There are several things to like about this argument.

- It is much easier than our original argument, which required some 'tricks' with the inequalities.
- The BW Theorem produces the limit value for us, so we need only verify that the whole sequence agrees w/ the limit of the subsequence!
- Since we have a complex BW Theorem, we can handle the real / complex Cauchy Completeness w/ a single argument.

I'll close the section on accumulation by mentioning a connection w/ lim sup:

<u>Theorem</u>: If s_n is a bounded sequence of real numbers, then $\limsup s_n$ is the largest accumulation point of s_n and $\liminf s_n$ is the smallest " " " .

<u>Proof</u> will be hw. (with sketch provided.)

<u>III. Infinite series</u>

<u>A. Definition and examples</u>

Recall from high school that if (a_n) is a sequence (real or complex), then

$$\sum_{n=\ell}^{m} a_n \quad \text{means} \quad a_\ell + a_{\ell+1} + a_{\ell+2} + \dots + a_{m-1} + a_m.$$

Eg: $$\sum_{n=1}^{10} n = 1+2+3+\_+9+10=55$$

$$\sum_{n=1}^{m} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

$$\sum_{k=1}^{10} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{100}.$$

An expression of the form
$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$
is called an <u>infinite series</u> based on the sequence (a_n).

We'll use limits to make sense of infinite series.
First, some important classes of examples.

1) Given $c, r \in \mathbb{R}$ where $c \neq 0$,
$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \ldots$$ is a <u>geometric series</u>.

2) $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$ is the <u>harmonic series</u>.

3) More generally, if $p \in \mathbb{R}$, then
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$ is a <u>Dirichlet p-series</u>.

<u>Remark</u> The Σ notation is useful for writing compactly and precisely. But if you get lost in the notation, it might help you to write out a few terms w/ a "+ ..." as we've done above.

Of course, we can only actually add up finitely many numbers. We must give precise meaning to these infinite "sums", with limits.

<u>Definition</u>: Given the infinite sum $\sum_{n=1}^{\infty} a_n$, we define the <u>sequence of partial sums</u> as
$$S_N := \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N.$$
That is, S_N is the truncation of the infinite series to the first N terms.

<u>Eg</u>: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has $S_1 = 1$, $S_2 = 1\,{}^1\!/\!{}_4$, $S_3 = 1\,{}^{13}\!/\!{}_{36}$, etc.

Definition (continued)

- It's also sometimes helpful to consider the <u>remainder</u>

$$R_N := \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \cdots$$

Thus, for any value of N,

$$\sum_{n=1}^{\infty} a_n = S_N + R_N.$$

- If $\lim_{N\to\infty} S_N$ converges to S, then we say the series <u>converges</u> to S and write $\sum_{n=1}^{\infty} a_n = S$.

Unsurprisingly, if S_N diverges, we say $\sum_{n=1}^{\infty} a_n$ diverges and if $\lim_{N\to\infty} S_N = \pm\infty$ we write $\sum_{n=1}^{\infty} a_n = \pm\infty$ (and say it diverges to $\pm\infty$).

<u>Examples:</u>

1) $\sum_{n=1}^{\infty} 1 = \infty$. The partial sum $S_N = N$, and $\lim_{N\to\infty} N = \infty$.

2) $\sum_{n=0}^{\infty} (-1)^n$ diverges (not to $\pm\infty$).
The partial sums S_N are 1 for N even and 0 for N odd so $\limsup S_N = 1$ but $\liminf S_N = 0$.

3) Let d_n be the nth digit of the decimal expansion of $\sqrt{2} = 1.414213\ldots$
so $d_0 = 1,\ d_1 = 4,\ d_2 = 1,\ d_3 = 4,\ d_4 = 2,\ d_5 = 1,\ d_6 = 3, \ldots$
Now $\sum_{n=0}^{\infty} \frac{d_n}{10^n}$ has S_N the Nth partial decimal expansion!
$S_0 = 1,\ S_1 = 1.4,\ S_2 = 1.41, \ldots$

By our previous discussion, we see that $\sum_{n=0}^{\infty} \frac{d_n}{10^n} = \sqrt{2}$.

4) (*** Important example ***)

Consider the geometric series $\sum_{n=0}^{\infty} c \cdot a^n$, where $c \neq 0$.

By a homework problem,

$$(1-a)(1+a+a^2+\dots+a^N) = 1-a^{N+1}$$

Thus, $S_N = c \cdot \frac{1-a^{N+1}}{1-a}$.

By another homework problem,

$$\lim_{N\to\infty} a^{N+1} = \begin{cases} 0 & \text{if } -1<a<1 \\ \infty & \text{if } a>1 \\ 1 & \text{if } a=1 \end{cases}$$

and does not exist if $a \leq -1$.

AoL gives us $\lim_{N\to\infty} S_N$ for $a \neq 1$:

$$\sum_{n=0}^{\infty} ca^n = \lim_{N\to\infty} S_N = \begin{cases} c \cdot \frac{1}{1-a} & \text{if } -1<a<1 \\ \infty & \text{if } a>1 \end{cases}$$

and diverges, not to ∞, for $a \leq -1$.

For $a=1$, we proceed directly:

$$\sum_{n=0}^{\infty} c \cdot 1^n = \sum_{n=0}^{\infty} c = c+c+c+\dots = \begin{cases} \infty & \text{if } c>0 \\ -\infty & \text{if } c<0 \end{cases}$$

<u>Concrete example:</u> $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$

where $S_N = \frac{1-\frac{1}{2^{N+1}}}{1-1/2} = 2 - \frac{1}{2^N}$.

5) $\sum_{n=0}^{\infty} \frac{n}{n+1} = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$

Since every term past $n=0$ satisfies $\frac{n}{n+1} \geq \frac{1}{2}$

we see that $S_N \geq \frac{1}{2}N$

so $\liminf S_N \geq \liminf \frac{1}{2}N = \infty$

so $\lim_{N\to\infty} S_N = \infty$, and the series diverges to ∞.

Let's go through our limit "toolbox" and apply the tools to $\lim_{N\to\infty} S_N$, the limit of partial sums.

Arithmetic of (Infinite) Limits is helpful:

<u>Theorem</u> (Arithmetic of Series)

If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ where A, B are real numbers or $\pm\infty$

then

i) $\sum_{n=1}^{\infty} ca_n = cA$ for any $c \neq 0$.

ii) $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

(But as usual, we give no definition to the symbol $\infty - \infty$.)

