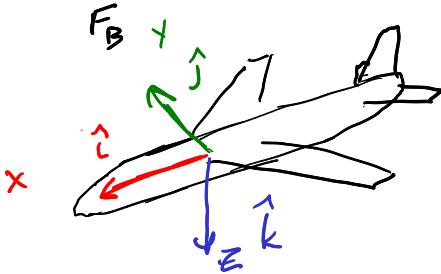
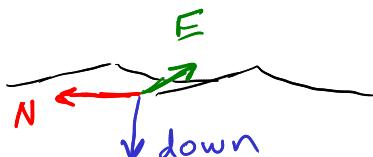


## Notation and Conventions

$F_I$  or  $F_E$



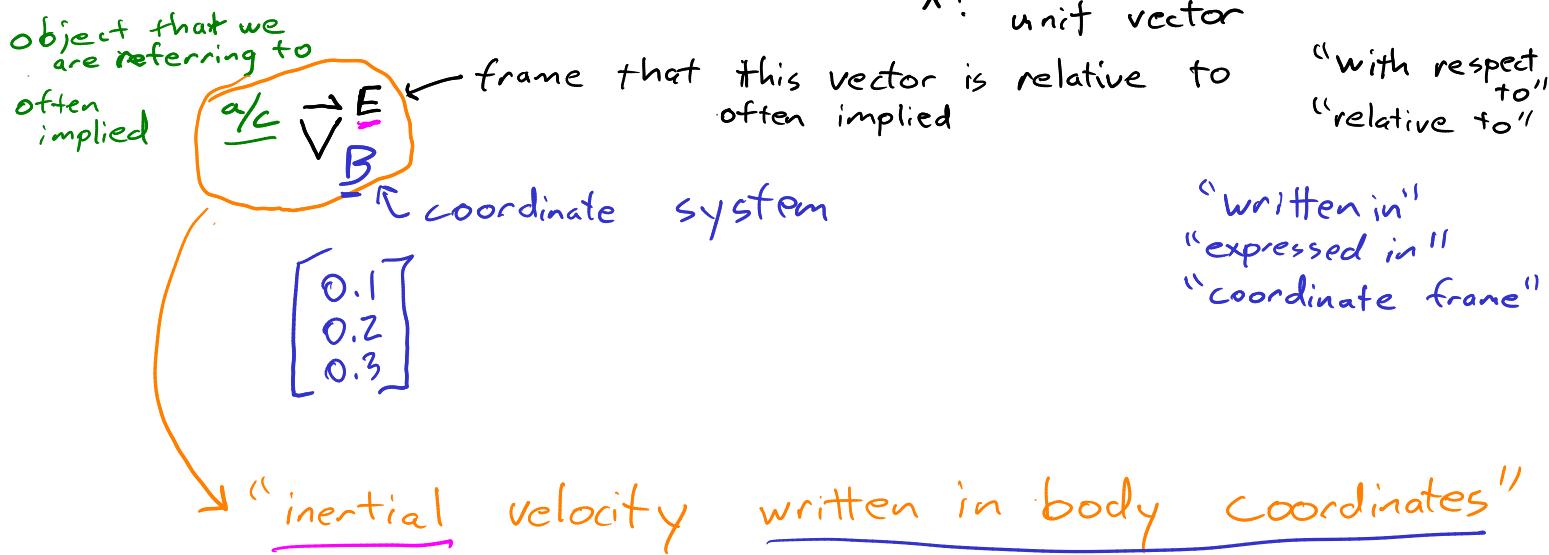
- Frame of reference: Collection of  $\geq 3$  points w/ constant distance between each other
- Inertial frame: A frame that translates with constant (possibly 0) velocity and does not rotate
  - Newton's second law is valid in an inertial frame
- Coordinate System: Three orthogonal unit vectors that allow measurement and vector representation

- Vector Notation

$\rightarrow$  or bold : vector

$^{\wedge}$  : unit vector

"with respect to"  
"relative to"



