7 The algebra of matrices

Exercises for beginners

1. In general a **matrix** is a rectangular array. If matrix has m horizontal rows and n vertical columns then is called an $\boldsymbol{m \times n}$ **matrix**.

(a) Let $A = \begin{bmatrix} 2 & -1 & 0 & 3 & 2 \\ 1 & -2 & 1 & -1 & 3 \\ 3 & 0 & 1 & 2 & -3 \end{bmatrix}$ be a 3×5 matrix. If we write $A = [a_{ij}]$ compute a_{11}, a_{31}, a_{13}, a_{35}. If we use the notation $A[i,j]$ compute $A[1,2]$, $A[2,1]$, $A[2,2]$, $A[3,4]$.

(b) Compute matrix B, if B is the 3×4 matrix defined by $B[i,j] = i - j$ (or in another notation $B = [i-j] \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$).

2. The **transpose** $A^\top$ of a matrix $A = [a_{ij}]$ in $\mathrm{Mat}_{m\times n}(\mathbb{R})$ is the matrix in $\mathrm{Mat}_{n\times m}(\mathbb{R})$ whose entry in the ith row and jth column is a_{ji}. That is, $A^\top[i,j] = A[j,i]$.

Compute $A^\top$ and $B^\top$, where A, B are matrices from problem 1 above.

3. (a) Consider the following three matrices $A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 2 \\ -3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & -2 \\ -5 & 0 & 2 \\ -2 & 4 & 1 \end{bmatrix}$. Compute $A + C$, $A + B$, $B + C$ and $B + B$.

(b) Consider the 1-row matrices $v_1 = \begin{bmatrix} -2 & 1 & 2 & 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 & 0 & 3 & -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$, and the 1-column matrices $v_4 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$, $v_5 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$, $v_6 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$. Compute $v_1 + v_2$, $v_1 + v3$, $v_2 + v_5$, $v_4 + v_6$, $v_5 + v_6$ and $v_6 + v_6$.

4. Matrices can be multiplied by real numbers, which in this context are often called **scalars**. Given A in $\mathrm{Mat}_{m\times n}(\mathbb{R})$ and c in $\mathbb{R}$, the matrix cA is the $m \times n$ matrix whose (i,j) entry is $c \cdot a_{ij}$ thus $(cA)[i,j] = c \cdot A[i,j]$. This multiplication is called **scalar multiplication** and cA is called the scalar product.

If $A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 4 \end{bmatrix}$, compute $2A$, $-7A$ and $-A$.

Standard exercises

1. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

2. Find x, y, z and w if $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$.

Product of 2 × 2 matrices

1. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ verify that $AB(C) = A(BC)$ and $A(B+C) = AB + AC$.

2. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = \boldsymbol{O}$, where I is an identity matrix of order 2.

3. If $A = \begin{bmatrix} 0 & -\operatorname{tg}\frac{\alpha}{2} \\ \operatorname{tg}\frac{\alpha}{2} & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

7.1 Matrix multiplication

Exercises for beginners

1. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ find AB or BA whichever exists.

2. Evaluate $A^2 - 3A + 9I$ if I is the unit matrix (the identity matrix) of order 3 and $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

3. Prove that $A^3 - 4A^2 - 3A + 11I = 0$, where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$. Find the products AB and BA. Show that $AB \neq BA$.

Standard exercises

1. Matrix A has x rows and $x+5$ columns. Matrix B has y rows and $11-y$ columns. Both AB and BA exist. Find x and y.

2. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ calculate the product AB.

3. Prove that the product of the matrices $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and $\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$ is a null matrix (a zero matrix), where θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Problems from exam

1. By Mathematical induction, prove that if $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

2. By Mathematical Induction, prove that if $A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$ then $A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$, where n is any positive integer.

3. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer.

7.2 Solutions

INTRODUCTION - EXERCISES FOR BEGINNERS

(a) The matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 3 & 2 \\ 1 & -2 & 1 & -1 & 3 \\ 3 & 0 & 1 & 2 & -3 \end{bmatrix}$$

is a 3×5 matrix. If we write $\mathbf{A} = [a_{ij}]$, then $a_{11} = 2$, $a_{31} = 3$, $a_{13} = 0$, $a_{35} = -3$, etc. If we use the notation $\mathbf{A}[i, j]$, then $\mathbf{A}[1, 2] = -1$, $\mathbf{A}[2, 1] = 1$, $\mathbf{A}[2, 2] = -2$, etc.