<u>Proof</u>: Arithmetic of Limits!

i) $\sum_{n=1}^{\infty} ca_n = \lim_{N\to\infty} ca_1 + ca_2 + \dots + ca_N = c \cdot \lim_{N\to\infty} a_1 + a_2 + \dots + a_N$

$= c \cdot A.$

ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{N\to\infty} a_1 + b_1 + a_2 + b_2 + \dots + a_N + b_N$

$= \lim_{N\to\infty} (a_1 + a_2 + \dots + a_N) + (b_1 + b_2 + \dots + b_N) = A + B.$ ∎

Less obviously, AoL gives us an easy (though "incomplete") way to show that some series diverge.

<u>Lemma:</u> If $\sum_{n=1}^{\infty} a_n$ converges (to <u>any</u> real number)

then $\lim_{n\to\infty} a_n = 0$.

<u>Pf:</u>

Since $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$ converges, by AoL, say to A,

$$\lim_{N\to\infty} S_N - S_{N-1} = \lim_{N\to\infty} (a_1+\dots+a_N)-(a_1+\dots+a_{N-1})$$
$$= \lim_{N\to\infty} a_N = A-A=0. \qquad \blacksquare$$

<u>Eg:</u> $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$, so we must (and do) have $\lim_{n\to\infty} \frac{1}{2^n} = 0$.

<u>Corollary</u> (the nth Term Test for divergence)

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Example:</u> $\sum_{n=1}^{\infty} \sqrt{1-\frac{1}{\sqrt{n}}}$ diverges, since $\lim_{n\to\infty} \sqrt{1-\frac{1}{\sqrt{n}}} = 1$ (by AoL).

*** <u>Caution:</u> If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$

may converge or

may diverge.

The nth Term Test gives us <u>no information</u> in this case. ***

<u>Remark:</u> Now you see clearly the difference between "if" and "if and only if".

<u>Example:</u> Although $\lim_{n\to\infty} \frac{1}{n} = 0$, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

We can see this by collecting terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \dots$$

(In general, $\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} \geq \frac{1}{2}$.)

Thus, $\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} 1/2 + 1/2 + \dots + 1/2 = \left(\lim_{N\to\infty} \lfloor \log_2 N \rfloor \cdot \frac{1}{2}\right)$

$= \infty$

the series diverges to ∞. ✓

The Cauchy Completeness Theorem let's us give an "if and only if" extension of the nth Term Test, at the cost of a considerable amount of complexity.

<u>Theorem</u> (Cauchy Convergence for Series)

Let $\sum_{k=1}^{\infty} a_k$ have sequence of partial sums $S_n \left(= \sum_{k=1}^{n} a_k.\right)$

The series converges if and only if

(*) $\forall \varepsilon > 0, \exists N$ s.t. $[n, m > N] \Rightarrow [|S_n - S_m| < \varepsilon]$

<u>Proof:</u> Follows immediately by specializing the Cauchy Completeness Theorem. ∎

What does Cauchy Convergence have to do with the nth Term Test?

If $n > m$, then in the above situation,

$$S_n - S_m = a_{m+1} + a_{m+2} + \dots + a_n.$$

So (*) says that

$\forall \varepsilon > 0, \exists N$ s.t. $[n > m > N] \Rightarrow [|a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon]$.

Thus, the nth Term Test may be seen as the special situation $m=n-1$ in Cauchy Convergence for Series: Obviously, if Cauchyness fails in this special case, then it fails (but the converse may not be true).

<u>Example</u> (Harmonic series, with Cauchy Convergence).
Since for any N, we can find $n>m>N$
s.t. $\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{n}>\frac{1}{2}$,
the Cauchy Convergence Criterion fails for $\sum_{n=1}^{\infty}\frac{1}{n}$ with $\varepsilon=\frac{1}{2}$.
So the series diverges.

The Monotone Sequence Theorem is also useful, as follows:

<u>Proposition</u>: If a_n is a nonnegative real sequence
then either $\sum_{n=1}^{\infty} a_n$ converges (and is bounded)
or else $\sum_{n=1}^{\infty} a_n=\infty$

<u>Proof</u>: Since (for all n) $a_n \geq 0$ and $S_{N+1}=S_N+a_{N+1}$,
the sequence S_N of partial sums is an increasing sequence.
By the MST, it is either bounded and convergent
or else unbounded and diverges to ∞. ∎

<u>Key point</u>: The Proposition tells us that checking convergence of a series w/ nonnegative terms is equivalent to finding an upper bound on its sequence of partial sums!

We'll develop this in detail, but let's first compute one more example.

Example: (Technique of Telescoping Sums)

Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, or explain why it diverges.

Solution:

We first notice that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(Use the technique of partial fractions:

Set $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$\Rightarrow 1 = A(n+1) + B\cdot n \Rightarrow 0n + 1 = (A+B)n + A$

and solve to find $A=1$, $B=-1$.)

So we're finding

$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$. Here $S_N = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$

We notice that the $\frac{-1}{n+1}$ from the nth term cancels w/ the $\frac{1}{n+1}$ " " $(n+1)$th term.

This leaves $S_N = 1 - \frac{1}{N+1}$. (Notice that 1 has nothing to cancel with, nor $\frac{1}{N+1}$ yet.)

Now $\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1 - 0 = 1 = \sum_{n=1}^{\infty} \frac{1}{n\cdot(n+1)}$.

Self-check/exercise: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (using a similar technique).

B. Tests for nonnegative series

For now, all the series we look at will have nonnegative terms.

We saw that the MST tells us such a series converges if and only if its partial sums are bounded.

Bounding partial sums is much easier than working with limits directly! (But a drawback is that we rarely will be able to calculate series' limits, only to say whether the limit converges.)

A basic tool compares two series directly.

<u>Theorem</u> (Comparison Test)

Let a_n, b_n be real sequences with $0 \le a_n \le b_n$ (for all n).

Then 1) $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

2) $\sum_{n=1}^{\infty} a_n = \infty \Rightarrow \sum_{n=1}^{\infty} b_n = \infty$

<u>Proof</u>: 1) It suffices by previous discussion to bound the partial sums. But if $\sum_{n=1}^{\infty} b_n = B$, then the partial sums of $\sum_{n=1}^{\infty} b_n$ are bounded by B, while the partial sums for $\sum_{n=1}^{\infty} a_n$ are bounded by those for $\sum b_n$.

In symbols,

$$\sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n \le B.$$

Thus $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ converges. ✓

2) Similarly,

$$\sum_{n=1}^{N} a_n \le \sum_{n=1}^{N} b_n$$

and since the LHS is unbounded, the RHS is also.

