Week 12

Gauss-Jordan method

- At each step, the pivot element is forced to be 1
- At each step, all terms above and below the pivot are eliminated

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_n \end{array}\right]$$

The solution then appears in the last column (i.e. $x_i = s_i$) so that this procedure circumvents the need to perform back substitution.

① Apply the Gauss-Jordan method to solve the following system

$$\begin{aligned} 2x_1 + 2x_2 + 6x_3 &= 4 \\ 2x_1 + x_2 + 7x_3 &= 6 \\ -2x_1 - 6x_2 - 7x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & -4 & -1 & 3 \end{array}\right] \xrightarrow[(-1)\, R_2]{\frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & -1 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -5 & -5 \end{array}\right] \xrightarrow{\frac{-1}{5} \cdot R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1 - 3R_3]{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

② Use the Gauss-Jordan method to solve the following system:

$$\begin{aligned} 4x_2 - 3x_3 &= 3 \\ -x_1 + 7x_2 - 5x_3 &= 4 \\ -x_1 + 8x_2 - 6x_3 &= 5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & -3 & 3 \\ -1 & 7 & -5 & 4 \\ -1 & 8 & -6 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 8 & -6 & 5 \\ -1 & 7 & -5 & 4 \\ 0 & 4 & -3 & 3 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} -1 & 8 & -6 & 5 \\ 0 & -1 & 1 & -1 \\ 0 & 4 & -3 & 3 \end{array}\right] \xrightarrow[(-1)R_2]{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -8 & 6 & -5 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -3 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & -8 & 6 & -5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_1 - 6R_3]{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -8 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 + 8R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\Longrightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

③ Apply the Gauss-Jordan method to the following system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 3x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \end{array}\right] \xrightarrow[\substack{R_3 - R_2 \\ R_4 - R_1}]{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 - R_3]{R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Longrightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}}$$

Week 12

Gauss-Jordan method

- At each step, the pivot element is forced to be 1
- At each step, all terms above & below the pivot are eliminated

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_n \end{array}\right]$$

here, $x_i = s_i$

④ Use the Gauss-Jordan method to solve the following three systems at the same time.

$$\begin{array}{rcl|c|c} 2x_1 - x_2 & = & 1 & 0 & 0 \\ -x_1 + 2x_2 - x_3 & = & 0 & 1 & 0 \\ -x_2 + x_3 & = & 0 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 3 & -2 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 + 3R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & -2 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & -2 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right] \xrightarrow[R_2 + R_3]{R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array}\right]$$

$$\boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{array}} \quad \boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 2 \end{array}} \quad \boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}}$$

Non-homogenous systems

① Determine the general solution of the following non-homogenous system and compare it with the general solution of the associated homogenous system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 4 \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 5 \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 7 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 3 & 6 & 1 & 4 & 7 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & -3 & -3 & -3 \\ 0 & 0 & -5 & -5 & -5 \end{array}\right] \xrightarrow[R_3/-5]{R_2/-3} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \boxed{x_2 = t \in \mathbb{R}}, \quad \boxed{x_4 = s \in \mathbb{R}} \qquad x_3 + s = 1 \quad \boxed{x_3 = 1 - s}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 4 \\ x_1 + 2t + 2(1-s) + 3s &= 4 \\ x_1 + 2t + 2 + s &= 4 \end{aligned} \qquad \boxed{x_1 = 2 - 2t - s}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} s$$

homogenous:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \boxed{x_2 = t \in \mathbb{R}}, \quad \boxed{x_4 = s \in \mathbb{R}} \qquad \boxed{x_3 = -s}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0 \\ x_1 + 2t - 2s + 3s &= 0 \end{aligned} \qquad \boxed{x_1 = -2t - s}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} s$$

Non-homogenous systems

② Determine the general solution of the following non-homogenous system and compare it with the general solution of the associated homogenous system.

