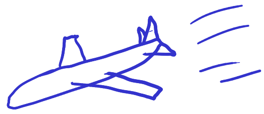
Welcome to ASEN 3728
Aircraft Dynamics!



Aeronautical Engineering

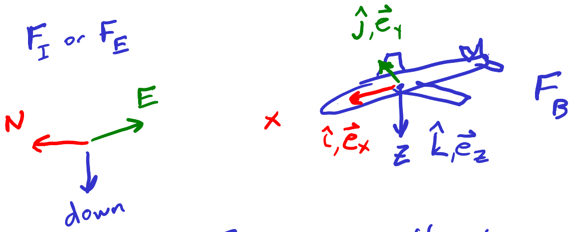
1. Aerodynamics
2. Structures / Materials
3. Propulsion
4. Dynamics + Control
   - Keep the pointy end forward
   - Get the aircraft to where we want to go

+ Mathematical description of A/C behavior
  + computer simulation model
+ Design of A/C and Control Systems to effect desirable dynamics

Notation + Conventions

Body Coordinate System

vectors: $\rightarrow$, $\hat{}$ ← unit, or **bold**



Frame: collection of ≥ 3 points (distance between points is constant)

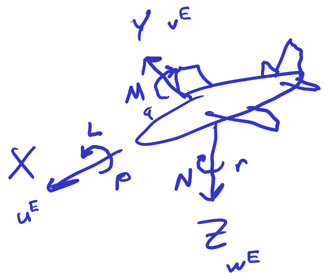
Inertial: translates with a constant velocity ~~acceleration~~ ~~rotation~~
└ Newton's laws valid

Coordinate System: 3 unit vectors that allow measurement

$\vec{V}_B^E$ ← if present, frame; ↖ coordinate system

"inertial velocity written in body coordinates"

Forces and Moments



$$\vec{f} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$V_g = |V_B^E| = \sqrt{u^{E^2} + v^{E^2} + w^{E^2}}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Control Surfaces

Aelerons δ_a

$+\delta_a$ = right down

$+\delta_a \Rightarrow -L$

Rudder

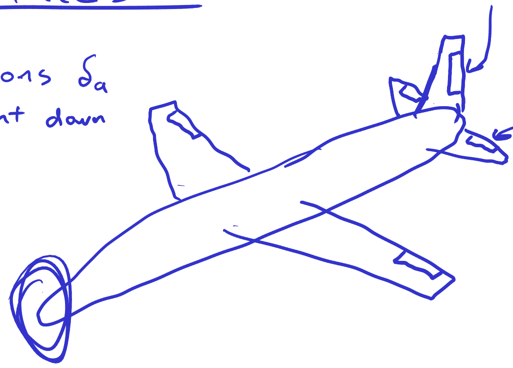
$+\delta_r$ = toward $-y$ direction

$+\delta_r \Rightarrow -N$, Y

Eelevator

$+\delta_e$ = down

$+\delta_e \Rightarrow -M$ $-Z$



Throttle: δ_t

$+\delta_t \Rightarrow +X$ force

no moment

Wind

(not gravity)

Aerodynamic Forces + Moments are functions of the A/C velocity w.r.t. the <u>air</u>

$$\vec{V}^E = \underset{\text{air-relative}}{\vec{V}} + \underset{\text{wind}}{\vec{W}}$$

When no wind $\vec{V}^E = \vec{V}$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Wind Angles

$u = V \cos\beta \cos\alpha$

$v = V \sin\beta$

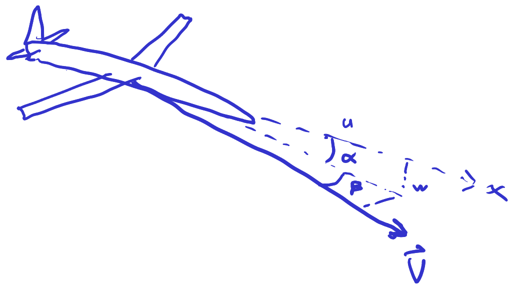
$w = V \cos\beta \sin\alpha$

angle of attack α

$\alpha \equiv \tan^{-1} \frac{w}{u}$

sideslip angle β

$\beta \equiv \sin^{-1} \frac{v}{V}$



Orientation

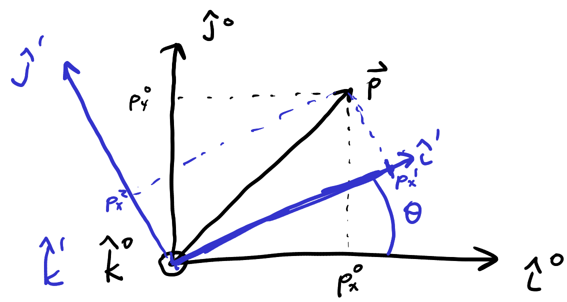
"1" ϕ roll

"2" Θ pitch $\qquad$ 3-2-1

"3" ψ yaw

know $\vec{V}^{E}_{B}$ (Frame: E; Coordinate: B) $\quad$ want $\vec{V}^{E}_{E}$

$\vec{p}_0 = \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix} \qquad \vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix}$



frame F_0

F_1

$\vec{p} = p_x^0 \hat{i}^0 + p_y^0 \hat{j}^0 + p_z^0 \hat{k}^0$

$\vec{p} = p_x^1 \hat{i}^1 + p_y^1 \hat{j}^1 + p_z^1 \hat{k}^1$

→ want p_x^1 in terms of $\vec{p}_0$

$p_x^1 = \vec{p} \cdot \hat{i}^1$

$= p_x^0 \hat{i}^0 \cdot \hat{i}^1 + p_y^0 \hat{j}^0 \cdot \hat{i}^1 + p_z^0 \hat{k}^0 \cdot \hat{i}^1$

$= \underline{\begin{bmatrix} \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \end{bmatrix}} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$

$p_x^1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \vec{p}_0$

$$\vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix} = \begin{bmatrix} \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \\ \hat{i}^0 \cdot \hat{j}^1 & \hat{j}^0 \cdot \hat{j}^1 & \hat{k}^0 \cdot \hat{j}^1 \\ \hat{i}^0 \cdot \hat{k}^1 & \hat{j}^0 \cdot \hat{k}^1 & \hat{k}^0 \cdot \hat{k}^1 \end{bmatrix} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$\vec{p}_1 = R_0^1 \, \vec{p}_0$

Direction Cosine Matrix $\qquad$ Rotation Matrix

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3: axis; θ: axis)

$$\boxed{\begin{matrix} c\theta & s\theta \\ -s\theta & c\theta \end{matrix}}$$

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \qquad R_2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

Properties of DCMS

$\vec{p}_2 = R_1^2 \vec{p}_1 \qquad \vec{p}_1 = R_0^1 \vec{p}_0 \qquad \vec{p}_2 = R_1^2 \vec{p}_1 = \vec{p}_2 = R_1^2 \underbrace{R_0^1 \vec{p}_0}$

Chaining $\qquad R_0^2 = R_1^2 R_0^1 \qquad \boxed{R_A^C = R_B^C R_A^B}$

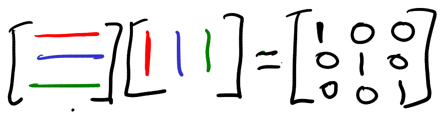
Inverse

$$\vec{p}_2 = R_1^2 \vec{p}_1$$

$$\underbrace{R_1^{2^{-1}} \vec{p}_2 = \cancel{R_1^{2^{-1}} R_1^2} \vec{p}_1 = R_2^1 \vec{p}_2}$$

$$R_2^1 = (R_1^2)^{-1} \qquad R_A^B = (R_B^A)^{-1}$$

since DCMs are orthonormal



$$(R_A^B)^{-1} = R_A^{B\,T}$$

$$\boxed{R_A^B = R_B^{A\,T}}$$

Euler Angles

Any ~~rotatation~~ rotation can be described by 3 Euler angles about 3 n-repeated axes

E.g. 3-1-3 rotation through α, β, γ

3-2-1 rotation through ψ, θ, ϕ

$$\vec{p}_B = R_E^B \vec{p}_E = \underbrace{R_1(\phi)}_{\text{roll}} \underbrace{R_2(\theta)}_{\text{pitch}} \underbrace{R_3(\psi)}_{\text{yaw}} \vec{p}_E$$

$$R_E^B = \begin{pmatrix} C_\theta C_\psi & C_\theta S_\psi & -S_\theta \\ S_\phi S_\theta C_\psi - C_\phi S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & S_\phi C_\theta \\ C_\phi S_\theta C_\psi + S_\phi S_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi & C_\phi C_\theta \end{pmatrix}$$

$\vec{p}^{\,E}_{pilot} = \vec{p}^{\,E}_{A/c} + \vec{p}^{\,B}_{pilot}$ $\leftarrow$ abstract vectors

$$\boxed{\vec{p}^{\,E}_{pilot\,E} = \vec{p}^{\,E}_{A/c\,E} + R_B^E \vec{p}^{\,B}_{pilot\,B}}$$

Vector Clarity

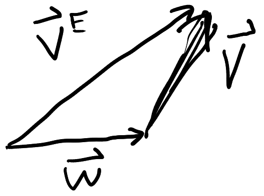
$$\vec{V}^{E}_{B}$$

E: "frame of reference", "with respect to", "relative to"

B: "coordinate frame", "expressed in", "written in"

$$\rightarrow \quad \vec{V}^{E} = \vec{V}^{(W)} + \vec{W}^{(E)}$$

by convention



True in <u>Any</u> coordinate system as long as all in <u>same</u> coordinate system

$$\vec{V}^{E}_{B} = \vec{V}_{B} + \vec{W}_{B} \qquad \vec{V}^{E}_{E} = \vec{V}_{E} + \vec{W}_{E}$$

$$\begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \cancel{\begin{bmatrix} \; \end{bmatrix}}$$

Kinematics

Kinematics : "Geometry of Motion" (no forces) $<$ Translational / Rotational

Dynamics : Effects of forces and moments on objects

Vector Derivatives

$\rightarrow \frac{d}{dt}\vec{p} \equiv$ time rate of change of $\vec{p}$

$\frac{d}{dt}\vec{p} = \vec{v}^E$

$$\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \quad \rightarrow \quad \dot{\vec{p}}_B \equiv \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Not Always True

$\hat{i}_B^B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\hat{i}_B \equiv \hat{i}_B^B$

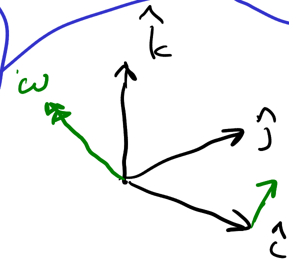
$$\vec{p}_B = p_x \hat{i}_B + p_y \hat{j}_B + p_z \hat{k}_B$$

$$\dot{\vec{p}}_B = \dot{p}_x \hat{i}_B + \dot{p}_y \hat{j}_B + \dot{p}_z \hat{k}_B$$

$\frac{d}{dt}(uv) = u\frac{d}{dt}v + v\frac{d}{dt}u$

$$\left(\frac{d}{dt}\vec{p}\right)_B = \dot{p}_x \hat{i}_B + p_x \frac{d}{dt}\hat{i}_B + \dot{p}_y \hat{j}_B + p_y \frac{d}{dt}\hat{j}_B + \dot{p}_z \hat{k}_B + p_z \frac{d}{dt}\hat{k}_B$$

What is $\frac{d}{dt}\hat{i}_B$



$\frac{d}{dt}\hat{i} = \vec{\omega} \times \hat{i}$

$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

$$= p_x(\vec{\omega}_B \times \hat{i}_B) + p_y(\vec{\omega}_B \times \hat{j}_B) + p_z(\vec{\omega}_B \times \hat{k}_B)$$

$$= \vec{\omega}_B \times \vec{p}_B$$

$$\boxed{\frac{d}{dt}\vec{p}_B = \dot{\vec{p}}_B + \vec{\omega}_B \times \vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Kinematic Transport Theorem

Equations of motion for a 3D body

$\dot{\vec{x}} = f(t, \vec{x})$ → Need

Aircraft State

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \quad \begin{matrix} \left.\begin{matrix} x_E \\ y_E \\ z_E \end{matrix}\right\} \vec{p}_E = \vec{p}_E^{\,E} \\ \left.\begin{matrix} \phi \\ \theta \\ \psi \end{matrix}\right\} \vec{\Theta} \text{ pseudo-vector} \\ \left.\begin{matrix} u^E \\ v^E \\ w^E \end{matrix}\right\} \vec{v}_B^{\,E} \\ \left.\begin{matrix} p \\ q \\ r \end{matrix}\right\} \vec{\omega}_B^{\,E} \end{matrix}$$

Need

$\dot{\vec{p}}_E$, $\dot{\vec{\Theta}}$ ← Kinematic

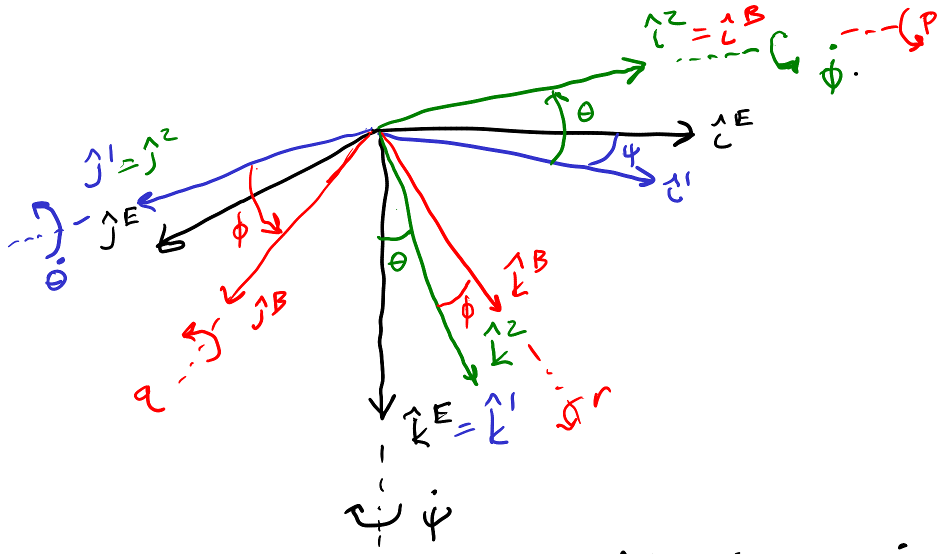
$\dot{\vec{v}}_B^{\,E}$, $\dot{\vec{\omega}}_B$ ← Dynamics

Translational Kinematics

$$\dot{\vec{p}}_E^{\,E} = \frac{d}{dt}\vec{p}_E - \cancel{{}^{E}\vec{\omega}_E}^{\,0} \times \vec{p}_E^{\,E} = \vec{v}_E^{\,E} = R_B^E \, \vec{v}_B^{\,E}$$

$$\boxed{\dot{\vec{p}}_E^{\,E} = R_B^E \, \vec{v}_B^{\,E}}$$

Rotational Kinematics



$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega} = \dot{\psi}\,\hat{k}^E + \dot{\theta}\,\hat{j}^1 + \dot{\phi}\,\hat{i}^B$$

$$\vec{\omega}_B = \dot{\psi}\,\hat{k}_B^E + \dot{\theta}\,\hat{j}_B^1 + \dot{\phi}\,\hat{i}_B^B$$

$$= \dot{\psi} R_E^B \underbrace{\hat{k}_E^E}_{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} + \dot{\theta} R_1^B \underbrace{\hat{j}_1^1}_{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + \dot{\phi}\,\hat{i}_B^B$$

$$= R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$R_E^B = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$R_1^B = R_1(\phi) R_2(\theta)$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi\, c\theta \\ 0 & -s\phi & c\phi\, c\theta \end{bmatrix}}_{} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

invert

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\sec x = \frac{1}{\cos x}$$

$$\dot{\vec{\sigma}} = T\,\vec{\omega}_B$$

attitude influence matrix

Dynamics $\vec{f}$ $\vec{G}$

Translational Dynamics

Newtons 2nd Law

$$\vec{f} = m\vec{a}$$

$$\vec{f} = m\underline{\frac{d}{dt}}\vec{V}^E$$

$$\frac{d}{dt}\vec{V}_B^E = \dot{\vec{V}}_B^E + \vec{\omega}_B \times \vec{V}_B^E$$

$$m\left(\dot{\vec{V}}_B^E + \tilde{\omega}_B \vec{V}_B^E\right) = \vec{f}_B$$

$$\boxed{\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \tilde{\omega}_B \vec{V}_B^E}$$ *

Rotational Dynamics

Euler's 2nd Law

angular momentum

$$\frac{d}{dt}\vec{h} = \vec{G}$$

moments

$$\vec{h} = I\vec{\omega}$$

moment of inertia

$$\mathbf{I} = \begin{pmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

$\dot{\vec{\omega}}_B$

$$\frac{d}{dt}\vec{h}_B = \dot{\vec{h}}_B + \tilde{\omega}_B \vec{h}_B = \vec{G}_B$$

$$I\dot{\vec{\omega}}_B + \tilde{\omega}_B I \vec{\omega}_B = \vec{G}_B$$

$$\boxed{\dot{\vec{\omega}}_B = I^{-1}\left[\vec{G}_B - \tilde{\omega}_B I \vec{\omega}_B\right]}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E^E \\ \dot{\vec{\Theta}} \\ \dot{\vec{V}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix}$$ *

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

$$\dot{\vec{\Theta}} = T\,\vec{\omega}_B$$

$$\vec{x} = \begin{bmatrix} x_E\\ y_E\\ z_E\\ \phi\\ \theta\\ \psi\\ u^E\\ v^E\\ w^E\\ p\\ q\\ r\end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{p}_E^E = \vec{p}_E \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{\Theta} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{V}_B^E \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{\omega}_B^E = \vec{\omega}_B \end{matrix}$$

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

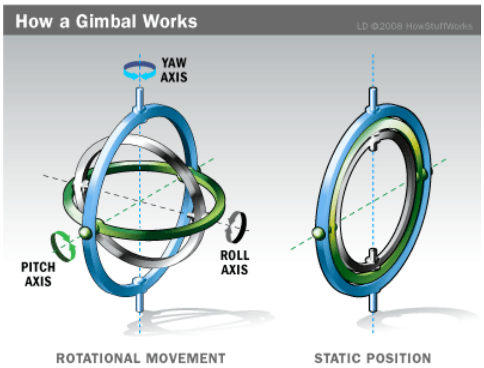
$$\dot{\vec{\Theta}} = T \vec{\omega}_B$$

attitude influence

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

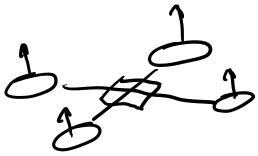
$$\dot{\vec{\omega}}_B = I_B^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B\right]$$

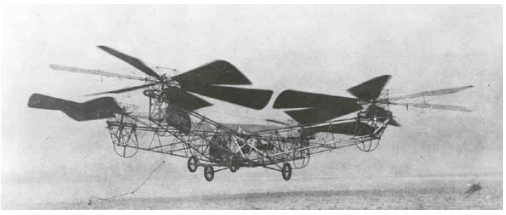
$$T = \begin{bmatrix} 1 & \sin\phi \underline{\tan\theta} & \cos\phi \underline{\tan\theta} \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \underline{\sec\theta} & \cos\phi \underline{\sec\theta} \end{bmatrix}$$



Multicopter

Quadrotor

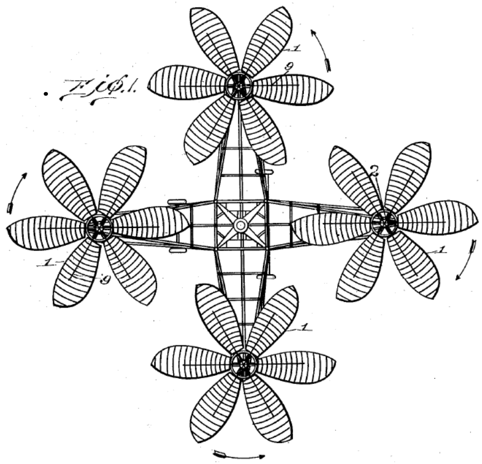


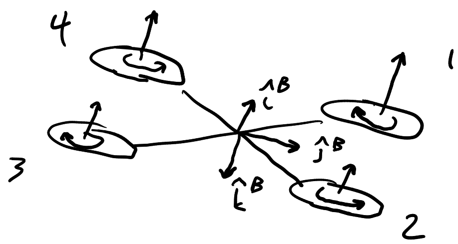


March 4, 1930. G. DE BOTHEZAT 1,749,471

HELICOPTER

Filed March 29, 1924 5 Sheets-Sheet 1





$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^c\vec{G} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

$$\vec{f} = {}^g\vec{f} + {}^a\vec{f} + {}^c\vec{f}$$
$$\vec{G} = {}^a\vec{G} + {}^c\vec{G}$$

Since symmetric about $\hat{i}^B - \hat{k}^B$ plane and $\hat{i}^B - \hat{j}^B$ plane

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \overbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}^{\vec{\omega}_B \times \vec{V}_B^E} + \overbrace{g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}}_{{}^a\vec{f}_B} + \frac{1}{m} \underbrace{\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}}_{{}^c\vec{f}_B}}^{\vec{f}_B}$$

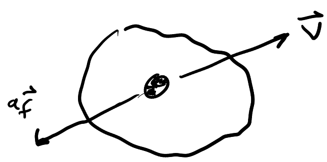
$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$I_B^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix}}^{I_B^{-1}(\vec{\omega}_B \times I\vec{\omega}_B)} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Aerodynamic Forces and Moments

$${}^a\vec{f} = -D\frac{\vec{V}}{|\vec{V}|}$$



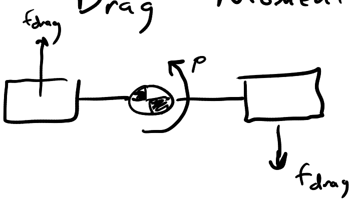
$V_a = |\vec{V}|$ airspeed

$$D = \overbrace{\frac{1}{2}\rho C_D A}^{\nu} V_a^2 = \nu V_a^2$$

density, coefficient of drag (shape), cross-sectional area

$${}^a\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = -\nu V_a^2 \frac{\vec{V}_B}{V_a}$$
$$= -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