By MST / Proposition, $\lim_{N\to\infty} \sum_{n=1}^{N} b_n = \infty$. ∎

<u>Important examples</u>:

1) Since $0 \le \frac{1}{n^2} \le \frac{2}{n(n+1)}$ for all $n \ge 1$ (as $n^2+n \le 2n^2$)

we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \le \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \quad \text{(by a previous example).}$$

By the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, to some value S.

The argument shows that $S \le 2$.

You can take partial sums to compute as many lower bounds as you like.

$\boxed{\text{Fact: } S = \frac{\pi^2}{6}}$

2) If $p \geq 2$, then $0 \leq \frac{1}{n^p} \leq \frac{1}{n^2}$ for all $n \geq 1$

so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges by the Comparison Test

(Comparison w/ $\sum_{n=1}^{\infty} \frac{1}{n^2}$.)

3) If $p < 1$, then $0 \leq \frac{1}{n} = \frac{1}{n^1} \leq \frac{1}{n^p}$.

Since we showed that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$,

also $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$, by the Comparison Test.

__Remarks:__ Like the nth Term Test, the Comparison Test will sometimes (often?) give __no information__.

__Eg:__ $0 \leq \frac{1}{n^2} \leq \frac{1}{n}$, but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The Comparison Test would give no information, since the "small" series converges and the "large" series diverges.

__Summary of Last "Important Example":__

The Dirichlet p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p \geq 2$

but diverges if $p \leq 1$.

We'll need another technique for $1 < p < 2$,

but first, let's do more examples of Comparison.

__Example:__ Either show $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, or that it diverges to ∞.

__Solution:__

Clearly, $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ for all $n \geq 1$,

so $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by Comparison w/ $\sum \frac{1}{n^2}$.

<u>Example</u>: Either show $\sum_{n=1}^{\infty} \frac{1}{n^2-5}$ converges, or that it diverges to ∞.

<u>Solution (w/ full discussion)</u>

Although the 1st two terms of $\frac{1}{n^2-5}$ are <0, it is equivalent to consider convergence of

$$\sum_{n=3}^{\infty} \frac{1}{n^2-5} \quad \text{or indeed} \quad \sum_{n=1000000}^{\infty} \frac{1}{n^2-5},$$

We try comparison w/ $\frac{1}{n^2}$: for $n \geq 3$, we have

$$0 \leq \frac{1}{n^2} \leq \frac{1}{n^2-5}$$

but unfortunately this is the "no information" situation.

Finally, we repeat the trick we used w/ $\frac{1}{n^2}$ and $\frac{1}{n^2+n}$ and instead compare

$$\frac{1}{n^2-5} \quad \text{with} \quad \frac{2}{n^2}.$$

Since $n^2 \leq 2(n^2-5)$

$\iff 10 \leq n^2$, as holds for $n \geq 4$,

we have $0 \leq \frac{1}{n^2-5} \leq \frac{2}{n^2}$ for $n \geq 4$,

and as $\sum \frac{2}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^2-5}$. ✓

<u>Solution (short)</u>:

Notice that $0 \leq \frac{1}{n^2-5} \leq \frac{2}{n^2}$

$\iff n^2 \leq 2n^2-10$

$\iff 10 \leq n^2$ holds for $n \geq 4$.

By Comparison, since $\sum \frac{1}{n^2}$ converges

$\sum_{n=1}^{\infty} \frac{1}{n^2-5}$ also converges. ✓

<u>Self-check</u>: Try the same w/ $\sum_{n=1}^{\infty} \frac{1}{n+\pi}$ and $\sum_{n=1}^{\infty} \frac{1}{n-\pi}$.

(Hint: both diverge.)

Example: Either show $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$ converges, or that it diverges to ∞.

Solution: We will use only that $\sin^2 n$ is a bounded, positive sequence (and no other properties of $\sin^2 n$). Indeed,

$$0 \leq \sin^2 n \leq 1$$

Thus, $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$. Since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series (as $\frac{1}{2} < 1$), $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$ also converges. ✓

Example: Either show $\sum_{n=1}^{\infty} \frac{n-2}{n^2}$ converges, or that it diverges to ∞.

Solution 1: Direct comparison. Since $\frac{n-2}{n^2} \geq \frac{1}{2n}$

$$\iff 2n^2 - 4n \geq n^2$$

$$\iff n^2 \geq 4n \text{, as happens for } n \geq 4$$

and as $\sum_{n=1}^{\infty} \frac{1}{2n} = \infty$, also $\sum_{n=1}^{\infty} \frac{n-2}{n^2} = \infty$. ✓

Selfcheck: Why didn't we compare w/ $\sum \frac{1}{n}$?

Solution 2: We have from prior examples that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = A \text{ for some real number } A \text{, while}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Apply Arithmetic of Series.

$$\sum_{n=1}^{\infty} \frac{n-2}{n^2} = \sum_{n=1}^{\infty} \frac{n}{n^2} - 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \infty - 2A = \infty.$$ ✓

We return to consider the Dirichlet p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $1<p<2$.
We use a trick similar to the one we used for $p=1$,
that is, for $\sum_{n=1}^{\infty} \frac{1}{n}$.

<u>Lemma:</u> If $1<p$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

<u>Proof:</u> We collect 2 terms, then 4 terms, then 8 terms, then...

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{< 2\cdot\frac{1}{2^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}}_{< 4\cdot\frac{1}{4^p}} + \cdots$$

$$< 1 + 2\cdot\frac{1}{2^p} + 4\cdot\frac{1}{4^p} + 8\cdot\frac{1}{8^p} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}} = \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k}.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is bounded above by the geometric series
with "ratio" $\frac{1}{2^{p-1}}$.

Now, since $p>1$, we see that $2^{p-1}>1$
so that $\frac{1}{2^{p-1}}<1$
so the geometric series converges,
forcing $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to also converge (since bounded). ∎

Let's collect what we know about Dirichlet p-series. These will
be an excellent source of series to compare with.

<u>Corollary</u> (Convergence/Divergence of Dirichlet p-series)
For a real number p, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
1) converges if $p>1$, and
2) diverges to ∞ if $p\le 1$.

<u>Example:</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$, $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$ all converge,
while $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{0.99}} = \infty$.

Big-Theta notation

Definition: If a_n and b_n are sequences so that,
for some positive constants α, β
and some N
we have that for all $n > N$, $\alpha b_n \le a_n \le \beta b_n$
then we say a_n is big theta of b_n
and write $a_n = \Theta(b_n)$.

