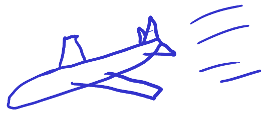
Welcome to ASEN 3728

Aircraft Dynamics!



Aeronautical Engineering

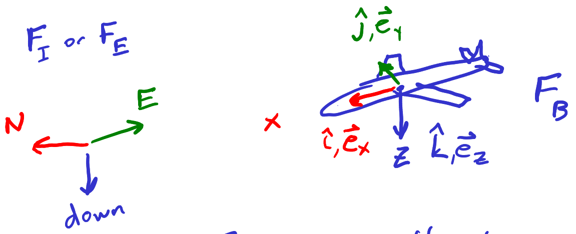
1. Aerodynamics
2. Structures / Materials
3. Propulsion
4. Dynamics + Control
   - Keep the pointy end forward
   - Get the aircraft to where we want to go

+ Mathematical description of A/C behavior
  + computer simulation model
+ Design of A/C and Control Systems to effect desirable dynamics

Notation + Conventions

Body Coordinate System

vectors: $\rightarrow$, $\hat{}$ (unit) or bold



Frame: collection of ≥ 3 points (distance between points is constant)

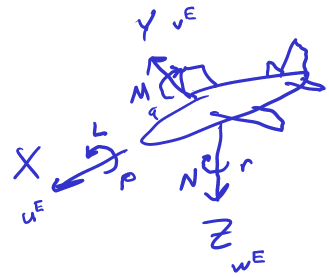
Inertial: translates with a constant velocity ~~acceleration, rotation~~
└ Newton's laws valid

Coordinate System: 3 unit vectors that allow measurement

$\vec{V}_B^E$ ← if present, frame; ← coordinate system

"inertial velocity written in body coordinates"

Forces and Moments



$$\vec{f} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$$

$$\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\vec{G} = L\hat{\imath} + M\hat{\jmath} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{\imath} + v^E\hat{\jmath} + w^E\hat{k}$$

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$V_g = |V_B^E| = \sqrt{u^{E^2} + v^{E^2} + w^{E^2}}$$

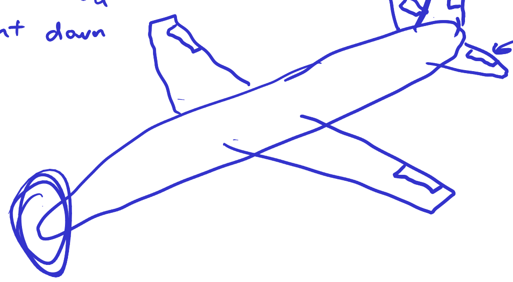
$$\vec{\omega}^E = p\hat{\imath} + q\hat{\jmath} + r\hat{k}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Control Surfaces

Rudder
$+\delta_r$ = toward $-y$ direction
$+\delta_r \Rightarrow -N$, Y

Aelerons δ_a
$+\delta_a$ = right down
$+\delta_a \Rightarrow -L$



Eelevator
$+\delta_e$ = down
$+\delta_e \Rightarrow -M \quad -Z$

Throttle: δ_t

$+\delta_t \Rightarrow +X$ force
no moment

Wind

Aerodynamic Forces + Moments (not gravity)
are functions of the A/C velocity w.r.t. the air

$$\vec{V}^E = \vec{V} + \vec{W}$$

$\vec{V}$: air-relative

$\vec{W}$: wind

When no wind $\vec{V}^E = \vec{V}$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Wind Angles

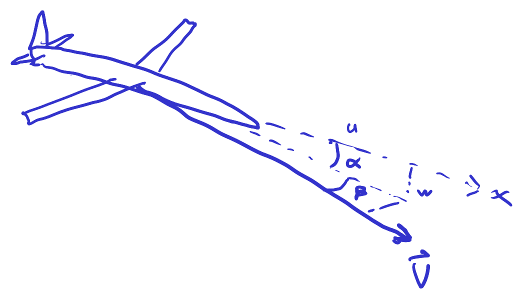
$u = V\cos\beta\cos\alpha$
$v = V\sin\beta$
$w = V\cos\beta\sin\alpha$

angle of attack α

$\alpha \equiv \tan^{-1}\frac{w}{u}$

sideslip angle β

$\beta \equiv \sin^{-1}\frac{v}{V}$



Orientation

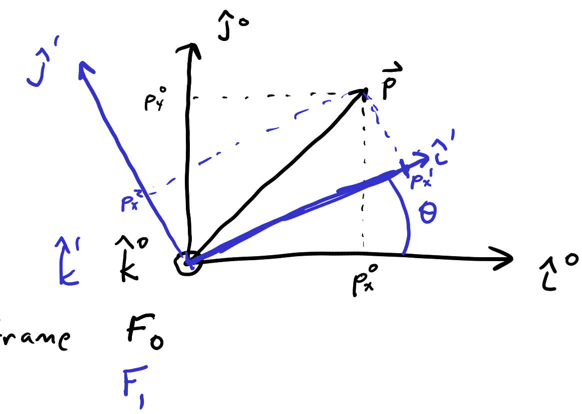
"1" ϕ roll

"2" θ pitch 3-2-1

"3" ψ yaw

know $\vec{V}^E_B$ (Frame: E, Coordinate: B) want $\vec{V}^E_E$

$\vec{p}_0 = \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$ $\vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix}$



$\vec{p} = p_x^0\hat{i}^0 + p_y^0\hat{j}^0 + p_z^0\hat{k}^0$

$\vec{p} = p_x^1\hat{i}^1 + p_y^1\hat{j}^1 + p_z^1\hat{k}^1$

→ want p_x^1 in terms of $\vec{p}_0$

$p_x^1 = \vec{p}\cdot\hat{i}^1$

$= p_x^0\hat{i}^0\cdot\hat{i}^1 + p_y^0\hat{j}^0\cdot\hat{i}^1 + p_z^0\hat{k}^0\cdot\hat{i}^1$

$= \underline{\begin{bmatrix} \hat{i}^0\cdot\hat{i}^1 & \hat{j}^0\cdot\hat{i}^1 & \hat{k}^0\cdot\hat{i}^1 \end{bmatrix}} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$

$p_x^1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \end{bmatrix} \vec{p}_0$

$$\vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix} = \begin{bmatrix} \hat{i}^0\cdot\hat{i}^1 & \hat{j}^0\cdot\hat{i}^1 & \hat{k}^0\cdot\hat{i}^1 \\ \hat{i}^0\cdot\hat{j}^1 & \hat{j}^0\cdot\hat{j}^1 & \hat{k}^0\cdot\hat{j}^1 \\ \hat{i}^0\cdot\hat{k}^1 & \hat{j}^0\cdot\hat{k}^1 & \hat{k}^0\cdot\hat{k}^1 \end{bmatrix} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$\vec{p}_1 = R^1_0\,\vec{p}_0$

Direction Cosine Matrix Rotation Matrix

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3: axis; θ: axis)

$\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \qquad R_2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

Properties of DCMS

$\vec{p}_2 = R_1^2 \vec{p}_1 \qquad \vec{p}_1 = R_0^1 \vec{p}_0 \qquad \vec{p}_2 = R_1^2 \vec{p}_1 = \vec{p}_2 = R_1^2 \underbrace{R_0^1 \vec{p}_0}$

Chaining $\qquad R_0^2 = R_1^2 R_0^1 \qquad \boxed{R_A^C = R_B^C R_A^B}$

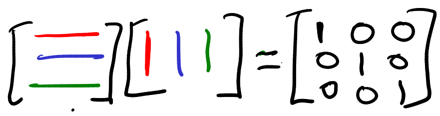
Inverse

$$\vec{p}_2 = R_1^2 \vec{p}_1$$

$$\underbrace{R_1^{2^{-1}} \vec{p}_2 = \cancel{R_1^{2^{-1}}} \cancel{R_1^2} \vec{p}_1 = R_2^1 \vec{p}_2}$$

$$R_2^1 = (R_1^2)^{-1} \qquad R_A^B = (R_B^A)^{-1}$$

since DCMs are orthonormal



$$(R_A^B)^{-1} = R_A^{B\,T}$$

$$\boxed{R_A^B = R_B^{A\,T}}$$

Euler Angles

Any ~~rotatation~~ rotation can be described by 3 Euler angles about 3 n-repeated axes

E.g. 3-1-3 rotation through α, β, γ
3-2-1 rotation through ψ, θ, ϕ

$$\vec{p}_B = R_E^B \vec{p}_E = \underbrace{R_1(\phi)}_{\text{roll}} \underbrace{R_2(\theta)}_{\text{pitch}} \underbrace{R_3(\psi)}_{\text{yaw}} \vec{p}_E$$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$\vec{P}_{pilot}^{\,E} = \vec{P}_{A/c}^{\,E} + \vec{P}_{pilot}^{\,B}$ $\qquad$ ← abstract vectors

$$\boxed{\vec{P}_{pilot\,E}^{\,E} = \vec{P}_{A/c\,E}^{\,E} + R_B^E \vec{P}_{pilot\,B}^{\,B}}$$

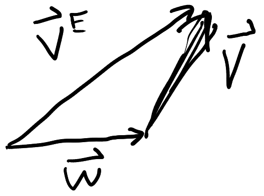
Vector Clarity

$\vec{V}^{E}_{B}$

E ← "frame of reference", "with respect to", "relative to"

B ← "coordinate frame", "expressed in", "written in"

→ $\vec{V}^{E} = \vec{V}^{(W)} + \vec{W}^{(E)}$ ← by convention



True in **Any** coordinate system as long as all in **same** coordinate system

$$\vec{V}^{E}_{B} = \vec{V}_{B} + \vec{W}_{B}$$

$$\begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \qquad \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \cancel{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}}$$

$$\vec{V}^{E}_{E} = \vec{V}_{E} + \vec{W}_{E}$$

Kinematics

Kinematics : "Geometry of Motion" (no forces) < Translational / Rotational

Dynamics : Effects of forces and moments on objects

Vector Derivatives

→ $\frac{d}{dt}\vec{p} \equiv$ time rate of change of $\vec{p}$

$\frac{d}{dt}\vec{p} = \vec{v}^E$

$\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$ ↗ $\dot{\vec{p}}_B \equiv \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

Not Always True

$\hat{i}^B$

$\hat{i}_B^B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\hat{i}_B \equiv \hat{i}_B^B$

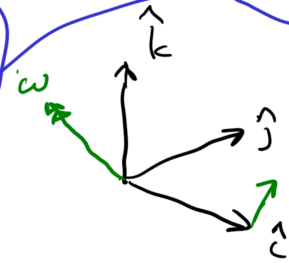
$\vec{p}_B = p_x \hat{i}_B + p_y \hat{j}_B + p_z \hat{k}_B$

$\dot{\vec{p}}_B = \dot{p}_x \hat{i}_B + \dot{p}_y \hat{j}_B + \dot{p}_z \hat{k}_B$

$\frac{d}{dt}(uv) = u\frac{d}{dt}v + v\frac{d}{dt}u$

$\left(\frac{d}{dt}\vec{p}\right)_B = \dot{p}_x \hat{i}_B + p_x \frac{d}{dt}\hat{i}_B + \dot{p}_y \hat{j}_B + p_y \frac{d}{dt}\hat{j}_B + \dot{p}_z \hat{k}_B + p_z \frac{d}{dt}\hat{k}_B$

What is $\frac{d}{dt}\hat{i}_B$



$\frac{d}{dt}\hat{i} = \vec{\omega} \times \hat{i}$

$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

$= p_x(\vec{\omega}_B \times \hat{i}_B) + p_y(\vec{\omega}_B \times \hat{j}_B) + p_z(\vec{\omega}_B \times \hat{k}_B)$

$= \vec{\omega}_B \times \vec{p}_B$

$\boxed{\frac{d}{dt}\vec{p}_B = \dot{\vec{p}}_B + \vec{\omega}_B \times \vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$

$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$

Kinematic Transport Theorem

Equations of motion for a 3D body

$\dot{\vec{x}} = f(t, \vec{x})$

Aircraft State

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \vec{p}_E = \vec{p}^{\,E}_E \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \vec{\Theta} \text{ pseudo-vector} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \vec{V}^E_B \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \vec{\omega}^E_B \end{matrix}$$

Need

$\dot{\vec{p}}_E$, $\dot{\vec{\Theta}}$ — Kinematic

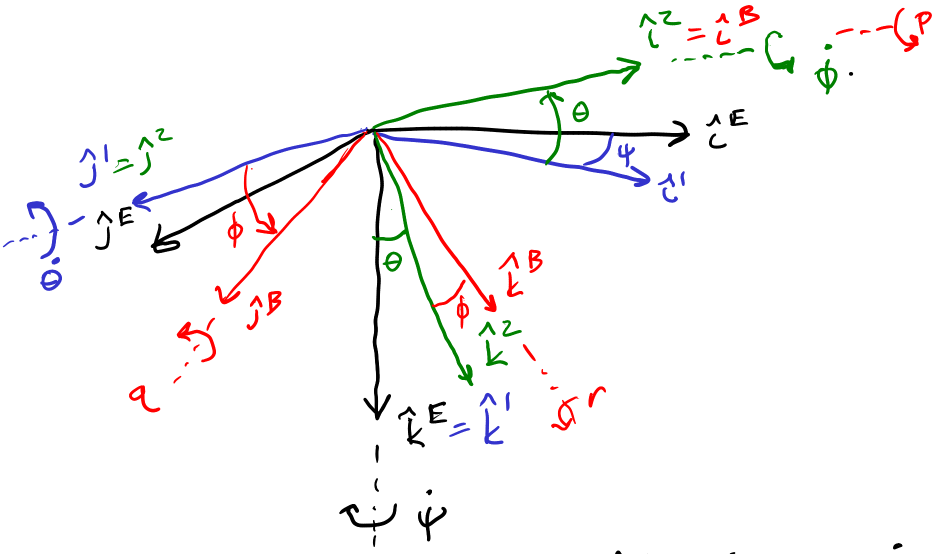
$\dot{\vec{V}}^E_B$, $\dot{\vec{\omega}}_B$ — Dynamics

Translational Kinematics

$$\dot{\vec{p}}^{\,E}_E = \frac{d}{dt}\vec{p}_E - \cancel{{}^E\vec{\omega}_E}^{\,0} \times \vec{p}^{\,E}_E = \vec{V}^E_E = R^E_B \vec{V}^E_B$$

$$\boxed{\dot{\vec{p}}^{\,E}_E = R^E_B \vec{V}^E_B}$$

Rotational Kinematics



$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega} = \dot\psi \hat{k}^E + \dot\theta \hat{j}^1 + \dot\phi \hat{i}^B$$

$$R^B_E = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$R^B_1 = R_1(\phi) R_2(\theta)$$

$$\vec{\omega}_B = \dot\psi \hat{k}^E_B + \dot\theta \hat{j}^1_B + \dot\phi \hat{i}^B_B$$

$$= \dot\psi R^B_E \underbrace{\hat{k}^E_E}_{\begin{bmatrix}0\\0\\1\end{bmatrix}} + \dot\theta R^B_1 \underbrace{\hat{j}^1_1}_{\begin{bmatrix}0\\1\\0\end{bmatrix}} + \dot\phi \hat{i}^B_B$$

$$= R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix} + \begin{bmatrix} \dot\phi \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix}} \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$$

invert

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\sec x = \frac{1}{\cos x}$$

$$\dot{\vec{o}} = T\,\vec{\omega}_B$$

attitude, influence matrix

Dynamics $\quad \vec{f} \quad \vec{G}$

Translational Dynamics

Newtons 2nd Law

$$\vec{f} = m\vec{a}$$

$$\vec{f} = m\underline{\frac{d}{dt}}\vec{V}^E$$

$$\frac{d}{dt}\vec{V}^E_B = \dot{\vec{V}}^E_B + \vec{\omega}_B \times \vec{V}^E_B$$

$$m\left(\dot{\vec{V}}^E_B + \tilde{\omega}_B \vec{V}^E_B\right) = \vec{f}_B$$

$$\boxed{\dot{\vec{V}}^E_B = \frac{\vec{f}_B}{m} - \tilde{\omega}_B \vec{V}^E_B} \quad *$$