(b) If $\mathbf{B}$ is the 3×4 matrix defined by $\mathbf{B}[i, j] = i - j$, then $\mathbf{B}[1, 1] = 1 - 1 = 0$, $\mathbf{B}[1, 2] = 1 - 2 = -1$, etc., so

$$\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}. \qquad \blacksquare$$

For positive integers m and n, we write $\mathfrak{M}_{m,n}$ for the set of all $m \times n$ matrices. Two matrices $\mathbf{A}$ and $\mathbf{B}$ in $\mathfrak{M}_{m,n}$ are **equal** provided all their corresponding entries are equal; i.e., $\mathbf{A} = \mathbf{B}$ provided that $a_{ij} = b_{ij}$ for all i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. Matrices that have the same number of rows as columns are called **square matrices**. Thus $\mathbf{A}$ is a square matrix if $\mathbf{A}$ belongs to $\mathfrak{M}_{n,n}$ for some $n \in \mathbb{P}$. The **transpose** $\mathbf{A}^T$ of a matrix $\mathbf{A} = [a_{ij}]$ in $\mathfrak{M}_{m,n}$ is the matrix in $\mathfrak{M}_{n,m}$ whose entry in the ith row and jth column is a_{ji}. That is, $\mathbf{A}^T[i, j] = \mathbf{A}[j, i]$. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}.$$

(a) Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 2 \\ -3 & 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & -2 \\ -5 & 0 & 2 \\ -2 & 4 & 1 \end{bmatrix}.$$

Then we have

$$\mathbf{A} + \mathbf{C} = \begin{bmatrix} 5 & 5 & -2 \\ -6 & 3 & 4 \\ -5 & 5 & 3 \end{bmatrix},$$

but $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{C}$ are not defined. Of course, the sums $\mathbf{A}+\mathbf{A}$, $\mathbf{B}+\mathbf{B}$ and $\mathbf{C}+\mathbf{C}$ are also defined; for example,

$$\mathbf{B}+\mathbf{B}=\begin{bmatrix} 2 & 0 & 10 & 6 \\ 4 & 6 & -4 & 2 \\ 8 & -4 & 0 & 4 \end{bmatrix}.$$

(b) Consider the 1-row matrices

$$\mathbf{v}_1=\begin{bmatrix} -2 & 1 & 2 & 3 \end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix} 4 & 0 & 3 & -2 \end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}$$

and the 1-column matrices

$$\mathbf{v}_4=\begin{bmatrix} 1 \\ 2 \\ -3 \\ 2 \end{bmatrix},\quad \mathbf{v}_5=\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix},\quad \mathbf{v}_6=\begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}.$$

The only sums of distinct matrices here that are defined are

$$\mathbf{v}_1+\mathbf{v}_2=\begin{bmatrix} 2 & 1 & 5 & 1 \end{bmatrix} \quad\text{and}\quad \mathbf{v}_5+\mathbf{v}_6=\begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}.$$

■

(a) If

$$\mathbf{A}=\begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 4 \end{bmatrix},$$

then

$$2\mathbf{A}=\begin{bmatrix} 4 & 2 & -6 \\ -2 & 0 & 8 \end{bmatrix} \quad\text{and}\quad -7\mathbf{A}=\begin{bmatrix} -14 & -7 & 21 \\ 7 & 0 & -28 \end{bmatrix}.$$

(b) In general, the scalar product $(-1)\mathbf{A}$ is the negative $-\mathbf{A}$ of $\mathbf{A}$. ■

Introduction - Standard exercises

Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix}=\begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

Sol. Equating the corresponding elements on both sides, we get

$x+3=0,\quad 2y+x=-7,\quad z-1=3,\quad 4a-6=2a$

$\Rightarrow\quad x=-3,\quad y=-2,\quad z=4,\quad a=3$

Find x, y, z and w if $3\begin{bmatrix} x & y \\ z & w \end{bmatrix}=\begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix}+\begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$.

Sol. The given equation is

$$\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix}=\begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

Equating the corresponding elements on both sides,

$3x=x+4,\quad 3y=6+x+y,\quad 3z=-1+z+w,\quad 3w=2w+3$

$\Rightarrow\quad x=2,\quad 2y=6+x,\quad 2z=-1+w,\quad w=3$

$\Rightarrow\quad x=2,\quad y=4,\quad z=1,\quad w=3$

Hence $x=2$, $y=4$, $z=1$ and $w=3$.

