Aerospace Dynamics and Control Systems:
Exam 1 Course Notes Summary

ASEN Course Notes

Abstract

This document presents a comprehensive summary of course notes on Aerospace Dynamics and Control Systems. The material covers fundamental concepts of reference frames, coordinate systems, aircraft kinematics and dynamics, derivation of equations of motion, linearization techniques, and control system design. Special attention is given to the representation of aircraft orientation, transformation matrices, and development of linear control systems for aircraft stabilization.

Contents

1 Notation and Conventions

1.1 Reference Frames and Coordinate Systems

A reference frame is defined as a collection of at least three points with constant distances between each other. Key frames include:

- **Inertial frame** (F_I or F_E): A frame that translates with constant (possibly zero) velocity and does not rotate. Newton's second law is valid in an inertial frame.
- **Body-fixed frame** (F_B): A frame attached to the aircraft.
- **Earth frame**: Often used as an approximation of an inertial frame for aircraft dynamics.

1.2 Vector Notation

The notation system used throughout these notes:

- Vector quantities: Represented with arrows ($\vec{V}$) or bold symbols ($\boldsymbol{V}$).
- Unit vectors: Denoted by a hat, such as $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.
- Superscripts: Indicate the frame of reference (e.g., $\vec{V}^E$ means velocity with respect to the Earth frame).

- Subscripts: Indicate the coordinate system used to express the vector components (e.g., $\vec{V}_B$ means velocity expressed in body coordinates).
- Vector components: Often written in matrix form, for example: $\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

2 Forces, Moments, and Velocities

2.1 Aerodynamic Forces and Moments

The aerodynamic forces (excluding gravity) acting on an aircraft are represented by:

$$\vec{A} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k} \tag{1}$$

In body coordinates:

$$\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{2}$$

Similarly, moments are represented by:

$$\vec{G} = L\hat{\imath} + M\hat{\jmath} + N\hat{k} \tag{3}$$

In body coordinates:

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \tag{4}$$

2.2 Velocity and Angular Velocity

The aircraft's velocity vector with respect to the Earth, expressed in the body frame:

$$\vec{V}^E = u^E\hat{\imath} + v^E\hat{\jmath} + w^E\hat{k} \tag{5}$$

In body coordinates:

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \tag{6}$$

The magnitude of velocity is:

$$V_a = |\vec{V}^E| = \sqrt{(u^E)^2 + (v^E)^2 + (w^E)^2} \tag{7}$$

Similarly, angular velocity:

$$\vec{\omega}^E = p\hat{\imath} + q\hat{\jmath} + r\hat{k} \tag{8}$$

In body coordinates:

$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{9}$$

3 Aircraft Anatomy and Control Surfaces

3.1 Conventional Aircraft Components

The key components of a conventional fixed-wing aircraft include:

- **Fuselage**: Main body of the aircraft
- **Wing**: Primary lifting surface
- **Empennage**: Tail assembly, including horizontal and vertical stabilizers
- **Propeller/Engine**: Thrust generation

3.2 Control Surfaces and Their Effects

- **Ailerons** (δ_a): Located on the wings, they create differential lift causing the aircraft to roll. Right aileron down creates positive roll (right-wing down). Effect: $\delta_a \Rightarrow -L$
- **Elevator** (δ_e): Located on the horizontal stabilizer, controls pitch. Deflecting downward creates a nose-down moment. Effect: $\delta_e \Rightarrow -M, -Z$
- **Rudder** (δ_r): Located on the vertical stabilizer, controls yaw. Deflecting rightward creates a rightward yaw. Effect: $\delta_r \Rightarrow +N, +Y$
- **Throttle/Propulsion** (δ_p): Controls thrust. Effect: $\delta_p \Rightarrow +X$

4 Wind and Aerodynamic Angles

4.1 Relative Wind

Aerodynamic forces and moments (excluding gravity) are functions of the aircraft's velocity with respect to the air:

$$\vec{V} \equiv \vec{V}^W \tag{10}$$

The relationship between Earth-relative velocity and wind-relative velocity:

$$\vec{V}^E = \vec{V}^{(W)} + \vec{W}^{(E)} \tag{11}$$

When the wind is zero, $\vec{W} = \vec{0}$, then $\vec{V} = \vec{V}^E$.

