Final Review

ASEN 3128 Aircraft Dynamics
UNIVERSITY OF COLORADO **BOULDER**

Announcements

- Final Exam: Saturday, May 3rd, 1:30-4pm

- FCQs, TA feedback due tonight (see Canvas Announcement)

Outline

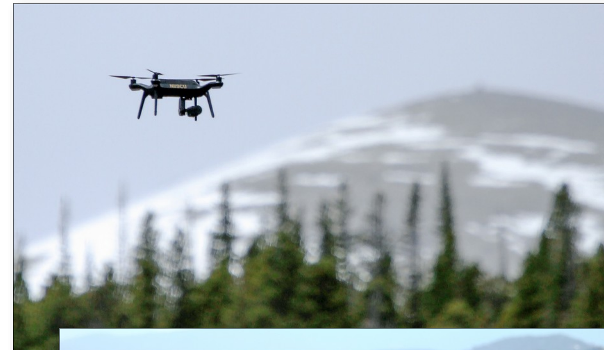
- Learning Goals
- Summary of Material

Aircraft Dynamics

Newton's Laws	(ASEN 2703)	*Inertial Reference Frame*
+ Aerodynamics	(ASEN 2702, ASEN 3711)	*Body Reference Frame*
Aircraft Dynamics	(ASEN 3128)	

Learning Goals

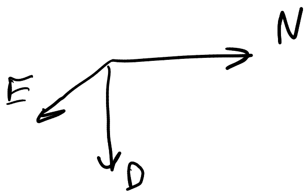
- How aircraft dynamics models are constructed
- How dynamics are simulated numerically
- How dynamical behavior is understood and specified
- How control is designed to achieve behavioral objectives
- Specific understanding of “how aircraft work” for a multirotor drone and a conventional (fixed-wing) airplane

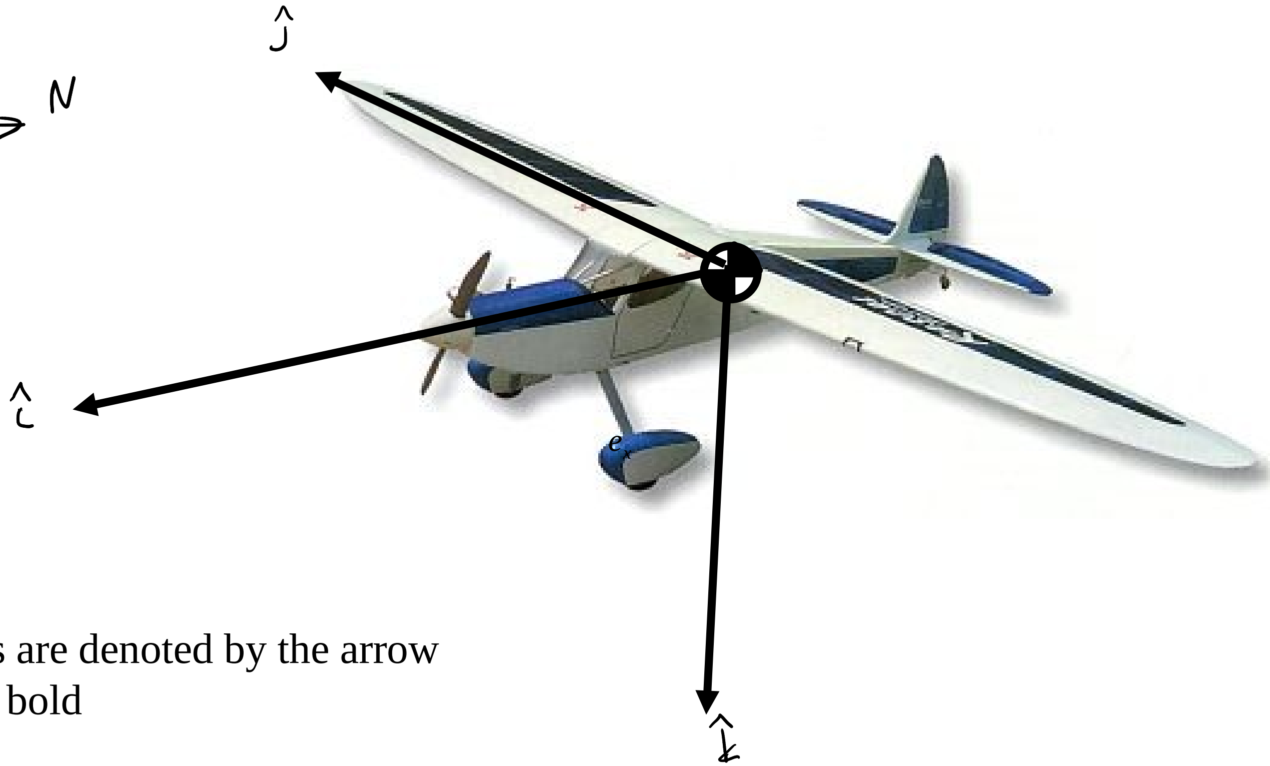


Course Material

1. **Nomenclature**
2. Aircraft Static Stability
3. **Euler Angles**
4. **Aircraft Equations of Motion (Kinematics + Dynamics)**
5. **Linearized Equations**
6. **Quadrotor Equations of Motion**
7. **Quadrotor Stability and Control**
8. Longitudinal Dynamics of Conventional (Fixed-Wing) Aircraft
9. Lateral Dynamics of Conventional (Fixed Wing) Aircraft
10. **State Space and Transfer Function-based Analysis**
11. Mode Approximations
12. Conventional Aircraft Control

Coordinate Systems

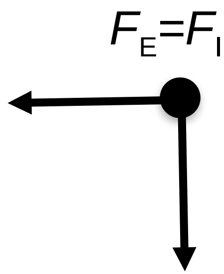


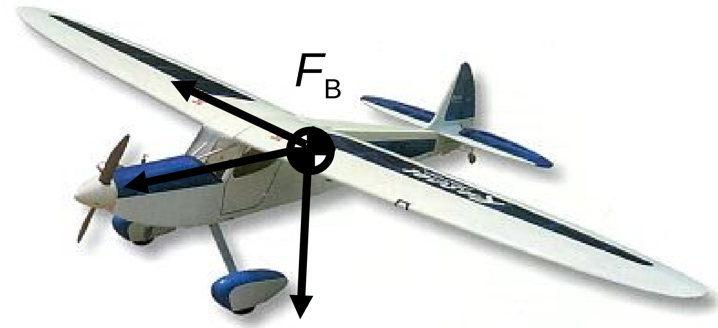


Note, vectors are denoted by the arrow (lectures) or bold

- Aircraft dynamics and control are best described in a coordinate system defined by the aircraft body
  - Often called the "<u>body coordinate system</u>"
  - Defines positive sign convention for all vectors (velocity, acceleration, force)