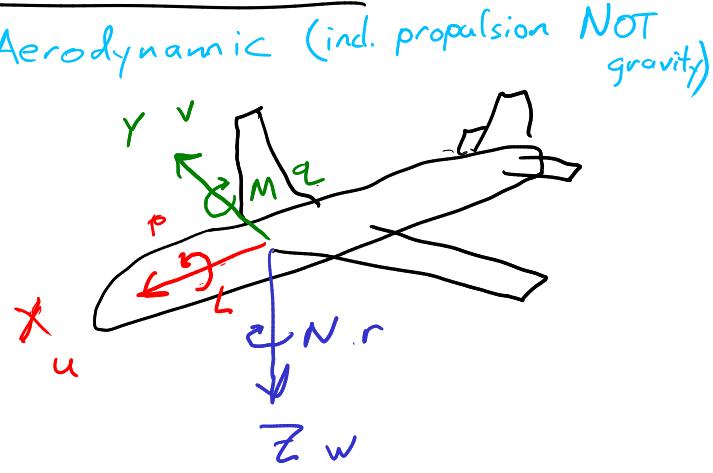
$\vec{v}_B^W$

$\vec{v}_E^E$

# Forces, Moments, and Velocities

$$\vec{A} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

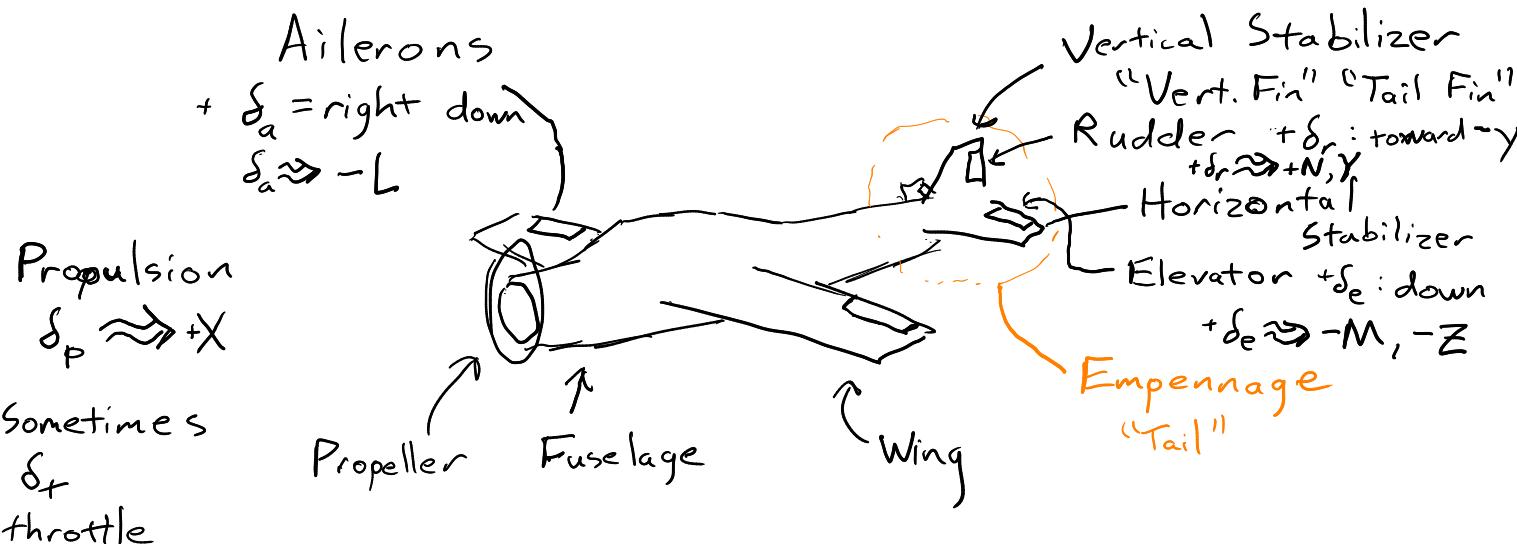
$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$V_g = |\vec{V}^E| = \sqrt{u^{E2} + v^{E2} + w^{E2}}$$

## Anatomy of an Airplane aka. "Conventional A/C"



## Wind

Aerodynamic Forces and Moment (i.e. not gravity) are functions of the A/C velocity w.r.t. the Air

$$\vec{V}^W$$

by convention

$$\vec{V} \equiv \vec{V}^W$$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\boxed{\vec{V}^E = \vec{V}^{(w)} + \vec{W}^{(E)}}$$

$$V = |\vec{V}|$$

$$\text{when } \vec{W} = \vec{0}, \vec{V} = \vec{V}^E$$

## Example

Wind blowing east @ 8m/s

A/C pointing north and traveling northward with respect to the wind at 60m/s

What is the A/C velocity vector relative to earth in NED

$$\begin{aligned}\vec{V}_E^E &= \vec{V}_E + \vec{W}_E \\ &= \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 8 \\ 0 \end{bmatrix}\end{aligned}$$

~~$\vec{V}_B + \vec{W}_E$~~   
Illegal!