4.2 Aerodynamic Angles

Two key angles define the orientation of the aircraft relative to the airflow:

- **Angle of Attack** (α): The angle between the body x-axis and the projection of the velocity vector onto the x-z plane.

$$\alpha = \arctan\left(\frac{w}{u}\right) \tag{12}$$

- **Sideslip Angle** (β): The angle between the velocity vector and the x-z plane.

$$\beta = \arcsin\left(\frac{v}{V}\right) \tag{13}$$

The velocity components can be expressed in terms of these angles:

$$u = V \cos\beta \cos\alpha \tag{14}$$
$$v = V \sin\beta \tag{15}$$
$$w = V \cos\beta \sin\alpha \tag{16}$$

5 Aircraft Orientation

5.1 Euler Angles

The aircraft's orientation is defined using three Euler angles:

- ϕ (roll): Rotation about the x-axis (1-axis)
- θ (pitch): Rotation about the y-axis (2-axis)
- ψ (yaw): Rotation about the z-axis (3-axis)

By convention, aircraft orientation is defined by a 3-2-1 sequence of rotations through ψ-θ-ϕ.

5.2 Direction Cosine Matrices (DCMs)

DCMs are used to transform vectors between different coordinate systems. The rotation matrices for the individual rotations are:

- Roll rotation matrix $\mathbf{R}_1(\phi)$:

$$\mathbf{R}_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \tag{17}$$

- Pitch rotation matrix $\mathbf{R}_2(\theta)$:

$$\mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{18}$$

- Yaw rotation matrix $\mathbf{R}_3(\psi)$:

$$\mathbf{R}_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The complete Earth-to-body DCM is given by:

$$\mathbf{R}_E^B = \mathbf{R}_1(\phi)\mathbf{R}_2(\theta)\mathbf{R}_3(\psi) \tag{20}$$

This yields:

$$\mathbf{R}_E^B = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \tag{21}$$

5.3 Properties of DCMs

Direction Cosine Matrices have important properties:

1. **Chaining property**: $\mathbf{R}_A^C = \mathbf{R}_B^C \mathbf{R}_A^B$
2. **Inverse property**: Since DCMs are orthonormal (columns are orthogonal unit vectors):

$$\mathbf{R}_B^A = (\mathbf{R}_A^B)^{-1} = (\mathbf{R}_A^B)^T \tag{22}$$

6 Kinematics

6.1 Vector Derivatives

The time derivative of a vector $\vec{P}$ is defined as:

$$\frac{d}{dt}\vec{P} = \text{time rate of change of } \vec{P} \tag{23}$$

For a vector $\vec{P}_B$ expressed in body coordinates:

$$\dot{\vec{P}}_B = \text{time rate of change of the elements (coordinates) of } \vec{P}_B \tag{24}$$

If $\vec{P}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$, then $\dot{\vec{P}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

6.2 Kinematic Transport Theorem

A fundamental relationship between vector derivatives in different reference frames:

$$\left(\frac{d\vec{P}}{dt}\right)_B = \dot{\vec{P}}_B + \vec{\omega}_B \times \vec{P}_B \tag{25}$$

Where $\vec{\omega}_B$ is the angular velocity of the body frame relative to the inertial frame, expressed in body coordinates.

For an inertial frame, this simplifies to:

$$\left(\frac{d\vec{P}}{dt}\right)_E = \dot{\vec{P}}_E \tag{26}$$

7 Aircraft Equations of Motion (EOM)

7.1 State Vector

The complete state vector for an aircraft can be represented as:

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \tag{27}$$

This includes position, orientation, velocity, and angular velocity components.

7.2 Translational Kinematics

The relationship between Earth-frame position derivatives and body-frame velocities:

$$\dot{\vec{P}}_E = (\mathbf{R}_B^E)^T \vec{V}_B^E \tag{28}$$

7.3 Rotational Kinematics

The relationship between Euler angle rates and body-frame angular velocities:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{29}$$

This is often written as:

$$\dot{\vec{o}} = \mathbf{T}\vec{\omega}_B \tag{30}$$

Where $\mathbf{T}$ is known as the "attitude influence matrix."

