12 Systems of linear equations - problems from exam

1. Solve the system of equations

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10. \end{aligned}$$

2. Show that the equations

$$\begin{aligned} 3x + 3y + 2z &= 1 \\ x + 2y + 0 \cdot z &= 4 \\ 0 \cdot x + 10y + 3z &= -2 \\ 2x - 3y - z &= 5 \end{aligned}$$

are consistent and hence obtain the solutions for x, y and z.

3. For what values of λ and μ do the system of equations

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

have

(i) no solution

(ii) unique solution

(iii) more than one solution?

4. Show that the system of equations:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2 \\ -x_1 + 2x_2 &= \lambda x_3 \end{aligned}$$

can possess a non-trivial solution only if $\lambda = 1$ or -3. Obtain the general solution in each case.

5. Discuss consistency of the system of equations:

$$\begin{aligned} 2x - 3y + 6z - 5w &= 3 \\ y - 4z + w &= 1 \\ 4x - 5y + 8z - 9w &= \lambda \end{aligned}$$

for various values of λ. If consistent, find the solution.

6. Test for consistency the system of linear equations:

$$\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}$$

where a, b, c are constants.

7. Find the values of λ and μ so that the equations

$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= \mu \end{aligned}$$

have

(i) no solution

(ii) a unique solution and

(iii) an infinite number of solutions.

8. Find the values of λ for which the equations

$$\begin{aligned} (\lambda - 1)x + (3\lambda + 1)y + 2\lambda z &= 0 \\ (\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z &= 0 \\ 2x + (3\lambda + 1)y + 3(\lambda - 1)z &= 0 \end{aligned}$$

are consistent, and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greater of these values.

9. For what values of k the equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= k \\ 4x + y + 10z &= k^2 \end{aligned}$$

have a solution and solve them completely in each case.

10. Find the values of a and b for which the equations:

$$\begin{aligned} x + ay + z &= 3 \\ x + 2y + 2z &= b \\ x + 5y + 3z &= 9 \end{aligned}$$

are consistent. When will these equations have a unique solution?

11. Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

12. Show that the equations

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

do not have a solution unless $a + c = 2b$.

12.1 Solutions

Solve the system of equations

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10.$$

Sol. In matrix notation, the given system of equations can be written as AX = B where

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 6 & -6 & 6 & 12 & : & 36 \\ 4 & 3 & 3 & -3 & : & -1 \\ 2 & 2 & -1 & 1 & : & 10 \end{bmatrix}$$

Operating $R_2 - 3R_1$, $R_3 - 2R_1$, $R_4 - R_1$,

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 0 & -9 & 0 & 9 & : & 18 \\ 0 & 1 & -1 & -5 & : & -13 \\ 0 & 1 & -3 & 0 & : & 4 \end{bmatrix}$$

Operating $-\frac{1}{9} R_2$,

$$\sim \begin{bmatrix} 2 & 1 & 2 & 1 & : & 6 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 1 & -1 & -5 & : & -13 \\ 0 & 1 & -3 & 0 & : & 4 \end{bmatrix}$$

Operating $R_1 - R_2$, $R_3 - R_2$, $R_4 - R_2$,

$$\sim \begin{bmatrix} 2 & 0 & 2 & 2 & : & 8 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & -3 & 1 & : & 6 \end{bmatrix}$$

Operating $R_4 - 3R_3$, $\frac{1}{2} R_1$,

$$\begin{bmatrix} 1 & 0 & 1 & 1 & : & 4 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & 0 & +13 & : & 39 \end{bmatrix}$$

Operating $R_1 + R_3$, $\frac{1}{13}R_4$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & : & 7 \\ 0 & 1 & 0 & -1 & : & -2 \\ 0 & 0 & -1 & -4 & : & -11 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Operating $R_1 + 3R_4$, $R_2 + R_4$, $R_3 + 4R_4$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & -1 & 0 & : & 1 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Operating $(-1)\, R_3$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & 0 & : & 1 \\ 0 & 0 & 1 & 0 & : & -1 \\ 0 & 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Hence $x_1 = 2$, $x_2 = 1$, $x_3 = -1$, $x_4 = 3$.

Using matrix method, show that the equations

$$3x + 3y + 2z = 1$$
$$x + 2y + 0.z = 4$$
$$0.x + 10y + 3z = -2$$
$$2x - 3y - z = 5$$

are consistent and hence obtain the solutions for x, y and z.