Coordinate Systems





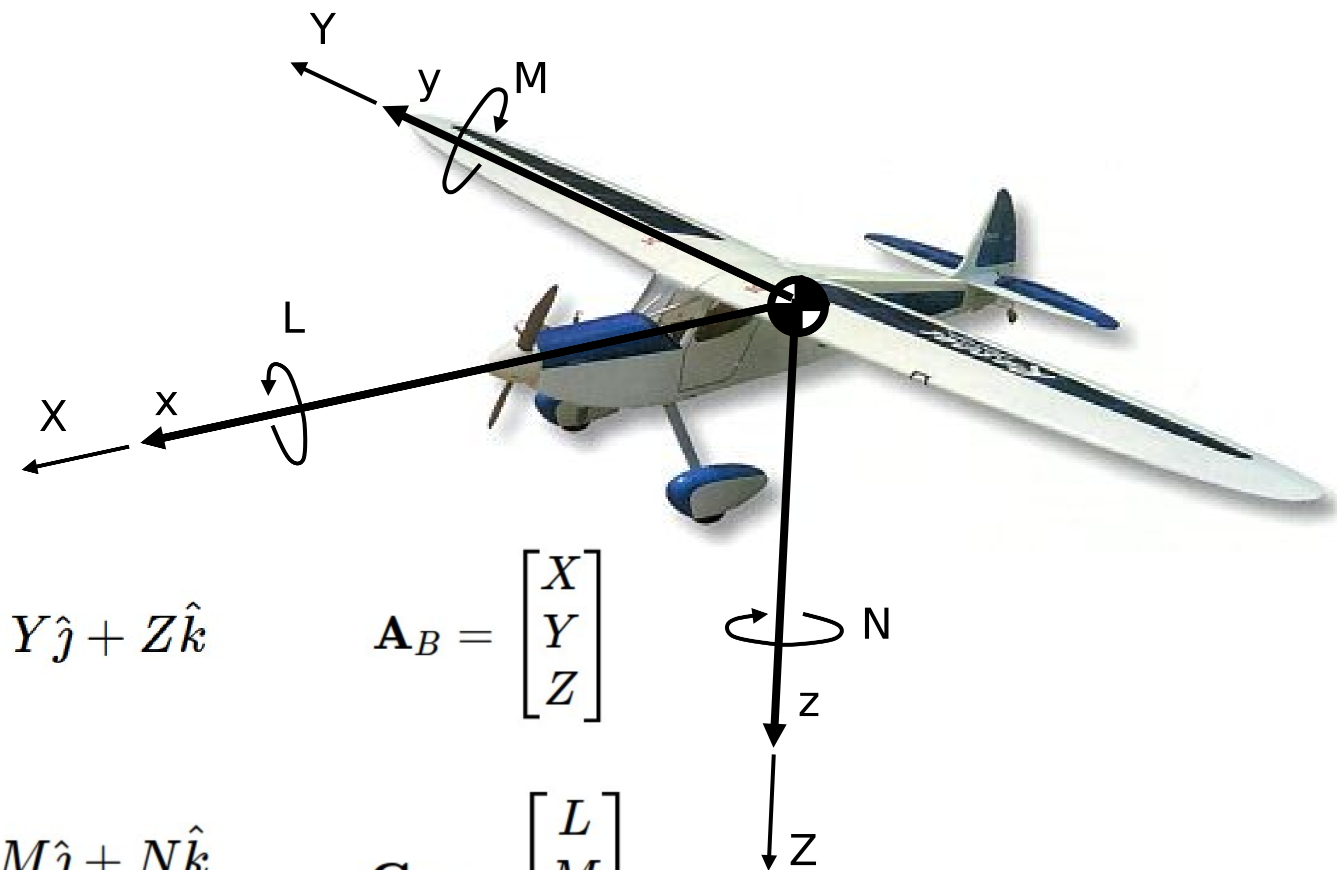
- Two main reference frames / coordinate systems
  - F_B = Body coordinate system: Aerodynamic forces determined best
  - $F_E = F_I$ = Inertial coordinate system: Newton's Laws apply

- Superscripts denote reference frame

$\mathbf{V}_B^E$ = "inertial velocity written in body coordinate system"

- Subscripts denote what coordinate system vector is written in

Aerodynamic Forces and Moments



$$\mathbf{A} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$$

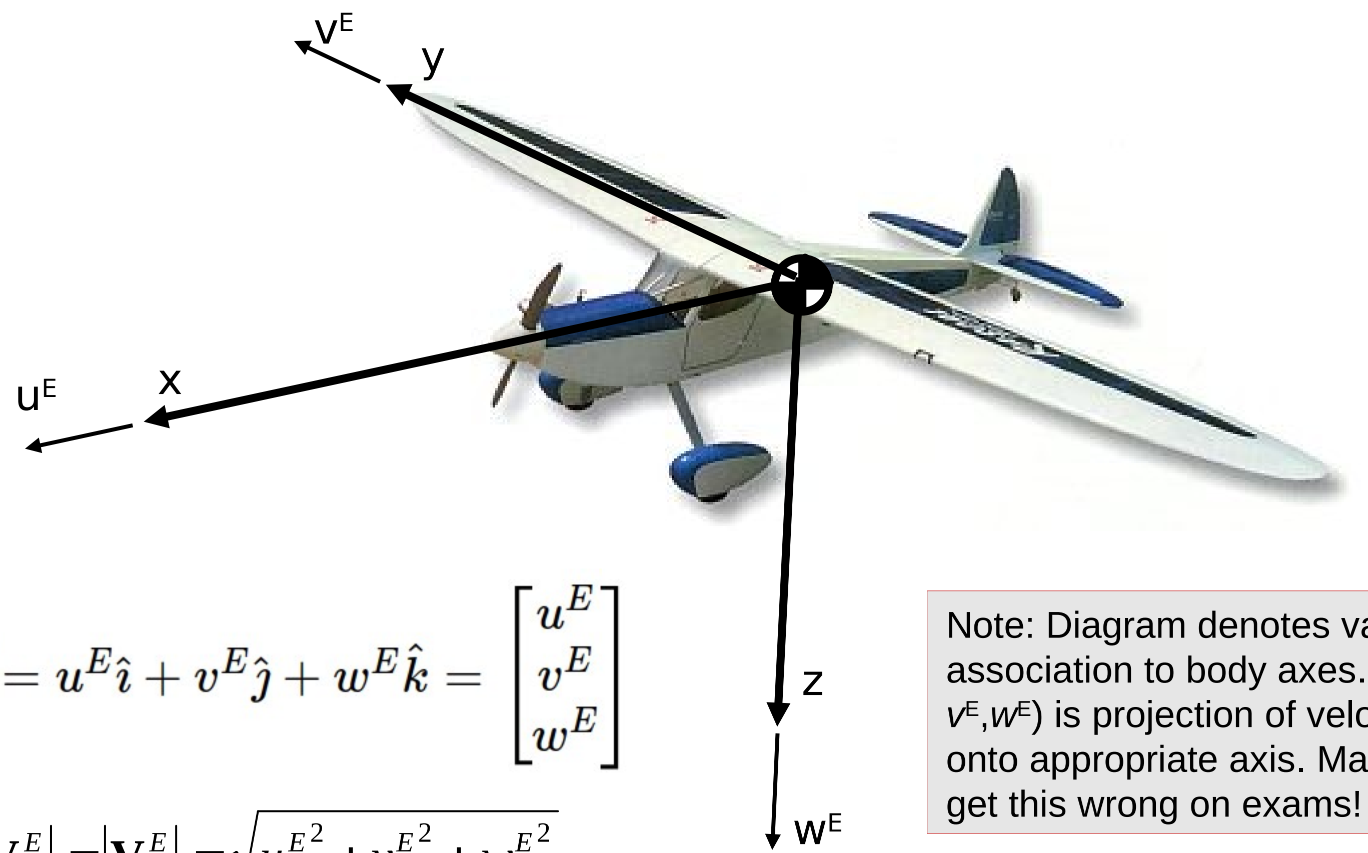
$$\mathbf{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\mathbf{G} = L\hat{\imath} + M\hat{\jmath} + N\hat{k}$$