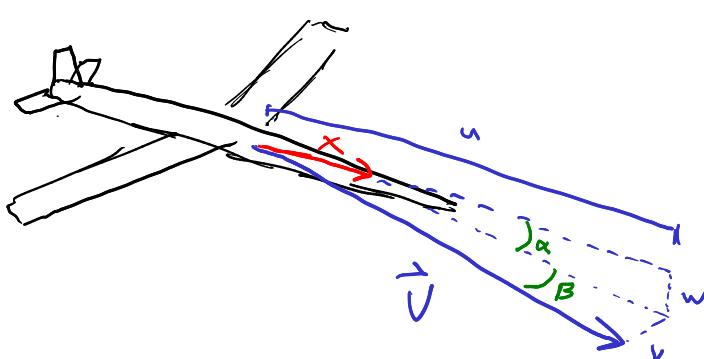
## Wind Angles

Angle of Attack

$$\alpha = \tan\left(\frac{w}{u}\right)$$

Sideslip Angle

$$\beta = \sin^{-1} \frac{v}{V}$$



$$\begin{aligned}u &= V \cos \beta \cos \alpha \\ v &= V \sin \beta \\ w &= V \cos \beta \sin \alpha\end{aligned}$$

## Orientation

axis

- " 1"  $\Phi$ : roll
- " 2"  $\Theta$ : pitch
- " 3"  $\Psi$ : yaw

E.G.  $45^\circ$

$90^\circ$

$45^\circ$

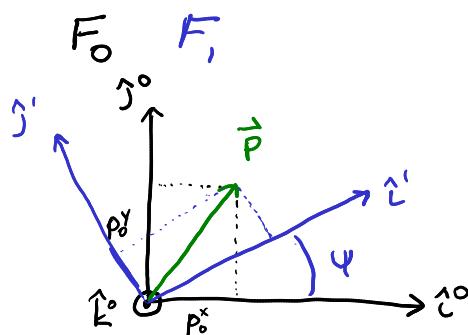
By convention A/C orientation is defined by a 3-2-1 sequence of rotations through  $\Psi-\Theta-\Phi$

Key task: changing the coordinate sys. that a vector is written in

Ex.

know:  $\vec{V}_B^E$

want:  $\vec{V}_E^E$



$$\vec{p}_0 = \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$$\vec{p}_1 = \begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix}$$

$$\vec{p} = p_0^x \hat{i}^0 + p_0^y \hat{j}^0 + p_0^z \hat{k}^0$$

$F_1$  is related to  $F_0$  by a 3-rotation through  $\Psi$

want  $p_1^x$  in terms of  $\vec{p}$

$$\begin{aligned} p_1^x &= \vec{p} \cdot \hat{i}^1 \\ &= p_0^x \hat{i}^0 \cdot \hat{i}^1 + p_0^y \hat{j}^0 \cdot \hat{i}^1 + p_0^z \hat{k}^0 \cdot \hat{i}^1 \\ &= p_0^x \cos \Psi + p_0^y \sin \Psi + p_0^z \cdot 0 \end{aligned}$$

$$p_1^x = [\cos \Psi \quad \sin \Psi \quad 0] \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$R_E^B$   $R_B^E$

$$R_1^1(\Psi) = \begin{bmatrix} \hat{i}^1 & \hat{j}^1 & \hat{k}^1 \\ \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \\ \hat{i}^0 \cdot \hat{j}^1 & \hat{j}^0 \cdot \hat{j}^1 & \hat{k}^0 \cdot \hat{j}^1 \\ \hat{i}^0 \cdot \hat{k}^1 & \hat{j}^0 \cdot \hat{k}^1 & \hat{k}^0 \cdot \hat{k}^1 \end{bmatrix}$$

$$\begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix} = \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$$\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$$

$R_3(\Psi)$   
c angle  
axis

$$R_1(\Phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

$$R_2(\Theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

# Properties of Direction Cosine Matrices (DCMs)

$$\vec{p}_B = R_A^B \vec{p}_A$$

Book:  $\vec{p}_B = L_{BA} \vec{p}_A$

## 1. Chaining

$$\begin{aligned}\vec{p}_C &= R_B^C \vec{p}_B \\ &= R_B^C R_A^B \vec{p}_A \\ &\quad \boxed{R_A^C = R_B^C R_A^B}\end{aligned}$$

