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THE DIPOLE DYNAMICAL SYSTEM

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Abstract. A dynamical system governing the collective interaction of N point-vortex dipoles is derived. Each dipole has an inherent orientation ψ and generates a velocity field that decays like $O(\mu/2\pi r^2)$ where μ is the dipole strength and r is the distance from the dipole. The system of N-complex ordinary differential equations plus N-real ordinary differential equations for the dipole positions and orientations are derived based on the assumption that each dipole moves with and tries to align itself with the local fluid velocity field.

1. **Introduction.** Models for two-dimensional, inviscid incompressible fluid flows based on point-vortex (monopole) decompositions of the Euler equations have played an important role in our understanding of the dynamical processes dominating a wide variety of flows, ranging from bluff body wakes [1], atmospheric and oceanographic flows [9, 10, 13, 14, 16], and two-dimenional turbulence [15, 19]. There is evidence, however, that in certain regimes where transient or non-equilibrium effects are important, physical structures such as dipoles, tripoles, and higher order multipoles can transport fluid mass and momentum in ways that individual monopoles cannot [2, 4, 5, 6, 18]. These structures can leave their imprint on long-time properties of the flow, such as the turbulent spectrum [15]. Because of this, several recent papers have attempted to explicitly use dipoles in formulating kinetic theory models for non-equilibrium processes [17, 20]. Theories for vorticity based two-dimensional turbulence [4, 6] frequently make reference to multipole structures [21]. In a separate exciting area of the modeling of Bose-Einstein condensates, frequent mention is made of the prevalence of dipole interactions, for example, in two-component BECs with dipole bound-states [11, 3, 12]. In all of these systems, it is known that finite-sized dipoles, in a sea of other dipoles, quickly lose their identity as the distances separating the individual point-vortices making up the dipoles are not conserved [19]. See, for example [7] which develops a 'scattering theory' for the collision of two finite-sized dipoles in detail. *Infinitesimal* dipoles or higher order multipoles do not suffer from this drawback as they are point particles that retain their identity upon interaction. It seems surprising that the basic system for interacting multipoles has not been derived, hence the goal of this paper is to derive such a system which governs the interaction of N-dipole particles in the unbounded plane. Our approach follows the simplest route towards the derivation of

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the N-vortex system of point vortices [19] in the plane. We formulate the dynamical equations for the N-dipoles of general strength $\mu_\alpha \in R$ in the complex plane at positions $z_\alpha(t) = x_\alpha(t) + iy_\alpha(t)$, $\alpha = 1, ..., N$ using two assumptions:

(i) Each dipole is advected with the local fluid velocity generated by the other $N-1$ dipoles;

(ii) Each dipole tries to align itself with the local velocity field generated by all the others.

In section 2 we derive the velocity field associated with a vortex-dipole as a limiting process $\epsilon \to 0$ from a finite-sized dipole made up of two equal and opposite strength point vortices $(\Gamma, -\Gamma)$ located a distance ϵ apart, having orientation ψ with respect to the horizontal. The point-dipole used as the basic ingredient in our model arises from passing to the limit $\epsilon = 0$ under the scaling $\Gamma \sim \mu/\epsilon$. This limit process leaves a dipole of strength μ and orientation angle ψ. In section 3 we derive the dynamical system governing the interaction of N dipoles under assumption (i) above, without the inclusion of the self-induced velocity. In this system, the dipole orientations retain their fixed initial values. Section 4 describes the dynamical system of N equations for the orientation angles ψ_β under assumption (ii) above. In section 5 we desingularize the system using a cut-off parameter $\delta > 0$, so that each dipole has a finite self-induced velocity. The dipole system developed in this paper is part of a more comprehensive approach to modeling complex flows using interacting multipoles such as monopoles, dipoles, and tripoles, as ballistic elements in order to more accurately capture nonequilibirum effects in two-dimensional turbulent flows.

