Homework 8

A. i) $\sum_{n=1}^{\infty} \cos^n(2)$

$\cos(2) + \cos^2(2) + \cos^3(2)$

$\cos^3(2) < \cos^2(2) < \cos(2)$, therefore the sum converges to 1

Ratio test

$$\lim_{n\to\infty} \frac{\cos^{n+1}(2)}{\cos^n(2)} = \lim_{n\to\infty} \frac{\cancel{\cos^n(2)}\cdot\cos(2)}{\cancel{\cos^n(2)}} = \lim_{n\to\infty} \cos(2) \approx 0{,}99939$$

$\cos(2) < 1$, $\therefore \sum_{n=1}^{\infty} \cos^n(2) \to$ converges (to 1)

ii) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

$$\frac{1}{n^2+3n} < \frac{1}{n^2}$$

Since, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by direct comparison test

$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$ also converges (to 1)

iii) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

$$\sqrt{n+1} - \sqrt{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \quad (\div\sqrt{n}) = \cancel{\lim_{n\to\infty} \frac{\sqrt{n}\cdot 1}{\sqrt{n+1}+\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{\frac{n}{n}+\frac{1}{n}} + \frac{\sqrt{n}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+0}+1} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

$0 < \frac{1}{2} < \infty$, $\therefore \sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, so both diverge

B) i) $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{2n^2}$

$0 \leq \sin^2(n) \leq 1$

$2n^2+1 \geq 2n^2 \Rightarrow \frac{1}{2n^2+1} \leq \frac{1}{2n^2} \leq \frac{1}{n^2}$

By direct comparison test, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2n^2+1}$ also converges

ii) $\sum_{n=1}^{\infty} \frac{1}{(n^3-\pi)}$

$\lim_{n\to\infty} \frac{\frac{1}{n^3-\pi}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3-\pi} \overset{\cdot n^3}{=} \lim_{n\to\infty} \frac{1}{1-\underbrace{\frac{\pi}{n^3}}_{\to 0}} = 1$

$0 < 1 < \pi$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then $\sum_{n=2}^{\infty} \frac{1}{n^3-\pi}$ also converges (they are equiconvergent)

iii) $\sum_{n=1}^{\infty} \frac{n-1}{n^2-1}$

$\frac{n-1}{n^2-1} = \frac{\cancel{n-1}}{(\cancel{n-1})(n+1)} = \frac{1}{n+1} < \frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{n-1}{n^2-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n(\cancel{n-1})}{(\cancel{n-1})(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} \overset{\cdot n}{=} \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\underbrace{\frac{1}{n}}_{\to 0}} =$

$= \lim_{n\to\infty} \frac{1}{1+0} = 1 \qquad 0 < 1 < \infty$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$ (they both diverge)

(C.) $\sum_{n=1}^{\infty} a_n \Rightarrow$ convergent series, b_n - bounded, nonnegative sequence

$0 \le b_n \le M$ (positive, bounded

$$\sum_{n=1}^{\infty} a_n \cdot b_n < \sum_{n=1}^{\infty} a_n \cdot M$$

$\sum_{n=1}^{\infty} a_n \cdot M \to$ converges (by Arithmetic of series for all $M \neq 0$)
(A.M)

Therefore by direct comparison test, $\sum_{n=1}^{\infty} a_n \cdot b_n$ also converges.

(D.) if $a_n = 4n^3 - 23n^2 + 4n - 1$, then $a_n = \Theta(n^3)$

$\frac{n^3}{2} \ominus 4n^3 \ominus 5n^3$

$\frac{n^3}{2} \ominus 23n^2 \ominus 5n^3$

$\frac{n^3}{2} \ominus 4n \ominus 5n^3$ for $n \ge 1$

$\frac{n^3}{2} \ominus 1 \ominus 5n^3$

$4n^3 \ominus n^3$

$23n^2 \ominus n^3$

$4n \ominus n^3$

$1 \ominus n^3$

(E.) if $\lim_{n \to \infty} a_n = 0$, then $\Theta(a_n + 1) = \Theta 1$

$\lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} 1 = 1$

$\Rightarrow \lim_{n\to\infty} a_n + 1 = 1$ (by AoL)

$\frac{a_n+1}{2} < a_n + 1 < 2(a_n+1)$

$\frac{1}{2} < 1 < 2$

$\alpha = \frac{1}{2}$ $\beta = 2$ $\Rightarrow$ for both

$\sum_{n=1}^{\infty} 2$ diverges, $\therefore$ $\sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty} a_n + 1$ both diverge, (are equiconvergent)