Univerza na Primorskem
UP FAMNIT
Študijsko leto 2021/2022

Algebra I

Izpit

– 18. Avgust 2022 –

Čas pisanja: 135 *minut. Maksimalno število točk:* 100. *Dovoljena je uporaba pisala in kalkulatorja. Pišite razločno in utemeljite vsak odgovor. Srečno!*

1. (a) Zapišite definicijo skalarnega produkta. Dokažite, da za vektorja $\vec{u}, \vec{v} \in \mathbb{R}^3$, ki sta pravokotna, velja $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$. (Namig: Uporabite dejstvo, da je $|\vec{u}|^2 = \langle u, u \rangle$). *(5 točk)*

   (b) Zapišite definicijo vektorskega produkta. Kako ga izračunamo z uporabo determinante? *(5 točk)*

   (c) Dokažite naslednjo trditev: Za poljubne vektorje $\vec{u}, \vec{v}$ in $\vec{w}$ velja, da je $|\langle u \times v, w \rangle|$ prostornina paralelepipeda, ki ga določajo vektorji $\vec{u}, \vec{v}$ in $\vec{w}$. *(5 točk)*

   (d) Zapišite enačbo ravnine v vektorski, splošni in parametrično vektorski obliki. Izpeljite enačbo ravnine skozi 3 točke z uporabo determinante. *(5 točk)*

2. Naj bodo A, B, C in D zaporedna oglišča paralelograma. Točka E leži na diagonali AC tako, da velja $|AE| : |EC| = 1 : 3$. Točka F leži na diagonali BD tako, da velja $|BD| : |BF| = 4 : 3$. Označimo s S presečišče daljic AF in ED.

   (a) Zapišite vektor $\overrightarrow{AS}$ kot linearno kombinacijo vektorjev $\vec{e} = \overrightarrow{AC}$ in $\vec{f} = \overrightarrow{BD}$. *(8 točk)*
   Hint: $|AS| = \frac{2}{3}|AF|$.

   (b) Če so $A(1, -2, 2)$, $B(3, -1, 4)$ in $C(2, 3, -3)$, poiščite kot med stranicama AB in BC ter določite ploščino paralelograma. *(12 točk)*
   Namig: Kot med dvema vektorjema lahko izračunamo s pomočjo enačbe $\cos\varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

3. Naj bo U ravnina z enačbo $x - y + 2z = 0$.

   (a) Zapišite enačbo premice ℓ, ki vsebuje točko $T(4, 0, 4)$ in je pravokotna na ravnino U. *(3 točke)*

   (b) Poiščite presečišče premice ℓ in ravnine U. *(5 točk)*

   (c) Določite koordinate točke A, ki leži na premici ℓ in je enako oddaljena od točke T in ravnine U. Kolikšna je ta razdalja? *(12 točk)*

4. Pokažite, da sistem linearnih enačb

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

   nima rešitve, razen če je $a + c = 2b$. V tem primeru rešitev tudi poiščite. *(20 točk)*

5. Za katere vrednosti $x \in \mathbb{R}$, bo naslednja matrika

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 2 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 2 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

   obrnljiva? *(20 točk)*

University of Primorska
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Academic year 2021/2022

Algebra I

Exam

– August 18, 2022 –

Time: 135 *minutes. Maximum number of points:* 100. *You are allowed to use a pen and a calculator. Write clearly, and justify all your answers. Good luck!*

1. (a) Write the definition of the scalar (dot) product. In addition, for two orthogonal vectors $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{R}^3$, prove that $|\overrightarrow{u} + \overrightarrow{v}|^2 = |\overrightarrow{u}|^2 + |\overrightarrow{v}|^2$. (Hint: Use the fact that $|\overrightarrow{u}|^2 = \langle u, u \rangle$). *(5 points)*

   (b) Write the definition of the cross (vector) product. How do we compute it in terms of determinants? *(5 points)*

   (c) Prove the following statement: For any vectors $\overrightarrow{u}, \overrightarrow{v}$ and $\overrightarrow{w}$, it holds that $|\langle u \times v, w \rangle|$ is the volume of the parallelepiped determined by $\overrightarrow{u}, \overrightarrow{v}$ and $\overrightarrow{w}$. *(5 points)*

   (d) Write the equation of a plane in vectorial, general and parametric vectorial form. Then, derive the equation of a plane through 3 non-collinear points in terms of determinants. *(5 points)*

2. Let A, B, C and D be consecutive points of a parallelogram. Point E divides the diagonal AC so that $|AE| : |EC| = 1 : 3$. Point F divides the diagonal BD so that $|BD| : |BF| = 4 : 3$. Let S be the point of intersection of line segments AF and ED.

   (a) Write the vector $\overrightarrow{AS}$ as a linear combination of vectors $\overrightarrow{e} = \overrightarrow{AC}$ and $\overrightarrow{f} = \overrightarrow{BD}$. *(8 points)*
   Hint: $|AS| = \frac{2}{3}|AF|$.

   (b) If $A(1, -2, 2)$, $B(3, -1, 4)$ and $C(2, 3, -3)$, find the angle between line segments AB and BC and determine the area of the parallelogram. *(12 points)*
   Hint: The angle between two vectors can be obtained from the equation $\cos\varphi = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|}$.

3. Let U be the plane defined by $x - y + 2z = 0$.

   (a) Find the line ℓ, that contains point $T(4, 0, 4)$ and is perpendicular to the plane U. *(3 points)*

   (b) Find the coordinates of the intersection of the line ℓ and the plane U. *(5 points)*

   (c) Determine the coordinates of the point A, that lies on the line ℓ and is equidistant from T and U. Also, determine this distance. *(12 points)*

4. Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

   does not have a solution unless $a + c = 2b$. In that case, find the solution of the system. *(20 points)*

5. For which values of $x \in \mathbb{R}$, will the matrix M given by

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & 1 & 2 & 1 \\ 1 & 1 & x & 1 & 1 \\ 1 & 2 & 1 & x & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

   be invertible? *(20 points)*