## 2. Inverse

$$\begin{aligned}\vec{p}_B &= R_A^B \vec{p}_A \\ (R_A^B)^{-1} \vec{p}_B &= (R_A^B)^{-1} R_A^B \vec{p}_A\end{aligned}$$

$$R_B^A \vec{p}_B = \vec{p}_A$$

$$R_B^A = (R_A^B)^{-1}$$

Since DCMs are orthonormal  
(columns are orthogonal unit vectors)

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_A^B (R_A^B)^T = I$$

$$(R_A^B)^{-1} = (R_A^B)^T$$

$$\boxed{R_B^A = R_A^B{}^T}$$

## Earth-to-body DCM

3-2-1 rotation through  $\psi, \theta, \phi$

$$\begin{aligned}\vec{p}_B &= R_1(\phi) R_2(\theta) R_3(\psi) \vec{p}_E \\ \boxed{\vec{p}_B = R_E^B \vec{p}_E}\end{aligned}$$

$$\begin{cases} c_\theta = \cos \theta \\ s_\theta = \sin \theta \end{cases}$$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

## Example:

Want: pilot  $\vec{p}_E$

know  $\vec{p}_E^E$ , pilot  $\vec{p}_B^B$ ,  $R_E^B$

$$\begin{aligned}\text{pilot } \vec{p}_E &= \vec{p}_E^E + \text{pilot } \vec{p}_B \\ \text{pilot } \vec{p}_E &= \vec{p}_E^E + \text{pilot } \vec{p}_B \times \boxed{(R_E^B)^T}\end{aligned}$$

# Kinematics

Kinematics: "Geometry of motion" (no forces)

Dynamics/Kinetics: Effects of forces and moments on an object

## Vector derivatives

$\dot{\vec{p}}$ : shorthand  $\dot{\vec{p}}^E$

$\frac{d}{dt} \vec{p} \equiv$  time rate of change of  $\vec{p}$  ← "Vector derivative"

$$\vec{v}^E \equiv \frac{d}{dt} \vec{p}$$

$\dot{\vec{p}}_B \equiv$  time rate of change of elements of  $\vec{p}_B$   
"coordinates"

if  $\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$  then  $\dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

Question

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \quad ? \quad \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix} \leftarrow$$

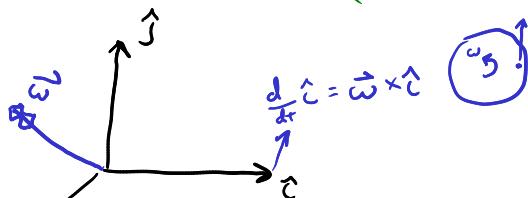
Not always true

$$\vec{p}_B = x_B \hat{i}_B + y_B \hat{j}_B + z_B \hat{k}_B$$

$$\rightarrow \dot{\vec{p}}_B = \dot{x}_B \hat{i}_B + \dot{y}_B \hat{j}_B + \dot{z}_B \hat{k}_B$$

$$\left( \frac{d}{dt} \vec{p} \right)_B = \dot{x}_B \hat{i}_B + \frac{x_B \frac{d}{dt} \hat{i}_B}{\cancel{+}} + \dot{y}_B \hat{j}_B + \frac{y_B \frac{d}{dt} \hat{j}_B}{\cancel{+}} + \dot{z}_B \hat{k}_B + \frac{z_B \frac{d}{dt} \hat{k}_B}{\cancel{+}}$$

what is  $\left( \frac{d}{dt} \hat{i} \right)_B$



$$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\text{Green terms} = x_B (\vec{\omega}_B \times \hat{i}_B) + y_B (\vec{\omega}_B \times \hat{j}_B) + z_B (\vec{\omega}_B \times \hat{k}_B) \\ = \vec{\omega}_B \times \vec{p}_B$$

$$\boxed{\left( \frac{d}{dt} \vec{p} \right)_B = \dot{\vec{p}}_B + \vec{\omega}_B \times \vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

"Kinematic Transport Theorem"

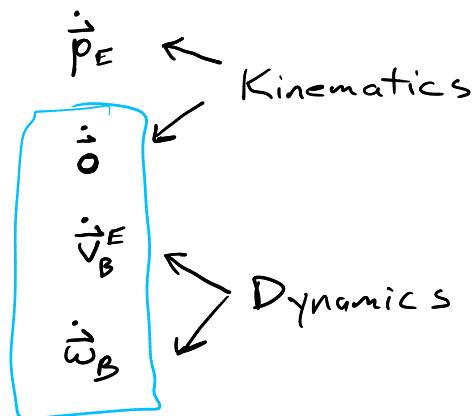
$$\left( \frac{d}{dt} \vec{p} \right)_E = \dot{\vec{p}}_E + \vec{\omega}_E \times \vec{p}_E = \dot{\vec{p}}_E$$