Notice: If $\alpha b_n \le a_n \le \beta b_n$, then also $\frac{1}{\beta} a_n \le b_n \le \frac{1}{\alpha} a_n$.
Thus, $a_n = \Theta(b_n) \Leftrightarrow b_n = \Theta(a_n)$. ✓

Remark: Big theta notation is extremely important in algorithm analysis, where it is the "gold standard" of complexity for an algorithm.

In algorithm analysis, we want to show the number of steps taken to be $\Theta(n^2)$ or $\Theta(n^3)$ or $\Theta(2^n)$ or similar.
For ANA-I, big theta notation will help us analyze series convergence. Here, we'll want to show the terms to be $\Theta(\frac{1}{n})$ or $\Theta(\frac{1}{n^2})$ or $\Theta(\frac{1}{2^n})$ or similar.

Definition: Two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are said to be equiconvergent if they both converge or both diverge.

Corollary (of Direct Comparison) If $a_n = \Theta(b_n)$ for positive sequences a_n, b_n then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are equiconvergent.

Pf: If $\sum_{n=1}^{\infty} b_n = \infty$, then also $\sum_{n=1}^{\infty} \alpha b_n = \infty$ for any constant α, hence $\sum_{n=1}^{\infty} a_n = \infty$.
Similarly, if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} \beta b_n$, (for any constant β) and hence so does $\sum_{n=1}^{\infty} a_n$. ∎

The following properties of Θ notation follow easily:

<u>Proposition</u>: If a_n, b_n, c_n, d_n are positive sequences with $a_n = \Theta(b_n)$, $c_n = \Theta(d_n)$

then

1) $\frac{1}{a_n} = \Theta(\frac{1}{b_n})$
2) $a_n + c_n = \Theta(b_n + d_n)$
3) $a_n \cdot c_n = \Theta(b_n \cdot d_n)$.
4) $\alpha \cdot a_n = \Theta(b_n)$ for any real $\alpha > 0$.

<u>Self-check</u>: Verify these!

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$

<u>Solution</u>:

Since $\frac{n}{2} \leq n-2 \leq n$ for $n \geq 4$ (so $\Theta(n-2) = n$)

and $n^2 \leq n^2+1 \leq 2n^2$ for $n \geq 1$, (so $\Theta(n^2+1) = n^2$)

we have $\Theta(\frac{n-2}{n^2+1}) = \Theta(\frac{n}{n^2}) = \Theta(\frac{1}{n})$,

so $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. ✓

Another approach, having about the same power, uses limits:

<u>Theorem</u> (Limit Comparison Test)

If a_n and b_n are positive real-valued sequences so that

$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for some real number L with $0 < L < \infty$

then

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are equiconvergent.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$, again.

Since $$\lim_{n\to\infty} \frac{\frac{n-2}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2-2n}{n^2+1} = \lim_{n\to\infty} \frac{1-2/n}{1+1/n^2} = \frac{1-0}{1+0} = 1,$$

$\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. ✓

Proof (of Limit Comparison Test)

By definition of Limit

$$\forall \varepsilon > 0, \ \exists N \text{ s.t. } [n > N] \Rightarrow \left[\left| \frac{a_n}{b_n} - L \right| < \varepsilon \right]$$

Thus, for $n > N$ we have

$$-\varepsilon < \frac{a_n}{b_n} - L < \varepsilon$$

$$\Leftrightarrow \quad (L-\varepsilon) b_n < a_n < (L+\varepsilon) b_n$$

Now, for small enough ε, we have $L - \varepsilon > 0$ (as $L > 0$), which gives us $\alpha = (L-\varepsilon)$

$\beta = (L+\varepsilon)$ (for this fixed small ε)

so that for $n > N$ we have $\alpha b_n < a_n < \beta b_n$

and in particular $a_n = \Theta(b_n)$.

Equiconvergence now follows by the previous Corollary. ■

Example: Discuss convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2 - 3n + 5}$

Solution 1: Since n^2 is the highest power of n in the denominator, we try Limit Comparison w/ $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{2n^2-3n+5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{2 - 3/n + 5/n^2} = \frac{1}{2-0+0} = \frac{1}{2}$$

and since $0 < \frac{1}{2} < \infty$, and since $\frac{1}{2n^2-3n+5}$ is eventually positive the series is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. ✓

Solution 2 (sketch) Show $\Theta(2n^2 - 3n + 5) = n^2$ directly,

so that $\Theta\left(\frac{1}{2n^2-3n+5}\right) = \Theta\left(\frac{1}{n^2}\right)$

so that $\sum \frac{1}{2n^2-3n+5}$ equiconvergent to $\sum \frac{1}{n^2}$

as in Solution 1. ✓

In series where an exponential or factorial term is present, it is often easier to use one of the following tests: the Root and Ratio Tests.

<u>Theorem</u>: (Ratio Test)

Let a_n be a positive real sequence.

1) If $\limsup \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $\liminf \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n = \infty$
3) Otherwise no information.

We'll prove this a little later. First, an example:

<u>Example</u>: Discuss convergence of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

<u>Solution</u>: Since 2^n is an exponential, it is worthwhile to try the Ratio Test. We evaluate

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2} \cdot (1+0)^2$$

$$= \frac{1}{2}$$

Since $\frac{1}{2} < 1$, the series converges. ✓

The Root Test has a similar flavor, and is sometimes easier to apply.

<u>Theorem</u>: (Root Test)

Let a_n be a positive real sequence.

1) If $\limsup (a_n)^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges
2) If $\limsup (a_n)^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n = \infty$
3) Otherwise, no information.

<u>Example</u>: Discuss convergence of $\sum_{n=1}^{\infty}\left(\frac{n+3}{2n}\right)^n$

<u>Solution</u>: Since we have an nth power, we try the Root Test.

Indeed,

$$\lim_{n\to\infty}\left(\left(\frac{n+3}{2n}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{n+3}{2n}\right) = \lim_{n\to\infty} \frac{1}{2}+\frac{3}{2n} = \frac{1}{2}+0$$

and since $1/2<1$, the series converges. ✓

<u>Proof (of Ratio Test)</u>:

1) Let $L=\limsup \frac{a_{n+1}}{a_n}$. By definition,

$$\forall \varepsilon>0,\ \exists N \text{ s.t. } [n>N] \Rightarrow \left[\frac{a_{n+1}}{a_n}<L+\varepsilon\right]$$

Thus, in this situation, we have

$$a_{N+k} < a_{N+k-1}\cdot(L+\varepsilon) < a_{N+k-2}(L+\varepsilon)^2 < \cdots < a_{N+1}(L+\varepsilon)^{k-1}$$

so the terms of the sequence are eventually bounded above by a geometric series with ratio $r=L+\varepsilon$.