2. **The vortex-dipole field.** Consider a pair of equal and opposite strength point vortices placed a distance ϵ apart as shown in figure 1. At position $z_1 = \frac{\epsilon}{2}\exp(i(\psi - \frac{\pi}{2}))$ in the complex plane we place a point vortex of strength $-\Gamma < 0$ (clockwise). At position $z_2 = \frac{\epsilon}{2}\exp(i(\psi + \frac{\pi}{2}))$ we place a point vortex of strength $\Gamma > 0$ (counterclockwise). We know that ϵ is constant [19] and that the finite-sized dipole moves along a ray $\theta = \psi$ perpendicular to the line connecting the two vortices, with constant velocity $v_\epsilon = \Gamma/2\pi\epsilon$. The complex potential for the pair, $F_\epsilon(z)$, is a linear combination of the complex potentials for each, $F_1(z)$ and $F_2(z)$, where

$$F_1(z) = \frac{i\Gamma}{2\pi}\log(z - z_1), \tag{1}$$

$$F_2(z) = -\frac{i\Gamma}{2\pi}\log(z - z_2). \tag{2}$$

Thus,

$$\begin{aligned} F_\epsilon(z) = F_1(z) + F_2(z) &= \frac{i\Gamma}{2\pi}\log(z - z_1) - \frac{i\Gamma}{2\pi}\log(z - z_2) \\ &= \frac{i\Gamma}{2\pi}\log\left(\frac{z - z_1}{z - z_2}\right) = \frac{i\Gamma}{2\pi}\log\left(\frac{1 - z_1/z}{1 - z_2/z}\right). \end{aligned} \tag{3}$$

Then

$$\begin{aligned} F_\epsilon(z) &= \frac{i\Gamma}{2\pi}\log\left[(1 - z_1/z)(1 - z_2/z)^{-1}\right] \\ &= \frac{i\Gamma}{2\pi}\log\left[(1 - \frac{\epsilon}{2z}\exp(i(\psi - \pi/2)))(1 - \frac{\epsilon}{2z}\exp(i(\psi + \pi/2)))^{-1}\right]. \end{aligned} \tag{4}$$

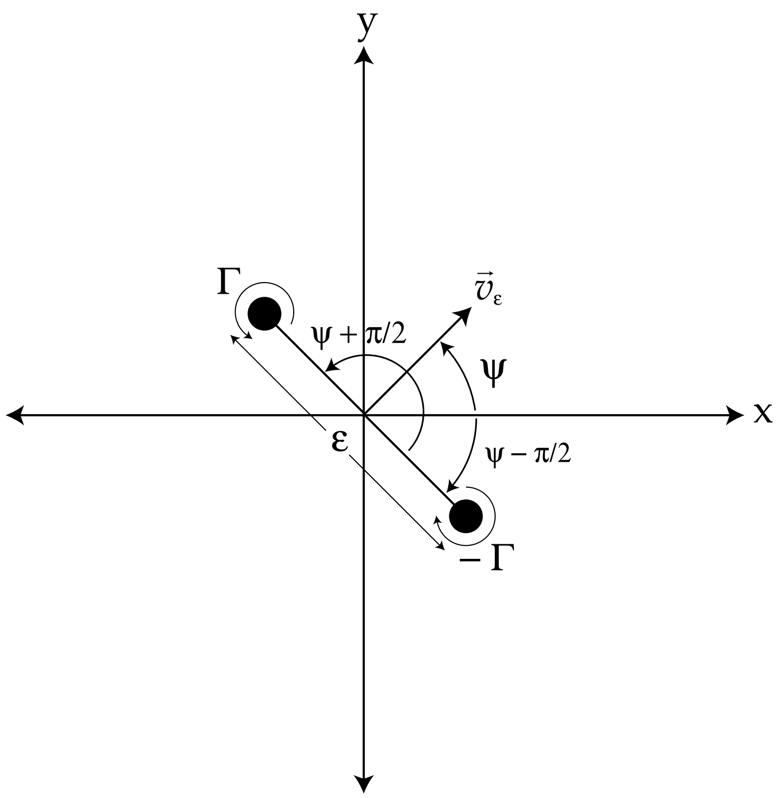

FIGURE 1. Finite vortex dipole based on equal and opposite strength point vortices Γ and $-\Gamma$, placed a distance ϵ apart, with orientation angle ψ based on the direction of motion with respect to the x-axis.

