Conventonal A/C Dynamics

Longitudinal

Altitude
Speed
Pitch

Lateral/Directional

Roll
Yaw
Sideslip

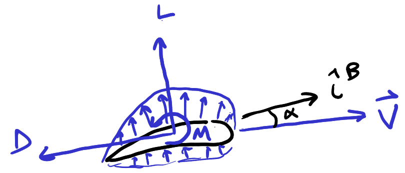
Long. Forces and Moments

Lift $\quad L = \frac{1}{2}\rho V_a^2 S C_L$ (density, Wing Area, airspeed)

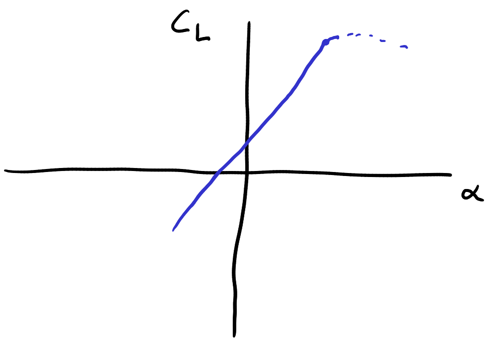
Drag $\quad D = \frac{1}{2}\rho V_a^2 S C_D$

Pitch Moment $\quad M = \frac{1}{2}\rho V_a^2 S \bar{c} C_m$

Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 S\bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$



Lift



$C_L(\alpha, q, \delta e)$

1st order Taylor series

$$L = \frac{1}{2}\rho V_a^2 S\left(C_{L_{zero}} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q}q + \frac{\partial C_L}{\partial \delta e}\delta_e\right)$$

Stability Derivatives

$$C_{a_b} \equiv \left.\frac{\partial \text{ nondimensionalized } a}{\partial \text{ nondimensionalized } b}\right|_{\substack{\text{condition}\\\text{(usually trim)}}}$$

- based on linear assumptions
- main tool for connecting aerodynamics + dynamics
- determined by A/C geometry
- only accurate in a linear region (e.g. small α)

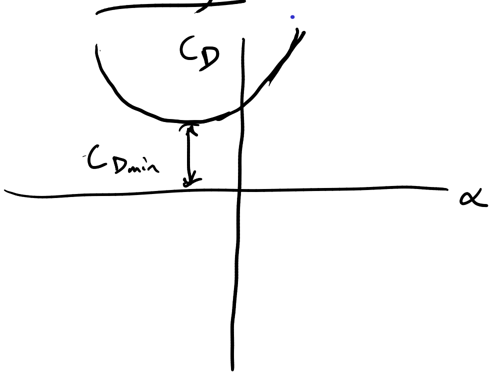
Estimated

- Geometric Data $\longrightarrow$ $C_{L_\alpha} \approx \dfrac{\pi AR}{1+\sqrt{1+\left(\frac{AR}{2}\right)^2}}$
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$\boxed{L = \frac{1}{2}\rho V_a^2 S\left(C_{L_{zero}} + C_{L_\alpha}\alpha + C_{L_q}\hat{q} + C_{L_{\delta e}}\delta e\right)}$$

$$\hat{q} = q\frac{\bar{c}}{2V_a}$$

Drag



parasitic + induced $\propto C_L^2$

$$\boxed{C_D = C_{D_{min}} + K\left(C_L(\alpha, q, \delta e) - C_{L_{min}}\right)^2}$$

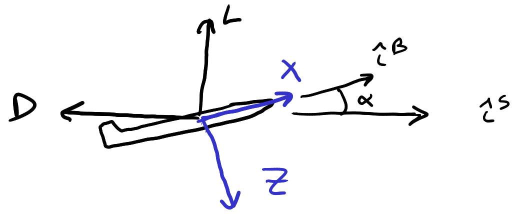
$$K = \frac{1}{\pi e AR}$$

e: Oswald's Efficiency

Pitch Moment

$$M \approx \frac{1}{2}\rho V_a^2 S\bar{c}\left(C_{m_{zero}} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right)$$

Lift + Drag $\longrightarrow$ Body coordinates



"stability coordinate system"

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} -D \\ -L \end{bmatrix}$$

Longitudinal Stability

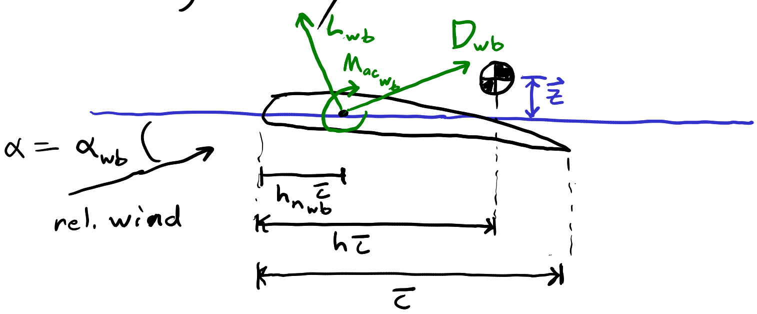
So far : C_L, C_D, C_m — Steady state forces

Now : C_{L_α}, C_{m_α}, $C_{L_{\delta_e}}$, $C_{m_{\delta_e}}$ — trim, static stability

static margin : $h_n - h$

Three contributors
1. Wing/body
2. Propulsion (often small)
3. Tail

Wing/body



Aerodynamic Center

$\frac{\partial \text{Moment}}{\partial \alpha} = 0$

Location stays constant

Moment usually < 0

Center of Pressure

Moment $= 0$

Changes w/ α

Neutral Point

C.G. location that yields $C_{m_\alpha} = 0$

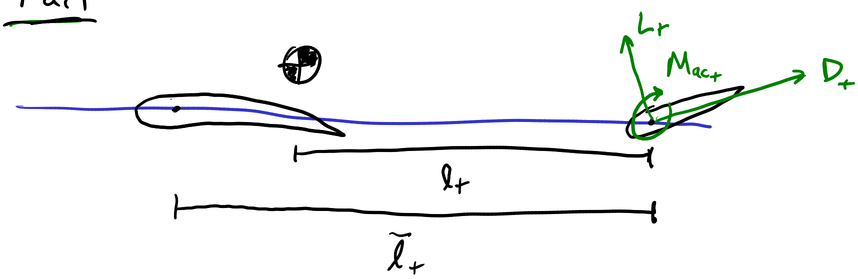
A.C. of entire A/c

$$M_{wb} = M_{ac_{wb}} + (L\cos\alpha + D\sin\alpha)(h - h_{n_{wb}})\bar{c} + (L\sin\alpha - D\cos\alpha)\bar{z}$$

only keep most important terms + nondimensionalize

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_{wb}})$$

Tail



$$L = L_{wb} + L_t$$
$$= C_{L_{wb}}\left(\tfrac{1}{2}\rho V^2 S\right) + C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right)$$
$$= C_L\left(\tfrac{1}{2}\rho V^2 S\right)$$

$$\Rightarrow C_L = C_{L_{wb}} + \frac{S_t}{S} C_{L_t}$$

$$M_t = -l_t L_t = -l_t C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right) \Rightarrow C_{m_t} = \underbrace{\frac{-l_t}{\bar{c}}\frac{S_t}{S}}\, C_{L_t}$$

$$= -V_H C_{L_t}$$

"volume ratio"

more convenient b/c C.G. can change

$$\bar{V}_H = \frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S} \Rightarrow V_H = \bar{V}_H - \frac{S_t}{S}(h - h_{n_{wb}})$$

$$C_{m_t} = -\bar{V}_H C_{L_t} + C_{L_t}\frac{S_t}{S}(h - h_{n_{wb}})$$

$$\rightarrow C_m = \overbrace{C_{m_{ac_{wb}}} + C_L(h - h_{n_{wb}})}^{\text{Wing/body}} - \bar{V}_H C_{L_t} + \underbrace{C_{m_p}}_{\text{propulsion}}$$

Tail

$$C_{m_\alpha} = \cancelto{0}{\frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha}} + C_{L_\alpha}(h - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

Want $h_n \equiv$ cg location where $C_{m_\alpha} = 0$

$$0 = C_{L_\alpha}(h_n - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

$$h_n = h_{n_{wb}} + \frac{1}{C_{L_\alpha}}\left(\underbrace{\bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha}}_{\text{tail correction}} - \frac{\partial C_{m_p}}{\partial \alpha}\right)$$

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$

$C_{m_\alpha} < 0$ for stability

static margin

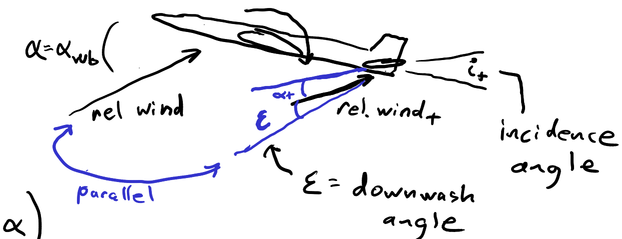
$K_n \equiv h_n - h$ must be > 0 for stability

back to h_n eq.

