Final Review – Part 2: Conventional Aircraft

ASEN 3128 Aircraft Dynamics
UNIVERSITY OF COLORADO **BOULDER**

Announcements

- Final Exam: Saturday, May 4th 7:30-10pm
- TAs will be at Office Hours and Study Hall to answer questions
- Sunberg Extra Office Hours: Friday 4-5 pm

Course Material – Bold in Review 1

(See Syllabus for Reading Assignments)

1. **Nomenclature**
2. Aircraft Static Stability
3. **Euler Angles**
4. **Aircraft Equations of Motion (Kinematics + Dynamics)**
5. **Linearized Equations**
6. **Quadrotor Equations of Motion**
7. **Quadrotor Stability and Control**
8. Longitudinal Dynamics of Conventional (Fixed-Wing) Aircraft
9. Lateral Dynamics of Conventional (Fixed Wing) Aircraft
10. **State Space and Transfer Function-based Analysis**
11. Mode Approximations
12. **Conventional Aircraft Control**

Course Material – Bold in Review 2

(See Syllabus for Reading Assignments)

1. Nomenclature
2. Aircraft Static Stability
3. Euler Angles
4. Aircraft Equations of Motion (Kinematics + Dynamics)
5. Linearized Equations
6. Quadrotor Equations of Motion
7. Quadrotor Stability and Control
8. **Longitudinal Dynamics of Conventional (Fixed-Wi Aircraft**
9. **Lateral Dynamics of Conventional (Fixed Wing) Aircraft**
10. State Space and Transfer Function-based Analysis
11. **Mode Approximations**
12. Conventional Aircraft Control

Conventional Aircraft Dynamics

- ConventionalAircraft dynamics and aerodynamics are commonly separated into two groups:
  - Longitudinal
    - Up-down, pitch plane, pitching motions
  - Lateral-directional
    - Side-to-side, turning motions (roll and yaw)

LONGITUDINAL DYNAMICS

Longitudinal Forces and Moments

$$\boxed{\begin{aligned} C_L &= C_{L_\alpha}\alpha + C_{L_q}\hat{q} + C_{L_{\delta_e}}\delta_e \\ C_D &= C_{D_{min}} + K\left(C_L - C_{L_{min}}\right)^2 \\ C_m &= C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta_e}}\delta_e \end{aligned}}$$

$$C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m0_p} + a_t \bar{V}_H\left(\epsilon_0 + i_t\right)\left[1 - \frac{a_t S_t}{aS}\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right)\right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial\epsilon}{\partial\alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial\alpha}$$

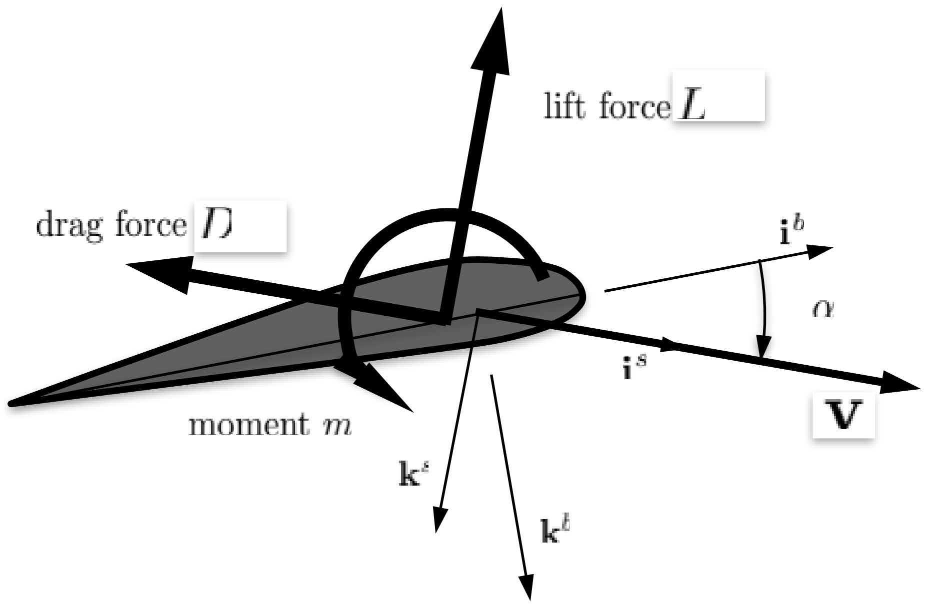
$$\underline{C_{m_\alpha} = C_{L_\alpha}\left(h - h_n\right)}$$

Direct dependence on CG location h

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial\delta_e}\frac{S_t}{S} = a_e\frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e\bar{V}_H + C_{L_{\delta_e}}\left(h - h_{n_{wb}}\right)$$

Longitudinal Forces – Body Frame



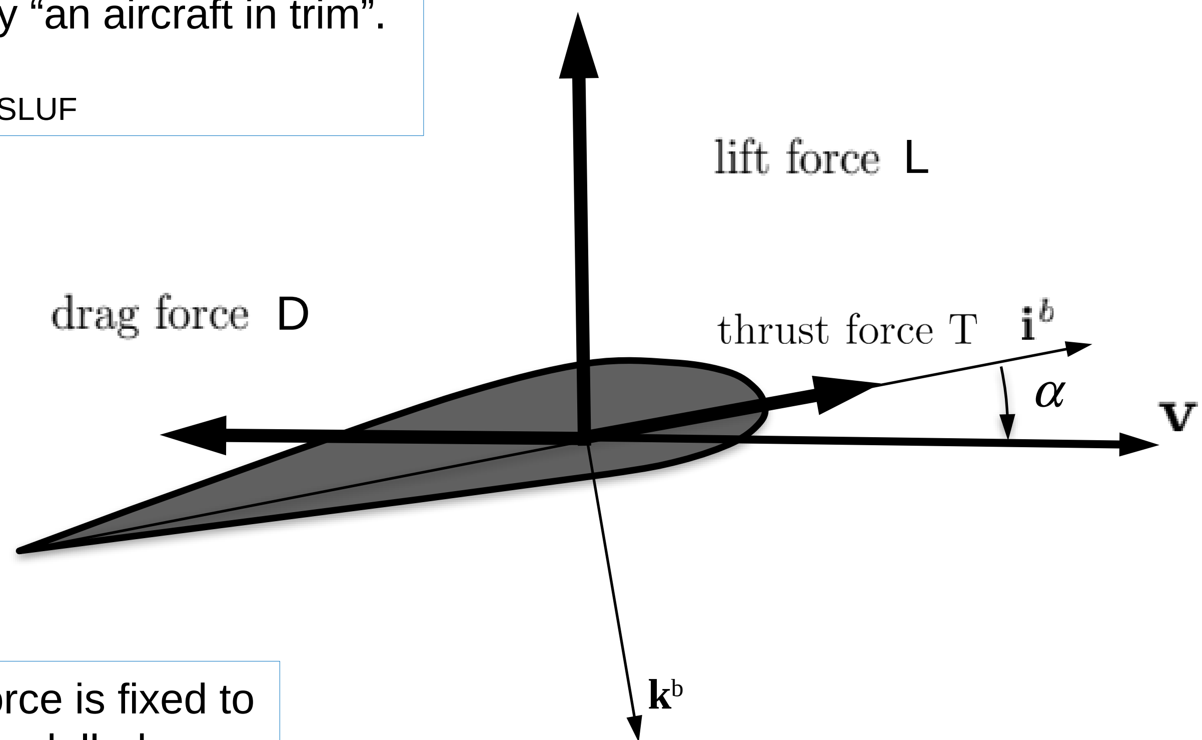
Lift and drag defined in stability (wind) frame, so rotate into body frame

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -D \\ -L \end{pmatrix}$$

Straight, level, unaccelerated flight (SLUF)

For a fixed-wing aircraft the main trim state will be straight, level, unaccelerated flight (SLUF)[1]. We will just say "an aircraft in trim".

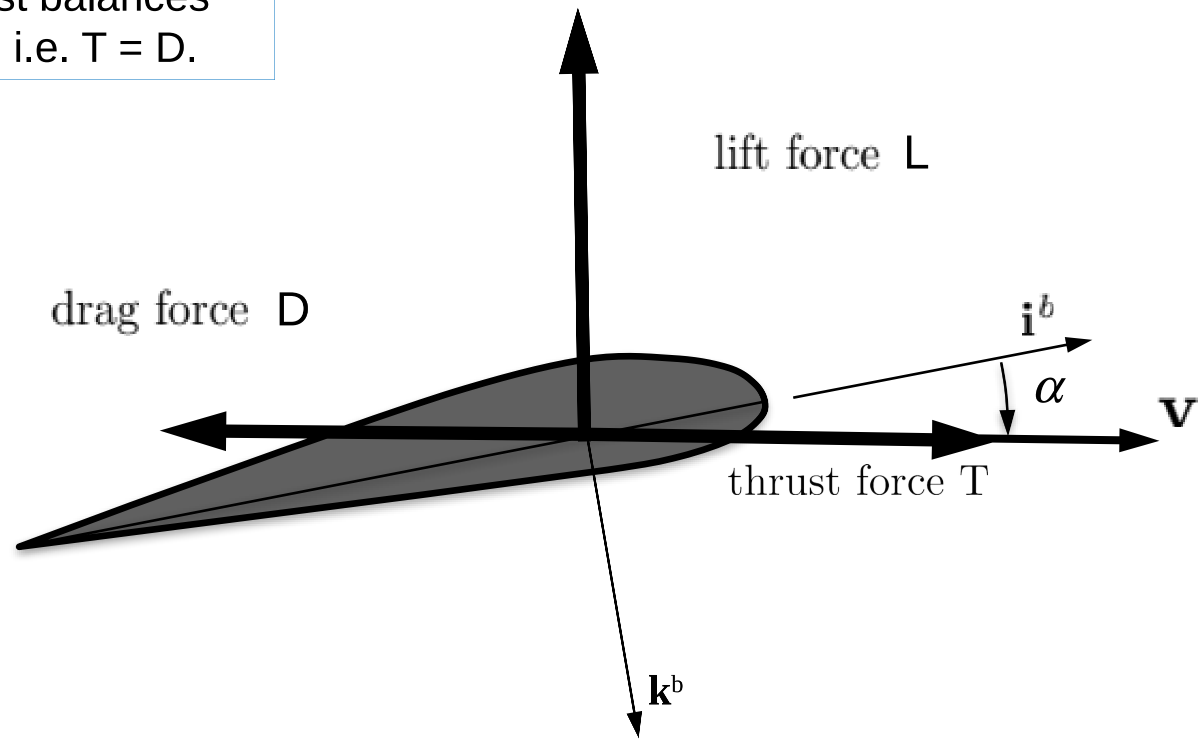
1. https://en.wiktionary.org/wiki/SLUF



In general, the thrust force is fixed to the body and is often modelled as pointing along the body x-axis.

Assumption: Thrust

For simple analysis of static stability (trim), we **assume** thrust balances drag and cancels it out, i.e. T = D.