(as $\sum_{n=N+1}^{\infty} a_n < \sum_{n=N+1}^{\infty} a_{N+1}\cdot(L+\varepsilon)^{n-(N+1)} = \sum_{n=0}^{\infty} a_{N+1}(L+\varepsilon)^n$.)

The result follows by taking ε small enough that $r=L+\varepsilon$ is <1, as we can do since $L<1$. ✓

2) By definition of liminf, $\forall\varepsilon>0, \exists N$ s.t. $[n>N]\Rightarrow\left[\frac{a_{n+1}}{a_n}>L-\varepsilon\right]$.

By taking ε s.t. $L-\varepsilon>1$, as we can do since $L>1$, we see that for large enough n, we have $a_{n+1}>a_n$.

Since $a_n>0$, it follows that $\lim_{n\to\infty} a_n>0$, hence that $\sum_{n=1}^{\infty} a_n$ diverges. ∎

The proof of the Root Test follows a similar idea.

(Omitted – see the clear Wikipedia article.)

Note that the general idea of both the Ratio and Root Tests is that the series "acts like" a geometric series with the associated test statistic (like $\limsup \frac{a_{n+1}}{a_n}$).

<u>Anticexamples</u> Consider the Ratio Test on $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

We calculate

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1^2 = 1.$$

Since $\sum \frac{1}{n} = \infty$ while $\sum \frac{1}{n^2}$ converges, we see that the Ratio Test statistic <u>cannot</u> give us useful information when we have a limit of 1. (Or $\limsup \geq 1$, $\liminf \leq 1$.)

Instead we try another test, like Direct Comparison. ✓

We'll do more examples with these tests shortly, after we extend to series with both positive and negative terms.

<u>C. Absolute convergence vs conditional convergence.</u>

For series with both positive and negative terms, the following allows us to apply the Comparison, Ratio, and Root tests to show convergence.

<u>Theorem:</u> For a sequence a_n of real numbers, if $\sum_{n=1}^{\infty} |a_n|$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.

<u>Proof:</u> We add $|a_n|$ to get from negative to nonnegative:

Since $-|a_n| \le a_n \le |a_n|$ for any n,

also $0 \le a_n + |a_n| \le 2|a_n|$ for any n.

Now since $\sum_{n=1}^{\infty} 2|a_n| = 2 \cdot \sum_{n=1}^{\infty} |a_n|$ converges,

by Direct Comparison so does $\sum_{n=1}^{\infty} a_n + |a_n|$

and by Arithmetic of Series,

so does $\sum_{n=1}^{\infty} a_n = \left(\sum_{n=1}^{\infty} a_n + |a_n|\right) - \left(\sum_{n=1}^{\infty} |a_n|\right)$. ∎

<u>Definition 1</u> If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say

$\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u> or <u>absolutely converges</u>.

(By the Theorem, an absolutely converging series converges.)

<u>Definition 2:</u> If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n| = \infty$,

then we say $\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u>.

*** All of our tests for convergence of series w/ nonnegative terms are now tests for absolute convergence! ***

<u>Example:</u> Discuss convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^{100}}{3^n}$

<u>Solution:</u> Since we have an exponential, try Ratio Test on

$\left| \frac{(-1)^n \cdot n^{100}}{3^n} \right| = \frac{n^{100}}{3^n}$.

Here $$\lim_{n\to\infty} \frac{\frac{(n+1)^{100}}{3^{n+1}}}{\frac{n^{100}}{3^n}} = \lim_{n\to\infty} \frac{\cancel{3^n}}{\cancel{3^n} \cdot 3} \cdot \left(\frac{n+1}{n}\right)^{100}$$

$$= 1/3 \cdot 1^{100} = 1/3$$

and since $1/3 < 1$, the series converges absolutely.

(that is, $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^{100}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{n^{100}}{3^n}$ converges.)

<u>Corollary:</u> $\lim_{n\to\infty} \frac{n^{100}}{3^n} = 0$.

Recall that $n!$ is recursively defined by $0! = 1$, $n! = n\cdot(n-1)!$.

<u>Example</u> Discuss convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n \cdot (n+2)}{n!}$.

<u>Solution</u> Since we have exponentials and factorials, the Ratio Test looks like a good test to try.

We apply to $\left|\frac{(-1)^n \cdot 5^n (n+2)}{n!}\right| = \frac{5^n \cdot (n+2)}{n!}$

$$\lim_{n\to\infty} \frac{\frac{5^{n+1}\cdot(n+3)}{(n+1)!}}{\frac{5^n\cdot(n+2)}{n!}} = \lim_{n\to\infty} \frac{5^n\cdot 5}{5^n} \cdot \frac{n+3}{n+2} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} 5 \cdot \frac{1+3/n}{1+2/n} \cdot \frac{1}{n+1} = 5\cdot 1 \cdot \tfrac{1}{\infty} = 0.$$

Since $0<1$, the series converges absolutely.

<u>Example</u> Discuss convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n+2}}{n^2+3}$

<u>Solution</u> We see clearly that $\sqrt{n+2} = \Theta(\sqrt{n})$

(as $\sqrt{n} \le \sqrt{n+2} \le \sqrt{3n}$)

and $n^2+3 = \Theta(n^2)$

(as $n^2 \le n^2+3 \le 4n^2$)

So $\frac{\sqrt{n+2}}{n^2+3} = \Theta\left(\frac{\sqrt{n}}{n^2}\right) = \Theta\left(\frac{1}{n^{3/2}}\right)$. Since $3/2 > 1$, the series is equiconvergent to a <u>convergent</u> series.

So $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n+2}}{n^2+3}$ converges absolutely.

Example: Discuss convergence of $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$.

Remark: We'll need to know only that $-1 \le \sin n \le 1$.

Non-solution: Although you may see the 2^n and think "Ratio Test", $\frac{|\sin(n+1)|}{|\sin n|}$ is not easy to deal with in limsup

and indeed, $\frac{\sin 2}{\sin 1} \approx 1.08 > 1$, while $\frac{\sin 3}{\sin 2} \approx 0.16$

Solution 1 (easy): Direct comparison! Since $-1 \le \sin n \le 1$,

also $0 \le |\sin n| \le 1$, so $0 \le \left|\frac{\sin n}{2^n}\right| \le \frac{1}{2^n}$.

As $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so this series converges absolutely.

Solution 2: The Ratio Test looks hopeless, but we try the Root Test.

Since $0 \le |\sin n| \le 1$, also $0 \le |\sin n|^{1/n} \le 1$

and so $0 \le \left(\frac{|\sin n|}{2^n}\right)^{1/n} \le \frac{1}{2}$.

