14 Cramer's Rule and some properties of determinants

The purpose of this section is to present Cramer's Rule and some properties of determinants that are helpful.

Transposition Doesn't Alter Determinants

- $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$ for all $n \times n$ matrices.

Effects of Row Operations

Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the three elementary row operations:

Type I: Interchange rows i and j.
Type II: Multiply row i by $\alpha \neq 0$.
Type III: Add α times row i to row j.

The value of $\det(\mathbf{B})$ is as follows:

- $\det(\mathbf{B}) = -\det(\mathbf{A})$ for Type I operations.
- $\det(\mathbf{B}) = \alpha \det(\mathbf{A})$ for Type II operations.
- $\det(\mathbf{B}) = \det(\mathbf{A})$ for Type III operations.

Invertibility and Determinants

- $\mathbf{A}_{n\times n}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$
  or, equivalently,
- $\mathbf{A}_{n\times n}$ is singular if and only if $\det(\mathbf{A}) = 0$.

Product Rules

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for all $n \times n$ matrices.
- $\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det(\mathbf{A})\det(\mathbf{D})$ if $\mathbf{A}$ and $\mathbf{D}$ are square.

Block Determinants

If $\mathbf{A}$ and $\mathbf{D}$ are square matrices, then

$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{cases} \det(\mathbf{A})\det\left(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) & \text{when } \mathbf{A}^{-1} \text{ exists,} \\ \det(\mathbf{D})\det\left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) & \text{when } \mathbf{D}^{-1} \text{ exists.} \end{cases}$$

The matrices $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are called the ***Schur complements*** of $\mathbf{A}$ and $\mathbf{D}$, respectively.

1. For

$$\mathbf{A} = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix},$$

$\lambda_i \neq 0$, find $\det(A)$.

2. Consider the block matrix $B = \begin{bmatrix} A_{r\times r} & C_{r\times s} \\ R_{s\times r} & B_{s\times s} \end{bmatrix}$. When the indicated inverses exist, the matrices defined by

$$S = B - RA^{-1}C \text{ and } T = A - CB^{-1}R$$

are called the Schur complements of A and B, respectively.

(a) If A and S are both nonsingular, verify that

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{R} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{R}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{R}\mathbf{A}^{-1} & \mathbf{S}^{-1} \end{pmatrix}.$$

(b) If B and T are nonsingular, verify that

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{R} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{T}^{-1} & -\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ -\mathbf{B}^{-1}\mathbf{R}\mathbf{T}^{-1} & \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{R}\mathbf{T}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix}.$$

Cramer's Rule

In a nonsingular system $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}$, the i^{th} unknown is

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i = \left[\mathbf{A}_{*1} \middle| \cdots \middle| \mathbf{A}_{*i-1} \middle| \mathbf{b} \middle| \mathbf{A}_{*i+1} \middle| \cdots \middle| \mathbf{A}_{*n}\right]$. That is, $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that column $\mathbf{A}_{*i}$ has been replaced by $\mathbf{b}$.

3. Determine the value of unknown x in

$$\begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$$

4. Determine the value of unknown y in the following system of equations:

$$\begin{cases} 2x + & 4y - & 5z = & -5 \\ -x - & y + & z = & 0 \\ 2x + & y - & z = & 1 \end{cases}.$$

5. Determine the value of t for which $x_3(t)$ is minimized in

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}.$$

6. By considering rank-one updated matrices, derive the following formulas.

(a)
$$\begin{vmatrix} \frac{1+\alpha_1}{\alpha_1} & 1 & \cdots & 1 \\ 1 & \frac{1+\alpha_2}{\alpha_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \frac{1+\alpha_n}{\alpha_n} \end{vmatrix} = \frac{1+\sum \alpha_i}{\prod \alpha_i}.$$

(b)
$$\begin{vmatrix} \alpha & \beta & \beta & \cdots & \beta \\ \beta & \alpha & \beta & \cdots & \beta \\ \beta & \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \beta & \cdots & \alpha \end{vmatrix}_{n\times n} = \begin{cases} (\alpha-\beta)^n\left(1+\frac{n\beta}{\alpha-\beta}\right) & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

(c)
$$\begin{vmatrix} 1+\alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & 1+\alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & 1+\alpha_n \end{vmatrix} = 1+\alpha_1+\alpha_2+\cdots+\alpha_n.$$

7. With respect to different values of parameter λ, find unknown x in the following system of equations:

$$\begin{cases} (\lambda-2)x- & 3y+ & 2z = & 1 \\ 3x- & 3y+ & (\lambda-3)z = & 1 \\ x- & y+ & 2z = & -1 \end{cases}.$$

8. With respect to different values of parameter λ, determine the value of unknown x in

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 2\lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$$

9. Find the value of paramter λ so that the following system

$$\begin{cases} 8z-3x & -6y = & \lambda x \\ 2x+y & +4z = & \lambda y \\ 4x+3y & +z = & \lambda z \end{cases}.$$

has infinitely many solutions. After that, for the largest value of λ, find a general solution of the system.

10. For the following tridiagonal matrix, A_n, let $D_n = \det(A_n)$. Derive the formula $D_n = 2D_{n-1} - D_{n-2}$ to deduce that $D_n = n+1$.

$$\mathbf{A}_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{n\times n}.$$

11. Determine all values of λ for which the matrix $A - \lambda I$ is singular, where

$$A = \begin{pmatrix} 0 & -3 & -2 \\ 2 & 5 & 2 \\ -2 & -3 & 0 \end{pmatrix}.$$

14.1 Solutions

1.

Problem: For $\mathbf{A} = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix}$, $\lambda_i \neq 0$, find $\det(\mathbf{A})$.