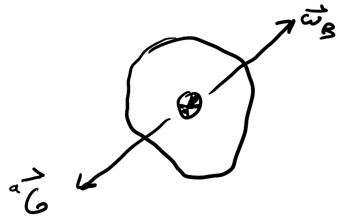
Drag Moment



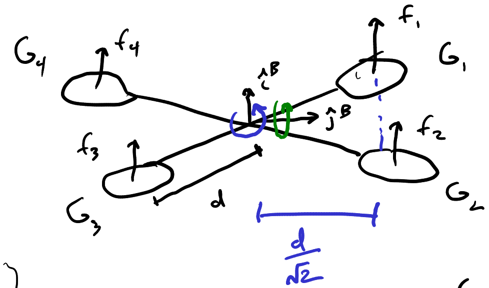
$$L_{drag} = -2l f_{drag}$$
$$= -2l\left(\tfrac{1}{2}\rho C_D A (l_p)^2\right)\operatorname{sign}(p)$$
$$= -\mu |p| p$$

$$-p^2 \operatorname{sign}(p) = -|p|p$$

$$^{a}\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = -\mu\sqrt{p^2+q^2+r^2}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



Control Forces + Moments



$$^{c}\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_1-f_2-f_3-f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$$^{c}G_B = \begin{bmatrix} \frac{d}{\sqrt{2}}(-f_1-f_2+f_3+f_4) \\ \frac{d}{\sqrt{2}}(f_1-f_2-f_3+f_4) \\ \underbrace{G_1 - G_2 + G_3 - G_4}_{k_m(f_1-f_2+f_3-f_4)} \end{bmatrix} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

$$G_i = k_G C_D \omega_r^2$$
$$f_i = k_f C_L \omega_r^2$$
$$k_m = \frac{k_G C_D}{k_f C_L}$$

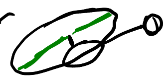
$$G_i = k_m f_i$$

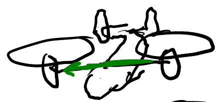
$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ k_m & -k_m & k_m & -k_m \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

invert

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

	Multirotor	vs	Conv Helicopter
Aerodynamic Efficient			✓
Mechanical Control Complexity	✓		
Power System Complexity: Turbine			✓
Electric	✓		





$\dot{\vec{x}} = f(\vec{x}, \vec{u})$

- differential
- 1st order
- ordinary
- nonlinear
- coupled

general form

$$\begin{cases} \dot{\vec{p}}^E_E = R^E_B \vec{V}^E_B \\ \dot{\vec{o}} = T \vec{\omega}_B \\ \dot{\vec{V}}^E_B = \dfrac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}^E_B \\ \dot{\vec{\omega}}_B = I_B^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B\right] \end{cases}$$

$$T = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}$$

$\phi = 90°$
$\theta = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 10°/s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q = 10°/s \\ r \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Quadrotors

$0 = \frac{1}{m} X + \frac{0}{m} \quad \therefore \; X = 0$
$Y = 0$

$0 = g + \frac{1}{m} Z^{0} + \frac{1}{m} Z_c$

$Z_{c,0} = -mg$

"Hover" trim condition

trim condition

$\dot{x}_{E,0} = 0$	$u^E_0 = 0$	$\dot{u}^E_0 = 0$
$\dot{y}_{E,0} = 0$	$v^E_0 = 0$	$\dot{v}^E_0 = 0$
$\dot{z}_{E,0} = 0$	$w^E_0 = 0$	$\dot{w}^E_0 = 0$
$p_0 = 0$	$\dot{p}_0 = 0$	$\phi_0 = 0$
$q_0 = 0$	$\dot{q}_0 = 0$	$\theta_0 = 0$
$r_0 = 0$	$\dot{r}_0 = 0$	$\psi_0 = \cdot$

any constant value

$f_{1,0} = \frac{mg}{4}$
$f_{2,0} = $ ″
$f_{3,0} = $ ″
$f_{4,0} = $ ″

$\boxed{Z_{c,0} = mg}$
$L_{c,0} = 0$
$M_{c,0} = 0$
$N_{c,0} = 0$

Linearazation

$\vec{x} = \vec{x}_0 + \Delta\vec{x} \qquad \vec{u} = \vec{u}_0 + \Delta\vec{u}$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta\phi \\ \Delta\theta \\ \Delta\psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$\Delta\dot{x} = A\Delta x + B\Delta u$

Taylor Series

$$y = f(x,u) \qquad x_0 + \Delta x \qquad y_0 = f(x_0, u_0)$$

$$y_0 + \Delta y = f(x_0 + \Delta x,\ u_0 + \Delta u)$$

$$= \underbrace{f(x_0, u_0)}_{y_0} + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + H.O.T.$$

$$\Delta y = \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + H.O.T.$$

since $\Delta x, \Delta u$ are small

$\Delta x \Delta u \approx 0$

$\sin(\Delta x) \approx \Delta x$

$\cos(\Delta x) \approx 1$

$$x + y = x_0 + \Delta x + y_0 + \Delta y$$

$$xu \approx x_0 u_0 + \left.\frac{\partial xu}{\partial x}\right|_0 \Delta x + \left.\frac{\partial xu}{\partial u}\right|_0 \Delta u$$

$$= x_0 u_0 + u_0 \Delta x + x_0 \Delta u$$

alternative

$$xu = (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + u_0 \Delta x + x_0 \Delta u + \cancel{\Delta x \Delta u} \to 0$$

$$\sin(\theta) \approx \sin\theta_0 + \left.\frac{\partial \sin\theta}{\partial \theta}\right|_0 \Delta\theta$$

$$= \sin\theta_0 + \cos\theta_0 \Delta\theta$$

alternatively

$$\sin(\theta) = \sin(\theta_0 + \Delta\theta) = \sin(\theta_0)\cancel{\cos(\Delta\theta)}^{1} + \cos(\theta_0)\cancel{\sin(\Delta\theta)}^{\Delta\theta} \approx \sin\theta_0 + \cos\theta_0 \Delta\theta$$

$$\cos(\theta) \approx \cos(\theta_0) - \sin(\theta_0)\Delta\theta$$

Linearize QR equations of motion

Examples

$$\dot\theta = \cos\phi\, q - \sin\phi\, r$$

$$\cancel{\dot\theta_0}^{0} + \Delta\dot\theta = \cos(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{q_0}^{0} + \Delta q) - \sin(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{r_0}^{0} + \Delta r)$$

$$\Delta\dot\theta \approx \cancel{\cos(\Delta\phi)}^{1}\Delta q - \cancel{\sin \Delta\phi}^{\Delta\phi}\Delta r$$

$$\approx \Delta q - \cancel{\Delta\phi\Delta r}^{0}$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

$$\dot w^E = q u^E - p v^E + g\cos\theta\cos\phi + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$\Delta \dot w^E = \cancel{\Delta q \Delta u^E}^{0} - \cancel{\Delta p \Delta v^E}^{0} + g + \frac{1}{m}\Delta Z + \frac{1}{m}(Z_{c,0} + \Delta Z_c)$$

cancel

$$\Delta \dot w^E = \frac{1}{m}\Delta Z + \frac{1}{m}\Delta Z_c$$

Linearized Aerodynamics

~~In real life $\nu(p,q,r)$~~

ν constant

$$Z = -\nu w \sqrt{u^2 + v^2 + w^2}$$

$$\Delta Z = Z_0 + \left.\frac{\partial Z}{\partial u}\right|_0 \Delta u + \left.\frac{\partial Z}{\partial v}\right|_0 \Delta v + \left.\frac{\partial Z}{\partial w}\right|_0 \Delta w$$

Looks like $\frac{0}{0}$ take limit as $u, v, w \to 0$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = -2\nu w u (u^2 + v^2 + w^2)^{-1/2}$$

$$\boxed{\left.\frac{\partial Z}{\partial u}\right|_0 = 0}$$

$$\boxed{\left.\frac{\partial Z}{\partial v}\right|_0 = 0}$$

$$\frac{\partial Z}{\partial w} = -\nu \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$$\boxed{\left.\frac{\partial Z}{\partial w}\right|_0 = 0}$$

$$\boxed{\Delta Z = 0}$$

$$\boxed{\begin{aligned}\Delta L &= 0\\ \Delta M &= 0\\ \Delta N &= 0\end{aligned}}$$

$$Z = -\nu w^2$$

$$\frac{\partial Z}{\partial w} = -\nu 2 \cancel{w}^{0}$$

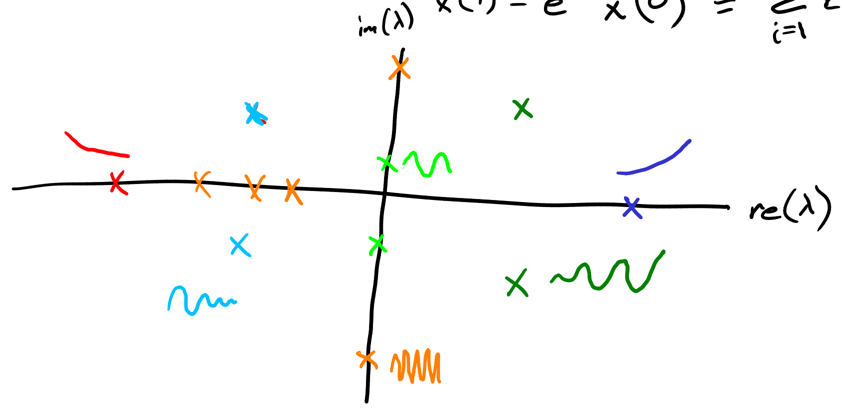
Review: State space representation of a Linear Dynamical System

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} + D\vec{u}$$

Eigenvectors Eigenvalues

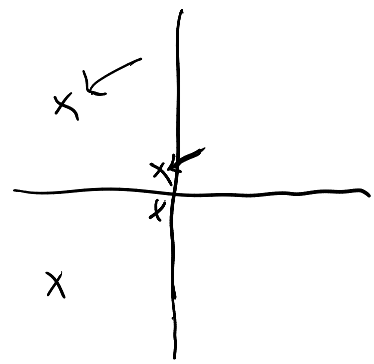
$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$\vec{x}(t) = e^{At}\vec{x}(0) = \sum_{i=1}^{n} k_i e^{\lambda_i t}\vec{v}_i$$



a+bi
a-bi

$$\vec{u} = -K\vec{x}$$
$$\dot{\vec{x}} = \underbrace{(A-BK)}_{A^{cl}}\vec{x}$$



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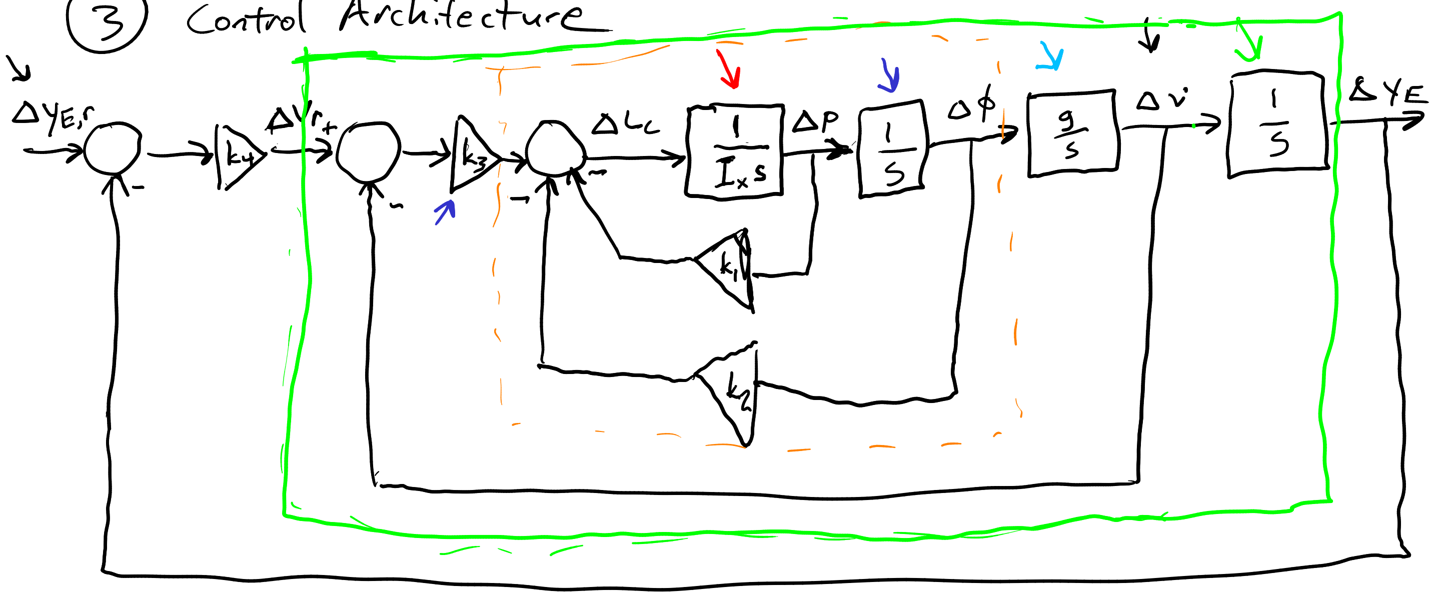
(1)
(2)

$$\underbrace{\begin{bmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{bmatrix}}_{\dot{\vec{x}}} = \begin{bmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta\phi \\ \Delta p \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_x} \end{bmatrix}}_{B} \underbrace{\Delta L_c}_{u}$$

(3) Control Architecture



(4) Choosing Gains

1) $k_1 + k_2$
2) k_4
3) k_3 Root Locus

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi - k_3 \Delta v - k_3 k_4 \Delta y_E + k_3 k_4 \Delta y_{E,r}$$

$$\begin{bmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix}}_{A^{cl}} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3 k_4}{I_x} \end{bmatrix} \Delta y_{E,r}$$

$$\rightarrow \quad \begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

$$\rightarrow \quad |A - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{2 I_x} + \sqrt{\frac{k_1^2}{4 I_x^2} - \frac{k_2}{I_x}}$$

Desired:

$\zeta = 0.7$

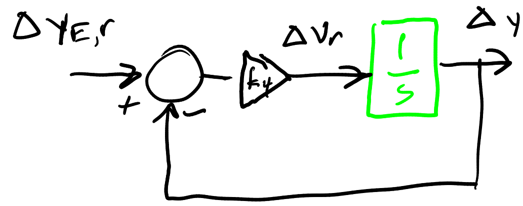
$\omega_n = 16 \text{ rad/s}$ $\quad T = \frac{2\pi}{\omega_n}$

$\frac{T}{2} = 0.2s$

$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$

$\lambda = -11.2 + 11.4i$

$$\boxed{\begin{aligned} k_1 &= 0.0016 \\ k_2 &= 0.0179 \end{aligned}}$$



$$\Delta \dot{y}_E = k_4 (y_{E,r} - \Delta y_E)$$

$$\Delta y_E(t) = \Delta y_{E,r}\left(1 - e^{-k_4 t}\right)$$

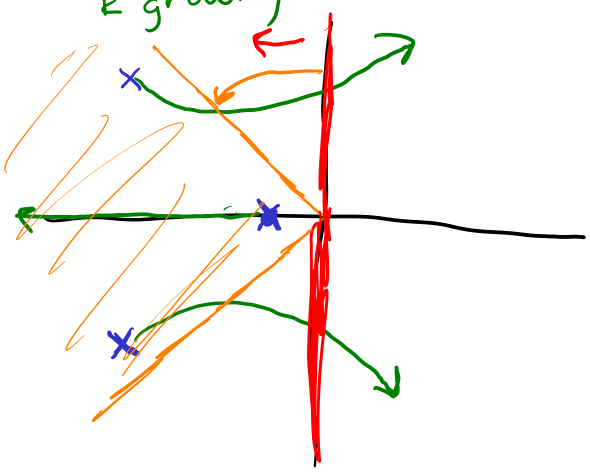
choose k_4 1 order of magnitude slower

$$\tau = \frac{1}{k_4} > 10 \cdot \frac{2\pi}{\zeta\omega_n} = 5.6 \qquad \boxed{k_4 = 0.17}$$

Choose k_3 via Root Locus

x: $k=0$

k growing



Root Locus: plot of the poles/eigen values of a system as a parameter (usually a gain) changes

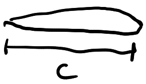
Conventional A/C Dynamics

Longitudinal

Altitude
Airspeed
Pitch

Lateral-Directional

Roll
Yaw
Sideslip
Turning



Lift
Drag
(Pitch Moment)

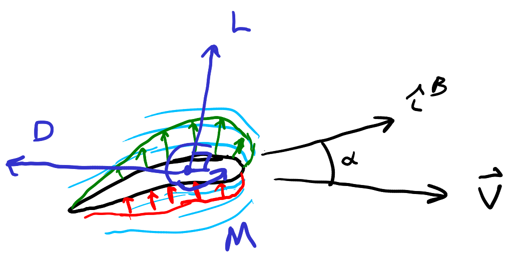
density, airspeed, wing Area

$$L = \frac{1}{2}\rho V_a^2 S C_L$$
$$D = \frac{1}{2}\rho V_a^2 S C_D$$
$$M = \frac{1}{2}\rho V_a^2 S c C_m$$

chord

$C_L(\alpha, q, \delta_e)$ nonlinear
$C_D(\alpha, q, \delta_e)$
$C_m(\alpha, q, \delta_e)$

Lift



1st order Taylor Series of $C_L(\alpha, q, \delta_e)$

$$\boxed{L = \frac{1}{2}\rho V_a^2 S\left(C_{L_0} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q}q + \frac{\partial C_L}{\partial \delta_e}\delta_e\right)}$$

$$= \frac{1}{2}\rho V_a^2 S\left(C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\underbrace{\frac{c}{2V_a}q}_{\hat{q}} + C_{L_{\delta_e}}\delta_e\right)$$

Nondimensional Stability Derivative

Nondim. control derivative

Stability Derivatives

- based linear assumptions / linearization
- Main tool connecting aerodynamics to dynamics
- Functions of A/C geometry

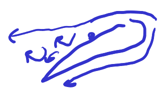
Example

Pitch Stiffness C_{m_α}

$$\ddot{\alpha} \approx \frac{k\, C_{m_\alpha}}{I}\alpha$$

$$\ddot{x} = -\frac{k_{spring}}{m}x$$

- Only valid in a linear region e.g. small α



$AR = \frac{b^2}{S}$ (span)

Estimated Using
- Geometric Data
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

eg. $C_{L_\alpha} = \frac{\pi AR}{1 + \sqrt{1 + (AR/2)^2}}$

Drag

nonlinear

α only

$$C_D(\alpha) = C_{D_p} + \frac{C_L(\alpha)^2}{\pi e AR}$$

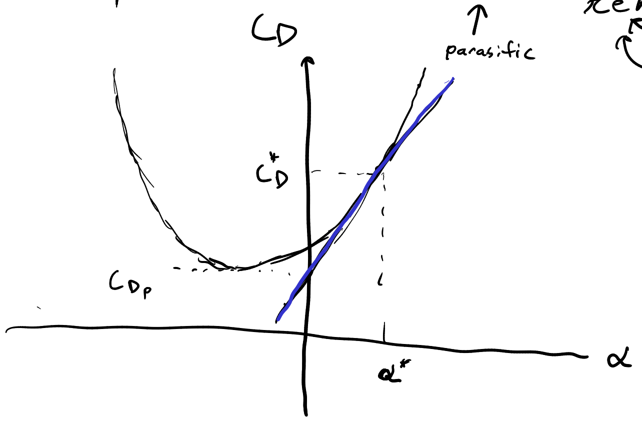
parasitic

Oswalds efficiency

induced drag

linearize

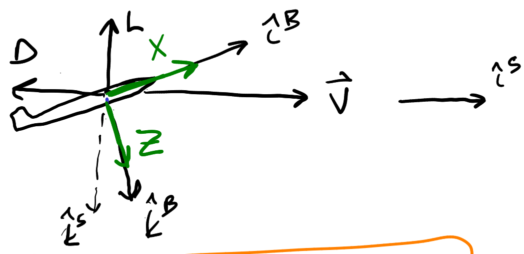
$$C_{D_0} + C_{D_\alpha}\alpha + C_{D_q}\hat{q} + C_{D_{\delta e}}\delta e$$



In this class:

$$K = \frac{1}{\pi e AR}$$

$$\boxed{C_D = C_{D_{min}} + K\left(C_L(\alpha, q, \delta e) - C_{L_{min}}\right)^2}$$



$\hat{i}^S$ "stability frame"

aerodynamic forces in $\hat{i}^B$ and $\hat{k}^B$

$$\boxed{\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -D \\ -L \end{bmatrix}}$$

Pitching Moment

$$\boxed{M = \frac{1}{2}\rho V_a^2 S c \left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right]}$$

trim coefficient of moment

pitch stiffness coefficient

pitch damping coefficient

Longitudinal Trim

Last time: Long. Forces + Moments

coordinate def. $\rightarrow 0$

$$L = \tfrac{1}{2}\rho V_a^2 S \left(C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\hat{q} + C_{L_{\delta e}}\delta e\right)$$