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2 \\ 3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1&1&2&2&1&1 \\ 2&2&4&4&3&1 \\ 2&2&4&4&2&2 \\ 3&5&8&6&5&3 \end{array}\right] \xrightarrow[R_3-2R_1,\ R_4-3R_1]{R_2-2R_1} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1 \\ 0&0&0&0&1&-1 \\ 0&0&0&0&0&0 \\ 0&2&2&0&2&0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1 \\ 0&2&2&0&2&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&-1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1 \\ 0&2&2&0&2&0 \\ 0&0&0&0&1&-1 \\ 0&0&0&0&0&0 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1 \\ 0&1&1&0&1&0 \\ 0&0&0&0&1&-1 \\ 0&0&0&0&0&0 \end{array}\right]$$

x_3, x_4 free

$\text{Rank}(A|b) = 3$
$\text{Rank}(A) = 3$

$x_3 = t \in \mathbb{R}$, $x_4 = s \in \mathbb{R}$

$\boxed{x_5 = -1}$

$x_2 + x_3 + x_5 = 0$
$x_2 + t - 1 = 0$
$\boxed{x_2 = 1-t}$

$x_1 + x_2 + 2x_3 + 2x_4 + x_5 = 1$
$x_1 + (1-t) + 2t + 2s - 1 = 1$
$x_1 + t + 2s = 1$
$\boxed{x_1 = 1-t-2s}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s$$

homogenous solution?

$\boxed{x_5 = 0}$ $\boxed{x_3 = t}$, $\boxed{x_4 = s}$

$x_1 - t + 2t + 2s + 0 = 0$

$\boxed{x_1 = -t-2s}$

$x_2 + t + 0 = 0$
$\boxed{x_2 = -t}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s$$

③ Determine the general solution of the following nonhomogenous system

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + \quad + x_3 &= -3 \\ 2x_1 + 4x_2 + \quad &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & -3 \\ 2 & 4 & 0 & 8 \end{array}\right] \xrightarrow[\substack{R_3 - R_2 \\ R_4 - 2R_2}]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & -6 & 6 \end{array}\right] \xrightarrow{\text{row swaps}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \ldots$$

$$\xrightarrow[-\frac{1}{6}R_3]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - 3R_3]{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Can be done here ! lets proceed with Gauss- Jordan

So, $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}$

Summary

Let $[A|b]$ be the augmented matrix for a consistent $m \times n$ non-homogenous system in which $\text{rank}(A) = r$.

- Reducing $[A|b]$ to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the general solution

$$\vec{x} = \vec{p} + x_{f_1}\vec{h}_1 + x_{f_2}\vec{h}_2 + \cdots + x_{f_{n-r}}\vec{h}_{n-r}$$

  As the free variables x_{f_i} range over all possible values, this general solution generates all possible solutions to the system.

- Column $\vec{p}$ is a particular solution of the non-homogenous system.

- The expression $x_{f_1}\vec{h}_1 + x_{f_2}\vec{h}_2 + \cdots + x_{f_{n-r}}\vec{h}_{n-r}$ is the general solution of the associated homogenous system

- Column $\vec{p}$ as well as the columns $\vec{h}_i$ are independent of the row echelon form to which $[A|b]$ is reduced

- The system possesses a unique solution if and only if any of the following is true:

  * $\text{rank}(A) = n$ = # of unknowns
  * There are no free variables
  * The associated homogenous system possesses only the trivial solution $(x_1=0, x_2=0, x_3=0, \ldots)$

Problems from exam

this is the system ☺ I tried saving space

① Among all the solutions that satisfy the system (#2 from non-homogenous systems)

find all those that satisfy the following two constraints:

$$(x_1 - x_2)^2 - 4x_5^2 = 0$$
$$x_3^2 - x_5^2 = 0$$

$x_1 = 1 - t - 2s$
$x_2 = 1 - t$
$x_3 = t$
$x_4 = s$
$x_5 = -1$

$x_3^2 - x_5^2 = t^2 - 1 = 0$
$(t+1)(t-1) = 0$

$\boxed{t = \pm 1}$

No s appears here

$(x_1 - x_2)^2 - 4x_5^2 = 0$
$(-2s)^2 - 4 = 0$
$4s^2 - 4 = 0$
$4(s^2 - 1) = 4(s+1)(s-1) = 0$