Lift curve slope $a \equiv C_{L_\alpha}$

$$C_{L_{wb}} = a_{wb}\alpha_{wb} = a_{wb}\alpha$$

$$C_{L_t} = a_t \alpha_t$$



$$\alpha_t = \alpha - i_t - \left(\varepsilon_{zero} + \frac{\partial \varepsilon}{\partial \alpha}\alpha\right)$$

$$\frac{\partial C_{L_t}}{\partial \alpha} = a_t\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)$$

$$\boxed{h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}}$$

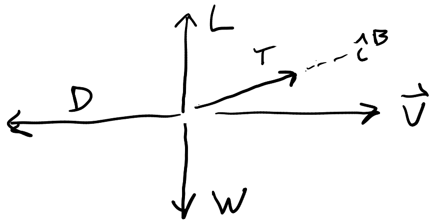
$$\boxed{C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\right]}$$

Longitudinal Control

δ_e changes C_{L_t}

$$\boxed{\begin{aligned} C_{L_{\delta_e}} &= \frac{\partial C_{L_t}}{\partial \delta_e}\frac{S_t}{S} = a_e\frac{S_t}{S} \\ C_{m_{\delta_e}} &= -a_e\bar{V}_H + C_{L_{\delta_e}}(h - h_{n_{wb}}) \end{aligned}}$$

Linear Trim Estimation



For Linear trim, $T = D$, $L = W$

0 for Linear Trim Estimation

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V^2 S} = \cancel{C_{L_{zero}}} + C_{L_\alpha}\alpha_{trim} + C_{L_{\delta e}}\delta_{e_{trim}}$$

$$C_{m_{trim}} = C_{m_{zero}} + C_{m_\alpha}\alpha_{trim} + C_{m_{\delta e}}\delta_{e_{trim}} = 0$$

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Cramer's rule)

$$\alpha_{trim} = \frac{C_{m_{zero}} C_{L_{\delta e}} + C_{m_{\delta e}} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha}$$

$$\delta_{e_{trim}} = -\frac{C_{m_{zero}} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta}$$

Longitudinal Linear Model

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$
$$\dot{\vec{\Theta}} = T \vec{\omega}_B$$
$$\dot{\vec{V}}_B^E = \frac{\vec{F}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$
$$\dot{\vec{\omega}} = I^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B\right]$$

2 differences
1. Aerodynamic Forces
2. I more complex

symmetry about x-z axis $\Rightarrow I_{xy} = I_{yz} = 0$

$I_{xz} \neq 0$

$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}\left(I_x - I_y + I_z\right)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x\left(I_x - I_y\right) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z\left(I_z - I_y\right) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$I_{yz} \neq 0$
$I_{xy} \neq 0$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

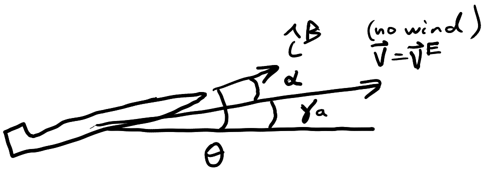
$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Linearize about trim state

Inputs: $u_0, h_0, \gamma_{a_0} \longrightarrow \alpha_0, \delta_{e_0}, \delta_{T_0}$

airspeed $= V = V_a$ (for u_0); altitude (for h_0); air-relative flight-path angle (for γ_{a_0})



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \Delta \vec{X} \qquad \vec{x}_0 = \begin{bmatrix} \cdot \\ \cdot \\ -h_0 \\ 0 \\ \theta_0 \\ \cdot \\ u_0^E \\ 0 \\ w_0^E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For linearization: $\vec{V} = \vec{V}^E$ (no wind)

assume $w_0^E = 0$

$$\vec{u} = \begin{bmatrix} \delta_e \\ \delta_a \\ \delta_r \\ \delta_T \end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix} \delta_{e_0} \\ 0 \\ 0 \\ \delta_{T_0} \end{bmatrix}$$

Examples to Linearize

$$\dot{\theta} = \cos\phi\, q - \sin\phi\, r$$

$$\cancel{\dot{\theta}_0}^{\,0} + \Delta\dot{\theta} = \cancel{\cos\phi_0 q_0}^{\,0} - \cancel{\sin\phi_0 r_0}^{\,0} + \frac{\partial}{\partial\phi}\left(\cos\phi\, q - \sin\phi\, r\right)\Big|_0 \Delta\phi$$
$$+ \frac{\partial}{\partial q}\left(\cos\phi\, q - \sin\phi\, r\right)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial r}\left(\cos\phi\, q - \sin\phi\, r\right)\Big|_0 \Delta r$$
$$= -\cancel{\sin\phi_0}^{\,0} q_0 - \cancel{\cos\phi_0 r_0}^{\,0} + \cancel{\cos\phi_0}^{\,1} \Delta q - \cancel{\sin\phi_0}^{\,0} \Delta r$$

$$\boxed{\Delta\dot{\theta} = \Delta q}$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{X}{m}$$

$$\cancel{\dot{u}_0}^{\,0} + \Delta\dot{u} = \cancel{r_0 v_0}^{\,0} - \cancel{q_0 w_0}^{\,0} \underbrace{\cancel{- g\sin\theta_0 + \frac{X_0}{m}}^{\,0}}_{\text{cancel}} + \frac{\partial}{\partial r} rv\Big|_0 \Delta r + \frac{\partial}{\partial v} rv\Big|_0 \Delta v + \frac{\partial}{\partial q}(-qw)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial w}(-qw)\Big|_0 \Delta w + \frac{\partial}{\partial\theta}(g\sin\theta)\Big|_0 \Delta\theta + \frac{\partial}{\partial X}\left(\frac{X}{m}\right)\Big|_0 \Delta X$$
$$= \cancel{v_0 \Delta r}^{\,0} + \cancel{r_0 \Delta v}^{\,0} - \cancel{w_0 \Delta q}^{\,0} - \cancel{q_0 \Delta w}^{\,0} - g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X$$

$$\boxed{\Delta\dot{u} = -g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X}$$

Lateral

Long.

$$\Delta\dot\phi = \Delta p + \Delta r\tan\theta_0$$

$$\Delta\dot\theta = \Delta q$$

$$\Delta\dot u = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot v = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot w = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot p = \Gamma_3\Delta L + \Gamma_4\Delta N$$

$$\Delta\dot q = \frac{\Delta M}{I_y}$$

$$\Delta\dot r = \Gamma_4\Delta L + \Gamma_8\Delta N$$

Dimensional stab derivs.

