

Algebra I
IZPIT
– 18. AVGUST 2022 –

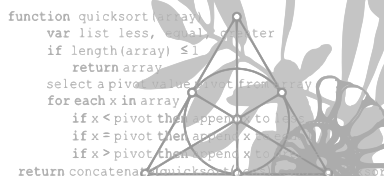
Čas pisanja: 135 minut. Maksimalno število točk: 100. Dovoljena je uporaba pisala in kalkulatorja. Pišite razločno in utemeljite vsak odgovor. Srečno!

1. (a) Zapišite definicijo skalarnega produkta. Dokažite, da za vektorja $\vec{u}, \vec{v} \in \mathbb{R}^3$, ki sta pravokotna, velja $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$. (Namig: Uporabite dejstvo, da je $|\vec{u}|^2 = \langle u, u \rangle$). (5 točk)
(b) Zapišite definicijo vektorskega produkta. Kako ga izračunamo z uporabo determinante? (5 točk)
(c) Dokažite naslednjo trditev: Za poljubne vektorje \vec{u}, \vec{v} in \vec{w} velja, da je $|\langle u \times v, w \rangle|$ prostornina paralelepipeda, ki ga določajo vektorji \vec{u}, \vec{v} in \vec{w} . (5 točk)
(d) Zapišite enačbo ravnine v vektorski, splošni in parametrični vektorski obliki. Izpeljite enačbo ravnine skozi 3 točke z uporabo determinante. (5 točk)
2. Naj bodo A, B, C in D zaporedna oglišča paralelograma. Točka E leži na diagonali AC tako, da velja $|AE| : |EC| = 1 : 3$. Točka F leži na diagonali BD tako, da velja $|BD| : |BF| = 4 : 3$. Označimo s S presečišče daljic AF in ED .
(a) Zapišite vektor \overrightarrow{AS} kot linearno kombinacijo vektorjev $\vec{e} = \overrightarrow{AC}$ in $\vec{f} = \overrightarrow{BD}$. (8 točk)
Hint: $|AS| = \frac{2}{3}|AF|$.
(b) Če so $A(1, -2, 2)$, $B(3, -1, 4)$ in $C(2, 3, -3)$, poiščite kot med stranicama AB in BC ter določite ploščino paralelograma. (12 točk)
Namig: Kot med dvema vektorjema lahko izračunamo s pomočjo enačbe $\cos \varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.
3. Naj bo U ravnina z enačbo $x - y + 2z = 0$.
(a) Zapišite enačbo premice ℓ , ki vsebuje točko $T(4, 0, 4)$ in je pravokotna na ravnino U . (3 točke)
(b) Poiščite presečišče premice ℓ in ravnine U . (5 točk)
(c) Določite koordinate točke A , ki leži na premici ℓ in je enako oddaljena od točke T in ravnine U . Kolikšna je ta razdalja? (12 točk)
4. Pokažite, da sistem linearnih enačb
$$\begin{aligned}3x + 4y + 5z &= a \\4x + 5y + 6z &= b \\5x + 6y + 7z &= c\end{aligned}$$
nima rešitve, razen če je $a + c = 2b$. V tem primeru rešitev tudi poiščite. (20 točk)
5. Za katere vrednosti $x \in \mathbb{R}$, bo naslednja matrika

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 2 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 2 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

obrnljiva?

(20 točk)



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Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

- Write the definition of the scalar (dot) product. In addition, for two orthogonal vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$, prove that $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$. (Hint: Use the fact that $|\vec{u}|^2 = \langle u, u \rangle$). (5 points)
 - Write the definition of the cross (vector) product. How do we compute it in terms of determinants? (5 points)
 - Prove the following statement: For any vectors \vec{u}, \vec{v} and \vec{w} , it holds that $|\langle u \times v, w \rangle|$ is the volume of the parallelepiped determined by \vec{u}, \vec{v} and \vec{w} . (5 points)
 - Write the equation of a plane in vectorial, general and parametric vectorial form. Then, derive the equation of a plane through 3 non-collinear points in terms of determinants. (5 points)

- Let A, B, C and D be consecutive points of a parallelogram. Point E divides the diagonal AC so that $|AE| : |EC| = 1 : 3$. Point F divides the diagonal BD so that $|BD| : |BF| = 4 : 3$. Let S be the point of intersection of line segments AF and ED .

- Write the vector \vec{AS} as a linear combination of vectors $\vec{e} = \vec{AC}$ and $\vec{f} = \vec{BD}$. (8 points)
Hint: $|AS| = \frac{2}{3}|AF|$.
- If $A(1, -2, 2)$, $B(3, -1, 4)$ and $C(2, 3, -3)$, find the angle between line segments AB and BC and determine the area of the parallelogram. (12 points)
Hint: The angle between two vectors can be obtained from the equation $\cos \varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

- Let U be the plane defined by $x - y + 2z = 0$.

- Find the line ℓ , that contains point $T(4, 0, 4)$ and is perpendicular to the plane U . (3 points)
- Find the coordinates of the intersection of the line ℓ and the plane U . (5 points)
- Determine the coordinates of the point A , that lies on the line ℓ and is equidistant from T and U . Also, determine this distance. (12 points)

- Show that the system of equations

$$3x + 4y + 5z = a$$

$$4x + 5y + 6z = b$$

$$5x + 6y + 7z = c$$

does not have a solution unless $a + c = 2b$. In that case, find the solution of the system.

(20 points)

- For which values of $x \in \mathbb{R}$, will the matrix M given by

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 2 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 2 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

be invertible?

(20 points)