# Aircraft Equations of Motion (EOM)

$$\dot{\vec{x}} = f(+, \vec{x}, \vec{u})$$

$$\begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \quad \left\{ \begin{array}{l} \vec{p}_E = \vec{p}_E^E \\ \vec{o} \text{ "pseudo-vector" array of numbers} \\ \vec{v}_B^E \\ \vec{\omega}_B^E \end{array} \right.$$

Need



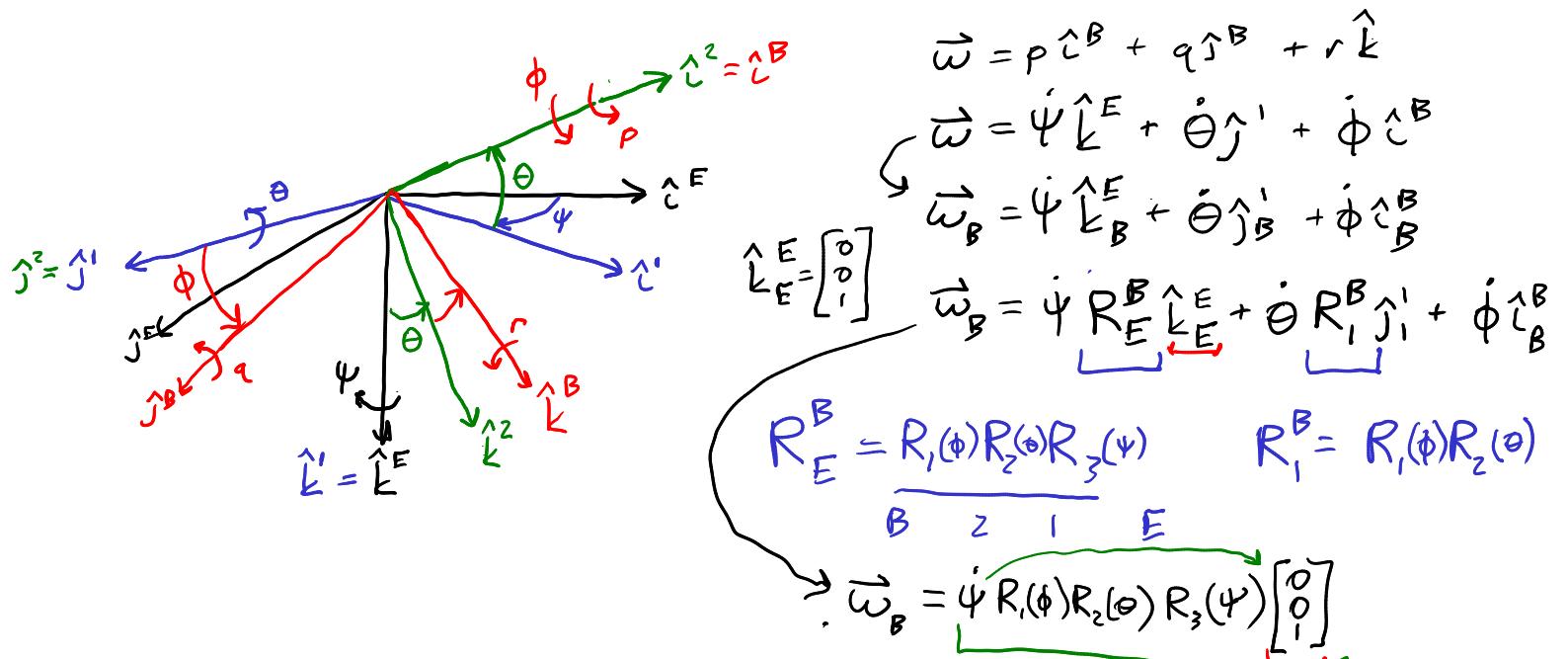
## Translational Kinematics

$$\dot{\vec{p}}_E^E = \frac{d}{dt} \vec{p}_E - \vec{\omega}_E^E \times \vec{p}_E^E = \frac{d}{dt} \vec{p}_E = \vec{v}_E^E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{p}}_E = (R_E^B)^T \vec{v}_B$$