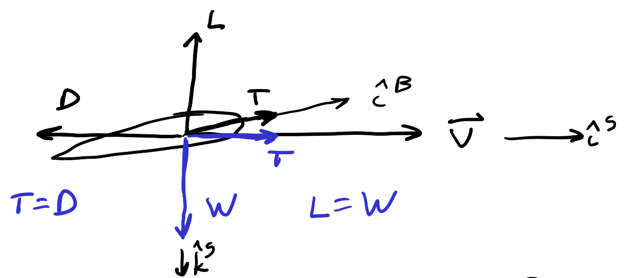
$$D = \tfrac{1}{2}\rho V_a^2 S \left(C_{D_{min}} + K\left(C_L(\alpha, q, \delta e) - C_{L_{min}}\right)^2\right)$$

$$M = \tfrac{1}{2}\rho V_a^2 S \bar{c} \left(C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right)$$

$$C_{a_b} \equiv \frac{\partial C_a}{b}$$

Trim: SLUF: Steady Level Unaccelerated Flight

just "in trim"



For linear trim calc

At trim, forces + moments about C.G. sum to 0

$\hat{i}^S$ dir: $T_{trim} = D_{trim}$

$\hat{k}^S$ dir: $L_{trim} = W$

$M_{trim} = 0$

V_a fixed

$$\cancel{C_{L_{trim}} = C_{L_\alpha}\alpha_{trim} = \frac{W}{\frac{1}{2}\rho V_a^2 S}}$$

$$\cancel{C_{m_{trim}} = C_{m_0} + C_{m_\alpha}\alpha_{trim} = 0}$$

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e = 0$$

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(special case of Cramer's Rule)

$$\boxed{\begin{aligned} \alpha_{trim} &= \frac{C_{m_0} C_{L_{\delta e}} + C_{m_{\delta e}} C_{L_{trim}}}{\Delta} \\ \delta_{e_{trim}} &= -\frac{C_{m_0} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta} \end{aligned} \qquad \Delta = C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha}}$$

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$

(from slides)

$$C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m0_p} + a_t \bar{V}_H \left(\epsilon_0 + i_t\right)\left[1 - \frac{a_t S_t}{a S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}$$

$$C_{m_\alpha} = C_{L_\alpha}\left(h - h_n\right)$$

Direct dependence on CG location h

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e}\frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}}\left(h - h_{n_{wb}}\right)$$

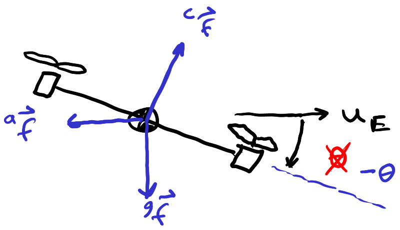
Some mistakes on hw

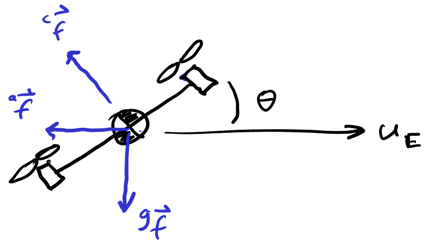
if $x(0) = \sum_i k_i \vec{v}_i$ $\qquad$ $x(t) = \sum_i k_i e^{\lambda_i t} \vec{v}_i$ not gains

Rate of climb $= \cancel{\dot{z}_E} = -\dot{z}_E$

$\dot{h} = -\dot{z}_E$

Diagram





Linearization

2 methods

Method 1: f.o. Taylor Series

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial y}\right|_0 \Delta y$$

$$xy = x_0 y_0 + y_0 \Delta x + x_0 \Delta y$$

$$\sin(\theta) = \sin(\theta_0) + \left.\frac{\partial \sin\theta}{\partial \theta}\right|_0 \Delta\theta$$

$$= \sin(\theta_0) + \cos(\theta_0)\Delta\theta$$

may include x_0

$$\Delta\dot{x} = \underbrace{a}\Delta x + b$$

Method 2

$f(x,y) \rightarrow f(x_0+\Delta x, y_0+\Delta y) \longrightarrow$ expand out/cancel out $\longrightarrow$ apply small disturbance approx

$$xy \rightarrow (x_0+\Delta x)(y_0+\Delta y) \rightarrow x_0 y_0 + y_0\Delta x + x_0 \Delta y + \cancelto{0}{\Delta y \Delta x}$$

$$\sin\theta \longrightarrow \sin(\theta_0+\Delta\theta) \longrightarrow \sin(\theta_0)\cancelto{1}{\cos(\Delta\theta)} + \cos(\theta_0)\cancelto{\Delta\theta}{\sin(\Delta\theta)}$$

$$\longrightarrow \sin(\theta_0) + \cos(\theta_0)\Delta\theta$$

W2 Q2

$\vec{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ $\qquad$ $\dot{\vec{V}}_B^E$ $\qquad$ $\vec{W}_E$

Longitudinal Dynamics

Last Time

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e = C_{L_{trim}}$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e = 0$$

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n) \leftarrow$$

Today: Linear Longitudinal EOM

Stability Derivatives

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

$$\dot{\vec{\Theta}} = T\vec{\omega}_B$$

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

$$\dot{\vec{\omega}}_B = I^{-1}\left[\vec{G}_B - \vec{\omega}_x^{\times} I \vec{\omega}_B\right]$$

Symmetry about x-z axis $\quad I_{xy} = I_{yz} = 0$

$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\begin{pmatrix}\dot x_E\\ \dot y_E\\ \dot z_E\end{pmatrix}=\begin{pmatrix}c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi\\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi\\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta\end{pmatrix}\begin{pmatrix}u^E\\ v^E\\ w^E\end{pmatrix}$$

$$\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix}=\begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\begin{pmatrix}\dot u^E\\ \dot v^E\\ \dot w^E\end{pmatrix}=\begin{pmatrix}rv^E - qw^E\\ pw^E - ru^E\\ qu^E - pv^E\end{pmatrix}+g\begin{pmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{pmatrix}+\frac{1}{m}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

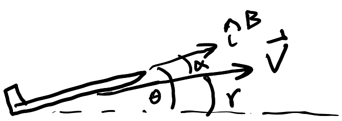
$$\begin{pmatrix}\dot p\\ \dot q\\ \dot r\end{pmatrix}=\begin{pmatrix}\Gamma_1 pq - \Gamma_2 qr\\ \Gamma_5 pr - \Gamma_6(p^2-r^2)\\ \Gamma_7 pq - \Gamma_1 qr\end{pmatrix}+\begin{pmatrix}\Gamma_3 L + \Gamma_4 N\\ \frac{1}{I_y}M\\ \Gamma_4 L + \Gamma_8 N\end{pmatrix}$$

$V = u_0$

Trim State

Inputs u_0, h_0, γ_{a_0} $\xrightarrow{\text{determine}}$ α_0, δ_{e_0}, δ_{t_0}

speed, altitude, air relative flight path angle



$$\vec x = \begin{bmatrix}x_E\\ y_E\\ z_E\\ \phi\\ \theta\\ \psi\\ u^E\\ v^E\\ w^E\\ p\\ q\\ r\end{bmatrix} = \vec x_0 + \begin{bmatrix}\Delta x_E\\ \Delta y_E\\ \Delta z_E\\ \Delta\phi\\ \Delta\theta\\ \Delta\psi\\ \Delta u^E\\ \Delta v^E\\ \Delta w^E\\ \Delta p\\ \Delta q\\ \Delta r\end{bmatrix} \qquad \vec x_0 = \begin{bmatrix}\cdot\\ \cdot\\ -h_0\\ 0\\ \theta_0\\ \cdot\\ u_0^E\\ 0\\ w_0^E\\ 0\\ 0\\ 0\end{bmatrix} \qquad \vec u_0 = \begin{bmatrix}\delta_{e_0}\\ 0\\ 0\\ \delta_{t_0}\end{bmatrix}$$

$$\cancelto{0}{\dot\theta_0} + \Delta\dot\theta = \cancelto{0}{\cos\theta_0 q_0} + \cancelto{0}{\frac{\partial\cos\phi q}{\partial\phi}}\Big|_0\Delta\phi + \cancelto{1}{\frac{\partial\cos\phi q}{q}}\Big|_0\Delta q - \cancelto{0}{\sin\theta_0 r_0} - \cancelto{0}{\frac{\partial\sin\phi r}{\partial\phi}}\Big|_0\Delta\phi - \cancelto{0}{\frac{\partial\sin\phi r}{\partial r}}\Big|_0\Delta r$$

$$\Delta\dot\theta = \Delta q$$

$$\cancelto{0}{\dot u_0} + \Delta\dot u = \cancelto{0}{r_0 v_0} + \cancelto{0}{r_0\Delta v} + \cancelto{0}{v_0\Delta r} + \cancelto{0}{\Delta r\Delta v} - \cancelto{0}{q_0 w_0} - \cancelto{0}{q_0\Delta w} - \cancelto{0}{w_0\Delta q} - \cancelto{0}{\Delta q\Delta w}$$

$$\underbrace{-g\sin\theta_0}_{\text{cancel}} - g\cos\theta_0\Delta\theta + \underbrace{\frac{1}{m}(X_0}_{\text{cancel}} + \Delta X)$$

$$\boxed{\Delta\dot u = -g\cos\theta_0\Delta\theta + \frac{1}{m}\Delta X}$$

Longitudinal

Lat-d

$$X_u \equiv \frac{\partial X}{\partial u}$$

$$\begin{aligned}
\Delta\dot\phi &= \Delta p + \Delta r\tan\theta_0\\
\Delta\dot\theta &= \Delta q\\
\Delta\dot u &= -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}\\
\Delta\dot v &= -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}\\
\Delta\dot w &= u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}\\
\Delta\dot p &= \Gamma_3\Delta L + \Gamma_4\Delta N\\
\Delta\dot q &= \frac{\Delta M}{I_y}\\
\Delta\dot r &= \Gamma_4\Delta L + \Gamma_8\Delta N
\end{aligned}$$

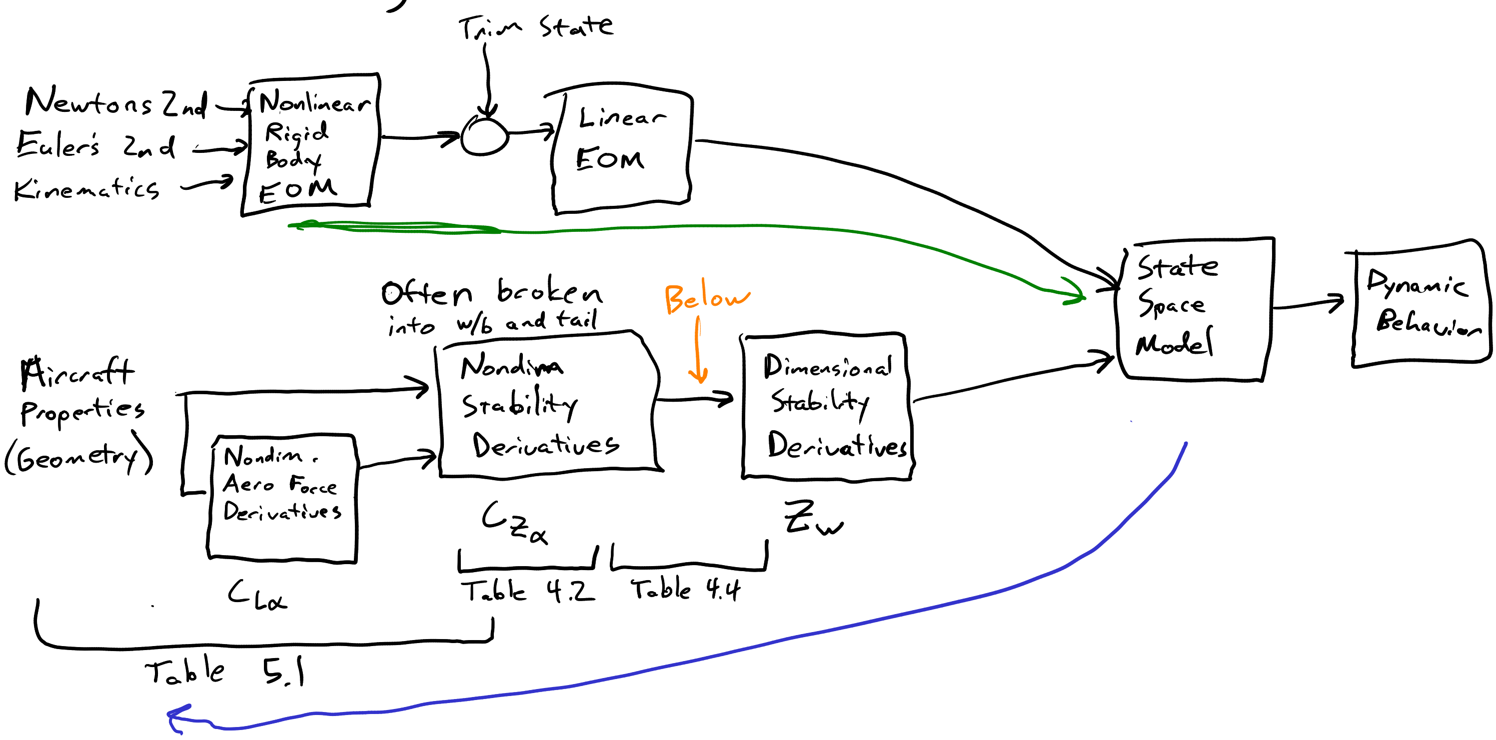
$$\begin{aligned}
\Delta X &= X_u\Delta u + X_w\Delta w + \Delta X_c\\
\Delta Y &= Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c\\
\Delta Z &= Z_u\Delta u + Z_w\Delta w + Z_{\dot w}\Delta\dot w + Z_q\Delta q + \Delta Z_c\\
\Delta L &= L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c\\
\Delta M &= M_u\Delta u + M_w\Delta w + M_{\dot w}\Delta\dot w + M_q\Delta q + \Delta M_c\\
\Delta N &= N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c
\end{aligned}$$

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Big Picture



Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 S\bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \underline{\hat{w}}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

$$C_{z_u} = \frac{\partial C_z}{\partial \hat{u}}$$

Wind-Angle Approximations

$$\Delta\alpha = \tan^{-1}\frac{\Delta w}{V} \approx \hat{w}$$

$$\Delta\beta = \sin^{-1}\frac{\Delta v}{V} \approx \hat{v}$$

Table 4.4
Longitudinal Dimensional Derivatives

Z_u

	X	Z	M
u	$\rho u_0 S C_{w_0} \sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$	$\boxed{-\rho u_0 S C_{w_0} \cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_u}$
w	$\frac{1}{2}\rho u_0 S C_{x_\alpha}$	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$
$\dot{w}$	$\frac{1}{4}\rho \bar{c} S C_{x_{\dot\alpha}}$	$\frac{1}{4}\rho \bar{c} S C_{z_{\dot\alpha}}$	$\frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot\alpha}}$

$Z_u \equiv \frac{\partial Z}{\partial u}\Big|_0 \qquad Z = \frac{1}{2}\rho V^2 S C_z$

$\frac{\partial Z}{\partial u}\Big|_0 = \frac{1}{2}\rho S\left(\frac{\partial \overset{u}{V}{}^2}{\partial u}\Big|_0 C_z + \frac{\partial C_z}{\partial u}\Big|_0 V^2\right)$

$= \frac{1}{2}\rho S\, 2u_0 C_{z_0} + \frac{1}{2}\rho u_0^2 S \frac{\partial C_z}{\partial u}\Big|_0$

$\boxed{Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}}$

$\frac{\partial f(x)g(x)}{\partial x} = f(x)\frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x}g(x)$

$C_{z_u} \equiv \frac{\partial C_z}{\partial \hat{u}}$

C_{z_u}

$\frac{\partial C_z}{\partial u} = \frac{\partial C_z}{\partial \hat{u} u_0} = \frac{1}{u_0}\frac{\partial C_z}{\partial \hat{u}}$

$C_{z_0} = -C_{w_0}\cos\theta_0$

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{\bar{c}}\frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{\bar{c}}$

Neg. means usually negligible.

*means contribution of the *tail only,* formula for wing-body not available.

†$C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0};\ C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

Nondimensional Longitudinal Stability Derivatives

Table 5.1
Summary—Longitudinal Derivatives

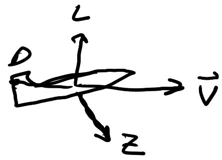
	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{\bar{c}} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{\bar{c}}$

Neg. means usually negligible.
*means contribution of the *tail only,* formula for wing-body not available.
$\dagger C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0};\ C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

α derivatives

$$C_{m_\alpha} = C_{L_\alpha}(h - h_N)$$

C_{z_α}



$$Z = -L\cos\alpha - D\sin\alpha$$

$$C_z = -(C_L\cos\alpha + C_D\sin\alpha)$$

$$= -(C_L + C_D\alpha)$$

$$C_{z_\alpha} \equiv \left.\frac{\partial C_z}{\partial \alpha}\right|_0 = -\left(C_{L_\alpha} + C_{D_0} + \cancel{\alpha_0 \frac{\partial C_D}{\partial \alpha}}^{\,0}\right)$$

$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

u derivatives

3 important factors:

- Compressibility : Mach Number
- Dynamic Pressure : $p_d = \frac{1}{2}\rho V^2$
- Thrust

$M \equiv \frac{V}{a}$ speed of sound

- Different from the dynamic pressure in nondimensionalization

Changes in C_L, C_D, etc. due to changes in dynamic pressure

$$C_{x_u} \equiv \left.\frac{\partial C_x}{\partial \hat{u}}\right|_0$$

$* \in \{x, z, m\}$

$$C_{*_u} = \left.\frac{\partial C_*}{\partial M}\right|_0 \underline{\left.\frac{\partial M}{\partial \hat{u}}\right|_0} + \left.\frac{\partial C_*}{\partial p_d}\right|_0 \underline{\left.\frac{\partial p_d}{\partial \hat{u}}\right|_0} + \left.\frac{C_*}{\partial C_T}\right|_0 \underline{\left.\frac{\partial C_T}{\partial \hat{u}}\right|_0}$$

$$\rightarrow \left.\frac{\partial M}{\partial \hat{u}}\right|_0 = u_0 \left.\frac{\partial M}{\partial u}\right|_0 = \frac{u_0}{a}\left.\frac{\partial V}{\partial u}\right|_0 = M_0$$

$$\frac{\partial p_d}{\partial \hat{u}}\bigg|_0 = u_0 \frac{\partial p_d}{\partial u}\bigg|_0 = u_0 \frac{1}{2}\rho \frac{\partial V^2}{\partial u}\bigg|_0 = u_0 \rho u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2}\rho V^2 S}$$

$$\frac{\partial C_T}{\partial \hat{u}}\bigg|_0 = u_0 \frac{\partial C_T}{\partial u}\bigg|_0 = u_0\left(\frac{\partial T/\partial u}{\frac{1}{2}\rho V^2 S} - \frac{2T}{\frac{1}{2}\rho V^3 S}\right)\bigg|_0 = \frac{\partial T/\partial u}{\frac{1}{2}\rho u_0 S}\bigg|_0 - 2C_{T_0}$$

$$\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$C_{T_0} = C_{D_0} + C_{W_0}\sin\theta_0$$

3 cases

Gliding case: $\boxed{C_{T_u} = 0}$

Constant Thrust (Jet): $\boxed{C_{T_u} = -2C_{T_0}}$

Constant Power (Prop): $\boxed{C_{T_u} = -3C_{T_0}}$

$TV = \text{constant}$

$$\frac{\partial T}{\partial u}\bigg|_0 = -\frac{T_0}{u_0}$$

C_{x_u} $\qquad C_x \approx C_T - C_D$

$$\frac{\partial C_x}{\partial M}\bigg|_0 = \frac{\partial C_T}{\partial M}\bigg|_0 - \frac{\partial C_D}{\partial M}\bigg|_0$$

$$\frac{\partial C_x}{\partial p_d}\bigg|_0 = \cancel{\frac{\partial C_T}{\partial p_d}\bigg|_0}^{\text{Neg}} - \frac{\partial C_D}{\partial p_d}\bigg|_0$$

$$\frac{\partial C_x}{\partial C_T}\bigg|_0 = 1 - \frac{\partial C_D}{\partial C_T}\bigg|_0$$

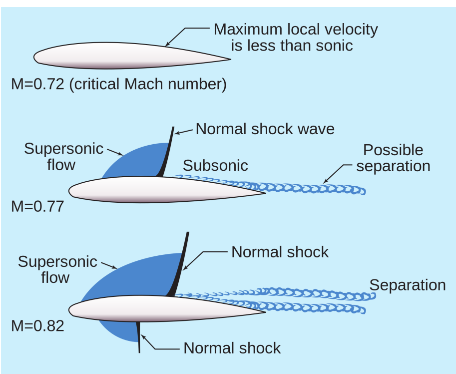
$$\boxed{C_{x_u} = M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right)\bigg|_0 - \rho u_0^2 \frac{\partial C_D}{\partial p_d}\bigg|_0 + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\bigg|_0\right)}$$

C_{z_u} $\qquad$ Assume $C_z = -C_L$

$$\boxed{C_{z_u} = -\underbrace{M_0 \frac{\partial C_L}{\partial M}\bigg|_0}_{\text{small except transonic}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d}\bigg|_0 - C_{T_u}\frac{\partial C_L}{\partial C_T}\bigg|_0}$$

C_{m_u}

$$\boxed{C_{m_u} = \underbrace{M_0 \frac{\partial C_m}{\partial M}\bigg|_0}_{\text{Mach Tuck}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d}\bigg|_0 + C_{T_u}\frac{\partial C_m}{\partial C_T}\bigg|_0}$$



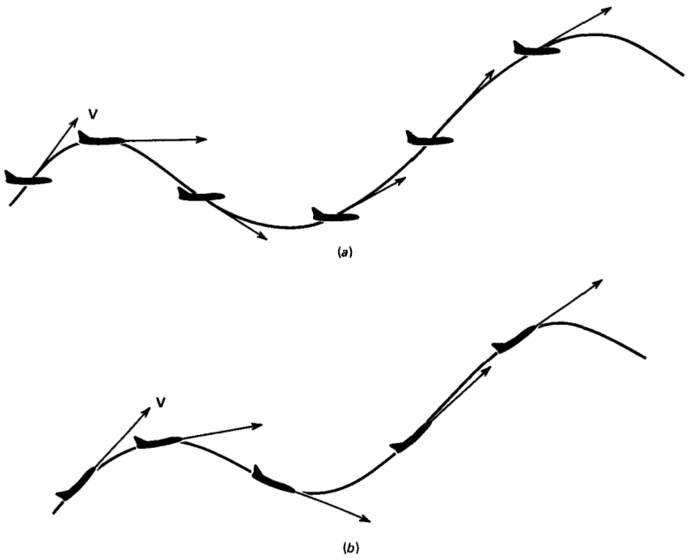


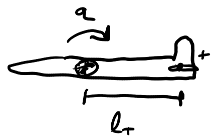
Figure 5.2 (a) Motion with zero q, but varying α_x. (b) Motion with zero α_x but varying q.

