Week 11

Consistency

Each of the following is equivalent to saying $[A|b]$ is consistent.

- In row reducing, a row of the following form never appears: $(0\ 0 \cdots 0 \mid \alpha)$ where $\alpha \neq 0$
- b is a nonbasic column in $[A|b]$
- $\text{rank}[A|b] = \text{rank}(A)$
- b is a combination of the basic columns in A

① Determine if the following system is consistent and solve it.

$$\begin{aligned}
x_1 + x_2 + 2x_3 + 2x_4 + x_5 &= 1\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 3x_5 &= 1\\
2x_1 + 2x_2 + 4x_3 + 4x_4 + 2x_5 &= 2\\
3x_1 + 5x_2 + 8x_3 + 6x_4 + 5x_5 &= 3
\end{aligned}$$

$$[A|b] = \left[\begin{array}{ccccc|c} 1&1&2&2&1&1\\ 2&2&4&4&3&1\\ 2&2&4&4&2&2\\ 3&5&8&6&5&3 \end{array}\right] \xrightarrow[\substack{R_3-2R_1\\ R_4-3R_1}]{R_2-2R_1} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1\\ 0&0&0&0&1&-1\\ 0&0&0&0&0&0\\ 0&2&2&0&2&0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1\\ 0&2&2&0&2&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&-1 \end{array}\right]$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{R_2/2} \left[\begin{array}{ccccc|c} 1&1&2&2&1&1\\ 0&1&1&0&1&0\\ 0&0&0&0&1&-1\\ 0&0&0&0&0&0 \end{array}\right]$$

$\text{rank}[A|b] = \text{rank}(A)$

So system is consistent! :)

Basic columns / basic variables are x_1, x_2, x_5

$\Rightarrow$ free variables are $x_3 = t \in \mathbb{R}$, $x_4 = s \in \mathbb{R}$

$\boxed{x_5 = -1}$

$x_2 + t - 1 = 0 \Rightarrow \boxed{x_2 = 1 - t}$

$x_1 + (1-t) + 2t + 2s - 1 = 1$

$x_1 + t + 2s = 1$

$\boxed{x_1 = 1 - t - 2s}$

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 1-t-2s\\ 1-t\\ t\\ s\\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\\ 1\\ 0\\ 0\\ -1 \end{bmatrix} + \begin{bmatrix} -1\\ -1\\ 1\\ 0\\ 0 \end{bmatrix} t + \begin{bmatrix} -2\\ 0\\ 0\\ 1\\ 0 \end{bmatrix} s$$

④ If A is an $m\times n$ matrix with $\text{rank}(A) = m$, explain why the system $[A|b]$ must be consistent for every right-hand side b.

$m = \text{rank}(A) \Rightarrow m \le n$ as $\text{rank}(A) \le \min(m,n)$.

As $[A|b]$ has $n+1$ columns and $n+1 > m$, the $\text{rank}[A|b] \le m$ (i.e. adding a column does not increase the rank)

Moreover, as $\text{rank}(A) = m$, $\text{rank}[A|b] \ge m$

So $\text{rank}[A|b] = m$, hence $[A|b]$ is always consistent

⑤ Consider two consistent systems whose augmented matrices are of the form $[A|b]$ and $[A|c]$. That is, they differ only on the right-hand side. Is the system associated with $[A\,|\,b+c]$ also consistent? Explain why or why not.

$[A|b]$ consistent $\Rightarrow$ any row of the form $(0\ 0 \cdots 0 \mid b_i)$ has $b_i = 0$

$[A|c]$ consistent $\Rightarrow$ any row of the form $(0\ 0 \cdots 0 \mid c_j)$ has $c_j = 0$

So, any row in $[A\,|\,b+c]$ with $(0\,0 \cdots 0 \mid b_k + c_k)$ has $b_k = 0$, $c_k = 0$ so $b_k + c_k = 0$

So, $[A\,|\,b+c]$ is consistent !

⑥ Is it possible for a parabola whose equation has the form $y = \alpha + \beta x + \gamma x^2$ to pass through the four points $(0,1)$, $(1,3)$, $(2,15)$, $(3,37)$? Why?

$(0,1) \rightarrow \alpha = 1$
$(1,3) \rightarrow \alpha + \beta + \gamma = 3$
$(2,15) \rightarrow \alpha + 2\beta + 4\gamma = 15$
$(3,37) \rightarrow \alpha + 3\beta + 9\gamma = 37$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 37 \end{array}\right] \xrightarrow[R_3 - R_1,\ R_4 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 4 & 14 \\ 0 & 3 & 9 & 36 \end{array}\right]$$

$$\xrightarrow[R_4 - 3R_2]{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 6 & 30 \end{array}\right] \xrightarrow[R_4/3]{R_3/2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 2 & 10 \end{array}\right] \xrightarrow{R_4 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent so it is possible ! :)

⑦ Suppose that an augmented matrix $[A|b]$ is reduced by means of Gaussian elimination to a row echelon form $[E|c]$. If a row of the form

$$(0\ 0\ \cdots\ 0\ |\ \alpha), \quad \alpha \neq 0$$

does not appear in $[E|c]$, is it possible that rows of this form could have appeared at earlier stages of the reduction process? Why or why not?

