Week 9

Row Echelon Form: An $m \times n$ matrix $E = [e_{ij}]$ with rows $E_{i*} = (e_{i1}, e_{i2}, \dots, e_{in})$ and columns $E_{*j} = (e_{1j}, e_{2j}, \dots, e_{mj})^T$ is said to be in row echelon form provided the following two conditions hold.

(i) If $E_{i*} = (e_{i1}, e_{i2}, \dots, e_{in})$ consists entirely of zeros, then all rows below E_{i*} are also entirely zero

(ii) If the first nonzero entry in $E_{i*} = (e_{i1}, e_{i2}, \dots, e_{in})$ lies in the j^{th} position, then all entries below the i^{th} position in columns $E_{*1}, E_{*2}, \dots, E_{*j}$ are zero

The first nonzero entry in a row is called a pivot.

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of a matrix: Suppose $A \in Mat_{m \times n}(\mathbb{R})$ is reduced by row operations to an Echelon form E. The rank of A is defined to be the number

$$\begin{aligned} \text{rank}(A) &= \text{number of pivots} \\ &= \text{number of nonzero rows in } E \\ &= \text{number of basic columns in } A \end{aligned}$$

Exercises for beginners

① Apply modified Gaussian elimination to the following matrix and circle the pivot positions.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - 2R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{bmatrix} \xrightarrow[R_4 + R_3]{R_2 + R_3} \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

② Determine the rank and identify the basic columns in

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 3 & 6 & 3 & 3 \end{bmatrix} \xrightarrow{R_2 - 2 \cdot R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - 3 \cdot R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{rank}(A) = 2}$

~~basic columns:~~

basic columns in A: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

③ Reduce each of the following matrices to row echelon form, determine the rank, and identify the basic columns.

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 9 \\ 2 & 6 & 7 & 6 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \text{rank}(A) = 3$$

$$\text{basic columns of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix} \right\}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 8 \\ 2 & 6 & 0 \\ 1 & 2 & 5 \\ 3 & 8 & 6 \end{bmatrix} \xrightarrow[\substack{R_4 - R_1 \\ R_5 - 3R_1}]{\substack{R_2 - 2R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & -6 \\ 0 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \xrightarrow[R_5 - R_2]{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -8 \\ 0 & 0 & 2 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow[R_5 + 2R_4]{R_3 + 4R_4} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 + 2R_5} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_5} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{rank}(B) = 3$$

$$\text{basic columns of } B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 6 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 0 \\ 5 \\ 6 \end{bmatrix} \right\}$$

④ Determine which of the following matrices are in row echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 2 & 3 & -4 \\ 0 & 0 & 7 & -8 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Standard exercises

① Suppose that A is an $m \times n$ matrix. Give a short explanation of why each of the following statements is true.

(a) $\operatorname{rank}(A) \leq \min\{m, n\}$
(b) $\operatorname{rank}(A) < m$ if one row of A is entirely zero
(c) $\operatorname{rank}(A) < m$ if one row of A is a multiple of another row
(d) $\operatorname{rank}(A) < m$ if one row of A is a combination of other rows
(e) $\operatorname{rank}(A) < n$ if one column of A is entirely zero

② How many different "forms" are possible for a 3×4 matrix that is in row echelon form?

$\boxed{15}$

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \quad \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

③ Suppose that $[A \mid b]$ is reduced to a matrix $[E \mid c]$.

(a) Is $[E \mid c]$ in row echelon form if E is?
(b) If $[E \mid c]$ is in row echelon form, must E be in row echelon form?

(a) No! Consider a matrix of the form

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{array}\right]$$

(b) Yes! E is a row echelon form obtainable from A.

def: Let $A = [a_{ij}]$ be a square matrix. A is said to be a symmetric matrix whenever $A^T = A$. i.e. $a_{ij} = a_{ji}$

A is said to be a hermitian matrix whenever $A = A^*$, i.e. $a_{ij} = \overline{a_{ji}}$ (this is the complex analog of symmetry)

A is said to be a skew-symmetric matrix whenever $A = -A^T$, i.e. $a_{ij} = -a_{ji}$

A is said to be a skew-hermitian matrix when $A = -A^*$, i.e. $a_{ij} = -\overline{a_{ji}}$

④ Construct an example of a 3x3 matrix A that satisfies the following conditions

* (a) A is both symmetric and skew-symmetric
(b) A is both hermitian and symmetric
(c) A is skew hermitian

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} i & 1+i & 1+i \\ i-1 & i & 1+i \\ i-1 & i-1 & i \end{bmatrix}$$

⑤ Explain why the set of all $n \times n$ symmetric matrices is closed under matrix addition. That is, explain why the ~~sd~~ sum of two $n \times n$ symmetric matrices is again an $n \times n$ symmetric matrix. Is the set of all $n \times n$ skew-symmetric matrices closed under matrix addition?