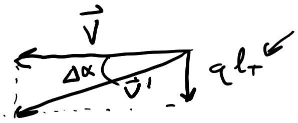
q- derivatives

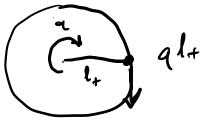
Wing-body || Tail

Tail

velocity observed by tail







$$\Delta C_{L_t} = a_t \Delta\alpha = a_t \tan^{-1}\frac{ql_t}{u_0} \approx a_t \frac{ql_t}{u_0}$$

$$\Delta C_L = \frac{S_t}{S}\Delta C_{L_t} = \frac{S_t}{S} a_t \frac{ql_t}{u_0}$$

$(C_{Z_q})_{tail}$

$$C_{Z_q} = \frac{\partial C_Z}{\partial \hat{q}}\Big|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_Z}{\partial q}\Big|_0 = -\frac{2u_0}{\bar{c}}\frac{\partial C_L}{\partial q}\Big|_0$$

$$(C_{Z_q})_{tail} = \frac{-2u_0}{\bar{c}} a_t \frac{S_t l_t}{S u_0} = \boxed{-2a_t V_H}$$

$$V_H = \frac{S_t l_t}{S\bar{c}}$$

$(C_{mq})_{tail}$

$$\Delta C_m = -V_H \Delta C_{L_T} = a_t V_H \frac{ql_t}{u_0}$$

$$C_{mq} = \frac{\partial C_m}{\partial \hat{q}}\Big|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_m}{\partial q}\Big|_0$$

$$\boxed{(C_{m_q})_{tail} = -2a_t V_H \frac{l_t}{\bar{c}}}$$

Wing-Body

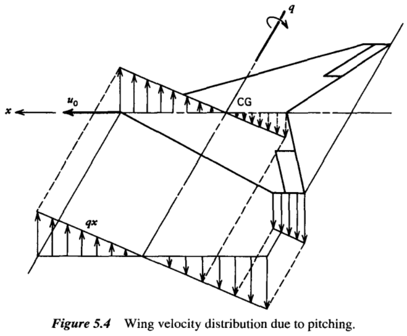


Figure 5.4 Wing velocity distribution due to pitching.

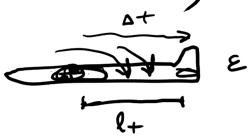
$\dot{\alpha}$ derivatives

Unsteady effects

Wing-Body $\rightarrow$ Determined by initial response
or oscillation of wing in wind tunnel
or flight test

Tail

Downwash Lag



$$\Delta\varepsilon = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\,\Delta t = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\frac{l_t}{u_0}$$

$$= -\Delta\alpha_t$$

$\left(C_{z_{\dot{\alpha}}}\right)_{tail}$

$$\Delta C_{L_t} = a_t\Delta\alpha_t = a_t\dot{\alpha}\frac{l_t}{u_0}\frac{\partial\varepsilon}{\partial\alpha}$$

$$\rightarrow \Delta C_L = a_t\dot{\alpha}\frac{l_t S_t}{u_0 S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$C_{z_{\dot{\alpha}}} = \left.\frac{\partial C_z}{\partial\frac{\dot{\alpha}\bar{c}}{2u_0}}\right|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_z}{\partial\dot{\alpha}} = -2a_t\frac{l_t S_t}{\bar{c}S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$\boxed{\left(C_{z_{\dot{\alpha}}}\right)_{tail} = -2a_t V_H\frac{\partial\varepsilon}{\partial\alpha}}$$

$$\boxed{\left(C_{m_{\dot{\alpha}}}\right)_{tail} = -2a_t V_H\frac{l_t}{\bar{c}}\frac{\partial\varepsilon}{\partial\alpha}}$$

Longitudinal Modes

$$x(t) = \sum_i k_i \vec{v}_i e^{\lambda_i t}$$

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$h_0 = 40k$ ft

$V_0 = 774$ ft/s

$\gamma_0 = \theta_0 = \alpha_0 = 0$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\omega_n = \sqrt{a^2 + b^2}$

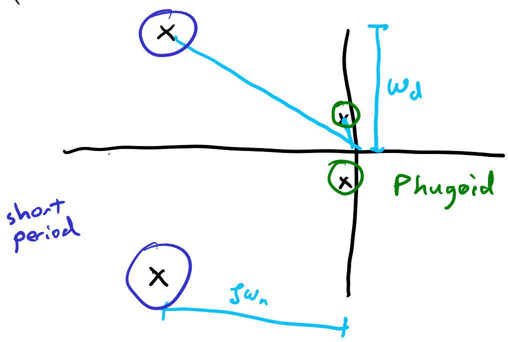
$$\lambda_{1,2} = \overset{a}{-0.37} \overset{\pm}{\pm} \overset{bi}{0.89i}$$

$\omega_n = 0.96, \; \zeta = 0.38$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$\omega_n = 0.067$

$\zeta = 0.049$



Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I)\, v_i = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \qquad \lambda_1 = -1, \; \lambda_2 = -2$$

$$(A - \lambda_1 I)\, v_1 = 0$$

$$\rightarrow \boxed{\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} v_1} = 0$$

let $v_1[1] = 1$

$v_1[2] = -1$

$$\boxed{v_1[1] + v_1[2] = 0}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{V}_{1,2} = \begin{bmatrix} 0.02 \pm 0.06i \\ 0.9996 \\ -0.0001 \pm 0.001i \\ 0.0011 \mp 0.0004i \end{bmatrix} \begin{array}{c} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{array} \qquad \vec{V}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097 \\ -0.0001 \mp 0.0001i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

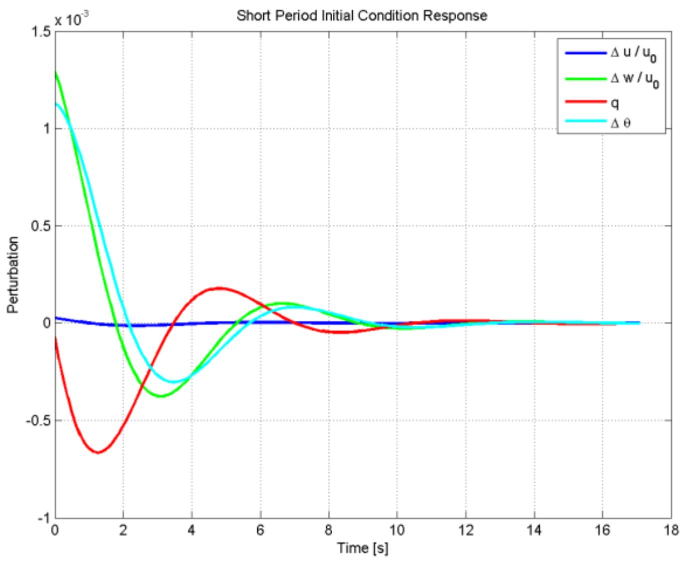
$$x(t) = \sum_i k_i \vec{v}_i e^{\lambda_i t}$$

$$x(0) = Re(\vec{v}_1) = \sum_i k_i \vec{V}_i \cancel{e^{\lambda_i 0}}^{1} = 0.5\vec{v}_1 + 0.5\vec{v}_2$$

$$x(t) = 0.5\vec{v}_1 e^{\lambda_1 t} + 0.5\vec{v}_2 e^{\lambda_2 t}$$

$\lambda_{1/2} = -.372 + .888i$
$\zeta = 0.387$
$\omega_n = 0.962$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$

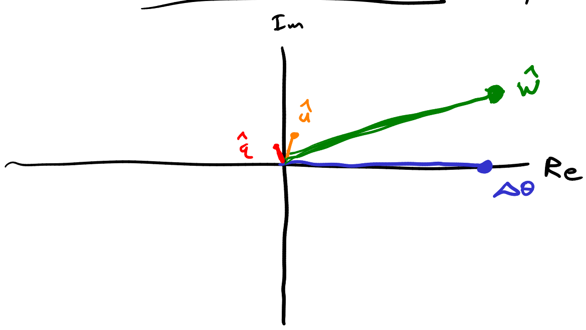


$$\vec{V}_{1,2}/\vec{V}_{1,2}[4] = \vec{V}'_{1,2} = \begin{bmatrix} 0.02 \pm 0.06i / \;\text{''}\; \;\text{''} \\ 0.9996 / \;\text{''}\; \;\text{''} \\ -0.0001 \pm 0.001i / 0.0011 - 0.0011i \\ 0.0011 \mp 0.0004i / 0.0011 - 0.0004i \end{bmatrix} = \begin{bmatrix} \\ \\ \\ 1.0 \end{bmatrix}$$

$$z_1 = r_1 e^{i\phi_1} = r_1 \angle \phi_1$$
$$z_2 = r_2 e^{i\phi_2}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{(i\phi_1 - i\phi_2)}$$

$$\hat{V}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.0066 \pm 0.016i \\ 1.0 \end{bmatrix} \qquad \begin{array}{l} \hat{u} = \frac{\Delta u}{u_0} \\ \hat{w} = \frac{\Delta w}{u_0} \approx \alpha \\ \hat{q} = \frac{\Delta q \bar{c}}{2u_0} \\ \Delta\theta \end{array}$$

Phasor Plot ($\hat{v}_1$, Short Period)



$\vec{v}$ $\begin{cases} \alpha \text{ and } \Delta\theta \text{ change in-phase} \\ \hat{u} \text{ relatively constant} \end{cases}$

λ $\begin{cases} \text{well-damped} \\ \text{high frequency} \end{cases}$

Phugoid Mode

phugoid = "flight"

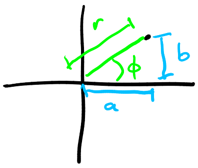
side note:

$z = a + bi$

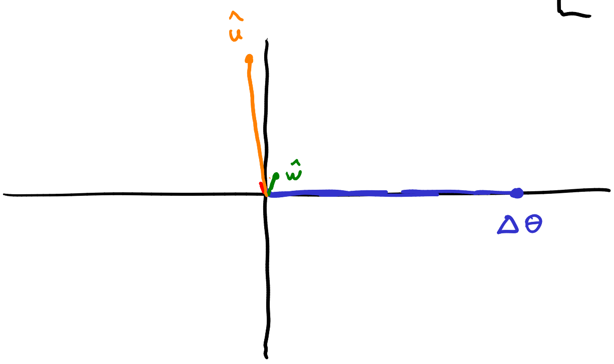
$= r \angle \phi$

$= r e^{i\phi}$

$r = \sqrt{a^2+b^2}$; $\phi = \text{atan2}(b, a)$



$$\hat{v}_{3,4} = \begin{array}{c} \begin{bmatrix} 0.62 \angle 92^\circ \\ 0.036 \angle 83^\circ \\ 0.0012 \angle 93^\circ \\ 1.0 \angle 0 \end{bmatrix} \end{array} \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ \hat{q} \\ \Delta\theta \end{array}$$



$\vec{v}$ $\begin{cases} \text{Large } \hat{u}, \Delta\theta \text{ out-of phase oscillations (offset} \sim 90^\circ) \\ \text{small } \alpha \end{cases}$

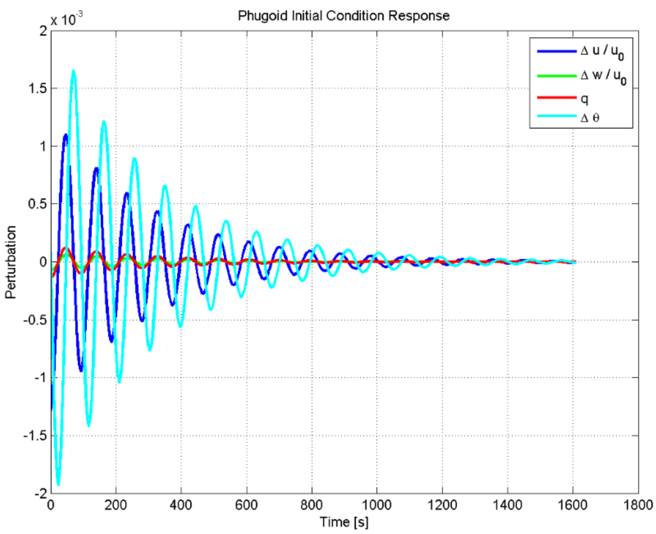
λ $\begin{cases} \text{low frequency} \\ \text{low damping} \end{cases}$

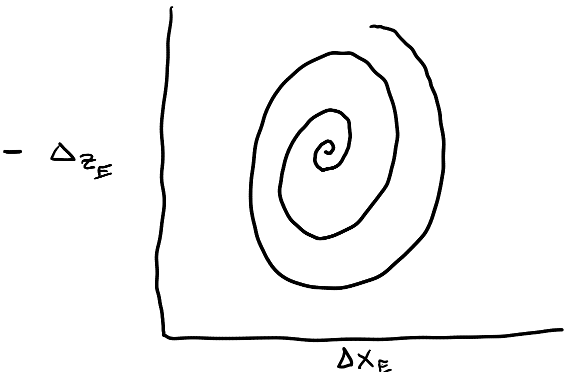
$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$

$\zeta = 0.0489$ ⟵ poorly damped

$\omega_n = 0.0673$ ⟵ slow response

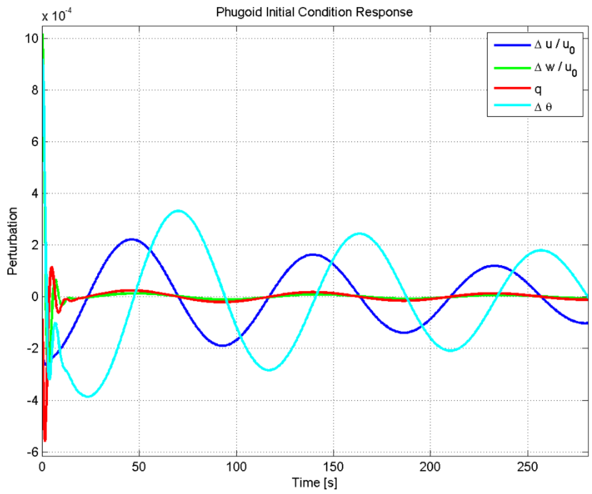
$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$





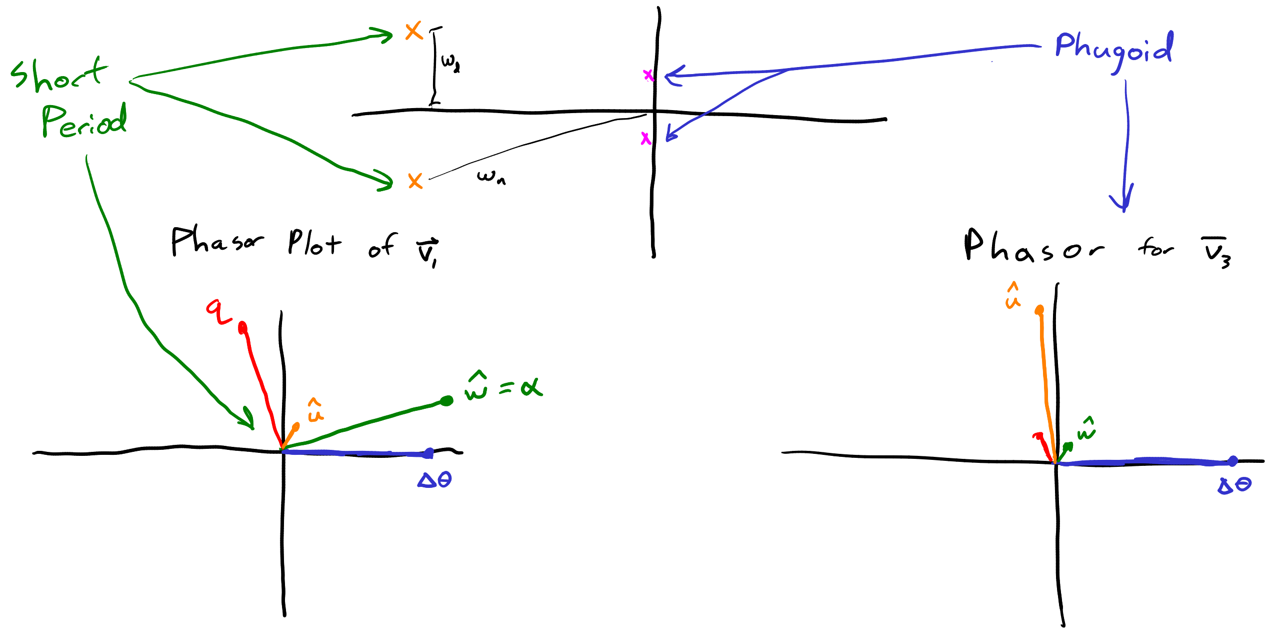
General Response

$$\mathbf{x}_0 = 0.8 \cdot Re(\mathbf{v}_{sp}) + 0.2 \cdot Re(\mathbf{v}_{ph})$$



Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon}}_{\text{Eigen}} \vec{x}_{lon} + \vec{c}_{lon} \qquad \vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$



Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume: $\Delta u = 0$
$\theta_0 = 0$
$Z_{\dot{w}} << m$
$Z_q << mu_0$

If we also assume no vertical motion, $\theta_0 = 0$ implies $\Delta\theta = \alpha \approx \frac{\Delta w}{u_0}$

$$\boxed{\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{Z_w}{m} & u_0 \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m}\right] & \dfrac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right]\right]}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

Dimensional Stab. Deriv.	Nondim. Stab. Deriv.	A/C Params
$Z_w \equiv \left.\frac{\partial Z}{\partial w}\right\|_0 = \frac{1}{2}\rho u_0 S C_{z_\alpha}$	C_{z_α}	$C_{z_\alpha} = -C_{D_0} - C_{L_\alpha}$
M_w	C_{m_α}	$= C_{L_\alpha}(h - h_n)$
$M_{\dot{w}}$	$C_{m_{\dot{\alpha}}}$	...
M_q	C_{m_q}	$-2a_t V_H \frac{l_t}{\bar{c}}$

How accurate is this approximation?