Rotational Dynamics

Euler's 2nd Law

angular momentum

$$\frac{d}{dt}\vec{h} = \vec{G}$$

moments

$$\vec{h} = I\vec{\omega}$$

moment of inertia

$$\mathbf{I} = \begin{pmatrix} \int (y^2+z^2)\,\mathrm{dm} & -\int xy\,\mathrm{dm} & -\int xz\,\mathrm{dm} \\ -\int xy\,\mathrm{dm} & \int (x^2+z^2)\,\mathrm{dm} & -\int yz\,\mathrm{dm} \\ -\int xz\,\mathrm{dm} & -\int yz\,\mathrm{dm} & \int (x^2+y^2)\,\mathrm{dm} \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

$\dot{\vec{\omega}}_B$

$$\frac{d}{dt}\vec{h}_B = \dot{\vec{h}}_B + \tilde{\omega}_B \vec{h}_B = \vec{G}_B$$

$$I\dot{\vec{\omega}}_B + \tilde{\omega}_B I \vec{\omega}_B = \vec{G}_B$$

$$\boxed{\dot{\vec{\omega}}_B = I^{-1}\left[\vec{G}_B - \tilde{\omega}_B I \vec{\omega}_B\right]}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}^E_E \\ \dot{\vec{\Theta}} \\ \dot{\vec{V}}^E_B \\ \dot{\vec{\omega}}^E_B \end{bmatrix} \qquad \begin{aligned} \dot{\vec{p}}_E &= R^E_B \vec{V}^E_B \\ \dot{\vec{\Theta}} &= T\,\vec{\omega}_B \end{aligned} \quad *$$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{p}_E^{\,E} = \vec{p}_E \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{\Theta} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{V}_B^{\,E} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \vec{\omega}_B^{\,E} = \vec{\omega}_B \end{matrix}$$

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

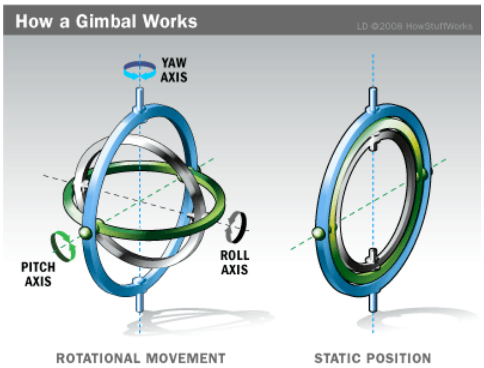
$$\dot{\vec{\Theta}} = T \vec{\omega}_B$$

attitude influence

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

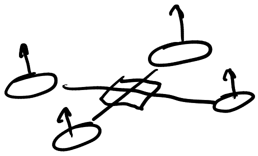
$$\dot{\vec{\omega}}_B = I_B^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B\right]$$

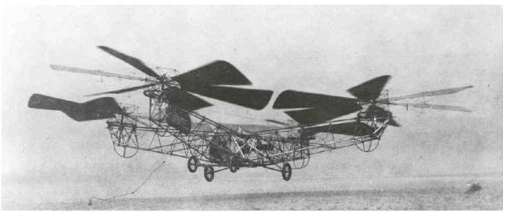
$$T = \begin{bmatrix} 1 & \sin\phi \, \underline{\tan\theta} & \cos\phi \, \underline{\tan\theta} \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \, \underline{\sec\theta} & \cos\phi \, \underline{\sec\theta} \end{bmatrix}$$



Multicopter

Quadrotor





March 4, 1930. G. DE BOTHEZAT 1,749,471

HELICOPTER

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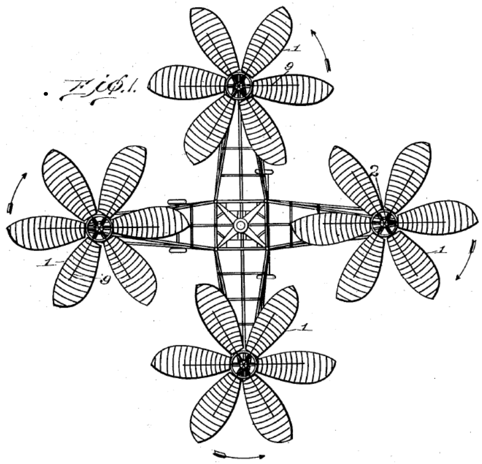
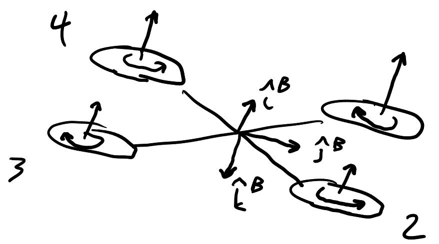




$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^c\vec{G} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

$$\vec{f} = {}^g\vec{f} + {}^a\vec{f} + {}^c\vec{f}$$
$$\vec{G} = {}^a\vec{G} + {}^c\vec{G}$$

Since symmetric about $\hat{i}^B - \hat{k}^B$ plane and $\hat{i}^B - \hat{j}^B$ plane

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \overbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}^{\vec{\omega}_B \times \vec{V}_B^E} + \overbrace{g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}}_{{}^a\vec{f}_B}}^{\vec{f}_B} + \frac{1}{m} \underbrace{\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}}_{{}^c\vec{f}_B}$$

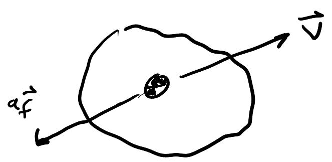
$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$I_B^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \overbrace{\begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix}}^{I_B^{-1}(\vec{\omega}_B \times I\vec{\omega}_B)} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Aerodynamic Forces and Moments

$${}^a\vec{f} = -D \frac{\vec{V}}{|\vec{V}|}$$



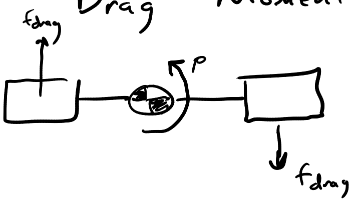
$V_a = |\vec{V}|$ airspeed

$$D = \overbrace{\tfrac{1}{2} \rho C_D A}^{\nu} V_a^2 = \nu V_a^2$$

density, coefficient of drag (shape), cross-sectional area

$${}^a\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = -\nu V_a^2 \frac{\vec{V}_B}{V_a}$$
$$= -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

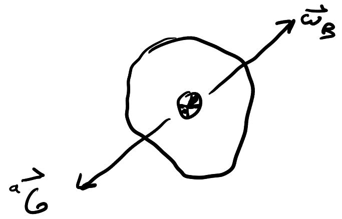
Drag Moment



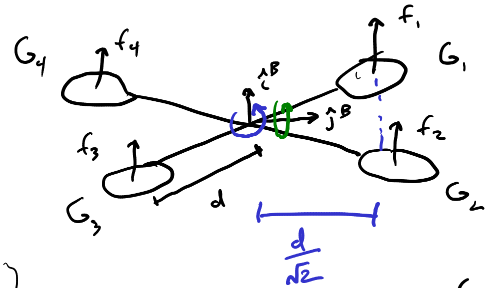
$$L_{drag} = -2l f_{drag}$$
$$= -2l\left(\tfrac{1}{2}\rho C_D A (l p)^2\right)\operatorname{sign}(p)$$
$$= -\mu |p| p$$

$$-p^2 \operatorname{sign}(p) = -|p|p$$

$${}^{a}\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = -\mu\sqrt{p^2+q^2+r^2}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



Control Forces + Moments



$${}^{c}\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_1-f_2-f_3-f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^{c}\vec{G}_B = \begin{bmatrix} \frac{d}{\sqrt{2}}(-f_1-f_2+f_3+f_4) \\ \frac{d}{\sqrt{2}}(f_1-f_2-f_3+f_4) \\ \underbrace{G_1-G_2+G_3-G_4}_{k_m(f_1-f_2+f_3-f_4)} \end{bmatrix} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

$$\left.\begin{aligned} G_i &= k_G C_D \omega_r^2 \\ f_i &= k_f C_L \omega_r^2 \end{aligned}\right\}$$

$$k_m = \frac{k_G C_D}{k_f C_L}$$

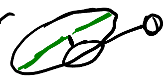
$$G_i = k_m f_i$$

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ k_m & -k_m & k_m & -k_m \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

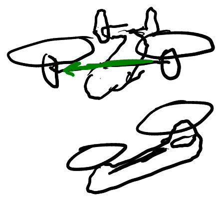
invert

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

Multirotor vs Conv Helicopter



	Multirotor	Conv Helicopter
Aerodynamic Efficient		✓
Mechanical Control Complexity	✓	
Power System Complexity Turbine		✓
Electric	✓	



$\dot{\vec{x}} = f(\vec{x}, \vec{u})$

- differential
- 1st order
- ordinary
- nonlinear
- coupled

general form

$$\begin{cases} \dot{\vec{p}}^E_E = R^E_B \, \vec{v}^E_B \\ \dot{\vec{o}} = T \, \vec{\omega}_B \\ \dot{\vec{V}}^E_B = \dfrac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}^E_B \\ \dot{\vec{\omega}}_B = I_B^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B\right] \end{cases}$$

$$T = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix}$$

$\phi = 90°$
$\theta = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 10°/s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q = 10°/s \\ r \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g\begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Quadrotors

$0 = \frac{1}{m}X + \frac{0}{m}$ $\therefore$ $X = 0$
$Y = 0$

$0 = g + \frac{1}{m}\cancel{Z} + \frac{1}{m}Z_c$

$Z_{c,0} = -mg$

"Hover" trim condition

trim condition

$\dot{x}_{E,0} = 0$ | $u^E_0 = 0$ | $\dot{u}^E_0 = 0$
$\dot{y}_{E,0} = 0$ | $v^E_0 = 0$ | $\dot{v}^E_0 = 0$
$\dot{z}_{E,0} = 0$ | $w^E_0 = 0$ | $\dot{w}^E_0 = 0$

$f_{1,0} = \frac{mg}{4}$
$f_{2,0} = $ "
$f_{3,0} = $ "
$f_{4,0} = $ "

$\boxed{Z_{c,0} = mg}$
$L_{c,0} = 0$
$M_{c,0} = 0$
$N_{c,0} = 0$

$p_0 = 0$ | $\dot{p}_0 = 0$ | $\phi_0 = 0$
$q_0 = 0$ | $\dot{q}_0 = 0$ | $\theta_0 = 0$
$r_0 = 0$ | $\dot{r}_0 = 0$ | $\psi_0 = \cdot$ any constant value

---

Linearazation

$\vec{x} = \vec{x}_0 + \Delta\vec{x}$ $\qquad$ $\vec{u} = \vec{u}_0 + \Delta\vec{u}$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} x_{E_0} + \Delta x_E \\ y_{E_0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta\phi \\ \Delta\theta \\ \Delta\psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$\Delta\dot{x} = A\Delta x + B\Delta u$

Taylor Series

$y = f(x,u)$ $\qquad x_0 + \Delta x$ $\qquad y_0 = f(x_0, u_0)$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$= \underbrace{f(x_0, u_0)}_{y_0} + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + H.O.T.$$

$$\Delta y = \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + H.O.T.$$

since $\Delta x, \Delta u$ are small

$\Delta x \Delta u \approx 0$

$\sin(\Delta x) \approx \Delta x$

$\cos(\Delta x) \approx 1$

$$x + y = x_0 + \Delta x + y_0 + \Delta y$$

$$xu \approx x_0 u_0 + \left.\frac{\partial xu}{\partial x}\right|_0 \Delta x + \left.\frac{\partial xu}{\partial u}\right|_0 \Delta u$$

$$= x_0 u_0 + u_0 \Delta x + x_0 \Delta u$$

alternative

$$xu = (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + u_0 \Delta x + x_0 \Delta u + \cancel{\Delta x \Delta u}^{\to 0}$$

$$\sin(\theta) \approx \sin\theta_0 + \left.\frac{\partial \sin\theta}{\partial \theta}\right|_0 \Delta\theta$$

$$= \sin\theta_0 + \cos\theta_0 \Delta\theta$$

alternatively

$$\sin(\theta) = \sin(\theta_0 + \Delta\theta) = \sin(\theta_0)\cancel{\cos(\Delta\theta)}^{1} + \cos(\theta_0)\cancel{\sin(\Delta\theta)}^{\Delta\theta} \approx \sin\theta_0 + \cos\theta_0 \Delta\theta$$

$$\cos(\theta) \approx \cos(\theta_0) - \sin(\theta_0)\Delta\theta$$

Linearize QR equations of motion

Examples

$$\dot\theta = \cos\phi\, q - \sin\phi\, r$$

$$\cancel{\dot\theta_0}^{0} + \Delta\dot\theta = \cos(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{q_0}^{0} + \Delta q) - \sin(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{r_0}^{0} + \Delta r)$$

$$\Delta\dot\theta \approx \cancel{\cos(\Delta\phi)}^{1}\Delta q - \cancel{\sin\Delta\phi}^{\Delta\phi}\Delta r$$

$$\approx \Delta q - \cancel{\Delta\phi\Delta r}^{0}$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

$$\dot w^E = q u^E - p v^E + g\cos\theta\cos\phi + \frac{1}{m}Z + \frac{1}{m}Z_c$$

$$\Delta\dot w^E = \cancel{\Delta q \Delta u^E}^{0} - \cancel{\Delta p \Delta v^E}^{0} + g + \frac{1}{m}\Delta Z + \frac{1}{m}(Z_{c,0} + \Delta Z_c)$$

cancel

$$\Delta\dot w^E = \frac{1}{m}\Delta Z + \frac{1}{m}\Delta Z_c$$

Linearized Aerodynamics

~~In real life $\nu(p,q,r)$~~ $\qquad \nu$ constant

$$Z = -\nu w\sqrt{u^2+v^2+w^2}$$

$$= Z_0 + \left.\frac{\partial Z}{\partial u}\right|_0 \Delta u + \left.\frac{\partial Z}{\partial v}\right|_0 \Delta v + \left.\frac{\partial Z}{\partial w}\right|_0 \Delta w$$

$\Delta Z =$ $\qquad$ " $\qquad$ " $\qquad$ "

Looks like $\frac{0}{0}$ $\qquad$ take limit as $u,v,w \to 0$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = -2\nu wu(u^2+v^2+w^2)^{-1/2}$$

$$\boxed{\left.\frac{\partial Z}{\partial u}\right|_0 = 0} \qquad \frac{\partial Z}{\partial w} = -\nu\frac{u^2+v^2+2w^2}{\sqrt{u^2+v^2+w^2}} \qquad \boxed{\left.\frac{\partial Z}{\partial w}\right|_0 = 0} \qquad \boxed{\Delta Z = 0}$$

$$\boxed{\left.\frac{\partial Z}{\partial v}\right|_0 = 0}$$

$$\boxed{\begin{aligned}\Delta L &= 0\\ \Delta M &= 0\\ \Delta N &= 0\end{aligned}}$$

$Z = -\nu w^2$

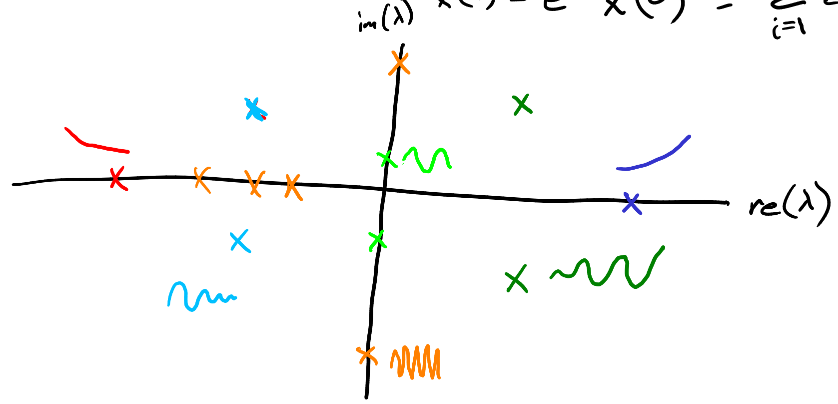
$\frac{\partial Z}{\partial w} = -\nu 2 \cancel{w}^{0}$

