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ANA-I Foundations of Analysis
Final Examination B – 9 Feb 2021

Name ____________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.
   (a) $\lim_{n\to\infty} \frac{\sqrt{2n^2 - n + 3}}{3n+4}$
   (b) $\lim_{n\to\infty} \sqrt[3]{\frac{n^2+1}{n^3-1}}$
   (c) $\lim_{x\to 3} \frac{x^2-9}{x^2-5}$
   (d) $\lim_{x\to 2} \frac{x^2-4}{x^2-3x+2}$
   (e) $\lim_{x\to 0} \frac{|x|}{x}$.

2. Series
   (a) (6 points) Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2 + 5n + 2}{(1+\frac{1}{10})^n}$ converges absolutely, converges conditionally, or diverges.
   (b) (7 points) Let $f(x) = \sum_{n=0}^{\infty} x^n$. For what values of x does the expression converge? For these values of x, write $f(x)$ in the form of an elementary function.

3. (6 points each) Examples. Justify your answers briefly.
   (a) There is a function $f : \mathbb{R} \to \mathbb{R}$ so that $f(-1) = -1$ and $f(1) = 1$, but so that f is never 0. What else can you say about f?
   (b) Explain briefly why $\{x \in \mathbb{Q}^{\geq 0} : x < 5\} \mid \{x \in \mathbb{Q}^{\geq 0} : x \geq 5\}$ is a Dedekind cut. What positive real number does it represent?
   (c) Give an example of a sequence a_n whose image set $\{a_n : n \in \mathbb{N}\}$ is not compact.
   (d) Give an example of a subset of $\mathbb{R}^2$ that is neither open nor closed.

4. (10 points) Let a_n be recursively defined by $a_0 = 2$, $a_{n+1} = \sqrt{5a_n}$ for $n \geq 0$. Show that the sequence converges, and find $\lim_{n\to\infty} a_n$.

5. (11 points) How many terms are needed to estimate $\sum_{n=0}^{\infty} \frac{10+(-1)^n \cdot n}{5^n}$ to within 0.1? Justify your answer!

6. (12 points) Show directly from definition that if a_n, b_n are sequences so that $\lim_{n\to\infty} a_n = 2$ and $\lim_{n\to\infty} b_n = 4$, then $\lim_{n\to\infty} a_n \cdot b_n = 8$.

7. (10 points) Prove that if f is a continuous function from $[2,5]$ to $[2,5]$, then f has a fixed point.

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ANA-I Foundations of Analysis
Final Examination A – 26 Jan, 2021

Name ________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.

   (a) $\lim\limits_{n\to\infty} \dfrac{3n^2+n-1}{-2n^2+n-4}$

   (b) $\lim\limits_{x\to 3-} \dfrac{x^2-9}{x+3}$

   (c) $\lim\limits_{x\to -3} \dfrac{x^2-9}{x+3}$

   (d) $\lim\limits_{x\to\infty} 2x+\sin 3x$

   (e) $\lim\limits_{n\to\infty} \dfrac{3n+6ni}{|2+3i|\,n}$

2. (6 points each) Series

   (a) Find the exact value that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+1}{4^n}$ converges to, or else conclude the series diverges.

   (b) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^{1/2}+3}{2n^2-1}$ converges absolutely, converges conditionally, or diverges.

3. (6 points each) Examples. Justify your answers briefly.

   (a) Give the Dedekind cut for $\sqrt[3]{5}$. Your answer should not refer directly to any irrational numbers.

   (b) Give an example of a sequence with exactly 3 accumulation points.

   (c) A function $\mathbb{R}\to\mathbb{R}$ that is strictly decreasing and continuous.

   (d) A sequentially compact subset of $\mathbb{R}^2$ that contains the point $(2,2)$.

4. (10 points) Let a_n be recursively defined by $a_0 = 15$, $a_{n+1} = \dfrac{1}{2}(a_n + \dfrac{9}{a_n})$ for $n \geq 0$. Show that the sequence converges, and find $\lim_{n\to\infty} a_n$.

5. (11 points) Using the method of bisection, estimate $\sqrt{10}$ to within 0.1.

6. (12 points) Show directly from definition that if a_n is a positive real sequence with $\lim_{n\to\infty} a_n = 0$ and $\lim_{x\to 0+} f(x) = L$,then $\lim_{n\to\infty} f(a_n) = L$.