For 747 @ cruise

Full A_{lon}

$\lambda_{1,2} = -0.372 \pm 0.888i$

$\zeta = 0.387$

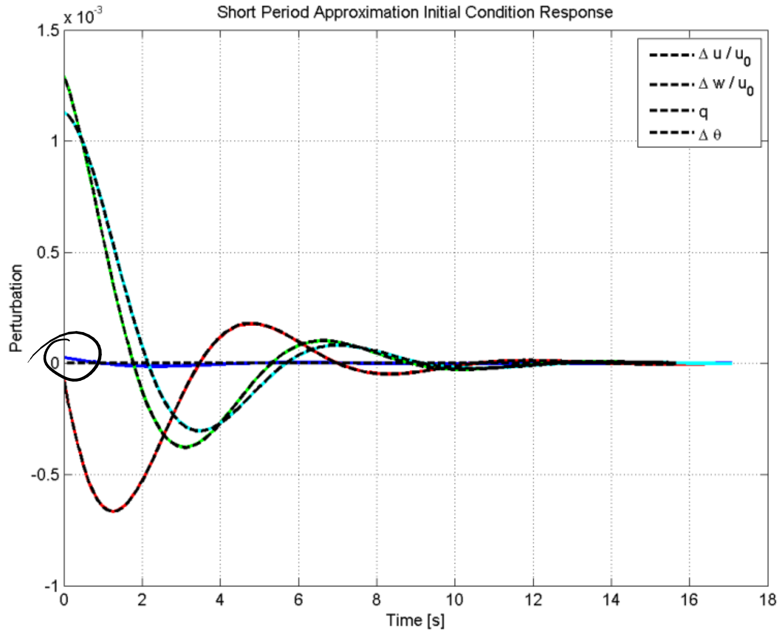
$\omega_n = 0.962$

S.P. approx

$\lambda_{sp} = -0.371 \pm .888i$

$\zeta = 0.385$

$\omega_n = 0.963$



(note: here since $\Delta\theta \neq \frac{\Delta w}{u_0}$, there is some vertical motion)

Phugoid Mode

Lanchester (1908) Assume conservation of energy

$$E = \frac{1}{2}mV^2 - mg\Delta z_E = \frac{1}{2}mu_0^2$$

$$V^2 = 2g\Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho u_0^2 S C_{W_0} + \rho g S C_{W_0}\Delta z_E = W + \rho g S C_{W_0}\Delta z_E$$

Newton's 2nd Law in z

$$W - L = m\Delta\ddot{z}_E$$

$$W - (W + \rho g S C_{W_0}\Delta z_E) = m\Delta\ddot{z}_E$$

$$\Delta\ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m}}_{\omega_n^2}\Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi\sqrt{2}\,\frac{u_0}{g}} \quad = \quad \begin{matrix} 0.138\,u_0 \leftarrow \text{if } u_0 \text{ in ft/s} \\ 0.453\,u_0 \leftarrow \text{if } u_0 \text{ in m/s} \end{matrix}$$

for 747

Full A_{lon}	Lanchester
$T = 93\,s$	$T = 107\,s$

"2×2" Phugoid Approx

Dynamics of Flight, Eq. (4.9,18) $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \cancel{\dfrac{X_w}{m}} & 0 & -g\cancel{\cos\theta_0}^{1} \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \cancel{\dfrac{Z_w}{m - Z_{\dot{w}}}} & \dfrac{Z_q + m u_0}{m - Z_{\dot{w}}} & \dfrac{-mg\cancel{\sin\theta_0}^{0}}{m - Z_{\dot{w}}} \\ \cancel{\dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right]} & \cancel{\dfrac{-M_{\dot{w}} m g \sin\theta_0}{I_y(m - Z_{\dot{w}})}} \\ 0 & \cancel{0} & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Assume $Z_{\dot{w}} \ll m$
$Z_q \ll m u_0$
$\Delta\dot{q}$ small
$\Delta\alpha = 0$
$\theta_0 = 0$

$$\begin{bmatrix} \Delta\dot{u} \\ \cancel{\Delta\dot{\alpha}} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m}\Delta u + u_0 \Delta q \longrightarrow \Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\boxed{\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}}$$

$$\lambda^2 - \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{Z_u g}{m u_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cancel{\cos\theta_0}^{1} + \tfrac{1}{2}\rho u_0 S C_{Z_u}$$

$$C_{Z_u} = -\underbrace{M_0 \frac{\partial C_L}{\partial M}\Big|_0}_{\text{small}} - \underbrace{\rho u_0^2 \frac{\partial C_L}{\partial p_d}\Big|_0}_{\text{small}} - \underbrace{C_{T_u}\frac{\partial C_L}{\partial C_T}\Big|_0}_{\text{small}} \qquad - C_{T_u}\frac{\partial C_X}{\partial C_T}\Big|_0$$

$$C_{Z_u} = 0 \qquad Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{w_0} \cancel{\sin\theta_0}^{\,0} + \frac{1}{2}\rho u_0 S \underline{C_{x_u}}$$

$$C_{x_u} = -2C_{T_0} \qquad \text{(constant thrust)}$$

$$C_{T_0} = C_{D_0} + C_{w_0}\cancel{\sin\theta_0}^{\,0}$$

$$X_u \simeq -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}$$ Same as Lanchester

$$\zeta = -\frac{X_u}{2}\sqrt{\frac{-u_0}{m Z_u g}} = \rho\frac{u_0 S C_{D_0}}{2}\sqrt{\frac{u_0}{\frac{1}{2}mg\rho u_0 S C_{L_0}}}$$

$$= C_{D_0}\sqrt{\frac{\frac{1}{4}\rho u_0^2 S}{\frac{1}{2}mg}\frac{1}{C_{L_0}}}$$

$$\zeta = \frac{C_{D_0}}{C_{L_0}}$$ High L/D = less energy loss = less damping

747

Full A_{lon}

$\lambda_{3,4} = -3.29\times10^{-3} \pm 6.72\times10^{-2} i$

$\zeta = 0.0489$

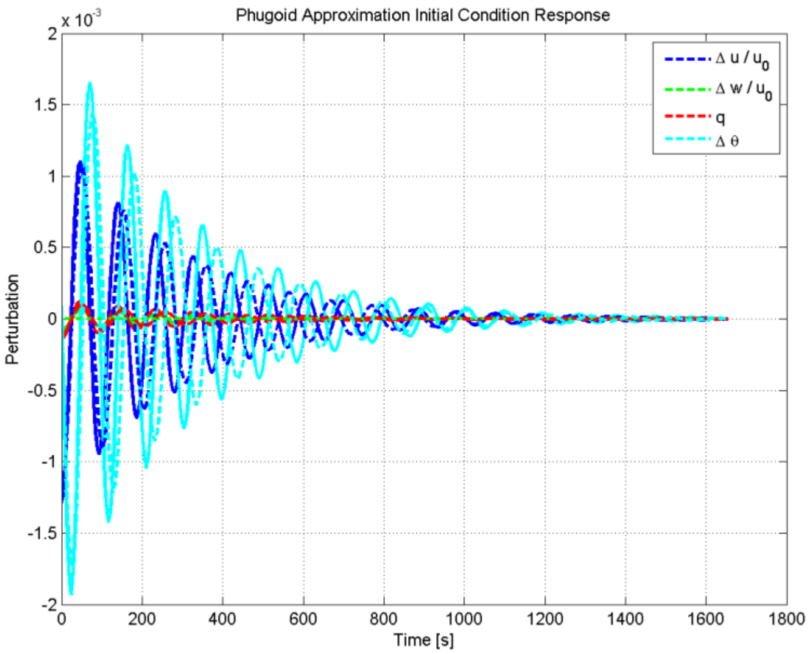
$\omega_n = 0.0673$

Ph Approx

$\lambda_{ph} = -3.43\times10^{-3} \pm 6.11\times10^{-2} i$

$\zeta = 0.0561$

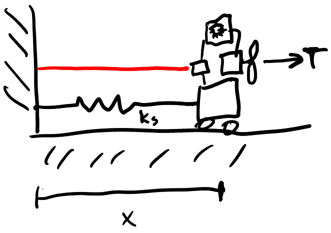
$\omega_n = 0.0612$



Longitudinal Control

State Space Review

$k_s \Leftrightarrow C_{m_\alpha}$



$T_0 = 0$
$T = \Delta T$

$f_s = -k_s(x - x_0)$

$m\ddot{x} = -k_s(x - x_0)$

$m\Delta\ddot{x} = -k_s\Delta x + \Delta T$

$\vec{u} = [\Delta T]$

ΔX_E, ΔY_E, ΔZ_E

$$\rightarrow \quad \dot{\vec{x}} = \begin{bmatrix}\Delta\dot{x}\\ \Delta\ddot{x}\end{bmatrix} = \overset{A}{\begin{bmatrix}0 & 1\\ -\frac{k_s}{m} & 0\end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix}\Delta x\\ \Delta\dot{x}\end{bmatrix}} + \overset{B}{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}} \vec{u}$$

u, v, w

$$\Delta x = \overset{C}{\begin{bmatrix}1 & 0\end{bmatrix}} \begin{bmatrix}\Delta x\\ \Delta\dot{x}\end{bmatrix} + \overset{D}{[0]}\,\vec{u}$$

$$|A - \lambda I| = 0 = \begin{vmatrix}-\lambda & 1\\ -\frac{k_s}{m} & -\lambda\end{vmatrix} = \lambda^2 + \frac{k_s}{m} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \sqrt{\frac{k_s}{m}}\, i$$

$$\lambda^2 + \underbrace{2\zeta\omega_n}_{0}\lambda + \underbrace{\omega_n^2}_{\frac{k_s}{m}} = 0$$

$$\Delta T = -k_p \Delta x - k_d \Delta\dot{x}$$

$\vec{u} = -K\vec{x} \qquad K = \begin{bmatrix}k_p & k_d\end{bmatrix} \qquad \vec{u} = \begin{bmatrix}-k_p & -k_d\end{bmatrix}\vec{x}$

$$\dot{\vec{x}} = A\vec{x} - BK\vec{x}$$

$$\dot{\vec{x}} = \underbrace{(A - BK)}_{A^{cl}}\vec{x}$$

$$A^{cl} = \begin{bmatrix}0 & 1\\ -\frac{(k_s + k_p)}{m} & -\frac{k_d}{m}\end{bmatrix}$$

Longitudinal Control

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix}\dfrac{\Delta X_c}{m}\\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}}\\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})}\\ 0\end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix}\dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0\\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}}\\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})}\\ 0 & 0 & 1 & 0\end{pmatrix}$$

δ_e = elevator   + downward deflection

δ_t = throttle   + more power

↑ δ_p in book

dimensional control derivatives

$$\Delta X_c = X_{\delta_e}\delta_e + X_{\delta_t}\delta_t$$

$$\Delta Z_c = Z_{\delta_e}\delta_e + \boxed{Z_{\delta_t}}\delta_t$$ often zero

$$\Delta M_c = M_{\delta_e}\delta_e + \boxed{M_{\delta_t}}\delta_t$$

$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + B_{lon}\vec{u}_{lon}$$

$$B_{lon} = \begin{bmatrix} \frac{X_{\delta_e}}{m} & \frac{X_{\delta_t}}{m} \\ \frac{Z_{\delta_e}}{m - Z_{\dot w}} & \frac{Z_{\delta_t}}{m - Z_{\dot w}} \\ \frac{M_{\delta_e}}{I_y} + \frac{M_{\dot w}}{I_y}\frac{Z_{\delta_e}}{(m-Z_{\dot w})} & \frac{M_{\delta_t}}{I_y} + \frac{M_{\dot w}Z_{\delta_e}}{I_y(m-Z_{\dot w})} \\ 0 & 0 \end{bmatrix} \qquad \vec{u}_{lon} = \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

747

$$B_{lon} = \begin{bmatrix} -0.000187 & 9.66 \\ -17.85 & 0 \\ -1.158 & 0 \\ 0 & 0 \end{bmatrix}$$

γ

$$\gamma = \underbrace{\begin{bmatrix} 0 & \frac{-1}{774} & 0 & 1 \end{bmatrix}}_{C_\gamma} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{D_\gamma} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

$\gamma = \Delta\theta \ \Delta\alpha$

$\longrightarrow C_\gamma$

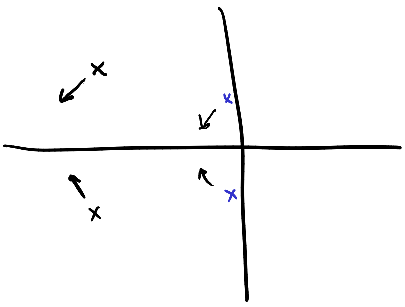
$$\delta_e = 10^\circ$$

$$\delta_t = \tfrac{1}{6} \approx 0.05\,w$$

Longitudinal Stability Augmentation

B 747

$$A_{sp} = \begin{bmatrix} -0.3151 & 773.98 \\ -0.0010 & -0.4285 \end{bmatrix}$$



$$|A - \lambda I| = (-0.3151 - \lambda)(-0.4285 - \lambda) - (-0.0010)(773.98)$$

$$\lambda = -0.372 \pm 0.889i$$

$$\omega_n = 0.964$$

$$\zeta = 0.386 \quad \times \quad 0.7$$

$$B_{sp} = \begin{bmatrix} -17.85 & 0 \\ -1.158 & 0 \end{bmatrix}$$

measured by rate gyro

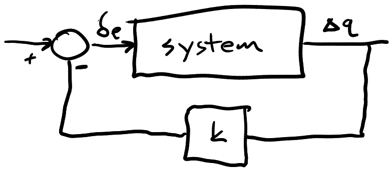
$$\delta_e = -k_s \Delta q$$

$$\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} = -\underbrace{\begin{bmatrix} 0 & k_s \\ 0 & 0 \end{bmatrix}}_{K_{sp}} \underbrace{\begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}}_{\vec{x}_{sp}}$$

$$A^{cl} = A_{sp} - B_{sp} K_{sp} = A_{sp} - \begin{bmatrix} 0 & -17.85 k_s \\ 0 & -1.158 k_s \end{bmatrix}$$

$$= \begin{bmatrix} -0.3151 & 773.98 + 17.85 k_s \\ -0.0010 & -0.4285 + 1.158 k_s \end{bmatrix}$$

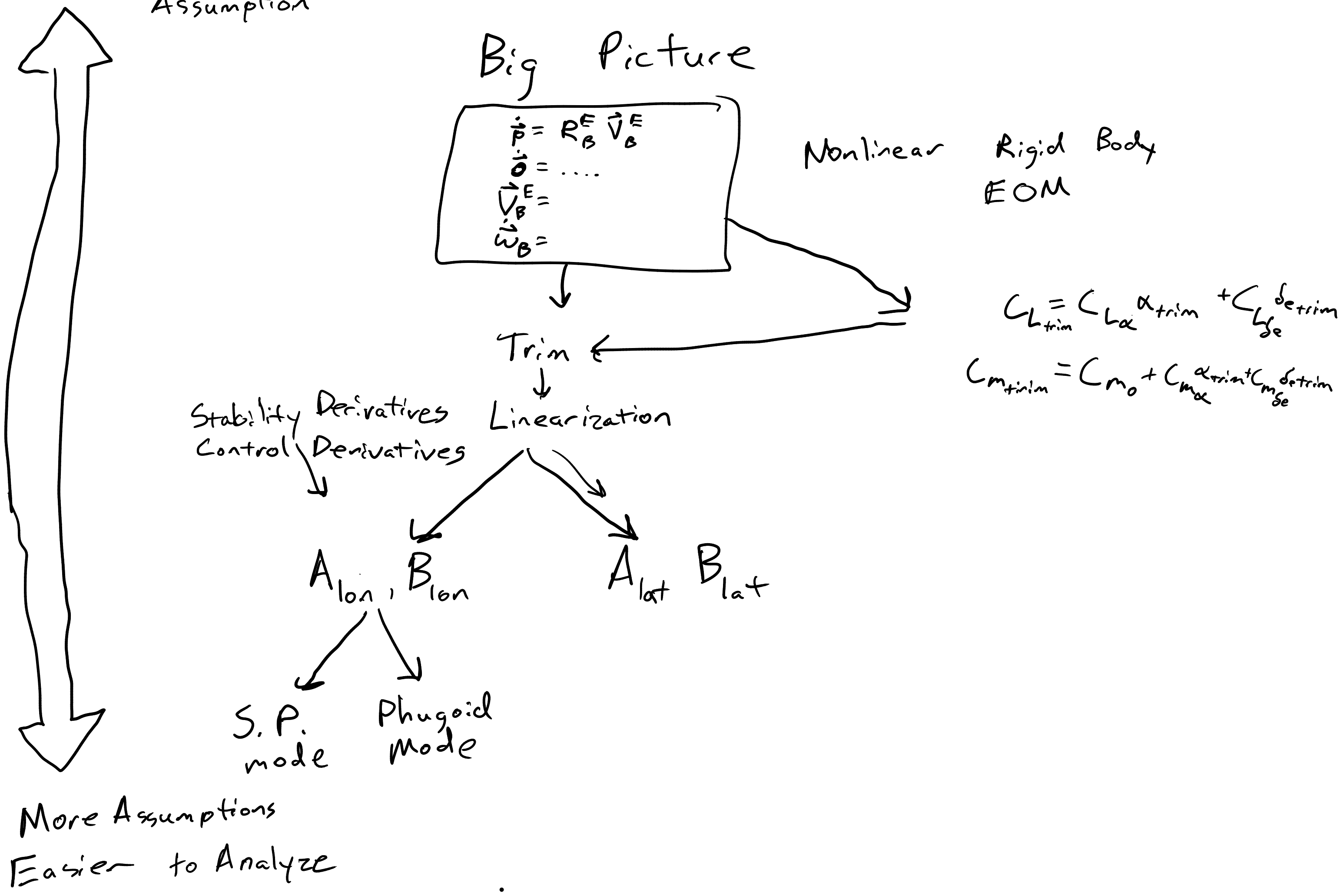
Matlab rlocus



$C_{\Delta q} = [0 \ 1]$ $D_{\Delta q} = [0]$

$B_{\delta e} = \begin{bmatrix} -17.85 \\ -1.158 \end{bmatrix}$

Lateral / Directional Dynamics



$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{u} = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3\Delta L + \Gamma_4\Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4\Delta L + \Gamma_8\Delta N$$

$$Y = \frac{1}{2}\rho V^2 S C_Y(\beta, p, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2}\rho V^2 S b C_l(\beta, p, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2}\rho V^2 S b C_n(\beta, p, r, \delta_a, \delta_r)$$

$$\hat{p} = \frac{pb}{2u_0}$$

$$Y \approx \frac{1}{2}\rho V_a^2 S\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$L \approx \frac{1}{2}\rho V_a^2 Sb\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right]$$

$$N \approx \frac{1}{2}\rho V_a^2 Sb\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right]$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3\Delta L_c + \Gamma_4\Delta N_c \\ \Gamma_4\Delta L_c + \Gamma_8\Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

Sideslip: β

$+\beta \Rightarrow$ wind coming from right

$$\beta = \sin^{-1}\frac{\Delta V}{V}$$

$$\beta \approx \frac{\Delta V}{u_0} = \hat{v}$$

$$L = \frac{1}{2}\rho V^2 S b C_\ell$$

$$C_{\ell_\beta} \equiv \frac{\partial C_\ell}{\partial \beta} = \frac{\partial C_\ell}{\partial \hat{v}}$$

$$L_v \equiv \frac{\partial L}{\partial v}\bigg|_0 = \frac{1}{2}\rho u_0^2 S b \frac{\partial C_\ell}{\partial v}\bigg|_0$$

$$= \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial \beta}\bigg|_0 \frac{\partial \beta}{\partial v}\bigg|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0}$$

$$\frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

$$\boxed{L_v = \frac{1}{2}\rho u_0 b S C_{\ell_\beta}}$$

$$C_{\ell_p} \equiv \frac{\partial C_\ell}{\partial \hat{p}}$$

$$L_p \equiv \frac{\partial L}{\partial p}\bigg|_0 = \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial p}\bigg|_0$$

$$\hat{p} = \frac{bp}{2u_0}$$

$$= \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial \hat{p}}\bigg|_0 \frac{\partial \hat{p}}{\partial p}\bigg|_0$$

$$\boxed{L_p = \frac{1}{4}\rho u_0 b^2 S C_{\ell_p}}$$

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$

Table 5.2
Summary—Lateral Derivatives

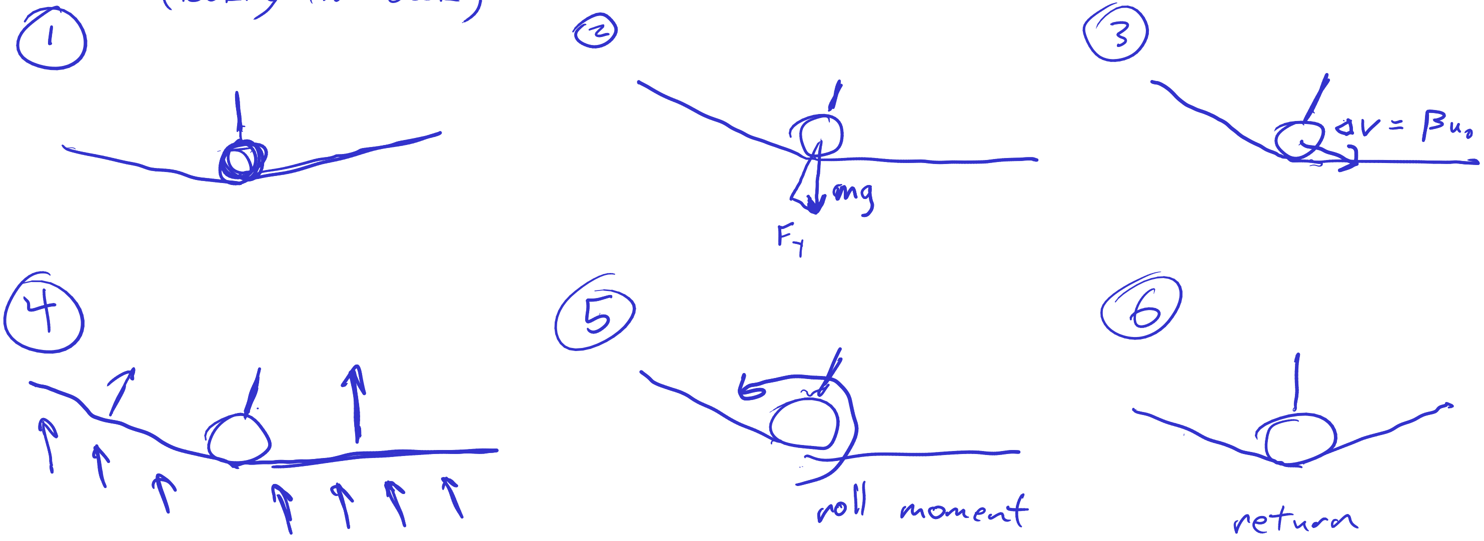
	C_y	C_l	C_n
β	$*-a_F \frac{S_F}{S}\left(1 - \frac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V \left(1 - \frac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F \frac{S_F}{S}\left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V \left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F \frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F \frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V \left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

C_{l_β} : Dihedral Effect

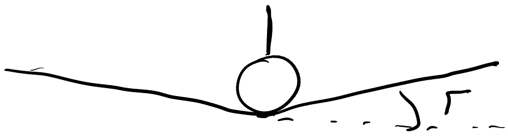
sign : negative

(looking from back)