Solution: Express $\mathbf{A}$ as a rank -oneupdated matrix $\mathbf{A} = \mathbf{D} + \mathbf{e}\mathbf{e}^T$, where $\mathbf{D} =$ diag $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{e}^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$. Apply second property of rank-oneupdated matrix to produce

$$\det(\mathbf{D} + \mathbf{e}\mathbf{e}^T) = \det(\mathbf{D})\left(1 + \mathbf{e}^T\mathbf{D}^{-1}\mathbf{e}\right) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right).$$

2. Use block multiplication to verify that the product of the matrix with its inverse is the identity matrix.

3. $x = 3$.

4. $y = 2$.

5.

Problem: Determine the value of t for which $x_3(t)$ is minimized in

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}.$$

Solution: Only one component of the solution is required, so it's wasted effort to solve the entire system. Use Cramer's rule to obtain

$$x_3(t) = \frac{\begin{vmatrix} t & 0 & 1 \\ 0 & t & 1/t \\ 1 & t^2 & 1/t^2 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = \frac{1 - t - t^2}{-1} = t^2 + t - 1, \quad \text{and set} \quad \frac{d\,x_3(t)}{dt} = 0$$

to conclude that $x_3(t)$ is minimized at $t = -1/2$.

6.

Let:

$$\det\left(\mathbf{I}+\mathbf{c}\mathbf{d}^T\right)=1+\mathbf{d}^T\mathbf{c}, \qquad (*)$$
$$\det\left(\mathbf{A}+\mathbf{c}\mathbf{d}^T\right)=\det(\mathbf{A})\left(1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\right). \qquad (**)$$

(a) Use the results of example from above with $\lambda_i = 1\,/\alpha_i$.

(b) Recognize that the matrix $\mathbf{A}$ is a rank-one updated matrix in the sense that

$$\mathbf{A}=(\alpha-\beta)\mathbf{I}+\beta\mathbf{e}\mathbf{e}^T, \quad \text{where} \quad \mathbf{e}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}.$$

If $\alpha=\beta$, then $\mathbf{A}$ is singular, so $\det(\mathbf{A})=0$. If $\alpha\neq\beta$, then (**) may be applied to obtain

$$\det(\mathbf{A})=\det\Big((\alpha-\beta)\mathbf{I}\Big)\left(1+\frac{\beta\mathbf{e}^T\mathbf{e}}{\alpha-\beta}\right)=(\alpha-\beta)^n\left(1+\frac{n\beta}{\alpha-\beta}\right).$$

(c) Recognize that the matrix is $\mathbf{I}+\mathbf{e}\mathbf{d}^T$, where

$$\mathbf{e}=\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix} \quad \text{and} \quad \mathbf{d}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}.$$

Apply (*) to produce the desired formula.

7. 1° $\lambda\neq 5$ and $\lambda\neq 9$, $x=\frac{4}{\lambda-9}$, 2° $\lambda=9$, $D=0$, $D_x\neq 0$, system does not have a solution. 3° $\lambda=5$, $D=D_x=0$, $(x,y,z)=(t+1,t,-1)$, $t\in\mathbb{R}$.

8. $D=0$, $D_x=1-2\lambda$ 1° $\lambda=\frac{1}{2}$, $D=D_x=0$, $(x,y,z)=(2-t,2,t)$, $t\in\mathbb{R}$. 2° $\lambda\neq\frac{1}{2}$, $D_x\neq 0$, system does not have a solution.

9. $D=(\lambda-1)(\lambda+7)(\lambda-5)=0$, $\lambda=5$: $(x,y,z)=(t,6t,11t/2)$, $t\in\mathbb{R}$.

10.

Expanding in terms of cofactors of the first row produces $D_n=2\mathring{A}_{11}-\mathring{A}_{12}$. But $\mathring{A}_{11}=D_{n-1}$ and expansion using the first column yields

$$\mathring{A}_{12}=(-1)\begin{vmatrix}-1&-1&0&\cdots&0\\0&2&-1&\cdots&0\\0&-1&2&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&2\end{vmatrix}=(-1)(-1)D_{n-2},$$

so $D_n=2D_{n-1}-D_{n-2}$. By recursion (or by direct substitution), it is easy to see that the solution of this equation is $D_n=n+1$.

11.

I method: Compute $\det(A - \lambda I)$ on a "smart way", so that you immediately get solution of the form $\det(A - \lambda I) = (\lambda - a)(\lambda - b)(\lambda - c)$ for some integers a, b, c.

II method:

Hint: If $p(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \lambda_1\alpha_1 + \alpha_0$ is a monic polynomial with integer coefficients, then the integer roots of $p(\lambda)$ are a subset of the factors of α_0.

$\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The cofactor expansion in terms of the first row yields

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= -\lambda \begin{vmatrix} 5-\lambda & 2 \\ -3 & -\lambda \end{vmatrix} + 3 \begin{vmatrix} 2 & 2 \\ -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & 5-\lambda \\ -2 & -3 \end{vmatrix} \\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4,\end{aligned}$$

so $\mathbf{A} - \lambda\mathbf{I}$ is singular if and only if $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$. According to the hint, the integer roots of $p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$ are a subset of $\{\pm 4, \pm 2, \pm 1\}$. Evaluating $p(\lambda)$ at these points reveals that $\lambda = 2$ is a root, and either ordinary or synthetic division produces

$$\frac{p(\lambda)}{\lambda - 2} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1).$$

Therefore, $p(\lambda) = (\lambda - 2)^2(\lambda - 1)$, so $\lambda = 2$ and $\lambda = 1$ are the roots of $p(\lambda)$, and these are the values for which $\mathbf{A} - \lambda\mathbf{I}$ is singular.