Sol. In matrix notation, the given system of equations can be written as AX = B, where

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 3 & 3 & 2 & : & 1 \\ 1 & 2 & 0 & : & 4 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ Operating } R_{12}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 3 & 3 & 2 & : & 1 \\ 0 & 10 & 3 & : & -2 \\ 2 & -3 & -1 & : & 5 \end{bmatrix} \text{ Operating } R_2 - 3R_1,\ R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 10 & 3 & : & -2 \\ 0 & -7 & -1 & : & -3 \end{bmatrix} \text{ Operating } R_3 + 3R_2,\ R_4 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & : & 4 \\ 0 & -3 & 2 & : & -11 \\ 0 & 1 & 9 & : & -35 \\ 0 & -1 & -5 & : & 19 \end{bmatrix} \text{ Operating } R_1 - 2R_3,\ R_2 + 3R_3,\ R_4 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -18 & 74 \\ 0 & 0 & 29 & -116 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 4 & -16 \end{bmatrix} \text{ Operating } R_{23}, \frac{1}{4} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & -18 & 74 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 29 & -116 \\ 0 & 0 & 1 & -4 \end{bmatrix} \text{ Operating } R_1 + 18R_4, R_2 - 9R_4, R_3 - 29R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & : & -4 \end{bmatrix} \text{ Operating } R_{34}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -4 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = \rho(A : B) = 3$ = number of unknowns.

$\Rightarrow$ The given system of equations is consistent and the unique solution is $x = 2, y = 1, z = -4$.

For what values of λ and μ do the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (*i*) *no solution*

(*ii*) *unique solution*

(*iii*) *more than one solution ?*

Sol. In matrix notation, the given system of equations can be written as AX = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix} \text{ Operating } R_2 - R_1, R_3 - R_1,$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix} \text{ Operating } R_1 - R_2, R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & : & 2 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

Case I. If $\lambda = 3, \mu \neq 10$

$\rho(A) = 2, \rho(A : B) = 3$

$\because$ $\rho(A) \neq \rho(A : B)$

$\therefore$ The system has no solution.

Case II. If $\lambda \neq 3$, μ may have any value.

$\rho(A) = \rho(A : B) = 3$ = a number of unknowns.

$\therefore$ The system has unique solution.

Case III. If $\lambda = 3, \mu = 10$

$\rho(A) = \rho(A : B) = 2 <$ number of unknowns.

$\therefore$ The system has an infinite number of solutions.

Show that the system of equations :

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

can possess a non-trivial solution only if $\lambda = 1$ or -3. Obtain the general solution in each case.

Sol. The given system of equations is

$$(2-\lambda)\, x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - (3+\lambda)\, x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - \lambda x_3 = 0$$

In matrix notation, it can be written as

$$AX = O$$

where $$A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

For non-trivial solution, $|A| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -(3+\lambda) & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)\,[\lambda(3+\lambda) - 4] + 2(-2\lambda + 2) + [4 - (3+\lambda)] = 0$$

$$\Rightarrow \quad \lambda^3 + \lambda^2 - 5\lambda + 3 = 0$$

$$(\lambda - 1)^2 (\lambda + 3) = 0$$

$$\therefore \quad \lambda = 1 \quad \text{or} \quad -3.$$

When $\lambda = 1$, the equations become

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 4x_2 + 2x_3 = 0$$
$$-x_1 + 2x_2 - x_3 = 0$$

which are identical.

The given system is equivalent to a single equation

$$x_1 - 2x_2 + x_3 = 0$$

Taking $x_2 = t$, $x_3 = s$, we get $x_1 = 2t - s$

$$\therefore \quad x_1 = 2t - s,\ x_2 = t,\ x_3 = s$$

which give an infinite number of non-trivial solutions, t and s being the parameters.

When $\lambda = -3$, the equations become

$$5x_1 - 2x_2 + x_3 = 0$$
$$2x_1 + 2x_3 = 0$$
$$-x_1 + 2x_2 + 3x_3 = 0$$

Solving the first two, we have

$$\frac{x_1}{-4} = \frac{x_2}{2-10} = \frac{x_3}{4} \quad \text{or} \quad x_1 = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\therefore \quad x_1 = t,\ x_2 = 2t,\ x_3 = -t$$

which give an infinite number of non-trivial solutions, t being the parameter.

Discuss consistency of the system of equations:

$$2x - 3y + 6z - 5w = 3$$
$$y - 4z + w = 1$$
$$4x - 5y + 8z - 9w = \lambda$$

for various values of λ. If consistent, find the solution.

Sol. Here

$$A = \begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 1 \\ \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The augmented matrix [A : B]

$$= \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 4 & -5 & 8 & -9 & : & \lambda \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_1$

$$= \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 1 & -4 & 1 & : & \lambda - 6 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 0 & 0 & 0 & : & \lambda - 7 \end{bmatrix}$$

(*i*) There is no solution if

$$\rho(A) \neq \rho(A : B)$$

i.e., if $\lambda - 7 \neq 0$ or $\lambda \neq 7$

$$\rho(A) = 2, \rho(A : B) = 3$$

(*ii*) There are infinite number of solutions.