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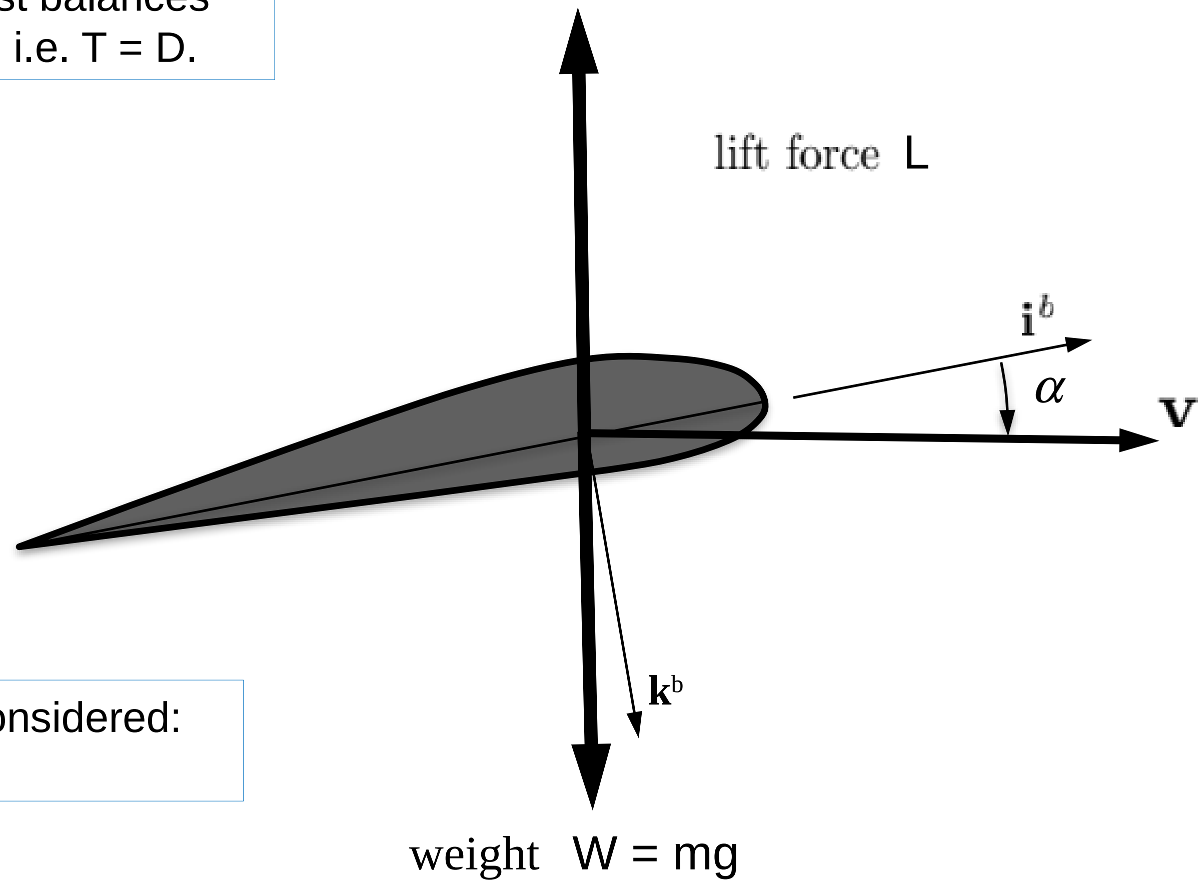


Only two forces are considered:
Lift and Weight

Equilibrium – Speed and Altitude

- At equilibrium we need

$$C_{m_{trim}} = 0$$

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V_a^2 S}$$

Density a function of (desired) altitude

Desired (air)speed

$$C_{L_{trim}} = C_{L_\alpha}\alpha_{trim} + C_{L_{\delta_e}}\delta_{e_{trim}}$$

$$C_{m_{trim}} = 0 = C_{m_0} + C_{m_\alpha}\alpha_{trim} + C_{m_{\delta_e}}\delta_{e_{trim}}$$

Trim Angle of Attack and Elevator

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta_e}} \\ C_{m_\alpha} & C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

solving

$$\alpha_{trim} = \frac{C_{m_0} C_{L_{\delta_e}} + C_{m_{\delta_e}} C_{L_{trim}}}{\Delta}$$

$$\delta_{e_{trim}} = -\frac{C_{m_0} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_\alpha}$$

General Trim Initial Condition

Straight, wings-level trim is defined by

u_0 • speed

γ_{a_0} • air-relative flight path angle

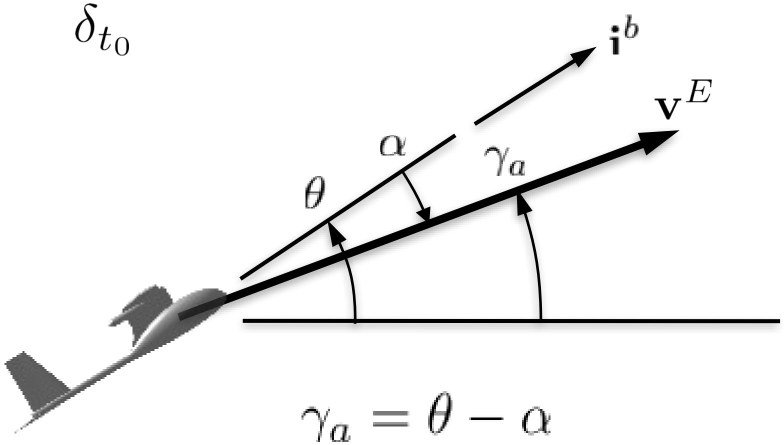
h_0 • altitude (height)

which is used to determine

α_0

δ_{e_0}

δ_{t_0}



$$\gamma_a = \theta - \alpha$$

Doesn't matter

Derived from γ_{a0} and α_0

$$\mathbf{x}_0 = \begin{pmatrix} \cdot \\ \cdot \\ -h_0 \\ 0 \\ \theta_0 \\ \cdot \\ u_0^E \\ 0 \\ w_0^E \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{u}_0 = \begin{pmatrix} \delta_{e_0} \\ 0 \\ 0 \\ \delta_{t_0} \end{pmatrix}$$

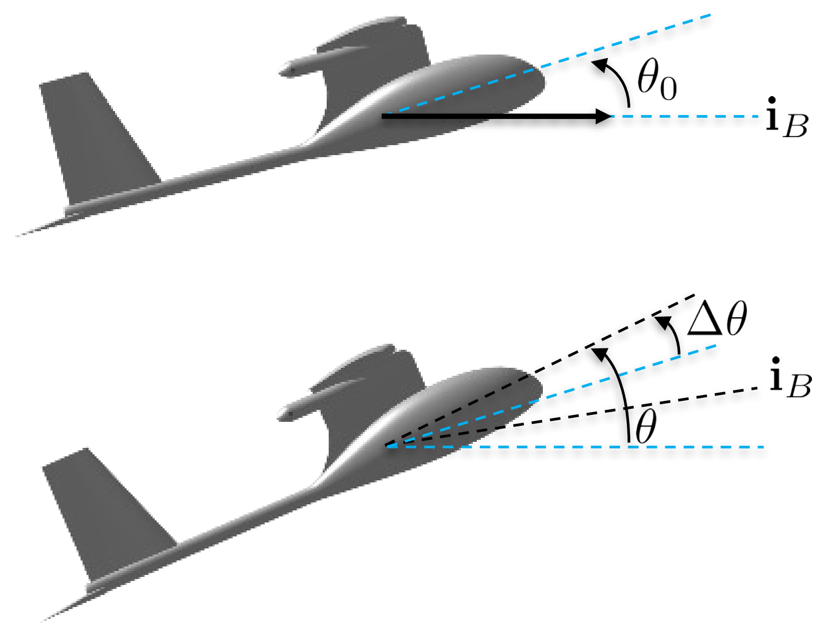
Derived from u_0 and α_0

This further assumes no background wind. What would be different…?

Variations

Can describe variables as reference condition ("trim condition") plus small variation, e.g $u^E = u = u_0 + \Delta u$

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x_{E_0} + \Delta x_E \\ y_{E_0} + \Delta y_E \\ z_{E_0} + \Delta z_E \\ \Delta\phi \\ \theta_0 + \Delta\theta \\ \Delta\psi \\ u_0 + \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$



Recall Nonlinear EOM

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} u^E \\ v^E \\ w^E \end{matrix}} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Which are kinematic and which are dynamic?

Linearized EOM

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{u} = -g\cos\theta_0 \Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0 \Delta r + g\cos\theta_0 \Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0 \Delta q - g\sin\theta_0 \Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

These variables depend on each other.

$$\Delta\dot{x}^E = \Delta u$$

$$\Delta\dot{y}^E = u_0 \cos\theta_0 \Delta\psi + \Delta v$$

$$\Delta\dot{z}^E = -u_0 \Delta\theta + \Delta w$$

$$\Delta\psi = \Delta r \sec\theta_0$$

They do not depend on these variables.

Linearized Aero Forces and Moments

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

Force due to control variables (more later)

Combine these equations with the linearized dynamic equations, group terms and put into matrix form

Longitudinal Equations

Dynamics of Flight, Eq. (4.9,18) $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + m u_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}} mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

6x6 (a.k.a "flight path") Dynamics Matrix

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \\ \Delta\dot{x} \\ \Delta\dot{z} \end{bmatrix} = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lon} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -u_0 & 0 & 0 \end{array}\right] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \\ \Delta x \\ \Delta z \end{bmatrix}$$

$$\Delta\dot{x} = \Delta u$$

$$\Delta\dot{z} = -u_0 \Delta\theta + \Delta w$$

Non-dimensionalization

Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 S\bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

$C_l = \frac{L}{\frac{1}{2}\rho V^2 S b}$

Why Dimensionless?

Dimensional derivatives change with

Trim condition

Size

Dimensionless derivatives stay ~~the same~~ close for an aircraft (with fixed CG)!

Longitudinal Dimensional Derivatives

$$Z_u = -\rho u_0 S C_{W_0} \cos\theta_0 + \tfrac{1}{2}\rho u_0 S C_{z_u}$$

$$X_u = \rho u_0 S C_{W_0} \sin\theta_0 + \tfrac{1}{2}\rho u_0 S C_{x_u}$$

$$Z_w = \frac{1}{2}\rho u_0 S C_{z_\alpha}$$

$$X_w = \frac{1}{2}\rho u_0 S C_{x_\alpha}$$

$$Z_{\dot{w}} = \frac{1}{4}\rho \bar{c} S C_{z_{\dot{\alpha}}}$$

$$X_{\dot{w}} = \frac{1}{4}\rho \bar{c} S C_{x_{\dot{\alpha}}}$$

$$Z_q = \frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$$

$$X_q = \frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$$

$$M_u = \tfrac{1}{2}\rho u_0 S \bar{c} C_{m_u}$$

$$M_w = \frac{1}{2}\rho u_0 S \bar{c} C_{m_\alpha}$$

$$M_{\dot{w}} = \frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot{\alpha}}}$$

$$M_q = \frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$$

Example Derivative Derivations:

$$\left(\frac{\partial V}{\partial w}\right)_0 = 0 \qquad \text{so} \qquad Z_w = \left(\frac{\partial Z}{\partial w}\right)_0 = \frac{1}{2}\rho u_0 S \left(\frac{\partial C_z}{\partial \hat{w}}\right)_0$$

$$\boxed{\hat{w} = \Delta\alpha}$$

$$Z_w = \frac{1}{2}\rho u_0 S C_{z_\alpha}$$

$$Z_{\dot{w}} = \left(\frac{\partial Z}{\partial \dot{w}}\right)_0 = \frac{1}{2}\rho u_0 \left(\frac{\partial C_z}{\partial \dot{\alpha}}\right)_0$$

$$= \frac{1}{2}\rho u_0 \left(\bar{c}/2u_0\right)\left(\frac{\partial C_z}{\partial \dot{\hat{\alpha}}}\right)_0$$

$$= \frac{1}{4}\rho \bar{c} S C_{z_{\dot{\alpha}}}$$

$$\underline{Z_q = \left(\frac{\partial Z}{\partial q}\right)_0} = \underline{\frac{1}{2}\rho u_0^2 \left(\frac{\partial C_z}{\partial q}\right)_0}$$

$$= \frac{1}{2}\rho u_0^2 \left(\bar{c}/2u_0\right)\left(\frac{\partial C_z}{\partial \hat{q}}\right)_0$$

$$= \frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$$

$$C_{z_q} = \frac{\partial C_z}{\partial \hat{q}} \qquad C_{z_q} \neq \frac{\partial C_z}{\partial q}$$

Example: Pitch Rate Derivatives

$$Z_q = \frac{1}{4}\rho u_0 \bar{c} S \boxed{C_{z_q}}$$

$$X_q = \frac{1}{4}\rho u_0 \bar{c} S \cancel{C_{x_q}}^{\,0}$$

by assumption

$$M_q = \frac{1}{4}\rho u_0 \bar{c}^2 S \boxed{C_{m_q}}$$

$$C_{Z_q} = -C_{L_q}$$

$$C_{m_q} = C_{m_q}$$

Stability frame (lift/drag) to body frame

$$= \underline{-2 a_t V_H}$$

$$= -2 a_t V_H \frac{l_t}{c}$$

Shape/geometry and mass to aerodynamic coefficients

Example: Pitch Rate Derivatives

$$Z_q = \frac{1}{4}\rho u_0 \bar{c} S C_{z_q} \qquad X_q = \frac{1}{4}\rho u_0 \bar{c} S \cancel{C_{x_q}} \; 0 \text{ by assumption} \qquad M_q = \frac{1}{4}\rho u_0 \bar{c}^2 S \boxed{C_{m_q}}$$