Review: State space representation of a Linear Dynamical System

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} + D\vec{u}$$

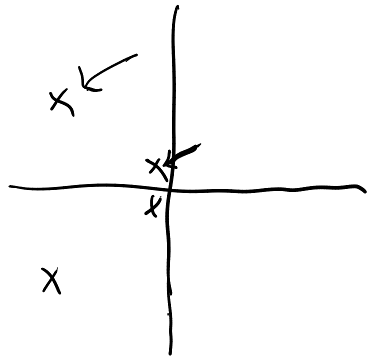
Eigenvectors Eigenvalues

$$A\vec{v}_i = \lambda_i \vec{v}_i$$
$$\vec{x}(t) = e^{At}\vec{x}(0) = \sum_{i=1}^{n} k_i e^{\lambda_i t}\vec{v}_i$$



a+bi
a-bi

$$\vec{u} = -K\vec{x}$$
$$\dot{\vec{x}} = \underbrace{(A-BK)}_{A^{cl}}\vec{x}$$



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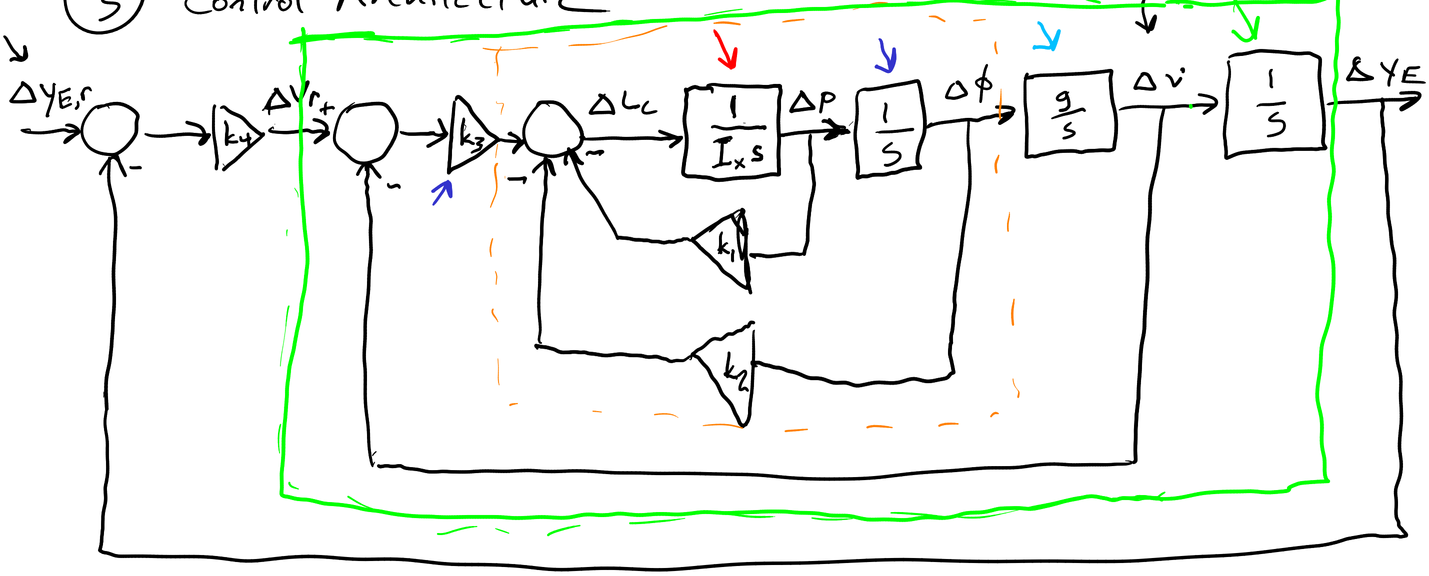
(1)
(2)

$$\underbrace{\begin{bmatrix}\Delta\dot{y}_E\\ \Delta\dot{v}\\ \Delta\dot{\phi}\\ \Delta\dot{p}\end{bmatrix}}_{\dot{\vec{x}}} = \begin{bmatrix}\Delta v\\ g\Delta\phi\\ \Delta p\\ \frac{1}{I_x}\Delta L_c\end{bmatrix} = \underbrace{\begin{bmatrix}0&1&0&0\\0&0&g&0\\0&0&0&1\\0&0&0&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\Delta y_E\\ \Delta v\\ \Delta\phi\\ \Delta p\end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix}0\\0\\0\\ \frac{1}{I_x}\end{bmatrix}}_{B}\underbrace{\Delta L_c}_{u}$$

(3) Control Architecture



(4) Choosing Gains

1) $k_1 + k_2$
2) k_4
3) k_3 Root Locus

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi - k_3 \Delta v - k_3 k_4 \Delta y_E + k_3 k_4 \Delta y_{E,r}$$

$$\begin{bmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix}}_{A^{cl}} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3 k_4}{I_x} \end{bmatrix} \Delta y_{E,r}$$

$$\rightarrow \quad \begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

$$\rightarrow \quad |A - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{2 I_x} + \sqrt{\frac{k_1^2}{4 I_x^2} - \frac{k_2}{I_x}}$$

Desired:

$\zeta = 0.7$

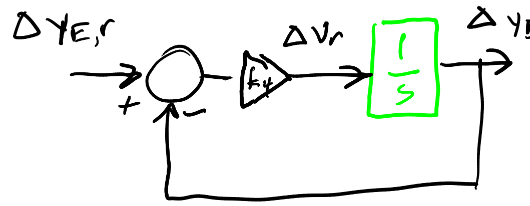
$\omega_n = 16 \text{ rad/s}$ $\quad T = \frac{2\pi}{\omega_n}$

$\frac{T}{2} = 0.2\,s$

$\lambda = -\zeta \omega_n \pm i \omega_n \sqrt{1-\zeta^2}$

$\lambda = -11.2 + 11.4 i$

$$\boxed{\begin{aligned} k_1 &= 0.0016 \\ k_2 &= 0.0179 \end{aligned}}$$



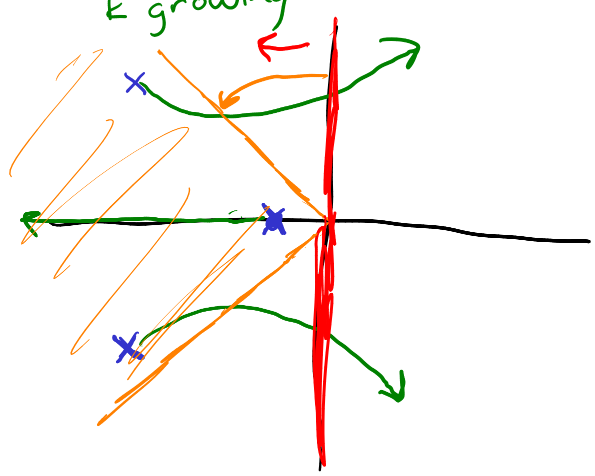
$$\Delta \dot{y}_E = k_4 (y_{E,r} - \Delta y_E)$$

$$\Delta y_E(t) = \Delta y_{E,r} \left(1 - e^{-k_4 t}\right)$$

choose k_4 1 order of magnitude slower

$$\tau = \frac{1}{k_4} > 10 \cdot \frac{2\pi}{\zeta \omega_n} = 5.6 \qquad \boxed{k_4 = 0.17}$$

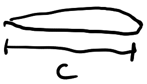
Choose k_3 via Root Locus



Root Locus: plot of the poles/eigen values of a system as a parameter (usually a gain) changes

Conventional A/C Dynamics

Longitudinal	Lateral-Directional
Altitude	Roll
Airspeed	Yaw
Pitch	Sideslip
	Turning



Lift

Drag

(Pitch Moment)

density, airspeed, wing Area

$$L = \frac{1}{2}\rho V_a^2 S C_L$$

$$D = \frac{1}{2}\rho V_a^2 S C_D$$

$$M = \frac{1}{2}\rho V_a^2 S c C_m$$

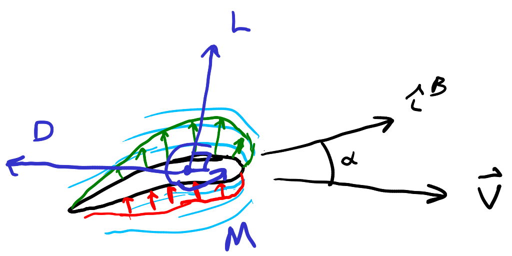
chord

$C_L(\alpha, q, \delta_e)$ nonlinear

$C_D(\alpha, q, \delta_e)$

$C_m(\alpha, q, \delta_e)$

Lift



1st order Taylor Series of $C_L(\alpha, q, \delta_e)$

$$\boxed{L = \frac{1}{2}\rho V_a^2 S\left(C_{L_0} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q}q + \frac{\partial C_L}{\partial \delta_e}\delta_e\right)}$$

$$= \frac{1}{2}\rho V_a^2 S\left(C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\underbrace{\frac{c}{2V_a}q}_{\hat{q}} + C_{L_{\delta_e}}\delta_e\right)$$

Nondimensional Stability Derivative

Nondim. control derivative

Stability Derivatives

- based linear assumptions / linearization
- Main tool connecting aerodynamics to dynamics
- Functions of A/C geometry

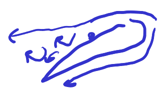
Example

Pitch Stiffness C_{m_α}

$$\ddot{\alpha} \approx \frac{k\, C_{m_\alpha}}{I}\alpha$$

$$\ddot{x} = -\frac{k_{spring}}{m}x$$

- Only valid in a linear region e.g. small α



$AR = \frac{b^2}{S}$ span

Estimated Using
- Geometric Data
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

eg. $$C_{L_\alpha} = \frac{\pi AR}{1 + \sqrt{1 + (AR/2)^2}}$$

Drag

nonlinear

α only

$$C_D(\alpha) = C_{D_p} + \frac{C_L(\alpha)^2}{\pi e AR}$$

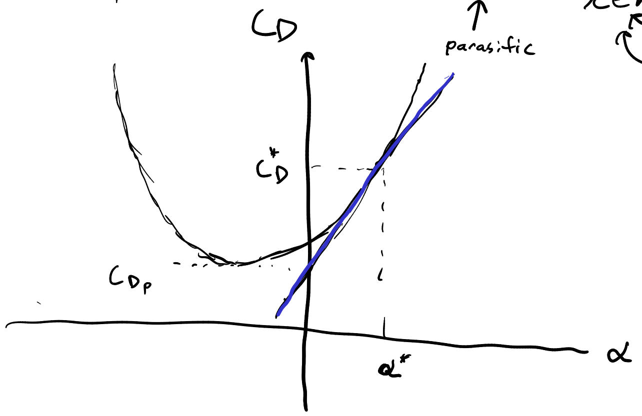
parasitic

Oswalds efficiency

induced drag

linearize

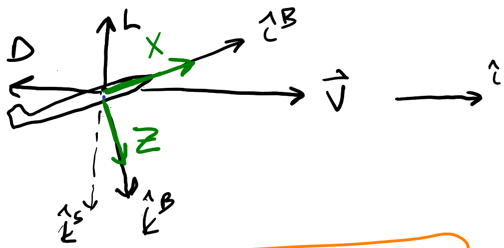
$$C_{D_0} + C_{D_\alpha}\alpha + C_{D_q}\hat{q} + C_{D_{\delta e}}\delta e$$



In this class:

$$K = \frac{1}{\pi e AR}$$

$$\boxed{C_D = C_{D_{min}} + K\left(C_L(\alpha, q, \delta e) - C_{L_{min}}\right)^2}$$



$\longrightarrow \hat{i}^S$ "stability frame"

aerodynamic forces in $\hat{i}^B$ and $\hat{k}^B$

$$\boxed{\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -D \\ -L \end{bmatrix}}$$

Pitching Moment

$$\boxed{M = \frac{1}{2}\rho V_a^2 S c \left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right]}$$

trim coefficient of moment

pitch stiffness coefficient

pitch damping coefficient

Longitudinal Trim

Last time: Long. Forces + Moments

coordinate def. $C_{L_0} \to 0$

$$L = \tfrac{1}{2}\rho V_a^2 S\left(C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\hat{q} + C_{L_{\delta e}}\delta e\right)$$

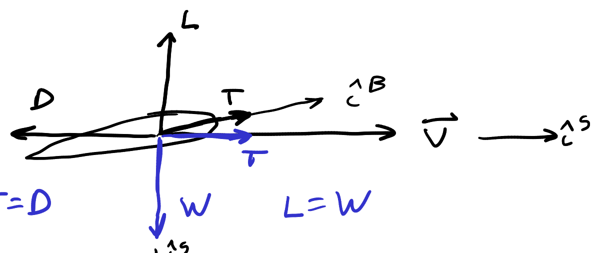
$$D = \tfrac{1}{2}\rho V_a^2 S\left(C_{D_{min}} + K\left(C_L(\alpha, q, \delta e) - C_{L_{min}}\right)^2\right)$$

$$M = \tfrac{1}{2}\rho V_a^2 S \bar{c}\left(C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right)$$

$$C_{a_b} \equiv \frac{\partial C_a}{b}$$

Trim: SLUF: Steady Level Unaccelerated Flight

just "in trim"



For linear trim calc

At trim, forces + moments about C.G. sum to 0

$\hat{i}^S$ dir: $T_{trim} = D_{trim}$

$\hat{k}^S$ dir: $L_{trim} = W$

$M_{trim} = 0$

V_a fixed

$$\cancel{C_{L_{trim}} = C_{L_\alpha}\alpha_{trim} = \frac{W}{\frac{1}{2}\rho V_a^2 S}}$$

$$\cancel{C_{m_{trim}} = C_{m_0} + C_{m_\alpha}\alpha_{trim} = 0}$$

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e = 0$$

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}\begin{bmatrix} \alpha_{trim} \\ \delta e_{trim} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta e_{trim} \end{bmatrix} = \begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}^{-1}\begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(special case of Cramer's Rule)

$$\boxed{\begin{aligned} \alpha_{trim} &= \frac{C_{m_0} C_{L_{\delta e}} + C_{m_{\delta e}} C_{L_{trim}}}{\Delta} \\ \delta e_{trim} &= -\frac{C_{m_0} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta} \end{aligned} \qquad \Delta = C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha}}$$

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$ (from slides)

$$C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m0_p} + a_t \bar{V}_H \left(\epsilon_0 + i_t\right)\left[1 - \frac{a_t S_t}{aS}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}$$

$$C_{m_\alpha} = C_{L_\alpha}\left(h - h_n\right)$$

Direct dependence on CG location h

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e}\frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}}\left(h - h_{n_{wb}}\right)$$

Some mistakes on hw

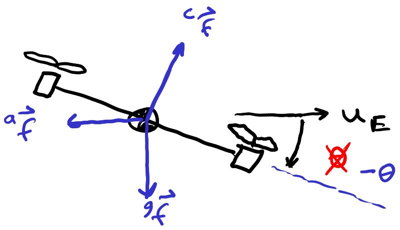
if $x(0) = \sum_i k_i \vec{v}_i$ $\qquad x(t) = \sum_i k_i e^{\lambda_i t} \vec{v}_i$

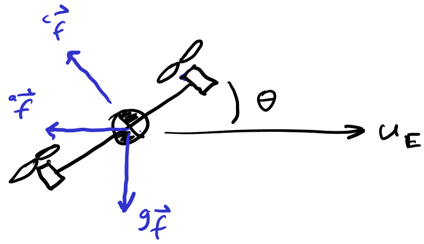
not gains

Rate of climb $= \cancel{\dot{z}_E} = -\dot{z}_E$

$\dot{h} = -\dot{z}_E$

Diagram





Linearization

2 methods

Method 1: f.o. Taylor Series

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial y}\right|_0 \Delta y$$

$$xy = x_0 y_0 + y_0 \Delta x + x_0 \Delta y$$

$$\sin(\theta) = \sin(\theta_0) + \left.\frac{\partial \sin\theta}{\partial\theta}\right|_0 \Delta\theta$$

$$= \sin(\theta_0) + \cos(\theta_0)\Delta\theta$$

may include x_0

$$\Delta\dot{x} = a\,\Delta x + b$$

Method 2

$f(x,y) \rightarrow f(x_0+\Delta x, y_0+\Delta y) \rightarrow$ expand out/cancel out $\rightarrow$ apply small disturbance approx

$$xy \rightarrow (x_0+\Delta x)(y_0+\Delta y) \rightarrow x_0 y_0 + y_0\Delta x + x_0 \Delta y + \cancel{\Delta y \Delta x}^{\,0}$$

$$\sin\theta \rightarrow \sin(\theta_0+\Delta\theta) \rightarrow \sin(\theta_0)\cancel{\cos(\Delta\theta)}^{\,1} + \cos(\theta_0)\cancel{\sin(\Delta\theta)}^{\,\Delta\theta}$$

$$\rightarrow \sin(\theta_0) + \cos(\theta_0)\Delta\theta$$

W2 Q2

$\vec{x} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ $\qquad \dot{\vec{V}}_B^E \qquad \vec{W}_E$