## Rotational Kinematics

want:  $\dot{\vec{o}} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$  have:  $\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$



$$R_E^B = \underbrace{R_1(\phi)}_B R_2(\theta) \underbrace{R_3(\psi)}_E$$

$$R_I^B = R_1(\phi) R_2(\theta)$$

$$\dot{\vec{\omega}}_B = \dot{\psi} R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$+ \dot{\theta} R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\phi} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}}_{\text{invert}} \underbrace{\begin{bmatrix} -s\theta & s\phi c\theta & c\phi c\theta \\ \dot{\phi} & \dot{\theta} & \dot{\psi} \end{bmatrix}}_{\dot{\vec{o}}}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & s\phi + \theta & c\phi + \theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\boxed{\dot{\vec{\omega}} = T \vec{\omega}_B}$

Eqn. 4.4, 7  
in book

$T$  = "attitude influence matrix"

# Dynamics

$$\vec{f} \quad \vec{G}$$

## Translational Dynamics

Newton's 2nd Law

$$\vec{f} = m \vec{a}$$

$$\vec{f} = m \frac{d}{dt} \vec{v}^E$$

want  $\dot{\vec{v}}_B^E$

$$\left( \frac{d}{dt} \vec{v}_B^E \right) = \dot{\vec{v}}_B^E + \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{v}}_B^E = \frac{d}{dt} \vec{v}_B^E - \vec{\omega}_B \times \vec{v}_B^E$$

$$\boxed{\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E}$$

$$\vec{\omega}_B \vec{v}_B^E$$

## Rotational Dynamics

"Euler's 2nd Law"

$$\frac{d}{dt} \vec{h} = \vec{G}$$

↑  
angular momentum

$$\vec{h} = I \vec{\omega}$$

$$I = I_B \text{ book}$$

$$\begin{aligned} I &= \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{pmatrix} \\ &= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix} \end{aligned}$$

Want  $\dot{\vec{\omega}}_B$

$$\frac{d}{dt} \vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \times \vec{h}_B = \vec{G}_B$$

$$I \dot{\vec{\omega}}_B + \vec{\omega}_B \times I \vec{\omega}_B = \vec{G}_B$$

$$\boxed{\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \dot{\vec{\omega}}_B = I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B)}$$

# A/C EOM

$$\dot{\vec{X}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B \\ \dot{\vec{\omega}}_B \end{bmatrix} = \begin{bmatrix} (R_E^B)^T \vec{v}_B^E \\ T \vec{\omega}_B \\ \vec{f}_B - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

## Quadrotors

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u_E \\ v_E \\ w_E \\ p \\ q \\ r \end{bmatrix} \quad \left\{ \begin{array}{l} \vec{p}_E \\ \vec{o} \\ \vec{v}_B^E \\ \vec{\omega}_B \end{array} \right\}$$

A/C EOM

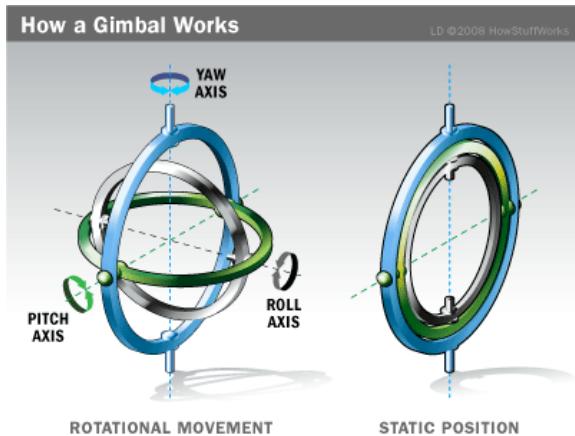
$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} (R_B^E)^T \vec{v}_B \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & s\phi + \theta & c\phi + \theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}$$

From Quiz MC2

$$T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{\omega}_B = \begin{bmatrix} 0 \\ 10^\circ/s \\ 0 \end{bmatrix}$$

$$\vec{o} = \begin{bmatrix} 0 \\ 0 \\ 10^\circ/s \end{bmatrix}$$



## Homework PI Monospinner

$\vec{F}$ : sum of all forces acting on A/C

$$\vec{F} = \vec{A} + g\vec{f}$$

aerodynamic forces      gravity

$\vec{G}$ : sum of all moments acting about the G.G. of A/C

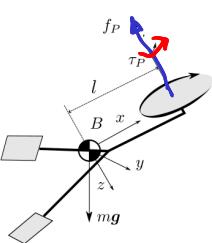
### Multirotor Case

$$\vec{f} = \vec{d} + \vec{c} + \vec{g}$$

drag      A      control

$$\vec{G} = \vec{d} + \vec{c}$$

drag      control



### Monospinner Assignment

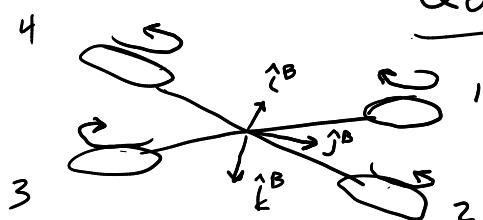
$$\vec{d} = 0 \quad (\text{by assumption})$$

