Homeworks

(A) Let S be a subset of the $\mathbb{R}^2$ given by all pairs (x,y), so that $x^2+y^2=2$. Show that S is a closed subset of $\mathbb{R}^2$.

If S^c is open $\Rightarrow$ S is closed, by def.

$S^c = x^2+y^2 \neq 2$

$$x^2+y^2>2 \quad \wedge \quad x^2+y^2<2$$

Since "<" and ">" are strict inequalities we have an open interval: $(-\infty, 2)\cup(2,\infty) / \mathbb{R}^2$

Hence, S is closed.

(B) By showing that any sequence in $A\cup L$ has the same limit sequence as some sequence in A, prove that $\bar{A} \subseteq A\cup L$, where L is the set of accumulation points of sequences in A.

We show that $\bar{A} \subseteq A \cup \{\lim a_n : a_n \text{ sequence in } A\}$ and

$$A\cup L \supseteq \bar{A}$$

($\supseteq$:) $\bar{A}$ is a closed set containing A.
As closed sets are sequentially closed, we have that $\bar{A}$ contains all limit points of sequences in A, hence all limit points in A.

($\subseteq$:) We show RHS is closed (since $\bar{A}$ is contained in any set containing A)

Let b_n be a sequence in $A \cup L$ s.t. $\lim_{n \to \infty} b_n \to b$ is a limit point in A.

Let a_n be a sequence s.t. $a_n \in A \cup L$.

Let $d(a_n, b) < \varepsilon$, $\varepsilon = \frac{1}{n}$

$$\Rightarrow \lim_{n \to \infty} d(a_n, b) = 0$$

$$\Rightarrow \lim_{n \to \infty} a_n = b$$

Since a_n is a sequence of points in $A \cup L$ and a_n converges to b it concludes that b is in $A \cup L$.

Hence closed.

C. Show that a set A in $\mathbb{R}^2$ is open in the Eucidian metric $\Longleftrightarrow$ it is open in the Manhattan metric.

$d_1 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ → Eucidian metric

$d_2 = |x_1 - y_1| + |x_2 - y_2|$ → Manhattan metric