$$\mathbf{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

- Aerodynamic Force and moment components are represented by capital letters

Inertial Velocity Components



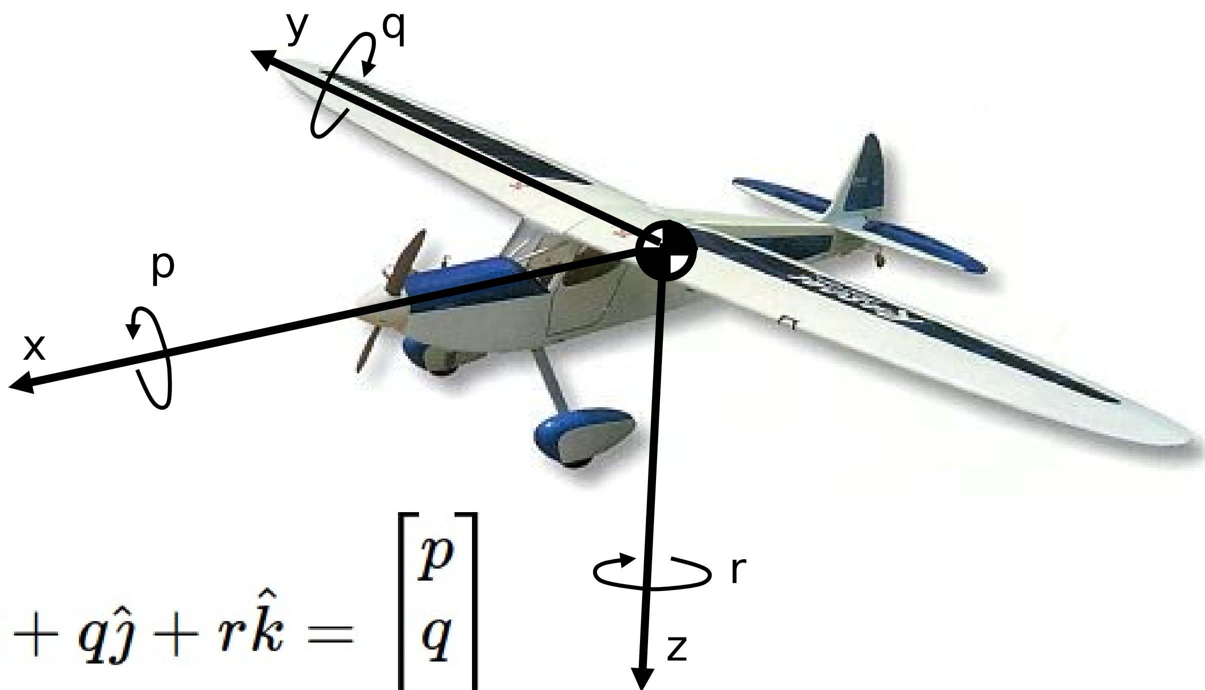
$$\mathbf{V}_B^E = u^E\hat{\imath} + v^E\hat{\jmath} + w^E\hat{k} = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$V_g = \left|\mathbf{V}_B^E\right| = \left|\mathbf{V}_E^E\right| = \sqrt{u^{E^2} + v^{E^2} + w^{E^2}}$$

Note: Diagram denotes variable association to body axes. Value of (u^E v^E,w^E) is projection of velocity vector onto appropriate axis. Many students get this wrong on exams!

- The inertial (linear) velocity is express in body coordinates
  - Aircraft ground speed V_g can be derived from GPS measurements

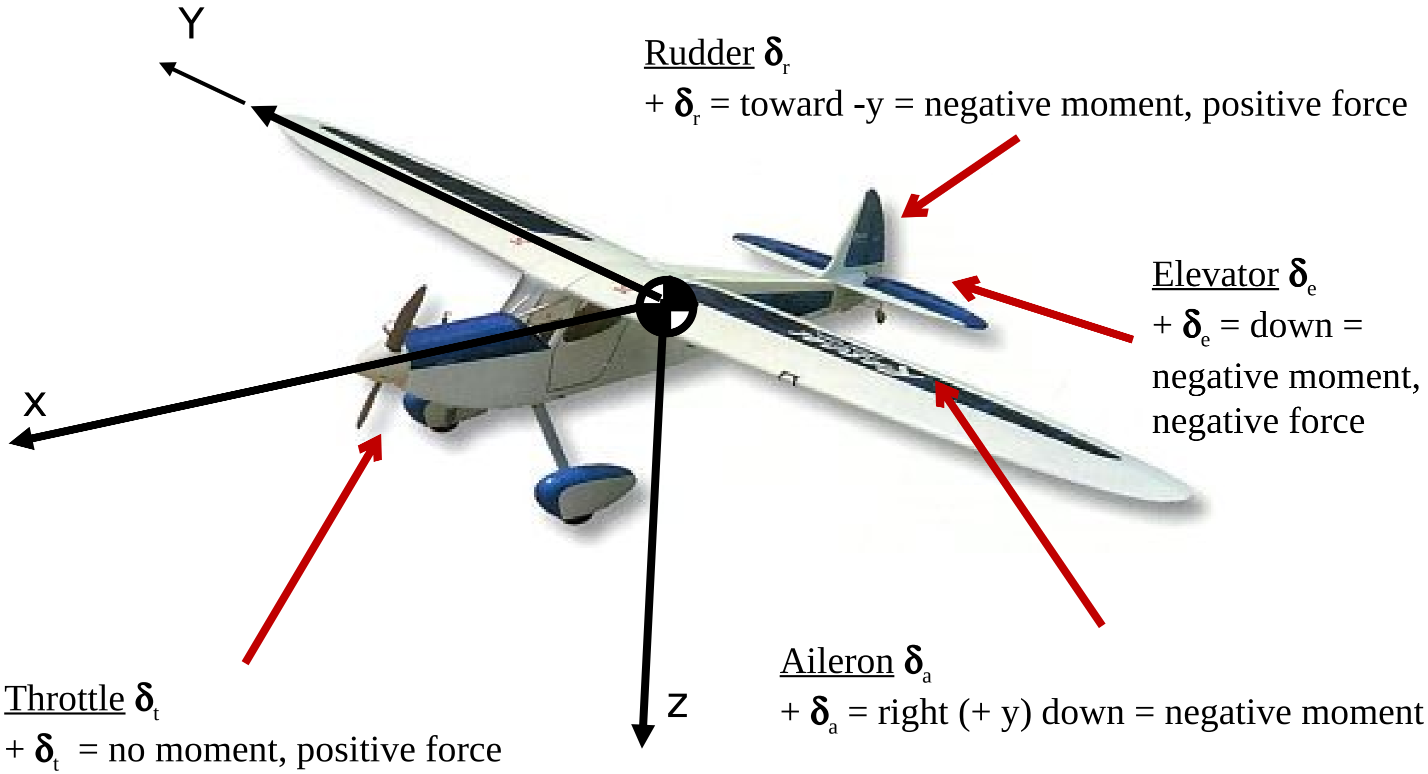
Inertial Angular Velocity



$$\boldsymbol{\omega}_B^E = p\hat{\imath} + q\hat{\jmath} + r\hat{k} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

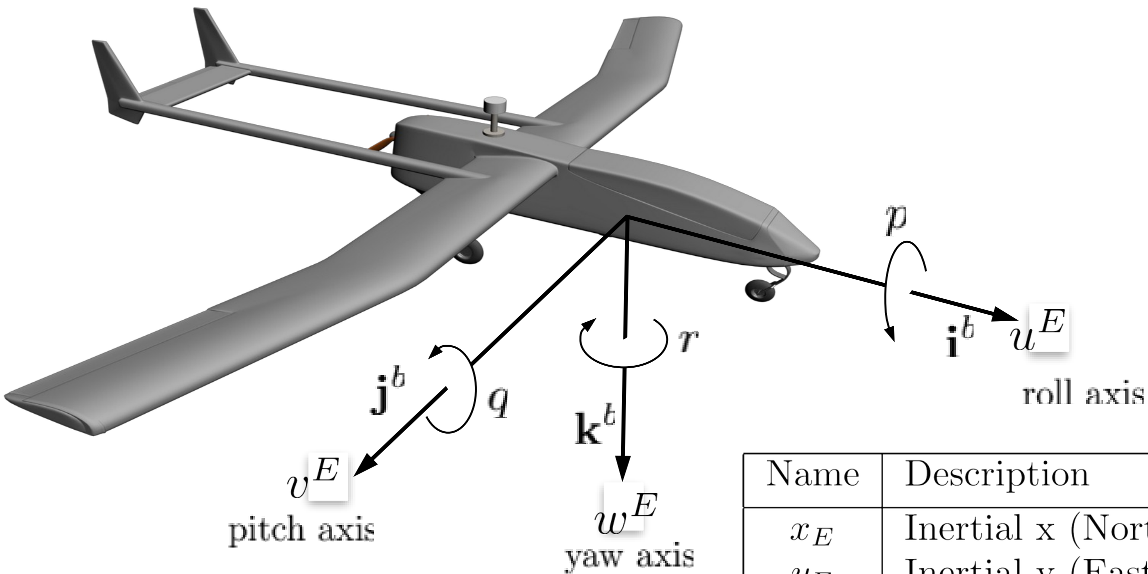
- The inertial angular velocity is also expressed in body coordinates
  - Components can be measured by rate gyros

