Homework 9

A. $p(n) = n^3 + C_2 n^2 + C_1 n + C_0 \qquad p(n) = \Theta n^3$

$0 \le \alpha n^3 \le n^3 + C_2 n^2 + C_1 n + C_0 \le \beta n^3 \; / : n^3$

$0 \le \alpha \le 1 + C_2/n + C_1/n^2 + C_0/n^3 \le \beta$

$\lim_{n\to\infty} \frac{C_2}{n} = 0 \quad \lim_{n\to\infty} \frac{C_1}{n^2} = 0 \quad \lim_{n\to\infty} \frac{C_0}{n^3} = 0 \quad \lim_{n\to\infty} 1 = 1$

$0 \le \alpha \le \lim_{n\to\infty} 1 + 0 + 0 + 0 = 1$

$\boxed{\alpha = \frac{1}{2}}$ as n get bigger

$0 \le 1 + C_2/n + C_1/n^2 + C_0/n^3 \le \beta$

$0 \le \lim_{n\to\infty} 1 + 0 + 0 + 0 \le \beta$

$0 \le 1 \le \beta$

$\boxed{\beta = 2}$ as n get bigger

→ Another approach for β and α is

$\beta = 1 + |C_2| + |C_1| + |C_0|$ → will work for all values positive or negative

$\alpha = \min\{C_2, C_1, C_0\}$ → if $C_2 < 0$, $C_1 < 0$, $C_0 < 0$

B.

i) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2 - 7} \le \sum_{n=1}^{\infty} \frac{1}{n^2-7} \qquad 0 \le \sin^2 n \le 1$

$\lim_{n\to\infty} \frac{\frac{1}{n^2-7}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-7} \; : n^2 = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} - \frac{7}{n^2}} = \frac{1}{1-0} = 1 \qquad 0 < 1 < \infty$

By limit comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{n^2-7}$, therefore they both converge. By direct comparison to $\sum_{10}^{\infty} \frac{1}{n^2-7}$ it shows that $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2-7}$ also converges.

B. ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}+2} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$

Ratio test

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{(n+1)!}}}{\frac{1}{\sqrt{n!}}} = \lim_{n\to\infty} \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \lim_{n\to\infty} \sqrt{\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}}} =$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} \; {:}\sqrt{n} = \frac{\cancel{\frac{1}{\sqrt{n}}}\to 0}{\sqrt{\frac{n}{n}+\cancel{\frac{1}{n}}}\to 0} = \frac{0}{1+0} = 0 < 1 \Rightarrow \text{Converges}$$

By the Ratio test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ Converges, and by direct comparison $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}+2}$ also converges.

iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n\cdot\sqrt{3n+1}}$

$$\lim_{n\to\infty} \frac{\frac{n-2}{n\sqrt{3n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}(n-2)}{n\sqrt{3n+1}} = \lim_{n\to\infty} \frac{(n-2)}{\sqrt{n}(\sqrt{3n+1})} =$$

$$= \lim_{n\to\infty} \frac{n-2}{\sqrt{3n^2+n}} \; {:}n = \lim_{n\to\infty} \frac{\frac{n}{n} - \frac{2}{n}\to 0}{\sqrt{\frac{3n^2}{n^2}+\frac{n}{n^2}}} = \frac{1}{\sqrt{3}}$$

$0 < \frac{1}{\sqrt{3}} < \infty$, $\therefore \displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is equiconvergent to $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n\cdot\sqrt{3n+1}}$ and they both diverge.

C. i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n^2}}{(n+1)!}$ $\qquad \displaystyle\sum_{n=1}^{\infty} \left|\frac{(-2)^{n^2}}{(n+1)!}\right| = \sum_{n=1}^{\infty} \frac{2^{n^2}}{(n+1)!}$

Ratio test

$$\lim_{n\to\infty} \frac{\frac{2^{n^2+2n+1}}{(n+2)!}}{\frac{2^{n^2}}{(n+1)!}} = \lim_{n\to\infty} \frac{\cancel{2^{n^2}}\cdot 4^n\cdot 2\cdot\cancel{(n+1)!}}{\cancel{2^{n^2}}\cdot(n+2)\cancel{(n+1)!}} = \lim_{n\to\infty} \frac{4^n\cdot 2}{(n+2)}$$

ii) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2+5^n}$ $\quad \sum_{n=1}^{\infty} \left|\frac{(-2)^n}{n^2+5^n}\right| = \sum_{n=1}^{\infty} \frac{2^n}{n^2+5^n} < \frac{2}{n^2} < \frac{1}{n^2}$ $\quad n^2+5^n > n^2$, $\frac{1}{n^2} > \frac{1}{n^2+5^n}$

By direct comparison, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=1}^{\infty} \frac{2^n}{n^2+5^n}$.
It converges absolutely.

iii) $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n \cdot \sqrt{n+2}}$ $\quad \sum_{n=1}^{\infty} \left|\frac{(-3)^n}{3^n \cdot \sqrt{n+2}}\right| = \sum_{n=1}^{\infty} \frac{\cancel{3^n}}{\cancel{3^n} \cdot \sqrt{n+2}} - \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+2}}^{\;\cdot\cdot\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{\frac{n}{n}}{\frac{n}{n}+\cancel{\frac{2}{n}}}} = \frac{1}{1} = 1$$

$0 < 1 < \infty$, therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is equiconvergent to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ and also $\sum_{n=1}^{\infty} \frac{3^n}{3^n \cdot \sqrt{n+2}}$, they all diverge.

AST

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}} > 0$, for all $n \geq 0$ (positive)

$$\frac{1}{\sqrt{n+3}} - \frac{1}{\sqrt{n+2}} = \frac{\sqrt{n+2} - \sqrt{n+3}}{\underbrace{(\sqrt{n+3})}_{>0}\underbrace{(\sqrt{n+2})}_{>0}} = \frac{-}{+} = -$$

$\sqrt{n+3} > \sqrt{n+2}$

the $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is decreasing.

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+2}}^{\;\cdot\cdot\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \to 0}{\sqrt{\frac{n}{n}+\frac{2}{n}}_{\to 0}} = \frac{0}{1} = 0$$

Therefore by AST, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ converges conditionally and by extension so does $\sum_{n=1}^{\infty} \frac{3^n}{3^n \cdot \sqrt{n+2}}$.

D. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}$ $\quad \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(n!)^2} \right| = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$

Ratio test

$$\lim_{n\to\infty} \frac{\frac{1}{((n+1)!)^2}}{\frac{1}{(n!)^2}} = \lim_{n\to\infty} \frac{(n!)^2}{((n+1)!)^2} = \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right)^2 = \lim_{n\to\infty} \left(\frac{\cancel{n!}}{(n+1)\cancel{(n!)}}\right)^2 =$$

$$= \lim_{n\to\infty} \frac{1}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{n^2+2n+1} = \lim_{n\to\infty} \frac{1/n^2}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}} = \frac{0}{1+0+0} = \frac{0}{1} = 0$$

$0 < 1 \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}$ Converges absolutely.

ASE $\rightarrow 0{,}01 \left(\frac{1}{100}\right)$

$\sum_{n=0}^{3} = 1 - 1 + \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$ is of 0,01 accuracy

$(n!)^2$

$n=1 \rightarrow (n!)^2 = 1 \leftarrow n=0$

$n=2 \rightarrow (n!)^2 = 4$

$n=3 \rightarrow (n!)^2 = 36$

$n=4 \rightarrow (4!)^2 = 576$