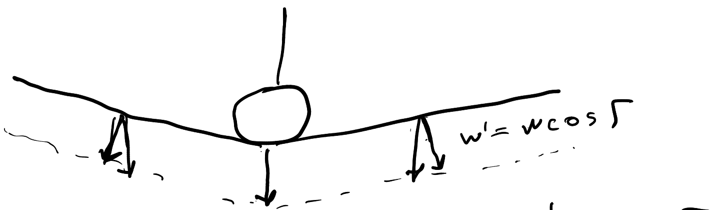


4 significant factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

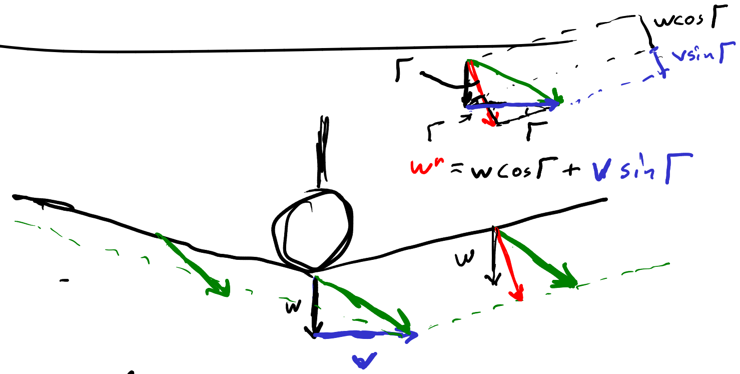


1. Dihedral Angle



$w = V\alpha$

$\alpha' = \frac{w'}{V} \alpha \cos\Gamma$



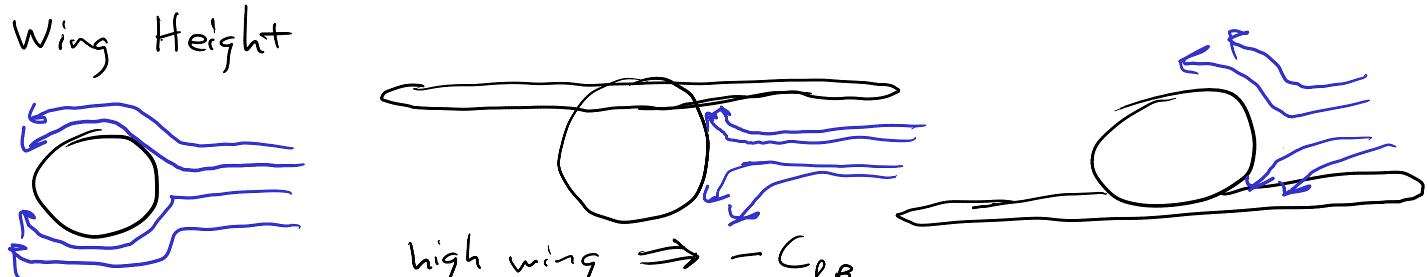
$\alpha^l = \alpha - \Gamma\beta$

$w^r = w + v\Gamma$

$\alpha^r = \alpha + \Gamma\beta$

$$\boxed{C^{\Gamma}_{l_\beta} \propto \Gamma}$$

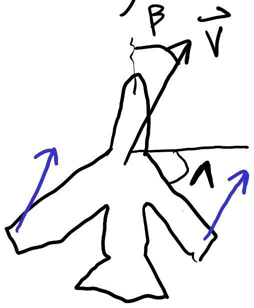
2. Wing Height



high wing $\Rightarrow -C_{l_\beta}$

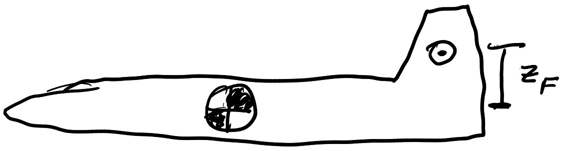
low wing $\Rightarrow +C_{l_\beta}$

3. Wing Sweep



$$C_{l_\beta}^{\Lambda} \propto 2 C_L V^2 \sin 2\Lambda$$

4. Tail



$$\Delta C_l^F = C_{L_F}\frac{S_F z_F}{Sb} = a_F(-\beta+\sigma)\frac{S_F z_F}{Sb}$$

$$C_{l_\beta}^F = -a_F\left(1-\frac{\partial\sigma}{\partial\beta}\right)\frac{S_F z_F}{Sb}\left(\frac{V_F}{V}\right)^2$$

Lateral Stability Derivatives
Sideslip, Coordinated Turn

Table 5.2
Summary—Lateral Derivatives

	C_y	C_l	C_n
β	$*-a_F\dfrac{S_F}{S}\left(1-\dfrac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V\left(1-\dfrac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F\dfrac{S_F}{S}\left(2\dfrac{z_F}{b}-\dfrac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V\left(2\dfrac{z_F}{b}-\dfrac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F\dfrac{S_F}{S}\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F\dfrac{S_F}{S}\dfrac{z_F}{b}\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

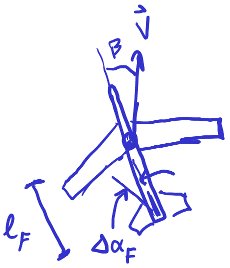
β derivatives

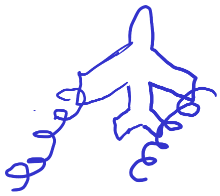
C_{l_β}: Dihedral Effect

1) Wing Height
2) Dihedral Angle
3) Vertical Tail
4) Wing Sweep

C_{n_β}: Weathervane Derivative Sign? (+)

Main Component: Vertical Tail





$$\alpha_F = -\beta + \sigma \quad \leftarrow \text{sidewash}$$

$$\rightarrow C_{L_F} = a_F(-\beta+\sigma)$$

$$N_F = -C_{L_F}\tfrac{1}{2}\rho V_F^2 S_F l_F$$

$$C_{n_F} = -C_{L_F}\frac{S_F}{S}\frac{l_F}{b}\left(\frac{V_F}{V}\right)^2$$

$$= -V_V C_{L_F}\left(\frac{V_F}{V}\right)^2$$

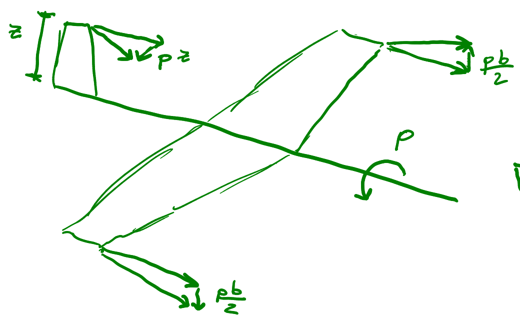
$$V_V \equiv \frac{S_F\, l_F}{S\, b}$$

$$C_{n_\beta} \approx \frac{\partial C_{n_F}}{\partial\beta}\bigg|_0 = -V_V\left(\frac{V_F}{V}\right)^2\frac{\partial C_{L_F}}{\partial\beta}\bigg|_0 = \boxed{V_V a_F\left(\frac{V_F}{V}\right)^2\left(1-\frac{\partial\sigma}{\partial\beta}\right)}$$

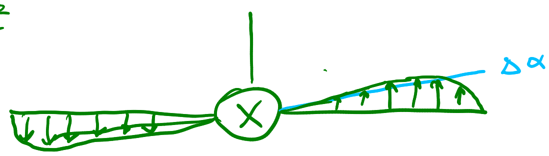
C_{y_β} (usually small) (similar derivation to C_{n_β})

p derivatives

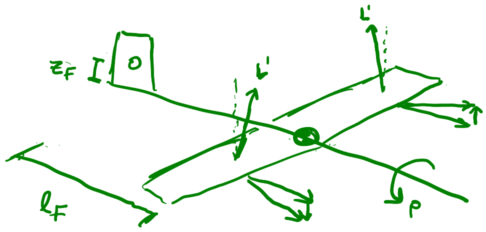
C_{l_p} roll damping derivative (−)



(From rear of aircraft)



C_{n_p} Wing Effects



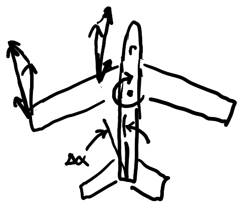
Tail Effect $\quad \Delta\alpha_F = -\frac{p z_F}{u_0} + p\frac{\partial\sigma}{\partial p} = -\hat{p}\left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$

$$(\Delta C_n)_{tail} = -\Delta C_{y_F}\frac{S_F}{S}\frac{l_F}{b} = a_F V_V \hat{p}\left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

$$(C_{n_p})_{tail} = a_F V_V\left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

C_{y_p} (usually small) (similar derivation to $(C_{n_p})_{tail}$)

r-derivatives



$$\Delta\alpha_F = \frac{r l_F}{u_0} + r\frac{\partial\sigma}{\partial r}$$

$$= \hat{r}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

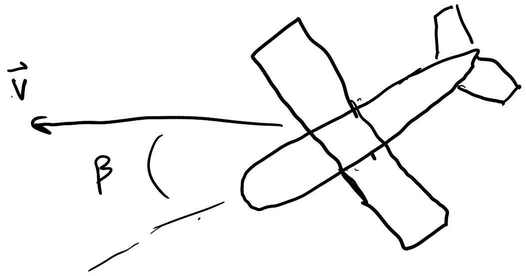
$$(C_{y_r})_{tail} = a_F\frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$(C_{l_r})_{tail} = a_F\frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$(C_{n_r})_{tail} = -a_F V_V\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$ ← yaw damping derivative

2 steady flight conditions (not "level")

Sideslip



$$\Delta Y + mg\,\Delta\phi = 0$$

$$\Delta L = 0$$
$$\Delta N = 0$$

can incorporate 1-engine out

$$\Delta Y = Y_v \Delta v + \cancel{Y_p p}^{\,0} + \cancel{Y_r r}^{\,0} + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r$$

$$\underbrace{\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix}}_{\text{invert}} \begin{bmatrix} \delta_r \\ \delta_a \\ \Delta\phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} \Delta v$$

$\Delta v \leftarrow \beta u_0$

Steady Sideslip

For Piper Cherokee

inv

$$\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix} \cancel{v} \quad (7.8,4)$$

βu_0

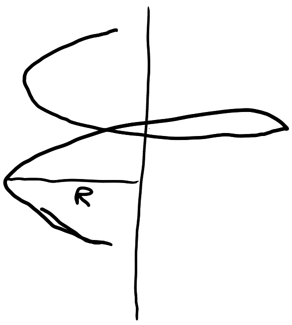
It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta \doteq v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

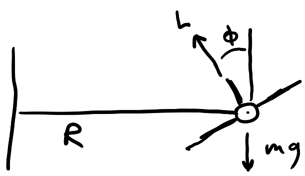
$$\delta_r/\beta = .303$$
$$\delta_a/\beta = -2.96$$
$$\phi/\beta = .104$$

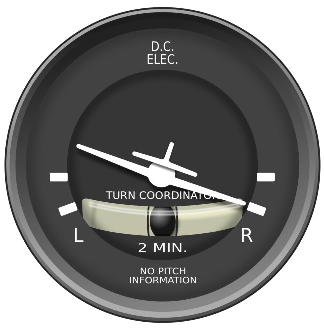
We see that a positive sideslip (to the right) of say 10° would entail $\boxed{\text{left rudder of 3°}}$ and $\boxed{\text{right aileron of 29.6°}}$ Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The $\boxed{\text{bank angle is seen to be only 1°}}$ to the right so the sideslip is almost flat.

Coordinated Turn

- angular velocity vector is constant and aligned with inertial z
- No aerodynamic forces in Y direction







$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\Theta \\ \sin\phi\cos\Theta \\ \cos\phi\cos\Theta \end{bmatrix}\omega$$

Since there is increased need for lift due to bank angle, involves both lat and lon dynamics

$$L\cos\phi = mg\cos\Theta$$
$$L\sin\phi = m\frac{u_0^2}{R}$$
$$= m\omega u_0$$

$$\rightarrow \quad \boxed{\tan\phi} = \frac{L\sin\phi}{L\cos\phi} = \frac{m\omega u_0}{mg\cos\Theta} = \boxed{\frac{\omega u_0}{g\cos\Theta}}$$

Assume: no wind, $\theta = \gamma$, $v \ll u$, $\cos\Theta = 1$, $\sin\Theta = \Theta$, $V = u = u_0$

From EOM:

$$Z = -mg\cos\phi - mqu$$

load factor "g's"

$$n = -\frac{Z}{mg} = \cos\phi + \frac{qu}{g}$$
$$= \cos\phi + \frac{\omega u_0 \sin\phi}{g}$$
$$= \cos\phi + \tan\phi\sin\phi$$
$$\boxed{n = \sec\phi}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\Theta\omega \\ \sin\phi\ \omega \\ \cos\phi\ \omega \end{bmatrix}$$

$$\rightarrow \quad n = \frac{L}{W}$$

$$\Delta C_L = \frac{L - mg}{\frac{1}{2}\rho V^2 S} = (n-1)C_W$$

Coordinated Turn

$$C_Y = 0 = C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_{\delta_r}}\delta_r + C_{Y_{\delta_a}}\delta_a$$
$$C_\ell = 0$$
$$C_n = 0$$
$$C_m = 0$$
$$C_L = (n-1)C_W$$

$$\begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{\ell_\beta} & C_{\ell_{\delta r}} & C_{\ell_{\delta a}} \\ C_{n_\beta} & C_{n_{\delta r}} & C_{n_{\delta a}} \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} C_{Y_p} & C_{Y_r} \\ C_{\ell_p} & C_{\ell_r} \\ C_{n_p} & C_{n_r} \end{bmatrix} \begin{bmatrix} \Theta \\ -\cos\phi \end{bmatrix} \frac{\omega b}{2u_0}$$

$$\begin{bmatrix} C_{m_\alpha} & C_{m_{\delta e}} \\ C_{L_\alpha} & C_{L_{\delta e}} \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta\delta e \end{bmatrix} = -\begin{bmatrix} C_{m_q} \\ C_{L_q} \end{bmatrix} \frac{\omega\bar{c}\sin\phi}{2u_0} + \begin{bmatrix} 0 \\ (n-1)C_W \end{bmatrix}$$

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$x(t) = \sum_i c_i e^{\lambda_i t} \vec{v}_i$$

$\uparrow$ determined by initial condition

λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02+9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02-9.47e-01i$	$3.49e-02$	$9.47e-01$

$$\begin{array}{c} \\ \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{array} \quad \begin{array}{c} \mathbf{v}_1 \\ \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathbf{v}_2 \\ \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathbf{v}_{3/4} \\ \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix} \end{array}$$

$\Delta\psi$

ΔY_E

Flight Path State Space Dynamics Matrix (also exists for lon)

ΔZ_E

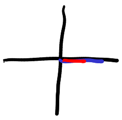
ΔX_E

$$\rightarrow \quad \Delta\dot{\psi} = \Delta r \sec\theta_0$$

$$\Delta\dot{Y}_E = u_0 \cos\theta_0 \underline{\Delta\psi} + \underline{\Delta v}$$

$$\begin{bmatrix} \Delta\dot{v} \\ \vdots \\ \\ \\ \\ \\ \end{bmatrix} = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{array}\right] \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta Y_E \end{bmatrix}$$

Spiral Mode



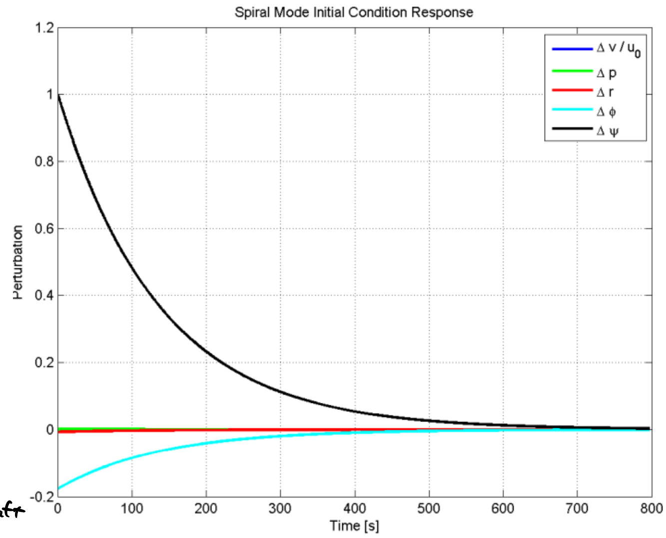
$$\tilde{V}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix} \begin{array}{ll} \hat{v} = \beta & \leftarrow \text{small} \\ p & \leftarrow \text{small} \\ r & \leftarrow \text{some} \\ \phi & \leftarrow \text{large} \\ \psi & \leftarrow \text{large} \end{array}$$

nondimensionalize velocities, normalize so that $\Delta\psi = 1$

$\lambda_1 = -0.0073$ ← stable for 747

$\tau = 137$ s

unstable for many aircraft

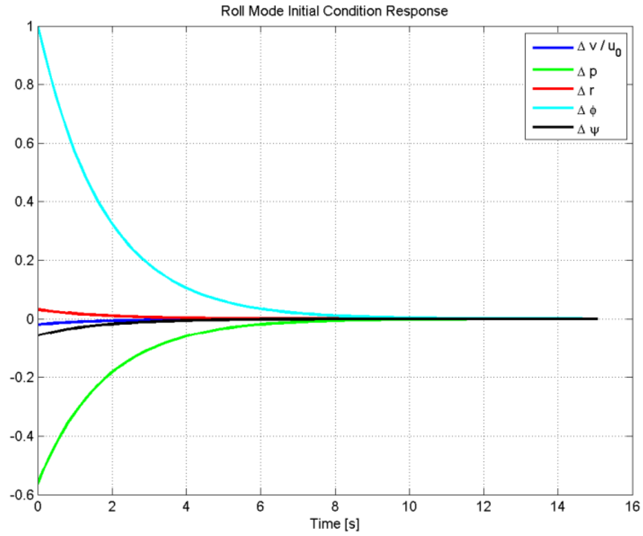


Roll

$$\tilde{V}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \begin{array}{ll} \hat{v} = \beta & \leftarrow \text{small} \\ p & \leftarrow \text{large} \\ r & \leftarrow \text{small} \\ \phi & \leftarrow \text{large} \\ \psi & \leftarrow \text{small} \end{array}$$

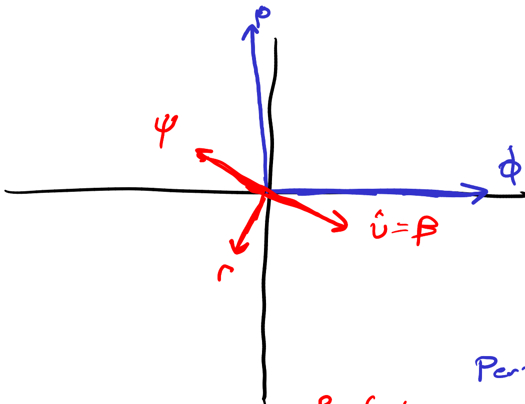
$\lambda_2 = -0.5625$

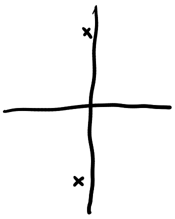
$\tau = 1.78$ s



Dutch Roll

$$\tilde{V}_3 = \begin{bmatrix} 0.3271 \angle -28^\circ \\ 0.9471 \angle 92^\circ \\ 0.2915 \angle -112.3^\circ \\ 1.0 \\ 0.3078 \angle 155^\circ \end{bmatrix} \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$

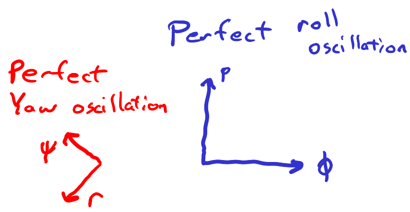




$\lambda_{3/4} = -0.033 \pm 0.947i$

$\zeta = 0.0349$ ← not well-damped

$\omega_n = 0.947$ ← fast

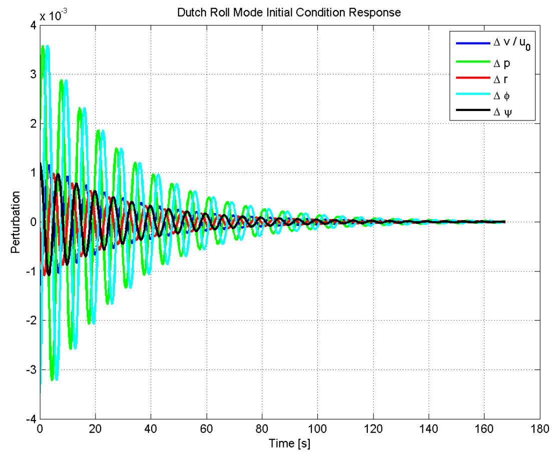


Phugoid (for comparison)

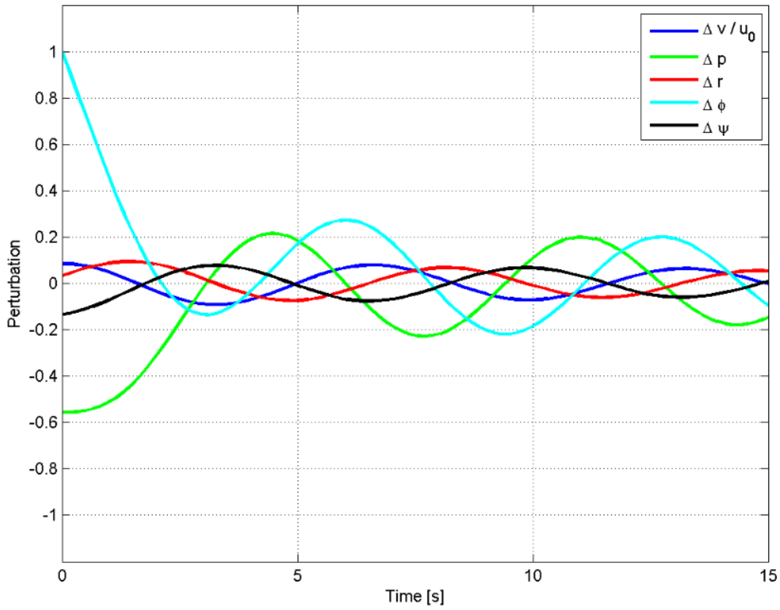
$\lambda = -0.00329 \pm 0.0672i$

$\zeta = 0.0489$

$\omega_n = 0.0673$



$$\mathbf{x}(0) = 0.4 \cdot Re(\mathbf{v}_r) + 0.4 \cdot Re(\mathbf{v}_{dr}) + 0.2 \cdot Re(\mathbf{v}_{spi})$$



Review: All modes

Name	Primary Variables	Fast / Slow	Damping
Short Period	α	Fast	Well-damped
Phugoid	speed/altitude	Slow	Poorly
Roll	Roll rate	Fast	(over)
Spiral	Yaw, Roll	Slow	(over)/unstable
Dutch Roll	β, Roll	Fast	Poorly

Trust the mathematical properties of A over your intuition!