$$\Delta C_m = -V_H \Delta C_{L_t} = -a_t V_H \frac{q l_t}{u_0}$$

$$C_{m_q} = \boxed{\left(\frac{\partial C_m}{\partial \hat{q}}\right)}_0$$

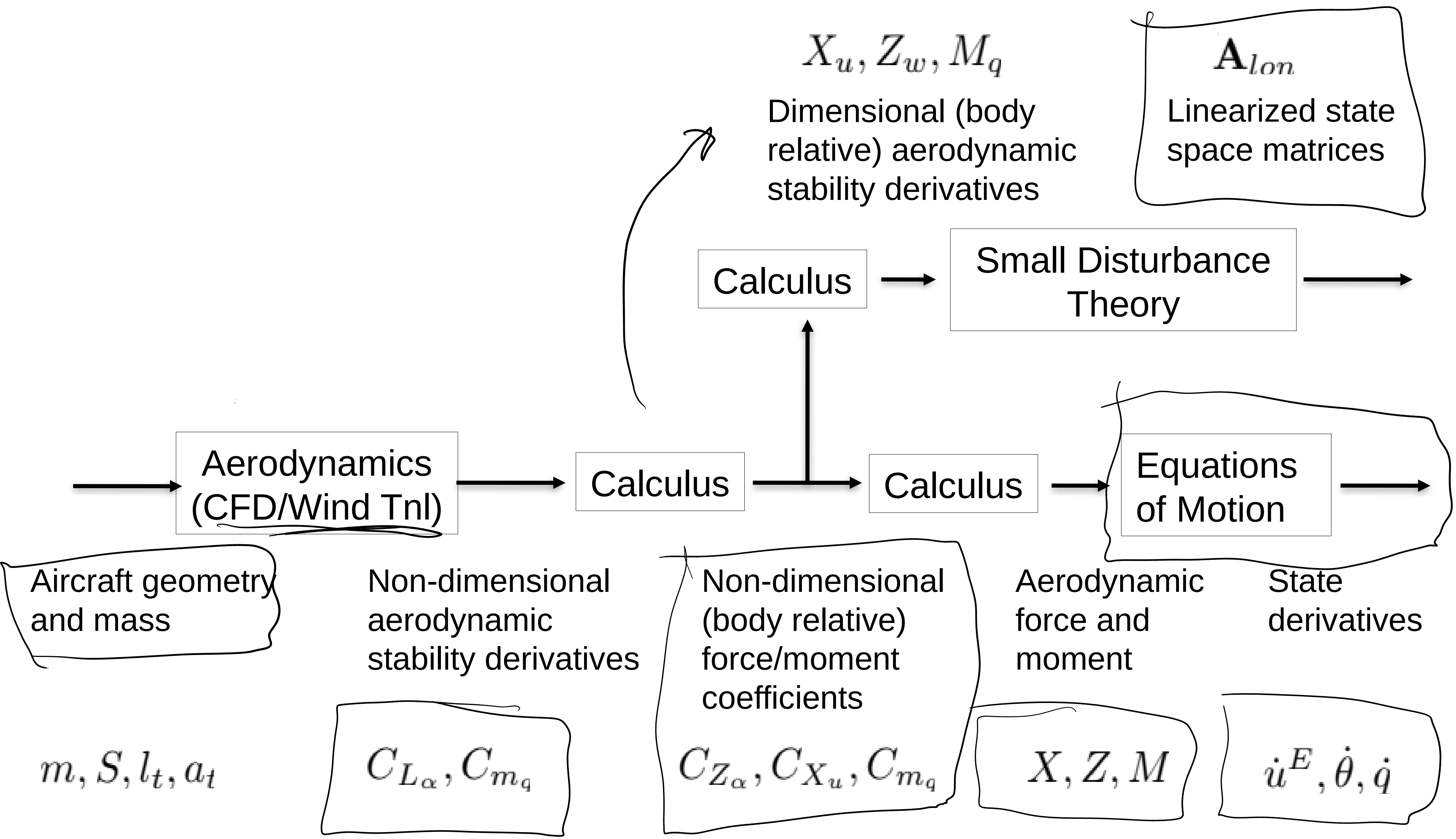
$$= \frac{2u_0}{c}\left(\frac{\partial C_m}{\partial q}\right)_0$$

$$= -\frac{2u_0}{c} a_t V_H \frac{l_t}{u_0}$$

$$\boxed{= -2 a_t V_H \frac{l_t}{c}}$$

Remember: derivatives are taken w.r.t. nondimensional values ()

Big Picture


X_u, Z_w, M_q
Dimensional (body relative) aerodynamic stability derivatives
$\mathbf{A}_{lon}$
Linearized state space matrices
Calculus
Small Disturbance Theory
Aerodynamics (CFD/Wind Tnl)
Calculus
Calculus
Equations of Motion
Aircraft geometry and mass
Non-dimensional aerodynamic stability derivatives
Non-dimensional (body relative) force/moment coefficients
Aerodynamic force and moment
State derivatives
m, S, l_t, a_t
C_{L_α}, C_{m_q}
$C_{Z_\alpha}, C_{X_u}, C_{m_q}$
X, Z, M
$\dot{u}^E, \dot{\theta}, \dot{q}$

Longitudinal Modes

Boeing 747 Jet Transport (Etkin 6.2)

$$h_0 = 40000 \text{ ft}$$
$$V_0 = 774 \text{ ft/s}$$
$$\gamma_0 = \theta_0 = \alpha_0 = 0^\circ$$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Longitudinal Modes

	λ_i	ζ	ω_n
Short period	$-3.72e-01+8.88e-01i$	$3.87e-01$	$9.62e-01$
	$-3.72e-01-8.88e-01i$	$3.87e-01$	$9.62e-01$
Phugoid	$-3.29e-03+6.72e-02i$	$4.89e-02$	$6.73e-02$
	$-3.29e-03-6.72e-02i$	$4.89e-02$	$6.73e-02$

Eigenvalues

With their damping ratio and natural frequency

α

$$\mathbf{v}_{1/2} \quad \begin{pmatrix} 0.0211 \pm 0.0166i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \mp 0.0004i \end{pmatrix}$$

u

$$\mathbf{v}_{3/4} \quad \begin{pmatrix} -0.9983 \\ -0.0573 \pm 0.0097i \\ -0.0001 \mp 0.0000i \\ 0.0001 \pm 0.0021i \end{pmatrix}$$

Associated Eigenvectors

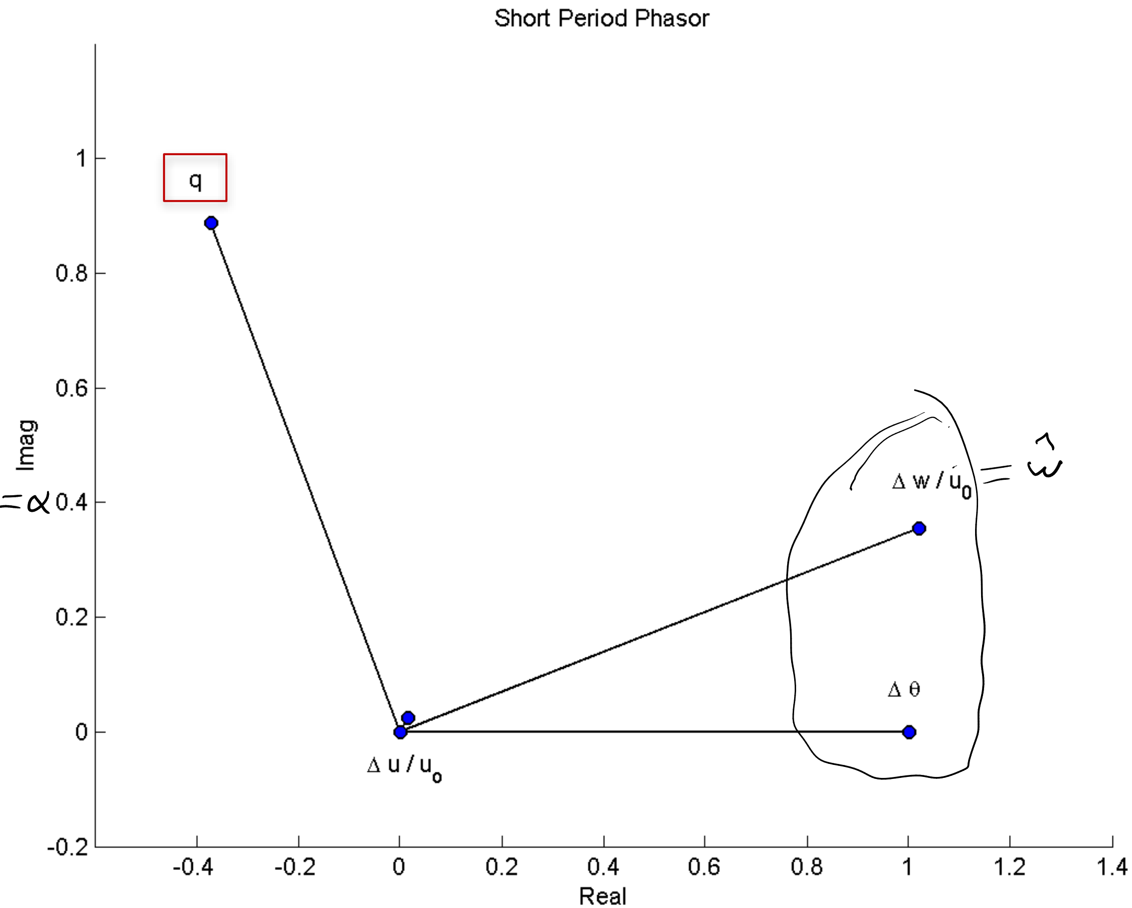
Short Period Mode

$$\lambda_{1/2} = -.372 + .888i$$

$\zeta = 0.387$ ← well damped

$\omega_n = 0.962$ ← fast response

$$\hat{\mathbf{v}}_1 = \begin{array}{cl} \left(\begin{array}{c} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ \boxed{0.9623\angle 112.74^\circ} \\ 1.0000\angle 0.0^\circ \end{array}\right) & \begin{array}{l} \hat{u} \\ \hat{w}=\alpha \\ \boxed{q} \\ \Delta\theta \end{array} \end{array}$$

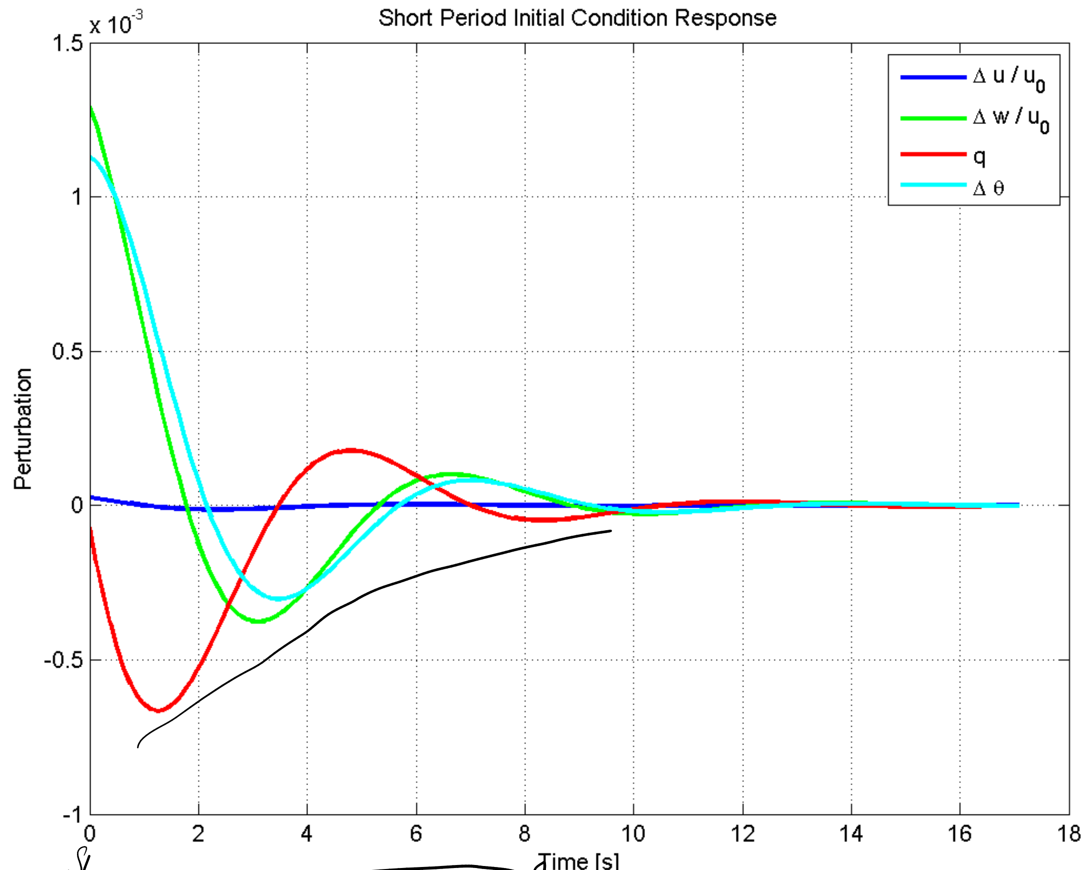


The short period mode is characterized by a fast oscillatory response in pitch. The airspeed perturbations are small and the flight path angle stays close to zero since pitch and angle of attack are almost equivalent.