If $\rho(A) = \rho(A : B) = 2$

i.e., $\lambda - 7 = 0$ or $\lambda = 7$

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$2x - 3y + 6z - 5w = 3 \qquad ...(1)$$
$$y - 4z + w = 1 \qquad ...(2)$$

Let $w = k_1, z = k_2$

From (2), $y - 4k_2 + k_1 = 1, y = 1 + 4k_2 - k_1$

From (1), $2x - 3 - 12k_2 + 3k_1 + 6k_2 - 5k_1 = 3$

or

$$2x = 6 + 6k_2 + 2k_1$$
$$x = 3 + 3k_2 + k_1$$
$$y = 1 + 4k_2 - k_1$$
$$z = k_2,$$
$$w = k_1$$

Test for consistency the system of linear equations:

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c \text{ where } a, b, c \text{ are constants.}$$

Sol. We have

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

operate $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The ranks of co-efficient matrix and augmented matrix for the last set of equations, are both 2. Hence, the equations are consistent. Also the given system is equivalent to

$$-2x + y + z = a \quad \text{...(1)}$$
$$3y = b \quad \text{...(2)}$$
$$3z = c \quad \text{...(3)}$$

$$\Rightarrow \qquad y = \frac{b}{3}, z = \frac{c}{3}$$

Put the values of y and z equation (1), we get

$$-2x + \frac{b}{3} + \frac{c}{3} = a$$

or
$$x = \frac{b + c - 3a}{6}$$

Find the values of λ and μ so that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$, have

(*i*) *no solution* (*ii*) *a unique solution and*

(*iii*) *an infinite number of solutions.*

Sol. We have

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The system admits of unique solution if, and only if, the co-efficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have no solution for those values of μ for which the matrices

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But co-efficient matrix is of rank 2 and augmented matrix is not of rank 2 unless $\mu = 9$.

Thus if $\lambda = 5$ and $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

Find the values of λ for which the equations

$$(\lambda - 1)\, x + (3\lambda + 1)\, y + 2\lambda z = 0$$
$$(\lambda - 1)\, x + (4\lambda - 2) y + (\lambda + 3)\, z = 0$$
$$2x + (3\lambda + 1)\, y + 3(\lambda - 1)\, z = 0$$

are consistent, and find the ratios of x : y : z when λ has the smallest of these values. What happens when λ has the greater of these values.

Sol. The given equations will be consistent

if $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$$

Operating $R_2 - R_1$

or, if $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0$$

Operating $C_3 + C_2$

or, if $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0$$

Expanding by R_2

or, if $$(\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda-1) \end{vmatrix} = 0$$

or, if $$2(\lambda - 3)\, [(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$$

or, if $$6\lambda\, (\lambda - 3)^2 = 0$$

or, if $$\lambda = 0 \quad \text{or} \quad 3.$$

(*a*) When $\lambda = 0$, the equations become

$$-x + y = 0 \qquad ...(1)$$
$$-x - 2y + 3z = 0 \qquad ...(2)$$
$$2x + y - 3z = 0 \qquad ...(3)$$

Solving (2) and (3), we get

$$\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4}.$$

Hence $x = y = z$

(*b*) When $\lambda = 3$, the equations become identical.

For what values of k the equations

$$x + y + z = 1$$
$$2x + y + 4z = k$$
$$4x + y + 10z = k^2$$

have a solution and solve them completely in each case.

Sol. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & k \\ 4 & 1 & 10 & : & k^2 \end{bmatrix} \text{ Operating } R_2 - 2R_1,\ R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -3 & 6 & : & k^2-4 \end{bmatrix} \text{ Operating } R_3/3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & -1 & 2 & : & \dfrac{k^2-4}{3} \end{bmatrix} \text{ Operating } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & k-2 \\ 0 & 0 & 0 & : & \dfrac{k^2-4}{3} - k + 2 \end{bmatrix}$$

The system of given equations will be consistent only if rank of augmented matrix is equal to the rank of co-efficient matrix. This means last element of row 3 = 0.

This can be possible only when

$$\frac{k^2-4}{3} - k + 2 = 0$$

or $$k^2 - 3k + 2 = 0$$

$$(k-2)(k-1) = 0$$

$\Rightarrow$ $$k = 1, 2.$$

When k = 1, We have $x + y + z = 1$ and $-y + 2z = -1$

or $x = -3\lambda,\ y = 1 + 2\lambda,\ z = \lambda$ is the solution.

When k = 2, $x = 1 - 3\lambda,\ y = 2\lambda,\ z = \lambda$ is the solution, where λ is arbitrary.

