University of Primorska
UP FAMNIT
Academic year 2021/2022

Algebra I
Midterm 2
– January 20, 2022 –

Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) Write the definition of a left and right inverse of an $m \times n$ matrix A. Then, prove the following statement: If a square matrix A has both a left inverse X and a right inverse Y, then $X = Y$. *(5 points)*

   (b) Prove the following statement: Let A be an $m \times n$ matrix. Then, A has a right inverse if and only if $rang(A) = m$. *(8 points)*

   (c) For an $n \times n$ matrix A, write the definition of the adjoint matrix $adj(A)$. Then, prove the following statements:

   i) $A \cdot adj(A) = det(A) \cdot I_n = adj(A) \cdot A$. *(4 points)*

   ii) If A is invertible, then $A^{-1} = \frac{1}{det(A)} adj(A)$. *(3 points)*

2. i) Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ by elementary row operations. *(10 points)*

   ii) Express $\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix}$ as the sum of a lower triangular matrix and an upper triangular matrix with zero leading (main) diagonal. *(10 points)*

3. Show that the system of equations

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2 \\ -x_1 + 2x_2 &= \lambda x_3 \end{aligned}$$

   can possess a non-trivial solution only if $\lambda = 1$ or $\lambda = -3$. Obtain the general solution in each case. *(20 points)*

4. (a) Determine $x \in \mathbb{R}$ such that $det(AB) = 0$, where

$$A = \begin{bmatrix} x & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

   (10 points)

(b) Let $A, B, X \in \mathbb{R}^{n\times n}$ be matrices such that $det(B) = det(A) - 1$ and $3A^2X = XB$. Determine $det(X)$.

5. Let $a, b \in \mathbb{R}$. Compute the determinant of the following $n \times n$ matrix

$$\begin{bmatrix} a & b & b & \cdots & b & b \\ b & a & 0 & \cdots & 0 & 0 \\ b & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & 0 & \cdots & a & 0 \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}.$$

(20 points)