Short Period Mode

$$\lambda_{1/2} = -.372 + .888i$$
$$\zeta = 0.387$$
$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



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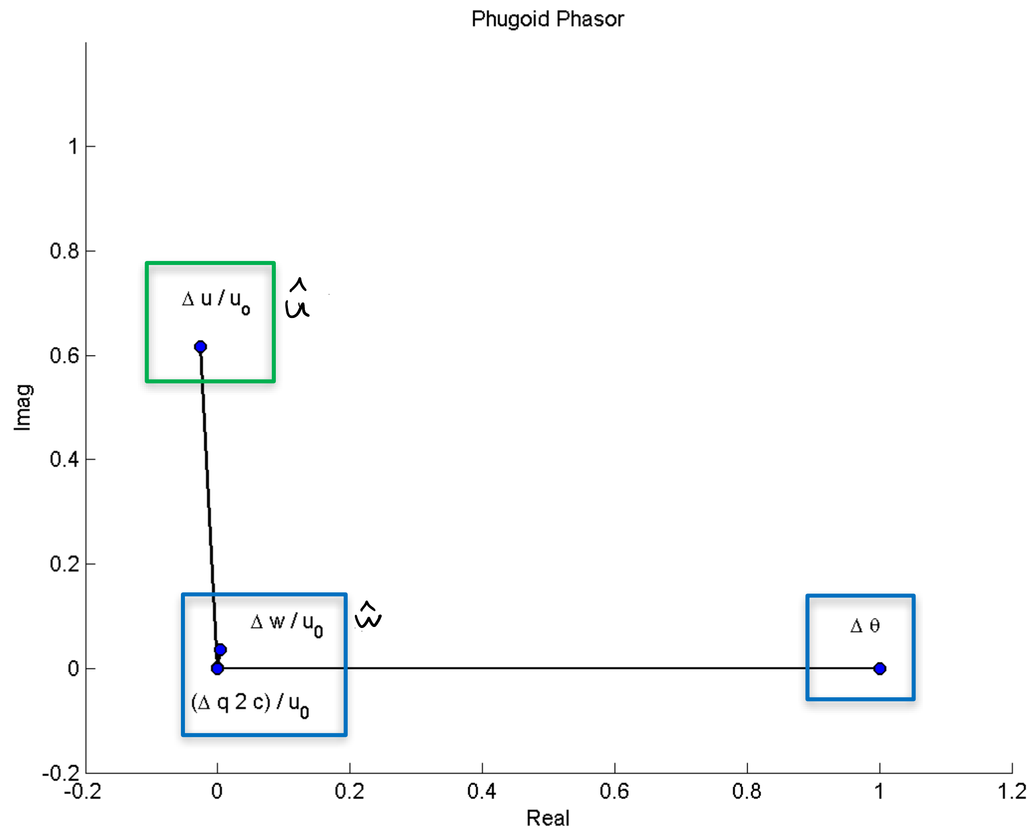
Phugoid Mode

$$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$$
$$\zeta = 0.0489$$ poorly damped
$$\omega_n = 0.0673$$ slow response

$$\begin{array}{cc} \mathbf{v}_{3/4} = \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0012\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} & \begin{array}{c} \hat{u} \\ \hat{\alpha} \\ \hat{q} \\ \Delta\theta \end{array} \end{array}$$

Non-dimensionalize and scale so Δθ term is 1



The phugoid mode is characterized by damped vertical oscillations of the aircraft’s trajectory about the nominal flight path with speed increasing (decreasing) as the height decreases (increases). It can be interpreted as an exchange of potential and kinetic energy

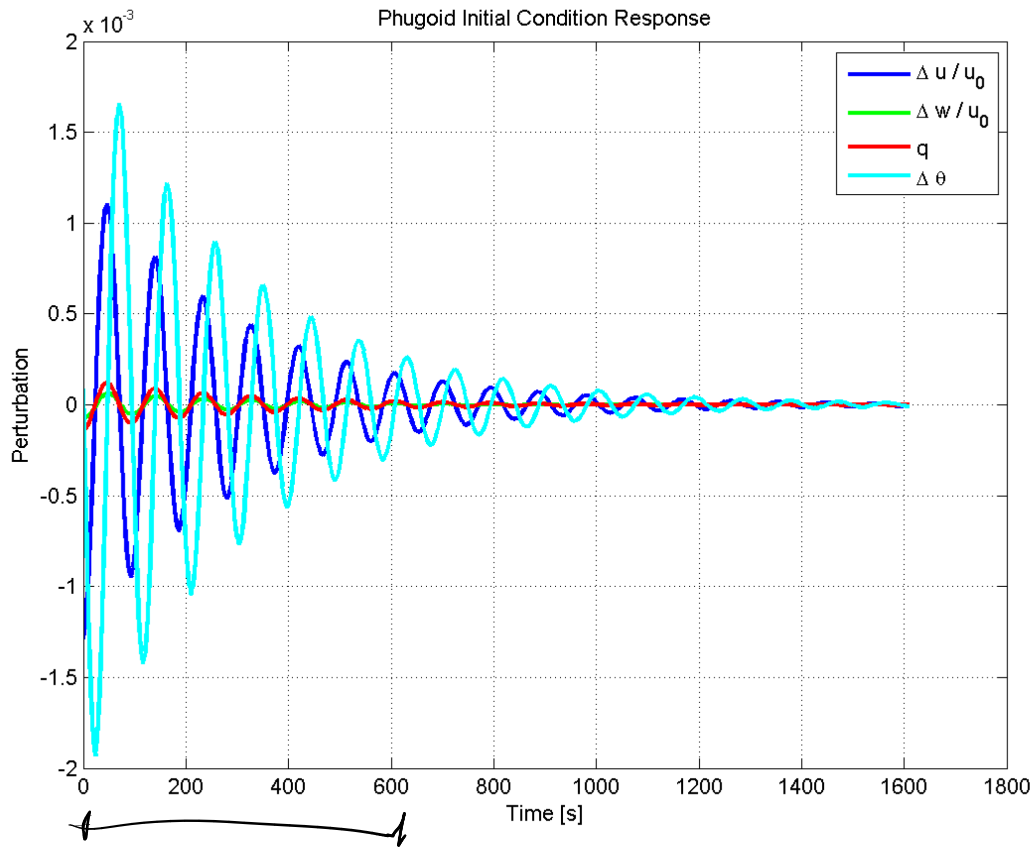
Phugoid Mode

$$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$$

$\zeta = 0.0489$ ← poorly damped

$\omega_n = 0.0673$ ← slow response

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



The phugoid mode is characterized by damped vertical oscillations of the aircraft's trajectory about the nominal flight path with speed increasing (decreasing) as the height decreases (increases). It can be interpreted as an exchange of potential and kinetic energy

Short Period Approximation

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ 0.9623\angle 112.74^\circ \\ 1.0000\angle 0.0^\circ \end{pmatrix} \begin{array}{c} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{array} \longrightarrow \begin{array}{l} \Delta u = 0 \\ \theta_0 = \alpha_0 = 0 \end{array}$$

$$\begin{pmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \dfrac{Z_w}{m} & u_0 \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m}\right] & \dfrac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] \end{pmatrix} \begin{pmatrix} \Delta w \\ \Delta q \end{pmatrix}$$

Analytical solution for eigenvalues of 2nd order system is straightforward.

$$\lambda^2 - \lambda \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}\left(M_q + M_{\dot{w}} u_0\right)\right]}_{-2\zeta\omega_n} \underbrace{- \frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

Short Period Approximation

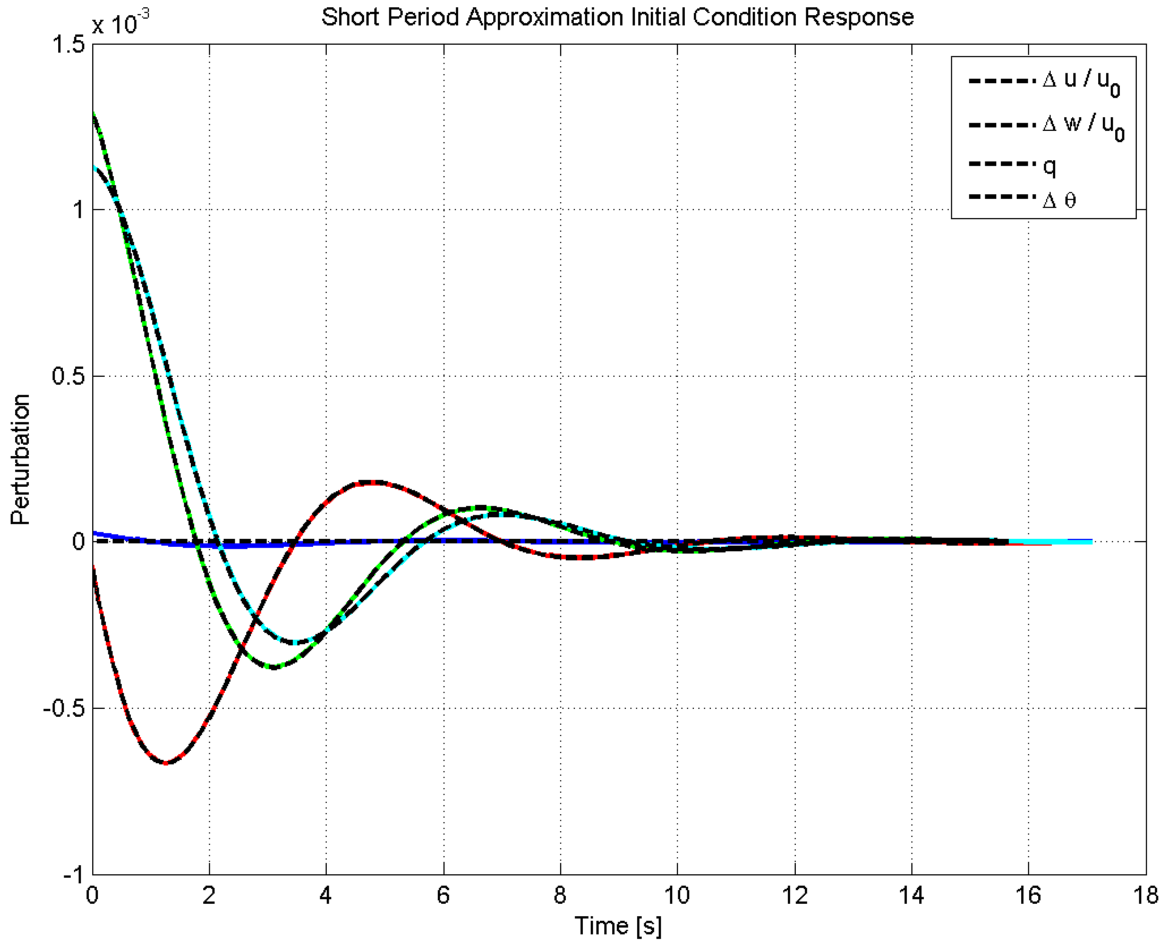
$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ 0.9623\angle 112.74^\circ \\ 1.0000\angle 0.0^\circ \end{pmatrix} \begin{array}{c} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{array}$$

Full longitudinal matrix

$$\lambda_{1/2} = -.372 + .888i$$
$$\zeta = 0.387$$
$$\omega_n = 0.962$$

Short period approximation

$$\lambda_{sp} = -.371 + .889i$$
$$\zeta = 0.385$$
$$\omega_n = 0.963$$



Phugoid Approximation

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0673\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} \begin{array}{l} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{array}$$

Lanchester (1908)

$$\Delta\alpha = 0$$
$$\theta_0 = \alpha_0 = 0$$
$$q = 0$$
$$T = D$$

Sum the forces in the Z-direction and apply Newton's Law

$$W - L = m\Delta\ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m\Delta\ddot{z}_E$$

$$m\Delta\ddot{z}_E + \rho g S C_{W_0} \Delta z_E = 0$$

$$\Delta\ddot{z}_E + \frac{\rho g S C_{W_0}}{m}\Delta z_E = 0$$

$$\omega_n = \sqrt{\frac{\rho g S C_{W_0}}{m}} \longrightarrow T = 2\pi/\omega_n = 2\pi\sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi\sqrt{2}\frac{u_0}{g}} = 0.138 u_0 = 0.453 u_0$$

when u_0 in feet per second

when u_0 in meters per second

Full longitudinal matrix

$$T = 93 \text{ s}$$

Lanchester phugoid approximation

$$\boxed{T = 107 \text{ s}}$$

Phugoid Approximation

$$\mathbf{v}_{3/4} = \begin{array}{c} \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0673\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} \end{array} \begin{array}{c} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{array}$$

$\longrightarrow$

$$\Delta\alpha = 0$$
$$\theta_0 = \alpha_0 = 0$$

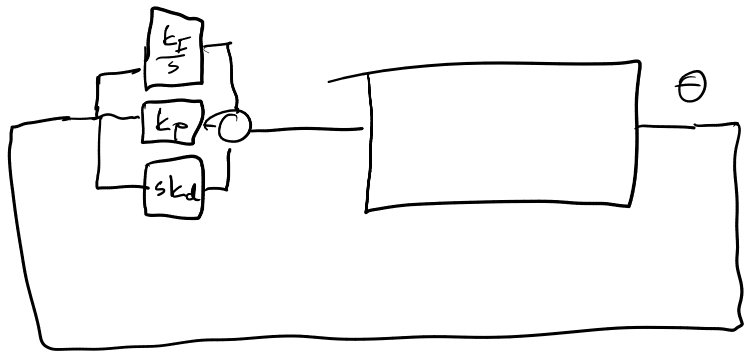
$$\begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{pmatrix} = \boxed{\begin{pmatrix} X_u/m & -g \\ -\dfrac{Z_u}{mu_0} & 0 \end{pmatrix}} \begin{pmatrix} \Delta u \\ \Delta\theta \end{pmatrix}$$

