Week 13

Identity matrix

The $n \times n$ matrix with 1's on the diagonal and 0's elsewhere

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

is called the <u>identity matrix</u> of order n. For every $m \times n$ matrix A,

$$A I_n = A \quad \text{and} \quad I_m A = A.$$

Matrix Inversion

For a given square matrix $A_{n \times n}$, the matrix $B_{n \times n}$ that satisfies the conditions

$$AB = I_n \quad \text{and} \quad BA = I_n$$

is called the <u>inverse</u> of A and is denoted by $B = A^{-1}$.

An invertible matrix is said to be <u>nonsingular</u>, and a square matrix with no inverse is called a <u>singular</u> matrix.

Reverse Order Law for Transpositions

For conformable matrices A and B,

$$(AB)^T = B^T A^T.$$

The case of conjugate transposition is similar. That is,

$$(AB)^* = B^* A^*$$

Existence of an Inverse

For an $n \times n$ matrix A, the following statements are equivalent:

- A^{-1} exists (A is nonsingular)
- $\text{rank}(A) = n$
- $A \longrightarrow I$ (via Gauss-Jordan)
- $A\vec{x} = \vec{0}$ implies that $\vec{x} = \vec{0}$

★ $\det(A) \neq 0$

Properties of Matrix Inversion

For nonsingular matrices A and B, the following properties hold:

- $(A^{-1})^{-1} = A$
- The product AB is also nonsingular
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$ and $(A^{-1})^* = (A^*)^{-1}$

Computing an Inverse

Gauss-Jordan elimination can be used to invert A by the reduction

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan}} [I \mid A^{-1}]$$

The only way for this reduction to fail is for a row of zeroes to emerge in the left-hand side of the augmented array, and this occurs if and only if A is a singular matrix.

① Possible pitfall ! For two $n \times n$ matrices, what is $(A+B)^2$?

$$(A+B)^2 = (A+B)(A+B) = \boxed{A^2 + AB + BA + B^2}$$

Note: not necessarily the case that $AB = BA$!

② Show that, for every matrix A, the products A^TA and AA^T are symmetric matrices.

$$A^TA_{i,j} = (A^T)_{i*} \cdot A_{*j} = a^T_{i1} \cdot a_{1j} + a^T_{i2} \cdot a_{2j} + \cdots + a^T_{in} \cdot a_{nj}$$

$$= a_{1i} \cdot a_{1j} + a_{2i} \cdot a_{2j} + \cdots + a_{ni} \cdot a_{nj} \quad ★$$

$$A^TA_{j,i} = (A^T)_{j*} \cdot A_{*i} = a^T_{j1} \cdot a_{1i} + a^T_{j2} \cdot a_{2i} + \cdots + a^T_{jn} \cdot a_{ni}$$

$$= a_{1j} \cdot a_{1i} + a_{2j} \cdot a_{2i} + \cdots + a_{nj} \cdot a_{ni} \quad ★★$$

$$★ = ★★ \Rightarrow A^TA_{i,j} = A^TA_{j,i}$$

:)

③ Trace of a product: The trace of a square matrix is the sum of its main diagonal entries.

Although matrix multiplication is not commutative, the trace function is one of the few cases where the order of the matrices can be changed without affecting the result.

For matrices $A \in Mat_{m \times n}(\mathbb{R})$ and $B \in Mat_{n \times m}(\mathbb{R})$, prove that

$$\text{trace}(AB) = \text{trace}(BA).$$

$$\text{trace}(AB) = \sum_{i=1}^{m} (AB)_{ii}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ji}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij}$$

$$= \sum_{j=1}^{n} (BA)_{jj} \qquad = \text{trace}(BA)$$

④ Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\delta = ad - bc \neq 0$,

then $A^{-1} = \frac{1}{\delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Recall that $AA^{-1} = I$ and $A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{\delta} & -b/\delta \\ -c/\delta & a/\delta \end{bmatrix} = \begin{bmatrix} \frac{1}{\delta}(ad - bc) & \frac{1}{\delta}(-ab + ba) \\ \frac{1}{\delta}(cd - dc) & \frac{1}{\delta}(-cb + da) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{as } \delta = ad - bc \neq 0$$

Also check $A^{-1}A$!

