16. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix},$$

Compute AB, BA, det A, det B, det AB and det BA. Verify that $AB \neq BA$ and det $AB = (\det A)(\det B)$.

First we compute the two matrix products AB and BA,

$$AB = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ 6 & 3 & 1 \\ 0 & 1 & 6 \end{pmatrix}.$$

Next, we evaluate the determinants,

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{vmatrix} = -1 + 2(15) = 29\,,$$

$$\det B = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 1 + 3 = 4\,,$$

$$\det AB = \begin{vmatrix} 7 & -1 & 0 \\ 3 & 1 & -1 \\ 3 & 9 & 5 \end{vmatrix} = 7(5+9) + 1(15+3) = 98 + 18 = 116\,,$$

$$\det BA = \begin{vmatrix} 4 & -1 & 2 \\ 6 & 3 & 1 \\ 0 & 1 & 6 \end{vmatrix} = 4(18-1) - 6(-6-2) = 68 + 48 = 116\,.$$

Note that even though $AB \neq BA$, one finds:

$$\det(AB) = \det(BA) = (\det\ A)(\det\ B)\,,$$

i.e. $116 = (29)(4)$, as expected.

17. Let A be a 3×3 matrix. The determinant of A is denoted by $\det A$.

(a) Is the equation $\det(3A) = 3 \det A$ true or false? Explain.

The determinant is multiplied by k if you multiply *one* of the rows by k. Here, $3A$ is a matrix obtained from A by multiplying each of the three rows of A by a factor of 3. Hence, $\det(3A) = 27 \det A$. In general, for an $n \times n$ matrix, $\det(kA) = k^n \det A$.

(b) Suppose that $\det A = 1$. Let B be a matrix obtained from A by permuting the order of the rows so that the first row of A is the second row of B, the second row of A is the third row of B and the third row of A is the first row of B. (This is called a *cyclic permutation*.) What is the value of $\det A$?

Each time you interchange a pair of rows, the determinant changes by an overall sign. In this case, one can obtain B from A by two pairwise interchanges. Thus, $\det B = 1$.

(c) Suppose that the 3×3 matrix $A \neq 0$ but $\det A = 0$. What can you say about the rank of A?

If $\det A = 0$, then the rank of A must be less than three. Since $A \neq 0$, the rank must be greater than zero. Thus, either the rank of A is one or two. No further deduction can be drawn without additional information.

7. Give a counterexample to show that for square matrices A and B of the same size, it is not always true that $|A + B| = |A| + |B|$.

Solution:

Almost any example works. Define

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, $|A| = 0$, $|B| = -1$, and since

$$A + B = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

we have that $|A + B| = -2$, which is not equal to $|A| + |B| = -1$, as required.

19. Evaluate the following determinant by hand:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$$

The simplest method is to apply a set of elementary row operations of the form $R_j \to R_j + kR_i$ (where k is some non-zero constant), which do not change the value of the determinant, such that the final resulting matrix is in upper triangular form.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_2\to R_2-R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_3\to R_3-R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 1 & 4 & 10 & 20 \end{vmatrix} \xrightarrow[R_4\to R_4-R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix}.$$

The first column below the main diagonal is now filled with zeros. We proceed similarly until all elements below the main diagonal consist of zeros:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} \xrightarrow[R_4\to R_4-R_1]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} \xrightarrow[R_3\to R_3-2R_2]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 3 & 9 & 19 \end{vmatrix}$$

$$\xrightarrow[R_4\to R_4-3R_2]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix} \xrightarrow[R_4\to R_4-3R_3]{} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1\,,$$

where in the last step, we used the fact that the determinant of an upper triangular matrix is equal to the product of the diagonal elements. Hence, we conclude that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} = 1\,.$$

2. Calculate the determinant of each of the following matrixces by using row reduction to produce an upper triangular form:

(a) $\begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & 12 \end{bmatrix}$.