If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ *and* $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ *verify that AB(C) = A(BC) and A(B + C) = AB + AC.*

Sol. $$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} \quad ...(1)$$

$$AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \quad ...(2)$$

$$AB(C) = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} = A(BC) \quad ...(3)$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$\therefore$ $$AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad ...(4)$$

Now $$A(B + C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2-3 & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} = AB + AC. \text{ Hence Proved.} \quad ...(5)$$

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, *show that* $A^2 - 5A + 7I = 0$, *where I is a matrix of order 2.*

Sol. $$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$\therefore$ $$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0. \text{ Hence shown.}$$

If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$, *show that* $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Sol. $$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$(I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \sin\alpha \,.\, \tan\frac{\alpha}{2} & -\sin\alpha + \cos\alpha \,.\, \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} \,.\, \cos\alpha + \sin\alpha & \sin\alpha \,.\, \tan\frac{\alpha}{2} + \cos\alpha \end{bmatrix}$$

Since we know that

$$\sin\alpha = \frac{2\tan\alpha/2}{1 + \tan^2\alpha/2} \text{ and } \cos\alpha = \frac{1 - \tan^2\alpha/2}{1 + \tan^2\alpha/2}$$

$$\therefore \quad \cos\alpha + \sin\alpha \,.\, \tan\alpha/2 = \frac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} + \frac{2\tan\alpha/2}{1+\tan^2\alpha/2}\,.\,\tan\alpha/2 = \frac{1+\tan^2\alpha/2}{1+\tan^2\alpha/2} = 1.$$

and
$$-\sin\alpha + \cos\alpha \,.\, \tan\alpha/2 = \frac{-2\tan\alpha/2}{1+\tan^2\alpha/2} + \frac{\tan\alpha/2\left(1-\tan^2\alpha/2\right)}{1+\tan^2\alpha/2}$$

$$= \frac{-2\tan\alpha/2 + \tan\alpha/2 - \tan^3\alpha/2}{1+\tan^2\alpha/2}$$

$$= \frac{-\tan\frac{\alpha}{2}\left[1+\tan^2\frac{\alpha}{2}\right]}{\left[1+\tan^2\frac{\alpha}{2}\right]} = -\tan\alpha/2.$$

Hence,

$$(I - A)\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1\end{bmatrix} = I + A. \text{ Hence shown.}$$

Matrix multiplication - Exercises for beginners

If $A = \begin{bmatrix}1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5\end{bmatrix}$ *and* $B = \begin{bmatrix}2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1\end{bmatrix}$ *find AB or BA whichever exists.*

Sol. It is observed that A is 3×4 matrix and B is 3×3 matrix. Hence AB does not exist.

$\Rightarrow$ BA exists.

$$\therefore \quad BA = \begin{bmatrix}2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5\end{bmatrix} \text{ or } BA = \begin{bmatrix}4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9\end{bmatrix}.$$

Evaluate $A^2 - 3A + 9I$ *if I is the unit matrix of order 3 and* $A = \begin{bmatrix}1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2\end{bmatrix}$.

Sol. $A^2 = A \times A$

$$= \begin{bmatrix}1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2\end{bmatrix} = \begin{bmatrix}-12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6\end{bmatrix}$$

$$\therefore \quad A^2 - 3A + 9I_3 = \begin{bmatrix}-12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6\end{bmatrix} - 3\begin{bmatrix}1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2\end{bmatrix} + 9\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6\end{bmatrix} - \begin{bmatrix}3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6\end{bmatrix} + \begin{bmatrix}9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9\end{bmatrix}$$

$$= \begin{bmatrix}-12-3+9 & -5-(-6)+0 & 11-9+0 \\ 11-6+0 & 4-9+9 & 1-(-3)+0 \\ -7-(-9)+0 & 11-3+0 & -6-6+9\end{bmatrix} = \begin{bmatrix}-6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3\end{bmatrix}.$$

Prove that $A^3 - 4A^2 - 3A + 11\,I = 0$*, where* $A = \begin{bmatrix}1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3\end{bmatrix}$.