Thus

$$\limsup \left(\frac{|\sin n|}{2^n}\right)^{1/n} \le \frac{1}{2} < 1$$

and so by the Root Test

$\sum_{n=1} \frac{\sin n}{2^n}$ converges absolutely.

General example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely for all $p > 1$

but either converges conditionally

or diverges when $p \le 1$.

(as $\left|\frac{(-1)^n}{n^p}\right| = \frac{1}{n^p}$; and by our result on Dirichlet p-series)

Fact (to be shown later): $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (conditionally).

In a later class, you may show $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln \frac{1}{2} \approx -0.69$.

The trouble with conditionally convergent series such as $\sum \frac{(-1)^n}{n}$ is that if we reorder the terms, we may change the sum of the series.

Although this is counter-intuitive — addition is commutative — it is a result of the observations that

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{-1}{2k+1} = -\infty \quad \leftarrow \text{odd terms}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{2k} = \sum_{k=1}^{\infty} \frac{1}{2k} = \infty \quad \leftarrow \text{even terms}$$

and that $\infty - \infty$ is an indeterminate form.

<u>Example:</u> Rearrange the terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ to add to 2018.

<u>Solution:</u> First: Take enough positive terms $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$ so that the sum is (just) over 2018.

Now take enough negative terms so that the sum is (just) under 2018.

Then take + terms over 2018 again

Then " − " under " .

Repeat in this process.

As the terms $\frac{1}{n}$ go to 0, this process will converge at 2018.

<u>Self-check:</u> Why does this argument fail for $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$?

Of course, 2018 can be replaced here with any other number.

<u>Moral:</u> The order of summation is important for a conditionally convergent series!!

D. Alternating series test and estimation

A series whose terms alternate in sign is called an alternating series.

Eg) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

There is an easy test for convergence of alternating series, but it will not tell us about absolute convergence.

The test also provides an easy-to-apply estimation result, which will be very helpful for numerical computation.

Theorem (Alternating Series Test and Estimation)

Let a_n be a positive, decreasing real-valued sequence with $\lim_{n\to\infty} a_n = 0$

Then

1) $\sum_{n=1}^{\infty} (-1)^n \cdot a_n$ converges (Test)

2) As usual, write R_N for $\sum_{n=N+1}^{\infty} (-1)^n \cdot a_n$ (so $\sum (-1)^n \cdot a_n = S_N + R_N$). for the remainder from the nth partial sum.

Then $|R_N| \leq a_{N+1}$ (Estimation).

Examples Since $1/n$ is a positive decreasing sequence w/ $\lim_{n\to\infty} 1/n = 0$, $\sum_{n=1}^{\infty} (-1)^n/n$ converges (by (1)).

Also, $-1 + \frac{1}{2} = -\frac{1}{2}$ is within $\frac{1}{3}$ of the limit sum

$-1 + \frac{1}{2} - \frac{1}{3} = -\frac{5}{6}$ is within $\frac{1}{4}$

$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = -\frac{7}{12}$ is within $\frac{1}{5}$, and so on.

Proof (of AST/E):

1) We split into even/odd subcases for the partial sums S_N.

Even: $S_{2k} = S_{2k-2} + \underbrace{a_{2k} - a_{2k-1}}_{\text{negative}} \leq S_{2k-2}$ (as $a_{2k} \leq a_{2k-1}$)

so the even subsequence S_{2k} is decreasing.

Odd: $S_{2k+1} = S_{2k-1} + \underbrace{a_{2k} - a_{2k+1}}_{\text{positive}} \geq S_{2k-1}$ similarly

so the odd subsequence S_{2k+1} is increasing.

Also, $-a_1 = S_1 \leq S_{2k+1} \leq S_{2k+2} \leq S_2 \leq -a_1 + a_2$

odd increasing

adding $+a_{2k+2}$

even decreasing

so both are bounded between $-a_1$ and $-a_1+a_2$.

Now the MST tells us that S_{2k} and S_{2k+1} converge to L_{even} and L_{odd}, respectively.

But $L_{even} - L_{odd} = \lim_{k\to\infty} S_{2k} - S_{2k+1} = \lim_{k\to\infty} a_{2k+1} = 0$ by hypothesis.

So $L_{even} = L_{odd} = L$. By a result proved in tutorial, the series converges to L.

2) Using the results of (1), since S_{2k+1} is increasing while S_{2k} is decreasing,

we have for any k that $S_{2k+1} \leq L \leq S_{2k}$.

Now $0 \leq L - S_{2k+1} \leq S_{2k+2} - S_{2k+1} = a_{2k+2}$ (odd remainders)

and $\underbrace{S_{2k+1} - S_{2k}}_{-a_{2k+1}} \leq L - S_{2k} \leq 0$ (even remainders),

In either case, we have $|L - S_N| = |R_N| \leq a_{N+1}$, as desired. ■

In practice, Alternating Series Estimation is even more useful than the Alternating Series Test.

- Since the AST does not tell us about absolute convergence, it is generally our <u>last resort</u>, for possibly-conditionally-convergent series.
- The ASE is useful and easy to apply even when we prefer another test for convergence.

<u>Example:</u> Discuss convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$. If it converges, estimate the sum to within an accuracy of 0.1.

<u>Solution:</u> We first check absolute convergence, using a Comparison Test.

Since $0 \leq \left| \frac{(-1)^n}{n^2+1} \right| = \frac{1}{n^2+1} \leq \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges (as $2>1$), we see that the series absolutely converges.

Now we apply ASE. The series is alternating for all n, and $\frac{1}{(n+1)^2+1} \leq \frac{1}{n^2+1}$ (as $(n+1)^2+1 \geq n^2+1$) for all $n \geq 0$.

Finally $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ since the series absolutely converges. (or by AOL).

So ASE applies.

Also, $\frac{1}{n^2+1} \leq 0.1 = \frac{1}{10}$ first at $n=3$.

Thus, $\sum_{n=0}^{3} \frac{(-1)^n}{n^2+1} = 1 - \frac{1}{2} + \frac{1}{5} = 0.7$ is accurate within 0.1 ✓

<u>Example:</u> Discuss convergence of $\sum_{n=1} \frac{(-1)^n}{n}$. If it converges, estimate the sum to within an accuracy of 0.1.

<u>Solution:</u> Since $\left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a main example of an interesting divergent series, $\sum \frac{(-1)^n}{n}$ does not converge absolutely.