7. (11 points) Show that if $S \times T$ is a closed subset of $\mathbb{R}^2$, then S and T are closed subsets of $\mathbb{R}$.

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ANA-I Foundations of Analysis
Final Examination B – 3 Feb, 2020

Name ________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.
   (a) $\lim_{n\to\infty} \frac{2n-2}{n-\sqrt{n}+3}$
   (b) $\lim_{x\to 0} \frac{x^2+2x+1}{x^2-1}$
   (c) $\lim_{x\to -1} \frac{x^2+2x+1}{x^2-1}$
   (d) $\lim_{x\to\infty} \frac{\sin 3x}{3x-9}$
   (e) $\lim_{n\to\infty} \frac{2n\cdot i^n}{n^2+1}$

2. (6 points each) Series
   (a) Determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n n+1}{n^2-1}$ converges absolutely, converges conditionally, or diverges.
   (b) Find the exact value that $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}$ converges to, or else conclude the series diverges.

3. (6 points each) Examples. Justify your answers briefly.
   (a) Explain from definition why if $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0$, then $\lim_{x\to 0} f(x) = 0$.
   (b) Give the Dedekind cut for 1/3. Your answer should not refer directly to any irrational numbers.
   (c) Either give an example of a conditionally convergent series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} |a_n| = \frac{1}{1000}$, or else explain why no such series exists.
   (d) Give an example of a compact subset of $\mathbb{R}$ that is not an interval.

(-see reverse side-)

4. (9 points) Let a_n be recursively defined by $a_0 = 128$, $a_n = \dfrac{n-\pi}{2n} \cdot a_{n-1}$for $n \geq 1$. Show that the sequence converges, and find $\lim\limits_{n\to\infty} a_n$.

5. (11 points) Estimate $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+2}{(-3)^n}$ to within an accuracy of 0.5.

6. (14 points) Show directly from definition that if $f(x)$ is a real function that is continuous everywhere, and a_n is a Cauchy sequence, then $f(a_n)$ is also a Cauchy sequence.

7. (10 points) Find a partial sum of $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ that gives an estimate of the sum that is accurate to within 0.01.

   Hint: One method is to compare the "tails" of the series with geometric series!

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ANA-I Foundations of Analysis
Final Examination A – 20 Jan, 2020

Name ______________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.

   (a) $\displaystyle\lim_{x\to 0+} \frac{|x| \cdot x}{x^2}$

   (b) $\displaystyle\lim_{n\to\infty} \frac{\cos n^2}{n}$

   (c) $\displaystyle\lim_{n\to\infty} \frac{n \cdot i^n}{n+1}$

   (d) $\displaystyle\lim_{x\to\infty} \frac{x^2}{x + \sqrt{3x^4+1}}$

   (e) $\displaystyle\lim_{x\to\infty} \frac{x^2 - 10x}{\sqrt{x^3/2+1}}$

2. (6 points each) Series

   (a) Determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n^3 + 2^n}{n^2 + 3^n}$ converges absolutely, converges conditionally, or diverges.

   (b) Find the exact value that $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$ converges to, or else conclude the series diverges.

3. (6 points each) Examples. Justify your answers briefly.

   (a) Suppose that real functions f and g have the same value on some open interval containing 0. Using the definition of limit, explain why $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$.

   (b) Show that if $a_2 > 0$, then $\Theta(a_2 n^2 + a_1 n + a_0) = \Theta(n^2)$.

   (c) Give the Dedekind cut for $1/\sqrt{2}$. Your answer should not refer directly to any irrational numbers.

   (d) Give a subset of $\mathbb{R}^2$ that is <u>both</u> open and closed, and give one that is <u>neither</u> open nor closed.

(-see reverse side-)

4. (13 points) Let a_n be a sequence so that $\lim_{n\to\infty} a_n/\sqrt{n} = 0$. Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n^2}$ converges.

5. (13 points) Let f and g be real functions. Show directly from the definitions that if f is everywhere continuous and $\lim_{x\to 1+} g(x) = L$, then $\lim_{x\to 1+} f(g(x)) = f(L)$.

6. (10 points) Let f be a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that if K is a compact subset of $\mathbb{R}^2$, then $f(K)$ is also compact.
   Hint: One approach is to use that the inverse image f^{-1} of an open set is open.

7. (8 points) Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show that $\lim_{k\to\infty} F_{2k+1}/F_{2k}$ converges, and find its limit.
   Use only facts that we have proved in Analysis I in this problem.
   Partial credit for solving for the limit without showing the ratio converges.