7.4 Translational Dynamics

From Newton's Second Law:

$$\vec{F} = m\vec{a} \tag{31}$$

For an aircraft:

$$\vec{F} = m\frac{d}{dt}\vec{V}^E \tag{32}$$

Using the kinematic transport theorem:

$$\dot{\vec{V}}_B^E = \frac{\vec{F}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E \tag{33}$$

The forces include aerodynamic forces and gravity:

$$\vec{F} = \vec{A} + m\vec{g} \tag{34}$$

7.5 Rotational Dynamics

From Euler's Second Law:

$$\frac{d\vec{h}}{dt} = \vec{G} \tag{35}$$

Where $\vec{h}$ is angular momentum and $\vec{G}$ is the total moment acting on the aircraft.
Angular momentum is related to angular velocity by:

$$\vec{h} = \mathbf{I}\vec{\omega} \tag{36}$$

Where $\mathbf{I}$ is the inertia tensor.
Using the kinematic transport theorem:

$$\mathbf{I}\dot{\vec{\omega}}_B + \vec{\omega}_B \times \mathbf{I}\vec{\omega}_B = \vec{G}_B \tag{37}$$

Solving for the angular acceleration:

$$\dot{\vec{\omega}}_B = \mathbf{I}^{-1}(\vec{G}_B - \vec{\omega}_B \times \mathbf{I}\vec{\omega}_B) \tag{38}$$

7.6 Complete Equations of Motion

The complete nonlinear aircraft equations of motion are:

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{P}}_E \\ \dot{\vec{o}} \\ \dot{\vec{V}}_B^E \\ \dot{\vec{\omega}}_B \end{bmatrix} = \begin{bmatrix} (\mathbf{R}_B^E)^T \vec{V}_B^E \\ \mathbf{T}\vec{\omega}_B \\ \frac{\vec{F}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E \\ \mathbf{I}^{-1}(\vec{G}_B - \vec{\omega}_B \times \mathbf{I}\vec{\omega}_B) \end{bmatrix} \tag{39}$$

8 Linearization and Control System Design

8.1 Linearization Process

The nonlinear equations of motion can be linearized around a trim condition:

$$\vec{x} \approx \vec{x}_o + \Delta\vec{x} \tag{40}$$

$$\dot{\vec{x}} \approx \Delta\dot{\vec{x}} \tag{41}$$

$$\vec{u} \approx \vec{u}_o + \Delta\vec{u} \tag{42}$$

For a hover trim condition of a quadrotor:

$$\vec{x}_o = \begin{bmatrix} x_{E,o} \\ y_{E,o} \\ z_{E,o} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{43}$$

$$\vec{u}_o = \begin{bmatrix} Z_{c,o} \\ L_{c,o} \\ M_{c,o} \\ N_{c,o} \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

8.2 Linear State-Space Model

The linearized equations of motion can be expressed in state-space form:

$$\Delta\dot{\vec{x}} = \mathbf{A}\Delta\vec{x} + \mathbf{B}\Delta\vec{u} \tag{45}$$

$$\Delta\vec{y} = \mathbf{C}\Delta\vec{x} + \mathbf{D}\Delta\vec{u} \tag{46}$$

Where:

- **A** is the state matrix
- **B** is the control input matrix
- **C** is the output matrix
- **D** is the feedthrough matrix

8.3 Linear Control Design Process

The process for designing a linear control system includes:

1. Derive the equations of motion
2. Linearize and separate the equations of motion
3. Design control architecture
4. Choose gain values
5. Test in linear simulation
6. Test in nonlinear simulation

8.4 Feedback Control

A typical control law for stabilization has the form:

$$\Delta\vec{u} = -\mathbf{K}\Delta\vec{x} \tag{47}$$

This creates a closed-loop system:

$$\Delta\dot{\vec{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\Delta\vec{x} = \mathbf{A}^{cl}\Delta\vec{x} \tag{48}$$

8.5 Gain Selection - Pole Placement

For a second-order system with characteristic equation:

$$\lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0 \tag{49}$$

Comparing to the standard form:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{50}$$

The gains can be selected as:

$$k_1 = 2\zeta\omega_n I_x \tag{51}$$

$$k_2 = \omega_n^2 I_x \tag{52}$$

For a system with desired damping ratio ζ and natural frequency ω_n.