Four Control Surfaces



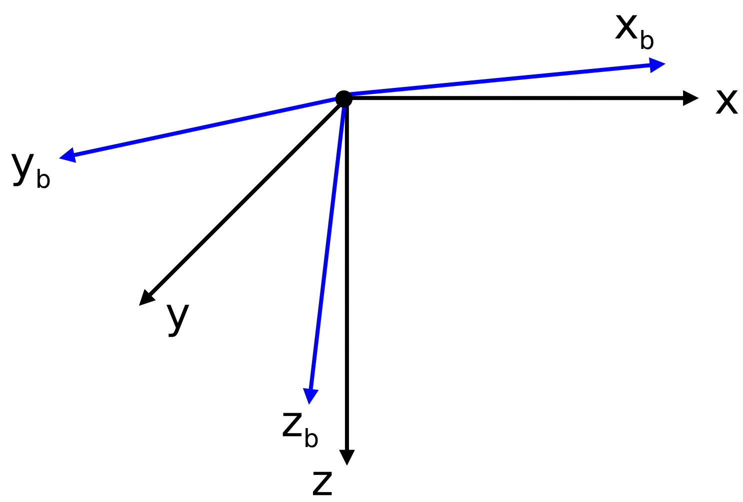
- We will focus on a standard aircraft with four control surfaces
  - 3 actual surfaces plus throttle

Aircraft State Variables



Name	Description
x_E	Inertial x (North) position
y_E	Inertial y (East) position
z_E	Inertial z (Down) position
ϕ	Roll angle
θ	Pitch angle
ψ	Yaw angle
u^E	Inertial velocity vector component along $\mathbf{i}_B$
v^E	Inertial velocity vector component along $\mathbf{j}_B$
w^E	Inertial velocity vector component along $\mathbf{k}_B$
p	Angular velocity vector component along $\mathbf{i}_B$ (Roll rate)
q	Angular velocity vector component along $\mathbf{j}_B$ (Pitch rate)
r	Angular velocity vector component along $\mathbf{k}_B$ (Yaw rate)

Inertial Frame to Body Frame Transformation



“1” “2” “3”

$$\mathcal{R}_E^B(\phi,\theta,\psi) = \mathcal{R}_{v2}^{B}(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_{E}^{v1}(\psi)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$c_\theta = \cos\theta$$
$$s_\theta = \sin\theta$$
etc.

Vector Derivatives

Consider the vector $\mathbf{p}$

$\frac{d\mathbf{p}}{dt}$ is the time rate of change of the vector $\mathbf{p}$

If $\mathbf{p}$ was the inertial position of an aircraft, then $\frac{d\mathbf{p}}{dt}$ would be the inertial velocity.

Vector Derivatives

Now consider the vector $\mathbf{p}_B$ written in a particular coordinate frame, e.g. the body coordinate system

$$\mathbf{p}_B = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$$

The time rates of change of the individual components of the vector is <u>defined to be</u>

$$\dot{\mathbf{p}}_B = \begin{pmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{pmatrix}$$

If the coordinate system F_B is rotating then

$$\frac{d\mathbf{p}_B}{dt} \neq \dot{\mathbf{p}}_B$$

Vector Derivatives

$$\frac{d}{dt}\mathbf{p}_B = \dot{\mathbf{p}}_B + (\omega_B \times \mathbf{p}_B) = \dot{\mathbf{p}}_B + \tilde{\omega}_B \mathbf{p}_B$$

$$\tilde{\omega}_B = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$$

Newton's Law

$$\frac{d\vec{v}^E}{dt} = \frac{\vec{f}}{m}$$

Rotational Kinematics

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + R_1(\phi) \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + R_1(\phi) R_2(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

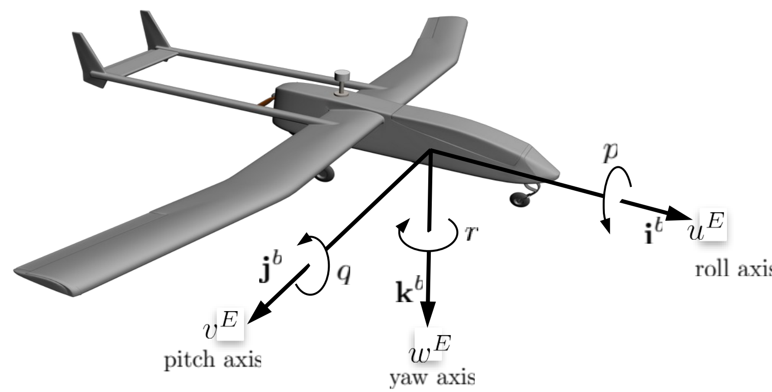
$$= \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}$$

Not rotation matrices (not orthonormal)

Inverting gives:

$$\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$



Vector form: $\dot{\mathbf{o}} = \mathbf{T} \cdot \omega_B$

Kinematic Equations

Six of the 12 state equations for the aircraft come from the kinematic equations relating positions and velocities:

$$\frac{d\mathbf{p}_E^E}{dt} = R_B^E \cdot \mathbf{v}_B^E$$

"Orientation vector" not in a coordinate system
(Also referred to as "attitude" or just "Euler angles")

$$\dot{\mathbf{o}} = \mathbf{T} \cdot \omega_B$$

The remaining six equations will come from applying Newton's 2nd law to the translational and rotational motion of the aircraft.

Dynamic Equations

The remaining six equations for the aircraft come from the dynamic equations relating derivatives of velocities and forces and moments:

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

Rotational Dynamics

$$\dot{\omega}_B = \mathbf{I}_B^{-1}\left[-\tilde{\omega}_B\left(\mathbf{I}_B\omega_B\right) + \mathbf{G}_B\right]$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \dfrac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

$$\Gamma_1 = \frac{I_{xz}\left(I_x - I_y + I_z\right)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x\left(I_x - I_y\right) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z\left(I_z - I_y\right) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

Same as Eq. 4.5,9, though book does not give the equations in this form

Equation of Motion in Vector Form

$$\dot{\mathbf{p}}_E^E = R_B^E \mathbf{v}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

Equation of Motion Summary

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Quadrotor Equations of Motion