Sol.
$$A^2 = \begin{bmatrix}1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3\end{bmatrix} = \begin{bmatrix}9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9\end{bmatrix}$$

$$A^3 = \begin{bmatrix}9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9\end{bmatrix}\begin{bmatrix}1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3\end{bmatrix} = \begin{bmatrix}28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34\end{bmatrix}$$

$$\therefore \quad A^3 - 4A^2 - 3A + 11\,I = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16-0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$*. Find the products AB and BA. Show that AB ≠ BA.*

Sol.

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

Orders of AB and BA are the same (3 × 3) but their corresponding elements are not equal.
Hence AB ≠ BA.

Matrix multiplication - Standard exercises

Matrix A has x rows and x + 5 columns. Matrix B has y rows and 11 – y columns. Both AB and BA exist. Find x and y.

Sol. As AB exists, $x + 5 = y$ ...(1)

If BA exists, number of columns in B should be equal to number of rows in A.

i.e., $11 - y = x$ ...(2)

Solving (1) and (2) for x and y ;

$$11 - (x + 5) = x \quad \text{or} \quad 6 - x = x$$

$$2x = 6 \quad \Rightarrow \quad x = 3$$

$$\therefore \quad y = x + 5 = 8$$

Hence $x = 3$, $y = 8$.

If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$, $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ *calculate the product AB.*

Sol. On adding the matrices A – B from A + B, we get

$$2A = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \quad i.e., \quad A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

On subtracting matrix A – B from A + B, we get

$$2B = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \quad i.e., \quad B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$$

Hence $$AB = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

Prove that the product of the matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix, where θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$.

Sol. $$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi + \sin\theta\sin\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$\because$ $\theta - \phi =$ an odd multiple of $\frac{\pi}{2}$

$\therefore$ $\cos(\theta-\phi) = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}$.

Matrix multiplication - Problems from exam

By Mathematical induction, prove that if $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

Sol. $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

When $n = 1$, $A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$

$\Rightarrow$ The result is true when $n = 1$.

Let us assume that the result is true for any positive integer k *i.e.,*

Let $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ ...(1)

Now $A^{k+1} = A^k . A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 3(1+2k)-4k & -4(1+2k)+4k \\ 3k+1-2k & -4k-(1-2k) \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$$

$\Rightarrow$ The result is true for $n = k + 1$. Hence by mathematical induction, the result is true for all positive integers n.

By Mathematical Induction, prove that if $A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$, *then* $A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$.

Sol. $$A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$$

When $n = 1$, $$A^1 = \begin{bmatrix} 1+10.1 & -25.1 \\ 4.1 & 1-10.1 \end{bmatrix} = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} = A$$

$\Rightarrow$ The result is true when $n = 1$

Let us assume that the result is true for any positive integer k *i.e.,*

Let $$A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

Now, $$A^{k+1} = A^k . A = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11(1+10k) - 100k & -25(1+10k) + 9(25k) \\ 44k + 4(1-10k) & -100k - 9(1-10k) \end{bmatrix}$$

$$= \begin{bmatrix} 11+10k & -25-25k \\ 4k+4 & -10k-9 \end{bmatrix} = \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

$$A^{k+1} = \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

$\Rightarrow$ The result is true for $n = k + 1$. Hence by mathematical induction, the result is true for all positive integers n.

If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, *show that* $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, *when n is a positive integer.*

Sol. $$A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

When $n = 1$, $$A^1 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = A$$

$\Rightarrow$ The result is true when $n = 1$.

Let us assume that the result is true for any positive integer k *i.e.,*

Let $$A^k = \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix}$$

Now, $$A^{k+1} = A^k . A$$

$$= \begin{bmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\alpha . \cos\alpha - \sin k\alpha \sin\alpha & \cos k\alpha \sin\alpha + \sin k\alpha . \cos\alpha \\ -\sin k\alpha \cos\alpha - \cos k\alpha \sin\alpha & -\sin k\alpha . \sin\alpha + \cos k\alpha . \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\alpha+\alpha) & \sin(k\alpha+\alpha) \\ -\sin(k\alpha+\alpha) & \cos(k\alpha+\alpha) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\alpha & \sin(k+1)\alpha \\ -\sin(k+1)\alpha & \cos(k+1)\alpha \end{bmatrix}$$

$\Rightarrow$ The result is true for $n = k + 1$.

Hence by mathematical induction, the result is true for all positive integers n.

Hence $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ is true.