Lateral Mode Approximations and Control Surfaces

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix} = Bu$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

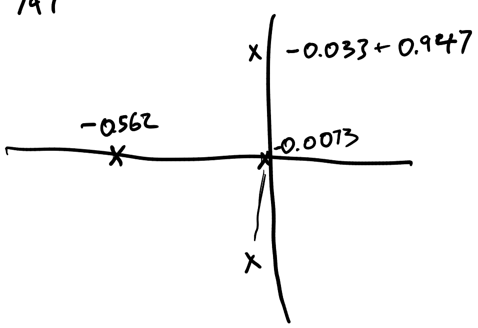
$$\begin{array}{c} \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{array}\right] \end{array} \begin{array}{c} v \\ p \\ r \\ \phi \\ \Delta\Psi \\ \Delta Y_E \end{array}$$

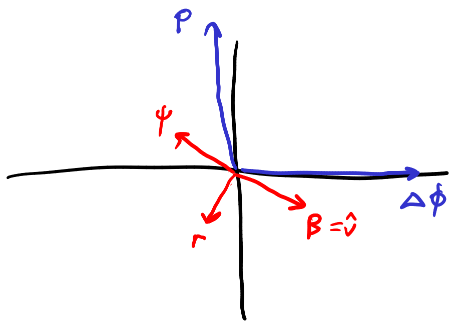
λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02+9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02-9.47e-01i$	$3.49e-02$	$9.47e-01$

$$\mathbf{v}_1 = \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \qquad \mathbf{v}_{3/4} = \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

Spiral   Roll   Dutch Roll

747





$$A_{lat} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\Theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

2x2 Spiral Approx.

$$\tilde{v}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix}$$

$p = 0$

$\dot{p} = 0$

Ignore side force

$$\begin{bmatrix} \dot{v} \\ \dot{p} = 0 \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\Theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = \mathcal{L}_v v + \mathcal{L}_r r \quad \therefore \quad v = -\frac{\mathcal{L}_r}{\mathcal{L}_v} r$$

$$\dot{r} = -\mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} r + \mathcal{N}_r r = \left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v}\right) r$$

$\dot{x} = A x$

$|A - \lambda I| = 0 \qquad \lambda = A \quad$ if A is a scalar

$$\boxed{\lambda_{s,approx} = \left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v}\right)} = -0.0296 \quad \text{for B747}$$

$$\tau = \frac{1}{\lambda} = 33.8\,s$$

-0.0073 $\quad$ Not a great approximation

$\tau = 137\,s$

Characteristic Eqn - Based Spiral Approx

$$|A_{lat} - \lambda I| = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

since $\lambda_s << 1$

$$D\lambda + E = 0$$

$$\boxed{\lambda_{s,approx} = -\frac{E}{D}}$$

$$\rightarrow E = g\left[(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r)\cos\Theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p)\sin\Theta_0\right]$$

$$D = -g(\mathcal{L}_v \cos\Theta_0 + \mathcal{N}_v \sin\Theta_0) + u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

for B747 $\quad \lambda_{s,approx} = -0.00725 \quad$ < 1% error

One necessary condition for stability is $E > 0$

$$(C_{l_\beta} C_{n_r} - C_{l_r} C_{n_\beta}) \cos\theta_0 + (C_{l_p} C_{n_\beta} - C_{l_\beta} C_{n_p}) \sin\theta_0 > 0$$

Roll Approximation

$$\dot{p} = \mathcal{L}_p p$$

$\lambda_{r,approx} = \mathcal{L}_p = -0.434$

$\lambda_r = -0.562$

23% difference

Roll + Spiral Approximation

Assume side force due to gravity produces same yaw rate that would exist with $\beta = 0$

$$0 = -u_0 r + g\phi \leftarrow$$

Also assume $Y_p = Y_r = 0$

$$\begin{bmatrix} 0 \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & -u_0 & g \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\left| \tilde{A} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right| = C\lambda^2 + D\lambda + E = 0$$

$$C = u_0 \mathcal{N}_v$$

$$D = u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v) - g\mathcal{L}_v$$

$$E = g(\mathcal{L}_v \mathcal{N}_r - \mathcal{L}_r \mathcal{N}_v)$$

B747 $\lambda_{s,approx} = -0.00734$ $\lambda_{r_{approx}} = -0.597$

"true" -0.0073 -0.562

Dutch Roll Approx

Assume $\phi = p = 0$ $Y_r = 0$

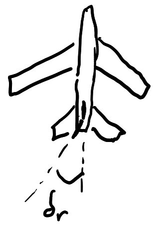
$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

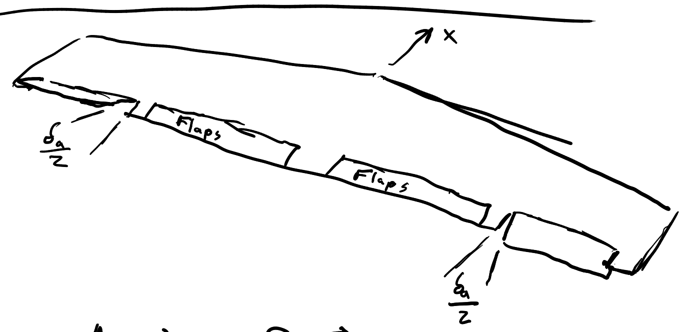
$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\lambda + (\mathcal{Y}_v \mathcal{N}_r + u_0 \mathcal{N}_v) = 0$$

$$\lambda_{dr,approx} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Lateral Control Surfaces





$$\dot{\vec{x}}_{lat} = A_{lat}\vec{x}_{lat} + \underbrace{B_{lat}\vec{u}_{lat}}_{\vec{z}_{lat}} \qquad \vec{u}_{lat} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

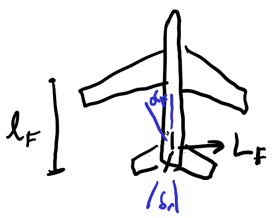
$\dot{p}$

$$B_{lat} = \begin{bmatrix} 0 & \frac{Y_{\delta_r}}{m} \\ \Gamma_3 L_{\delta_a} + \Gamma_4 N_{\delta_a} & \Gamma_3 L_{\delta_r} + \Gamma_4 N_{\delta_r} \\ \Gamma_4 L_{\delta_a} + \Gamma_8 N_{\delta_a} & \Gamma_4 L_{\delta_r} + \Gamma_8 N_{\delta_r} \\ 0 & 0 \end{bmatrix}$$

Table 7.1
Dimensional Control Derivatives

	X	Z	M
δ_e	$C_{x_{\delta_e}}\frac{1}{2}\rho u_0^2 S$	$C_{z_{\delta_e}}\frac{1}{2}\rho u_0^2 S$	$C_{m_{\delta_e}}\frac{1}{2}\rho u_0^2 S\bar{c}$
δ_p	$C_{x_{\delta_p}}\frac{1}{2}\rho u_0^2 S$	$C_{z_{\delta_p}}\frac{1}{2}\rho u_0^2 S$	$C_{m_{\delta_p}}\frac{1}{2}\rho u_0^2 S\bar{c}$

	Y	L	N
δ_a	$C_{y_{\delta_a}}\frac{1}{2}\rho u_0^2 S$	$C_{l_{\delta_a}}\frac{1}{2}\rho u_0^2 Sb$	$C_{n_{\delta_a}}\frac{1}{2}\rho u_0^2 Sb$
δ_r	$C_{y_{\delta_r}}\frac{1}{2}\rho u_0^2 S$	$C_{l_{\delta_r}}\frac{1}{2}\rho u_0^2 Sb$	$C_{n_{\delta_r}}\frac{1}{2}\rho u_0^2 Sb$



$$N_F = -l_F L_F = -l_F \tfrac{1}{2}\rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$

$$C_{n_F} = \frac{N_F}{\frac{1}{2}\rho V^2 S b} = -\underbrace{\frac{l_F S_F}{S b}}_{V_V}\left(\frac{V_F^2}{V^2}\right) C_{L_F} = -V_V C_{L_F}\left(\frac{V_F^2}{V^2}\right)$$

$$C_{n_{\delta_r}} \equiv \frac{\partial C_{n_F}}{\partial \delta_r}\bigg|_0 = -V_V\left(\frac{V_F^2}{V^2}\right)\underbrace{\frac{\partial C_{L_F}}{\partial \delta_r}\bigg|_0}_{a_r} = \boxed{-a_r V_V\left(\frac{V_F^2}{V^2}\right)}$$

Y_{δ_r}, N_{δ_r}, L_{δ_r}

Ailerons

L_{δ_a} - Negative ← Aileron reversal can occur due to twist in the wings

N_{δ_a} - Can be either sign — if $N_{\delta_a} > 0$ this is called adverse yaw

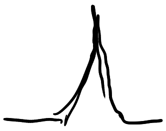
Y_{δ_a} - usually small

State Space $\leftrightarrow$ Laplace

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$G_{yu}(s)$

Review Properties of Laplace Transforms



$$\mathcal{L}[x(t)](s) = \int_0^{\infty} e^{-st} x(t)\, dt$$

Appendix A1

$$x(t) \Longleftrightarrow x(s)$$

$$\dot{x}(t) \Longleftrightarrow sx(s) - x(0)$$

$$\int_0^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{1}{s} x(s)$$

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha x(s) + \beta y(s)$$

Transfer Function

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 u$$

$$\ddot{x} = -2\zeta\omega_n \dot{x} - \omega_n^2 x + \omega_n^2 u$$

$$s^2 x(s) = -2\zeta\omega_n s x(s) - \omega_n^2 x(s) + \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\, x(s) = \omega_n^2 u(s)$$

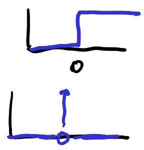
$$G_{xu}(s) \equiv \frac{X(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

X $\ddot{x} = -\dot{x}^2 + u$ No transfer function for nonlinear systems

---

step

impulse



$\delta(0)$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$u(t) \longrightarrow u(s)$

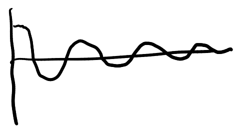
step $\quad u(s) = \frac{1}{s}$

impulse $\quad u(s) = 1$

$$x(s) = G_{xu}(s)\, u(s)$$

$x(s) \longrightarrow x(t)$ table, method of partial fractions

$$\frac{s+b}{s^3 + a_1 s^2 + a_2 s + a_3} \longrightarrow \frac{c_1}{s^2 + d_1} + \frac{c_2}{s^2 + d_2}$$



Final Value Theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\, x(s)$$

Only if $s\,x(s)$ is stable

Two representations of a Linear System

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$G_{yu}(s)$

$|A - \lambda I|$

State space to TF — ss2tf

given SS want G_{yu}

$$s\,x(s) = A\,x(s) + B\,u(s)$$

$$(sI - A)\,x(s) = B\,u(s)$$

$$x(s) = (sI - A)^{-1} B\,u(s)$$

$$y(s) = \left(C(sI - A)^{-1}B + D\right) u(s)$$

(from here on assume $D = 0$)

$$M^{-1} = \frac{\operatorname{adj}(M)}{|M|} \qquad (sI - A)^{-1} = \frac{\operatorname{adj}(sI - A)}{|sI - A|}$$

Adjugate: transpose of cofactor matrix F

$$F_{ij} = (-1)^{i+j}\left|M_{-i-j}\right|$$

$-i$ = all rows except i

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \\ & & \\ & & \end{bmatrix}$

$$\operatorname{adj}(M) = F^T$$

$$G_{yu}(s) = \frac{y(s)}{u(s)} = C(sI - A)^{-1}B = \boxed{\frac{C\,\operatorname{adj}(sI - A)\,B}{|sI - A|}} = \frac{N(s)}{D(s)}$$

$|sI - A| = 0$

Roots of $D(s)$ are the eigenvalues of A

stability $\Longleftrightarrow$ all roots of $D(s)$ in LHP
all eigenvalues of A in LHP

TF to SS

tf2ss

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^n + b_1 s^{n-1} \dots + b_{n-1} s + b_n}{a_0 s^n + \dots a_{n-1} s + a_n}$$

e.g. $\frac{Y(s)}{U(s)} = G(s) = \frac{b_0 s + b_1}{s^2 + a_1 + a_2}$

multiply by $\frac{x(s)}{x(s)}$

$$\frac{y}{u} = \frac{b_0 s x + b_1 x}{s^2 x + a_1 s x + a_2 x}$$

$\rightarrow y(t) = b_0 \dot{x}(t) + b_1 x(t)$

$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$

$\rightarrow \ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u$

$$\dot{\vec{x}} = A \vec{x} + B u$$

$$\rightarrow \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = C \vec{x} + D u$$

$$\rightarrow y = \begin{bmatrix} b_1 & b_0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + [0] u$$

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

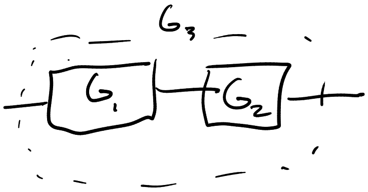
$$\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \\ a_n & \dots & & & a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

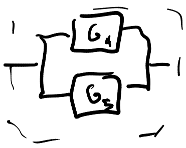
$$C = \begin{bmatrix} b_n - a_n b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \qquad D = [0]$$

$\vec{x}$ may not correspond to interpretable physical variables

Why transfer function

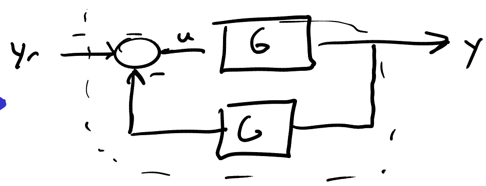


$G_3 = G_1 G_2$



$G_6 = G_4 + G_5$

rlocus



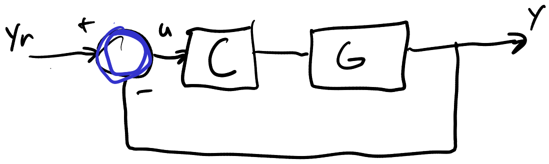
$y = Gu$

$u = y_r - Cy$

$y = G(y_r - Cy)$

$(1 + GC) y = G y_r$

$$\boxed{\frac{y}{y_r} = \frac{G}{1 + GC}}$$



$y = GCu$

$u = y_r - y$

$y = GC(y_r - y)$

$y(1+GC) = GCy_r$

$$\boxed{\frac{Y}{y_r} = \frac{GC}{1+GC}}$$

Example

$$A_{ph} = \begin{bmatrix} -0.0025 & -30 \\ 0.0001 & 0 \end{bmatrix} \qquad x = \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$B_{ph,\delta_T} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \qquad C_{ph,\theta} = [0 \quad 1] \qquad D = [0]$$

1) find $G_{\theta\delta_t}(s)$

$$G_{\theta\delta_t}(s) = \frac{C\,\mathrm{adj}(sI-A)\,B}{|sI-A|} \qquad sI - A = \begin{bmatrix} s+0.0025 & 30 \\ -0.0001 & s \end{bmatrix}$$

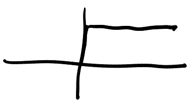
F^T

$$F = \begin{bmatrix} & 0.0001 \\ & \end{bmatrix}$$

$$G_{\theta\delta_t}(s) = \frac{[0 \quad 1]\begin{bmatrix} & \\ 0.0001 & \end{bmatrix}\begin{bmatrix} 10 \\ 0 \end{bmatrix}}{(s+0.0025)s - 0.0001\cdot(-30)}$$

$$\boxed{= \frac{0.001}{s^2 + 0.0025s + 0.003}}$$

2) $\lim_{t\to\infty} \Delta\theta(t)$ in response to a δ_t step function with magnitude ε



$\delta_t(t) = \mathbb{1}(t)\,\varepsilon$

$\delta_t(s) = \frac{1}{s}\varepsilon$

$$\Delta\theta(s) = G_{\theta\delta_t}(s)\,\delta_t(s) = \frac{0.001\,\varepsilon}{s(s^2 + 0.0025s + 0.003)}$$

$$\lim_{s\to 0} s\,\Delta\theta(s) = \lim_{s\to 0} \frac{\cancel{s}\,0.001\varepsilon}{\cancel{s}(s^2 + 0.0025s + 0.003)} = \frac{0.001\varepsilon}{0.003} = \frac{\varepsilon}{3}$$

$$\begin{bmatrix} G_{y_1u_1}(s) & G_{y_1u_2}(s) \\ G_{y_2u_1}(s) & G_{y_2u_2}(s) \end{bmatrix} = \frac{C\,\mathrm{adj}(sI-A)B}{|sI-A|}$$

$$= \frac{\begin{bmatrix} N_{y_1u_1}(s) & N_{y_1u_2}(s) \\ N_{y_2u_1}(s) & N_{y_2u_2}(s) \end{bmatrix}}{|sI-A| = D(s)}$$

$$G_{y_1u_2}(s) = \frac{N_{y_1u_2}(s)}{D(s)}$$

Automatic Flight Control Systems

- Autopilot
  - Controls a/c trajectory w/o requiring input from pilot
- Stability Augmentation System (SAS) ← adjusting stability derivatives C_{n_β}
  - Designed to improve dynamic stability
  - Relies on sensors, not on pilot inputs
- Control Augmentation System (CAS) ← adjusting control derivatives L_{δ_a}
  - Pilot input has two paths to control surfaces
    - Mechanical system
    - through the CAS electrical path
- Fly by Wire (FBW) ← choose your own derivatives
  - No mechanical link from pilot to actuators (a CAS with complete authority)

Autopilot: 1910s (A/C: Lockheed Vega (1930s))

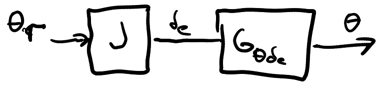
SAS (F-104) 1950s

CAS (F-15) Early '70s

FBW (F-16) Late '70s

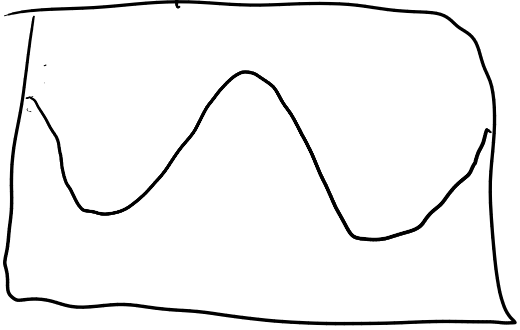
Control Design Task 1: Pitch Attitude Controller

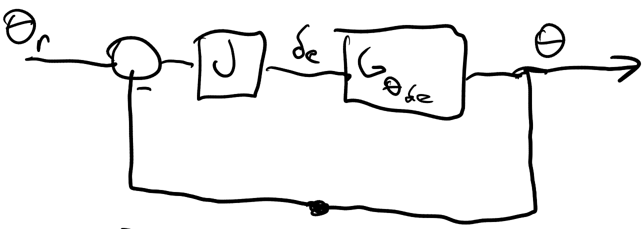
Input: θ_r Goal: $\theta \to \theta_r$ quickly (minimal Phugoid)



$J = 1$

step

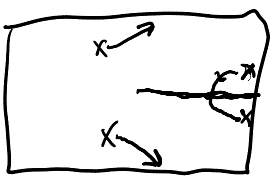


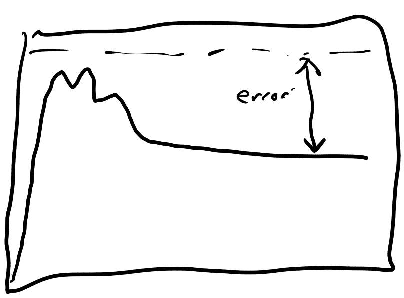


$J = K$

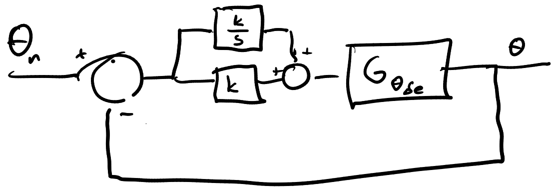
$K = -0.5$

RL

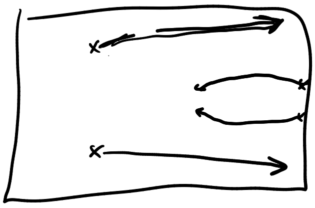




$J = K\left(1 + \frac{1}{s}\right)$



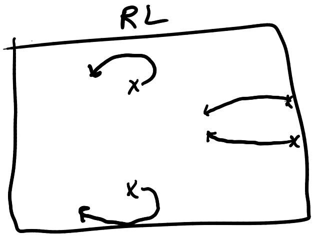
RL



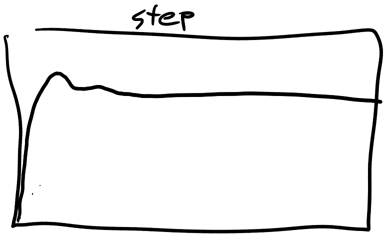
step

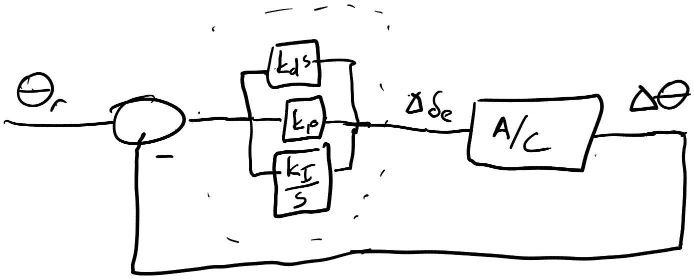
$J = K\left(1 + \frac{1}{s} + s\right)$

RL



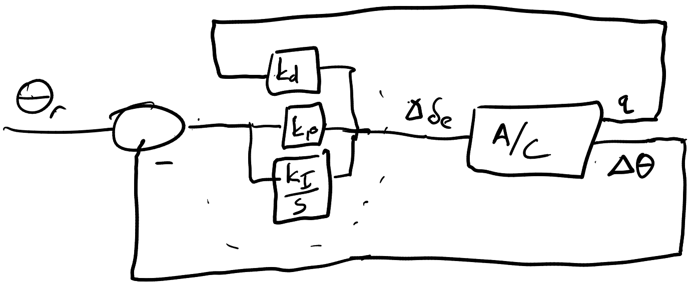
step





$\frac{d\Delta\theta}{dt}$ very large

$\Delta\theta$



$\Delta\delta_e = -k_p \Delta\theta$

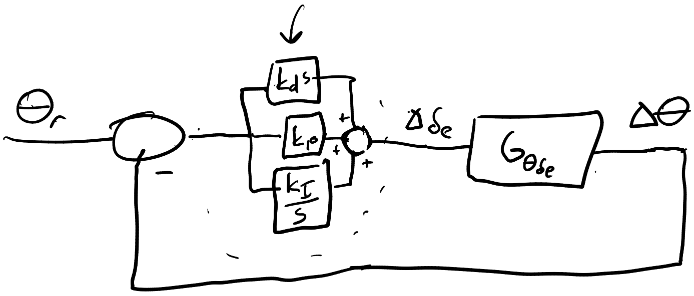
$\vec{u} = -K\vec{x}$

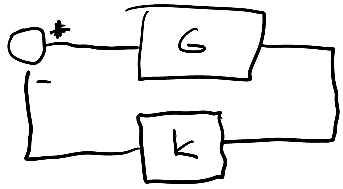
$$\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & k_p \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ q \\ \Delta\theta \end{bmatrix}$$