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

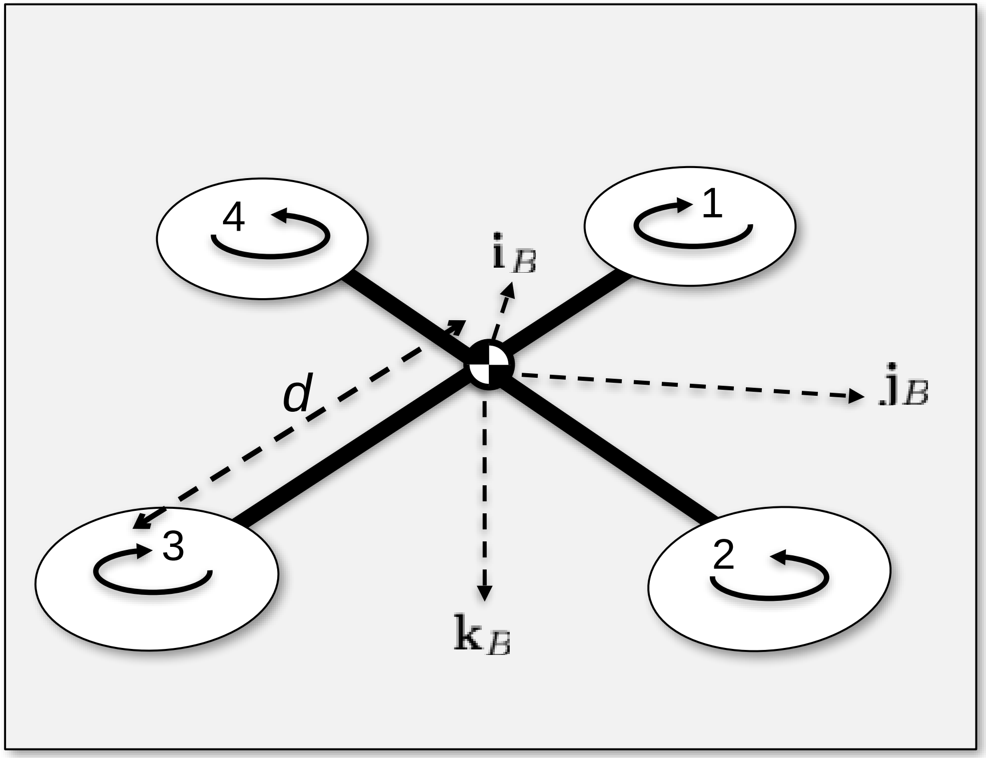
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Control Forces and Moments

$$
{}^c\mathbf{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}
$$

$$
{}^c\mathbf{G}_B = \begin{bmatrix} \frac{d}{\sqrt{2}}(-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{\sqrt{2}}(f_1 - f_2 - f_3 + f_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{bmatrix}
$$



$\tau_i = k_m f_i$ (check in notes)

Equilibrium and Static Stability

- Equilibrium = The states do not change without a disturbance or control action
  - Derivative of the state vector is zero (almost)
- Static stability = Consider the equilibrium condition and whether aircraft returns to condition after small perturbation
- Dynamic Stability = Details of how a system returns to equilibrium
- For aircraft we call the equilibrium condition the "**trim condition**"
- For fixed-wing aircraft the trim condition is
  - "Straight, level, unaccelerated flight (SLUF)"
- For a quadrotor the trim state is
  - "Hovering"
  - "Steady hover"

Trim Condition

In a “**steady hover**” equilibrium condition

$$\dot{x}_{E,0} = 0 \qquad u_0^E = 0 \qquad \dot{u}_0^E = 0$$
$$\dot{y}_{E,0} = 0 \qquad v_0^E = 0 \qquad \dot{v}_0^E = 0$$
$$\dot{z}_{E,0} = 0 \qquad w_0^E = 0 \qquad \dot{w}_0^E = 0$$

Subscript 0 denotes trim value

$$p_0 = 0 \qquad \dot{p}_0 = 0 \qquad \phi_0 = 0$$
$$q_0 = 0 \qquad \dot{q}_0 = 0 \qquad \theta_0 = 0$$
$$r_0 = 0 \qquad \dot{r}_0 = 0 \qquad \psi_0 = \cdot$$

Any constant value

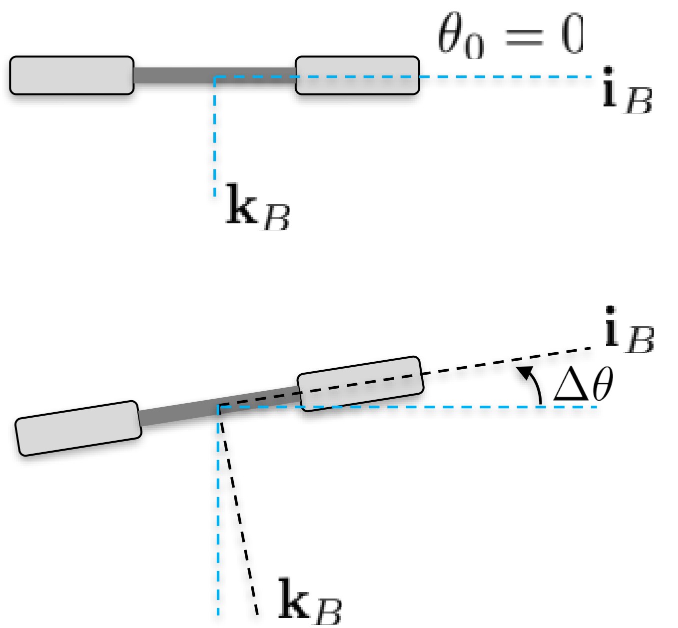
Small Disturbances

Can describe variables as reference condition ("trim condition") plus small disturbance, e.g $u^E = u = u_0 + \Delta u$

only for linearization

Assume that there is no wind

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta\phi \\ \Delta\theta \\ \psi_0 + \Delta\psi \\ \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$



Linearization

any variables that change

$$y = f(x, u)$$

No!

$$\cancel{\Delta y = f(\Delta x, \Delta u)}$$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$y_0 + \Delta y \approx f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u$$

$$\Delta y \approx \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u$$

3 steps

1. Take partial derivative
2. Evaluate at trim
3. Multiply by disturbance

$$f(x, \dot{x}, \theta, \dot{\theta})$$

Linearized Quadrotor EOM

Longitudinal

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{u} \\ \Delta \dot{\theta} \\ \Delta \dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g\Delta\theta \\ \Delta q \\ \frac{1}{I_y}\Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

Vertical

$$\begin{pmatrix} \Delta \dot{z}_E \\ \Delta \dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m}\Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta \dot{\psi} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

Unleash the Memes!

Longest Flight Ever?

Longest Flight Ever?

64 days, 22 hours, 19 minutes

Longest Flight Ever?

64 days, 22 hours, 19 minutes

- Oil changes
- Several hours when both pilots were asleep

Break

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Linear ODEs

$$\dot{x} = A\,x + B\,u$$
$$y = C\,x + D\,u$$

Aircraft dynamics (mostly) uses this one
It is called a “state space” model.

These contain (roughly) the same information!

Always keep this in mind!

$$\ddot{y} + 2\zeta\,\omega_n\,\dot{y} + \omega_n^2\,y = \omega_n^2 u$$

$$u \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\,\omega_n\,s + \omega_n^2}} \longrightarrow y$$

$G(s)$

Laplace Transform Properties

$$x(t) \Longleftrightarrow X(s)$$

$$\dot{x}(t) \Longleftrightarrow sX(s)$$ (assuming initial conditions zero)

$$\int_0^{\tau} x(\tau)\,d\tau \Longleftrightarrow \frac{1}{s}X(s)$$

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha X(s) + \beta Y(s)$$

$$G_{xu}(s) = \frac{X(s)}{U(s)}$$ (assuming zero initial conditions)

step: $\frac{1}{s}$

impulse: 1

$$X(s) = G_{xu}(s)\,U(s)$$

Transfer Function Uses

1. $u(t) \rightarrow u(s) \rightarrow x(s) = G_{xu}(s)\, u(s) \rightarrow x(t)$

Not in this course

2. Stability / Transient Behavior

Roots of TF denominator (poles)

= Eigenvalues of A matrix

3. Steady-state behavior

a) Final Value Theorem

$$\boxed{\begin{array}{l}\text{If } sX(s) \text{ is stable} \\ \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)\end{array}}$$

Special Case: step input

$$\lim_{s\to 0} \cancel{s}\, G(s) \frac{1}{\cancel{s}} \longrightarrow \lim_{s\to 0} G(s)$$

b) Frequency Response

$|G(i\omega)|$ $\qquad \angle G(i\omega)$

(see notes)