Longitudinal Dynamics

Last Time

$$C_L = C_{L_\alpha}\alpha + C_{L_{\delta e}}\delta e = C_{L_{trim}}$$

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta e}}\delta e = 0$$

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n) \leftarrow$$

Today: Linear Longitudinal EOM

Stability Derivatives

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

$$\dot{\vec{O}} = T\vec{\omega}_B$$

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

$$\dot{\vec{\omega}}_B = I^{-1}\left[\vec{G}_B - \vec{\omega}_x^{\times} I \vec{\omega}_B\right]$$

Symmetry about x-z axis $\qquad I_{xy} = I_{yz} = 0$

$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$\begin{pmatrix}\dot x_E\\ \dot y_E\\ \dot z_E\end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix}\begin{pmatrix}u^E\\ v^E\\ w^E\end{pmatrix}$$

$$\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix} = \begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\begin{pmatrix}\dot u^E\\ \dot v^E\\ \dot w^E\end{pmatrix} = \begin{pmatrix}rv^E - qw^E\\ pw^E - ru^E\\ qu^E - pv^E\end{pmatrix} + g\begin{pmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{pmatrix} + \frac{1}{m}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

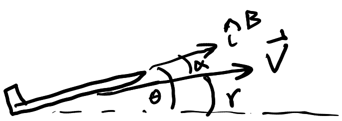
$$\begin{pmatrix}\dot p\\ \dot q\\ \dot r\end{pmatrix} = \begin{pmatrix}\Gamma_1 pq - \Gamma_2 qr\\ \Gamma_5 pr - \Gamma_6(p^2 - r^2)\\ \Gamma_7 pq - \Gamma_1 qr\end{pmatrix} + \begin{pmatrix}\Gamma_3 L + \Gamma_4 N\\ \frac{1}{I_y} M\\ \Gamma_4 L + \Gamma_8 N\end{pmatrix}$$

$V = u_0$

Trim State

Inputs $\underline{u_0}$, h_0, γ_{a_0} $\xrightarrow{\text{determine}}$ α_0, δe_0, δ_{t_0}

speed, altitude, air relative flight path angle



$$\vec{x} = \begin{bmatrix} x_E\\ y_E\\ z_E\\ \phi\\ \theta\\ \psi\\ u^E\\ v^E\\ w^E\\ p\\ q\\ r\end{bmatrix} = \vec{x}_0 + \begin{bmatrix}\Delta x_E\\ \Delta y_E\\ \Delta z_E\\ \Delta\phi\\ \Delta\theta\\ \Delta\psi\\ \Delta u^E\\ \Delta v^E\\ \Delta w^E\\ \Delta p\\ \Delta q\\ \Delta r\end{bmatrix} \qquad \vec{x}_0 = \begin{bmatrix}\cdot\\ \cdot\\ -h_0\\ 0\\ \theta_0\\ \cdot\\ u_0^E\\ 0\\ w_0^E\\ 0\\ 0\\ 0\end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix}\delta e_0\\ 0\\ 0\\ \delta_{t_0}\end{bmatrix}$$

$$\cancelto{0}{\dot\theta_0} + \Delta\dot\theta = \cancelto{0}{\cos\theta_0 q_0} + \cancelto{0}{\frac{\partial\cos\phi q}{\partial\phi}\Big|_0}\Delta\phi + \cancelto{1}{\frac{\partial\cos\phi q}{q}\Big|_0}\Delta q - \cancelto{0}{\sin\theta_0 r_0} - \cancelto{0}{\frac{\partial\sin\phi r}{\partial\phi}\Big|_0}\Delta\phi - \cancelto{0}{\frac{\partial\sin\phi r}{\partial r}\Big|_0}\Delta r$$

$$\Delta\dot\theta = \Delta q$$

$$\cancelto{0}{\dot u_0} + \Delta\dot u = \cancelto{0}{r_0 v_0} + \cancelto{0}{r_0\Delta v} + \cancelto{0}{v_0\Delta r} + \cancelto{0}{\Delta r\Delta v} - \cancelto{0}{q_0 w_0} - \cancelto{0}{q_0\Delta w} - \cancelto{0}{w_0\Delta q} - \cancelto{0}{\Delta q\Delta w}$$
$$\underbrace{-g\sin\theta_0}_{\text{cancel}} - g\cos\theta_0\Delta\theta + \frac{1}{m}(\underbrace{X_0}_{\text{cancel}} + \Delta X)$$

$$\boxed{\Delta\dot u = -g\cos\theta_0\Delta\theta + \frac{1}{m}\Delta X}$$

Longitudinal

Lat-d

$X_u \equiv \frac{\partial X}{\partial u}$

$$\begin{aligned}
\Delta\dot\phi &= \Delta p + \Delta r\tan\theta_0\\
\Delta\dot\theta &= \Delta q\\
\Delta\dot u &= -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}\\
\Delta\dot v &= -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}\\
\Delta\dot w &= u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}\\
\Delta\dot p &= \Gamma_3\Delta L + \Gamma_4\Delta N\\
\Delta\dot q &= \frac{\Delta M}{I_y}\\
\Delta\dot r &= \Gamma_4\Delta L + \Gamma_8\Delta N
\end{aligned}$$

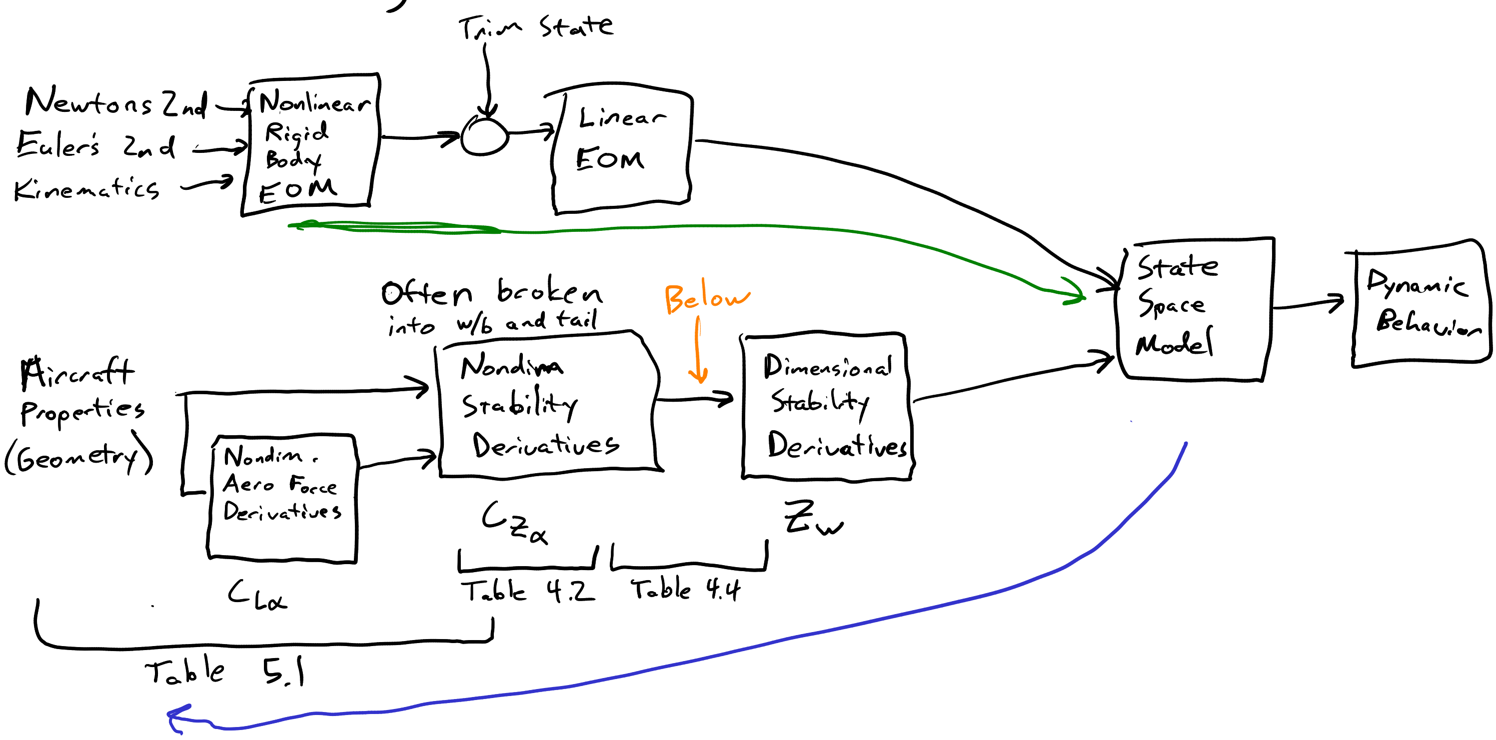
$$\begin{aligned}
\Delta X &= X_u\Delta u + X_w\Delta w + \Delta X_c\\
\Delta Y &= Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c\\
\Delta Z &= Z_u\Delta u + Z_w\Delta w + Z_{\dot w}\Delta\dot w + Z_q\Delta q + \Delta Z_c\\
\Delta L &= L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c\\
\Delta M &= M_u\Delta u + M_w\Delta w + M_{\dot w}\Delta\dot w + M_q\Delta q + \Delta M_c\\
\Delta N &= N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c
\end{aligned}$$

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot w}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot w}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot w})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot w}} & \dfrac{Z_w}{m - Z_{\dot w}} & \dfrac{Z_q + mu_0}{m - Z_{\dot w}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot w}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot w} Z_u}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot w} Z_w}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot w}(Z_q + mu_0)}{m - Z_{\dot w}}\right] & \dfrac{-M_{\dot w} mg\sin\theta_0}{I_y(m - Z_{\dot w})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Big Picture



Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 S\bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

$$C_{z_u} = \frac{\partial C_z}{\partial \hat{u}}$$

Wind-Angle Approximations

$$\Delta\alpha = \tan^{-1}\frac{\Delta w}{V} \approx \hat{w}$$

$$\Delta\beta = \sin^{-1}\frac{\Delta v}{V} \approx \hat{v}$$

Table 4.4
Longitudinal Dimensional Derivatives

	X	Z	M
u	$\rho u_0 S C_{w_0}\sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$	$\boxed{-\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}}$ Z_u	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_u}$
w	$\frac{1}{2}\rho u_0 S C_{x_\alpha}$	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar{c} S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar{c}^2 S C_{m_q}$
$\dot{w}$	$\frac{1}{4}\rho \bar{c} S C_{x_{\dot\alpha}}$	$\frac{1}{4}\rho \bar{c} S C_{z_{\dot\alpha}}$	$\frac{1}{4}\rho \bar{c}^2 S C_{m_{\dot\alpha}}$

$$Z_u \equiv \frac{\partial Z}{\partial u}\bigg|_0 \qquad Z = \frac{1}{2}\rho V^2 S C_Z$$

$$\frac{\partial Z}{\partial u}\bigg|_0 = \frac{1}{2}\rho S\left(\frac{\partial V^2}{\partial u}\bigg|_0 C_Z + \frac{\partial C_Z}{\partial u}\bigg|_0 V^2\right)$$

$$= \frac{1}{2}\rho S\, 2u_0 C_{Z_0} + \frac{1}{2}\rho u_0^2 S \frac{\partial C_Z}{\partial u}\bigg|_0$$

$$\boxed{Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{Z_u}}$$

$$\frac{\partial f(x)g(x)}{\partial x} = f(x)\frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x}g(x)$$

$$C_{Z_u} \equiv \frac{\partial C_Z}{\partial \hat{u}}$$

C_{Z_u}

$$\frac{\partial C_Z}{\partial u} = \frac{\partial C_Z}{\partial \hat{u} u_0} = \frac{1}{u_0}\frac{\partial C_Z}{\partial \hat{u}}$$

$$C_{Z_0} = -C_{w_0}\cos\theta_0$$

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot\alpha$	Neg.	$*-2a_t V_H \frac{\partial\epsilon}{\partial\alpha}$	$*-2a_t V_H \frac{l_t}{\bar{c}}\frac{\partial\epsilon}{\partial\alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{\bar{c}}$

Neg. means usually negligible.
*means contribution of the *tail only*, formula for wing-body not available.
†$C_{T_u} = \dfrac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0}$; $C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

Nondimensional Longitudinal Stability Derivatives

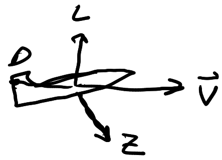
Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{\bar{c}}\frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{\bar{c}}$

Neg. means usually negligible.
*means contribution of the *tail only*, formula for wing-body not available.
†$C_{T_u} = \dfrac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0}$; $C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

α derivatives

$$C_{m_\alpha} = C_{L_\alpha}(h - h_N)$$



$C_{Z\alpha}$

$$Z = -L\cos\alpha - D\sin\alpha$$

$$C_Z = -(C_L\cos\alpha + C_D\sin\alpha)$$

$$= -(C_L + C_D\alpha)$$

$$C_{Z_\alpha} \equiv \left.\frac{\partial C_Z}{\partial \alpha}\right|_0 = -\left(C_{L_\alpha} + C_{D_0} + \cancel{\alpha_0 \frac{\partial C_D}{\partial \alpha}}\right)$$

$$\boxed{C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

u derivatives

3 important factors:

- Compressibility : Mach Number $\quad M \equiv \frac{V}{a}$ ← speed of sound
- Dynamic Pressure : $p_d = \frac{1}{2}\rho V^2$
- Thrust

- Different from the dynamic pressure in nondimensionalization
- Changes in C_L, C_D, etc. due to changes in dynamic pressure

$$C_{X_u} \equiv \left.\frac{\partial C_X}{\partial \hat{u}}\right|_0$$

$* \in \{X, Z, m\}$

$$C_{*_u} = \left.\frac{\partial C_*}{\partial M}\right|_0 \left.\frac{\partial M}{\partial \hat{u}}\right|_0 + \left.\frac{\partial C_*}{\partial p_d}\right|_0 \left.\frac{\partial p_d}{\partial \hat{u}}\right|_0 + \left.\frac{C_*}{\partial C_T}\right|_0 \left.\frac{\partial C_T}{\partial \hat{u}}\right|_0$$

→ $$\left.\frac{\partial M}{\partial \hat{u}}\right|_0 = u_0 \left.\frac{\partial M}{\partial u}\right|_0 = \frac{u_0}{a}\left.\frac{\partial V}{\partial u}\right|_0 = M_0$$

$$\left.\frac{\partial p_d}{\partial \hat{u}}\right|_0 = u_0 \left.\frac{\partial p_d}{\partial u}\right|_0 = u_0 \frac{1}{2}\rho \left.\frac{\partial V^2}{\partial u}\right|_0 = u_0 \rho\, u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2}\rho V^2 S}$$

$$\left.\frac{\partial C_T}{\partial \hat{u}}\right|_0 = u_0 \left.\frac{\partial C_T}{\partial u}\right|_0 = u_0 \left.\left(\frac{\partial T/\partial u}{\frac{1}{2}\rho V^2 S} - \frac{2T}{\frac{1}{2}\rho V^3 S}\right)\right|_0 = \left.\frac{\partial T/\partial u}{\frac{1}{2}\rho u_0 S}\right|_0 - 2 C_{T_0}$$

$$\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$C_{T_0} = C_{D_0} + C_{W_0}\sin\theta_0$$

3 cases

Gliding case $\boxed{C_{T_u} = 0}$

Constant Thrust (Jet) : $\boxed{C_{T_u} = -2C_{T_0}}$

Constant Power (Prop) : $\boxed{C_{T_u} = -3C_{T_0}}$

$TV = \text{constant}$

$$\left.\frac{\partial T}{\partial u}\right|_0 = -\frac{T_0}{u_0}$$

C_{x_u} $\qquad C_x \approx C_T - C_D$

$$\left.\frac{\partial C_x}{\partial M}\right|_0 = \left.\frac{\partial C_T}{\partial M}\right|_0 - \left.\frac{\partial C_D}{\partial M}\right|_0$$

$$\left.\frac{\partial C_x}{\partial p_d}\right|_0 = \cancel{\left.\frac{\partial C_T}{\partial p_d}\right|_0}^{\text{Neg}} - \left.\frac{\partial C_D}{\partial p_d}\right|_0$$

$$\left.\frac{\partial C_x}{\partial C_T}\right|_0 = 1 - \left.\frac{\partial C_D}{\partial C_T}\right|_0$$

