hw 7 by Zhivko Stoimchev 89221056

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1.1

1. Show that any seq. of positive real numbers either has a subsequence that converges, or else a subseq. that diverges to ∞

- Let a_n be a seq. of positive real numbers.

<u>Need to prove:</u>

a) a_n has a subseq. that converge

1.2

b) b_n has a subseq. that diverge to ∞

<u>Proof of a)</u> Suppose that a_n is bounded. Then there exists $\exists$ real number $M > 0$ s.t. $a_n \leq M$ for all $n \in \mathbb{N}$.
In that case, the seq. a_n is bounded of positive real numbers. ~~B~~ With this, by Bolzano-Weierstrass theorem, a_n has a convergent subseq. a_{n_k}. This concludes that the subseq. (a_{n_k}) converges (in this case).

BW Thm: every bounded seq. has a convergent subsequence -(from Tutorials)

<u>Proof of b)</u> Suppose that a_n is unbounded. Then
$\exists$ subseq. a_{n_k} of a_n s.t. $a_{n_k} \to \infty$ as $k \to \infty$.
This concludes that the subseq. (a_{n_k}) diverges to ∞. (in this case).

1.3

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B. For each of the following series, determine whether the series converges or diverges. (Sum from 1 to ∞) For those that converge, find the value that they converge to.

i) $\sum (4^n+2^n)/3^n$ $\cancel{= \sum \frac{4^n}{3^n} + \sum \frac{2^n}{3^n}}$ $\quad a_n = \frac{4^n+2^n}{3^n}$

By applying ratio test, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{3^{n+1}}}{\frac{4^n}{3^n}} + \frac{\frac{2^{n+1}}{3^{n+1}}}{\frac{2^n}{3^n}} = \frac{1}{1} + \frac{1}{1} = 2 > 1 \Rightarrow \text{diverges}$$

ii) $\sum \sin^n(3)$

The seq. $\sin^n(3)$ converges to $\frac{1}{1-\sin(3)}$ because

$\sum r^n = \frac{1}{1+r}$ if $|r|<1$ and $\sin 3 = 0,14$

iii) $\sum 1/(n^2+2n)$, we conclude that $\frac{1}{n^2+2n} < \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ converges, by <u>direct comparison</u> test we have that $\sum \frac{1}{n^2+2n}$ also converges, to 1.

iv) $\sum(\sqrt{n+2}-\sqrt{n})$ $\Rightarrow$ first we rationalise:

$$\Rightarrow \sqrt{n+2}-\sqrt{n} \cdot \frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = \frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}}$$

From this we have

$$2\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = 2\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} = 2\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\sqrt{\frac{n}{n}+\frac{2}{n}}+\frac{\sqrt{n}}{\sqrt{n}}} = 2\lim \frac{1}{\sqrt{1+0}+1} = 2\lim\frac{1}{2} = \frac{2}{2} = 1$$

$\Rightarrow$ we have $0<1<\infty$, $\sum(\sqrt{n+2}-\sqrt{n})$ is equiconvergent to $\sum\frac{1}{\sqrt{n}}$, so they converge both.

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3.1

3.2

C. For each of the following series, determine whether the series converges, diverges to $\pm\infty$, or diverges, not to $\pm\infty$. Don't try to find what the series converge to.

i) $\sum \sin^2(3n+2)/(2n^2-3)$, $a_n = \sin^2(3n+2)/(2n^2-3)$

we define $b_n = \frac{1}{n^2}$ and it converges, so we have

3.3 3.4

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{\boxed{\sin^2 \frac{3n+2}{2n^2-3}}}{\frac{1}{n^2}\left(\frac{3n+2}{2n^2-3}\right)^2} \cdot \left(\frac{3n+2}{2n^2-3}\right)^2 \boxed{=} \lim \frac{n^2(9n^2+12n+4)}{4n^4-12n^2+9} = \frac{9}{4} \in \mathbb{R}^+$$

3.5 3.6

that means that $\sum a_n$ and $\sum b_n$ have the same nature, and $\sum a_n$ is convergent.

ii) $\sum 1/(n-2\pi)$, $\underbrace{a_n = 1/(n-2\pi) > 1/n = b_n}$, and $\sum \frac{1}{n}$ is divergent

$a_n = \frac{1}{n-2\pi}$

$b_n = \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{n}{n-2\pi} = 1 \in \mathbb{R}^+$ $\Rightarrow$ this means that $\sum a_n$ and $\sum b_n$ have same nature, and this shows that $\Rightarrow \sum a_n$ is divergent

iii) $\sum (n+1)/(n^3+2)$, $a_n = n+1/(n^3+2)$ and $b_n = n/n^3$, or $1/n^2$

$\Rightarrow \sum b_n$ is convergent (because $\alpha = 2 > 1$), and we have

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+n^2} = 1 \in \mathbb{R}^+$ $\Rightarrow$ this means that $\sum a_n$ and $\sum b_n$ have same nature, and this shows that