Step Response Example

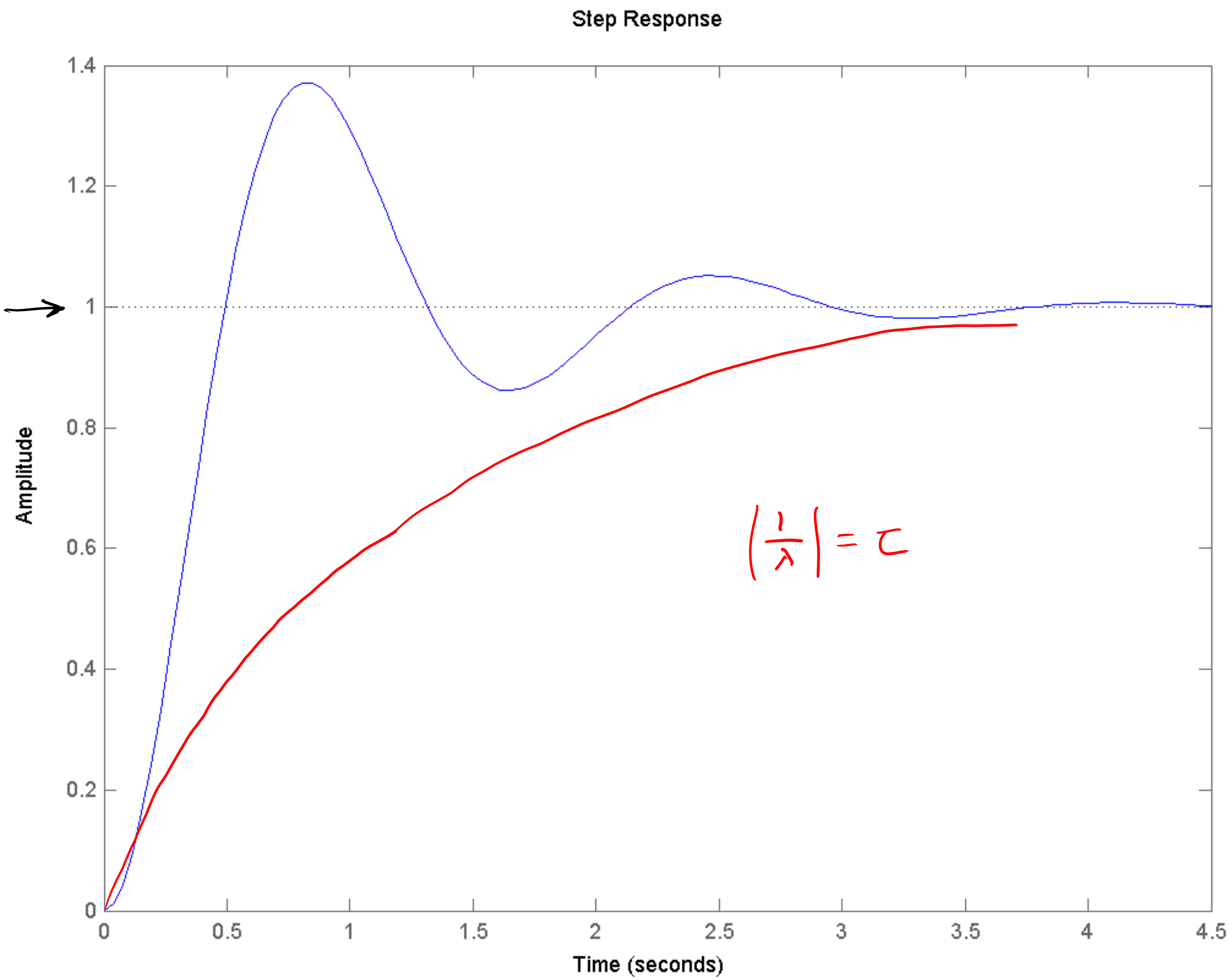
$G(0)$

Second order system

$$G(s) = \frac{16}{s^2 + 2.4s + 16}$$

$\zeta = 0.3$

$\omega_n = 4$



Frequency Response Example

Second order system

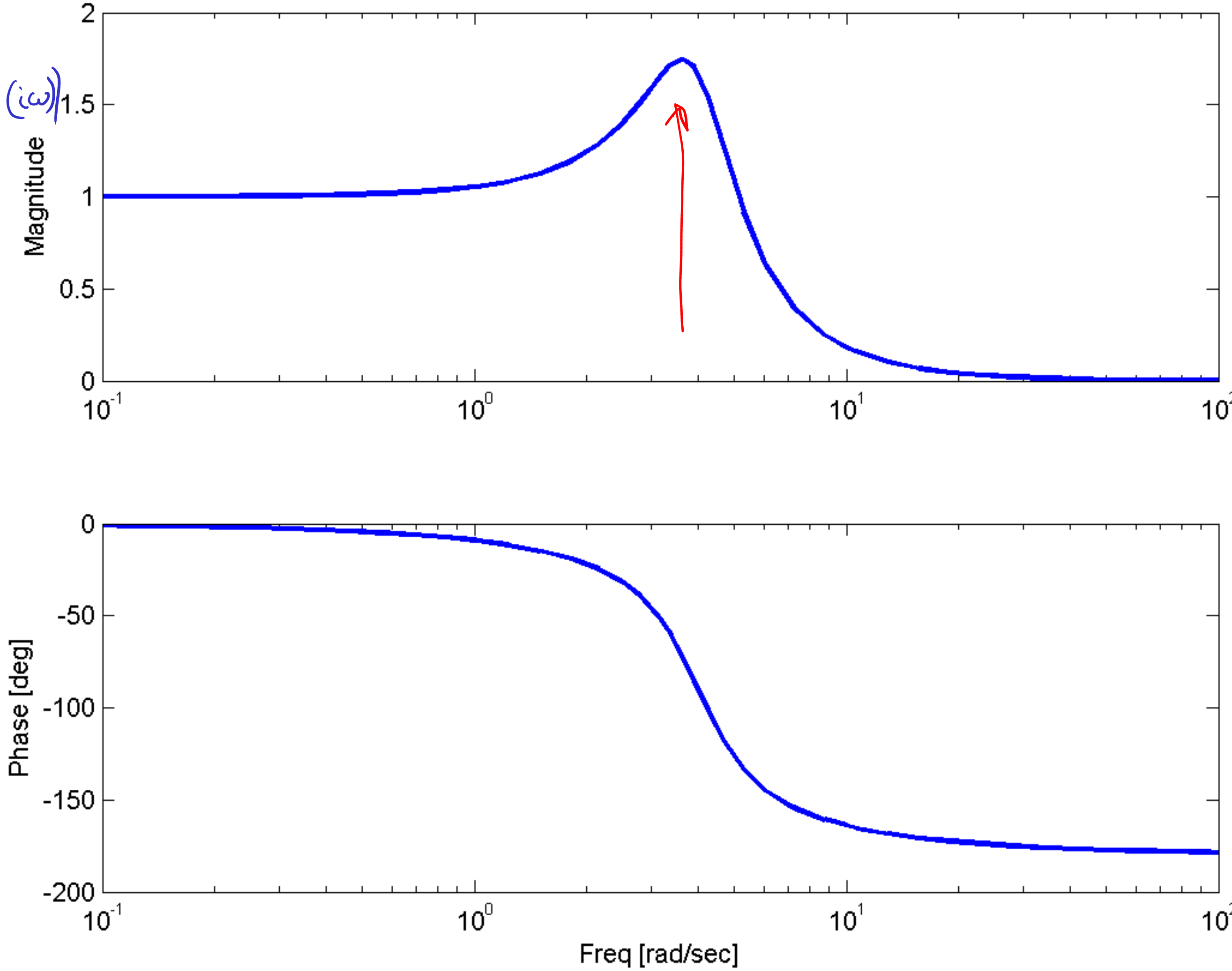
$$G(s) = \frac{16}{s^2 + 2.4s + 16}$$

$$\zeta = 0.3$$

$$\omega_n = 4$$

$|G(i\omega)|$

$\angle G(i\omega)$



Translating between representations

$$G_{yu}(s) = \frac{N(s)}{D(s)} = \cancel{\frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + \dots a_{n-1} s + a_n}} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

strictly proper
$n > m$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ & & & \ddots & & & \vdots \\ & & & & \ddots & & 0 \\ & & & & & & 1 \\ -a_n & \cdots & & & & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_m & \cdots & b_0 & \underbrace{0 \; 0 \; 0}_{\text{if } n > m+1} \end{bmatrix} \qquad D = [0]$$

