University of Primorska
UP FAMNIT
Academic year 2021/2022

Algebra I

Exam

– January 26, 2022 –

Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) Write the equation of a plane in vectorial, general and parametric vectorial form. Then, derive the equation of a plane through 3 non-collinear points in terms of determinants. *(6 points)*

   (b) Write the characterization of:

      i. Two parallel lines in terms of the cross product. *(2 points)*

      ii. Two parallel planes (given in general forms) in terms of the cross product. *(2 points)*

   Prove at least one of these characterizations. *(3 points)*

   (c) i. Write the definition of rank of a matrix in terms of linear independency. *(3 points)*

      ii. Prove the statement: A non-homogeneous system $AX = B$ has a solution if and only if $rank(A) = rank(A|B)$. *(4 points)*

2. Let A, B, C and D be consecutive points of a parallelogram. Point E divides the diagonal AC so that $|AE| : |EC| = 1 : 3$. Point F divides the diagonal BD so that $|BD| : |BF| = 4 : 3$. Let S be the point of intersection of line segments AF and ED.

   (a) Write the vector $\overrightarrow{AS}$ as a linear combination of vectors $\overrightarrow{e} = \overrightarrow{AC}$ and $\overrightarrow{f} = \overrightarrow{BD}$. *(8 points)*
   Hint: $|AS| = \frac{2}{3}|AF|$.

   (b) If $A(1, -2, 2)$, $B(3, -1, 4)$ and $C(2, 3, -3)$, find the angle between line segments AB and BC and determine the area of the parallelogram. *(12 points)*
   Hint: The angle between two vectors can be obtained from the equation $\cos\varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

3. Let $\Pi : x - 4y + 2z - 7 = 0$ be a plane in space and let ℓ be the line given by the intersection of the planes $x - 2y - 4z = -3$ and $2x + y - 3z + 1 = 0$. Find the line p on the plane Π that is orthogonal to ℓ and contains the point given by the intersection between ℓ and Π. *(20 points)*

4. Discuss the solutions of the following system of linear equations with respect to parameters $\lambda, \mu \in \mathbb{R}$:

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

(20 points)

5. Compute the determinant

$$det\left(\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\cdot\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\cdots\begin{bmatrix}n & n-1\\ n+1 & n\end{bmatrix}\right)$$

for every $n \in \mathbb{N}$. *(20 points)*