Algebra I
MIDTERM 1
– NOVEMBER 25, 2021 –

Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) Write the definition of the scalar (dot) product. In addition, for two orthogonal vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$, prove that $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$. (Hint: Use the fact that $|\vec{u}|^2 = \langle u, u \rangle$). *(5 points)*

   (b) Write the definition of the cross (vector) product. How do we compute it in terms of determinants? *(5 points)*

   (c) Prove the following statement: For any vectors $\vec{u}, \vec{v}$ and $\vec{w}$, it holds that $|\langle u \times v, w \rangle|$ is the volume of the parallelepiped determined by $\vec{u}, \vec{v}$ and $\vec{w}$. *(5 points)*

   (d) Derive the formula for computing the distance between a point and a line in $\mathbb{R}^3$ (Hint: Formula looks as follows: $d = \frac{|(\vec{r_1} - \vec{r_0}) \times v|}{|\vec{v}|}$). *(5 points)*

2. Let the points $A(3, 2, 1)$, $B(4, 1, -2)$, $C(-5, -4, 8)$ and $D(6, 3, 7)$ constitute the triangular pyramid $ABCD$.

   (a) Compute the volume of the pyramid. *(10 points)*

   (b) Compute the height of the pyramid, assuming that the triangle ABC is a basis of the pyramid. *(10 points)*

3. Find the parameter λ in the parametric form of the line ℓ given by

$$\frac{x-3}{1} = \frac{y-1}{\lambda} = \frac{z+2}{1}$$

   so that line ℓ intersects the line $p: \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{-1}$. For such λ, find the point of intersection and the angle between the two given lines ℓ and p.

   Hint: The angle between two vectors can be obtained from the equation $\cos \varphi = \frac{\vec{v_1} \cdot \vec{v_2}}{|\vec{v_1}| \cdot |\vec{v_2}|}$. *(20 points)*

4. Let U be the plane defined by $x - y + 2z = 0$.

   (a) Find the line ℓ, that contains point $T(4, 0, 4)$ and is perpendicular to the plane U. *(3 points)*

   (b) Find the coordinates of the intersection of the line ℓ and the plane U. *(5 points)*

   (c) Determine the coordinates of the point A, that lies on the line ℓ and is equidistant from T and U. Also, determine this distance. *(12 points)*

5. Let $\ell : \frac{x+1}{-1} = \frac{y-2}{-2} = \frac{z}{1}$ be a given line, and let $A(1, 1, 0)$ be a given point. Find a point B which is symmetric to the point A with respect to the line ℓ, and find the plane which contains the line ℓ and points A and B. *(20 points)*