$$\Delta X = X_u\Delta u + X_w\Delta w + \Delta X_c$$

$$\Delta Y = Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c$$

$$\Delta Z = Z_u\Delta u + Z_w\Delta w + Z_{\dot w}\Delta\dot w + Z_q\Delta q + \Delta Z_c$$

$$\Delta L = L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c$$

$$\Delta M = M_u\Delta u + M_w\Delta w + M_{\dot w}\Delta\dot w + M_q\Delta q + \Delta M_c$$

$$\Delta N = N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c$$

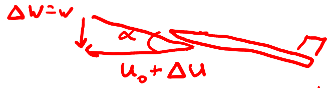
Dimensional stability Derivative

$$X_u \equiv \left.\frac{\partial X}{\partial u}\right|_0$$

Nondim Stab Deriv.

$$C_{x_u} \equiv \left.\frac{\partial C_x}{\partial \hat u}\right|_0$$

$$\hat w = \frac{w}{u_0} \approx \alpha$$



$$\alpha = \operatorname{atan}\left(\frac{\Delta w}{u_0+\Delta u}\right) \approx \frac{w}{u_0}$$

Table 4.4
Longitudinal Dimensional Derivatives

	X (X_u)	Z	M
u	$\boxed{\rho u_0 S C_{w_0}\sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}}$	$-\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}$	$\frac{1}{2}\rho u_0 \bar c S C_{m_u}$
w	$\boxed{\frac{1}{2}\rho u_0 S C_{x_\alpha}}$ X_w	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar c S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar c S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar c S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar c^2 S C_{m_q}$
$\dot w$	$\frac{1}{4}\rho \bar c S C_{x_{\dot\alpha}}$	$\frac{1}{4}\rho \bar c S C_{z_{\dot\alpha}}$	$\frac{1}{4}\rho \bar c^2 S C_{m_{\dot\alpha}}$

$$Z_u \equiv \left.\frac{\partial Z}{\partial u}\right|_0 \qquad Z = \frac{1}{2}\rho V^2 S C_Z$$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = \frac{1}{2}\rho S\left(\left.\frac{\partial V^2}{\partial u}\right|_0 C_Z + V^2\left.\frac{\partial C_Z}{\partial u}\right|_0\right)$$

$$= \frac{1}{2}\rho S\left(2u_0 C_{Z_0} + u_0^2\left.\frac{\partial C_Z}{\partial u}\right|_0\right)$$

$$\boxed{Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{Z_u}}$$

$$\hat u = \frac{u}{V} = \frac{u}{u_0} \qquad u = \hat u u_0$$

$$\left.\frac{\partial C_Z}{\partial u}\right|_0 = \left.\frac{\partial C_Z}{u_0\partial\hat u}\right|_0 = \frac{1}{u_0}C_{Z_u}$$

$$C_{Z_0} = -C_{w_0}\cos\theta_0$$

Table 5.1
Summary—Longitudinal Derivatives

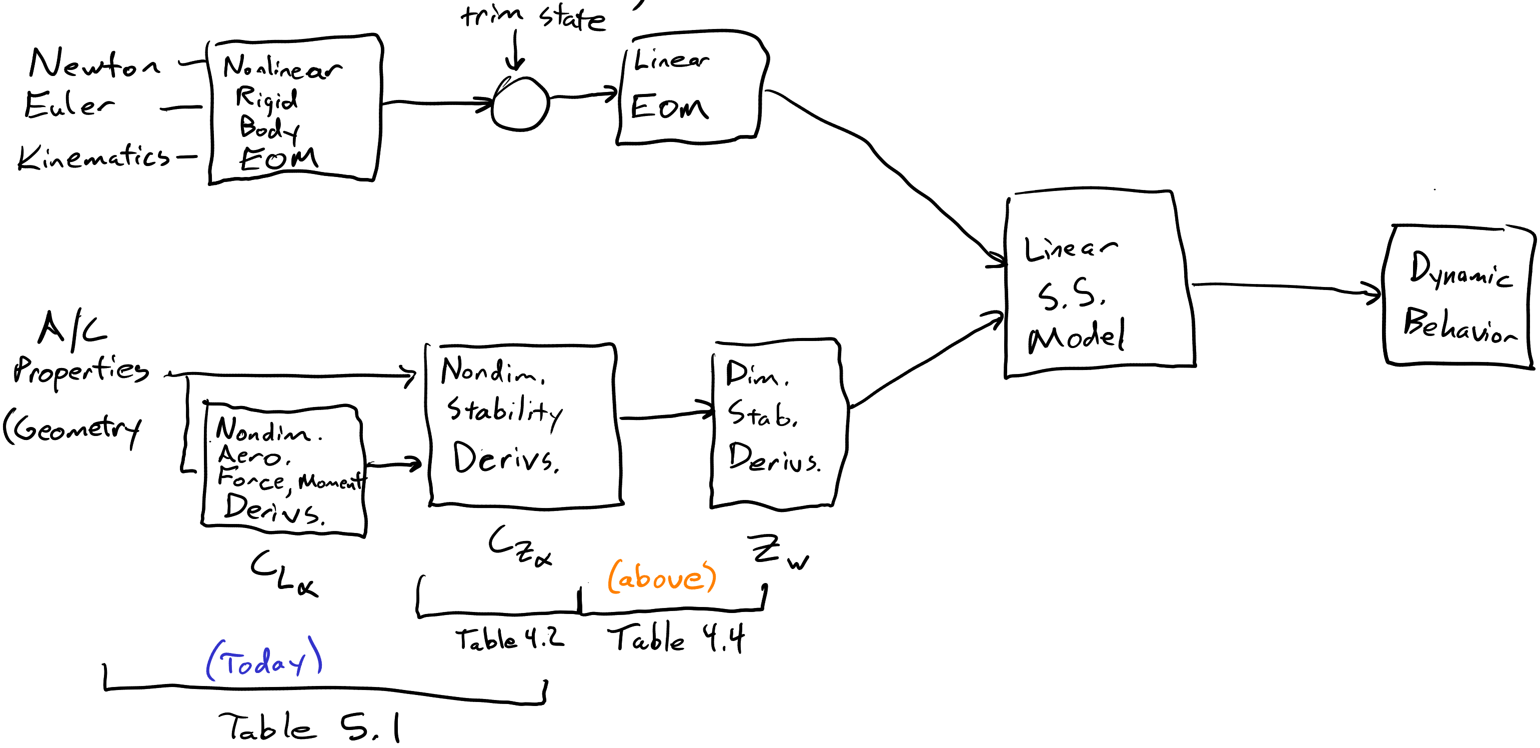
	C_x	C_z	C_m
$\hat u$†	$M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right) - \rho u_0^2\frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-M_0\frac{\partial C_L}{\partial M} - \rho u_0^2\frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$M_0\frac{\partial C_m}{\partial M} + \rho u_0^2\frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{L_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot\alpha$	Neg.	$*-2a_t V_H\frac{\partial\epsilon}{\partial\alpha}$	$*-2a_t V_H\frac{l_t}{\bar c}\frac{\partial\epsilon}{\partial\alpha}$
$\hat q$	Neg.	$*-2a_t V_H$	$*-2a_t V_H\frac{l_t}{\bar c}$

Neg. means usually negligible.