$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ -f_P \end{bmatrix}$$

$$\vec{d} = 0 \quad (\text{by assumption})$$

$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ -\tau_P \end{bmatrix} + \vec{P}_B \times \vec{f}_B$$

### Quadrotor Case



$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$$\vec{c}_G_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since the quadrotor is symmetric about the  $\hat{x}_B - \hat{z}_B$  and  $\hat{y}_B - \hat{z}_B$  planes

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$I^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

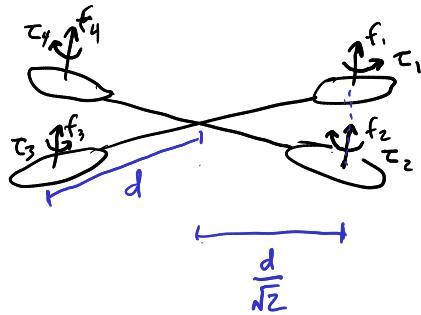
$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \underbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}_{\text{gravity and Coriolis}} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

$$\vec{d}_B \quad \vec{c}_B$$

# Control Forces and Moments



$${}^C \vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix}$$

$${}^C \vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} \frac{d}{N\sqrt{2}} (-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{N\sqrt{2}} (f_1 - f_2 - f_3 + f_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{bmatrix}$$

$w_i$

$$f_i = k_f C_L(w_i)^2$$

$$\tau_i = k_\tau C_D(w_i)^2$$

$$\boxed{\tau_i = k_m f_i}$$

$$k_m = \frac{k_\tau C_D}{k_f C_L}$$

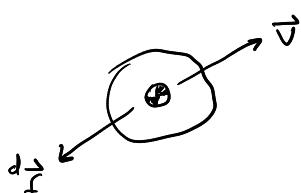
control forces + Moments  $\iff$  individual rotor forces

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} \\ \frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} \\ -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$\downarrow$  invert

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

## Drag Forces

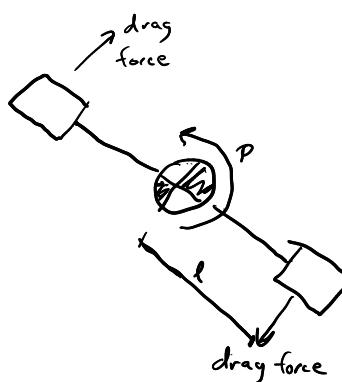


$$d_f = -D \frac{\vec{v}}{V_a} \quad V_a = |\vec{v}|$$

$$D = \frac{1}{2} \rho V_a^2 C_D A = \nu V_a^2$$

$$d_f^B = \begin{bmatrix} X_d \\ Y_d \\ Z_d \end{bmatrix} = -\nu V_a^2 \frac{\vec{v}_B}{V_a} = -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

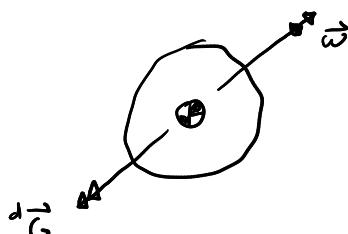
## Drag Moments



$$\begin{aligned} L_{drag} &= -2l f_{drag} \\ &= -2l \underbrace{\frac{1}{2} \rho C_D A (l_p)^2}_{\mu} \text{sign}(p) \\ &= -\mu p l p l \end{aligned}$$

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

General case



$$d_G^B = \begin{bmatrix} L_d \\ M_d \\ N_d \end{bmatrix} = -\mu |\vec{\omega}| \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

# Quadrotor Linear Model

$$\dot{\vec{x}} = f(\vec{x}, \vec{u})$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} (\vec{R}_E^B)^T \vec{v}_B^E \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

- differential
  - first order
  - ordinary (not partial)
  - coupled
  - nonlinear
- simulate