When expanded for small ϵ (fixed z), using the binomial expansion and $\log(1+z) \sim z - O(z^2)$, we obtain at leading order

$$F_\epsilon(z) \sim -\frac{\Gamma\epsilon}{2\pi z}\exp(i\psi). \tag{5}$$

Then, taking the limit $\epsilon \to 0$, and scaling $\Gamma \sim \mu/\epsilon$ gives

$$lim_{\epsilon\to 0}F_\epsilon(z) \equiv F_0(z) = -\frac{\mu}{2\pi z}\exp(i\psi). \tag{6}$$

$F_0(z)$ is the complex potential for a point-vortex dipole located at $z = 0$, having strength μ, and orientation ψ as shown in figure 2. The complex velocity is given by

$$\frac{dF_0}{dz} = u - iv = \dot{z}^* = \frac{\mu}{2\pi z^2}\exp(i\psi). \tag{7}$$

It is useful to express this in polar coordinates where $z = r\exp(i\theta)$

$$\dot{r} = \frac{\mu}{2\pi r^2}\cos(\psi - \theta) \tag{8}$$

$$r\dot{\theta} = \frac{\mu}{2\pi r^2}\sin(\psi - \theta). \tag{9}$$

If we locate the dipole at an arbitrary point $z = z_\beta$ and allow it to have strength μ_β, with orientation ψ_β, the complex velocity becomes

$$\dot{z}^* = \frac{dF_0}{dz} = \frac{\mu_\beta}{2\pi(z - z_\beta)^2}\exp(i\psi_\beta). \tag{10}$$

The streamline pattern generated by the point-vortex dipole is shown in figure 2.

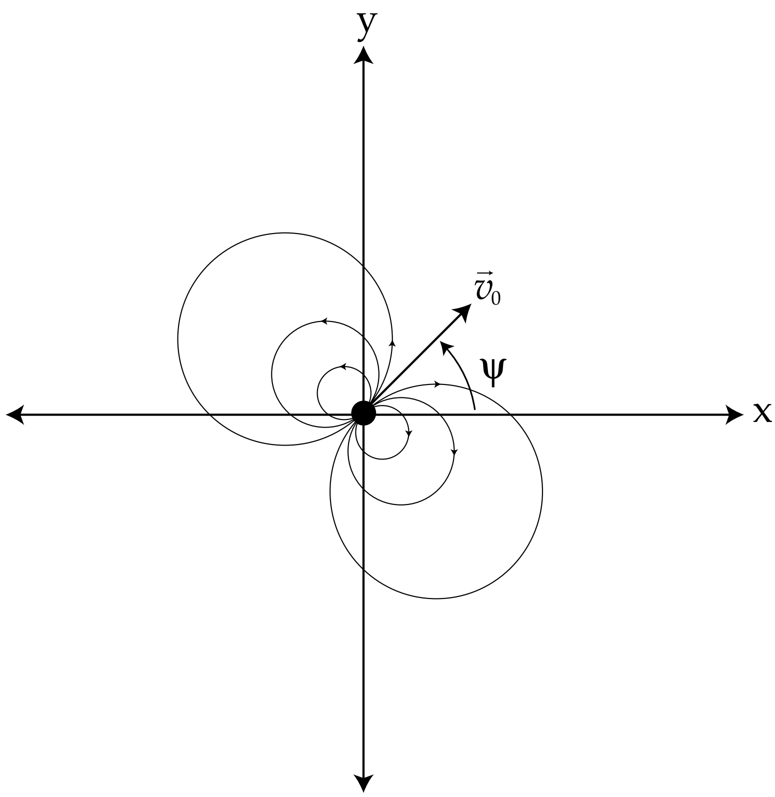

FIGURE 2. Streamline pattern for a point-dipole arising from the finite point-vortex dipole in limit $\epsilon \to 0$, $\Gamma = \mu/\epsilon$.