*means contribution of the *tail only*, formula for wing-body not available.

$$\dagger C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0};\ C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$$

Big Picture


Newton
Euler
Kinematics
Nonlinear Rigid Body EOM
trim state
Linear EOM
A/C Properties
(Geometry
Nondim. Aero. Force, Moment Derivs.
Nondim. Stability Derivs.
Dim. Stab. Derivs.
Linear S.S. Model
Dynamic Behavior
C_{L_α}
C_{Z_α}
Z_w
(above)
Table 4.2
Table 4.4
(Today)
Table 5.1

Nondimensional Stability Derivatives

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1-\frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2\frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0\frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha}+C_{D0})$	$-a(h_n-h)$
$\dot{\alpha}$	Neg.	$*-2a_tV_H\frac{\partial\epsilon}{\partial\alpha}$	$*-2a_tV_H\frac{l_t}{\bar{c}}\frac{\partial\epsilon}{\partial\alpha}$
$\hat{q}$	Neg.	$*-2a_tV_H$	$*-2a_tV_H\frac{l_t}{\bar{c}}$

Neg. means usually negligible.
*means contribution of the *tail only*, formula for wing-body not available.
$\dagger C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0};\ C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

α-derivatives

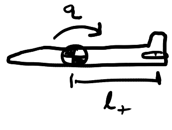
$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h-h_n)}$$

C_{z_α}

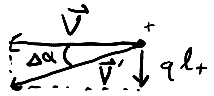
$$Z = -L\cos\alpha - D\sin\alpha$$
$$C_z = -(C_L\cos\alpha + C_D\sin\alpha)$$
$$\approx -(C_L + \underline{C_D}\,\alpha)$$
$$C_{z_\alpha} \equiv \left.\frac{\partial C_z}{\partial\alpha}\right|_0 = -(C_{L\alpha} + C_{D_0} + \cancel{\alpha_0}^{\,0} C_{D\alpha})$$
$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

q-derivates

Wing-body || $\boxed{\text{Tail}}$



velocity observed by tail



Lift curve slope of tail
$$\Delta C_{L_+} = a_+\Delta\alpha = a_+\tan^{-1}\left(\frac{ql_+}{u_0}\right) \approx a_+\frac{ql_+}{u_0}$$
$$\Delta C_L = \frac{S_+}{S}\Delta C_{L_+}$$
$$= \frac{S_+}{S}a_+\frac{ql_+}{u_0}$$

$(C_{z_q})_{tail}$

$$C_{z_q} = \left.\frac{\partial C_z}{\partial\hat{q}}\right|_0 = \left.\frac{2u_0}{\bar{c}}\frac{\partial C_z}{\partial q}\right|_0 = -\left.\frac{2u_0}{\bar{c}}\frac{\partial C_L}{\partial q}\right|_0$$
$$(C_{z_q})_{tail} = -\frac{2u_0}{\bar{c}}a_+\frac{S_+}{S}\frac{l_+}{u_0} = \boxed{-2a_+V_H}$$

$$V_H = \frac{S_+ l_+}{S\bar{c}}$$

$(C_{m_q})_{tail}$

$$\Delta C_m = -V_H \Delta C_{L_t} = a_t V_H \frac{q\, l_t}{u_0}$$

$$C_{m_q} \equiv \frac{\partial C_m}{\partial \hat{q}}\Big|_0 = \frac{2u_0}{\bar{c}} \frac{\partial C_m}{\partial q}\Big|_0$$

$$(C_{m_q})_{tail} = \boxed{-2 a_t V_H \frac{l_t}{\bar{c}}}$$

Wing-Body

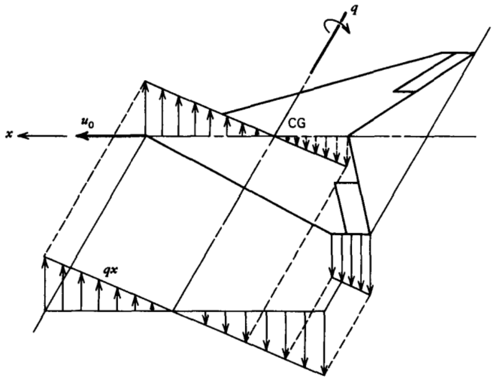


Figure 5.4 Wing velocity distribution due to pitching.

measure in wind tunnel or CFD

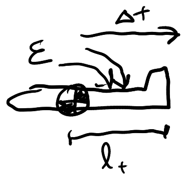
$\dot{\alpha}$ derivatives

Unsteady effects

Wing-body $\longrightarrow$ determined by initial response or oscillation of wing in wind tunnel or flight test

Tail

Downwash lag



$$-\Delta\alpha_t = \Delta\varepsilon = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \Delta t$$

$$= -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \frac{l_t}{u_0}$$

$$\Delta C_{L_t} = a_t \Delta\alpha_t = a_t \dot{\alpha} \frac{l_t}{u_0} \frac{\partial \varepsilon}{\partial \alpha}$$

$$\Delta C_L = a_t \dot{\alpha} \frac{l_t S_t}{u_0 S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{z_{\dot{\alpha}}})_{tail} \equiv \frac{\partial C_z}{\partial \frac{\dot{\alpha}\bar{c}}{2u_0}}\Bigg|_0 = \frac{2u_0}{\bar{c}} \frac{\partial C_z}{\partial \dot{\alpha}}\Bigg|_0 = \boxed{-2 a_t \frac{l_t S_t}{\bar{c} S} \frac{\partial \varepsilon}{\partial \alpha}}$$

$$\boxed{(C_{m_{\dot{\alpha}}})_{tail} = -2 a_t V_H \frac{l_t}{\bar{c}} \frac{\partial \varepsilon}{\partial \alpha}}$$

u derivatives

3 important factors:

- Compressibility: Mach Number
- Dynamic Pressure: $p_d = \frac{1}{2}\rho V^2$
- Thrust

— Different from the dynamic pressure in nondimensionalization. Changes in C_L, C_D etc. due to changes in dynamic pressure.

$$C_{X_u} \equiv \frac{\partial C_X}{\partial \hat{u}}\bigg|_0$$

$* \in \{X, Z, M\}$

$$C_{*_u} = \underline{\frac{\partial C_*}{\partial M}\bigg|_0 \frac{\partial M}{\partial \hat{u}}\bigg|_0} + \underline{\frac{\partial C_*}{\partial p_d}\bigg|_0 \frac{\partial p_d}{\partial \hat{u}}\bigg|_0} + \underline{\frac{\partial C_*}{\partial C_T}\bigg|_0 \frac{\partial C_T}{\partial \hat{u}}\bigg|_0}$$

$M = \frac{V}{a}$

$$\frac{\partial M}{\partial \hat{u}}\bigg|_0 = u_0 \frac{\partial M}{\partial u}\bigg|_0 = \frac{u_0}{a}\frac{\partial V}{\partial u}\bigg|_0 = M_0$$

Mach number at trim

$$\frac{\partial p_d}{\partial \hat{u}}\bigg|_0 = u_0 \frac{\partial p_d}{\partial u}\bigg|_0 = u_0 \frac{1}{2}\rho \frac{\partial V^2}{\partial u}\bigg|_0 = u_0 \rho u_0 = \rho u_0^2$$

$C_T = \frac{T}{\frac{1}{2}\rho V^2 S}$

$$\frac{\partial C_T}{\partial \hat{u}}\bigg|_0 = u_0 \frac{\partial C_T}{\partial u}\bigg|_0 = u_0\left(\frac{\partial T/\partial u}{\frac{1}{2}\rho V^2 S} - \frac{2T}{\frac{1}{2}\rho V^3 S}\right)\bigg|_0$$

$$= \frac{\partial T/\partial u}{\frac{1}{2}\rho u_0 S}\bigg|_0 - 2C_{T_0}$$

$C_{T_0} = C_{D_0} + C_{W_0}\sin\theta_0$

3 Cases

Gliding: $C_{T_u} = 0$

Constant Thrust (Jet): $\boxed{C_{T_u} = -2C_{T_0}}$

Constant Power (Prop):

$TV = \text{constant}$

$$\frac{\partial T}{\partial u}\bigg|_0 = -\frac{T_0}{u_0}$$

$$\boxed{C_{T_u} = -3C_{T_0}}$$

C_{x_u}

$$C_x \approx C_T - C_D$$

$$\left.\frac{\partial C_x}{\partial M}\right|_0 = \left.\frac{\partial C_T}{\partial M}\right|_0 - \left.\frac{\partial C_D}{\partial M}\right|_0$$

$$\left.\frac{\partial C_x}{\partial p_d}\right|_0 = \left.\frac{\partial C_T}{\partial p_d}\right|_0 - \left.\frac{\partial C_D}{\partial p_d}\right|_0$$

$$\left.\frac{\partial C_x}{\partial C_T}\right|_0 = 1 - \left.\frac{\partial C_D}{\partial C_T}\right|_0$$

$$\boxed{C_{x_u} = M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right)\Big|_0 - \rho u_0^2 \left.\frac{\partial C_D}{\partial p_d}\right|_0 + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)\Big|_0}$$