$\Rightarrow \sum a_n$ converges

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D. Determine whether each of the following series are absolutely convergent, conditionally convergent, or divergent.

i) ~~$\frac{1}{2}$~~ $\sum (-2)^{n^2} / (n+2)!$ $\Rightarrow \sum \left| \frac{(-2)^{n^2}}{(n+2)!} \right| = \sum \frac{2^{n^2}}{(n+2)!} \Rightarrow a_n = \frac{2^{n^2}}{(n+2)!}$

by applying ratio test we get:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{(n+1)^2}}{(n+3)!}}{\frac{2^{n^2}}{(n+2)!}} = \frac{2^{n^2+2n+1}}{(n+3) 2^{n^2}} = \frac{2^{2n+1}}{n+3}$$

because we can't go any further, we apply another ratio test!

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{\frac{2^{2n+1}}{2^{2n}}}{n+2}} = \frac{2(n+2)}{n+3} = \frac{2n+4}{n+3}$$ Now we compute the limit:

$$\Rightarrow \lim \frac{2n+4}{n+3} = \lim \frac{\frac{2n}{n} + \frac{4}{n}}{\frac{n}{n} + \frac{3}{n}} = \frac{2}{1} = 2 > 0$$, which means $\sum a_n = +\infty$

The seq. is divergent

ii) $\sum (-3)^n / (3n^2 + 6^n)$: First we use ~~comparison test~~, we get:

$|a_n| = \left| \frac{(-3)^n}{3n^2+6^n} \right| = \frac{3^n}{3n^2+6^n}$ for $3n^2+6^n > 6^n /$: $\Rightarrow \frac{1}{3n^2+6n} < \frac{1}{6^n} / \cdot 3^n \Rightarrow \frac{3^n}{3n^2+6^n} < \frac{3^n}{6^n}$

$a_n \quad b_n$

$b_n = \frac{3^n}{6^n} = \left(\frac{1}{2}\right)^n / \frac{1}{4}$

Now we use route test (from Tutorials), and get

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$, which means the seq. is convergent

$\Rightarrow$ Since b_n is convergent, $|a_n|$ absolutely converges by comparison test.

iii) $\sum (-4)^n / (4^n \sqrt{2n+1})$ $\sim \sum \frac{(-1)^n \cancel{4^n}}{\cancel{4^n}\sqrt{2n+1}} = \sum \frac{(-1)^n}{\sqrt{2n+1}}$

with absolute value, we have $\sum \frac{1}{\sqrt{2n+1}} \sim \sum b_n \Rightarrow b_n = \frac{1}{\sqrt{n}}$

we now compute $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \lim\limits_{n\to\infty} \sqrt{\frac{n}{2n+1}} = \sqrt{\lim\limits_{n\to\infty} \frac{n}{n(2+\frac{1}{n})}} = \frac{1}{\sqrt{2}} \in \mathbb{R}$

now we can see that $\sum a_n$ and $\sum b_n$ have same nature, which means they are divergent.

$$\sum \frac{(-1)^n 4^n}{4^n \sqrt{2n+1}} = \sum \frac{(-1)^n}{\sqrt{2n+1}}$$

$a_n = \frac{1}{\sqrt{2n+1}}$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0$, and $a_{n+1} = \frac{1}{\sqrt{2n+3}} < a_n \frac{1}{\sqrt{2n+1}}$

$\Rightarrow$ from there we conclude that $\sum (-1)^n a_n$ is divergent hw7 by Zhivko Stoimchev 89221056

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E. Show that if $a_n = 5n^3 - 19n^2 + 7n - 3$, then $a_n = \Theta(n^3)$

$\exists \alpha > 0$ and $\exists \beta > 0$, $\exists N$ s.t. $\forall n > N$

$\alpha n^3 \leq 5n^3 - 19n^2 + 7n - 3 \leq \beta n^3$

$1 \cdot n^3 \leq 5n^3 - 19n^2 + 7n - 3 \leq 5n^2 + 19n^3 + 7n^3 + 3n^2 = 29n^3$

that is equivalent with $0 \leq 4n^3 - 19n^2 + 7n - 3$

from what we get $19n^2 - 7n + 3 \leq 4n^3$

Now we show that $\exists N = 10$ and $\forall n > N$, it holds.

$\exists \alpha = 1$ and $\exists \beta = 29$

$\exists N = 10 \; \forall n > N$

$\alpha \cdot \beta_n \leq a_n \leq \beta \cdot b_n$

Index of comments

1.1 A: 4.5/5

1.2 or

1.3 Please elaborate here.

3.1 One of your colleagues has very similar homework.

3.2 C: 4/5

3.3 You had here just sin^2(3n+2)

3.4 This is not correct.

3.5 this limit would be zero.

3.6 We prefer using 'are equiconvergent'

4.1 D: 3.5/5

4.2 Something is not okay here.

4.3 1?

4.4 The test you are referring to is called 'root' test, however the test you are using is 'ratio' test

4.5 What about conditional convergence? Try using AST. This series in fact converges conditionally.