44.3. Find vectors $\vec{x}$ that solve the equation $\vec{a} \times (\vec{a} \times \vec{x}) = (\vec{a} \times \vec{x}) \times \vec{x}$

Solution: Let $\vec{a} = (a, b, c)$ and $\vec{x} = (x, y, z)$

then:

$$\vec{a} \times (\vec{a} \times \vec{x}) - (\vec{a} \times \vec{x}) \times \vec{x} = \vec{0}$$

$$\vec{a} \times (\vec{a} \times \vec{x}) + \vec{x} \times (\vec{a} \times \vec{x}) = \vec{0}$$

$$(\vec{a} + \vec{x}) \times (\vec{a} \times \vec{x}) = \vec{0}$$

now we compute $(\vec{a} \times \vec{x})$:

$$\vec{a} \times \vec{x} = \begin{vmatrix} i & j & k \\ a & b & c \\ x & y & z \end{vmatrix} =$$

$$i \begin{vmatrix} b & c \\ y & z \end{vmatrix} - j \begin{vmatrix} a & c \\ x & z \end{vmatrix} + k \begin{vmatrix} a & b \\ x & y \end{vmatrix} =$$

$$i(bz - cy) - j(az - cx) + k(ay - bx) =$$

$$((bz - cy), (cx - az), (ay - bx))$$

then:

$$(\vec{a} + \vec{x}) \times (\vec{a} \times \vec{x}) =$$

$$(a+x, b+y, c+z) \times ((bz - cy), (cx - az), (ay - bx)) =$$

$$\begin{vmatrix} i & j & k \\ a+x & b+y & c+z \\ bz - cy & cx - az & ay - bx \end{vmatrix} =$$

$$i \begin{vmatrix} b+y & c+z \\ cx - az & ay - bx \end{vmatrix} - j \begin{vmatrix} a+x & c+z \\ bz - cy & ay - bx \end{vmatrix} + k \begin{vmatrix} a+x & b+y \\ bz - cy & cx - az \end{vmatrix}$$

so we get the following system:

$$(b+y)(ay-bx)-(c+z)(cx-az)=0 \quad (1)$$

$$(c+z)(bz-cy)-(a+x)(ay-bx)=0 \quad (2)$$

$$(a+x)(cx-az)-(b+y)(bz-cy)=0 \quad (3)$$

$(1)+(2)$ gives us:

$$(b+y)(ay-bx)-(a+x)(ay-bx)=0$$

$$\Rightarrow (b+y-a-x)(ay-bx)=0 \quad (1^*)$$

$(1)+(3)$ gives us:

$$(a+x)(cx-az)-(c+z)(cx-az)=0$$

$$\Rightarrow (a+x-c-z)(cx-az)=0 \quad (2^*)$$

And finally $(2)+(3)$ gives us:

$$(c+z)(bz-cy)-(b+y)(bz-cy)=0$$

$$\Rightarrow (c+z-b-y)(bz-cy)=0 \quad (3^*)$$

So we have system that contains 1^*, 2^* and 3^*

from 1^* and 2^*, we conclude

$$y=x+a-b \ \vee \ y=\frac{b}{a}x \quad \text{and} \quad z=x+a-c \ \vee \ z=\frac{c}{a}x$$

from $y = x + a - b$ and $z = x + a - c$

by plugging in (3^*), we get

$$(c + z - b - y) = c + (x + a - c) - b - (x + a - b) = 0$$

$$c + x + a - c - b - x - a + b = 0$$

$$0 = 0 \checkmark$$

so $\boxed{(t, t + a - b, t + a - c)}$

is one solution of the system.

and if $y = \frac{b}{a} x$ and $z = \frac{c}{a} x$

and plug it in third equation

we get:

$$(bz - cy) = \left(b \cdot \frac{c}{a} x - c \cdot \frac{b}{a} x\right) = 0$$

$$0 = 0 \checkmark$$

so $\boxed{\left(t, \frac{b}{a} t, \frac{c}{a} t\right)}$ is

another solution of the system.