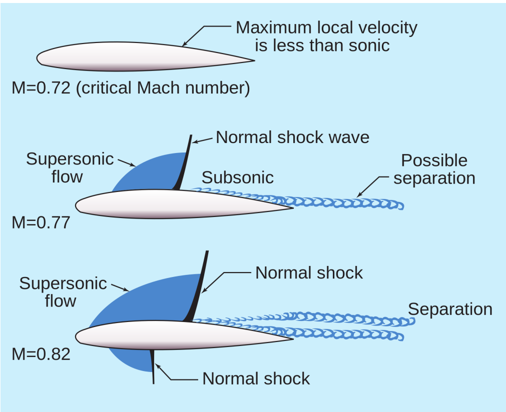
C_{z_u}

Assume $C_z = -C_L$

$$\boxed{C_{z_u} = \underbrace{-M_0 \left.\frac{\partial C_L}{\partial M}\right|_0}_{\text{small except for transonic}} - \rho u_0^2 \left.\frac{\partial C_L}{\partial p_d}\right|_0 - C_{T_u} \left.\frac{\partial C_L}{\partial C_T}\right|_0}$$

C_{m_u}

$$\boxed{C_{m_u} = \underbrace{M_0 \left.\frac{\partial C_m}{\partial M}\right|_0}_{\text{Mach Tuck}} + \rho u_0^2 \left.\frac{\partial C_m}{\partial p_d}\right|_0 + C_{T_u} \left.\frac{\partial C_m}{\partial C_T}\right|_0}$$



Longitudinal Modes

Full State Space

$$\dot{\vec{x}} = \underset{n \times n}{A}\vec{x}$$

$$\boxed{\dot{\vec{x}} = A\vec{x}} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

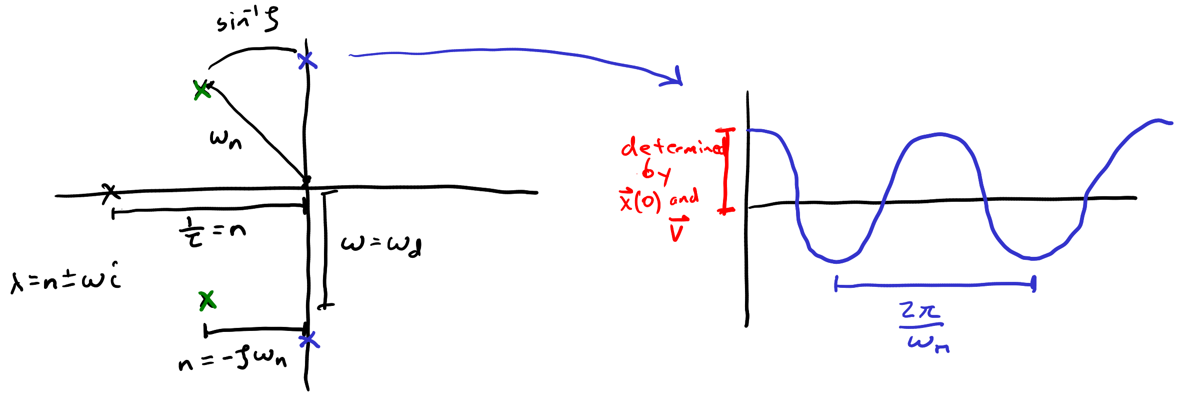
(assume that A has n distinct non zero eigenvalues)

$$\boxed{\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}}$$

What does this mean?

A single real-valued $(\lambda, \vec{v})$ pair, or a pair of complex-valued $((\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2))$ describes a <u>mode</u>.

λ: "speed" of mode
$\vec{v}$: "shape" of mode



Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$(A - \lambda_i I)\vec{v}_i = 0$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \qquad \lambda_1 = -1, \quad \lambda_2 = -2$$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0 \qquad (\vec{v}_1)_1 + (\vec{v}_1)_2 = 0$$

let $(\vec{v}_1)_1 = 1$
$(\vec{v}_1)_2 = -1$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$(A - \lambda_2 I)\vec{v}_2 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \vec{v}_2 = 0 \qquad \therefore \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \alpha$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i \cancel{e^{\lambda 0}}^{1} = \sum_i q_i \vec{v}_i = q_1\vec{v}_1 + q_2\vec{v}_2 + \dots q_n\vec{v}_n = V\vec{q}$$

$$V = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_n] \qquad \vec{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$\boxed{\begin{aligned} \vec{x}(0) &= V\vec{q} \\ \vec{q} &= V^{-1}\vec{x}(0) \end{aligned}}$$

Conventional A/C Long. Modes

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$h_0 = 40\text{k ft}$

$u_0 = 774 \text{ ft/s}$

$\gamma_0 = \theta_0 = \alpha_0 = 0$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

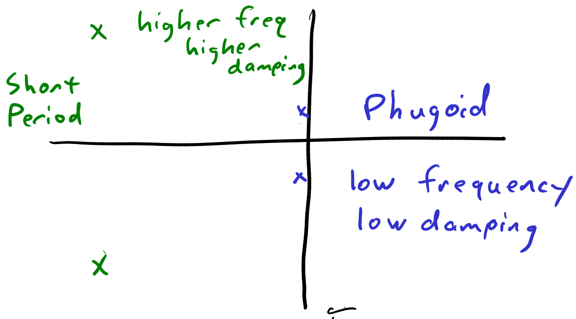
$\lambda_{1,2} = -0.37 \pm 0.89i$

$\omega_n = 0.96 \quad \zeta = 0.38$

$\lambda_{3,4} = -0.0033 \pm 0.067i$

$\omega_n = 0.067$

$\zeta = 0.049$

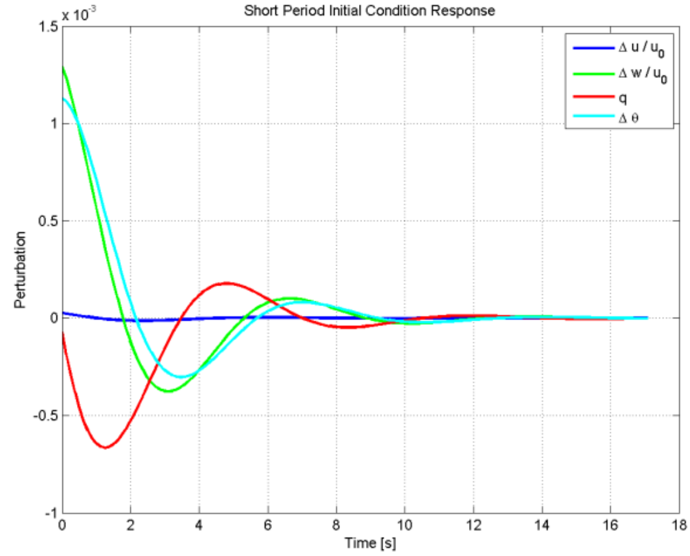


$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.001i \\ 0.0011 \mp 0.0004i \end{bmatrix} \begin{matrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{matrix} \qquad \vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097 \\ -0.0001 \mp 0i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

$\vec{x}(0) = 0.5\,\vec{v}_1 + 0.5\,\vec{v}_2 = \mathrm{Re}(\vec{v}_{1,2})$

$$\lambda_{1/2} = -.372 + .888i$$
$$\zeta = 0.387$$
$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



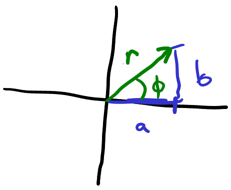
Phasor Plot: plot of eigenvector in complex plane