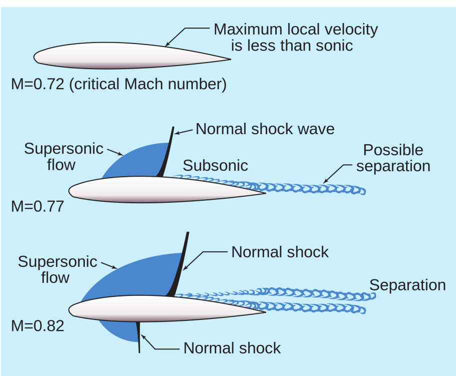
$$\boxed{C_{x_u} = M_0\left.\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right)\right|_0 - \rho u_0^2 \left.\frac{\partial C_D}{\partial p_d}\right|_0 + C_{T_u}\left(1 - \left.\frac{\partial C_D}{\partial C_T}\right|_0\right)}$$

C_{z_u} $\qquad$ Assume $\quad C_z = -C_L$

$$\boxed{C_{z_u} = -\underbrace{M_0 \left.\frac{\partial C_L}{\partial M}\right|_0}_{\text{small except transonic}} - \rho u_0^2 \left.\frac{\partial C_L}{\partial p_d}\right|_0 - C_{T_u}\left.\frac{\partial C_L}{\partial C_T}\right|_0}$$

C_{m_u}

$$\boxed{C_{m_u} = \underbrace{M_0 \left.\frac{\partial C_m}{\partial M}\right|_0}_{\text{Mach Tuck}} + \rho u_0^2 \left.\frac{\partial C_m}{\partial p_d}\right|_0 + C_{T_u}\left.\frac{\partial C_m}{\partial C_T}\right|_0}$$



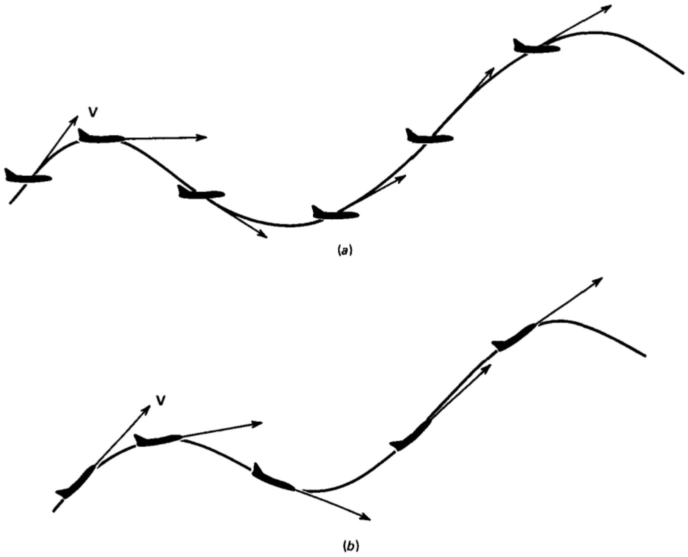


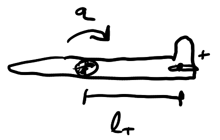
Figure 5.2 (*a*) Motion with zero q, but varying α_x. (*b*) Motion with zero α_x but varying q.

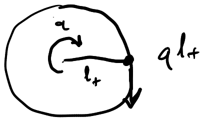
q - derivatives

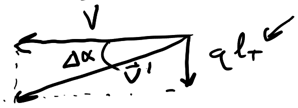
Wing-body || Tail

Tail

velocity observed by tail







$$\Delta C_{L_t} = a_t \Delta\alpha = a_t \tan^{-1}\frac{q l_t}{u_0} \approx a_t \frac{q l_t}{u_0}$$

$$\Delta C_L = \frac{S_t}{S}\Delta C_{L_t} = \frac{S_t}{S} a_t \frac{q l_t}{u_0}$$

$(C_{z_q})_{tail}$

$$C_{z_q} = \frac{\partial C_z}{\partial \hat{q}}\Big|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_z}{\partial q}\Big|_0 = -\frac{2u_0}{\bar{c}}\frac{\partial C_L}{\partial q}\Big|_0$$

$$(C_{z_q})_{tail} = \frac{-2u_0}{\bar{c}} a_t \frac{S_t l_t}{S u_0} = \boxed{-2 a_t V_H}$$

$$V_H = \frac{S_t l_t}{S\bar{c}}$$

$(C_{m_q})_{tail}$

$$\Delta C_m = -V_H \Delta C_{L_t} = a_t V_H \frac{q l_t}{u_0}$$

$$C_{mq} = \frac{\partial C_m}{\partial \hat{q}}\Big|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_m}{\partial q}\Big|_0$$

$$\boxed{(C_{m_q})_{tail} = -2 a_t V_H \frac{l_t}{\bar{c}}}$$

Wing - Body

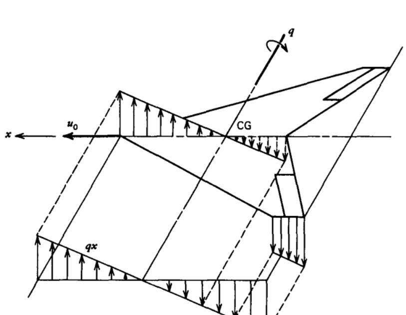


Figure 5.4 Wing velocity distribution due to pitching.

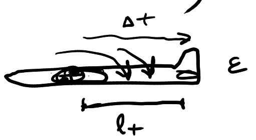
$\dot{\alpha}$ derivatives

Unsteady effects

Wing - Body $\rightarrow$ Determined by initial response or oscillation of wing in wind tunnel or flight test

Tail

Downwash Lag



$$\Delta\varepsilon = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\,\Delta t = -\frac{\partial\varepsilon}{\partial\alpha}\dot{\alpha}\frac{l_t}{u_0}$$
$$= -\Delta\alpha_t$$

$(C_{Z_{\dot{\alpha}}})_{tail}$

$$\Delta C_{L_t} = a_t \Delta\alpha_t = a_t \dot{\alpha}\frac{l_t}{u_0}\frac{\partial\varepsilon}{\partial\alpha}$$
$$\rightarrow \Delta C_L = a_t \dot{\alpha}\frac{l_t S_t}{u_0 S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$C_{Z_{\dot{\alpha}}} = \left.\frac{\partial C_Z}{\partial \frac{\dot{\alpha}\bar{c}}{2u_0}}\right|_0 = \frac{2u_0}{\bar{c}}\frac{\partial C_Z}{\partial\dot{\alpha}} = -2a_t\frac{l_t S_t}{\bar{c} S}\frac{\partial\varepsilon}{\partial\alpha}$$

$$\boxed{(C_{Z_{\dot{\alpha}}})_{tail} = -2a_t V_H \frac{\partial\varepsilon}{\partial\alpha}}$$

$$\boxed{(C_{m_{\dot{\alpha}}})_{tail} = -2a_t V_H \frac{l_t}{\bar{c}}\frac{\partial\varepsilon}{\partial\alpha}}$$

Longitudinal Modes

$$x(t) = \sum_i k_i \vec{v}_i e^{\lambda_i t}$$

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot w}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot w}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot w})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot w}} & \dfrac{Z_w}{m - Z_{\dot w}} & \dfrac{Z_q + mu_0}{m - Z_{\dot w}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot w}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot w}Z_u}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot w}Z_w}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot w}(Z_q + mu_0)}{m - Z_{\dot w}}\right] & \dfrac{-M_{\dot w} mg\sin\theta_0}{I_y(m - Z_{\dot w})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$h_0 = 40k$ ft

$V_0 = 774$ ft/s

$\gamma_0 = \theta_0 = \alpha_0 = 0$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\omega_n = \sqrt{a^2 + b^2}$

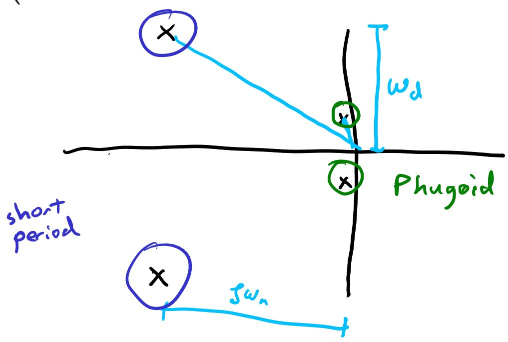
$$\lambda_{1,2} = \overset{a}{-0.37} \overset{\pm}{\pm} \overset{bi}{0.89i}$$

$\omega_n = 0.96, \ \zeta = 0.38$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$\omega_n = 0.067$

$\zeta = 0.049$



Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I) v_i = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \qquad \lambda_1 = -1, \ \lambda_2 = -2$$

$$(A - \lambda_1 I) v_1 = 0$$

$$\rightarrow \boxed{\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}} \boxed{v_1} = 0$$

let $v_1[1] = 1$

$v_1[2] = -1$

$$\boxed{v_1[1] + v_1[2] = 0}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.06i \\ 0.9996 \\ -0.0001 \pm 0.001i \\ 0.0011 \mp 0.0004i \end{bmatrix} \begin{array}{l} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{array} \qquad \vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097 \\ -0.0001 \mp 0.0001i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

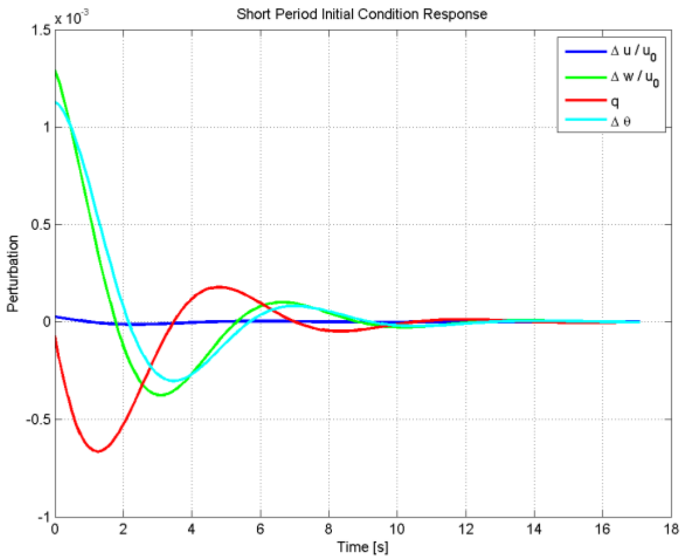
$$x(t) = \sum_i k_i \, \vec{v}_i e^{\lambda_i t}$$

$$x(0) = Re(\vec{v}_1) = \sum_i k_i \vec{v}_i \cancel{e^{\lambda_i 0}} = 0.5\vec{v}_1 + 0.5\vec{v}_2$$

$$x(t) = 0.5\vec{v}_1 e^{\lambda_1 t} + 0.5\,\vec{v}_2 e^{\lambda_2 t}$$

$$\lambda_{1/2} = -.372 + .888i$$
$$\zeta = 0.387$$
$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



$$\vec{v}_{1,2} / \vec{v}_{1,2}[4] = \vec{v}_{1,2}' = \begin{bmatrix} 0.02 \pm 0.06i / \quad '' \quad '' \\ 0.9996 / \quad '' \quad '' \\ -0.0001 \pm 0.001i / 0.0011 - 0.0011i \\ 0.0011 \mp 0.0004i / 0.0011 - 0.0004i \end{bmatrix} = \begin{bmatrix} \\ \\ \\ 1.0 \end{bmatrix}$$

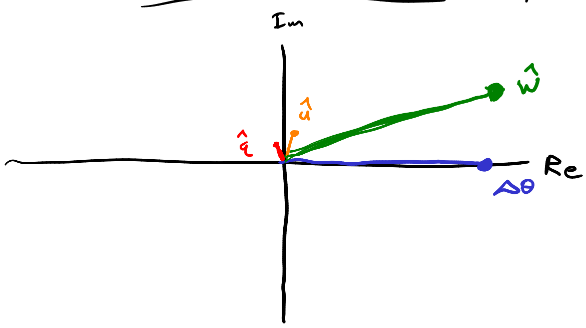
$$z_1 = r_1 e^{i\phi_1} = r_1 \angle \phi_1$$
$$z_2 = r_2 e^{i\phi_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{(i\phi_1 - i\phi_2)}$$

$$\hat{v}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.0066 \pm 0.016i \\ 1.0 \end{bmatrix}$$

$$\hat{u} = \frac{\Delta u}{u_0}$$
$$\hat{w} = \frac{\Delta w}{u_0} \approx \alpha$$
$$\hat{q} = \frac{\Delta q \bar{c}}{2u_0}$$
$$\Delta\theta$$

Phasor Plot ($\hat{v}_1$, Short Period)



$\vec{v}$ $\begin{cases} \alpha \text{ and } \Delta\theta \text{ change in-phase} \\ \hat{u} \text{ relatively constant} \end{cases}$

λ $\begin{cases} \text{well-damped} \\ \text{high frequency} \end{cases}$

Phugoid Mode

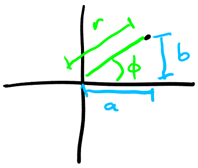
phugoid = "flight"

side note:

$z = a + bi$

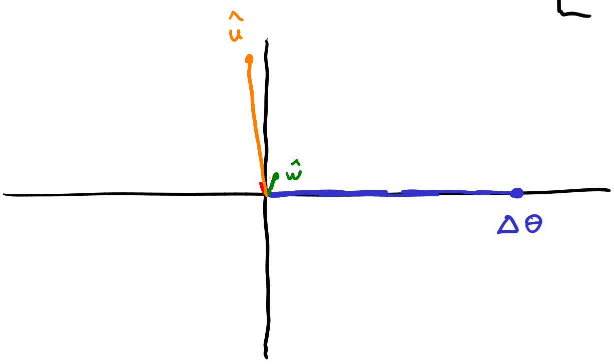
$= r \angle \phi$

$= re^{i\phi}$



$r = \sqrt{a^2+b^2}\ ;\ \phi = \text{atan2}(b, a)$

$$\hat{v}_{3,4} = \begin{array}{cc} \begin{bmatrix} 0.62 \angle 92^\circ \\ 0.036 \angle 83^\circ \\ 0.0012 \angle 93^\circ \\ 1.0 \angle 0 \end{bmatrix} & \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ \hat{q} \\ \Delta\theta \end{array} \end{array}$$



$\vec{v}$ { Large $\hat{u}$, $\Delta\theta$ out-of phase oscillations (offset ~90°); small α }

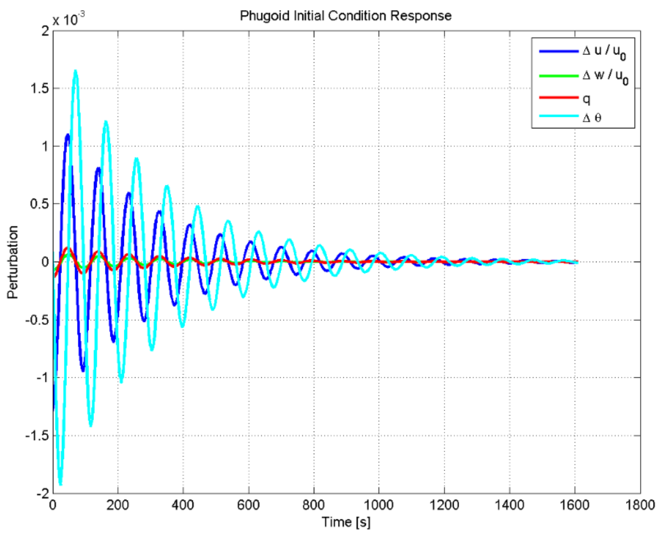
λ { low frequency; low damping }

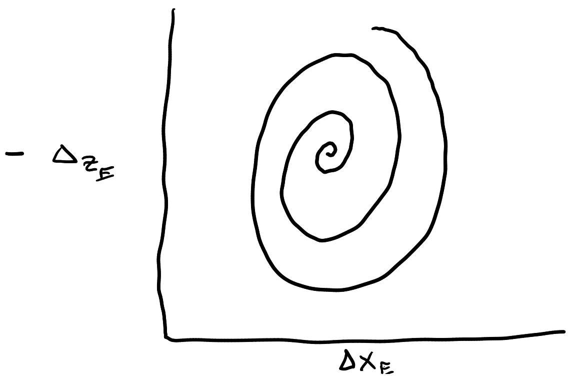
$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$

