Homework 1

Ⓐ Prove that for all positive natural numbers n,

$$1\cdot 2+2\cdot 3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

base case

$n=0$

$$0\cdot 1=\frac{0\cdot 1\cdot 2}{3}$$

$$0=\frac{0}{3}$$

$$0=0$$

$n=1$

$$1\cdot 2=\frac{1\cdot 2\cdot 3}{3}$$

$$2=\frac{6}{3}$$

$$2=2$$

Inductive hypothesis

$n=k$

$$1\cdot 2+2\cdot 3+\ldots+k(k+1)=\frac{k(k+1)(k+2)}{3}$$

$n=k+1$

$$\underbrace{1\cdot 2+2\cdot 3+\ldots+k(k+1)}+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$$

$$\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$$

$$\frac{k(k+1)(k+2)}{3}+\frac{3(k+1)(k+2)}{3}=\frac{(k+1)(k+2)(k+3)}{3}$$

$$\frac{(k+1)(k+2)(k+3)}{3}=\frac{(k+1)(k+2)(k+3)}{3}$$

By proof of Mathematical Induction we conclude that $P_n \Rightarrow P_{n+1}$ and therefore the statement $1\cdot 2+2\cdot 3+\ldots n(n+1)=\frac{n(n+1)(n+2)}{3}$ is T.

B) For any natural number n, let P_n be the statement n^3+n+1 which is even.

i) Show that for every natural number $n \in \mathbb{N}$, we have that $P_n \Rightarrow P_{n+1}$

ii) Show that P_n is always false.

i) We assume that $P_n = n^3+n+1 = 2k$

$(n+1)^3+(n+1)+1 =$

$n^3+3n^2+3n+1+n+2 =$

$\underbrace{(n^3+n+1)}_{\text{first statement assume it's true}} + \underbrace{3n(n+1)}_{\text{always even}} + 2$

ii) P_n is always false

$n=2n+1 \Leftarrow n^3+n+1=2k \Rightarrow n=2n$

$(2n+1)^3+(2n+1)+1 =$

$= 8n^3+12n^2+6n+1+2n+2 =$

$= 8n^3+12n^2+8n+3$

odd (because of 3)

$8n+2n+1$

$10n+1 \to$ odd

base case

$n=1$

$1^3+1+1=3$

3 is an odd number

C) Using the ordered pairs definition of the integers $\mathbb{Z}$, verify the distributive property for $\mathbb{Z}$.

Property of $\mathbb{N}$

$l, m, n \in \mathbb{N}$

$l(m+n) = lm+ln$

$x_1, x_2, x_3 \in \mathbb{Z}$

~~$x_1 = (m_1 - n_1)$~~

$x_1 = (m_1, n_1) = m_1 - n_1 \quad n_1, m_1 \in \mathbb{N}$

$x_2 = (m_2, n_2) = m_2 - n_2 \quad n_2, m_2 \in \mathbb{N}$

$x_3 = (m_3, n_3) = m_3 - n_3 \quad n_3, m_3 \in \mathbb{N}$

Therefore we have

$[(m_1,n_1)+(m_2,n_2)] \cdot (m_3,n_3) =$

$= (m_1,n_1)\cdot(m_3,n_3) + (m_2,n_2)(m_3,n_3) =$

$= x_1 \cdot x_3 + x_2 \cdot x_3 =$

$= (x_1+x_2)\cdot x_3$

Distributive property in $\mathbb{Z}$.

Sidenote B

Mathematical induction consists of two steps base case and induction hypothesis. Here if we go by the hypothesis with the assumption that the first statement is true we prove the hypothesis to be true. But the base case is that which shows us that the first statement is indeed false.

(D) We extend $\mathbb{Q}^{\geq 0}$ to $\mathbb{Q}$ by another application of the method of ordered pairs, exactly like extending from $\mathbb{N}$ to $\mathbb{Z}$. Explain how to construct the standard ordering $\leq$ of the rational numbers in terms of the coordinates in these ordered pairs.

$n, m \in \mathbb{N}$

$n_{1/2}, m_{1/2}, p_{1/2}, q_{1/2} \in \mathbb{N}$

$\underline{\mathbb{Q}^{\geq 0}}$

$(n,m) = \frac{n}{m}$

$\underline{\mathbb{Q}^{\geq 0} \Rightarrow \mathbb{Q}}$

$((n_1, m_1), (n_2, m_2)) = \frac{n_1}{m_1} - \frac{n_2}{m_2}$

$((p_1, q_1), (p_2, q_2)) = \frac{p_1}{q_1} - \frac{p_2}{q_2}$

$$((n_1, m_1), (n_2, m_2)) \leq ((p_1, q_1), (p_2, q_2))$$

$$\frac{n_1}{m_1} - \frac{n_2}{m_2} \leq \frac{p_1}{q_1} - \frac{p_2}{q_2}$$

$$\boxed{\frac{n_1}{m_1} + \frac{p_2}{q_2} \leq \frac{p_1}{q_1} + \frac{n_2}{m_2}}$$

(E) Let G be the set $\{0, ♡\}$. Find a binary operation $\oplus$ so that G is a group. Prove that your operation really yields a group. (Can use an addition table!)

$\boxed{\text{Operation: "+"}}$

Properties of a group

1. closed for $\oplus$
2. $\oplus$ is associative
3. has an identity for $\oplus$
4. every element has an inverse under $\oplus$

2. $0+(0+♡) = (0+0)+♡$

$♡+(0+♡) = (♡+0)+♡$

+	0	♡
0	0	♡
♡	♡	0

4. $♡+♡ = 0$, which proves the inverse under "+"

$0+0=0$

! 1, 3, 4 can be proved from the table! (seen)

1 Is closed for "+" because all results are in G.

3 Additive identity is 0, because adding 0 gives back the same element.