$\vec{x}$ may not correspond to physical states

$$G_{yu}(s) = C(sI - A)^{-1}B = \frac{C\mathrm{adj}(sI - A)B}{|sI - A|}$$

2x2 inverse

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

adj

Modal Analysis

For n-by-n A with n distinct nonzero eigenvalues (the only case you will be tested on)

The solution is

$$\mathbf{x}(t) = \sum_i q_i \mathbf{v}_i e^{\lambda_i t}$$

where

$$\mathbf{x}(0) = \sum_i q_i \mathbf{v}_i$$

$\vec{q} = V^{-1}\vec{x}(0)$

$\vec{x}(0) = 2\vec{v}_1 + 3\vec{v}_2$

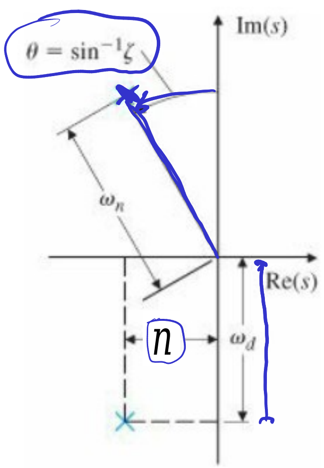


Image from Franklin, Powell, and Emami-Naeini

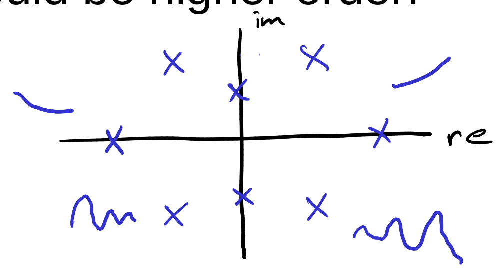
Eigenvalues are solutions to the characteristic equation,

$$|A - \lambda I| = 0$$

For second order system,

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

but the characteristic equation could be higher order!



I will assume that your calculator cannot work with complex numbers. However, I will expect you to be able to do basic math.

- Converting between polar and cartesian
- Addition, subtraction, multiplication, division

$r \angle \phi \quad re^{i\phi} \quad a+bi$

$$\frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac - adi + cbi + bd}{c^2 + d^2}$$

Roots of the transfer function denominator are the eigenvalues of A!

Feedback control

Let’s consider a **feedback control law**

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi$$

which we can implement by measuring the roll rate and roll angle and "feeding them back” into the command for the roll moment.

Closed Loop Behavior

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Original system, “plant”

$$+$$

$$\mathbf{u} = -\mathbf{K}\mathbf{x}$$

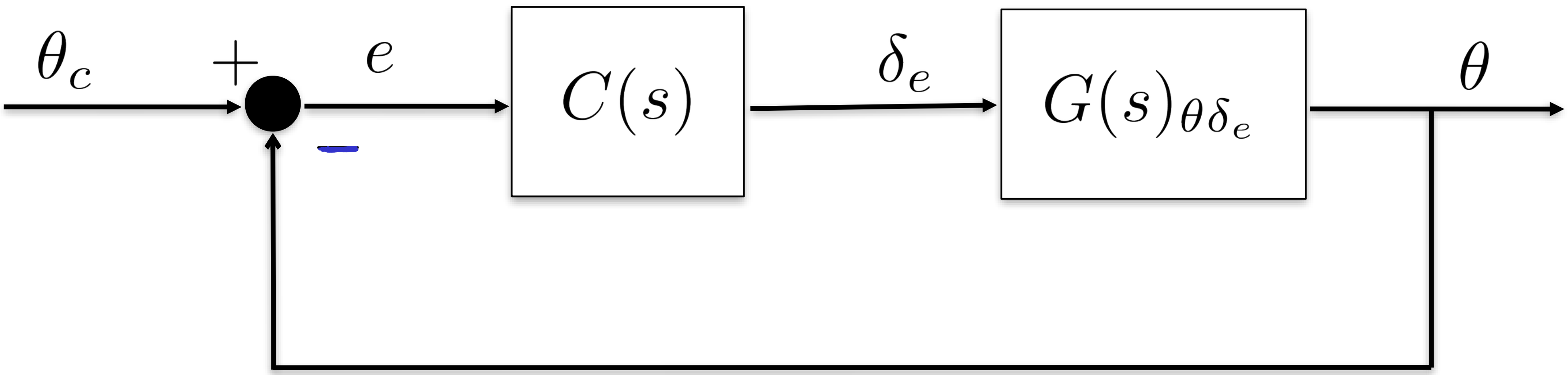
Control law

$$\dot{\mathbf{x}} = \underbrace{(\mathbf{A} - \mathbf{B}\mathbf{K})}_{A^{cl}}\mathbf{x}$$

Closed loop dynamics

The eigenvalues and eigenvectors of this closed loop state space matrix describe the “closed loop behavior”.

Feedback Control: Pitch



$$G_{\theta\theta_c} = \frac{C\,G_{\theta\delta_e}}{1 + C\,G_{\theta\delta_e}}$$

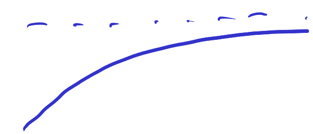
Should be able to

- derive transfer functions for simple block diagrams
- pick gains for simple controllers

General Approach to Control Design by Hand

1. Write down the control law
2. Determine the closed-loop representation of the system

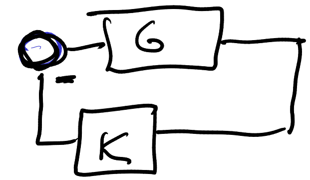
$$\boxed{A - BK} \text{ or } \boxed{\frac{CG}{1+CG}}$$



3. Decide what performance goals should be (damping ratio, rise time, steady-state error, etc.)
4. Calculate gains to achieve goal

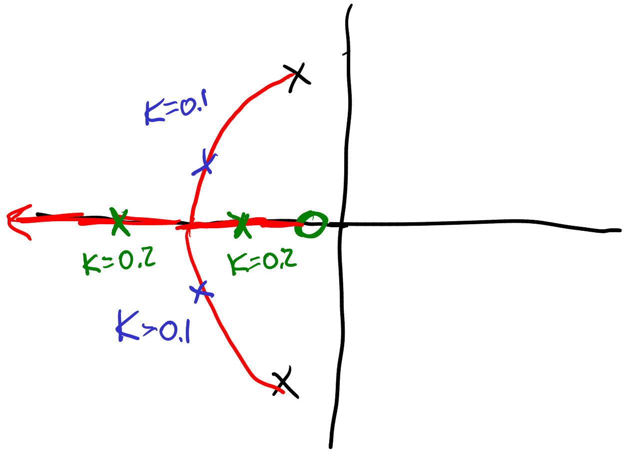
Questions

Root Loci



$$\frac{G}{1+KG}$$

$$G(s) = \frac{N(s)}{D(s)}$$



$K=0 \Rightarrow G$

start at O.L. poles

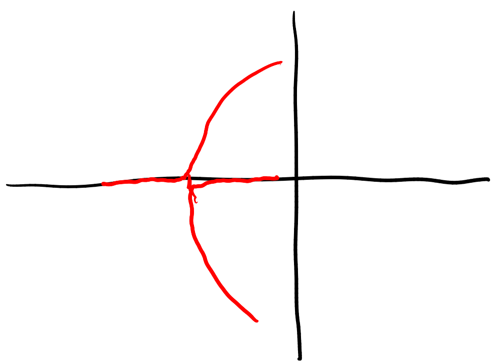
denoted by x

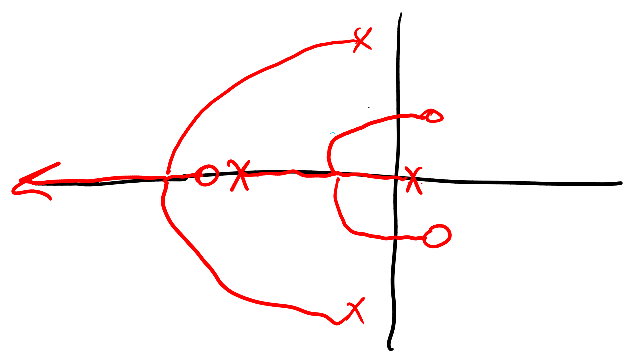
roots of D(s)