$\zeta = 0.0489$ ⟵ poorly damped

$\omega_n = 0.0673$ ⟵ slow response

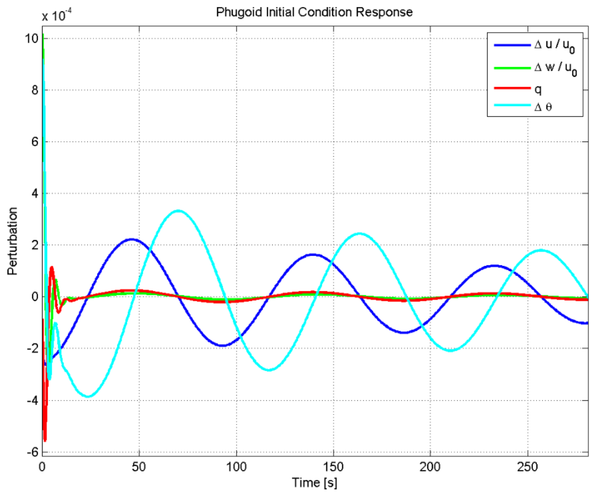
$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$





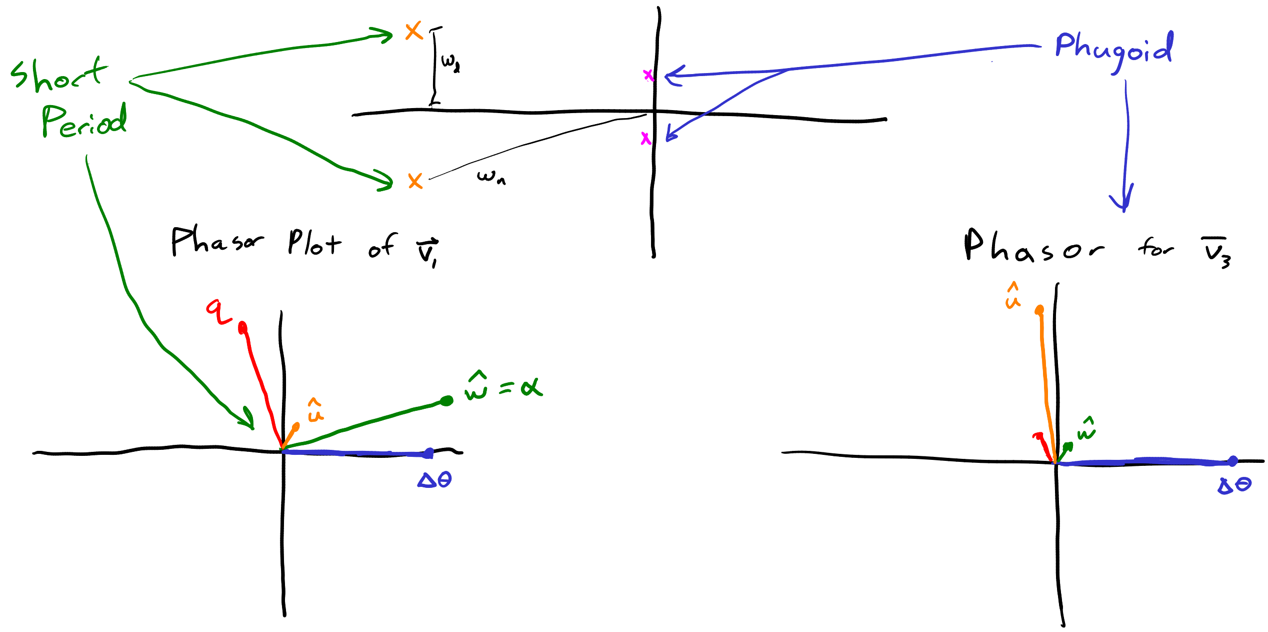
General Response

$$\mathbf{x}_0 = 0.8 \cdot Re(\mathbf{v}_{sp}) + 0.2 \cdot Re(\mathbf{v}_{ph})$$



Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon}}_{\text{Eigen}} \vec{x}_{lon} + \vec{c}_{lon} \qquad \vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$



Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Assume: $\Delta u = 0$
$\theta_0 = 0$
$Z_{\dot{w}} << m$
$Z_q << m u_0$

If we also assume no vertical motion, $\theta_0 = 0$ implies $\Delta\theta = \alpha \approx \frac{\Delta w}{u_0}$

$$\boxed{\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{Z_w}{m} & u_0 \\ \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m}\right] & \dfrac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}\left[M_q + M_{\dot{w}}u_0\right]\right]}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

Dimensional Stab. Deriv.	Nondim. Stab. Deriv.	A/C Params
$Z_w \equiv \left.\frac{\partial Z}{\partial w}\right\|_0 = \frac{1}{2}\rho u_0 S C_{z_\alpha}$	C_{z_α}	$C_{z_\alpha} = -C_{D_0} - C_{L_\alpha}$
M_w	C_{m_α}	$= C_{L_\alpha}(h - h_n)$
$M_{\dot{w}}$	$C_{m_{\dot\alpha}}$	...
M_q	C_{m_q}	$-2 a_t V_H \frac{l_t}{\bar c}$

How accurate is this approximation?

For 747 @ cruise

Full A_{lon}

$\lambda_{1,2} = -0.372 \pm 0.888i$

$\zeta = 0.387$

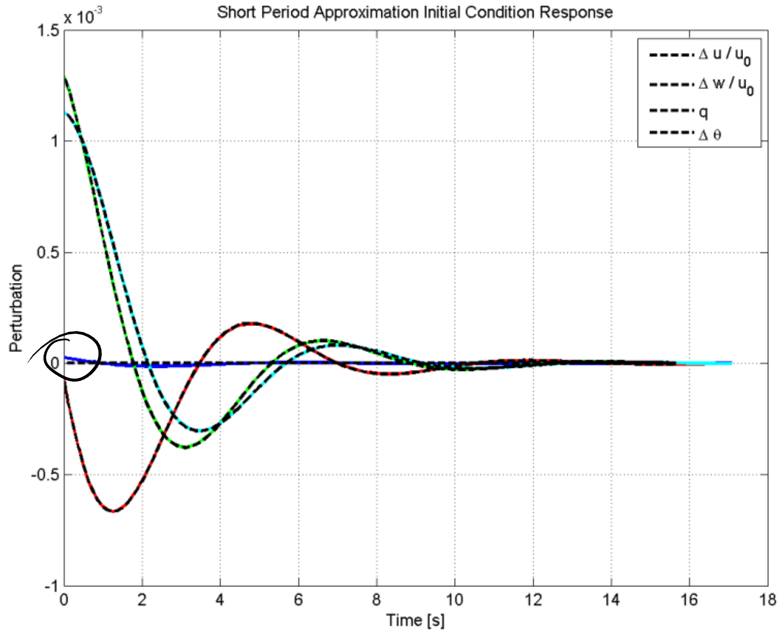
$\omega_n = 0.962$

S.P. approx

$\lambda_{sp} = -0.371 \pm .889i$

$\zeta = 0.385$

$\omega_n = 0.963$



(note: here since $\Delta\theta \neq \frac{\Delta w}{u_0}$, there is some vertical motion)

Phugoid Mode

Lanchester (1908) Assume conservation of energy

$$E = \frac{1}{2} m V^2 - m g \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g\Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho u_0^2 S C_{W_0} + \rho g S C_{W_0} \Delta z_E = W + \rho g S C_{W_0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m\Delta\ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m\Delta\ddot{z}_E$$

$$\Delta\ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m}}_{\omega_n^2} \Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi\sqrt{2}\,\frac{u_0}{g}} = \begin{matrix} 0.138\,u_0 \leftarrow \text{if } u_0 \text{ in ft/s} \\ 0.453\,u_0 \leftarrow \text{if } u_0 \text{ in m/s} \end{matrix}$$

for 747

Full A_{lon}

$T = 93\,s$

Lanchester

$T = 107\,s$

"2×2" Phugoid Approx

Dynamics of Flight, Eq. (4.9,18) $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cancel{\cos\theta_0}^{\,1} \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\cancel{\sin\theta_0}^{\,0}}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

Assume $Z_{\dot{w}} \ll m$

$Z_q \ll mu_0$

$\Delta\dot{q}$ small

$\Delta\alpha = 0$

$\theta_0 = 0$

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\alpha} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m}\Delta u + u_0\Delta q \longrightarrow \Delta q = -\frac{Z_u}{mu_0}\Delta u$$

$$\boxed{\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}}$$

$$\lambda^2 - \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{Z_u g}{mu_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0}\cancel{\cos\theta_0}^{\,1} + \tfrac{1}{2}\rho u_0 S C_{Z_u}$$

$$C_{Z_u} = \underbrace{-M_0\frac{\partial C_L}{\partial M}\Big|_0}_{\text{small}} - \underbrace{\rho u_0^2\frac{\partial C_L}{\partial p_d}\Big|_0}_{\text{small}} - \underbrace{C_{T_u}\frac{\partial C_L}{\partial C_T}\Big|_0}_{\text{small}} \qquad -C_{T_u}\frac{\partial C_X}{\partial C_T}\Big|_0$$

$C_{Z_u} = 0$

$$Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{w_0} \cancel{\sin\theta_0}^{\,0} + \frac{1}{2}\rho u_0 S C_{x_u}$$

$$C_{x_u} = -2C_{T_0} \qquad \text{(constant thrust)}$$

$$C_{T_0} = C_{D_0} + C_{w_0}\cancel{\sin\theta_0}^{\,0}$$

$$X_u \simeq -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}} \qquad \text{Same as Lanchester}$$

$$\zeta = -\frac{X_u}{2}\sqrt{\frac{-u_0}{m Z_u g}} = \rho\frac{u_0 S C_{D_0}}{2}\sqrt{\frac{u_0}{\frac{1}{2} m g \rho u_0 S C_{L_0}}}$$

$$= C_{D_0}\sqrt{\frac{\frac{1}{4}\rho u_0^2 S}{\frac{1}{2} m g}\frac{1}{C_{L_0}}}$$

$$\zeta = \frac{C_{D_0}}{C_{L_0}} \qquad \text{High } L/D = \text{less energy loss} = \text{less damping}$$

747

Full A_{lon}

$\lambda_{3,4} = -3.29\times10^{-3} \pm 6.72\times10^{-2} i$

$\zeta = 0.0489$

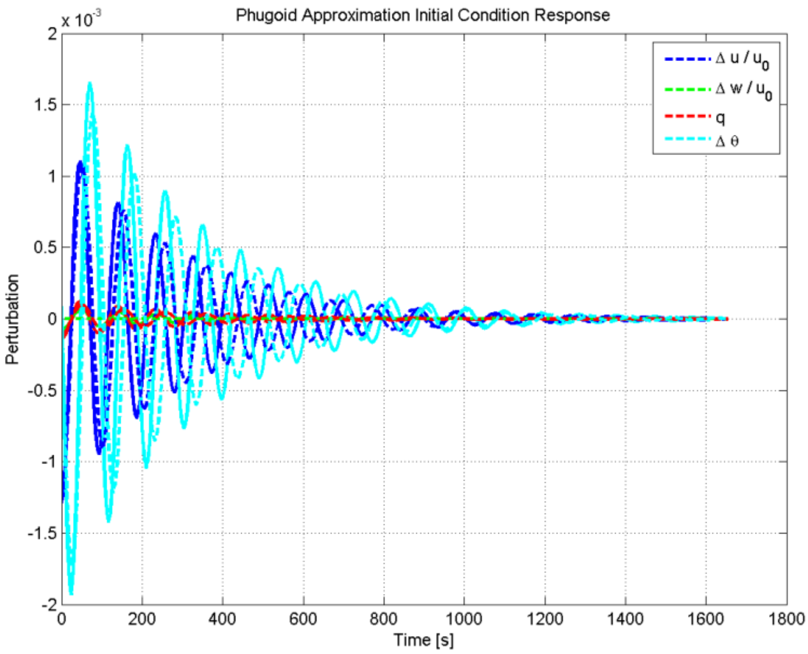
$\omega_n = 0.0673$

Ph Approx

$\lambda_{ph} = -3.43\times10^{-3} \pm 6.11\times10^{-2} i$

$\zeta = 0.0561$

$\omega_n = 0.0612$

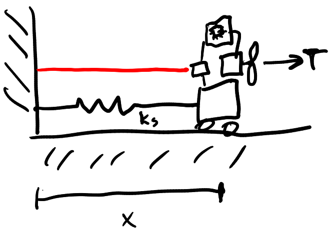


Longitudinal Control

State Space Review

$T_0 = 0$
$T = \Delta T$

$k_s \Leftrightarrow C_{m_\alpha}$



$$^{s}f = -k_s(x - x_0)$$
$$m\ddot{x} = -k_s(x - x_0)$$
$$m\Delta\ddot{x} = -k_s\Delta x + \Delta T$$

$$\vec{u} = [\Delta T]$$

$$\rightarrow \quad \dot{\vec{x}} = \begin{bmatrix}\Delta\dot{x}\\ \Delta\ddot{x}\end{bmatrix} = \overset{A}{\begin{bmatrix}0 & 1\\ -\frac{k_s}{m} & 0\end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix}\Delta x\\ \Delta\dot{x}\end{bmatrix}} + \overset{B}{\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}}\vec{u}$$

ΔX_E, ΔY_E, ΔZ_E

u, v, w

$$\Delta x = \overset{C}{\begin{bmatrix}1 & 0\end{bmatrix}} \begin{bmatrix}\Delta x\\ \Delta\dot{x}\end{bmatrix} + \overset{D}{[0]}\vec{u}$$

$$|A - \lambda I| = 0 = \begin{vmatrix}-\lambda & 1\\ -\frac{k_s}{m} & -\lambda\end{vmatrix} = \lambda^2 + \frac{k_s}{m} = 0$$

$$\lambda^2 + \underbrace{2\zeta\omega_n}_{0}\lambda + \underbrace{\omega_n^2}_{\frac{k_s}{m}} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \sqrt{\frac{k_s}{m}}\,i$$

$$\Delta T = -k_p\Delta x - k_d\Delta\dot{x}$$

$$\vec{u} = -K\vec{x} \qquad K = [k_p \quad k_d] \qquad \vec{u} = [-k_p \quad -k_d]\vec{x}$$

$$\dot{\vec{x}} = A\vec{x} - BK\vec{x}$$
$$\dot{\vec{x}} = \underbrace{(A - BK)}_{A^{cl}}\vec{x}$$

$$A^{cl} = \begin{bmatrix}0 & 1\\ -\frac{(k_s + k_p)}{m} & -\frac{k_d}{m}\end{bmatrix}$$

Longitudinal Control

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix}\dfrac{\Delta X_c}{m}\\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}}\\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})}\\ 0\end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix}\dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0\\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}}\\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})}\\ 0 & 0 & 1 & 0\end{pmatrix}$$

δ_e = elevator + downward deflection

δ_t = throttle + more power

δ_p in book

dimensional control derivatives

$$\Delta X_c = X_{\delta_e}\delta_e + X_{\delta_t}\delta_t$$

$$\Delta Z_c = Z_{\delta_e}\delta_e + \boxed{Z_{\delta_t}}\delta_t$$

often zero

$$\Delta M_c = M_{\delta_e}\delta_e + \boxed{M_{\delta_t}}\delta_t$$

$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + B_{lon}\vec{u}_{lon}$$

$$B_{lon} = \begin{bmatrix} \frac{X_{\delta_e}}{m} & \frac{X_{\delta_t}}{m} \\ \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta_t}}{m-Z_{\dot{w}}} \\ \frac{M_{\delta_e}}{I_y}+\frac{M_{\dot{w}}}{I_y}\frac{Z_{\delta_e}}{(m-Z_{\dot{w}})} & \frac{M_{\delta_t}}{I_y}+\frac{M_{\dot{w}}Z_{\delta_e}}{I_y(m-Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix} \qquad \vec{u}_{lon} = \begin{bmatrix}\delta_e \\ \delta_t\end{bmatrix}$$

747

$$B_{lon} = \begin{bmatrix} -0.000187 & 9.66 \\ -17.85 & 0 \\ -1.158 & 0 \\ 0 & 0 \end{bmatrix}$$

γ

$$\gamma = \underbrace{\begin{bmatrix} 0 & \frac{-1}{774} & 0 & 1 \end{bmatrix}}_{C_\gamma} \begin{bmatrix}\Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta\end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{D_\gamma}\begin{bmatrix}\delta_e \\ \delta_t\end{bmatrix}$$