Solution:

Row-reducing to make the matrix upper triangular:

$$\begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & 12 \end{bmatrix} \xrightarrow{R_3:R_3+1/2R_1} \begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ 0 & 1 & 45/2 \end{bmatrix}$$

$$\xrightarrow{R_3:R_3+1/4R_2} \begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ 0 & 0 & 93/4 \end{bmatrix}$$

The determinant of the resulting upper triangular matrix is the product of the diagonals, and hence is $10 \cdot (-4) \cdot (93/4) = -930$. Since none of the row operations changed the determinant, $\boxed{-930}$ is the determinant of the original matrix.

(c) $\begin{bmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{bmatrix}$.

Solution:

Row-reducing into upper triangular form again,

$$\begin{bmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{bmatrix} \xrightarrow{R_2:R_2+2R_1, R_3:R_3+3R_1} \begin{bmatrix} 1 & -1 & 5 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & -1 & 9 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4-2R_1, R_3:R_3-R_2} \begin{bmatrix} 1 & -1 & 5 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4:R_4+R_2} \begin{bmatrix} 1 & -1 & 5 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{Swap } R_3 \text{ and } R_4} \begin{bmatrix} 1 & -1 & 5 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The determinant of the resulting matrix is 4. Tracing back the steps, the only step that changed the determinant was the last one, which multiplied it by -1. Therefore, the determinant of the original matrix is $\boxed{-4}$.

13. (a) If A is an $n \times n$ matrix, prove that $|cA| = c^n|A|$. (Hint: Use a proof by induction on n.)

Proof:

Base case:
Show that the statement holds for $n = 1$.

Asssume: A is a 1×1 matrix, c is a scalar.
Need to show: $|cA| = c|A|$.

Let A be a 1×1 matrix. Then, we can say that $A = [a_{11}]$. In that case,

$$|cA| = |[ca_{11}]| = ca_{11} = c|A|$$

so the equation holds.

Inductive step:
Here, we show that the statement for $n = k$ implies the statement for $n = k + 1$.

Asssume: For any $k \times k$ matrix A, $|cA| = c^k|A|$. *Need to show:* For any $(k+1) \times (k+1)$ matrix A, $|cA| = c^{k+1}|A|$.

Let A be a $(k+1) \times (k+1)$ matrix with entries a_{ij}. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(k+1)} \\ a_{21} & a_{22} & \cdots & a_{2(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ a_{(k+1)1} & a_{(k+1)1} & \cdots & a_{(k+1)(k+1)} \end{bmatrix}$$

Now, let $B = cA$. We need to show that $|B| = c^{k+1}|A|$. Expanding out along the first row,

$$|B| = b_{11}|B_{11}| + b_{12}|B_{12}| + \cdots + b_{1(k+1)}|B_{1(k+1)}|$$

Since $B = cA$, we have that

$$B = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1(k+1)} \\ ca_{21} & ca_{22} & \cdots & ca_{2(k+1)} \\ \vdots & \vdots & \vdots & \vdots \\ ca_{(k+1)1} & ca_{(k+1)1} & \cdots & ca_{(k+1)(k+1)} \end{bmatrix}$$

In other words, we have that $b_{ij} = ca_{ij}$. Furthermore, B_{ij} is B with row i and column j crossed out, and it should clear that this is just equal to cA_{ij}. Therefore, we see that

$$|B| = ca_{11}|cA_{11}| + ca_{12}|cA_{12}| + \cdots + ca_{1(k+1)}|cA_{1(k+1)}|$$

Now, A_{ij} is a $k \times k$ matrix. Therefore, by the inductive hypothesis, $|cA_{ij}| = c^k|A_{ij}|$. Plugging that in,

$$\begin{aligned} |B| &= ca_{11}c^k|A_{11}| + ca_{12}c^k|A_{12}| + \cdots + ca_{1(k+1)}c^k|A_{1(k+1)}| \\ &= c^{k+1}\left(a_{11}|A_{11}| + a_{12}|A_{12}| + \cdots + a_{1(k+1)}|A_{1(k+1)}|\right) \end{aligned}$$

