Univerza na Primorskem
UP FAMNIT
Študijsko leto 2021/2022

Algebra I

2. kolokvij

– 18. Januar 2022 –

Čas pisanja: 135 *minut. Maksimalno število točk:* 100. *Dovoljena je uporaba pisala in kalkulatorja. Pišite razločno in utemeljite vsak odgovor. Srečno!*

1. (a) Kdaj je matrka simetrična? Zapišite definicijo transponirane matrike in naštejte vsaj tri lastnosti transponiranja. Dokažite naslednjo izjavo: Za vsako kvadratno matriko A je $A + A^T$ simetrična matrika. *(6 točk)*

   (b) Zapišite definicijo $n \times n$ determinante (z uporabo permutacij) in naštejte vsaj štiri primere uporabe determinant. *(6 točk)*

   (c) Zapišite in dokažite Cramerjevo pravilo za reševanje sistema linearnih enačb. *(8 točk)*

2. Za katere vrednosti $\beta \in \mathbb{R}$ bo imel naslednji sistem linearnih enačb neskončno mnogo rešitev? Za vse dobljene vrednosti tudi poiščite rešitve.

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= \beta \\ 4x + y + 10z &= \beta^2 \end{aligned}$$

(20 točk)

3. Za katere vrednosti $a \in \mathbb{R}$ bo determinanta matrike

$$A = \begin{bmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{bmatrix}$$

enaka 30? *(20 točk)*

4. (a) Za matrike $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ in $C = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$, rešite matrično enačbo $(A + 3I)(X - I) = B$. *(10 točk)*

   (b) Naj bo $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Pokažite da za matriko $A \in \mathbb{R}^{2\times 2}$ velja

$$det(A) = 1 \;\Leftrightarrow\; A^T E A = E.$$

(10 točk)

5. Zapišite matriko $A = \begin{bmatrix} 2 & -2 & 1 \\ 6 & -1 & 5 \\ 3 & 7 & 4 \end{bmatrix}$ v obliki $A = LU$, kje je L spodnje trikotna matrika ki ima na glavni diagonali vse elemente enake 1, in U zgornje trikotna matrika. *(20 točk)*

THE LU FACTORIZATION

We have now come full circle, and we are back to where the text began—solving a nonsingular system of linear equations using Gaussian elimination with back substitution. This time, however, the goal is to describe and understand the process in the context of matrices.

If $\mathbf{Ax} = \mathbf{b}$ is a nonsingular system, then the object of Gaussian elimination is to reduce $\mathbf{A}$ to an upper-triangular matrix using elementary row operations. If no zero pivots are encountered, then row interchanges are not necessary, and the reduction can be accomplished by using only elementary row operations of Type III. For example, consider reducing the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$$

to upper-triangular form as shown below:

$$\begin{aligned} \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} &\longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{pmatrix} \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix} \\ &\longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \mathbf{U}. \end{aligned} \tag{3.10.1}$$

We learned in the previous section that each of these Type III operations can be executed by means of a left-hand multiplication with the corresponding elementary matrix $\mathbf{G}_i$, and the product of all of these $\mathbf{G}_i$'s is

$$\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}.$$

In other words, $\mathbf{G}_3\mathbf{G}_2\mathbf{G}_1\mathbf{A} = \mathbf{U}$, so that $\mathbf{A} = \mathbf{G}_1^{-1}\mathbf{G}_2^{-1}\mathbf{G}_3^{-1}\mathbf{U} = \mathbf{LU}$, where $\mathbf{L}$ is the lower-triangular matrix

$$\mathbf{L} = \mathbf{G}_1^{-1}\mathbf{G}_2^{-1}\mathbf{G}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}.$$

Thus $\mathbf{A} = \mathbf{LU}$ is a product of a lower-triangular matrix $\mathbf{L}$ and an upper-triangular matrix $\mathbf{U}$. Naturally, this is called an $\mathbf{LU}$ ***factorization*** of $\mathbf{A}$.

Example 3.10.1

Once $\mathbf{L}$ and $\mathbf{U}$ are known, there is usually no need to manipulate with $\mathbf{A}$. This together with the fact that the multipliers used in Gaussian elimination occur in just the right places in $\mathbf{L}$ means that $\mathbf{A}$ can be successively overwritten with the information in $\mathbf{L}$ and $\mathbf{U}$ as Gaussian elimination evolves. The rule is to store the multiplier ℓ_{ij} in the position it annihilates—namely, the (i,j)-position of the array. For a 3×3 matrix, the result looks like this:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\textit{Type III operations}} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21} & u_{22} & u_{23} \\ \ell_{31} & \ell_{32} & u_{33} \end{pmatrix}.$$

For example, generating the LU factorization of

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$$

by successively overwriting a single 3×3 array would evolve as shown below:

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 12 & 16 \end{pmatrix} \; R_3 - 4R_2 \longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \end{pmatrix}.$$

Thus

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

This is an important feature in practical computation because it guarantees that an LU factorization requires no more computer memory than that required to store the original matrix $\mathbf{A}$.

Example 3.10.2

Problem 1: Use the LU factorization of $\mathbf{A}$ to solve $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}.$$

Problem 2: Suppose that after solving the original system new information is received that changes $\mathbf{b}$ to

$$\tilde{\mathbf{b}} = \begin{pmatrix} 6 \\ 24 \\ 70 \end{pmatrix}.$$

Use the LU factors of $\mathbf{A}$ to solve the updated system $\mathbf{Ax} = \tilde{\mathbf{b}}$.

