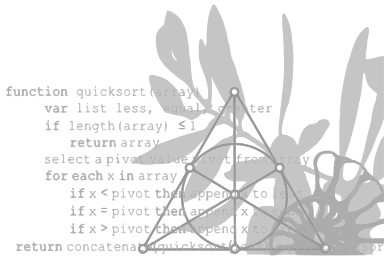

University of Primorska
UP FAMNIT
Academic year 2022/2023

Algebra I

Midterm 2

– January 12, 2023 –

Time: 120 *minutes. Maximum number of points:* 50. *You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!*

1. (a) Write the definition of a left and right inverse of an $m \times n$ matrix A. Then, prove the following statement: If a square matrix A has both a left inverse X and a right inverse Y, then $X = Y$. *(3 points)*

   (b) Prove the following statement: Let A be an $m \times n$ matrix. Then, A has a right inverse if and only if $rang(A) = m$. *(3 points)*

   (c) For an $n \times n$ matrix A, write the definition of the adjoint matrix $adj(A)$. Then, prove the following statements:

   i) $A \cdot adj(A) = det(A) \cdot I_n = adj(A) \cdot A$. *(2 points)*

   ii) If A is invertible, then $A^{-1} = \frac{1}{det(A)} adj(A)$. *(2 points)*

2. We are given matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 1 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & -1 \\ 5 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find matrix X from the equation $AX - BT = X + AB$. *(10 points)*

3. Write the matrix $A = \begin{bmatrix} 2 & -2 & 1 \\ 6 & -1 & 5 \\ 3 & 7 & 4 \end{bmatrix}$ in the form $A = LU$, where L is a lower triangular matrix with all coefficient on the main diagonal equal to 1, and U is an upper triangular matrix. *(10 points)*

4. Show that the system of equations

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2 \\ -x_1 + 2x_2 &= \lambda x_3 \end{aligned}$$

   can possess a non-trivial solution only if $\lambda = 1$ or $\lambda = -3$. Obtain the general solution in each case. *(10 points)*

5. Let $a, b \in \mathbb{R}$. Compute the determinant of the following $n \times n$ matrix

$$\begin{bmatrix} a & b & b & \cdots & b & b \\ b & a & 0 & \cdots & 0 & 0 \\ b & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & 0 & 0 & \cdots & a & 0 \\ b & 0 & 0 & \cdots & 0 & a \end{bmatrix}.$$

(10 points)