$\big\downarrow$

Analytical solution for eigenvalues of 2nd order system is straightforward.

$$\lambda^2 - \lambda\underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n} - \underbrace{\frac{Z_u g}{mu_0}}_{-\omega_n^2} = 0$$

Pitch Hold Controller



Break

Warning: Video shows Mild Injury

positive_kp_gain_skiMeme.mp4

Frew.1

Warning! Motion sickness possible.

https://www.youtube.com/watch?v=pGFnz9ffjOI

Break

LATERAL DYNAMICS

Big Picture

More Accurate,
Fewer Assumptions

$$\dot{\mathbf{p}}^E = R_B^E \cdot \mathbf{V}_B^E$$
$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$
$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$
$$\dot{\omega}_B = \mathbf{I}_B^{-1}\left[-\tilde{\omega}_B\left(\mathbf{I}_B \omega_B\right) + \mathbf{G}_B\right]$$

Nonlinear EOM

Linearization ← Trim ← $C_{L_{trim}} = C_{L_\alpha}\alpha_{trim} + C_{L_{\delta_e}}\delta_{e_{trim}}$

A_lon, B_lon   A_lat, B_lat

Longitudinal and Lateral
Linearized EOM

S.P.   Phugoid   Roll   Spiral   Dutch Roll

Mode Approximations

More Assumptions
Easier to Analyze

Lateral Equations

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \boxed{\Gamma_3 L_v + \Gamma_4 N_v} & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

are functions of the moments of inertia from lecture 4

Lateral Matrix

Define matrix elements in order to simplify notation

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix} \quad \mathbf{B}_{lat} = \begin{pmatrix} 0 & \mathcal{Y}_{\delta_r} \\ \mathcal{L}_{\delta_a} & \mathcal{L}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \\ 0 & 0 \end{pmatrix}$$

6x6 (a.k.a "flight path") Dynamics Matrix

$$
\left[\begin{array}{cccc|cc}
 & & A_{lat} & & & \\
\hline
0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0
\end{array}\right]
\begin{bmatrix} v \\ p \\ r \\ \phi \\ \psi \\ y \end{bmatrix}
$$

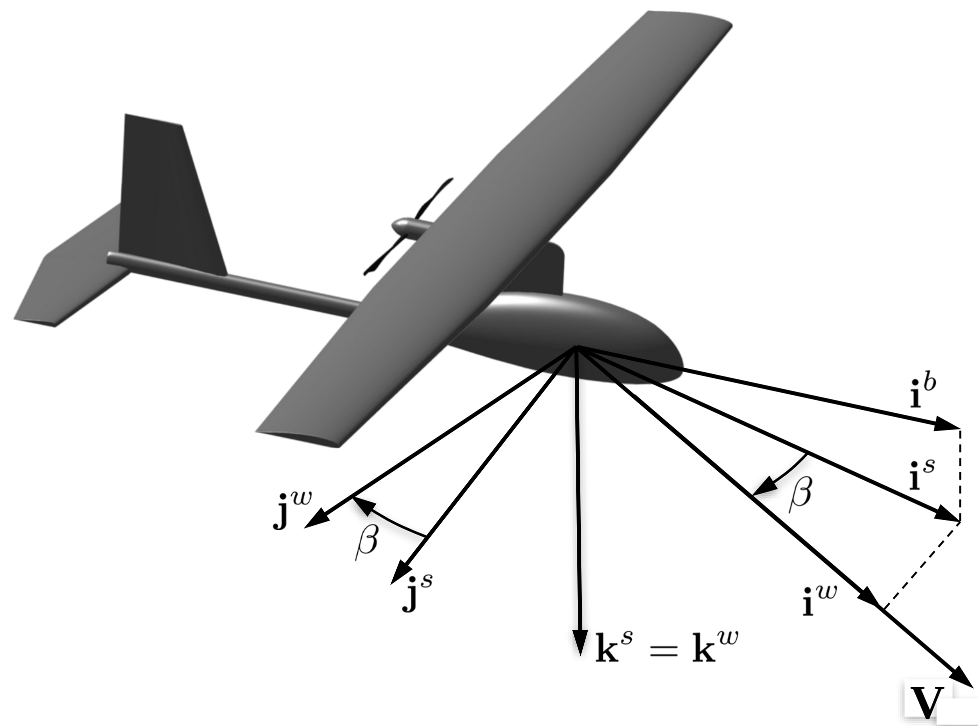
$$\Delta\dot{\psi} = \sec\theta_0 \Delta r$$

$$\Delta\dot{y} = \Delta v + u_0 \Delta\psi \cos\theta_0$$

Sideslip

Aerodynamics a function of the sideslip angle, not roll or yaw

$$\Delta\beta = \sin^{-1}\frac{\Delta v}{V} \approx \frac{\Delta v}{u_0} = \hat{v}$$



Dimensional Derivatives

$$Y_v = \frac{1}{2}\rho u_0 S C_{y_\beta} \qquad L_v = \frac{1}{2}\rho u_0 b S C_{l_\beta} \qquad N_v = \frac{1}{2}\rho u_0 b S C_{n_\beta}$$

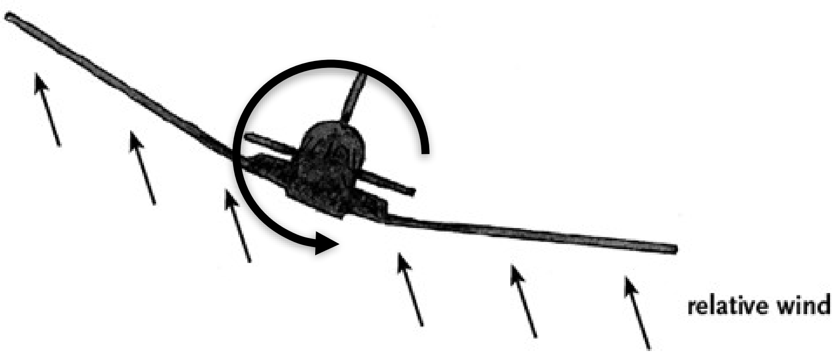
$$Y_p = \frac{1}{4}\rho u_0 b S C_{y_p} \qquad L_p = \frac{1}{4}\rho u_0 b^2 S C_{l_p} \qquad N_p = \frac{1}{4}\rho u_0 b^2 S C_{n_p}$$

$$Y_r = \frac{1}{4}\rho u_0 b S C_{y_r} \qquad L_r = \frac{1}{4}\rho u_0 b^2 S C_{l_r} \qquad N_r = \frac{1}{4}\rho u_0 b^2 S C_{n_r}$$

Beta Derivatives

$$Y_v = \frac{1}{2}\rho u_0 S C_{y_\beta} \qquad L_v = \frac{1}{2}\rho u_0 b \boxed{S C_{l_\beta}} \qquad N_v = \frac{1}{2}\rho u_0 b S C_{n_\beta}$$

C_{l_β} = dihedral effect

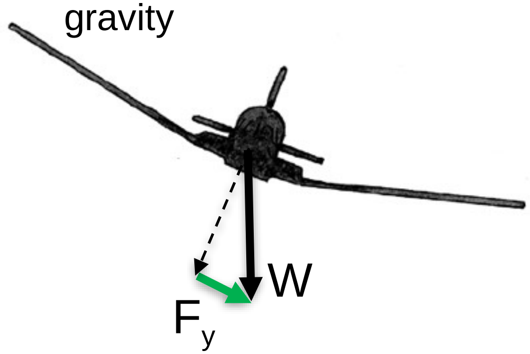


- Even though roll stiffness is small, aircraft are often still stable in roll
- A roll angle induces a side force due to gravity, which induces a sideslip
- Based on $C_{l\beta}$ a negative (restoring) moment is created
- The derivative $C_{l\beta}$ is called "dihedral effect" because it tends to depend heavily on wing dihedral

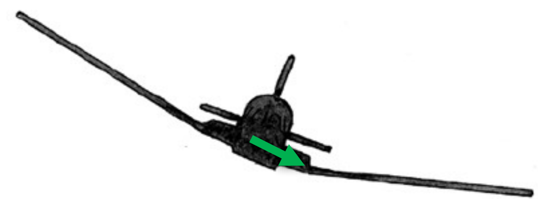
Dihedral effect

Aircraft flying into the page, straight and level

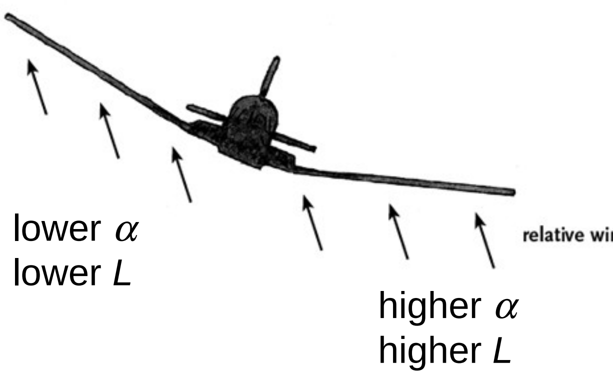
Positive roll induces positive side-force due to gravity



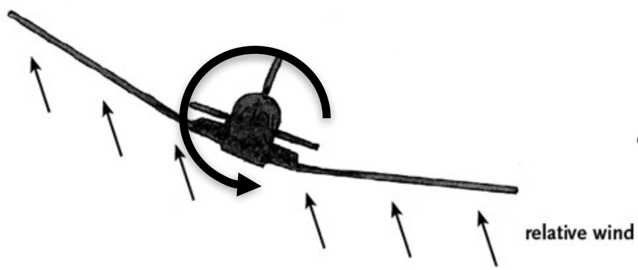
Side-force induces positive sideslip



Sideslip creates imbalance of angle of attack, lift

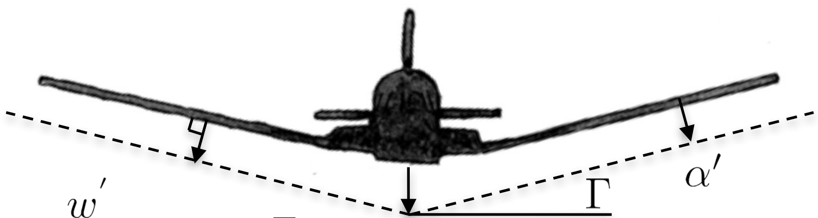


Imbalance in lift creates negative roll moment



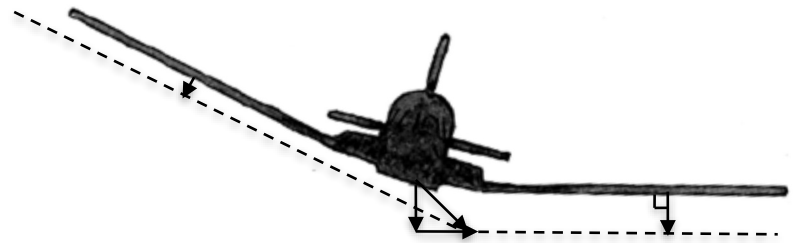
Aircraft returns to straight and level flight

Dihedral Effect – Dihedral Angle



$$\alpha' = \frac{w'}{V} = \alpha \cos \Gamma$$

$$w = V\alpha$$



$$v = V\beta$$

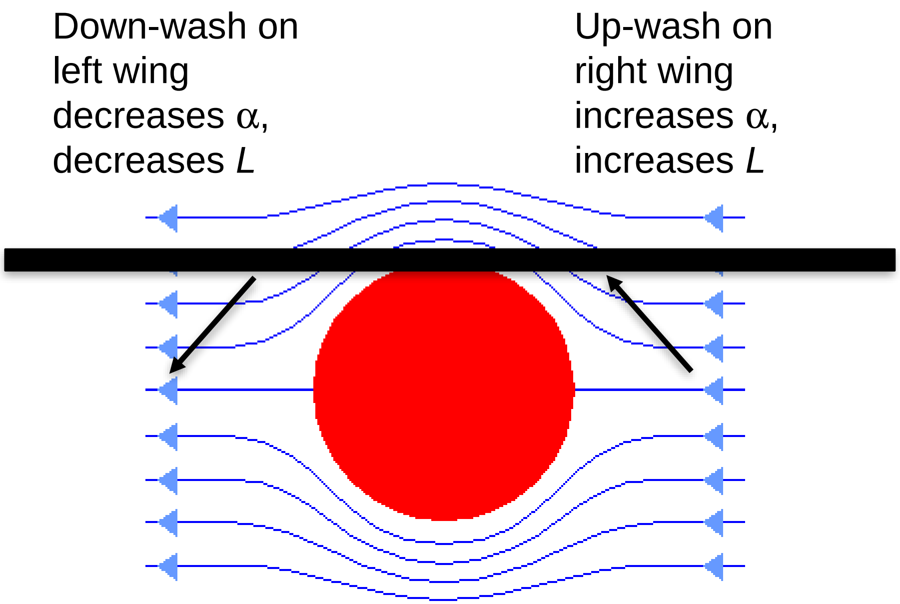
$$\Delta C_l^\Gamma \propto \Delta\alpha C_L$$