⑤ Let A, B, and C denote three given matrices

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}.$$

Solve (for X) the following matrix equations:

(a) $3X + A = B$
(b) $AX = B$
(c) $(2-X)B = XA^T + 3I$
(d) $XA = B$
(e) $AXB = C$
(f) $X^{-1}A = B^{-1}$
(g) $AX + I = X - 2I$
(h) $A^{-1}X = X - I$
(i) $(A+3I)(X-I) = B$
(j) $B^{-1}XA = (3B+2I)^{-1}$
(k) $(AXB)^{-1} = B^{-1}(X^{-1}+B)$

e.g. (h) ....

$$A^{-1}X - X = X - I - X$$

$$(A^{-1} - I)X = -I$$

$$(A^{-1}-I)^{-1}(A^{-1}-I)X = (A^{-1}-I)^{-1}(-I)$$

$$X = (A^{-1}-I)^{-1}(-I)$$

Recall that matrix multiplication is <u>not</u> commutative !

Multiplying on the left vs. multiplying on the right matters !

<u>Strategy:</u> Solve algebraically for X, then compute <u>only once</u>

(e) $AXB = C$

$$A^{-1}AXBB^{-1} = A^{-1}CB^{-1}$$

$$\boxed{X = A^{-1}CB^{-1}}$$:)

Compute A^{-1}, B^{-1}, and the product $A^{-1}CB^{-1}$

A^{-1}: $\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2/3} \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|cc} 2 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & 1/3 \end{array}\right] \xrightarrow{R_1/2}$

$$\boxed{A^{-1} = \begin{bmatrix} 1/2 & 1/6 \\ 0 & 1/3 \end{bmatrix}}, \qquad \boxed{B^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 3/5 \end{bmatrix}}$$

B^{-1}: $\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|cc} 5 & 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{array}\right]$

$\xrightarrow{R_1/5} \left[\begin{array}{cc|cc} 1 & 0 & 2/5 & 1/5 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2/5 & 1/5 \\ 0 & 1 & 1/5 & 3/5 \end{array}\right]$

Compute: $X = A^{-1}CB^{-1} = \begin{bmatrix} 1/2 & 1/6 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 3/5 \end{bmatrix}$

⑥ For all matrices $A \in Mat_{n \times k}(\mathbb{R})$ and $B \in Mat_{k \times n}(\mathbb{R})$,

Show that the block matrix $L = \begin{bmatrix} I - BA & B \\ 2A - ABA & AB - I \end{bmatrix}$ has the property $L^2 = I$.

* Matrices with this property are said to be <u>involutary</u> and they occur in the science of cryptography.

$$L^2 = \begin{bmatrix} I - BA & B \\ 2A - ABA & AB - I \end{bmatrix} \begin{bmatrix} I - BA & B \\ 2A - ABA & AB - I \end{bmatrix}$$

$$= \begin{bmatrix} (I - BA)^2 + B(2A - ABA) & \underline{(I - BA)B + B(AB - I)} \\ (2A - ABA)(I - BA) + (AB - I)(2A - ABA) & \underline{(2A - ABA)B + (AB - I)^2} \end{bmatrix}$$

$$= \begin{bmatrix} I - 2BA + BABA + 2BA - BABA & \cdots \\ 2A - 2ABA - ABA + ABABA + 2ABA - ABABA - 2A + ABA & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} I & \cdots \\ 0 & \cdots \end{bmatrix} = \begin{bmatrix} I & (I - BA)B + B(AB - I) \\ 0 & (2A - ABA)B + (AB - I)^2 \end{bmatrix}$$

$$= \begin{bmatrix} I & B - BAB + BAB - B \\ 0 & 2AB - ABAB + ABAB - 2AB + I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

So $L^2 = I$:)

⑦ If possible, find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}$$

Problems from exam

① Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by Gauss-Jordan.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - 3\cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + 5\cdot R_2]{R_1 - 2\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3]{R_2 - 2\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & 5/2 & -3/2 & 1/2 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}}$$

② If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ find A^{-1} and show that $A^3 = A^{-1}$.

