A1) Let A be any open set, and $a \in A$.
Then a has an open ~~set~~ interval around it contained in A
So we get $(a-\delta, a+\delta) \subseteq A$ for $\delta > 0$.
So there exist rationals s.t $a-\delta < x < y < a+\delta$
$\Rightarrow (x,y) \subseteq (a-\delta, a+\delta) \subseteq A$
So every open set contains an open interval with rational endpoints
Because $\mathbb{Q}$ is contable $x,y \in \mathbb{Q}$, so it follows that $\mathbb{R}$ is covered by countably many open bals, because we have basis for standard topology on $\mathbb{R}$

B_n be an open cover in $\mathbb{R}^2$, by considering the union of all rationals.
So $\{B_n\} = \{N_p(n) : p, n \in \mathbb{Q}\}$
Now let m be an ~~arbitara~~ arbitrary point s.t $m \in \mathbb{R}^2$ and let M be an arbitrary open set s.t $m \in M$.
Since M is open, we have neighbourhoods around m with radius ε s.t $K_\varepsilon(m) \Rightarrow K_\varepsilon(m) \subset M$
Next we take a point $q \in \mathbb{Q}^2$ s.t $d(q,m) < \frac{\varepsilon}{100}$
wich exists since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$. So now we take a point z in $\mathbb{Q}$ s.t. $\varepsilon/100 < z < \varepsilon/10$.
So $M_z(k) \in \{B_n\}$
So we can see that $m \in M_z(k) \subset M_\varepsilon(m) \subset$.

So $\{B_n\}$ is contable since it is union of a countable collection of countable sets.
So it follows that $\mathbb{R}^2$ is covere by countably many open balls ■

B i) $\lim\limits_{x\to 1} \frac{(x-2)}{(x-1)^2}$

a) $f(x) = x-2 \to \lim\limits_{x\to 1} f(x) \quad 1-2=-1<0$ so $L_f<0$ (negative)

b) $g(x) = (x-1)^2 \to \lim\limits_{x\to 1} (x-1)^2 =$

$$\begin{cases} L_g \to 0 & \text{for } x=1 \\ L_g \to 0 & \text{for } x\to 1^+ \\ L_g \to 0 & \text{for } x \to 1^- \end{cases} \Rightarrow L_g \pm\infty$$

So form all of this we can see that $\lim\limits_{x\to 1} \frac{(x-2)}{(x-1)^2}$ doesn't converge and Because L_f is negative and L_g is $\pm\infty$ we get $\lim\limits_{x\to 1} = -\infty$

ii) $\lim\limits_{x\to 1} \frac{(x^2-1)}{x-1}$

$\lim\limits_{x\to 1} \frac{(x^2-1)}{x-1} = \lim\limits_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim\limits_{x\to 1} (x+1) = 1+1 = 2$ (By AoL)

$L=2$ so it converges

~~iii)~~

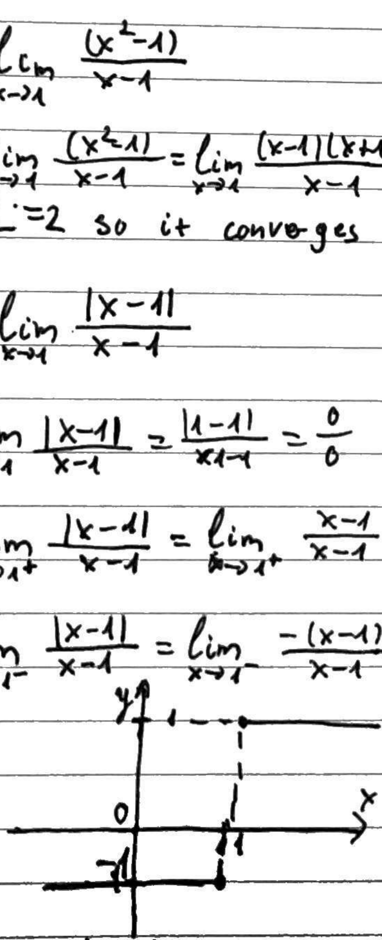
iii) $\lim\limits_{x\to 1} \frac{|x-1|}{x-1}$

$\lim\limits_{x\to 1} \frac{|x-1|}{x-1} = \frac{|1-1|}{1-1} = \frac{0}{0}$ (We have to use another approach)

$\lim\limits_{x\to 1^+} \frac{|x-1|}{x-1} = \lim\limits_{x\to 1^+} \frac{x-1}{x-1} = \lim\limits_{x\to 1^+} 1 = 1$

$\lim\limits_{x\to 1^-} \frac{|x-1|}{x-1} = \lim\limits_{x\to 1^-} \frac{-(x-1)}{x-1} = \lim\limits_{x\to 1^-} -1 = -1$ So

$$\begin{cases} x\to 1 \Rightarrow x-1 \to 0 \\ x\to 1^+ \Rightarrow x-1 \to 0^+ \\ x \to 1^- \Rightarrow x-1\to 0^- \end{cases}$$



So we can see that $\lim\limits_{x\to 1} \frac{(x^2-1)}{x-1}$ doesn't converge

$\mathcal{C}$

$f(x) = x^3+1$

$\lim_{x \to c} f(x) = L_f$

or $\lim_{x\to c} f(x) = c^3+1 \Rightarrow f(c) = c^3+1$

Let $\varepsilon > 0$, and we use the def. of limit, ~~so we get~~ (requires to produce δ s.t. $\delta > 0$).

$0 < |x-c| < \delta$

so $|f(x) - L_f| < \varepsilon$ or $|f(x) - f(c)| < \varepsilon$

$|x^3+1-(c^3+1)| = |x^3-c^3| = |x-c||x^2+cx+c^2| \leq (|x|^2+|cx|+|c|^2)|x-c|$

We take $\delta \leq 1$ then we have

$0 < |x-c| < 1$

$|x| = |x-c+c| \overset{\text{by } \Delta \text{ ineq}}{\leq} |x-c|+|c| < 1+|c|$

We have

$|x-c|(|x|^2+|cx|+|c|^2) \leq |x-c|((1+|c|)^2+|c|(1+|c|)+|c|^2)$

$= |x-c|(1+2|c|+|c|^2+|c|+|c|^2+|c|^2)$

$= |x-c|(3|c|^2+3|c|+1)$

So we have that $|x-c|(3|c|^2+3|c|+1) < \varepsilon$

so $|x-c| < \dfrac{\varepsilon}{3|c|^2+3|c|+1}$, we choose

$$\delta = \frac{\varepsilon}{3|c|^2+3|c|+1}$$

$|x^3-c^3| < |x-c|(3|c|^2+3|c|+1)$

$|x^3-c^3| < \dfrac{\varepsilon}{3|c|^2+3|c|+1} \cdot (3|c|^2+3|c|+1$

$|x^3-c^3| < \varepsilon$

We get that x^3+1 is continuous and has a limit $f(c)$ at every real number c.