Asymmetric lift cause a negative (restoring) roll moment

$$\boxed{C_{l_\beta}^\Gamma \propto \Gamma}$$

$$C_{l_\beta}^\Gamma \propto C_L$$

Dihedral Effect – Wing Location



high wing

Asymmetric lift cause a negative (restoring) roll moment

Aircraft flying into the page

Dihedral Effect – Wing Sweep

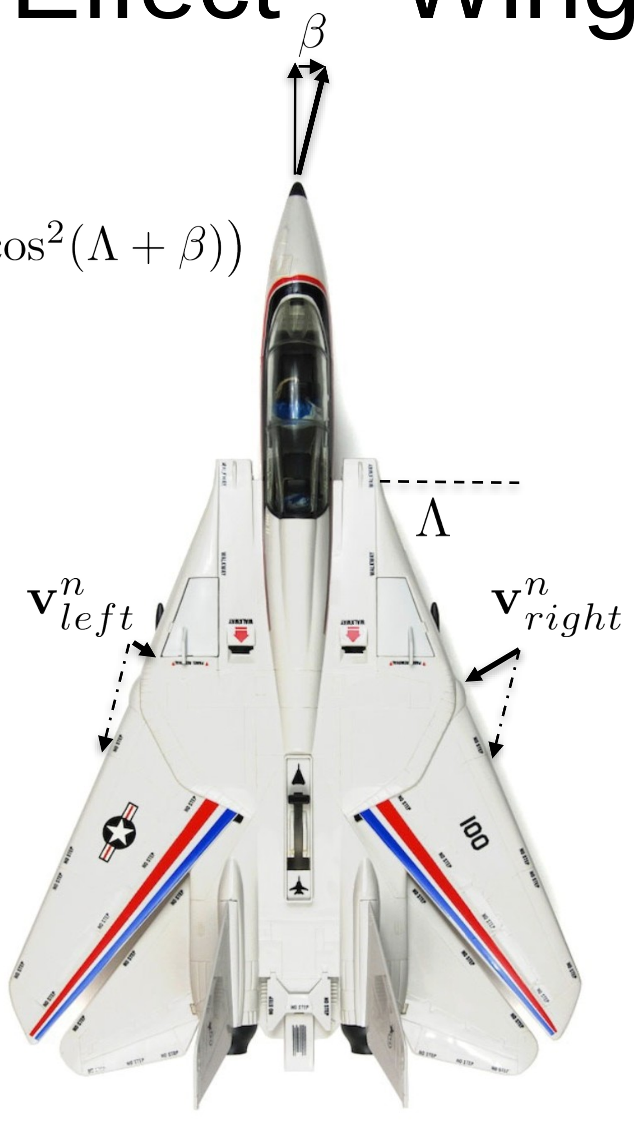
$$\Delta C_l^{\Lambda} \propto C_L \left(V_{left}^2 - V_{right}^2 \right)$$

$$\Delta C_l^{\Lambda} \propto C_L V^2 \left(\cos^2(\Lambda - \beta) - \cos^2(\Lambda + \beta) \right)$$

$$\Delta C_l^{\Lambda} \propto 2\beta C_L V^2 \sin 2\Lambda$$

$$C_{l_\beta}^{\Lambda} \propto 2 C_L V^2 \sin 2\Lambda$$

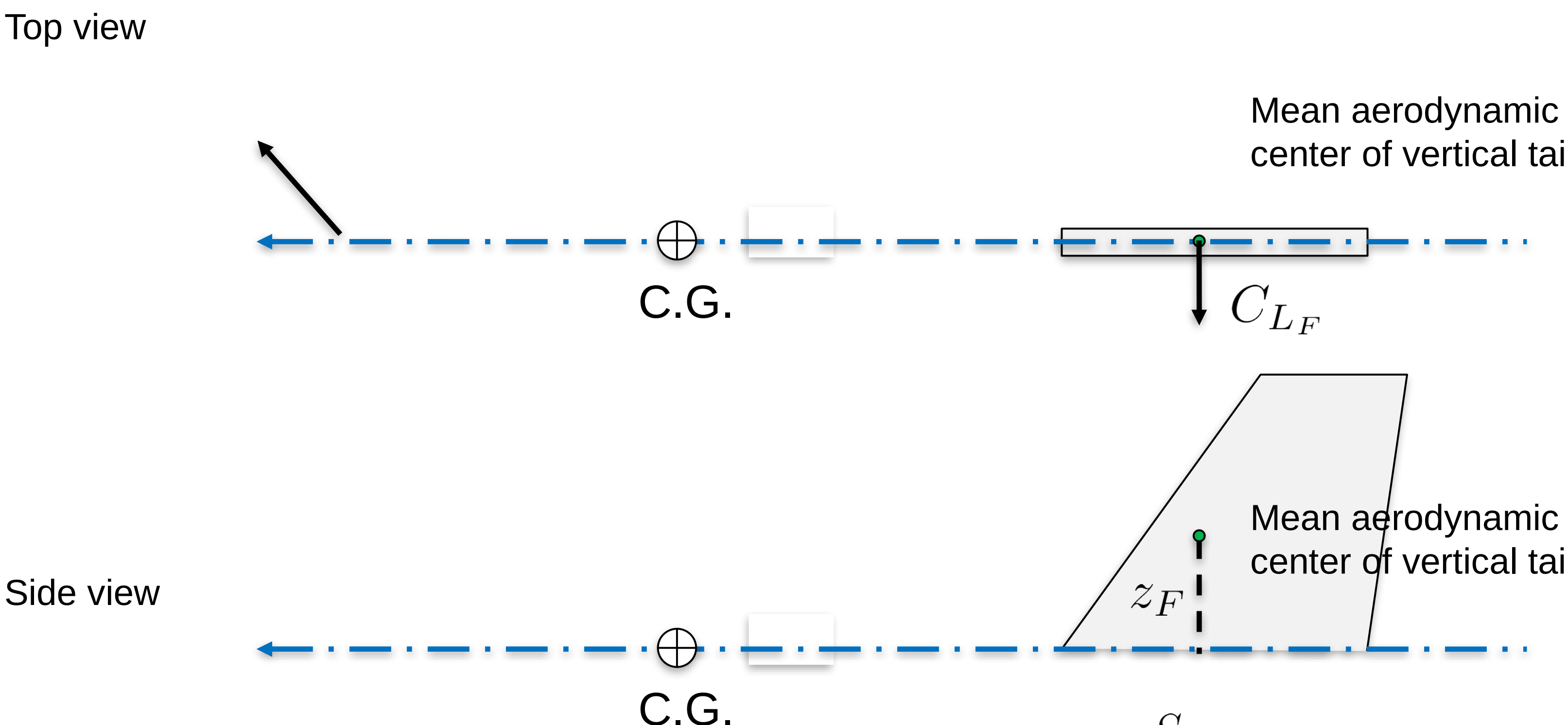
Increased lift on the right, decreased lift on the left leads to negative (restoring) roll moment



Lift generated on a swept wing is a function of the normal component of wind velocity

http://www.yojoe.com/images/resize/w/MAX/vehicles/83/skystriker/skystriker_top_swept.jpg

Dihedral Effect – Vertical Tail



$$\Delta C_l^F = C_{L_F} \frac{S_F z_F}{Sb} = a_F(-\beta + \sigma) \frac{S_F z_F}{Sb}$$

$$C_{l_\beta}^F = -a_F(1 - \frac{\partial \sigma}{\partial \beta}) \frac{S_F z_F}{Sb}$$

Beta Derivatives

$$Y_v = \frac{1}{2}\rho u_0 S \boxed{C_{y_\beta}}$$

$$L_v = \frac{1}{2}\rho u_0 b S \boxed{C_{l_\beta}}$$

$$N_v = \frac{1}{2}\rho u_0 b S \boxed{C_{n_\beta}}$$

$$C_{Y_\beta} = C_{Y_\beta}$$

$$C_{l_\beta} = C_{l_\beta}$$

$$C_{n_\beta} = C_{n_\beta}$$

$$= -a_F\left(1 - \frac{\partial\sigma}{\partial\beta}\right)\frac{S_F}{S}$$

Function of i.) dihedral; ii.) sweep; iii.) high/low wing; iv.) high/low tail

$$= V_V a_F\left(1 - \frac{\partial\sigma}{\partial\beta}\right)$$

Lateral Modes

Boeing 747 Jet Transport (Etkin 6.2)

$$h_0 = 40000 \text{ ft}$$
$$V_0 = 774 \text{ ft/s}$$
$$\gamma_0 = \theta_0 = \alpha_0 = 0^\circ$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Lateral Modes

λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02+9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02-9.47e-01i$	$3.49e-02$	$9.47e-01$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \begin{array}{c} \mathbf{v}_1 \\ \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathbf{v}_2 \\ \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \end{array} \begin{array}{c} \\ p \\ \hat{} \\ \\ \end{array} \quad \begin{array}{c} \mathbf{v}_{3/4} \\ \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix} \end{array}$$

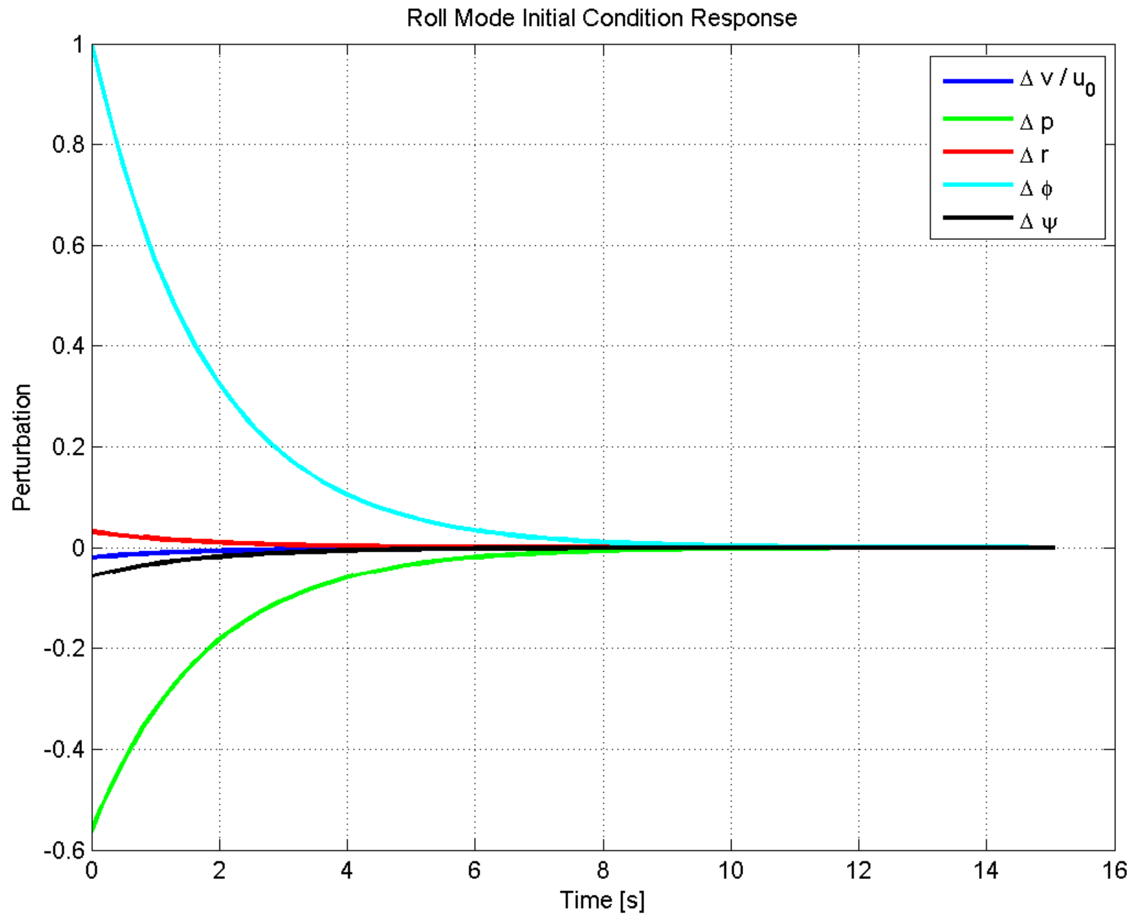
Spiral **Roll** **Dutch Roll**

Roll Mode

$$\lambda_2 = -.56248$$
$$t_{half} = 1.23 \text{ s}$$

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0000 \\ -0.0562 \end{pmatrix} \begin{array}{l} \hat{v} = \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{array}$$

