Homework 6

(A.) Show that any convergent sequence of complex numbers is bounded.

$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} a_n + ib_n = a + ib$

z_n is convergent and therefore

$\lim_{n\to\infty} a_n = a \qquad \lim_{n\to\infty} b_n = b$

By the MS Theorem every real-valued convergent sequence is monotone and bounded.

$\therefore \exists M_a$ such that $M_a = \max = \{a+\varepsilon, a_1, a_2, \dots a_N\}$ and
$\exists M_b$ such that $M_b = \max = \{b+\varepsilon, b_1, b_2, \dots b_N\}$

$|a_n| \leq M_a$ and $|b_n| \leq M_b$

$|a_n + b_n \cdot i| < M_a + iM_b = M$

$|z_n| < M$

By definition a complex sequence is bounded if it's range is bounded by M.

(B) $\lim_{n\to\infty} z_n = 0$ iff $\lim_{n\to\infty} |z_n| = 0$

$\Leftarrow$ $\lim z_n = 0$

$\forall \varepsilon > 0, \exists N \; [n > N] \Rightarrow [|z_n - 0| < \varepsilon]$

$|z_n| < \varepsilon$

$\Rightarrow$ $\lim |z_n| = 0$

$\forall \varepsilon > 0, \exists N \; [n > N] \Rightarrow [||z_n| - 0| < \varepsilon]$

$||z_n|| < \varepsilon \qquad z_n < \varepsilon$

$|\sqrt{a_n^2 + b_n^2}| < \varepsilon$

$\sqrt{a_n^2 + b_n^2} < \varepsilon$

C. $\forall \varepsilon > 0 \ \exists N$ such that $[n \leq N] \Rightarrow [d(s_n, t_n) - d(s,t) < \varepsilon]$

d_2

$$d(s_n, t_n) - d(s,t) < \varepsilon$$

$$\sqrt{(s_n - t_n)^2} - \sqrt{(s-t)^2} < \varepsilon \qquad \varepsilon/2 - \varepsilon/2 < \varepsilon$$

$$s_n - t_n - s + t < \varepsilon \qquad 0 < \varepsilon$$

$$(s_n - s) - (t_n - t) < \varepsilon$$

$$d(s_n, s) = \sqrt{(s_n - s)^2} = s_n - s < \varepsilon/2$$
$$d(t_n, t) = \sqrt{(t_n - t)^2} = t_n - t < \varepsilon/2$$

d_∞ $\max(s_n, t_n) = \{|s_n - t_n|\} = \{|s_n - s + s - t_n - t + t|\} \leq$

$\max\{|s_n - s|, |t_n - t|, |s - t|\} \rightarrow$ by Comparison of cases

D. Construct a sequence having $\{0, 1, 3, 6, 20\}$ as it's set of accumulation points.

$$a_n \begin{cases} 0 & : n \pmod 5 = 0 \\ 1 & : n \pmod 5 = 1 \\ 3 & : n \pmod 5 = 2 \\ 6 & : n \pmod 5 = 3 \\ 20 & : n \pmod 5 = 4 \end{cases}$$

$$a_n = (0, 1, 3, 6, 20, 0, 1, 3, 6, 20, \ldots)$$