$K \to \infty$

Closed loop poles will either go to infinity in some direction, or O.L. zeros

roots of N(s)





Question from Piazza

Problem 4 (10 pts)

Consider a controller for a quadrotor aircraft that feeds back roll angle error $\phi_c - \phi$, where ϕ_c is the commanded roll angle, and roll rate p to the roll control moment L_c. The mass and moments of inertia of the quadrotor are: $m = 0.1$ kg, $I_x =$ 2e-4 kg m^2, $I_y =$ 1.6e-4 kg m^2, $I_z =$ 8e-4 kg m^2.

a. (4 pts) Use the linearized equations to design a feedback control law such that the closed loop behavior of the roll angle has natural frequency $\omega_n = 3$ rad/sec and damping ratio $\zeta = 0.5$.

b. (4 pts) Consider the outer loop feedback control law $\phi_c = k(y_c - y_E)$ where ϕ_c is the roll command passed to the controller designed for Problem 4.a and y_c is the lateral position command. What are the eigenvalues of the closed loop state matrix for the case when $k = -2$.

Most Missed Quiz Questions

Q4
1 Point

If $\phi = 90^\circ$ and $\theta = 0^\circ$ and the aircraft is rotating at 10 degrees per second about the body y-axis, which of the following is true?

- ○ $\dot{\theta} = -10^\circ/s$
- ◉ $\dot{\psi} = 10^\circ/s$
- ○ $\dot{\theta} = 10^\circ/s$
- ○ $\dot{\psi} = -10^\circ/s$

Most Missed Quiz Questions

Q4
1 Point

A quadrotor is in steady hover. A small perturbation causes a small forward motion (+x inertial direction). The linearized equations give a resulting drag force on the aircraft that is

- ○ a small value in the negative x direction.
- ◉ zero.
- ○ a small value in the positive x direction.
- ○ a small value upward.

Most Missed Quiz Questions

Q1
1 Point

At trim, the quad copter is stable.

- ○ True
- ◉ False

Most Missed Quiz Questions

Q2
1 Point

A student team designed a conventional aircraft with a horizontal tail behind the main wing. The team collected data in a wind tunnel and derived the stability and control derivatives for the aircraft. The team forget the sign convention they used for the elevator angle but determined that $C_{m_{\delta_e}} > 0$. What is the sign convention used by the team for the elevator deflection?

- (●) A positive elevator corresponds to an upward deflection.
- ( ) A positive elevator corresponds to a downward deflection.
- ( ) The angle of attack is needed to determine the sign convention.
- ( ) The area of the horizontal tail is needed to determine the sign convention.

Most Missed Quiz Questions

Q2
1 Point

An aircraft is flying along in trim at height h with speed V. At time t_0 the pilot introduces a positive step input to the elevator angle. Which statement is TRUE?

- ○ The natural frequency of the phugoid mode is increased by the step change in the elevator angle.
- ○ The aircraft will oscillate, but it will eventually decay and the aircraft will return to trim at the same height and same speed.
- ○ The short period mode, but not the phugoid mode, will be excited in the aircraft.
- ● The short period and phugoid modes will be excited in the aircraft.

Most Missed Quiz Questions

Q3
1 Point

Which feature of an aircraft will increase (make more positive/less negative) the dihedral effect stability derivative?

- ○ A positive dihedral angle on the main wings.
- ◉ The main wing mounted below the center of the fuselage.
- ○ A positive sweep angle on the main wing.
- ○ A vertical tail above the center of the fuselage.

Most Missed Quiz Questions

Q2
1 Point

In addition to damping the yaw rate of an aircraft, the yaw damper has the effect of speeding up (i.e. making the eigenvalue more negative) which other aircraft mode?

- (●) Spiral mode
- ( ) Roll mode
- ( ) Short period mode
- ( ) Phugoid mode

Most Missed Quiz Questions

A control system designer wants to use LQR with the standard longitudinal linearized dynamics to reduce phugoid oscillations. Which Q matrix would be most appropriate for the cost function?

- (selected) $$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- $$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Outline

- Example Problems
- Questions

From 2019 Exam

TRUE or FALSE?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

From 2019 Exam

TRUE or **FALSE**?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

The spiral mode has a real eigenvalue, so it does not have a damping ratio.

From 2019 Exam

TRUE or FALSE?

For the augmented lateral dynamical system with $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$, the control law $\mathbf{u} = -\mathbf{K}^* \, \mathbf{x}_{\mathrm{lat}}$ may have no effect on certain eigenvalues of the system.

From 2019 Exam

TRUE or FALSE?

For the augmented lateral dynamical system with $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$, the control law $\mathbf{u} = -\mathbf{K}^* \, \mathbf{x}_{\text{lat}}$ may have no effect on certain eigenvalues of the system.

Feedback of the first four terms does not change the eigenvalues associated with the last two terms, ie the zero eigenvalues.

From 2019 Exam

TRUE or FALSE?

Near trim for straight and level airplane flight, the rate of change in bank angle at time *t* depends on the bank angle at time *t*.

From 2019 Exam

TRUE or **FALSE**?

Near trim for straight and level airplane flight, the rate of change in bank angle at time *t* depends on the bank angle at time *t*.

Bank angle not in the linear equation for time rate of change of bank angle

$$\boxed{\Delta\dot{\phi} \quad = \Delta p + \Delta r \tan\theta_0}$$

From 2019 Exam

TRUE or FALSE?

For straight and level flight, an airplane's change in roll rate near trim at time *t* generally depends on the change in sideslip angle near trim at time *t*.

From 2019 Exam

TRUE or FALSE?

For straight and level flight, an airplane's change in roll rate near trim at time *t* generally depends on the change in sideslip angle near trim at time *t*.

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \boxed{\Gamma_3 L_v + \Gamma_4 N_v} & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

From 2019 Exam

TRUE or FALSE?

Aircraft 1 and Aircraft 2 are identical except for one feature: the vertical tail for Aircraft 1 is above the centerline of the fuselage whereas the vertical tail for Aircraft 2 is below. $C_{n_{\beta,1}} = C_{n_{\beta,2}}$.

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Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ($\lambda_5 = 0$, $\lambda_6 = 0$). Further, they both have the same eigenvector $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^T$ associated with one of these eigenvalues. Why are the other eigenvectors associated with $\lambda_5 = 0$ different?

Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ($\lambda_5 = 0$, $\lambda_6 = 0$). Further, they both have the same eigenvector $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^T$ associated with one of these eigenvalues. Why are the other eigenvectors associated with $\lambda_5 = 0$ different?

$$\boxed{\Delta \dot{x}^E = \Delta u}$$

$$\boxed{\Delta \dot{z}^E = -u_0 \Delta \theta + \Delta w}$$

$$\boxed{\Delta \dot{\psi} = \Delta r \sec \theta_0}$$

$$\boxed{\Delta \dot{y}^E = u_0 \cos \theta_0 \Delta \psi + \Delta v}$$

Questions?