Review for Test 1

① Compute the distance between:

(a) Point $T(2,3,1)$ and line given by $x = 1+2t$, $y = 2+t$, $z = 1-2t$.

$$d(T,p) = \frac{|(\vec{r}_T - \vec{r}_p) \times \vec{v}|}{|\vec{v}|}$$

$\vec{r}_T = (2,3,1)$, $\cancel{p = (1,2,1) + t(2,1,-2)}$

$\vec{r}_p = (1,2,1)$ $\quad p = (1,2,1) + t(2,1,-2)$

$\hookrightarrow \vec{v} = (2,1,-2)$

$\downarrow$

$\vec{r}_T - \vec{r}_p = (1,1,0)$

$$(\vec{r}_T - \vec{r}_p) \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} = \cdots = \boxed{(-2,2,-1)}$$

$$d(T,p) = \frac{|(-2,2,-1)|}{|(2,1,-2)|}$$

$$= \frac{\sqrt{9}}{\sqrt{9}} = \boxed{1}$$

(b) Point $T(2,3,1)$ and plane $\Sigma: 3x + 2y - 6z = -1$

$$d(T,\Sigma) = \frac{|(\vec{r}_T - \vec{r}_0) \cdot \vec{n}|}{|\vec{n}|}$$

Choose any point on Σ to be $r_0 = (x,y,z)$, as long as it satisfies $3x + 2y - 6z = -1$.

e.g. $\boxed{\vec{r}_0 = (1,1,1)}$

$$d(T,\Sigma) = \frac{7}{7} = \boxed{1}$$

$\vec{r}_T - \vec{r}_0 = (1,2,0)$, $\vec{n} = (3,2,-6)$

$(\vec{r}_T - \vec{r}_0) \cdot \vec{n} = \cdots = 7$

$|\vec{n}| = \cdots = 7$

(c) Lines $p = (1,1,1) + \lambda(1,2,0)$ and $q = (1,0,3) + \lambda(2,4,0)$.

$\vec{v}_p = (1,2,0)$, $\vec{v}_q = (2,4,0)$ $\Rightarrow$ $\vec{v}_q = 2 \cdot \vec{v}_p$ $\Rightarrow$ $p \parallel q$

$$d(p,q) = \frac{|(\vec{r}_p - \vec{r}_q) \times \vec{v}_q|}{|\vec{v}_q|}$$

$\vec{r}_p = (1,1,1)$, $\vec{r}_q = (1,0,3)$

$\boxed{\vec{r}_p - \vec{r}_q = (0,1,-2)}$

$$(\vec{r}_p - \vec{r}_q) \times \vec{v}_q = \begin{vmatrix} i & j & k \\ 0 & 1 & -2 \\ 2 & 4 & 0 \end{vmatrix} = \cdots = \boxed{(8,-4,-2)}$$

$$d(p,q) = \frac{|(8,-4,-2)|}{|(2,4,0)|}$$

$$= \cdots = \frac{\sqrt{84}}{\sqrt{20}} \approx \boxed{2.05}$$

(d) Lines $\rho = (3,-4,4) + \lambda(-2,7,2)$ and $q = (-3,4,1) + \lambda(1,-2,-1)$.

$\vec{v}_\rho = (-2,7,2)$, $\vec{v}_q = (1,-2,-1)$ $\Rightarrow$ $\vec{v}_\rho \nparallel \vec{v}_q$!

$$d(\rho, q) = \frac{|(\vec{v}_\rho \times \vec{v}_q)\cdot(\vec{r}_q - \vec{r}_\rho)|}{|\vec{v}_\rho \times \vec{v}_q|}$$

$$\vec{v}_\rho \times \vec{v}_q = \begin{vmatrix} i & j & k \\ -2 & 7 & 2 \\ 1 & -2 & -1 \end{vmatrix} = \cdots = \boxed{(-3,0,-3)}$$

$$\vec{r}_q - \vec{r}_\rho = (-3,4,1) - (3,-4,4) = \boxed{(-6,8,-3)}$$

$$d(\rho, q) = \frac{|(-3,0,-3)\cdot(-6,8,-3)|}{|(-3,0,-3)|}$$

$$= \cdots = \frac{27}{\sqrt{18}} \approx \boxed{6.36}$$

(e) Line $\rho = (2,-1,0) + \lambda(2,0,1)$ and plane $\Sigma: x - 2y - 2z = 0$.

$\boxed{d=0}$

Note: $d(\rho,\Sigma) \neq 0$ iff $\rho \parallel \Sigma$?

$\rho \parallel \Sigma$ iff $\vec{v}_\rho \cdot \vec{n} = 0$ (i.e. $\vec{v}_\rho \perp \vec{n}$)

So, $\rho \parallel \Sigma$?