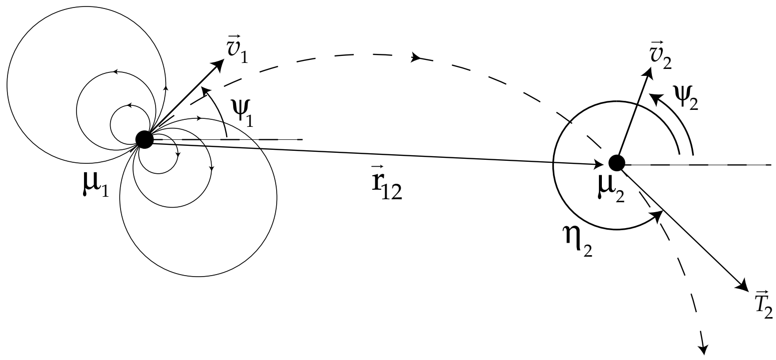

FIGURE 3. Dipole interaction diagram. Two dipoles of strengths μ_1 and μ_2 and orientations ψ_1 and ψ_2 are separated instantaneously by distance $\vec{r}_{12}$. $\vec{T}_2$ is the tangent vector at z_2 associated with the velocity field generated by μ_1. Dipole μ_2 tries to align itself with the local velocity field, hence $\psi_2 \rightarrow \eta_2$.

3. **Dipole interaction model without self-induced velocity.** At location $z = z_\alpha$, the local complex velocity $\dot{z}^* = \dot{z}^*_\alpha$ is given by

$$\dot{z}^*_\alpha = \frac{dF_0}{dz_\alpha} = \frac{\mu_\beta}{2\pi(z_\alpha - z_\beta)^2}\exp(i\psi_\beta). \tag{11}$$

Hence, each dipole generates an infinite self-induced velocity, i.e.

$$\frac{dF_0}{dz_\alpha}|_{z_\beta = z_\alpha} = \infty. \tag{12}$$

We exclude this self-induced term, and then under the assumption that each dipole is advected by the local fluid velocity, along with linear superposition, one obtains the coupled system of N-complex equations for a collection of dipoles at positions $z_\alpha(t)$

$$\dot{z}^*_\alpha = {\sum_{\beta=1}^{N}}' \frac{\mu_\beta \exp(i\psi_\beta)}{2\pi(z_\alpha - z_\beta)^2}, \quad (\alpha = 1, ..., N) \tag{13}$$

where $'$ indicates that we omit the singular term $\beta = \alpha$. Along with this system, one needs the initial dipole locations $z_\alpha(0)$, as well as their initial orientations $\psi_\alpha(0)$, $\alpha = 1, ..., N$.

4. **Dynamical alignment.** We now let the orientations of each of the dipoles, ψ_β, evolve dynamically. Our assumption is that each dipole tries to align itself with the local velocity field $\vec{T}_\alpha$ generated by the others. This assumption is potentially most applicable to two-component Bose-Einstein condensates whose mean field equations are the Gross-Pitaevskii (NLS) equations [11, 3, 12] which couples the dipoles coherently to an external field. How the coupling dynamically aligns the dipoles is not well understood, nor are the details of reducing the governing mean-field pde to interacting particle systems. We mention also the very interesting recent work [8] in which active Brownian particle theories are used to explain the certain pattern-forming characteristics of biological swarms. In this theory, there is a term (see their eqn (11)) which acts to align velocities in a many-particle formulation. It

is useful to first consider just the two dipole interaction model depicted in figure 3. The two orientation angles are ψ_1 and ψ_2, and the angles associated with the tangent vectors $\vec{T}_1$, $\vec{T}_2$, to the local fluid velocities are η_1 and η_2.

To obtain the local velocity vector $\vec{T}_\alpha \equiv (u_\alpha, v_\alpha)$ at each dipole site, we use system (13) and the fact that $\dot{z}^*_\alpha = u_\alpha - iv_\alpha$. Then

$$u_\alpha = Re(\dot{z}^*_\alpha) = Re\left[\sum_{\beta=1}^{N}{}' \frac{\mu_\beta \exp(i\psi_\beta)}{2\pi(z_\alpha - z_\beta)^2}\right] \tag{14}$$

$$v_\alpha = -Im(\dot{z}^*_\alpha) = -Im\left[\sum_{\beta=1}^{N}{}' \frac{\mu_\beta \exp(i\psi_\beta)}{2\pi(z_\alpha - z_\beta)^2}\right], \tag{15}$$