8.6 Response Characteristics

For a second-order system with complex eigenvalues $\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$, the solution has the form:

$$\Delta\phi(t) = e^{-\zeta\omega_n t}(C_3\sin(\omega_d t) + C_4\cos(\omega_d t)) \tag{53}$$

Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.
Performance specifications include:

- Rise time: $t_r = \frac{1.8}{\omega_n}$
- Settling time: $t_s = \frac{4.6}{\zeta\omega_n}$
- Overshoot: $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

9 Special Cases and Applications

9.1 Quadrotor Dynamics

For a quadrotor, the control forces and moments are related to individual rotor forces by:

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} \\ -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{54}$$

Where:

- d is the distance from the center to each rotor
- k_m is the ratio of thrust to moment coefficient
- f_i is the force generated by rotor i

9.2 Drag Forces and Moments

Drag forces on an aircraft can be modeled as:

$$\vec{F}_d = -D\frac{\vec{V}}{V_a} \tag{55}$$

Where:

$$D = \frac{1}{2}\rho V_a^2 C_D A = \nu V_a^2 \tag{56}$$

Similarly, drag moments:

$$\vec{G}_B^d = \begin{bmatrix} L_d \\ M_d \\ N_d \end{bmatrix} = -\mu|\vec{\omega}| \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{57}$$

10 Solutions to Linear ODEs

10.1 Scalar Case

For a first-order system:

$$\dot{x} = ax \Rightarrow x(t) = x_0 e^{at} \tag{58}$$

For a second-order system:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{59}$$

The characteristic equation is:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{60}$$

With solutions:

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{61}$$

10.2 Vector Case

For a state-space system:

$$\dot{\vec{x}} = \mathbf{A}\vec{x} \Rightarrow \vec{x}(t) = e^{\mathbf{A}t}\vec{x}(0) \tag{62}$$

Where:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \dots \tag{63}$$

Using eigendecomposition, if $\vec{x}(0) = \sum_i a_i \vec{v}_i$ where $\vec{v}_i$ are eigenvectors of $\mathbf{A}$ with eigenvalues λ_i, then:

$$\vec{x}(t) = \sum_i a_i \vec{v}_i e^{\lambda_i t} \tag{64}$$

11 Modal Analysis

11.1 Eigenvalues and Eigenvectors

For a matrix $\mathbf{A}$, the eigenvalue problem is:

$$\mathbf{A}\vec{v}_i = \lambda_i \vec{v}_i \tag{65}$$

This can be rewritten as:

$$(\mathbf{A} - \lambda_i \mathbf{I})\vec{v}_i = \vec{0} \tag{66}$$

For non-trivial solutions, the determinant must be zero:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{67}$$

For a 2×2 matrix:

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc \tag{68}$$

11.2 Stability Analysis

The stability of a linear system depends on the eigenvalues of the state matrix:

- If all eigenvalues have negative real parts, the system is stable
- If any eigenvalue has a positive real part, the system is unstable
- If any eigenvalue has a zero real part, the system is marginally stable

For complex eigenvalues $\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$:

- The damping ratio ζ determines the decay rate
- The natural frequency ω_n determines the oscillation rate
- $\zeta < 0$ indicates an unstable system
- $0 < \zeta < 1$ indicates an underdamped system (oscillatory)
- $\zeta = 1$ indicates a critically damped system
- $\zeta > 1$ indicates an overdamped system

12 Summary

This document has covered the fundamental concepts of aerospace dynamics and control systems, including:

- Reference frames and coordinate systems
- Vector notation and transformations
- Aircraft forces, moments, and kinematics
- Direction Cosine Matrices for orientation representation
- Derivation of nonlinear equations of motion
- Linearization techniques
- State-space representation
- Control system design using pole placement
- Stability analysis using eigenvalues

These principles form the foundation for understanding aircraft dynamics and developing control systems for aircraft stabilization and navigation.