Aside: Polar Coordinates of a complex number

$z = a + bi$

$z = re^{i\phi}$

$z = r\angle\phi$



$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

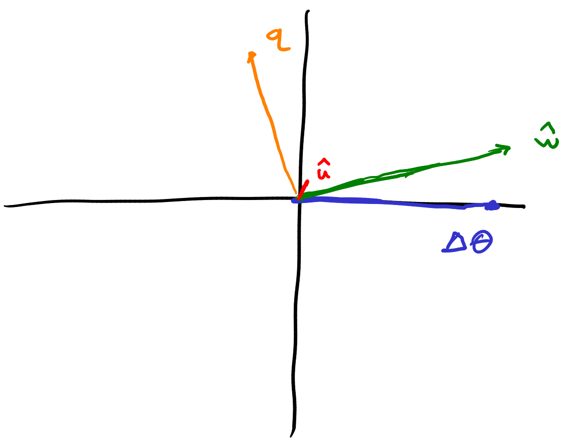
$r = \sqrt{a^2+b^2} \quad \phi = \mathrm{atan2}(b,a)$

For consistent phasor plots,
- normalize so that $\Delta\theta = 1$
- nondimensionalize u and w

Short Period

$$\vec{V}'_{1,2} = \vec{V}_{1,2} / (\vec{V}_{1,2})_4 = \begin{bmatrix} \vdots \\ \vdots \\ 1.0 \end{bmatrix}$$

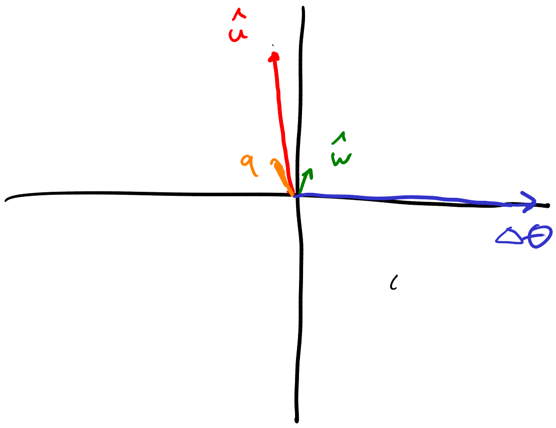
$$\hat{V}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.37 \pm 0.89i \\ 1.0 \end{bmatrix} \quad \begin{matrix} \hat{u} = \frac{\Delta u}{u_0} \\ \hat{w} = \frac{\Delta w}{u_0} \approx \alpha \\ q \\ \Delta\theta \end{matrix} \Rightarrow$$



Phugoid Mode

phugoid = “flight”

$$\hat{V}_{3,4} = \begin{array}{cc} \left[\begin{array}{c} 0.62\angle 92^\circ \\ 0.036\angle 83^\circ \\ 0.067\angle 93^\circ \\ 1.0\angle 0 \end{array}\right] & \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ q \\ \Delta\theta \end{array} \end{array}$$

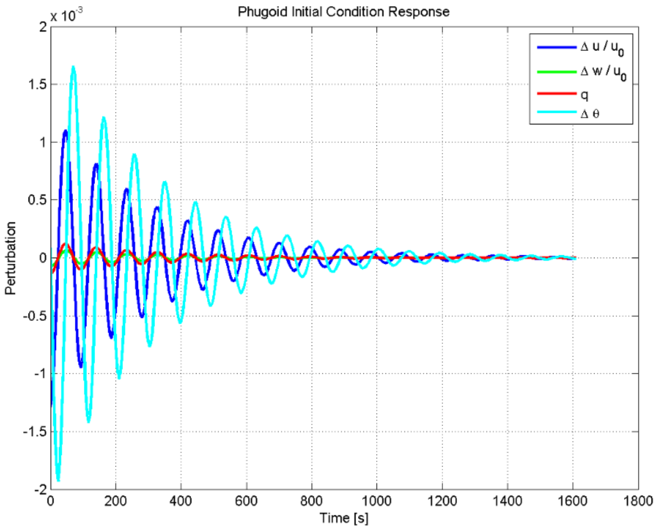


$$\lambda_{3,4} \Rightarrow \begin{cases} \text{low frequency} \\ \text{low damping} \end{cases}$$

$$\vec{V}_{3,4} \Rightarrow \begin{cases} \text{Large } \hat{u} \text{ and } \theta \text{ oscillations out of phase } (90^\circ \text{ offset}) \\ \text{small } \alpha \end{cases}$$

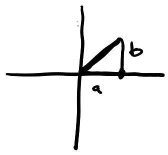
$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$
$\zeta = 0.0489 \leftarrow$ poorly damped
$\omega_n = 0.0673 \leftarrow$ slow response

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



Longitudinal Mode Approximations

$\vec{v} = \begin{bmatrix} a+bi \\ \\ \\ \end{bmatrix}$



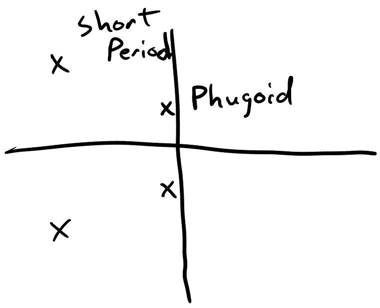
$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + \underbrace{B_{lon}\vec{u}_{lon}}_{\vec{c}_{lon}}$$

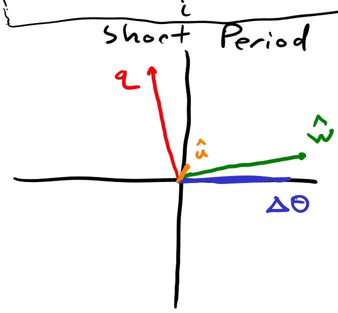
$$\vec{y} = \underset{\downarrow\ I}{C}\,\vec{x}_{lon} + \underset{\downarrow\ 0}{D}\,\vec{u}_{lon}$$

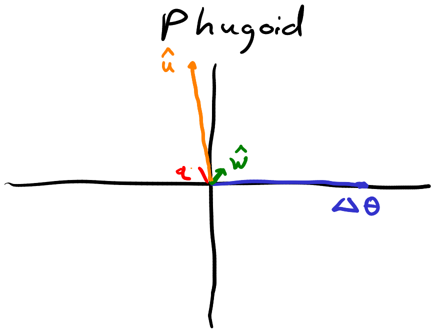
$$\vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$\boxed{\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}}$$

$\hat{w} \equiv \dfrac{\Delta w}{u_0}$







Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

Assume: $\Delta u = 0$

$Z_{\dot{w}} \ll m$

$Z_q \ll m u_0$

$\theta_0 = 0$

No vertical motion

$\Delta\theta = \Delta\alpha = \dfrac{\Delta w}{u_0}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - \cancel{Z_{\dot{w}}}} & \dfrac{\cancel{Z_q} + mu_0}{m - \cancel{Z_{\dot{w}}}} & \dfrac{-mg\sin\cancel{\theta_0}}{m - \cancel{Z_{\dot{w}}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - \cancel{Z_{\dot{w}}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(\cancel{Z_q} + mu_0)}{m - \cancel{Z_{\dot{w}}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\cancel{\theta_0}}{I_y(m - \cancel{Z_{\dot{w}}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

~ 0

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{Z_w}{m} & u_0 \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m}\right] & \dfrac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}\left(M_q + M_{\dot{w}}u_0\right)\right]}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

Dimensional Stab. Deriv.	Nondim. Stab. Deriv.	A/C Properties
$Z_w \equiv \frac{\partial Z}{\partial w}\Big\vert_0 = \underbrace{\frac{1}{2}\rho u_0 S C_{Z_\alpha}}_{\text{Table 4.4}}$	C_{Z_α}	$= \underbrace{-C_{L_\alpha} - C_{D_0}}_{\text{Table 5.1}}$
M_w	C_{m_α}	$= C_{L_\alpha}(h - h_n)$
$M_{\dot{w}}$	$C_{m_{\dot{\alpha}}}$	....
M_q	C_{m_q}	$\left(C_{m_q}\right)_{tail} = -2a_t V_H \frac{l_t}{\bar{c}}$

How accurate is this approximation?

