Univerza na Primorskem
UP FAMNIT
Študijsko leto 2021/2022

Algebra I

Izpit

– 26. Januar 2022 –

Čas pisanja: 135 *minut. Maksimalno število točk:* 100*. Dovoljena je uporaba pisala in kalkulatorja. Pišite razločno in utemeljite vsak odgovor. Srečno!*

1. (a) Kako izračunamo pravokotno projekcijo vektorja $\overrightarrow{u}$ na vektor $\overrightarrow{v}$, z uporabo skalarnega produkta? Dokažite izjavo: Če je $\overrightarrow{w}$ pravokotna projekcija vektorja $\overrightarrow{u}$ na vektor $\overrightarrow{v}$, potem sta vektorja $\overrightarrow{u} - \overrightarrow{w}$ in $\overrightarrow{v}$ pravokotna. *(7 točk)*

   (b) Zapišite karakterizacijo:

      i. Vzporednosti dve premice v $\mathbb{R}^3$ v smislu linearne odvisnosti. *(3 točke)*

      ii. Vzporednosti premice in ravnine v smislu skalarnega produkta. *(3 točke)*

   (c) i. Zapišite definicijo ranga matrike v smislu vrtičnega ešalona (tj. matrike stopničaste oblike). *(3 točke)*

      ii. Dokažite izjavo: Homogeni sistem linearnih enačb $A_{m\times n}X = 0$ ima netrivijalno rešitev če in samo če je $rang(A) < n$. *(4 točke)*

2. V previlnem šestkotniku $ABCDEF$ je točka G razpolovišče stranice EF, točka S pa presečišče premic AC in BG ter velja $|AS| : |SC| = 3 : 4$. Stranica šestkotnika meri 2 enoti. Označimo $\overrightarrow{a} = \overrightarrow{BA}$ in $\overrightarrow{b} = \overrightarrow{BC}$.

   (a) Izrazite vektor $\overrightarrow{SG}$ kot linearno kombinacijo vektorjev $\overrightarrow{a}$ in $\overrightarrow{b}$. *(10 točk)*

   (b) Izračunajte dolžino vektorja $\overrightarrow{AC}$. *(5 točk)*

   (c) Izračunajte kot φ med vektorjema $\overrightarrow{AC}$ in $\overrightarrow{AF}$. *(5 točk)*

3. Naj bosta $\ell : \frac{x}{6} = \frac{y-3}{-2} = -z - 5$ in $p = (1, 7, -4) + \lambda(1, -2, 2)$ premici v prostoru.

   (a) Poiščite vse točke T na premici p, ki so na razdalji 6 od točke $A(1, 7, -4) \in p$. *(10 točk)*

   (b) Zapišite splošno obliko enačbe ravnine Σ, ki vsebuje premico ℓ in je vzporedna s premico p. *(10 točk)*

4. Za katere vrednosti $\lambda \in \mathbb{R}$ ima sistem

$$\begin{aligned} 8z - 3x - 6y &= \lambda x \\ 2x + y + 4z &= \lambda y \\ 4x + 3y + z &= \lambda z \end{aligned}$$

   neskončno mnogo rešitev? Za največjo dobljeno vrednost parametra λ poiščite tudi rešitev sistema. *(20 točk)*

5. Z uporabo elementarnih operacij pokažite da velja

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b),$$

   pri čemer so vrednosti $a, b, c \in \mathbb{R}$ paroma različne. *(20 točk)*

University of Primorska
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Academic year 2021/2022

Algebra I

Exam

– January 26, 2022 –

Time: 135 minutes. Maximum number of points: 100. You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!

1. (a) How do we compute the vector projection of vector $\vec{u}$ on vector $\vec{v}$ using the dot product? Prove the next statement: If $\vec{w}$ is the vector projection of vector $\vec{u}$ on vector $\vec{v}$, then vectors $\vec{u}-\vec{w}$ and $\vec{v}$ are orthogonal. *(7 points)*

   (b) Give the characterisation for:

      i. two parallel lines in $\mathbb{R}^3$ using linear dependence. *(3 points)*

      ii. a line parallel to a plane using the dot product. *(3 points)*

   (c) i. Give the definition of the rank of a matrix that involves the matrix echelon form. *(3 points)*

      ii. Prove the statement: A homogeneous system of linear equations $A_{m\times n}X = 0$ has a nontrivial solution if and only if $rang(A) < n$. *(4 points)*

2. In a regular hexagon $ABCDEF$ point G is the midpoint of line segment EF, point S is the intersection of lines AC and BG, and $|AS| : |SC| = 3 : 4$. The side of the hexagon has length 2. Denote $\vec{a} = \overrightarrow{BA}$ and $\vec{b} = \overrightarrow{BC}$.

   (a) Write the vector $\overrightarrow{SG}$ as a linear combination of vectors $\vec{a}$ and $\vec{b}$. *(10 points)*

   (b) Compute the length of the vector $\overrightarrow{AC}$. *(5 points)*

   (c) Compute the angle φ between vectors $\overrightarrow{AC}$ and $\overrightarrow{AF}$. *(5 points)*

3. Let $\ell : \frac{x}{6} = \frac{y-3}{-2} = -z-5$ and $p = (1,7,-4) + \lambda(1,-2,2)$ be two lines in space.

   (a) Find all points T in the line p, that are at distance 6 from the point $A(1,7,-4) \in p$. *(10 points)*

   (b) Write the general form equation of the plane Σ, that contains line ℓ and is parallel to line p. *(10 points)*

4. For which values $\lambda \in \mathbb{R}$ has the system

$$8z - 3x - 6y = \lambda x$$
$$2x + y + 4z = \lambda y$$
$$4x + 3y + z = \lambda z$$

   infinitely many solutions? Find the solution of the system for the greatest obtained value λ. *(20 points)*

5. Using elementary row operations show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b),$$

   where $a,b,c \in \mathbb{R}$ are all different. *(20 points)*