Find the values of a and b for which the equations :

$$x + ay + z = 3$$
$$x + 2y + 2z = b$$
$$x + 5y + 3z = 9$$

are consistent. When will these equations have a unique solution ?

Sol. The augmented matrix is

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 1 & 2 & 2 & : & b \\ 1 & 5 & 3 & : & 9 \end{bmatrix}$$

Operating $R_2 - R_1$, $R_3 - R_1$,

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 0 & 2-a & 1 & : & b-3 \\ 0 & 5-a & 2 & : & 6 \end{bmatrix}$$

Operating $R_3 - 2R_2$,

$$[A : B] \sim \begin{bmatrix} 1 & a & 1 & : & 3 \\ 0 & 2-a & 1 & : & b-3 \\ 0 & 1+a & 0 & : & 12-2b \end{bmatrix}$$

Here the co-efficient matrix is

$$A \sim \begin{bmatrix} 1 & a & 1 \\ 0 & 2-a & 1 \\ 0 & 1+a & 0 \end{bmatrix}$$

The given system of equations will be inconsistent if rank of A and rank of augmented matrix (A : B) are not equal.

$\Rightarrow$ When $1 + a = 0$ but $12 - 2b \neq 0$

or if $a = -1, b \neq 6$

Equations will be consistent if

$a = -1$ and $b = 6$

In this case, $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

which gives $x - y + z = 3$

and $3y + z = 3$ or $y = 1 - \dfrac{z}{3}$

$x = 4y$ or $x = 4 - \dfrac{4z}{3}$

i.e., $x = 4 - \dfrac{4\lambda}{3}, y = 1 - \dfrac{\lambda}{3}, z = \lambda$

where λ is a parameter. Thus, it has an infinite number of solutions.

When $a \neq -1$ and b has any value, the augmented matrix and co-efficient matrix will have same rank showing the system of equations is consistent but it has a unique solution since rank = 3 = number of unknowns (3).

Show that if $\lambda \neq -5$, the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

Sol. The augmented matrix is

$$\begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 1 & 2 & -3 & : & -2 \\ 6 & 5 & \lambda & : & -3 \end{bmatrix} \text{ Operating } R_{12}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & -2 \\ 3 & -1 & 4 & : & 3 \\ 6 & 5 & \lambda & : & -3 \end{bmatrix} \qquad \text{Operating } R_2 - 3R_1,\ R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & -2 \\ 0 & -7 & 13 & : & 9 \\ 0 & -7 & \lambda + 18 & : & 9 \end{bmatrix} \qquad \text{Operating } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & -2 \\ 0 & -7 & 13 & : & 9 \\ 0 & 0 & \lambda + 5 & : & 0 \end{bmatrix}$$

Case I. If $\lambda + 5 = 0$ or $\lambda = -5$

Rank (A) = 2

Rank (A : B) = 2

Thus rank (A) = Rank (A : B) = 2 < number of unknowns.

$\therefore$ The system is consistent and has an infinite number of solutions.

The equations become :

$$x + 2y - 3z = -2$$
$$-7y + 13z = 9$$

$$\Rightarrow \qquad y = \frac{13z - 9}{7},\ x = \frac{4 - 5z}{7}$$

or $$x = \frac{4 - 5k}{7},\quad y = \frac{13\,k - 9}{7},\ z = k, \text{ where } k \text{ is arbitrary.}$$

$$\lambda = -5.$$

Case II. When $\lambda \neq -5$

Rank (A) = 3

Rank (B) = 3 = number of unknowns.

$\therefore$ The system has a unique solution.

This gives $x + 2y - 3z = -2$

$-7\,y + 13z = 9$

Put $z = 0$

$-7y = 9$

$$y = -\frac{9}{7},\ x = \frac{4}{7}.$$

Hence $\lambda \neq -5$, $x = \frac{4}{7}, y = -\frac{9}{7}, z = 0$ is the solution of given system of equations.

Show that the equations

$$3x + 4y + 5z = a$$
$$4x + 5y + 6z = b$$
$$5x + 6y + 7z = c$$

do not have a solution unless $a + c = 2b$.

Sol. The augmented matrix is

$$\begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix} \text{ Operating } R_2 - R_1, R_3 - R_2$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & b-a \\ 1 & 1 & 1 & : & c-b \end{bmatrix} \text{ Operating } R_3 - R_2$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 1 & 1 & 1 & : & b-a \\ 0 & 0 & 0 & : & a+c-2b \end{bmatrix}$$

Here rank of A : 2

Rank of (A : B) *i.e.*, augmented matrix = 3

i.e., Rank of A and (A : B) are not equal.

The system has no solution in such situation.

However if $a + c - 2b = 0$ or $a + c = 2b$

$$\rho(A : B) = \rho(A) = 2 < \text{number of unknowns.}$$

The system will have an infinite number of solutions.