$\vec{v}_\rho = (2,0,1)$, $\vec{n} = (1,-2,-2)$

$\vec{v}_\rho \cdot \vec{n} = \cdots = 0$

$\cancel{d(\rho,\Sigma) = |\vec{n}\cdot\vec{r}_\rho}$

$$d(\rho,\Sigma) = \frac{|(\vec{n}\cdot\vec{r}_\rho) - d|}{|\vec{n}|}$$

$\vec{n}\cdot\vec{r}_\rho = (1,-2,-2)\cdot(2,-1,0)$

$= \cdots = \boxed{4}$

$$d(\rho,\Sigma) = \frac{|4-0|}{|\vec{n}|} = \boxed{\frac{4}{3}}$$

(f) Planes $\Sigma: x + y + z = 7$ and $\Pi: x + y + z - 6 = 0$.

As $\Sigma \parallel \Pi$,

$\vec{n}_\Sigma = (1,1,1)$, $\vec{n}_\Pi = (1,1,1)$ $\Rightarrow$ $\Sigma \parallel \Pi$

$$d(\Sigma,\Pi) = \frac{|\langle \vec{r}_\Pi - \vec{r}_\Sigma, \vec{n}\rangle|}{|\vec{n}|}$$

Note: $\vec{r}_\Sigma = (x,y,z)$ satisfies equation $x + y + z = 7$ when $\boxed{\vec{r}_\Sigma = (1,1,5)}$

Similarly, let $\boxed{\vec{r}_\Pi = (1,1,4)}$

$$d(\Sigma,\Pi) = \frac{1}{|\vec{n}|}$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

$\vec{r}_\Pi - \vec{r}_\Sigma = (0,0,-1)$, $\vec{n} = (1,1,1)$

$\boxed{\langle \vec{r}_\Pi - \vec{r}_\Sigma, \vec{n}\rangle = -1}$

② Find the point T on the line given by $2x-y=2$ and $x-y-z=1$ that is equidistant from points $A(4,1,1)$ and $B(2,1,1)$

$$2x-y=2 \Rightarrow y = 2x-2$$

$$x-y-z=1 \Rightarrow z = x-y-1 = x-(2x-2)-1 = -x+1$$

$$(x,\ 2x-2,\ -x+1) \overset{x\to\lambda}{=} (0,-2,1) + \lambda(1,2,-1)$$

$$\Rightarrow \quad p = (0,-2,1) + \lambda(1,2,-1)$$

Use point $T(\lambda,\ -2+2\lambda,\ 1-\lambda)$

$$d(T,A) = d(T,B), \quad A(4,1,1),\ B(2,1,1)$$

$$\left(\sqrt{(x_A-x_T)^2+(y_A-y_T)^2+(z_A-z_T)^2}\right)^2 = \left(\sqrt{(x_B-x_T)^2+(y_B-y_T)^2+(z_B-z_T)^2}\right)^2$$

$$(4-\lambda)^2 + \underline{(1-(-2+2\lambda))}^2 + (1-(1-\lambda))^2 = (2-\lambda)^2 + \underline{(1-(-2+2\lambda))}^2 + (1-(1-\lambda))^2$$

$$(4-\lambda)^2 = (2-\lambda)^2$$

$$16-8\lambda+\lambda^2 = 4-4\lambda+\lambda^2$$

$$16-4 = 8\lambda-4\lambda$$

$$12 = 4\lambda$$

$$\lambda=3 \Rightarrow \boxed{T(3,4,-2)}$$

③ Find point T on the line $\ell = (1,0,0) + \lambda(1,1,1)$ that is equidistant from planes $x+y-z=-1$ and $x-y+z=5$.

$T(1+\lambda, \lambda, \lambda)$

$\vec{n}_\Sigma = (1,1,-1)$ $\quad$ $\vec{n}_\pi = (1,-1,1)$

$d_\Sigma = -1$ $\quad$ $d_\pi = 5$

$d(T,\Sigma) = d(T,\pi)$

$$\frac{|\vec{n}_\Sigma \cdot T - d_\Sigma|}{|\vec{n}_\Sigma|} = \frac{|\vec{n}_\pi \cdot T - d_\pi|}{|\vec{n}_\pi|}$$

$$\sqrt{3}\cdot \quad \frac{|1\cdot(1+\lambda) + 1\cdot\lambda + (-1)\cdot\lambda + 1|}{\sqrt{1+1+1}} = \frac{|1(1+\lambda) + (-1)\lambda + 1\cdot\lambda - 5|}{\sqrt{1+1+1}} \cdot \sqrt{3}$$

$$|1+\lambda + \lambda - \lambda + 1| = |1+\lambda - \lambda + \lambda - 5|$$

$$|\lambda + 2| = |\lambda - 4|$$

$$\lambda + 2 = -\lambda + 4 \qquad \left(\text{else, get } \lambda+2 = \lambda - 4 \Rightarrow 2 = -4 \text{ ↯}\right)$$

$$2\lambda = 2$$

$$\lambda = 1$$

$$\boxed{T(2,1,1)}$$