It should be clear that by definition, $|A| = a_{11}|A_{11}| + a_{12}|A_{12}| + \cdots + a_{1(k+1)}|A_{1(k+1)}|$, using expansion along the first row. Therefore, the above expression simplifies to $|B| = c^{k+1}|A|$, as required. $\square$

15. Solve the following determinant equations for $x \in \mathbb{R}$:

(a)

$$\begin{vmatrix} x & 2 \\ 5 & x+3 \end{vmatrix} = 0$$

Solution:

Expanding out the determinant results in the equation

$$0 = x(x+3) - 10 = x^2 + 3x - 10$$

Factoring, we see that $x^2 + 3x - 10 = (x+5)(x-2)$. Thus, the equation is

$$0 = (x+5)(x-2)$$

and therefore the solutions are $\boxed{x = -5}$ and $\boxed{x = 2}$.

(c)

$$\begin{vmatrix} x-3 & 5 & -19 \\ 0 & x-1 & 6 \\ 0 & 0 & x-2 \end{vmatrix} = 0$$

Solution:

Since the determinant of an upper triangular matrix is the product of the entries on the diagonal, the equation turns out to be

$$(x-3)(x-1)(x-2) = 0$$

Therefore, the solutions are $\boxed{x = 1, 2, 3}$.

5. Let A be an upper triangular matrix. Prove that $|A| \neq 0$ if and only if all the main diagonal elements of A are nonzero.

Proof: Since this is an if and only if proof, we break it up into two parts. For the rest of the proof, assume that A is $n \times n$ with (i, j) entry a_{ij}.

Part 1:
Assumptions: A is upper triangular, $|A| \neq 0$.
Need to show: All of the diagonal entries of A are nonzero.

Since A is upper triangular,

$$|A| = a_{11}a_{22} \cdots a_{nn}$$

Since $|A| \neq 0$, we can't have any $a_{ii} = 0$, as that would make the determinant 0. Thus, we're done.

Part 2:
Assumptions:A is upper triangular, all the diagonal entries of A are nonzero.
Need to show: $|A| \neq 0$.

Just like above, $|A| = a_{11} \cdots a_{nn}$. Since none of the a_{ii} are 0, the product is nonzero. Hence, we're done. $\square$

6. Find the determinant of the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Solution:

The row operations Swap Row 1 and Row 6, then Swap Row 2 and Row 5, then Swap Row 3 and Row 4 result in the following matrix:

$$\begin{bmatrix} a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & a_{16} \end{bmatrix}$$

This matrix is upper triangular and has determinant $a_{61}a_{52}a_{43}a_{34}a_{25}a_{16}$. Since each row swap multiplied the determinant by -1, and there were 3 of them, our row operations multiplied the determinant by -1. Therefore, the determinant of the original matrix is $\boxed{-a_{61}a_{52}a_{43}a_{34}a_{25}a_{16}}$.

12. Let A be an $n \times n$ matrix having two identical rows.

(a) Use part (3) of Theorem 3.3 to prove that $|A| = 0$.

Proof:
Assume that A has Row i = Row j. Now, consider the row operation R which swaps Row i and Row j. From class, (and Theorem 3.3) in the book, we know that $|R(A)| = -|A|$. However, since Row i and j are identical, it's clear that $R(A)$ is exactly the same as A, and therefore we get that

$$|A| = -|A|$$

Solving, it follows that $|A| = 0$, so we're done. □

6. Let A and B be $n \times n$ matrices.

(a) Show that $|AB| = 0$ if and only if $|A| = 0$ or $|B| = 0$.

Proof:
Since this is an if and only if proof, we break it up into two parts.

Part 1:
Assumptions: $|AB| = 0$.
Need to show: $|A| = 0$ or $|B| = 0$.

