Date Analysis I - hw 1
Subject

A: Prove that for all positive natural numbers n
$$1\cdot 2+2\cdot 3+\ldots+n\cdot(n+1)=n\cdot(n+1)(n+2)/3.$$

Proof: Base step: $n=1$
$$1\cdot(1+1)=1\cdot(1+1)(1+2)/3$$
$$2=2$$
$$LHS=RHS \checkmark$$

Induction step: $n=k$
$$1\cdot 2+2\cdot 3+\ldots+k(k+1)=k(k+1)(k+2)/3$$
now assume that $n=k+1$ is true
$$1\cdot 2+2\cdot 3+\ldots+k(k+1)+(k+1)(k+2)=k(k+1)(k+2)/3+$$
$$+(k+1)(k+2)$$
$$RHS: \frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$$
$$=(k+1)(k+2)\cdot\left[\frac{k}{3}+1\right]$$
$$=\frac{(k+1)(k+2)(k+3)}{3} \Rightarrow n=k+1$$
$$\Rightarrow \frac{n(n+1)(n+2)}{3} \Rightarrow LHS=RHS$$

B: For each of the following subsets of $\mathbb{Z}$, explain whether the subset is well-ordered or not (by the usual ordering on $\mathbb{Z}$.

i) even numbers:

Proof: For a set to be well-ordered, it has to have a smallest integer, which $\mathbb{Z}$ doesn't have. So, we conclude that the subset of even numbers from $\mathbb{Z}$ is not well-ordered

$-\infty$ -2 -1 0 1 2 $+\infty$

ii) perfect squares:

$0^2=0$
$1^2=1=(-1)^2$
$2^2=4=(-2)^2$

Proof: The subset of perfect squares of $\mathbb{Z}$, which has smallest perfect square, 0, or 1 is ~~x~~ well-ordered, because of the condition that a ~~b~~ subset needs to have a least integer to be well-ordered.

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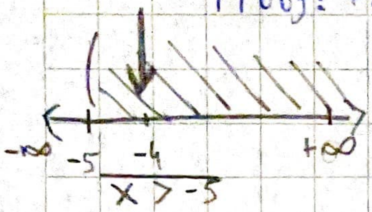
iii) the integers that are strictly greater than -5



Proof: for the subset of strictly greater integers than -5, whose smallest integer is -4, it holds that this subset is well-ordered subset.

C: Using the ordered pairs definition of the integers $\mathbb{Z}$, verify the associativity property for $\mathbb{Z}$. (You may use the properties of the naturals $\mathbb{N}$, including associativity, as stated in the notes.)

Task: prove that $a+(b+c)=(a+b)+c$

Proof: we'll be using ordered pairs, so that

$x_1 = 2$ $\quad y_1 = 0$

$x_2 = 5$ $\quad y_2 = 1$

$x_3 = 7$ $\quad y_3 = 3$

with the definition $(x_1, y_1)+((x_2,y_2)+(x_3,y_3))=((x_1,y_1)+(x_2,y_2))+(x_3,y_3)$

we replace for x_n: $(2,0)+((5,1)+(7,3))=((2,0)+(5,1))+(7,3)$

$=(2,0)+(5+7,1+3)=(2+5,0+1)+(7,3)$

$=(2,0)+(12,4)=(7,1)+(7,3)$

$=(2+12,0+4)=(7+7,1+3)$

$=(14,4)=(14,4)$ ✓ it holds

we proved that $a+(b+c)=(a+b)+c$ is true (holds)

* same holds for multiplication

$(2,0)\cdot((5,1)\cdot(7,3))=((2,0)\cdot(5,1))\cdot(7,3)$

$=(2,0)\cdot(35,3)=(10,0)\cdot\cancel{35}\cdot(7,3)$

$=(70,0)=(70,0)$ ✓

it holds!

Identity proved △

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D: Let G be the set $\{0, \heartsuit\}$. Find a binary operation $\oplus$ so that G is a group. Prove that your operation really yields a group!
(One effjicent way to specify $\oplus$ is to write out an addition table.)

Task: prove that our operation really yields a group

Proof:

i, ii) Let's assume that axioms for closure and associativity under (*) multiplication hold.

iii) Now, we give our operation an ~~element~~ identity, which means that it leaves unchanged every element of the set when the operation is applied.

The identity for multiplication is 1. Let's assume that our identity is described as h. So, this means that $0 \cdot e = 0$, as so $\heartsuit \cdot e = \heartsuit$.

iv) Every element has an inverse element, so when multiplied with itself, it gives us 0.

*	0	♡
0	0	♡
♡	♡	0

i) closure under multiplication ✓
ii) associativity under multiplication ✓
iii) existence of Identity ✓
iv) existence of Inverses ✓

■ It holds that my our operation yields a group!