Univerza na Primorskem
UP FAMNIT
Študijsko leto 2021/2022

Algebra I
Izpit
– 9. Februar 2022 –

Čas pisanja: 135 *minut. Maksimalno število točk:* 100. *Dovoljena je uporaba pisala in kalkulatorja. Pišite razločno in utemeljite vsak odgovor. Srečno!*

1. (a) Zapišite definicijo skalarnega produkta in naštejte vsaj tri njegove lastnosti. Nato dokažite naslednjo trditev: Za poljubna vektorja $\vec{u}, \vec{v} \in \mathbb{R}^3$, velja $\langle \vec{u}, \vec{v} \rangle = |\vec{u}||\vec{v}| \cos \varphi$, pri čemer je φ kot med vektorjema $\vec{u}$ in $\vec{v}$. *(7 točk)*
   (b) Izpeljite enačbo za premico v $\mathbb{R}^3$, v vektorski, parametrični in kanonični obliki. *(6 točk)*
   (c) Zapišite in dokažite Cramerjevo pravilo za reševanje sistema linearnih enačb. *(7 točk)*

2. V kocki $ABCDA'B'C'D'$ (točka A' leži nad točko A) označimo z $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AD}$ in $\vec{c} = \overrightarrow{AA'}$. Točka T leži na stranici AB tako, da velja $|AT| : |TB| = 1 : 3$, točka P deli stranico $B'C'$ v razmerju $|B'P| : |B'C'| = 1 : 5$ in točka S leži na presečišču telesnih diagonal.
   (a) Zapišite vektorja $\overrightarrow{D'T}$ in $\overrightarrow{SP}$ kot linearno kombinacijo vektorjev $\vec{a}$, $\vec{b}$ in $\vec{c}$. *(10 točk)*
   (b) Določite razmerje $|CR| : |RB'|$, če je R presečišče daljic CB' in BP. *(10 točk)*
   Namig: Zapišite vektor $\overrightarrow{CR}$ kot linearno kombinacijo vektorjev $\vec{a}$, $\vec{b}$ in $\vec{c}$ na dva načina.

3. Dani imamo premici $\ell = (7, 0, 1) + \lambda(2, 1, -2)$ in $q : x + 3 = 4 - 4y = 20 - 4z$.
   (a) Poiščite presečišče premic ℓ in q. *(7 točk)*
   (b) Zapišite enačbo ravnine, ki vsebuje premici ℓ in q. *(7 točk)*
   (c) Izračunajte kot med premicama ℓ in q. *(6 točk)*
   Namig: Kot med dvema vektorjema izračunamo s pomočjo enačbe $\cos \varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

4. Pokažite, da sistem linearnih enačb

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

   nima rešitve, razen če je $a + c = 2b$. V tem primeru rešitev tudi poiščite. *(20 točk)*

5. Z uporabo osnovnih operacij nad vrsticami pokažite, da je

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix} = 0.$$

(20 točk)

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Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) Write the definition of the scalar (dot) product and state at least 3 of its properties. Then, prove the following statement: For any two vectors $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^3$, it holds that $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = |\overrightarrow{u}||\overrightarrow{v}| \cos\varphi$, where φ is the angle between vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. *(7 points)*

   (b) In $\mathbb{R}^3$, derive the equation of a line in vectorial, parametric and canonical form. *(6 points)*

   (c) Write down and prove Cramer's rule for solving systems of linear equations. *(7 points)*

2. In a cube $ABCDA'B'C'D'$ (point A' is above point A) let $\overrightarrow{a} = \overrightarrow{AB}$, $\overrightarrow{b} = \overrightarrow{AD}$ and $\overrightarrow{c} = \overrightarrow{AA'}$. Point T lays on the line segment AB so that $|AT| : |TB| = 1 : 3$, point P divides the line segment $B'C'$ so that $|B'P| : |B'C'| = 1 : 5$ and point S is the intersection of space diagonals (i.e. segments AC' and BD').

   (a) Write vectors $\overrightarrow{D'T}$ and $\overrightarrow{SP}$ as a linear combination of vectors $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$. *(10 points)*

   (b) Find the ratio $|CR| : |RB'|$, if R is the intersection of line segments CB' and BP. *(10 points)*
   Hint: Express the vector $\overrightarrow{CR}$ as a linear combination of $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ in two ways.

3. We are given lines $\ell = (7, 0, 1) + \lambda(2, 1, -2)$ and $q : x + 3 = 4 - 4y = 20 - 4z$.

   (a) Find the intersection of lines ℓ and q. *(7 points)*

   (b) Find the equation of the plane containing lines ℓ and q. *(7 points)*

   (c) Compute the angle between lines ℓ and q. *(6 points)*
   Hint: The angle between two vectors can be obtained from the equation $\cos\varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

4. Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

   does not have a solution unless $a + c = 2b$. In that case, write the solution of the system. *(20 points)*

5. Using elementary row operations show that

$$\begin{vmatrix} a+2 & b+2 & c+2 \\ x+1 & y+1 & z+1 \\ 2x-a & 2y-b & 2z-c \end{vmatrix} = 0.$$

(20 points)