$$A^2 = A \cdot A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

★ as $A^3 \cdot A = A^4$ and $A \cdot A^3 = A^4$ it is sufficient to end here

$$A^4 = A^3 \cdot A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + 2R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3/_{-4}} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{array}\right] \xrightarrow[R_1 - 3R_3]{R_2 + 6R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/4 & -3/4 & -3/4 \\ 0 & 2 & 0 & 1/2 & 5/2 & 3/2 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/4 & -3/4 & -3/4 \\ 0 & 1 & 0 & 1/4 & 5/4 & 3/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & -3/2 \\ 0 & 1 & 0 & 1/4 & 5/4 & 3/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/4 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & -2 & -3/2 \\ 1/4 & 5/4 & 3/4 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}}$$

⑤ Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by elementary row operations.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & -5 & -15 & 0 & 1 & -4 \\ 0 & -1 & -4 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & -1 & -4 & 1 & 0 & -2 \\ 0 & -5 & -15 & 0 & 1 & -4 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 4 & -1 & 0 & 2 \\ 0 & -5 & -15 & 0 & 1 & -4 \end{array}\right] \xrightarrow{R_3 + 5R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 4 & -1 & 0 & 2 \\ 0 & 0 & 5 & -5 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5} \cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 4 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & \frac{1}{5} & \frac{6}{5} \end{array}\right] \xrightarrow[R_1 - 4R_3]{R_2 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -\frac{4}{5} & -\frac{19}{5} \\ 0 & 1 & 0 & 3 & -\frac{4}{5} & -\frac{14}{5} \\ 0 & 0 & 1 & -1 & \frac{1}{5} & \frac{6}{5} \end{array}\right]$$

$$\xrightarrow{R_1 - 2 \cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{4}{5} & \frac{9}{5} \\ 0 & 1 & 0 & 3 & -\frac{4}{5} & -\frac{14}{5} \\ 0 & 0 & 1 & -1 & \frac{1}{5} & \frac{6}{5} \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix}}$$

⑥ Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular with each diagonal element equal to 1 and U is an upper triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, \quad U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & p \end{bmatrix} = \begin{bmatrix} d & e & f \\ ad & ae+g & af+h \\ bd & be+cg & bf+ch+p \end{bmatrix}$$

Comparing RHS with LHS, we get the following:

$$\underbrace{\begin{matrix} 5 = d \\ 7 = ad \\ 3 = bd \end{matrix}} \qquad \underbrace{\begin{matrix} -2 = e \\ 1 = ae+g \\ 7 = be+cg \end{matrix}} \qquad \underbrace{\begin{matrix} 1 = f \\ -5 = af+h \\ 4 = bf+ch+p \end{matrix}}$$

$\boxed{d = 5}$ $\boxed{e = -2}$ $\boxed{f = 1}$

$\boxed{a = 7/5}$ $\quad g = 1 - ae = \boxed{19/5}$

$\boxed{b = 3/5}$ $\quad c = \frac{1}{g}(7 - be) = \boxed{41/19}$

$h = -5 - af = \boxed{-32/5}$

$p = 4 - bf - ch = \boxed{327/19}$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 7/5 & 1 & 0 \\ 3/5 & 41/19 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 19/5 & -32/5 \\ 0 & 0 & 327/19 \end{bmatrix}}$$

(7) For which values of x is the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ a singular matrix?

Call the given matrix A. We want to find values of x such that $\det(A) = 0$!

Recall: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$$\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} \xrightarrow{R_2+R_3} \begin{vmatrix} 3-x & 2 & 2 \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix}$$

$$\det(A) = (-x)\cdot\begin{vmatrix} 3-x & 2 \\ -2 & -1-x \end{vmatrix} + x\cdot\begin{vmatrix} 3-x & 2 \\ -2 & -4 \end{vmatrix}$$

$$= (-x)\Big((3-x)(-1-x)+4\Big) + x\Big((3-x)\cdot 4+4\Big)$$

$$= x\Big(4(3-x) - (-1-x)(3-x)\Big)$$

$$= x(3-x)(4+1+x) = x(3-x)(5+x)$$

Letting $\det(A) = 0$ yields $x(3-x)(5+x) = 0$

So A is singular for $\boxed{x=0, \quad x=3, \quad x=-5}$