$\gamma = \Delta\theta - \Delta\alpha \longrightarrow C_\gamma$

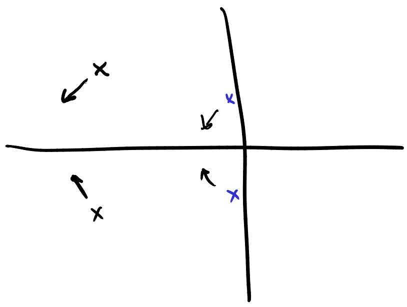
$$\delta_e = 10^\circ$$

$$\delta_t = \frac{1}{6} \approx 0.05\,W$$

Longitudinal Stability Augmentation

B 747

$$A_{sp} = \begin{bmatrix} -0.3151 & 773.98 \\ -0.0010 & -0.4285 \end{bmatrix}$$



$$|A-\lambda I| = (-0.3151-\lambda)(-0.4285-\lambda) - (-0.0010)(773.98)$$

$$\lambda = -0.372 \pm 0.889i$$

$$\omega_n = 0.964$$

$$\zeta = 0.386 \quad \times \quad 0.7$$

$$B_{sp} = \begin{bmatrix} -17.85 & 0 \\ -1.158 & 0 \end{bmatrix}$$

measured by rate gyro

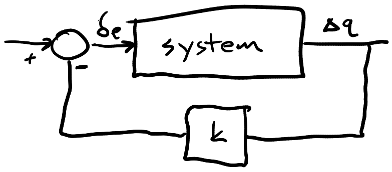
$$\delta_e = -k_s\Delta q$$

$$\begin{bmatrix}\delta_e \\ \delta_t\end{bmatrix} = -\underbrace{\begin{bmatrix} 0 & k_s \\ 0 & 0 \end{bmatrix}}_{K_{sp}} \underbrace{\begin{bmatrix}\Delta w \\ \Delta q\end{bmatrix}}_{\vec{x}_{sp}}$$

$$A^{cl} = A_{sp} - B_{sp}K_{sp} = A_{sp} - \begin{bmatrix} 0 & -17.85k_s \\ 0 & -1.158k_s \end{bmatrix}$$

$$= \begin{bmatrix} -0.3151 & 773.98 + 17.85k_s \\ -0.0010 & -0.4285 + 1.158k_s \end{bmatrix}$$

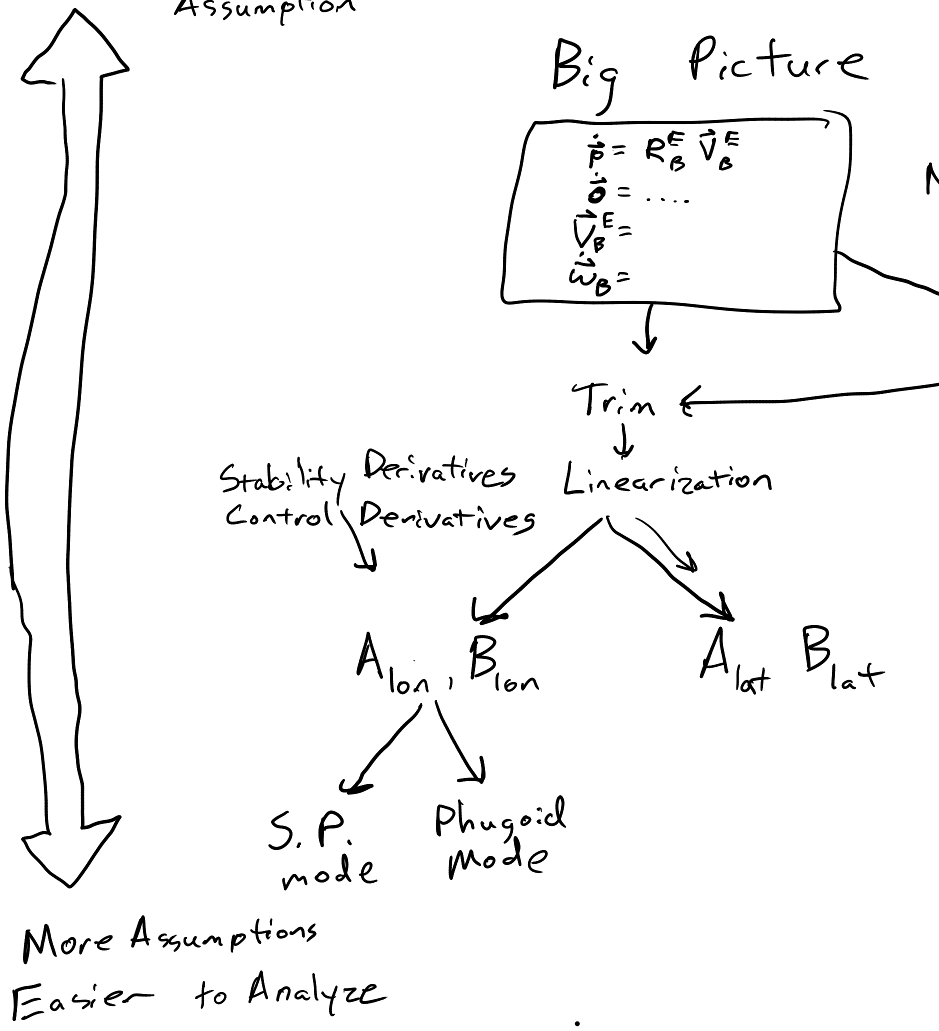
Matlab rlocus



$C_{\Delta q} = [0 \ 1]$ $D_{\Delta q} = [0]$

$B_{\delta e} = \begin{bmatrix} -17.85 \\ -1.158 \end{bmatrix}$

Lateral / Directional Dynamics



$$C_{L_{trim}} = C_{L_\alpha} \alpha_{trim} + C_{L_{\delta_e}} \delta_{e_{trim}}$$

$$C_{m_{trim}} = C_{m_0} + C_{m_\alpha} \alpha_{trim} + C_{m_{\delta_e}} \delta_{e_{trim}}$$

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{u} = -g\cos\theta_0 \Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0 \Delta r + g\cos\theta_0 \Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0 \Delta q - g\sin\theta_0 \Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

$$Y = \frac{1}{2}\rho V^2 S C_y(\beta, p, r, \delta_a, \delta_r)$$
$$L = \frac{1}{2}\rho V^2 S b C_l(\beta, p, r, \delta_a, \delta_r)$$
$$N = \frac{1}{2}\rho V^2 S b C_n(\beta, p, r, \delta_a, \delta_r)$$

$$\hat{p} = \frac{pb}{2u_0}$$

$$Y \approx \frac{1}{2}\rho V_a^2 S\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$
$$L \approx \frac{1}{2}\rho V_a^2 Sb\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right]$$
$$N \approx \frac{1}{2}\rho V_a^2 Sb\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right]$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

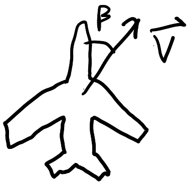
$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3\Delta L_c + \Gamma_4\Delta N_c \\ \Gamma_4\Delta L_c + \Gamma_8\Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$
$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$
$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

Sideslip: β

$+\beta \Rightarrow$ wind coming from right



$$\beta = \sin^{-1}\frac{\Delta V}{V}$$
$$\beta \approx \frac{\Delta V}{u_0} = \hat{V}$$

$$L = \frac{1}{2}\rho V^2 S b C_\ell$$

$$C_{\ell_\beta} \equiv \frac{\partial C_\ell}{\partial \beta} = \frac{\partial C_\ell}{\partial \hat{v}}$$

$$L_v \equiv \frac{\partial L}{\partial v}\Big|_0 = \frac{1}{2}\rho u_0^2 S b \frac{\partial C_\ell}{\partial v}\Big|_0$$

$$= \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial \beta}\Big|_0 \frac{\partial \beta}{\partial v}\Big|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0}$$

$$\frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

$$\boxed{L_v = \frac{1}{2}\rho u_0 b S C_{\ell_\beta}}$$

$$C_{\ell_p} \equiv \frac{\partial C_\ell}{\partial \hat{p}}$$

$$\hat{p} = \frac{bp}{2u_0}$$

$$L_p \equiv \frac{\partial L}{\partial p}\Big|_0 = \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial p}\Big|_0$$

$$= \frac{1}{2}\rho u_0^2 b S \frac{\partial C_\ell}{\partial \hat{p}}\Big|_0 \frac{\partial \hat{p}}{\partial p}\Big|_0$$

$$\boxed{L_p = \frac{1}{4}\rho u_0 b^2 S C_{\ell_p}}$$

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$

Table 5.2
Summary—Lateral Derivatives

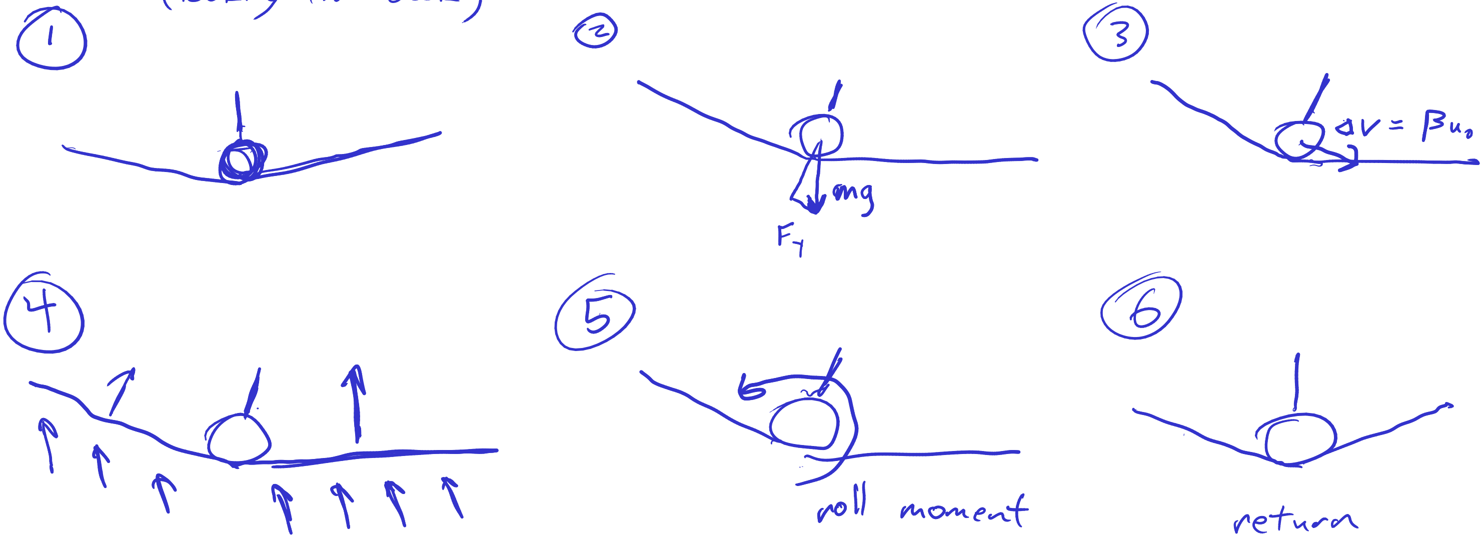
	C_y	C_l	C_n
β	$*-a_F \frac{S_F}{S}\left(1 - \frac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V \left(1 - \frac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F \frac{S_F}{S}\left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V \left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F \frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F \frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V \left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

C_{l_β} : Dihedral Effect

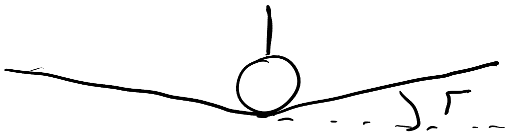
sign : negative

(looking from back)

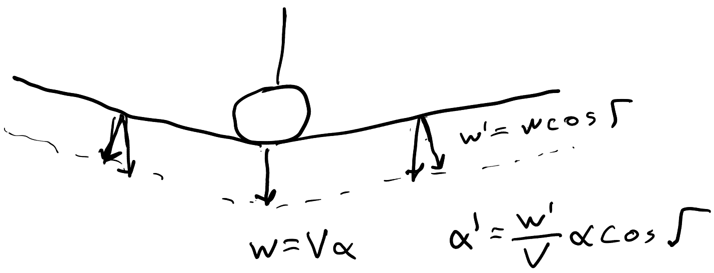


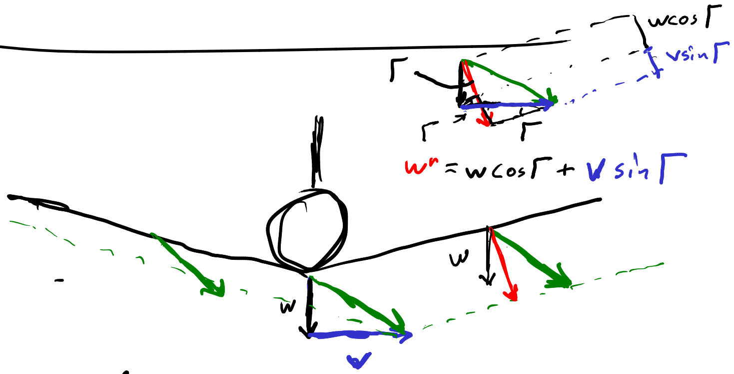
4 significant factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail



1. Dihedral Angle





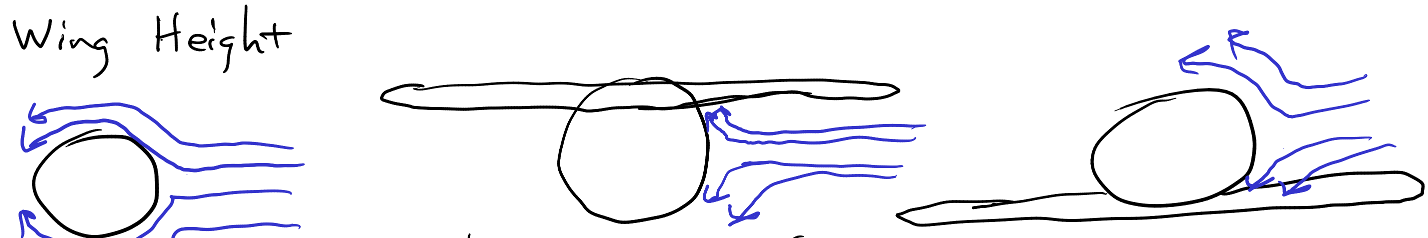
$\alpha^l = \alpha - \Gamma\beta$

$w^r = w + v\Gamma$

$\alpha^r = \alpha + \Gamma\beta$

$$\boxed{C_{l_\beta}^{\Gamma} \propto \Gamma}$$

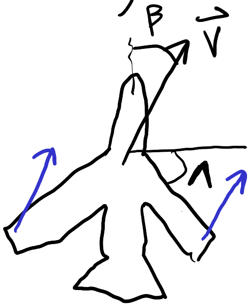
2. Wing Height



high wing $\Rightarrow -C_{l_\beta}$

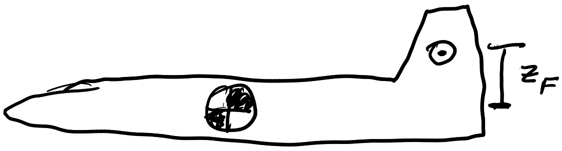
low wing $\Rightarrow +C_{l_\beta}$

3. Wing Sweep



$$C_{l_\beta}^{\Lambda} \propto 2 C_L V^2 \sin 2\Lambda$$

4. Tail



$$\Delta C_l^F = C_{L_F}\frac{S_F z_F}{Sb} = a_F(-\beta+\sigma)\frac{S_F z_F}{Sb}$$

$$C_{l_\beta}^F = -a_F\left(1-\frac{\partial\sigma}{\partial\beta}\right)\frac{S_F z_F}{Sb}\left(\frac{V_F}{V}\right)^2$$

Lateral Stability Derivatives
Sideslip, Coordinated Turn

Table 5.2
Summary—Lateral Derivatives

	C_y	C_l	C_n
β	$*-a_F \frac{S_F}{S}\left(1-\frac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V\left(1-\frac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F \frac{S_F}{S}\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F \frac{S_F}{S}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F \frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

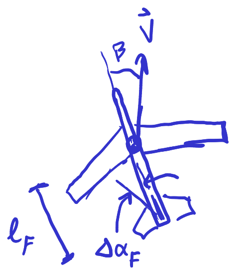
β derivatives

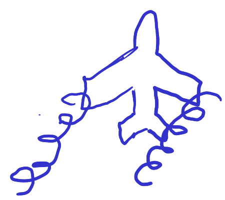
C_{l_β}: Dihedral Effect

1) Wing Height
2) Dihedral Angle
3) Vertical Tail
4) Wing Sweep

C_{n_β}: Weathervane Derivative Sign? (+)