Non-dimensionalize and scale so Δϕ term is 1



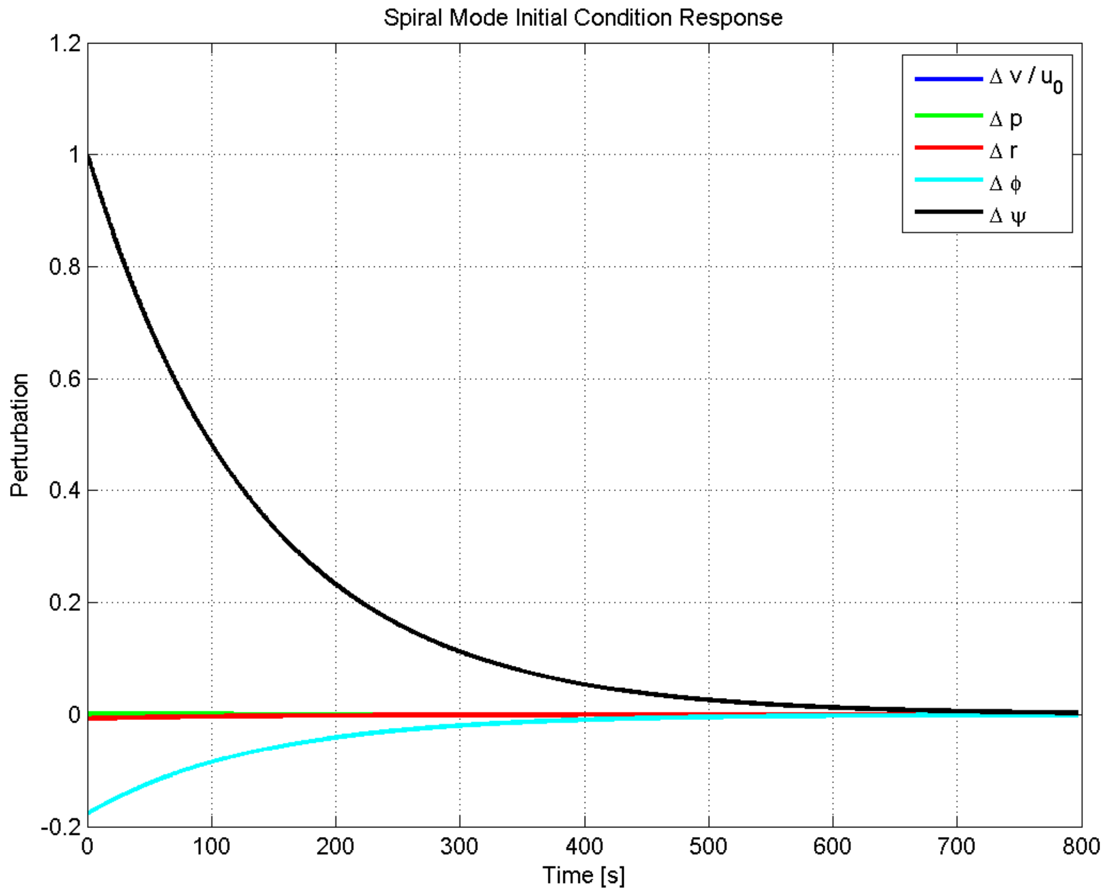
The roll mode is characterized by a **fast roll rate** with very small sideslip, yaw rate, and roll angle.

Spiral Mode

$$\lambda_1 = -.0073$$
$$t_{half} = 95 \text{ s}$$

$$\hat{\mathbf{v}}_2 = \begin{array}{cc} \begin{pmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0000 \end{pmatrix} & \begin{array}{l} \hat{v} = \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{array} \end{array}$$

Non-dimensionalize and scale so $\Delta\psi$ term is 1



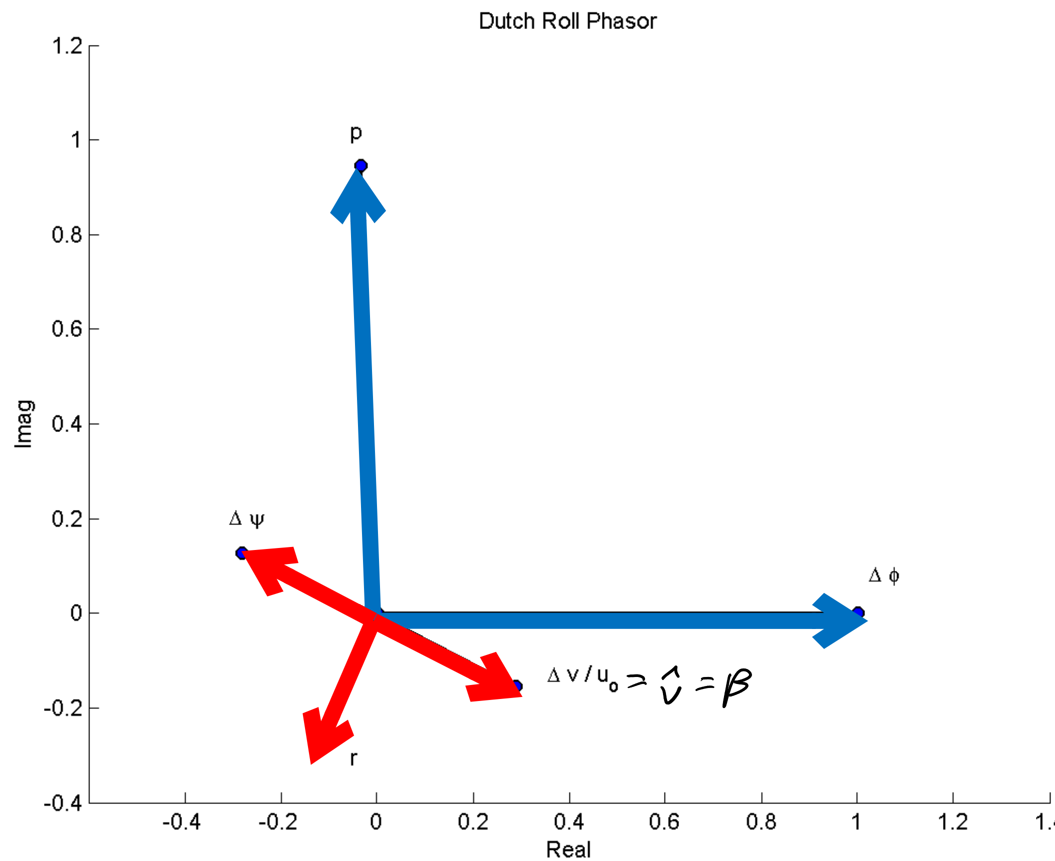
The spiral mode is characterized by strong coupling in roll and yaw, with sideslip remaining relatively fixed. The spiral mode is **often unstable**.

Dutch Roll Mode

$$\lambda_{3/4} = -0.033 + 0.947i$$
$$\zeta = 0.0349$$ ← Not well damped
$$\omega_n = 0.947$$ ← fast response

Phugoid (for comparison)

$$\mathbf{v}_{3/4} = \begin{array}{cc} \begin{pmatrix} 0.3271\angle -28.0^\circ \\ 0.9471\angle 92.0^\circ \\ 0.2915\angle -112.3^\circ \\ 1.0000\angle 0.0^\circ \\ 0.3078\angle 155.7^\circ \end{pmatrix} & \begin{array}{c} \hat{\beta} \\ p \\ r \\ \Delta\phi \\ \Delta\psi \end{array} \end{array}$$

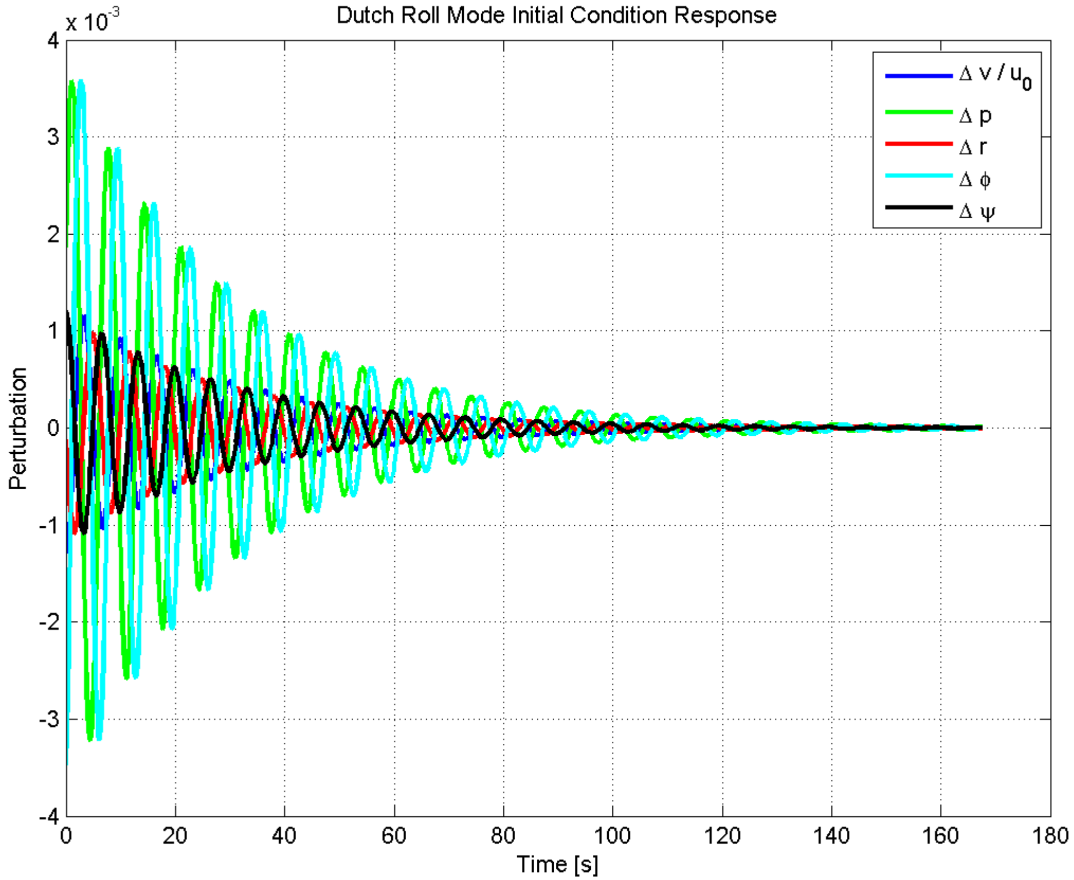


The Dutch roll mode is characterized by a **"tail-wagging" motion** whereby the horizontal flight path stays relatively straight (sideslip and yaw tend oscillate with opposite sign and balance each other out).

Dutch Roll Mode

$$\lambda_{3/4} = -0.033 + 0.947i$$
$$\zeta = 0.0349$$
$$\omega_n = 0.947$$

$$\mathbf{x}(0) = Re(\mathbf{v}_{3/4}) = \begin{pmatrix} -0.9958 \\ 0.0019 \\ -0.0001 \\ -0.0035 \\ 0.0012 \end{pmatrix}$$



The Dutch roll mode is characterized by a **“tail-wagging” motion** whereby the horizontal flight path stays relatively straight (sideslip and yaw tend oscillate with opposite sign and balance each other out).

Review: All the modes

Name	*Primary* variables	Fast/Slow	Damping
Short Period	Angle of attack	Fast	Well-damped
Phugoid	Speed, Altitude	Slow	Poorly-Damped
Roll	Roll rate	Fast	Over-Damped
Spiral	Roll, Yaw	Slow	Over-Damped OR Unstable
Dutch Roll	All lateral Degrees of Freedom	Fast	Poorly-Damped

Trust the mathematical properties of the matrix over your intuition!