$\boxed{s = \pm 1}$

t cancels regardless of value

So... there are 4 solutions? $t=1, s=1$; $t=1, s=-1$; $t=-1, s=1$; $t=-1, s=-1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{cccc} \begin{bmatrix} -2 & 2 & 0 & 4 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \\ t=s=1 \quad t=1,\ s=-1 \quad t=-1,\ s=1 \quad t=s=-1 \end{array}$$

② Consider the following system:

$$\begin{aligned} 2x + 2y + 3z &= 0 \\ 4x + 8y + 12z &= -4 \\ 6x + 2y + \alpha z &= 4 \end{aligned}$$

(a) Determine all values of α for which the system is consistent

(b) Determine all values of α for which there is a unique solution, and compute the solution for these cases

(c) Determine all values of α for which there are infinitely many different solutions, and give the general solution for these cases.

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 4 & 8 & 12 & -4 \\ 6 & 2 & \alpha & 4 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 4 & 6 & -4 \\ 0 & -4 & \alpha-9 & 4 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 4 & 6 & -4 \\ 0 & 0 & \alpha-3 & 0 \end{array}\right]$$

(a) ~~[illegible]~~ Consistent for all values of α !

(b) $\boxed{\alpha \neq 3}$ $\Rightarrow z = 0$. Solution: $\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$

(c) $\boxed{\alpha = 3}$: $z = t \in \mathbb{R}$

$$\begin{aligned} 4y + 6t &= -4 \\ 4y &= -4 - 6t \\ y &= -1 - \tfrac{3}{2}t \end{aligned}$$

$$\begin{aligned} 2x + 2(-1 - \tfrac{3}{2}t) + 3t &= 0 \\ 2x - 2 - 3t + 3t &= 0 \\ 2x - 2 &= 0 \\ 2x &= 2 \quad \Rightarrow \quad x = 1 \end{aligned}$$

Solution: $\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix} t}$

③ Construct a nonhomogenous system of three equations in four unknowns that has

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-3\\0\\2\\1\end{bmatrix}$$

as its general solution.

$x_1 = 1 - 2x_2 - 3x_4 \rightsquigarrow x_1 + 2x_2 + 3x_4 = 1$

x_2 free ! (can start with a row of all 0's)

$x_3 = 1 + 2x_4 \rightsquigarrow x_3 - 2x_4 = 1$

x_4 free ! (start with another row of all 0's)

$$\left[\begin{array}{cccc|c}1 & -2 & 0 & -3 & 1\\0 & 0 & 1 & -2 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Now, perform some row operations to construct a system :)

Example:

$$\xrightarrow[R_4 + R_2]{R_3 + R_1} \left[\begin{array}{cccc|c}1 & -2 & 0 & -3 & 1\\0 & 0 & 1 & -2 & 1\\1 & -2 & 0 & -3 & 1\\0 & 0 & 1 & -2 & 1\end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{cccc|c}1 & -2 & 0 & -3 & 1\\0 & 0 & 1 & -2 & 1\\1 & -2 & 2 & -7 & 3\\0 & 0 & 1 & -2 & 1\end{array}\right] \xrightarrow{R_4 + R_3} \left[\begin{array}{cccc|c}1 & -2 & 0 & -3 & 1\\0 & 0 & 1 & -2 & 1\\1 & -2 & 2 & -7 & 3\\1 & -2 & 3 & -9 & 4\end{array}\right]$$

$$\boxed{\begin{aligned} x_1 - 2x_2 \qquad\quad - 3x_4 &= 1\\ x_3 - 2x_4 &= 1\\ x_1 - 2x_2 + 2x_3 - 7x_4 &= 3\\ x_1 - 2x_2 + 3x_3 - 9x_4 &= 4\end{aligned}}$$