Homework 6:

(A.) Show that any convergent sequence of real numbers is bounded:

Since $\lim_{n\to\infty} z_n = z$ we can take $\varepsilon = 1$ and we can find an N s.t. $|z_n - z| < 1$

Now by Δ inequality

$$|z_n| = |z_n + z - z| \leq |z_n - z| + |z| < 1 + |z|$$

$$M = \max\{1+|z|, |z_1|, |z_2| \dots |z_N|\}$$

$\Rightarrow |z_n| \leq M$, for all values of M defined.

(B) For a sequence of complex numbers (z_n), show that $\lim_{n\to\infty} z_n = 0$ iff $\lim_{n\to\infty} |z_n| = 0$.

$$\lim_{n\to\infty} z_n = 0 \iff \lim_{n\to\infty} |z_n| = 0$$

($\Rightarrow$) Let $\lim_{n\to\infty} z_n = 0$, then by definition we have

$\forall \varepsilon > 0 \ \exists N$ s.t $[n > N] \Rightarrow [|z_n - 0| < \varepsilon]$

$|z_n| < \varepsilon$

$\Rightarrow |z_n| \to 0$

$\therefore \lim_{n\to\infty} |z_n| = 0$

($\Leftarrow$) Let $\lim_{n\to\infty} |z_n| = 0$, then by definition we have

$\forall \varepsilon > 0 \ \exists N$ s.t. $[n > N] \Rightarrow [|\sqrt{a^2 + (bi)^2} - 0| < \varepsilon] \Rightarrow$ $\boxtimes$

$\Rightarrow [|\sqrt{a^2 + (bi)^2}| < \varepsilon] \Rightarrow [\sqrt{a^2 + (bi)^2} < \varepsilon] \Rightarrow z_n < \varepsilon$

ⓒ Show that if (s_n, t_n) converges to (s,t) in $\mathbb{R}^2$ with the usual d_2 metric, then the sequence converges to the same limit in the d_0 ("max") metric.

$(\mathbb{R}^2, d_2) = (s_n, t_n)$ converges

1. $d(s_n, s_n) = 0$
2. $d(s_n, t_n) \neq d(t_n, s_n)$
3. $\sqrt{(s_n-t_n)^2 + (s-t)^2} = \sqrt{s_n^2 + 2s_nt_n + t_n^2 + s^2 - 2st + t^2}$

$$\leq \sqrt{s_n^2+t_n^2} - \sqrt{2s_nt_n - 2st} + \sqrt{s^2+t^2}$$

$$|s_n - t_n| - \sqrt{2s_nt_n + 2st} + |s-t|$$

Since the sequence converges in d_2 we have:

$$|s_n - t_n| - \sqrt{2s_nt_n + 2st} + |s-t| < \varepsilon$$

$$|s_n - t_n| + |s-t| < \varepsilon + \sqrt{2s_nt_n + 2st}$$

Notice that this is just the max possible value of the distances between the components $d_0 = \{|s_n - t_n|, |s-t|\}$, so we have proven that if (s_n, t_n) holds in d_2 it also holds in d_0

~~(D) Let~~

(D) Construct a sepuence having $\{0, 1, 3, 6, 20\}$ as accumulation points.

Such is the sepuence:

$0, 1, 3, 6, 20, 0, 1, 3, 6, 20, 0, 1, 3, 6, 20$

Where:

$a_1 = 0 \rightarrow n = K$

$a_2 = 1 \rightarrow n = 2K$

$a_3 = 3 \rightarrow n = 3K$

$a_4 = 6 \rightarrow n = 4K$

$a_5 = 20 \rightarrow n = 5K$

(E) Let S_n be a (bounded) sepuence of real numbers:

i) Show that for any subsepuence S_{n_k} of S_n, we have $\limsup S_{n_k} \leq \limsup S_n$. Conclude that if a is an accumulation point of S_n $\boxed{\text{it is an accumulation point of } S_{n_k}}$ then $a \leq \limsup S_n$.

ii) Show that for any ε, there are infinicely many values of S_n whitin ε of $\limsup S_n$. Conclude that $\limsup S_n$ is an accumulation point of S_n.

(Hint: One way to proceed is to consider seperately the case where there are ∞ many points at least ε greater than the $\limsup S_n$, and the case where all but finitely many points are at least smaller than the $\limsup$.)