Spiral Approximation

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0000 \end{pmatrix} \longrightarrow$$

$$\Delta\dot{p} = \Delta p = 0$$

Assume no side-force
(ignore Y-force equation)

$$\begin{pmatrix} 0 \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta r \end{pmatrix}$$

$$\Delta\dot{r} = \left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right) \Delta r$$

$$\lambda_{spi} = \frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v}$$

Full lateral matrix

$$\lambda_1 = -.0073$$
$$t_{half} = 95 \text{ s}$$

Spiral mode approximation

$$\lambda_{spi} = -0.0296$$
$$t_{half} = 23.4 \text{ s}$$

Spiral Approximation

$$\lambda_1 = -.0073$$
$$t_{half} = 95 \text{ s}$$

Since spiral eigenvalue is so small, characteristic equation can be approximated by last two terms

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

$$\downarrow$$

$$D\lambda + E = 0$$

$$\lambda = -\frac{E}{D}$$

$$E = g\left[(\mathcal{N}_r\mathcal{L}_v - \mathcal{N}_v\mathcal{L}_r)\cos\theta_0 + (\mathcal{N}_v\mathcal{L}_p - \mathcal{L}_v\mathcal{N}_p)\sin\theta_0\right]$$

$$D = -g\left(\mathcal{L}_v\cos\theta_0 + \mathcal{N}_v\sin\theta_0\right) + u_0\left(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v\right)$$

Spiral Approximation

$$\lambda_1 = -.0073$$
$$t_{half} = 95 \text{ s}$$

Since spiral eigenvalue is so small, characteristic equation can be approximated by last two terms

$$D\lambda + E = 0$$

$$\lambda = -\frac{E}{D}$$

$$E = g\left[\left(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r\right)\cos\theta_0 + \left(\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p\right)\sin\theta_0\right]$$

$$D = -g\left(\mathcal{L}_v \cos\theta_0 + \mathcal{N}_v \sin\theta_0\right) + u_0\left(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v\right)$$

Full lateral matrix

$$\lambda_1 = -.0073$$
$$t_{half} = 95 \text{ s}$$

Spiral mode approximation

$$\lambda_{spi} = -0.00725$$
$$t_{half} = 96 \text{ s}$$

Roll Approximation

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0000 \\ -0.0562 \end{pmatrix}$$

Ignore everything except roll rate

$$\Delta \dot{p} = \boxed{\mathcal{L}_p} \Delta p$$
$$\left(\Delta \dot{\phi} = \Delta p \right)$$

$$\lambda_r = \mathcal{L}_p$$

Full lateral matrix

$$\lambda_2 = -.56248$$
$$t_{half} = 1.23 \text{ s}$$

Roll mode approximation

$$\lambda_r = -.4342$$
$$t_{half} = 1.60 \text{ s}$$

Roll and Spiral Approximation

Assume side force produces same yaw rate as β = 0.
Also assume $Y_p = Y_r = 0$.

$$\begin{pmatrix} 0 \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -u_0 & g \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix}$$

$$\downarrow$$

$$C\lambda^2 + D\lambda + E = 0$$

$$E = g\left(\mathcal{N}_r\mathcal{L}_v - \mathcal{N}_v\mathcal{L}_r\right)$$

$$D = u_0\left(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v\right) - g\mathcal{L}_v$$

$$C = u_o\mathcal{N}_v$$

Full lateral matrix

$$\lambda_1 = -.0073$$
$$\lambda_2 = -.56248$$

Approximation

$$\lambda_{spi} = -0.00734$$
$$\lambda_r = -0.597$$

Dutch Roll Approximation

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.3271\angle -28.0^\circ \\ 0.9471\angle 92.0^\circ \\ 0.2915\angle -112.3^\circ \\ 1.0000\angle 0.0^\circ \\ 0.3078\angle 155.7^\circ \end{pmatrix}$$

Assume a flat, snaking motion

$$\Delta p = \Delta\phi = 0$$
$$Y_r = 0$$

$$\begin{pmatrix} \Delta\dot{v} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta r \end{pmatrix}$$

$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\,\lambda + (\mathcal{Y}_v\mathcal{N}_r + u_0\mathcal{N}_v) = 0$$

Full lateral matrix

$$\lambda_{3/4} = -0.033 + 0.947i$$

Dutch roll mode approximation

$$\lambda_{dr} = -0.1008 \pm 0.9157i$$

Eigenvalues of Mode Approximations

$$\dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta a} \delta_a$$

$$\dot{x} = A x + B u$$

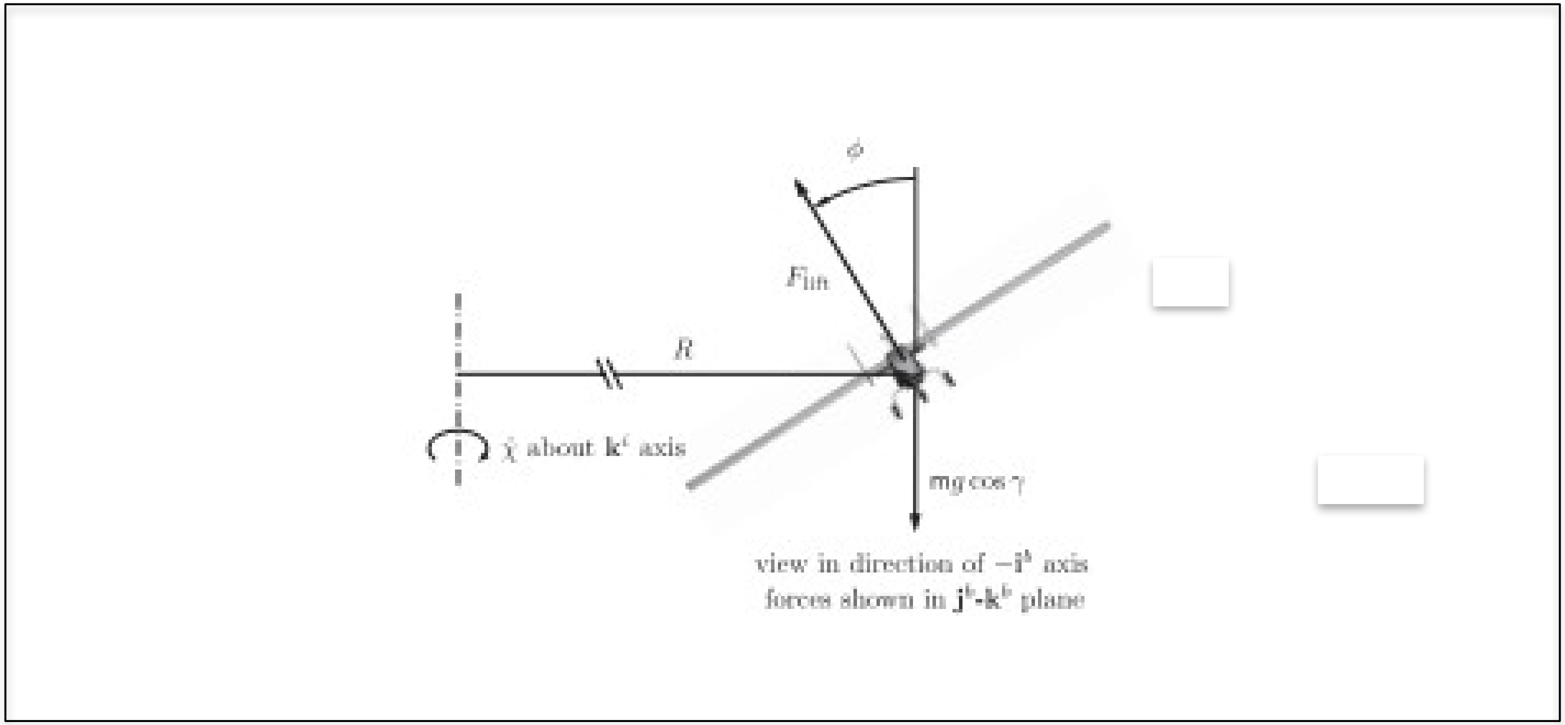
$$|A - \lambda I| = 0$$

$\lambda = A$ if A is scalar

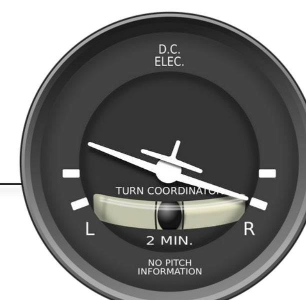
$\delta_a = -kp$

$$\dot{p} = (\mathcal{L}_p - \mathcal{L}_{\delta a} k) p$$

Coordinated Turn



- A "banked" or "coordinated" turn is one in which
  - Angular velocity vector is a constant and aligned with inertial z direction
  - No Aerodynamic forces in the y direction ()
    - Pilot "flying the turn on the turn-and-bank indicator"



Coordinated Turn

$$\begin{pmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_\beta} & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{pmatrix} \begin{pmatrix} \beta \\ \delta_r \\ \delta_a \end{pmatrix} = \begin{pmatrix} C_{Y_p} & C_{Y_r} \\ C_{l_p} & C_{l_r} \\ C_{n_p} & C_{n_r} \end{pmatrix} \begin{pmatrix} \theta \\ -\cos\phi \end{pmatrix} \frac{\omega b}{2u_0}$$

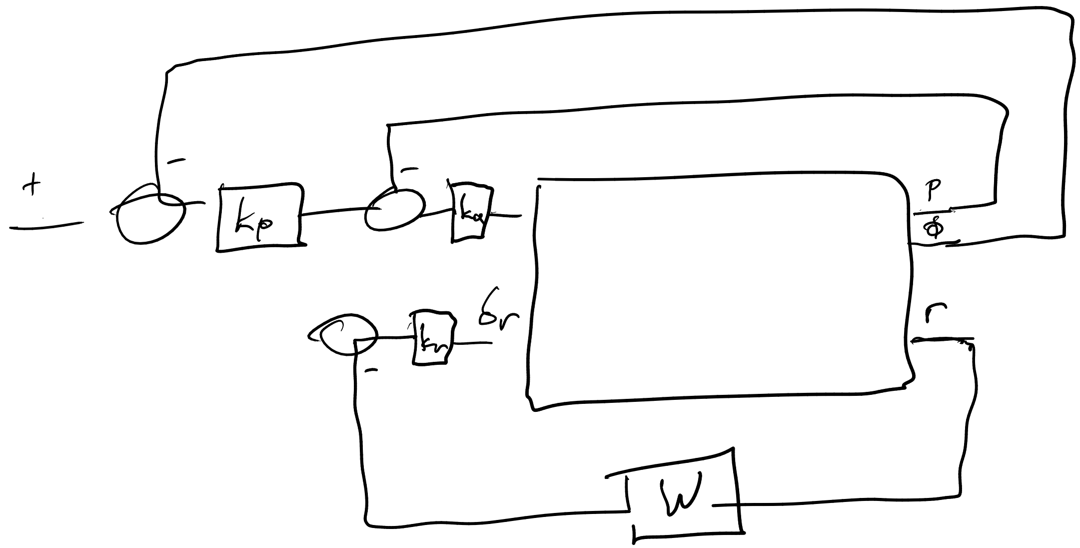
$$\begin{pmatrix} C_{m_\alpha} & C_{m_{\delta_e}} \\ C_{L_\alpha} & C_{L_{\delta_e}} \end{pmatrix} \begin{pmatrix} \Delta\alpha \\ \Delta\delta_e \end{pmatrix} = -\begin{pmatrix} C_{m_q} \\ C_{L_q} \end{pmatrix} \frac{\omega\bar{c}\sin\phi}{2u_0} + \begin{pmatrix} 0 \\ (n-1)C_W \end{pmatrix}$$

All these variables are perturbations, but only these two are added to non-zero trim values

These are based on small climb rate

May not work perfectly in nonlinear simulation – why?

Roll Controller



Outline

- Example Problems
- Questions

From 2019 Exam

TRUE or FALSE?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

From 2019 Exam

TRUE or **FALSE**?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

The spiral mode has a real eigenvalue, so it does not have a damping ratio.

From 2019 Exam

TRUE or FALSE?

For the augmented lateral dynamical system with $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$, the control law $\mathbf{u} = -\mathbf{K}^* \, \mathbf{x}_{\mathrm{lat}}$ may have no effect on certain eigenvalues of the system.

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Feedback of the first four terms does not change the eigenvalues associated with the last two terms, ie the zero eigenvalues.

From 2019 Exam

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Near trim for straight and level airplane flight, the rate of change in bank angle at time *t* depends on the bank angle at time *t*.

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Bank angle not in the linear equation for time rate of change of bank angle

$$\boxed{\Delta\dot{\phi} \quad = \Delta p + \Delta r \tan\theta_0}$$

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TRUE or FALSE?

For straight and level flight, an airplane's change in roll rate near trim at time *t* generally depends on the change in sideslip angle near trim at time *t*.

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$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \boxed{\Gamma_3 L_v + \Gamma_4 N_v} & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

From 2019 Exam

TRUE or FALSE?

Aircraft 1 and Aircraft 2 are identical except for one feature: the vertical tail for Aircraft 1 is above the centerline of the fuselage whereas the vertical tail for Aircraft 2 is below. $C_{n_{\beta,1}} = C_{n_{\beta,2}}$.

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Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ($\lambda_5 = 0$, $\lambda_6 = 0$). Further, they both have the same eigenvector $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^T$ associated with one of these eigenvalues. Why are the other eigenvectors associated with $\lambda_5 = 0$ different?

Advanced Bonus Question

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$$\Delta\dot{x}^E = \Delta u$$

$$\Delta\dot{z}^E = -u_0\Delta\theta + \Delta w$$

$$\Delta\psi = \Delta r \sec\theta_0$$

$$\Delta\dot{y}^E = u_0\cos\theta_0\Delta\psi + \Delta v$$

Questions?