For 747 @ cruise

Full A_{lon}

$\lambda_{1,2} = -0.372 \pm 0.888i$

$\zeta = 0.387$

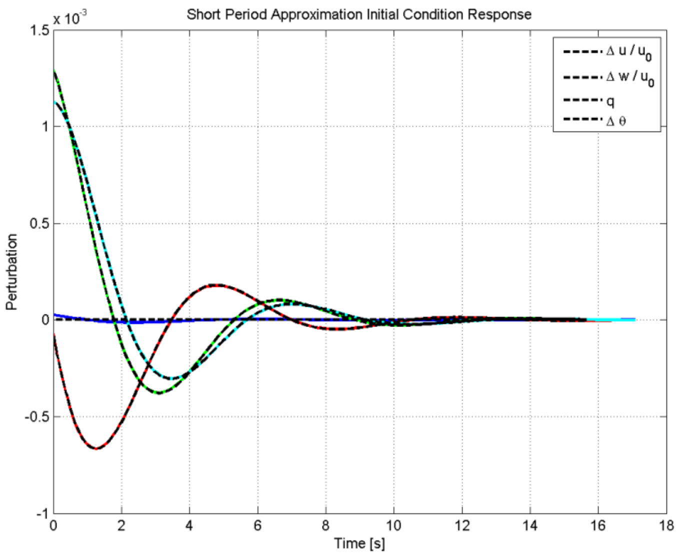
$\omega_n = 0.962$

S.P. Approx

$\lambda_{sp} = -0.371 \pm 0.889i$

$\zeta = 0.385$

$\omega_n = 0.963$



Phugoid Mode

Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - mg\Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g\Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho u_0^2 S C_{W_0} + \rho g S C_{W_0}\Delta z_E = W + \rho g S C_{W_0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m\Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m\Delta\ddot{z}_E$$

$$\Delta\ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m}}_{\omega_n^2}\Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{\rho g S C_{w_0}}} = 2\pi\sqrt{\frac{\overbrace{\tfrac{1}{2}u_0^2 m g}^{W}}{g^2 \underbrace{\tfrac{1}{2}\rho u_0^2 S C_{w_0}}_{W}}} = \boxed{\pi\sqrt{2}\,\frac{u_0}{g}}$$

$$\boxed{\begin{aligned} T &= 0.138\,u_0 \quad \text{if } u_0 \text{ in ft/s} \\ &= 0.453\,u_0 \quad \text{if } u_0 \text{ in m/s} \end{aligned}}$$

for 747	Full A_{lon}	Lanchester
	$T = 93\,s$	$T = 107\,s$

"2x2" Phugoid Approximation

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} \qquad \mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - \cancel{Z_{\dot{w}}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{\cancel{Z_q} + m u_0}{m - \cancel{Z_{\dot{w}}}} & \dfrac{-mg\sin\theta_0}{m - \cancel{Z_{\dot{w}}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}} m g \sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$-g$ $\quad$ 0

$\Delta\alpha = 0$

$\Delta\dot{q}$ small

$Z_{\dot{w}} \ll m$

$Z_q \ll m u_0$

$\theta_0 \to 0$

$$\rightarrow \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} = \Delta\dot{\alpha} u_0 = 0 \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m}\Delta u + u_0 \Delta q$$

$$\Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\Delta\dot{\theta} = \Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\boxed{\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_0} & 0 \end{bmatrix}}_{A_{ph}} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}}$$

$$|A_{ph} - \lambda I| = \lambda^2 - \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{Z_u g}{m u_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{w_0} \cancel{\cos\theta_0}^{\,1} + \frac{1}{2}\rho u_0 S C_{z_u}$$

$$C_{z_u} = \underbrace{-M_0 \frac{\partial C_L}{\partial M}\Big|_0}_{\text{small}} \underbrace{- \rho u_0^2 \frac{\partial C_L}{\partial p_d}\Big|_0}_{\text{small}} \underbrace{- C_{T_u}\frac{\partial C_L}{\partial C_T}\Big|_0}_{\text{small}}$$

assume

$$C_{z_u} = 0 \qquad \therefore \; \underline{Z_u = -\rho u_0 S C_{w_0}}$$

$$X_u = \rho u_0 S C_{w_0} \cancel{\sin\theta_0}^{\,0} + \frac{1}{2}\rho u_0 S C_{x_u}$$

$$C_{x_u} = -2C_{T_0}$$

$$C_{T_0} = C_{D_0} + C_{w_0}\cancel{\sin\theta_0}^{\,0}$$

$$\underline{X_u = -\rho u_0 S C_{D_0}}$$

$$\boxed{\omega_n = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}}$$

Same as Lanchester

$$\zeta = \frac{-X_u}{2}\sqrt{-\frac{u_0}{m Z_u g}} = \frac{\rho u_0 S C_{D_0}}{2}\sqrt{\frac{u_0}{\cancel{2} m g \rho u_0 S C_{L_0}}}$$

Note: in class I included this $\frac{1}{2}$ incorrectly. The final expression for ζ is still correct.

substitute $mg = \frac{1}{2}\rho u_0^2 S C_{L_0}$ and cancel

$$\boxed{\zeta = \frac{1}{\sqrt{2}}\frac{C_{D_0}}{C_{L_0}}}$$

High $\frac{L}{D} \Rightarrow$ less damping

(b/c less energy loss)

747

Full A_{lon}

$\lambda_{3,4} = -3.29\times10^{-3} \pm 6.72\times10^{-2} i$

$\zeta = 0.0489$

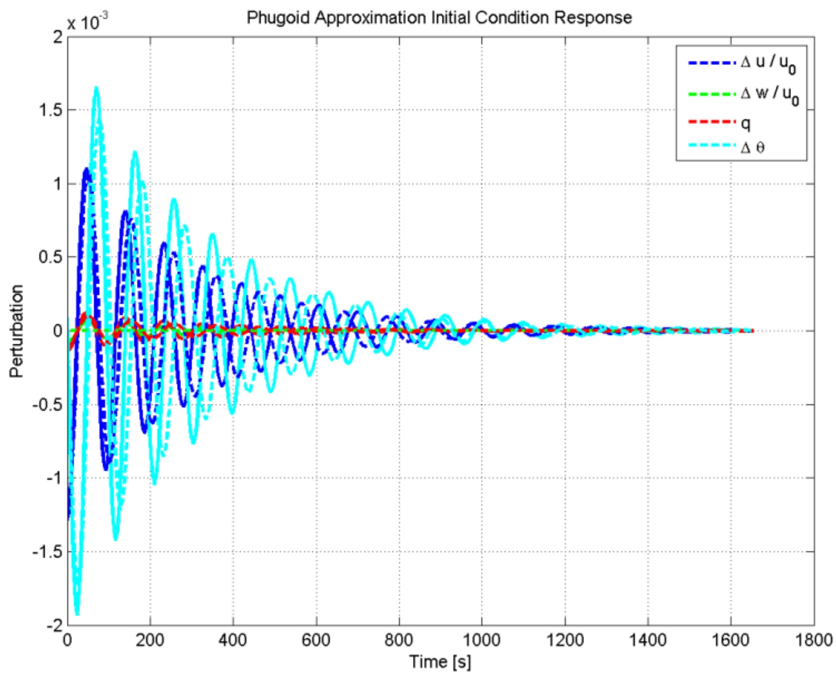
$\omega_n = 0.0673$

2×2 Ph. Approx

$\lambda_{ph} = -3.43\times10^{-3} \pm 6.11\times10^{-2} i$

$\zeta = 0.0561$

$\omega_n = 0.0612$



Longitudinal Control

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}.$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\vec{c}_{lon} = B_{lon}\vec{u}_{lon} \qquad \vec{u}_{lon} = \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}$$