However, $\frac{1}{n}$ is clearly a positive decreasing sequence with $\lim_{n\to\infty} \frac{1}{n} = 0$!!

Thus, the AST applies, and

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

To estimate to within $0.1 = 1/10$, we stop just before the $n=10$ term of $1/10$.

Our estimate is $\sum_{n=1}^{9} \frac{(-1)^n}{n} = -1 + 1/2 - 1/3 + 1/4 - 1/5 + 1/6 - 1/7 + 1/8 - 1/9$
$= \frac{-1879}{2520} \approx -0.75$ ✓

<u>Example</u>: Discuss convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1}$. If it converges, describe how many terms are required to estimate the series within 0.1.

<u>Solution</u>: Since (by the homework) $2n+1 = \Theta(n)$ and $4 = \Theta(1)$, we have $\Theta\left(\frac{4}{2n+1}\right) = \Theta\left(\frac{1}{n}\right)$, and as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series does not converge absolutely.

But as $\frac{4}{2(n+1)+1} \le \frac{4}{2n+1}$ (since $2(n+1)+1 \ge 2n+1$) the terms are decreasing. Clearly $\frac{1}{2n+1}$ is positive for all n.

And $\lim_{n\to\infty} \frac{4}{2n+1} = 0$ by AoL.

Thus, the AST applies and $\sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}$ converges.

To estimate within 0.1, we use the ASE. As

$\frac{4}{2n+1} \le .1 \iff 40 \le 2n+1 \iff 19.5 \le n$

So our estimate is with $\sum_{n=0}^{19} \frac{(-1)^n \cdot 4}{2n+1}$. ✓

<u>Fact</u>: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1} = \pi$, although this might not be so easy to see with the first few partial sums!

N	0	1	2	3	4	5	---
S_N	4	$2\frac{2}{3}$	$3\frac{7}{15}$	$2\frac{94}{105}$	$3\frac{107}{315}$	$2\frac{3382}{3465}$	---
a_{N+1}	$4/3$	$4/5$	$4/7$	$4/9$	$4/11$	$4/13$	

There's also a series that turns out to converge to $1/e$.

<u>Example:</u> Discuss convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. If it converges, estimate the sum to within accuracy of 0.01.

<u>Solution:</u> To test absolute convergence, we use the Ratio Test:

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

So the series converges absolutely.

To estimate, we notice that $\frac{1}{n!} > 0$ for all n and that, since $n!$ is increasing, $\frac{1}{n!}$ is decreasing.

That $\lim_{n\to\infty} \frac{1}{n!} = 0$ follows from absolute convergence.

So the ASE applies.

To get accuracy within $0.01 = \frac{1}{100}$, we notice that $\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$. So summing through $n=4$ suffices!

$$\sum_{n=0}^{4} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = 3/8 = 0.375. \checkmark$$

<u>Self-check:</u> How many terms would you need for accuracy within $\frac{1}{1000} = 0.001$?

<u>Fact:</u> $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1/e \approx 0.367879$

<u>Example:</u> Discuss convergence of $\sum_{n=1}^{\infty} \left(-1-\frac{1}{n}\right)^n$

<u>Solution:</u> Notice that $\left(-1-\frac{1}{n}\right)^n = (-1)^n \cdot \left(1+\frac{1}{n}\right)^n$, so the series is alternating. But since $\liminf \left(1+\frac{1}{n}\right)^n \geq 1$ (as the terms are ≥ 1 in absolute value), the AST does <u>not</u> apply.

Indeed, since the even terms are ≥ 1 and the odd " " ≤ -1, we see that $\lim_{n\to\infty} (-1)^n \cdot (1+1/n)^n$ does not converge, and so by the nth term test, the series does not converge either.

Remark: The AST is almost the converse of the nth term test that you might have wished for at some points. We need also that the series is alternating and that the terms are (eventually) decreasing; and of course, it doesn't tell us anything about absolute convergence.

Strategies for testing series convergence:

There is no algorithm to determine whether a series converges or diverges, but there are strategies for using our tests. Let's summarize (For series $\sum_{n=0} a_n$.)

- The nth term test says if $\lim_{n\to\infty} a_n \neq 0$ then the series diverges. "Usually" unhelpful, but can save a lot of work when it helps. Easy to check if you're fluid w/ limits.
- The Ratio (or Root) Test will only help if there's some kind of exponential/factorial/similar in the series terms. It's easy to check: if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ < 1 then series converges, > 1 " " diverges

  and is good to try if you have some exponential/factorial and the test statistic is not difficult to compute.

  (Recall also limsup / lim inf versions!)

- Direct / Limit Comparison Tests are powerful, but difficult to use in that you must "tune" the test by finding a series to compare with. The "big theta" approach will often yield insight to this problem (and occasionally will answer the problem entirely.)
- The Alternating Series Test, since it does not prove or disprove absolute convergence, is the test of last resort for an alternating series, to be used after you have shown the series not to converge absolutely. (It is generally easy to use in this case.)

Interlude: Countable + Uncountable sets

Infinite sets can have surprising properties.

For instance, although

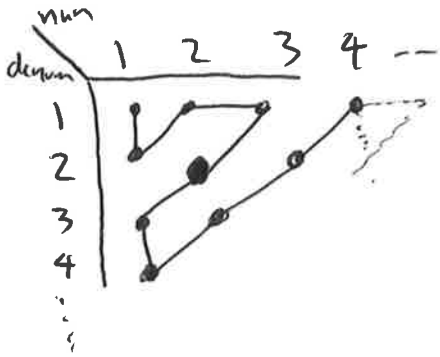
$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C},$$

we have exhibited a sequence a_n which takes on every integer value exactly once

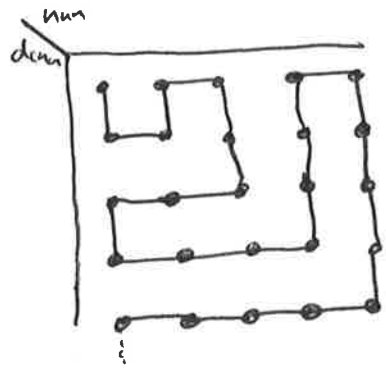
$$(a_n) = (0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots)$$

and sketched a sequence r_n that takes on every rational value exactly once.

Recall this was constructed by alternating positive/negative as we did w/ $\mathbb{Z}$, and "walking through" a table of numerator/denominator pairs in a manner such as



or



etc.

Thus, although $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$,
we can find a 1-1 correspondence (a "bijection")
between $\mathbb{N}$ and $\mathbb{Z}$
and between $\mathbb{N}$ and $\mathbb{Q}$.