Suppose we have rows $R_1, \dots, R_k$ of the form $(0\ 0\ \cdots\ 0\ 1\ \alpha_i)$ for $1 \le i \le k$, $\alpha_i \neq 0$ during the reduction process.

We could reduce R_i to $(0\ 0\ \cdots\ 0\ |\ 0)$ by

$$R_i - \frac{\alpha_i}{\alpha_1} R_1 \quad \text{for all} \quad 1 < i \le k$$

but then $R_1 = (0\ 0\ \cdots\ 0\ |\ \alpha_1)$, $\alpha_1 \neq 0$ remains. As $[E|c]$ is consistent and there is no means to reduce R_1 to $(0\ 0\ \cdots\ 0\ |\ 0)$, then the rows $R_1, \dots, R_k$ could not exist in A, unless $\alpha_i = 0$.

Short answer: $[E|c]$ row echelon and no row of form $(0\ 0 \cdots 0 | \alpha)$ $\Rightarrow$ b is a nonbasic column in $[A|b]$ $\Rightarrow$ consistency $\Rightarrow$ such a row never appears, by definition

Homogenous systems

A system of m linear equations with n unknowns in which the right-hand side consists entirely of 0's is said to be a homogenous system.

* Consistency is never an issue with homogenous systems as $x_1 = x_2 = \cdots = x_n = 0$ is always a solution. This is called the trivial solution.

① Explain why the following homogenous ~~Taks~~ system has infinitely many solutions and exhibit the general solution.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0 \end{aligned}$$

Short answer: consistent system w/ more unknowns than equations (i.e. $y = mx + b$ has inf. many solutions)

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 1 & 4 & 0 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 0 & -5 & -5 & 0 \end{array}\right] \xrightarrow[R_3/(-5)]{R_2/(-3)} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivots for x_1 & x_3 $\Rightarrow$ x_2 & x_4 free variables

$x_2 = t \in \mathbb{R}$, $x_4 = s \in \mathbb{R}$

$x_3 + s = 0 \Rightarrow \boxed{x_3 = -s}$

$x_1 + 2t + 2(-s) + 3s = 0$

$x_1 + 2t + s = 0$

$\boxed{x_1 = -2t - s}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t-s \\ t \\ -s \\ s \end{bmatrix}}$$

② Explain why the following homogenous system has only the trivial solution

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + 7x_3 &= 0 \\ 3x_1 + 6x_2 + 8x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 5 & 7 & 0 \\ 3 & 6 & 8 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \Rightarrow 2x_3 = 0 \Rightarrow \boxed{x_3 = 0}$$

$x_2 + 3x_3 = 0 \Rightarrow \boxed{x_2 = 0}$

$x_1 + 2x_2 + 2x_3 = 0 \Rightarrow \boxed{x_1 = 0}$

③ Explain why the following homogenous system has infinitely many solutions and exhibit the general solution

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + 7x_3 &= 0 \\ 3x_1 + 6x_2 + 6x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 5 & 7 & 0 \\ 3 & 6 & 6 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$, as x_3 is not a pivot x_3 is free :

Let $\boxed{x_3 = t}$

$x_2 + 3x_3 = 0 \Rightarrow \boxed{x_2 = -3t}$

$x_1 + 2x_2 + 2x_3 = 0$
$x_1 - 6t + 2t = 0 \Rightarrow \boxed{x_1 = 4t}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \cdot \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}$$

④ Determine the general solution for each of the following homogenous systems

(a)
$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 2x + y + z &= 0 \\ 4x + 2y + z &= 0 \\ 6x + 3y + z &= 0 \\ 8x + 4y + z &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_3 + 3x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 - x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 - x_4 &= 0 \end{aligned}$$

(d)
$$\begin{aligned} 2x + y + z &= 0 \\ 4x + 2y + z &= 0 \\ 6x + 3y + z &= 0 \\ 8x + 5y + z &= 0 \end{aligned}$$

(a)
$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 1 & 4 & 0 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -2 & -2 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4 free : $\boxed{x_2 = t}$, $\boxed{x_4 = s}$

$R_3 \Rightarrow -x_3 - x_4 = 0$

$-x_3 - s = 0 \Rightarrow \boxed{x_3 = s}$

$R_1 \Rightarrow x_1 + 2x_2 + x_3 + 2x_4 = 0$
$x_1 + 2t + 3s = 0$
$\Rightarrow \boxed{x_1 = -3s - 2t}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

5. Among all the solutions that satisfy the homogenous system

$$\begin{aligned} x + 2y + z &= 0 \\ 2x + 4y + z &= 0 \\ x + 2y - z &= 0 \end{aligned}$$

determine those that also satisfy the nonlinear constraint

$$y - xy = 2z.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \Rightarrow \boxed{z = 0}, \quad \boxed{y = t} \in \mathbb{R} \text{ free !}$$

$$\begin{aligned} x + 2y + z &= 0 \\ x + 2t + 0 &= 0 \end{aligned}$$

$$\Rightarrow \boxed{x = -2t}$$

$$y - xy = 2z$$

$$t - (-2t) \cdot t = 0 \Rightarrow t + 2t^2 = 0$$

$$t(1 + 2t) = 0 \Rightarrow t = 0 \text{ OR } (1 + 2t) = 0$$

- $t = 0$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

- $1 + 2t = 0 \Rightarrow t = -1/2$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}}$$