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ANA-I Foundations of Analysis
2nd Midterm Examination – 14 Jan, 2020

Name ______________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
75 total points.

1. (6 points each) Calculations. For each sequence, explain whether it is convergent, divergent to $\pm\infty$, or otherwise divergent (not to $\pm\infty$). If it is convergent, find its limit. You may use any theorems we have proved in class or on homework.

   (a) $\lim_{n\to\infty} \frac{3}{2-ni}$

   (b) $\lim_{x\to 2} \sqrt{\frac{x^2-4}{x+4}}$

   (c) $\lim_{x\to 2} \sqrt{\frac{x^2-4}{x-2}}$

2. (6 points each) For each series, determine whether it converges absolutely, converges conditionally, or diverges.

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \sqrt{n}+1}{n^2-3n+1}$

3. (6 points each) Examples. Justify your answers briefly.

   (a) Let $\mathcal{A}$ be the family of functions defined everywhere on $\mathbb{R}$ so that $\lim_{x\to\infty} f(x)$ converges. Show that $\mathcal{A}$ is an algebra of functions.

   (b) Give an example of a set with a finite number of points that is not sequentially compact, or else explain why no such set exists.

   (c) Give an example of series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ that are divergent, but so that $\left(\sum_{n=0}^{\infty} a_n\right) + \left(\sum_{n=0}^{\infty} b_n\right)$ converges.

(-see reverse side-)

4. (6 points) Find the exact value that the sequence $\sum_{n=0}^{\infty} \sin^n 1$ converges to. (Note that as usual, $\sin^n x$ means $(\sin x)^n$.)

5. (11 points) Let f be a real function, defined everywhere on $\mathbb{R}$. Prove that if $\lim_{x\to 2-} f(x) = L$, then also $\lim_{x\to -2+} f(-x) = L$. (Here of course $\lim_{x\to -2+}$ denotes the limit as x goes to -2 from the right.)
Hint: Your proof is likely to use the letters ϵ and δ.

6. Consider the "sawtooth" function

$$f(x) = \begin{cases} x - n & \text{if } n \leq x < n+1 \text{ for } n \text{ even} \\ 1 + n - x & \text{if } n \leq x < n+1 \text{ for } n \text{ odd} \end{cases}$$

   (a) (1 points) Sketch a graph of $f(x)$.

   (b) (9 points) Prove that $f(x)$ is continuous everywhere on $\mathbb{R}$.

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ANA-I Foundations of Analysis
1st Midterm Examination – 18 Nov, 2019

Name ________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
72 total points.

1. (6 points each) Limit calculations. For each real-valued sequence, explain whether it is convergent, divergent to $\pm\infty$, or otherwise divergent (not to $\pm\infty$). If it is convergent, find its limit. If it is divergent, find its lim sup and lim inf. You may use any theorems we have proved in class or on homework.
   (a) $s_n = \dfrac{2n-1}{n-\sqrt{2}}$
   (b) $s_n = \dfrac{(-1)^n \cdot 2n - 1}{4-5n}$
   (c) $s_n = \dfrac{(-1)^n \cdot 2\sqrt{n} + 12}{2n-3}$
   (d) $s_n = n - n^2$

2. (6 points each) Examples. Justify your answers briefly.
   (a) Give an example of a bounded sequence that diverges.
   (b) Using our construction of $\mathbb{Z}$ by the method of order pairs, explain why $2-3=-1$.
   (c) Give the Dedekind cut for $\sqrt{3}+\sqrt{2}$. (Of course, your answer should not directly refer to irrational numbers such as $\sqrt{3}$ or $\sqrt{2}$.)

3. (12 points) Let (r_n) be a bounded sequence of real numbers from $[1/10, \infty)$ (that is, each entry r_n is $\geq 1/10$), and (s_n) be a sequence of real numbers with $\lim_{n\to\infty} s_n = \infty$. Working directly from the definitions, show that also $\lim_{n\to\infty} r_n \cdot s_n = \infty$.

4. (10 points) Consider the sequence recursively defined by the rule $s_n = 2s_{n-1}/n$ for $n \geq 1$, with the initial value of $s_0 = 1$. Show that s_n converges, and find its limit.

5. (8 points) Using any technique from this class that you like, show that if (s_n) is a Cauchy sequence of positive real numbers, then the sequence $\left(\sqrt{s_n}\right)$ is also Cauchy.