<u>Self-check:</u> Can you find a 1-1 correspondence between $\mathbb{Z}$ and $\mathbb{Q}$?

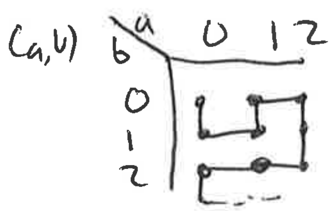
<u>Definition:</u> If a set S is the range of some sequence then we say S is <u>countable</u>.

<u>Prop:</u> $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable, as is any finite set.

<u>Pf:</u> We've seen $\mathbb{Z}, \mathbb{Q}$ already.
$\mathbb{N}$ follows from the sequence $a_n = n$.
A finite set is the range of the sequence which enumerates the set in some order, then repeats the last element infinitely often
e.g. $\{0,2,3,5\}$ and the sequence $(0,2,3,5,5,5,\ldots)$ $\boxed{\equiv}$

<u>Prop:</u> The set $\mathbb{N}^2$ = all ordered pairs of natural numbers is countable.

<u>Pf:</u> Apply the same argument as for $\mathbb{Q}$!



$\boxed{\equiv}$

Remark: It is easy to see that a set is countable and infinite (countably infinite) if and only if it is in 1-1 correspondence with $\mathbb{N}$.

A set that is not countable is (unsurprisingly) called <u>uncountable</u>.
The main thing I want to tell you in this Inter-lude is:

<u>Theorem</u> $\mathbb{R}$ is uncountable.

Thus, although $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all of the "same size", $\mathbb{R}$ is much bigger.
I'll give you 2 proofs.

<u>Proof 1</u>: (By Cantor Diagonalization).
Suppose for contradiction that (a_n) is a sequence whose range is $\mathbb{R}$.
Then for each n, the number a_n has a decimal expansion. (following the .) [redacted]
Let d_n be the nth digit of a_n, so that if e.g.

$$a_0 = 1.414132\ldots \qquad\qquad d_0 = 4$$
$$a_1 = 0.2356\ldots \qquad \text{then} \qquad d_1 = 3$$
$$a_2 = 7.71717\ldots \qquad\qquad d_2 = 7$$
$$\vdots \qquad\qquad\qquad \vdots$$

If d is a number 0–9, then let $\bar{d} = \begin{cases} 0 & \text{if } d \neq 0 \\ 1 & \text{if } d = 0 \end{cases}$

Finally, let a be the number $0.\ldots$ whose nth digit following the . is $\bar{d}_n$. So in the above e.g., we'd have

$$a = 0.\bar{4}\bar{3}\bar{7}\ldots = 0.000\ldots$$

Now we ask: where is a in our sequence?

$a \neq a_0$, since differs at 0th decimal place.

$a \neq a_1$ — — — — — 1st — —

$a \neq a_2$ — — — — — 2nd — —

$\vdots$

Since a differs from each a_n in (at least) the nth decimal place, a is not in the range of our sequence.

As a is a real number, and the range of a_n was $\mathbb{R}$, this is a contradiction! #

Remarks: A similar proof approach w/ Cantor Diagonalization is used in high-level Computer Science classes to show that no computer program can be written to check if another computer program will terminate.

While Cantor Diagonalization yields a nice proof, the other proof that I'll show you ties in better to ANA-I. I'll start with a Lemma, whose proof I'll defer.

Lemma*: If the interval $[0,3]$ is contained in the union of a sequence (a_i, b_i) of open intervals (that is, $[0,3] \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$) then

$$3 \leq \sum_{i=0}^{\infty} b_i - a_i$$

(that is, length of $[0,3]$ is $\leq$ the sum of lengths of covering intervals.)

Proof: (that $\mathbb{R}$ is uncountable, based on the Lemma).

Suppose not, that c_n is a sequence having range of $\mathbb{R}$.

Recall that $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$,

So let d_n be the subsequence of c_n having values in $[0,3]$, and for each n, let (a_n, b_n) be an interval of length $1/2^n$, so that $b_n - a_n = 1/2^n$. and so that d_n lies on the interval.

(E.g., we could take $a_n = d_n - 1/2^{n+1}$, $b_n = d_n + 1/2^{n+1}$.)

But now each element of $[0,3]$ is some d_n, so lies on some interval.

Thus, $[0,3] \subseteq \bigcup_{n=0}^{\infty} (a_n, b_n)$

so the Lemma tells us that $3 \leq \sum_{n=0}^{\infty} b_n - a_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$,

yielding our desired contradiction. # $\square$

To prove the Lemma, I'll first show that it holds if, instead of an infinite # of intervals, I have some finite number of them.

Lemma *': If the interval $[0,a]$ is contained in the union of open intervals $(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)$ then $a \leq \sum_{i=0}^{n} b_i - a_i$.

Proof: We proceed by induction on n.

Base case: $n=0$. Here we have $[0,a] \subseteq (a_0, b_0)$

so $a_0 < 0$ and $b_0 > a$,

so $b_0 - a_0 > a - 0$ ✓

<u>Induction step:</u> We assume the result for n intervals (and any a) and prove it for $n+1$ intervals.

Since $[0,a] \subseteq \bigcup_{i=0}^{n+1} (a_i, b_i)$,

in particular a is in one of the intervals.

Wlog, let a be in (a_{n+1}, b_{n+1})

(otherwise, renumber the intervals.)

Notice that $[0,a] \setminus (a_{n+1}, b_{n+1}) = [0, a_{n+1}]$.

Thus,

$$[0, a_{n+1}] \subseteq \bigcup_{i=0}^{n} (a_i, b_i)$$

so by induction,

$$a_{n+1} \leq \sum_{i=0}^{n} b_i - a_i.$$ As also $a \leq b_{n+1}$,

we get

$$a \leq \left(\sum_{i=0}^{n} b_i - a_i\right) + b_{n+1} - a_{n+1},$$ as desired. $\blacksquare$

What about an infinite collection of intervals?

Lemma * will follow from Lemma *' and the following theorem (which we defer further):

<u>Theorem</u> If a closed interval $[a,b]$ is contained in a union of open intervals (a_i, b_i)

(that is, $[a,b] \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$)

then we can select some finite set of indices S

so that $[a,b] \subseteq \bigcup_{i \in S} (a_i, b_i)$

<u>Example:</u> $[0,2] \subseteq (-1, 1/10) \cup \bigcup_{i=1}^{\infty} (1/2^i, 3)$.

Here $(-1, 1/10)$ and $(1/16, 3)$ will suffice.

This Theorem and its proof will be a main topic of the next section.