Main Component: Vertical Tail





$$\alpha_F = -\beta + \sigma \quad \leftarrow \text{sidewash}$$

$$\rightarrow C_{L_F} = a_F(-\beta+\sigma)$$

$$N_F = -C_{L_F}\tfrac{1}{2}\rho V_F^2 S_F l_F$$

$$C_{n_F} = -C_{L_F}\frac{S_F}{S}\frac{l_F}{b}\left(\frac{V_F}{V}\right)^2$$

$$= -V_V C_{L_F}\left(\frac{V_F}{V}\right)^2$$

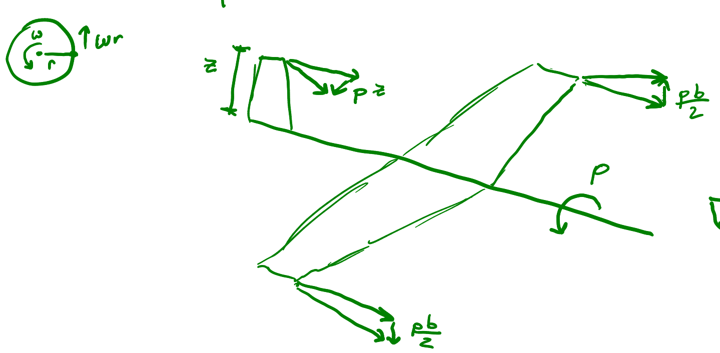
$$V_V \equiv \frac{S_F l_F}{S b}$$

$$C_{n_\beta} \approx \left.\frac{\partial C_{n_F}}{\partial\beta}\right|_0 = -V_V\left(\frac{V_F}{V}\right)^2\left.\frac{\partial C_{L_F}}{\partial\beta}\right|_0 = \boxed{V_V a_F\left(\frac{V_F}{V}\right)^2\left(1-\frac{\partial\sigma}{\partial\beta}\right)}$$

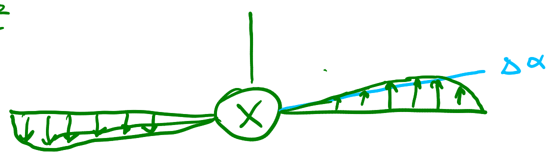
C_{y_β} (usually small) (similar derivation to C_{n_β})

p derivatives

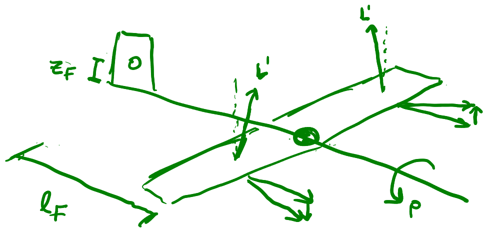
C_{l_p} roll damping derivative (-)



(From rear of aircraft)



C_{n_p} Wing Effects



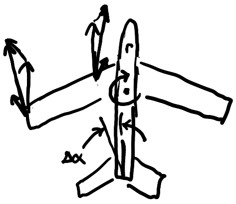
Tail Effect $\Delta\alpha_F = -\frac{p z_F}{u_0} + p\frac{\partial\sigma}{\partial p} = -\hat{p}\left(2\frac{z_F}{b} - \frac{\partial\sigma}{\partial\hat{p}}\right)$

$$(\Delta C_n)_{tail} = -\Delta C_{Y_F}\frac{S_F}{S}\frac{l_F}{b} = a_F V_V \hat{p}\left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

$$(C_{n_p})_{tail} = a_F V_V\left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

C_{y_p} (usually small) (similar derivation to $(C_{n_p})_{tail}$)

<u>r-derivatives</u>



$$\Delta\alpha_F = \frac{r l_F}{u_0} + r\frac{\partial\sigma}{\partial r}$$
$$= \hat{r}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

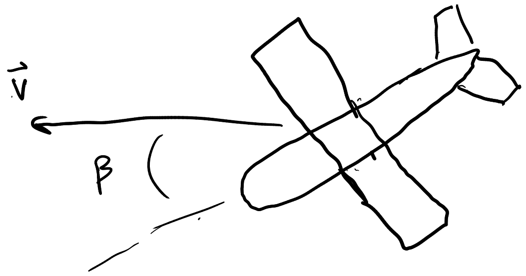
$$(C_{y_r})_{tail} = a_F\frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$(C_{l_r})_{tail} = a_F\frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$(C_{n_r})_{tail} = -a_F V_V\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right) \quad \leftarrow \text{yaw damping derivative}$$

2 steady flight conditions (not "level")

Sideslip



$$\Delta Y + mg\,\Delta\phi = 0$$

$$\Delta L = 0$$
$$\Delta N = 0$$

can incorporate 1-engine out

$$\Delta Y = Y_v \Delta v + \cancel{Y_p p}^{\,0} + \cancel{Y_r r}^{\,0} + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r$$

$$\underbrace{\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix}}_{\text{invert}} \begin{bmatrix} \delta_r \\ \delta_a \\ \Delta\phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} \Delta v \quad \leftarrow \beta u_0$$

Steady Sideslip

For Piper Cherokee

inv

$$\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix} \cancel{v}\ \beta u_0 \qquad (7.8,4)$$

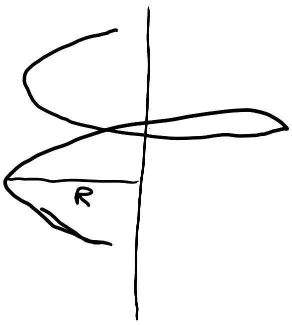
It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta \doteq v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

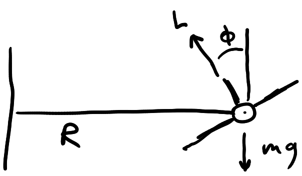
$$\delta_r/\beta = .303$$
$$\delta_a/\beta = -2.96$$
$$\phi/\beta = .104$$

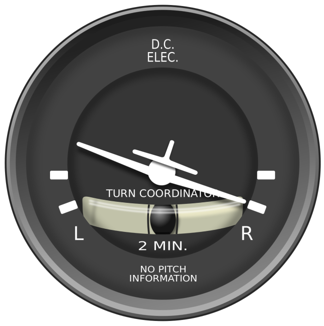
We see that a positive sideslip (to the right) of say 10° would entail $\boxed{\text{left rudder of 3°}}$ and $\boxed{\text{right aileron of 29.6°}}$. Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The $\boxed{\text{bank angle is seen to be only 1°}}$ to the right so the sideslip is almost flat.

Coordinated Turn

- angular velocity vector is constant and aligned with inertial z
- No aerodynamic forces in Y direction







$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\Theta \\ \sin\phi\cos\Theta \\ \cos\phi\cos\Theta \end{bmatrix}\omega$$

Since there is increased need for lift due to bank angle, involves both lat and lon dynamics

$$L\cos\phi = mg\cos\theta$$

$$L\sin\phi = m\frac{u_0^2}{R}$$

$$= m\omega u_0$$

$$\boxed{\tan\phi} = \frac{L\sin\phi}{L\cos\phi} = \frac{m\omega u_0}{mg\cos\theta} = \boxed{\frac{\omega u_0}{g\cos\theta}}$$

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$x(t) = \sum_i c_i e^{\lambda_i t} \vec{v}_i$$

c_i determined by initial condition

λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02 + 9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02 - 9.47e-01i$	$3.49e-02$	$9.47e-01$

$$\begin{array}{c} \\ \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{array} \quad \begin{array}{c} \mathbf{v}_1 \\ \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathbf{v}_2 \\ \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \end{array} \quad \begin{array}{c} \mathbf{v}_{3/4} \\ \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix} \end{array}$$

$\Delta\psi$

Δy_E

Flight Path State Space Dynamics Matrix (also exists for lon)

Δz_E

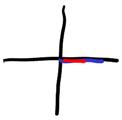
Δx_E

$$\rightarrow \quad \Delta\dot{\psi} = \Delta r \sec\theta_0$$

$$\Delta\dot{y}_E = u_0 \cos\theta_0 \underline{\Delta\psi} + \underline{\Delta v}$$

$$\begin{bmatrix} \Delta\dot{v} \\ \vdots \\ \vdots \end{bmatrix} = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{array}\right] \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta y_E \end{bmatrix}$$

Spiral Mode



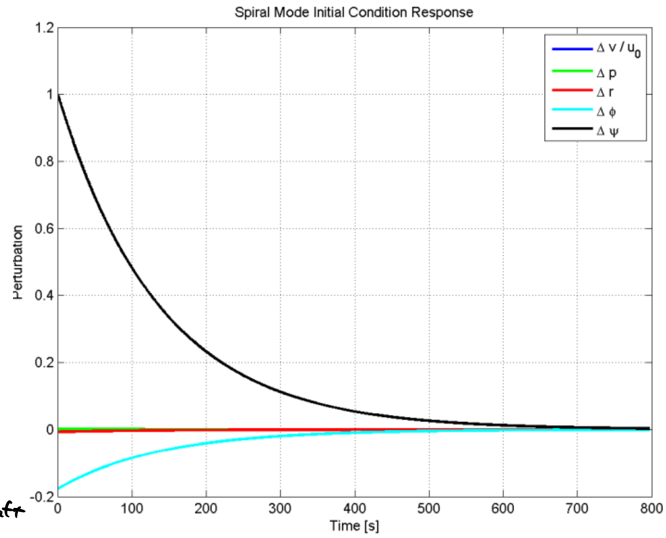
$$\tilde{V}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix} \begin{array}{ll} \hat{v} = \beta & \leftarrow \text{small} \\ p & \leftarrow \text{small} \\ r & \leftarrow \text{some} \\ \phi & \leftarrow \text{large} \\ \psi & \leftarrow \text{large} \end{array}$$

nondimensionalize velocities, normalize so that $\Delta\psi = 1$

$\lambda_1 = -0.0073$ ← stable for 747

$\tau = 137$ s

unstable for many aircraft

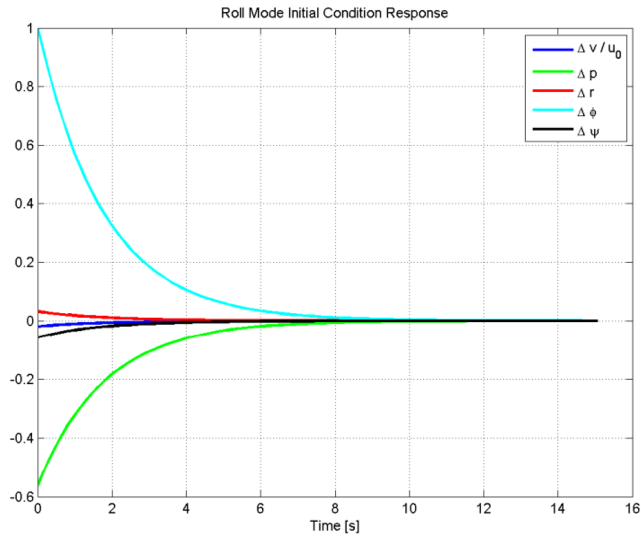


Roll

$$\tilde{V}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \begin{array}{ll} \hat{v} = \beta & \leftarrow \text{small} \\ p & \leftarrow \text{large} \\ r & \leftarrow \text{small} \\ \phi & \leftarrow \text{large} \\ \psi & \leftarrow \text{small} \end{array}$$

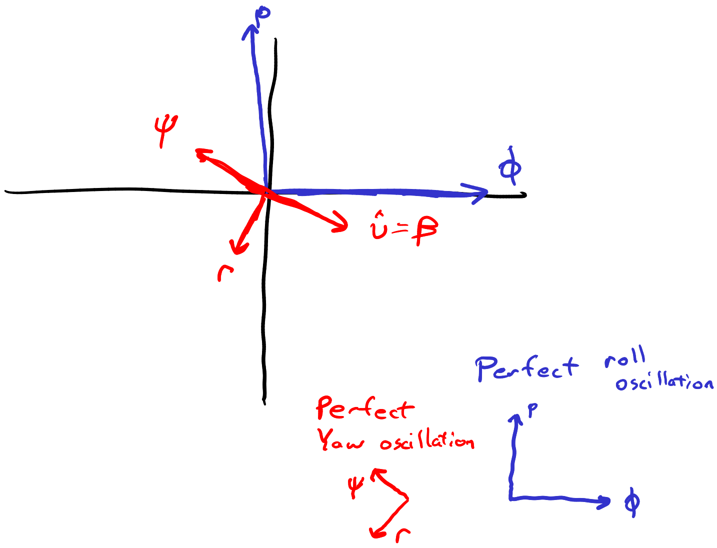
$\lambda_2 = -0.5625$

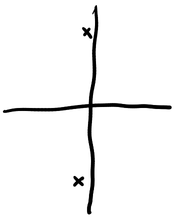
$\tau = 1.78$ s



Dutch Roll

$$\tilde{V}_3 = \begin{bmatrix} 0.3271 \angle -28^\circ \\ 0.9471 \angle 92^\circ \\ 0.2915 \angle -112.3^\circ \\ 1.0 \\ 0.3078 \angle 155^\circ \end{bmatrix} \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$





$\lambda_{3/4} = -0.033 \pm 0.947i$

$\zeta = 0.0349$ ← not well-damped

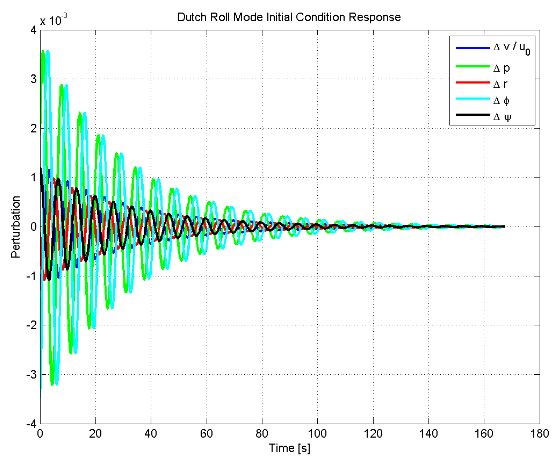
$\omega_n = 0.947$ ← fast

Phugoid (for comparison)

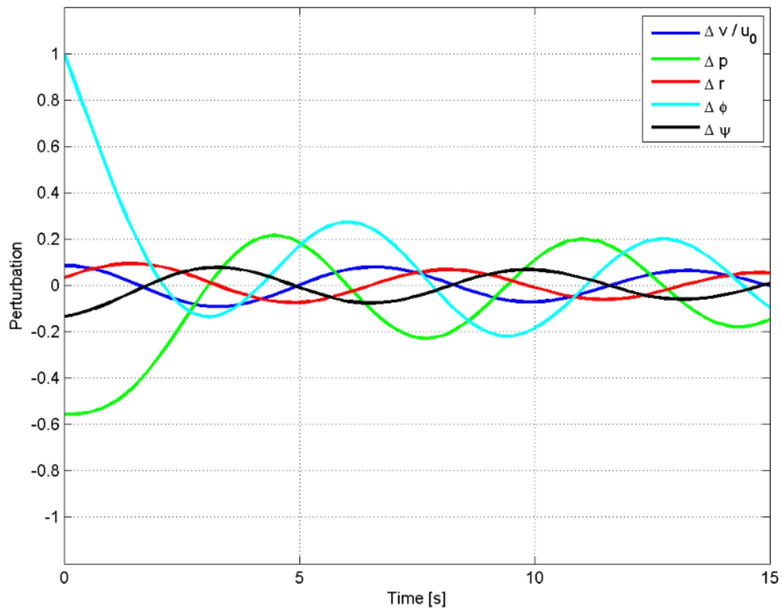
$\lambda = -0.00329 \pm 0.0672i$

$\zeta = 0.0489$

$\omega_n = 0.0673$



$$\mathbf{x}(0) = 0.4 \cdot Re(\mathbf{v}_r) + 0.4 \cdot Re(\mathbf{v}_{dr}) + 0.2 \cdot Re(\mathbf{v}_{spi})$$



Review: All modes

Name	Primary Variables	Fast / Slow	Damping
Short Period	α	Fast	Well-damped
Phugoid	speed / altitude	Slow	Poorly
Roll	Roll rate	Fast	(over)
Spiral	Yaw, Roll	Slow	(over) / unstable
Dutch Roll	β, Roll	Fast	Poorly

Trust the mathematical properties of A over your intuition!