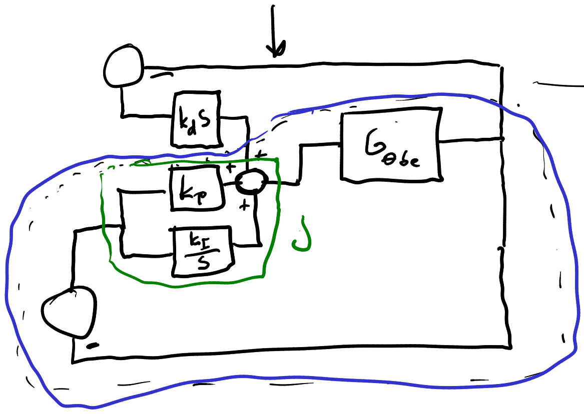
$A - BK$

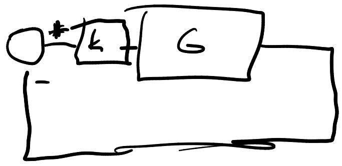
$$A_{lon} - \begin{bmatrix} -0.0002 \\ -17.85 \\ -1.158 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & k_p \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{lon} - \begin{bmatrix} 0 & 0 & 0 & -0.0002k_p \\ 0 & 0 & 0 & -17.85k_p \\ 0 & 0 & 0 & -1.158k_p \\ 0 & 0 & 0 & 0 \end{bmatrix}$$









$$G = \frac{G_{\theta\delta_e}}{1 + J\,G_{\theta\delta_e}}$$

$$= \frac{G_{\theta\delta_e}}{1 + \left(k_p + \frac{k_I}{s}\right) G_{\theta\delta_e}}$$

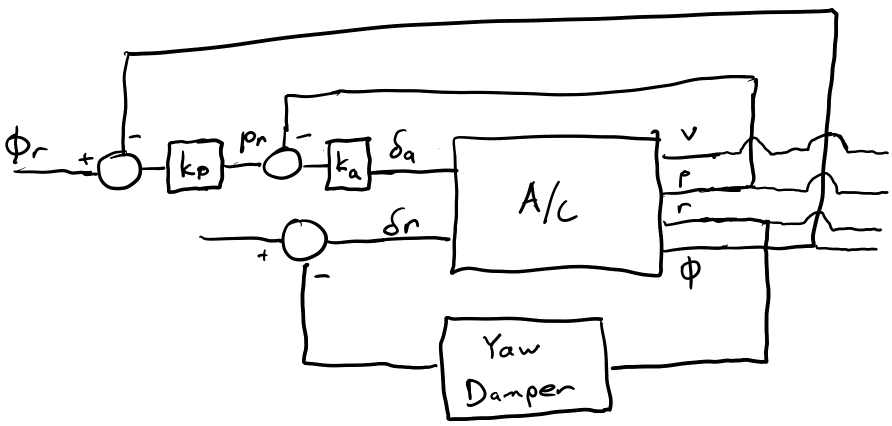
Control Task 2: Roll Controller

Input: $\phi_r = \phi_c$

In Book

Goals: $\phi \rightarrow \phi_r$ quickly

Alleviate Dutch Roll



Part 1: Yaw Damper

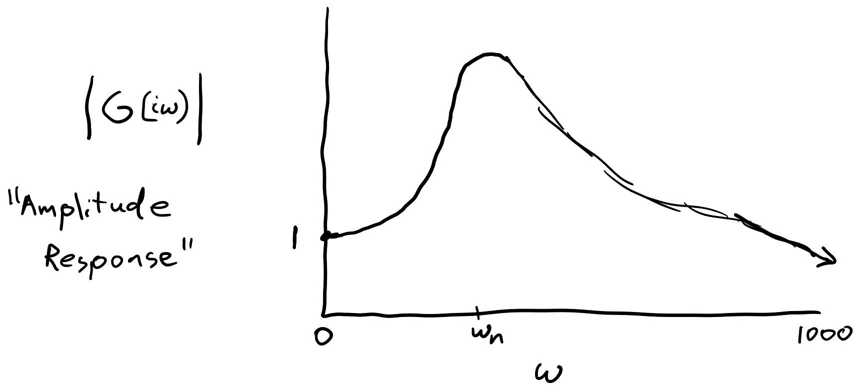
$\delta_r = -k_r r$ from RL, choose $k_r = -1.9$

Problem: In steady right turn for $k_r = -1.9$, rudder will be positive (wrong way for coordinated turn)

Want: $\delta_r \longrightarrow 0$ at low frequency

$\delta_r = -k_r r$ near dutch roll freq.

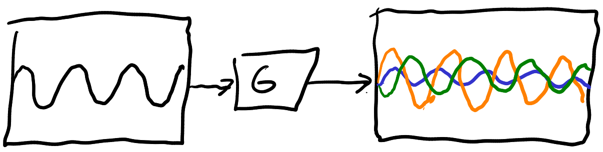
Review: Frequency Response



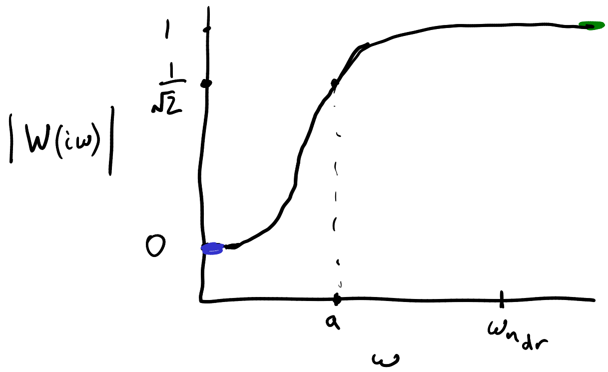
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$|G(0)| = 1$$

$$\lim_{\omega\to\infty} |G(i\omega)| = 0$$



"Washout"/high pass filter



$$W(s) = \frac{s}{s+a}$$

$$|W(0)| = 0$$

$$\lim_{\omega\to\infty} |W(\omega i)| = \lim_{\omega\to\infty} \left|\frac{\omega i}{\omega i}\right| = 1$$

$$|W(ai)| = \left|\frac{ai}{a+ai}\right| = \left|\frac{a e^{\frac{\pi}{2}i}}{\sqrt{2}\, a e^{\frac{\pi}{4}i}}\right| = \frac{1}{\sqrt{2}}$$

ai, √2, a

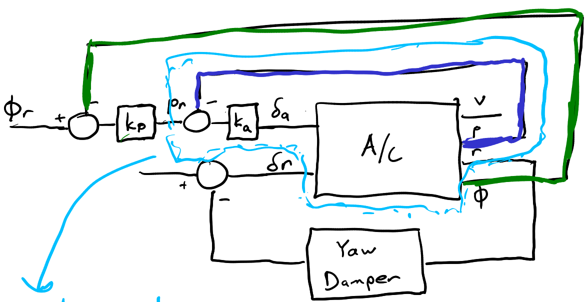
for 747 $\lambda_{dr} = -0.033 \pm 0.947i$

ω_d $\quad \omega_{n_{dr}} \approx 1$

choose $a = 0.1$

$$\boxed{\delta_r = -k_r \frac{s}{s+0.1}}$$

Part 2: Inner and Outer loop Aileron Controller



with $k_a = -1$ this looks like 0.25

$$\delta_a = (p_r - p)k_a$$

$$\dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta a}\delta_a$$

$$-0.4342 = \frac{1}{\tau} \qquad -0.1431$$

$$\dot{p} = \underbrace{(\mathcal{L}_p - \mathcal{L}_{\delta a}k_a)}_{= \frac{1}{\tau_{cl}}} p + \mathcal{L}_{\delta a}k_a p_r$$

from A_{lat}

$\lambda_r = -0.5625$

at steady state, $\dot{p} = 0$

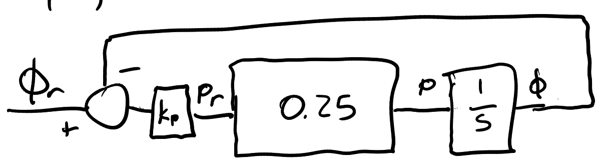
$$p_\infty = -\underbrace{\frac{\mathcal{L}_{\delta a}k_a}{\mathcal{L}_p - \mathcal{L}_{\delta a}k_a}}_{\text{Not}=1} p_r$$

Choose $k_a = -1$

$$\hookrightarrow -\frac{\mathcal{L}_{\delta a}k_a}{\mathcal{L}_p - \mathcal{L}_{\delta a}k_a} = 0.25$$

Outer Loop

Assume: $\dot{\phi} = p$, roll rate dynamics (p, above) are much faster than ϕ response that we want

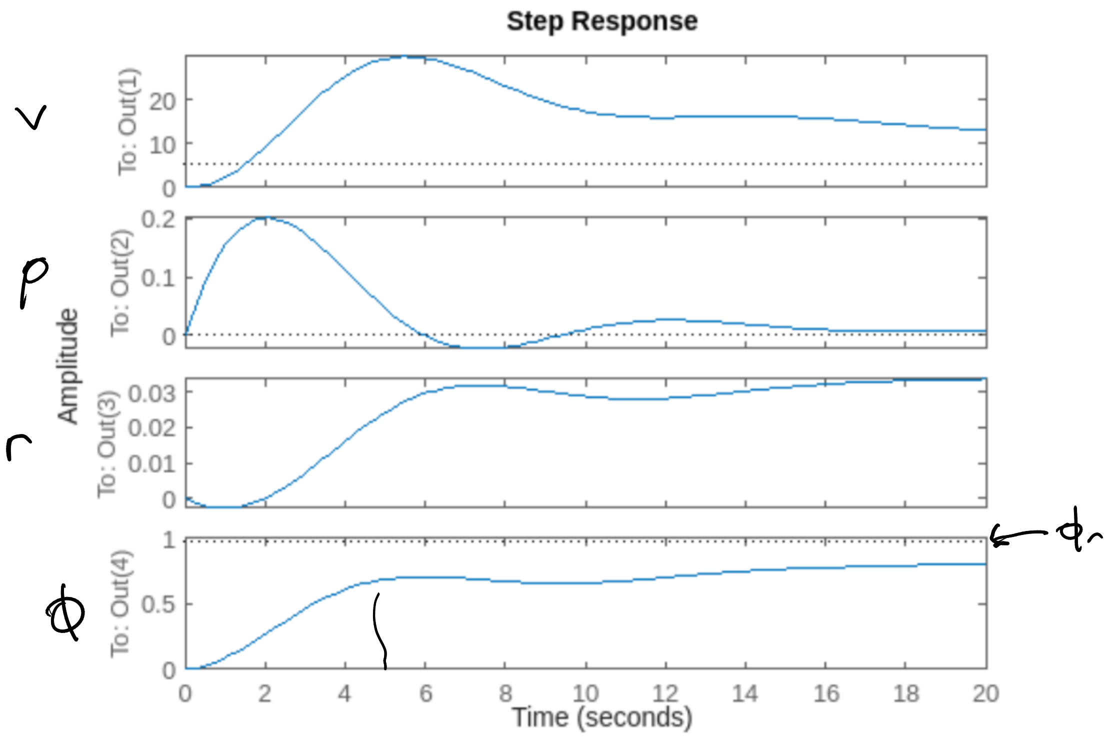


$$\dot{\phi} = 0.25\, k_p (\phi_r - \phi)$$

$$\dot{\phi} = -0.25\, k_p \phi + 0.25 k_p \phi_r$$

Based on Root Locus

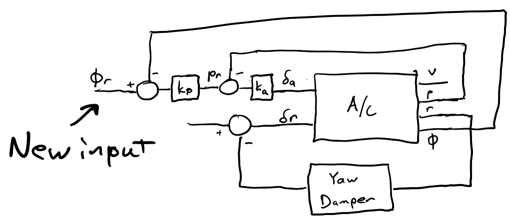
choose $k_p = 1.5$



State Space Gain Matrix for this Controller (without washout)

$$\vec{u} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = -K\vec{x} = -\underbrace{\begin{bmatrix} 0 & k_a & 0 & k_a k_p \\ 0 & 0 & k_r & 0 \end{bmatrix}}_{K} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\delta_a = -k_a p + k_a p_r$$
$$= -k_a p - k_a k_p \phi$$
$$\delta_r = -k_r r$$



$$A_{roll} = A_{lat} - B_{lat} K$$

$$\dot{\vec{x}} = A_{roll}\,\vec{x} + \underbrace{B_{lat}\begin{bmatrix} k_p k_a \\ 0 \end{bmatrix}}_{B_{roll}} \phi_c$$

Taste of Optimal Control

Techniques for choosing gains

1. Poll Assignment

   $A^{cl} = A - BK$ solve for entries in K to achieve desired A^{cl} eigenvalues

2. Hand Tuning (often PID)
3. Root Locus
4. Optimal Control $\longrightarrow$

Optimization Problem

$$\underset{x}{\text{minimize}} \quad f(x) \leftarrow \text{objective}$$

optimization variables

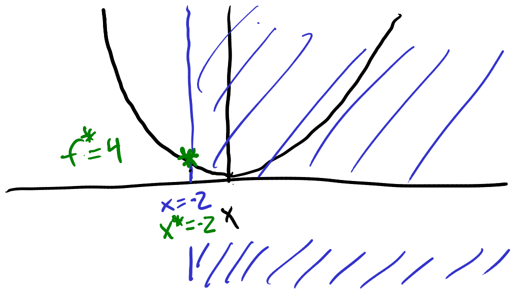
$$\text{subject to} \quad g(x) \leq 0 \leftarrow \text{constraints}$$

Engineer specifies f and g; computer finds x^* that minimizes $f(x)$

Example

$$\underset{x}{\text{minimize}} \quad x^2$$

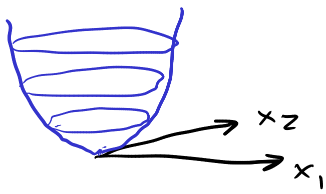
$$\text{subject to} \quad x + 2 \leq 0 \qquad x \leq -2$$



Example

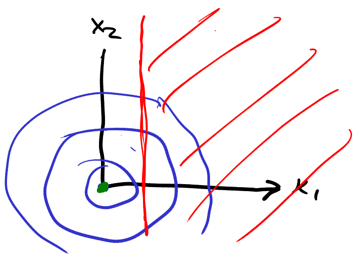
$$\underset{\vec{x}}{\text{minimize}} \quad x_1^2 + x_2^2$$

$$\text{subject to} \quad x_1 - 1 \leq 0$$



$$\nabla f = 0$$

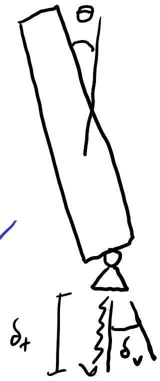
$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Optimal Control

$$\underset{\vec{x}(t),\vec{u}(t)}{\text{minimize}} \; J(\vec{x},\vec{t})$$

$$\text{subject to} \quad \dot{\vec{x}} = f(\vec{x},\vec{u})$$
$$\vec{x}_{min} \le \vec{x} \le \vec{x}_{max}$$
$$\vec{u}_{min} \le \vec{u} \le \vec{u}_{max}$$



$$\underset{x(t)\;u(t)}{\text{minimize}} \; \|\vec{x}(T) - \vec{x}_{target}\|$$

$$\text{subject to} \quad fuel(t) \ge 0$$
$$\theta_{min} \le \theta(t) \le \theta_{max}$$

Special Problem: LQR

"Linear Quadratic Regulator"

Linear Analytic Solution!

$u(t) = -K\,x(t)$

$$\underset{\vec{u}(t),\vec{x}(t)}{\text{minimize}} \; J(\vec{x},\vec{u}) = \int_0^\infty \vec{x}(t)^T Q\, \vec{x}(t) + \vec{u}(t)^T R\, \vec{u}(t)\, dt$$

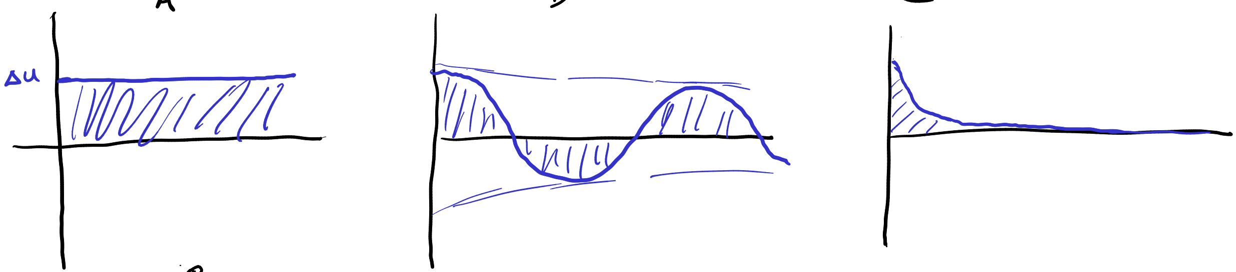
$$\text{subject to} \quad \dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t)$$
$$\vec{x}(0) = x_0$$

Ex: Long Dynamics

$$Q = \begin{bmatrix} c_u & 0 & 0 & 0 \\ 0 & c_w & 0 & 0 \\ 0 & 0 & c_q & 0 \\ 0 & 0 & 0 & c_\theta \end{bmatrix} \Rightarrow \vec{x}^T Q \vec{x} = c_u \Delta u^2 + c_w \Delta w^2 + c_q q^2 + c_\theta \Delta\theta^2$$

$$R = \begin{bmatrix} c_{\delta_e} & 0 \\ 0 & c_{\delta_t} \end{bmatrix} \Rightarrow \vec{u}^T R \vec{u} = c_{\delta_e} \Delta\delta_e^2 + c_{\delta_t} \Delta\delta_t^2$$

A B C



$$J(\vec{x},\vec{u}) = \int_0^\infty x^T Q x + u^T R u \, dt$$

$c_u \Delta u^2$

Relatively Large $Q \Rightarrow$ smaller state deviations

Relatively Large $R \Rightarrow$ smaller control deviations

$$\underset{x\ u}{\text{minimize}}\ J(x,u) = \int_0^\infty x^TQx + u^TRu\,dt$$

can't change

$$J_0(x,u) = x(0)^TQx(0) + u(0)^TRu(0)$$

if this were the cost function

$$V^*(x(\tau)) = \min \int_\tau^\infty x^TQx + u^TRu\,dt$$

$$J^* = J^*(x^*,u^*) = V^*(x(0))$$

$$V^*(x(\tau)) = \min\Big\{ \underbrace{\int_\tau^{\tau+d\tau} x^TQx + u^TRu\,dt}_{\text{immediate}} + \underbrace{\int_{\tau+d\tau}^\infty x^TQx + u^TRu\,dt}_{\text{future}} \Big\}$$

$$= \min\Big\{ \big(x(\tau)^TQx(\tau) + u(\tau)^TRu(\tau)\big)d\tau + V^*(x(\tau+d\tau)) \Big\}$$

$$\cancel{V^*(x(\tau))} = \min\Big\{ \text{''} \quad \text{''} \quad \text{''} + \cancel{V^*(x(\tau))} + \frac{dx}{dt}\cdot\nabla V^*\Big|_{x(\tau)} d\tau \Big\}$$

$$Ax + Bu$$

$$0 = \min_u\{x^TQx + u^TRu + (Ax+Bu)\cdot\nabla V^*\}$$

assumption $\quad V^*(x(\tau)) = x(\tau)^TSx(\tau)$

$$\nabla V^* = 2Sx$$

$$0 = \min_{u(\tau)} \underbrace{\big(x(\tau)^TQx(\tau) + u(\tau)^TRu(\tau) + (Ax(\tau) + Bu(\tau))\cdot 2Sx(\tau)\big)}_{f(u)}$$

minimized when $\nabla f = 0$

$$\nabla f = 2Ru + B^T2Sx = 0$$

$$u = -\underbrace{R^{-1}B^TS}_{K}\,x$$

$$\boxed{u = -Kx}$$

S is a solution to

$$0 = Q + A^TS + SA + SBR^{-1}B^TS$$

In matlab

$$\boxed{K = \mathrm{lqr}(A,B,Q,R)}$$

Traditional Methods:

Use RL, hand tuning, Pole assignment $\rightarrow$ choose K directly

LQR

Choose Q, R $\rightarrow$ computer calculates K