and

$$\eta_\alpha = \tan^{-1}(v_\alpha/u_\alpha). \tag{16}$$

We seek 'relaxation' equations for the evolution of $\psi_1(t)$ and $\psi_2(t)$ which guarantee that $(\psi_1 - \eta_1) \to 2n\pi$ and $(\psi_2 - \eta_2) \to 2n\pi$ as $t \to \infty$. Hence, we write the system

$$\frac{d}{dt}(\psi_\beta - \eta_\beta) = V(\psi_\beta - \eta_\beta), \quad (\beta = 1, 2) \tag{17}$$

for potential V. If we choose

$$V(\psi_\beta - \eta_\beta) = -\sin(\psi_\beta - \eta_\beta) \tag{18}$$

then the phase diagram in figure 4 shows that the orientations align dynamically with the local velocity fields. The dynamical equations for ψ_1 and ψ_2 are then

$$\dot{\psi}_\beta = \dot{\eta}_\beta - \sin(\psi_\beta - \eta_\beta), \quad (\beta = 1, 2) \tag{19}$$

Similarly, for the N-dipole system, the equations for the orientations can be written

$$\dot{\psi}_\beta = \dot{\eta}_\beta - \sin(\psi_\beta - \eta_\beta), \quad (\beta = 1, ..., N) \tag{20}$$

Differentiating (16) gives

$$\dot{\eta}_\beta = \left(\frac{\dot{v}_\beta u_\beta - \dot{u}_\beta v_\beta}{u_\beta^2}\right)\cos^2(\eta_\beta). \tag{21}$$

Hence, the full nonlinear system of equations governing the alignment process is

$$\dot{\psi}_\beta = \left(\frac{\dot{v}_\beta u_\beta - \dot{u}_\beta v_\beta}{u_\beta^2}\right)\cos^2(\eta_\beta) - \sin(\psi_\beta - \eta_\beta), \tag{22}$$

along with (14), (15), and (16).

5. **Desingularized model with finite self-induced velocity.** A desingularized model is easily obtained which includes self-induction. Introducing a *core* parameter $\delta > 0$, we write the desingularized system as

$$\dot{z}^*_\alpha = \sum_{\beta=1}^{N} \frac{\mu_\beta \exp(i\psi_\beta)}{2\pi[(z_\alpha - z_\beta)^2 + \delta^2]}, \quad (\alpha = 1, ..., N) \tag{23}$$

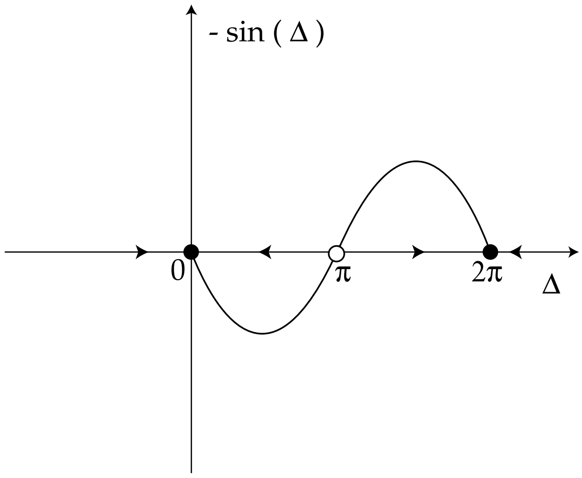

FIGURE 4. Sin potential drives the angles toward alignment. $\Delta_\beta \equiv \psi_\beta - \eta_\beta$, $\Delta_\beta \to 0, 2\pi$ as $t \to \infty$.

where there is no longer a need to exclude the term $\alpha = \beta$. The complex velocity field is now given by

$$\frac{dF_0}{dz_\alpha} = \frac{\psi_\beta \exp(i\psi_\beta)}{2\pi[(z_\alpha - z_\beta)^2 + \delta^2]}, \tag{24}$$

giving rise to a finite self-induced velocity at each dipole site

$$\frac{dF_0}{dz_\alpha}|_{z_\alpha = z_\beta} = \frac{\mu_\beta \exp(i\psi_\beta)}{2\pi\delta^2} < \infty. \tag{25}$$

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