Solution 1: The LU factors of the coefficient matrix were determined in Example 3.10.1 to be

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}.$$

The strategy is to set $\mathbf{Ux} = \mathbf{y}$ and solve $\mathbf{Ax} = \mathbf{L}(\mathbf{Ux}) = \mathbf{b}$ by solving the two triangular systems

$$\mathbf{Ly} = \mathbf{b} \quad \text{and} \quad \mathbf{Ux} = \mathbf{y}.$$

First solve the lower-triangular system $\mathbf{Ly} = \mathbf{b}$ by using forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix} \implies \begin{array}{l} y_1 = 12, \\ y_2 = 24 - 2y_1 = 0, \\ y_3 = 12 - 3y_1 - 4y_2 = -24. \end{array}$$

Now use back substitution to solve the upper-triangular system $\mathbf{Ux} = \mathbf{y}$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -24 \end{pmatrix} \implies \begin{array}{l} x_1 = (12 - 2x_2 - 2x_3)/2 = 6, \\ x_2 = (0 - 3x_3)/3 = 6, \\ x_3 = -24/4 = -6. \end{array}$$

Solution 2: To solve the updated system $\mathbf{Ax} = \tilde{\mathbf{b}}$, simply repeat the forward and backward substitution steps with $\mathbf{b}$ replaced by $\tilde{\mathbf{b}}$. Solving $\mathbf{Ly} = \tilde{\mathbf{b}}$ with forward substitution gives the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \\ 70 \end{pmatrix} \implies \begin{array}{l} y_1 = 6, \\ y_2 = 24 - 2y_1 = 12, \\ y_3 = 70 - 3y_1 - 4y_2 = 4. \end{array}$$

Using back substitution to solve $\mathbf{Ux} = \mathbf{y}$ gives the following updated solution:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \\ 4 \end{pmatrix} \implies \begin{array}{l} x_1 = (6 - 2x_2 - 2x_3)/2 = -1, \\ x_2 = (12 - 3x_3)/3 = 3, \\ x_3 = 4/4 = 1. \end{array}$$

Example 3.10.3

Computing $\mathbf{A}^{-1}$. Although matrix inversion is not used for solving $\mathbf{Ax} = \mathbf{b}$, there are a few applications where explicit knowledge of $\mathbf{A}^{-1}$ is desirable.

Problem: Explain how to use the LU factors of a nonsingular matrix $\mathbf{A}_{n\times n}$ to compute $\mathbf{A}^{-1}$ efficiently.

Solution: The strategy is to solve the matrix equation $\mathbf{AX} = \mathbf{I}$. Recall from (3.5.5) that $\mathbf{AA}^{-1} = \mathbf{I}$ implies $\mathbf{A}[\mathbf{A}^{-1}]_{*j} = \mathbf{e}_j$, so the j^{th} column of $\mathbf{A}^{-1}$ is the solution of a system $\mathbf{Ax}_j = \mathbf{e}_j$. Each of these n systems has the same coefficient matrix, so, once the LU factors for $\mathbf{A}$ are known, each system $\mathbf{Ax}_j = \mathbf{LUx}_j = \mathbf{e}_j$ can be solved by the standard two-step process.

(1) Set $\mathbf{y}_j = \mathbf{Ux}_j$, and solve $\mathbf{Ly}_j = \mathbf{e}_j$ for $\mathbf{y}_j$ by forward substitution.

(2) Solve $\mathbf{Ux}_j = \mathbf{y}_j$ for $\mathbf{x}_j = [\mathbf{A}^{-1}]_{*j}$ by back substitution.

This method has at least two advantages: it's efficient, and any code written to solve $\mathbf{Ax} = \mathbf{b}$ can also be used to compute $\mathbf{A}^{-1}$.

Note: A tempting alternate solution might be to use the fact $\mathbf{A}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$. But computing $\mathbf{U}^{-1}$ and $\mathbf{L}^{-1}$ explicitly and then multiplying the results is not as computationally efficient as the method just described.

Not all nonsingular matrices possess an LU factorization. For example, there is clearly no nonzero value of u_{11} that will satisfy

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}.$$

Algebra I

Midterm 2

– January 18, 2022 –

Time: 135 *minutes. Maximum number of points:* 100. *You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!*

1. (a) When is a matrix symmetric? Give the definition of a transpose of a matrix and write at least three properties of transposing. Prove the next statement: For every square matrix A the matrix $A + A^T$ is symmetric as well. *(6 points)*

   (b) State the definition of a $n \times n$ determinant (using permutations) and give at least four examples of use of the determinant. *(6 points)*

   (c) Write down and prove Cramer's rule for solving systems of linear equations. *(8 points)*

2. For which values of $\beta \in \mathbb{R}$ will the next system have infinitely many solutions? For each value that you got compute also the solutions.

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= \beta \\ 4x + y + 10z &= \beta^2 \end{aligned}$$

(20 points)

3. For which values $a \in \mathbb{R}$ will the determinant of the matrix

$$A = \begin{bmatrix} 1 & a & 3 & 2 \\ 2 & 2 & -2 & 1 \\ 3 & 3 & -5 & 1 \\ 4 & 4 & -7 & 5 \end{bmatrix}$$

equal 30? *(20 points)*

4. (a) For matrices $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$, solve the matrix equation $(A + 3I)(X - I) = B$. *(10 points)*

   (b) Let $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Show that for matrix $A \in \mathbb{R}^{2\times 2}$ it holds that

$$det(A) = 1 \;\Leftrightarrow\; A^T E A = E.$$

(10 points)

5. Write the matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ 6 & -1 & 5 \\ 3 & 7 & 4 \end{bmatrix}$ in the form $A = LU$, where L is a lower triangular matrix with all coefficient on the main diagonal equal to 1, and U is an upper triangular matrix. *(20 points)*