← elevator, + = down

← throttle, + = more thrust

dimensional control derivatives

$$\Delta X_c = X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t$$

$$\Delta Z_c = Z_{\delta_e}\Delta\delta_e + \underline{Z_{\delta_t}}\Delta\delta_t$$

$$\Delta M_c = M_{\delta_e}\Delta\delta_e + \underline{M_{\delta_t}}\Delta\delta_t$$

often 0

$$B_{lon} = \begin{bmatrix} \dfrac{X_{\delta_e}}{m} & \dfrac{X_{\delta_t}}{m} \\ \dfrac{Z_{\delta_e}}{m - Z_{\dot{w}}} & \dfrac{Z_{\delta_t}}{m - Z_{\dot{w}}} \\ \dfrac{M_{\delta_e}}{I_y} + \dfrac{M_{\dot{w}}Z_{\delta_e}}{I_y(m - Z_{\dot{w}})} & \dfrac{M_{\delta_t}}{I_y} + \dfrac{M_{\dot{w}}Z_{\delta_t}}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix}$$

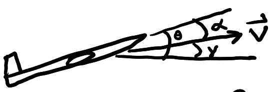
for 747

$$\begin{array}{c|cc} & \delta_e & \delta_t \\ \hline \Delta u & \boxed{-0.000187} & 9.66 \\ \Delta w & -17.85 & 0 \\ \Delta q & -1.158 & 0 \\ \Delta\theta & \boxed{0} & 0 \end{array}$$

$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + B_{lon}\vec{u}_{lon}$$

C and D depend on output

E.g. flight path angle γ



$$\Delta\gamma = \Delta\theta - \Delta\alpha$$

$$\Delta\gamma = \underbrace{\begin{bmatrix} 0 & -\frac{1}{u_0} & 0 & 1 \end{bmatrix}}_{C_\gamma} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$\vec{y} = [\Delta\gamma] = C_\gamma \vec{x}_{lon} + \underbrace{[0]}_{D}\vec{u}_{lon}$$

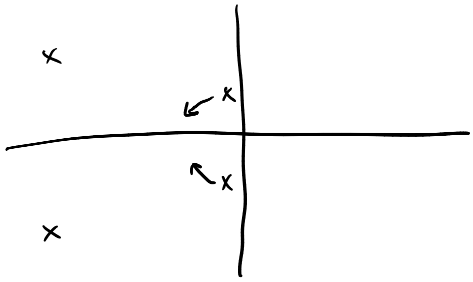
Open-loop step response to control inputs

$$B_{\delta_e} = B_{lon}(:,1) \qquad B_{\delta_t} = B_{lon}(:,2)$$

$\delta_e \quad 1°$ $\qquad \delta_t: \frac{1}{6} = 0.05\,W$

$\delta_e \Rightarrow \gamma$ little change $\qquad \delta_t \Rightarrow \gamma$ siginificant increase

Longitudinal stability augmentation



Realistic main goal:

- increase phugoid damping

On homework: change s.p. damping ratio

- not usually a real-life goal
- easy to do by hand

Increase Phugoid damping with

$$\Delta\delta_e = -k_\theta \Delta\theta$$

$$\dot{\vec{x}}_{ph} = \begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \underbrace{A_{ph}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix}}_{\text{previous}} + B_{ph\delta_e}\Delta\delta_e \qquad B_{lon} = \begin{bmatrix} & \\ & \end{bmatrix}$$

for 747

Derivation of A_{ph} and $B_{ph,\delta e}$

Assume $\dot{q}$ only depends on α, q, and δe

solve for $\dot{q}=0$ (after short period has damped out)

$$\cancelto{0}{\Delta\dot{q}} = \frac{M_w}{I_Y}\Delta w + \frac{M_{\delta e}}{I_Y}\Delta\delta e + \frac{M_q}{I_Y}\cancelto{0}{\Delta q}$$

$$\Rightarrow \Delta w = -\frac{M_{\delta e}}{M_w}\Delta\delta e$$

solve for $\Delta\dot{w} = 0$ (after s.p. oscillations damp out)

$$\Delta\dot{w} = 0 = \frac{Z_u}{m}\Delta u + \frac{Z_w}{m}\Delta w + \frac{m u_0}{m}\cancelto{\Delta\dot{\theta}}{\Delta q} + \frac{Z_{\delta e}}{m}\Delta\delta e$$

solve for $\Delta\dot{\theta}$, substitute Δw above

$$\frac{m u_0}{m}\Delta\dot{\theta} = -\frac{Z_u}{m}\Delta u - \frac{Z_w}{m}\left(-\frac{M_{\delta e}}{M_w}\Delta\delta e\right) - \frac{Z_{\delta e}}{m}\Delta\delta e$$

$$\boxed{\Delta\dot{\theta} = -\frac{Z_u}{m u_0}\Delta u + \left(\frac{Z_w}{m u_0}\frac{M_{\delta e}}{M_w} - \frac{Z_{\delta e}}{m u_0}\right)\Delta\delta e}$$

$$\boxed{\Delta\dot{u} = \frac{X_u}{m}\Delta u + \frac{X_{\delta e}}{m}\Delta\delta e}$$

$$\boxed{\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{X_u}{m} & -g\\ -\frac{Z_u}{m u_0} & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}\frac{X_{\delta e}}{m}\\ \frac{Z_w\frac{M_{\delta e}}{M_w} - Z_{\delta e}}{m u_0}\end{bmatrix}\Delta\delta e}$$

$$A_{ph} = \begin{bmatrix}-0.0069 & -32.2\\ 0.0001 & 0\end{bmatrix} \qquad B_{ph\delta e} = \begin{bmatrix}\cancelto{0}{-0.0002}\\ -0.44\end{bmatrix}$$

$$\Delta\delta e = -k_\theta\Delta\theta$$

$$\vec{u} = -K_{ph\theta}\vec{x}_{ph}$$

$$= -\begin{bmatrix}0 & k_\theta\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix}$$

$$A^{cl} = A_{ph} - B_{ph\delta_e} K_{ph\theta}$$

$$= A_{ph} - \begin{bmatrix} 0 \\ -0.44 \end{bmatrix} \begin{bmatrix} 0 & k_\theta \end{bmatrix}$$

$$= A_{ph} - \begin{bmatrix} 0 & 0 \\ 0 & -0.44 k_\theta \end{bmatrix}$$

$$= \begin{bmatrix} -0.0069 & -32.2 \\ -0.0001 & 0.44 k_\theta \end{bmatrix}$$

$$|A^{cl} - \lambda I| = (-0.0069 - \lambda)(0.44 k_\theta - \lambda) - (-0.0001)(-32.2) = 0$$

$$= \lambda^2 + \underbrace{(-0.44 k_\theta + 0.0069)}_{2\zeta\omega_n}\lambda \underbrace{- 0.003 k_\theta - 0.00322}_{-\omega_n^2} = 0$$

$$\zeta = 0.7 \implies k_\theta = -0.2$$

Plug back into 4x4

$$\Delta\delta_e = -K_\theta \vec{x}_{lon}$$

$$= -\begin{bmatrix} 0 & 0 & 0 & k_\theta \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$A^{cl} = A_{lon} - \underline{B_{\delta e}} K_\theta$$

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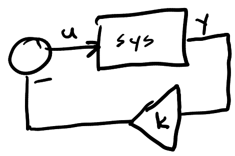
rlocus assumes $\underline{u} = -k y$



$$\Delta\delta_e = -k_\theta \Delta\theta$$

$$y = \Delta\theta = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{C_\theta} \overbrace{\begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}}^{\vec{x}_{lon}}$$

$D = 0$

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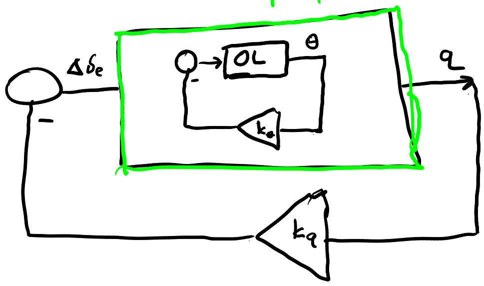
derivative gain

$$\Delta\delta_e = \underbrace{-k_\theta \Delta\theta}_{\text{proportional}} \underbrace{- k_q \Delta q}_{\text{derivative}}$$

$k_\theta = -0.5$

proportional feedback



$$\Delta q = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

from root locus choose $k_q = -1$