Using the properties of the determinant, $|AB| = |A||B|$. Thus, we have that

$$0 = |AB| = |A||B|$$

Since the product of the two numbers $|A|$ and $|B|$ is 0, one of the numbers must be 0. Thus, $|A| = 0$ or $|B| = 0$, as required.

Part 2:
Assumptions:$|A| = 0$ or $|B| = 0$.
Need to show: $|AB| = 0$.

Again using the same properties, $|AB| = |A||B| = 0$, since one of $|A|$ and $|B|$ is 0, and therefore their product is 0. □

(b) Show that if $AB = -BA$ and n is odd, then A or B is singular.

Proof:
Assumptions: $AB = -BA$.
Need to show: A or B is singular.

Since $AB = -BA$, we can take determinants of both sides to arrive at:

$$|AB| = |-BA|$$

Now, from an earlier homework problem we know that $|cA| = c^n|A|$ for any $n\times n$ matrix A. Therefore, we have that $|-BA| = |(-1)BA| = (-1)^n|BA| = -|BA|$ using the fact that n is odd. Thus,

$$\begin{aligned} |AB| &= -|BA| \\ \Rightarrow |A||B| &= -|B||A| \\ \Rightarrow 2|A||B| &= 0 \end{aligned}$$

Thus, we must have $|A| = 0$ or $|B| = 0$. Since the determinant of a matrix is 0 precisely when it's singular, that shows that either A or B is singular. □

13. We say that a matrix B is similar to a matrix A if there exists some (nonsingular) matrix P such that $P^{-1}AP = B$.

   (a) Show that if B is similar to A, then they are both square matrices of the same size.

   Proof:
   Assumptions: B is similar to A.
   Need to show: A and B are both square matrices of the same size.

   Since B is similar to A, there exists a nonsingular matrix P such that $P^{-1}AP = B$. The only matrices with inverses are square, so assume that P is $n \times n$. Then clearly P^{-1} is also $n \times n$.
   Now, let A be $l \times m$. Since the product $P^{-1}AP$ is possible, we see that from the fact that P^{-1} and A can be multiplied, we get that $n = l$, and from the fact that A and P can be multiplied, we get that $m = n$. Therefore, we see that A is $n \times n$.
   Finally, working out the dimensions of the product $P^{-1}AP$ shows that B is $n \times n$. Hence, A and B are square matrices of the same size, and we're done. $\square$

   (b) Find two different matrices B similar to

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   Solution:

   This just requires choosing two matrices P. First, let

$$P = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   In that case, clearly, $B = P^{-1}AP = A$, so

$$B = A = \boxed{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

   is one example.
   Now, pick another P:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

   In that case, using the formula for the inverse of a 2×2 matrix, we see that

$$\begin{aligned} B = P^{-1}AP &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \boxed{\begin{bmatrix} -2 & -4 \\ 3 & 7 \end{bmatrix}} \end{aligned}$$

(f) Prove that if A is similar to I_n, then $A = I_n$.

Proof:
Assumptions: A is similar to I_n.
Need to show: $A = I_n$.
Since A is similar to I_n, there exists a matrix P such that $A = P^{-1}I_nP$. Since multiplying by the identity doesn't do anything, that means that $A = P^{-1}P = I_n$, as required. $\square$

(g) Prove that if A and B are similar, then $|A| = |B|$.

Proof:
Assumptions: A is similar to B.
Need to show: $|A| = |B|$.
Since A is similar to I_n, there exists a matrix P such that

$$B = P^{-1}AP$$

Taking determinants of both sides and using properties of determinants,

$$\begin{aligned} |B| &= |P^{-1}AP| = |P^{-1}||A||P| \\ &= |P^{-1}||P||A| = |P^{-1}P||A| \\ &= |I_n||A| = |A| \end{aligned}$$

as required. (We could rearrange the product of the determinants because these are just numbers, and hence order of multiplication doesn't matter.) $\square$