}} \sim \sum b_n \Rightarrow b_n = \frac{1}{\sqrt{n}}$

we now compute $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{2n+1}} = \sqrt{\lim_{n\to\infty} \frac{n}{n(2+\frac{1}{n}^{\to 0})}} = \frac{1}{\sqrt{2}} \in \mathbb{R}^+$

now we can see that $\sum a_n$ and $\sum b_n$ have same nature, which means they are divergent.

$$\sum \frac{(-1)^n 4^n}{4^n \sqrt{2n+1}} = \sum \frac{(-1)^n}{\sqrt{2n+1}}$$

$a_n = \frac{1}{\sqrt{2n+1}}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$, and $a_{n+1} = \frac{1}{\sqrt{2n+3}} < a_n \frac{1}{\sqrt{2n+1}}$

$\Rightarrow$ from there we conclude that $\sum (-1)^n a_n$ is divergent hw7 by Zhivko Stoimchev 89221056

E. Show that if $a_n = 5n^3 - 19n^2 + 7n - 3$, then $a_n = \Theta(n^3)$

$\exists \alpha > 0$ and $\exists \beta > 0$, $\exists N$ s.t. $\forall n > N$

$\alpha n^3 \leq 5n^3 - 19n^2 + 7n - 3 \leq \beta n^3$

$1 \cdot n^3 \leq 5n^3 - 19n^2 + 7n - 3 \leq 5n^2 + 19n^3 + 7n^3 + 3n^2 = 29n^3$

that is equivalent with $0 \leq 4n^3 - 19n^2 + 7n - 3$

from what we get $19n^2 - 7n + 3 \leq 4n^3$

Now we show that $\exists N = 10$ and $\forall n > N$, it holds.

$\exists \alpha = 1$ and $\exists \beta = 29$

$\exists N = 10 \; \forall n > N$

$\alpha \cdot \beta_n \leq a_n \leq \beta \cdot b_n$