A, B $n \times n$ symmetric matrices $\Rightarrow a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$

$A + B = [a_{ij} + b_{ij}]$ and $a_{ij} + b_{ij} = a_{ji} + b_{ji}$ ☺

A, B skew symmetric matrices $\Rightarrow a_{ij} = -a_{ji}$, $b_{ij} = -b_{ji}$

$a_{ij} + b_{ij} = (-a_{ji}) + (-b_{ji}) = -(a_{ji} + b_{ji})$ $\Rightarrow$ Yes! ☺

Problems from Exam

① Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 - 3\cdot R_1 \\ R_4 - 6\cdot R_1}]{R_2 - 2\cdot R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix} \xrightarrow{R_4 - R_2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\text{Rank}(A) = 3}$$

$$\begin{array}{l} \text{even} \\ \text{odd} \\ \text{even} \end{array} \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix} \xrightarrow{2\cdot R_2} \begin{bmatrix} 2 & 3 & 4 & -1 \\ 10 & 4 & 0 & -2 \\ -4 & 5 & 12 & -1 \end{bmatrix} \xrightarrow[R_3 + 2\cdot R_1]{R_2 - 5\cdot R_1} \begin{bmatrix} 2 & 3 & 4 & -1 \\ 0 & -11 & -20 & 3 \\ 0 & 11 & 20 & -3 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 2 & 3 & 4 & -1 \\ 0 & -11 & -20 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

none equal to 1 ...
bad :(

$$\boxed{\text{Rank}(B) = 2}$$

② Determine the rank of $A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$

$$\xrightarrow[\substack{R_3 - 2R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_4 + R_2} \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \boxed{\text{rank}(A) = 4}$$

③ Find the rank of the following matrices: $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ ← First column of B is bad again ☹

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & 5 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$\boxed{\text{Rank}(A) = 3}$

Let's get a 2 in the first column ☺

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 2 & -1 & -3 & 9 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 2 & -1 & -3 & 9 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 6 & 1 & 3 & 8 \end{bmatrix} \xrightarrow[R_4 - R_2]{R_3 - 2 \cdot R_2} \begin{bmatrix} 2 & -1 & -3 & 9 \\ 4 & 2 & 6 & -1 \\ 2 & -1 & -3 & 9 \\ 2 & -1 & -3 & 9 \end{bmatrix}$$

Now, first column is easier to deal with ☺

$$\xrightarrow[R_3 - R_1]{R_3 - R_1} \begin{bmatrix} 2 & -1 & -3 & 9 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 - 2 \cdot R_1} \begin{bmatrix} 2 & -1 & -3 & 9 \\ 0 & 4 & 12 & -19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank}(B) = 2}$

④ Determine the rank of the following matrices: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank}(A) = 2}$

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[R_3 - 3 \cdot R_2]{R_1 + R_4} \begin{bmatrix} 1 & 2 & -5 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \end{bmatrix} \xrightarrow[R_4 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & -2 & 6 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & -1 & 3 & 1 \end{bmatrix} \xrightarrow[R_4 + R_3]{R_2 + 2 \cdot R_3} \begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -5 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank}(B) = 2}$

⑤ Prove that each of the following statements is true.

(a) If $A = [a_{ij}]$ is skew symmetric, then $a_{jj} = 0$ for each j.

(b) If $A = [a_{ij}]$ is skew hermitian, then each a_{jj} is a pure imaginary number, i.e. a multiple of the imaginary unit i

(c) If A is real and symmetric, then $B = iA$ is skew hermitian.

(a) Skew-symmetric $\Rightarrow$ $a_{ij} = -a_{ji}$ $\Rightarrow$ $a_{jj} = -a_{jj}$ $\Rightarrow$ $a_{jj} = 0$

(b) skew hermitian $\Rightarrow$ $a_{ij} = -\overline{a_{ij}}$ $\Rightarrow$ $a_{jj} = -\overline{a_{jj}}$

$a + bi = -(a - bi)$

$a + bi = -a + bi$

$2a = 0 \Rightarrow a = 0 \Rightarrow a_{jj} = bi$

(c) A symmetric and real $\Rightarrow$ $a_{ij} = a_{ji}$

$\Rightarrow i a_{ij} = i a_{ji} = -(\overline{a_{ji} i})$ $\Rightarrow$ skew-hermitian !!