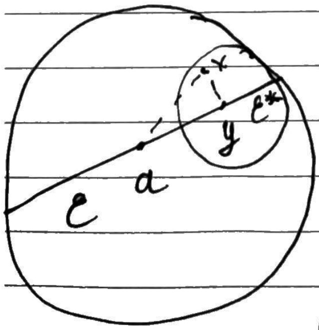
Homework 10

A. Let H be a set of all points (x, y) in $\mathbb{R}^2$, we also have $H = \{(x,y) \in \mathbb{R} \mid x^2 + 3y^2 = 12\}$.
Now in H, we have a seq (x_n, y_n) s.t. it converges in $\mathbb{R}^2$

$x_n \to x$ $\quad x, y \in \mathbb{R}^2$, so $\quad x_n + y_n \to x + y$ (*)
$y_n \to y$

Since $(x_n, y_n) \in H$, we get $x_n^2 + 3y_n^2 = 12$
from (*) and if we square both sides we get:
$x_n^2 + y_n^2 \to x^2 + y$. Since this holds we get that $x^2 + 3y^2 = 12$
$(x,y) \in H$, it also follows that the limit of the seq.
is also in the set, so from this and the definition of close set
$\Rightarrow H$ is closed set.
We know that H is subset of $\mathbb{R}^2$, so H is a closed subset of $\mathbb{R}^2$.
$x^2 + 3y^2 = 12$ is a circle, and we know that the max. ~~point~~ distance between two points is the diameter $\Rightarrow x^2 + 3y^2 = 12$ is bounded.
Since this two points are in H we get that H is bounded.

B. We have metric space M with metric $d(M, d)$
W.T.S: any ε-ball is open set



$a \in M$, where a is a point
From the def for ε-ball we have
$B_\varepsilon(a) = \{x \in M : d(x,a) < \varepsilon\}$ or $B_\varepsilon^{(d)}(a)$.
Now we take ε^* s.t.
$\left.\begin{array}{l} \varepsilon^* = \varepsilon - d(y,a) \\ \text{and } x \in B_{\varepsilon^*}(y) \end{array}\right\}$ so we get $d(x,y) < \varepsilon^*$

Using the Δ-inequality we get:
$d(x,a) \le d(x,y) + d(y,a) < \varepsilon - d(y,a) = \varepsilon^*$
$d(x,a) \le d(x,y) + d(y,a) < \varepsilon^* + d(y,a) = \varepsilon$
or $d(x,a) < \varepsilon$

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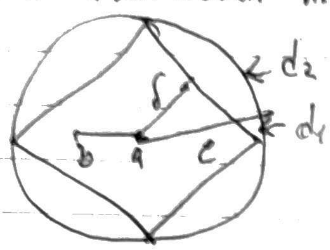
So we get that $B_\varepsilon^{(d)}$ a is open ball, also we have that $B_{\varepsilon^*}^{(d)}(y) \subseteq B_\varepsilon^{(d)}(a)$.
So from the definition for open set
$\Rightarrow B_\varepsilon^{(d)}(a)$ is open set
so any ε-ball is an open set.

C. Eveledean metric $d_2 = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$
Manhattan metric $d_1 = |x_1-y_1| + |x_2-y_2|$



Let A be a set and for any point $a \in A$ and $\varepsilon > 0$, there is δ s.t. $\delta > 0$ and we apply def for open sets. We choose $\delta < \varepsilon$, so we have: $d_1(a,b) < \delta \Rightarrow d_2(a,b) < \varepsilon$ (i)
and $d_1(a,b) > \delta \Leftarrow d_2(a,b) < \varepsilon$ (ii)

WTS: That these metrics define the same open set A.
i) $d_1(a,b) < d_2(a,b)$ then $B_1(a,\delta) \subseteq B_2(a,\varepsilon)$

So if A is an open set in d_1, for each $a \in A$, $\delta > 0$ and $\varepsilon > 0$
$B_1(a,\delta) \subseteq A \Rightarrow B_2(a,\varepsilon) \subseteq A$
$\Rightarrow A$ is open set.
ii) Now we have $d_2(a,b) \le d_1(a,b)$ then $B_2(a,\varepsilon) \subseteq B_1(a,\delta)$
So if A is an open set in d_2, for each $a \in A$, $\delta > 0$ and $\varepsilon > 0$
$B_2(a,\varepsilon) \subseteq A \Rightarrow B_1(a,\delta) \subseteq A$
$\Rightarrow A$ is open set ∎

D. We know that L is the set of acc points of a seq. in A
so $A \cup L$ contains all of these points or
$A \cup L$ contains the limit of a seq in A.
Since $A \cup L$ contains all limit points of A, $A \cup L$ is closed
WTP: $\bar{A} \subseteq A \cup L$
Let $a \in \bar{A}$, so we have $a \in A$ or $a \in \bar{A} \setminus A$.
So $a \in A$ is trivial so we only work wit $a \in \bar{A} \setminus A$
We get that $a \notin A$ and $a \in \bar{A}$
Because A doesn't contain limit points $\Rightarrow$ a is a limit point of A, $a \in L$
So from $a \in \bar{A}$ and $a \in L$ we get that $\bar{A} \subseteq A \cup L$

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E. Let A be a set which contains two point x, y with $d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$ and let $A \subseteq \mathbb{R}^3$. ~~A is seq. ... A is seq. compact~~

WTP: $A \subseteq \mathbb{R}^3$ is seq. comp $\Leftrightarrow$ A closed + A bounded

i) For $A \subseteq \mathbb{R}^3$ is seq.-comp $\Rightarrow$ A closed + A bounded

Let (a_n) be a seq. in A, whic converges to a, $a \in \mathbb{R}$, so every subseq of A converges to a.

Since a is limit and $a \in A$, we can use the proposition of closed.

If the limit of every convergent seq. of set belongs to the set, then the set is closed $\Rightarrow$ A is closed

Next suppose that A is unbounded, which is contradiction.

$\Rightarrow \exists (a_n)$ in A s.t. $|a_n| \to \infty$ as $n \to \infty$.

So every subseq. of (a_n) is unbounded and diverges, and this implies that (a_n) has no convergent sebseq. $\rightarrow\leftarrow$

Since this contradicts the A is seq.-compact $\Rightarrow$ A is bounded

ii) For $A \subseteq \mathbb{R}^3$ is seq.-comp $\Leftarrow$ A cosed + A bounded

For (a_n) being a seq. in A, we get that (a_n) is bounded since A is bounded

Since every bounded seq. of real numbers has a convergent subseq. $\Rightarrow$ (a_n) has a convergent subseq.

We know that A is closed and ~~the~~ it contains the limit of the subseq. $\Rightarrow$ A is seq. comp

$\blacksquare$