## Linearization

$$\vec{x} \approx \vec{x}_0 + \Delta \vec{x}$$

$$\dot{\vec{x}} \approx \dot{\vec{x}}_0$$

$$\vec{u} \approx \vec{u}_0 + \Delta \vec{u}$$

$\overset{\text{trim}}{\text{condition}}$

For a quadrotor, "Hover" trim condition  
dot means any value

$$\vec{x}_0 = \begin{bmatrix} x_{E,0} \\ y_{E,0} \\ z_{E,0} \\ \phi_0 \\ \theta_0 \\ \psi_0 \\ u_E^0 \\ v_E^0 \\ w_E^0 \\ p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} z_{c,0} \\ l_{c,0} \\ m_{c,0} \\ n_{c,0} \end{bmatrix} = \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Want Linear EOM

$$\dot{\vec{x}} = A \vec{x} + B \vec{u}$$

Approach: Use first-order Taylor Series approx

$$y = f(x, u)$$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$\cancel{y_0 + \Delta y \approx f(x_0, u_0) + \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}}$$

ignore

## 2 Approaches for finding Taylor series

1. Calculate partial derivatives (always work)

2. Substitute  $x = x_0 + \Delta x$  use small number approximations (sometimes faster)

$$\sin(\Delta x) \approx \Delta x$$

$$\cos(\Delta x) \approx 1$$

$$\Delta x \Delta u \approx 0$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\psi & \sin\theta \cos\psi - \cos\phi \sin\psi & \cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi \\ \cos\theta \sin\psi & \sin\theta \cos\psi + \cos\phi \sin\psi & \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi \\ -\sin\theta & \sin\phi \cos\theta & \cos\phi \cos\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Example  $f(\phi, q, r)$

$$\dot{\theta} = \cos(\phi)q - \sin(\phi)r$$

Approach 1:  $\dot{\theta}_0 + \Delta\dot{\theta} = \cos(\phi_0)q_0 - \sin(\phi_0)r_0 + \frac{\partial f}{\partial \phi}\Big|_0 \Delta\phi + \frac{\partial f}{\partial q}\Big|_0 \Delta q + \frac{\partial f}{\partial r}\Big|_0 \Delta r$

$$= \left( \frac{\partial \cos(\phi)}{\partial \phi} q_0 - \frac{\partial \sin(\phi)}{\partial \phi} r_0 \right) \Delta\phi + \cos(\phi_0) \Delta q - \sin(\phi_0) \Delta r$$

$\Delta\dot{\theta} = \Delta q$

Approach 2:  $\dot{\theta}_0 + \Delta\dot{\theta} = \cos(\phi_0 + \Delta\phi)(q_0 + \Delta q) - \sin(\phi_0 + \Delta\phi)(r_0 + \Delta r)$

$$= 1 \Delta q - \Delta\phi \Delta r$$

$\Delta\dot{\theta} = \Delta q$

Harder

$$\dot{w}^E = \underbrace{qu^E - pv^E + g \cos\theta \cos\phi}_{f(\phi, \theta, u, v, w, p, q, Z_c)} + \frac{1}{m} Z_d + \frac{1}{m} Z_c$$

Assume no wind  
 $u=u^E, v=v^E, w=w^E$

$$\Delta\dot{w} = \frac{\partial f}{\partial \phi}\Big|_0 \Delta\phi + \underbrace{\frac{\partial f}{\partial \theta}\Big|_0 \frac{\partial f}{\partial u}\Big|_0 \Delta u + \frac{\partial f}{\partial v}\Big|_0 \Delta v + \frac{\partial f}{\partial w}\Big|_0 \Delta w}_{0} + \frac{\partial f}{\partial p}\Big|_0 \Delta p + \frac{\partial f}{\partial q}\Big|_0 \Delta q + \frac{\partial f}{\partial Z_c}\Big|_0 \Delta Z_c$$

$$Z_d = -\sqrt{w \sqrt{u^2 + v^2 + w^2}}$$

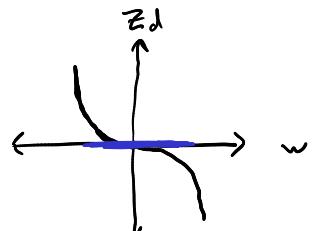
Simple case: assume  $u, v = 0$

$$Z_d = -\sqrt{w} |w| = -\sqrt{w^2} \text{sign}(w)$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\sqrt{w} \left( 2w \text{sign}(w) + w^2 \frac{\partial}{\partial w} \text{sign}(w) \right)\Big|_0$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = 0$$

No drag force in Linear model!



# Simple "Non hover" example

EOM  $\dot{u} = \frac{e_f}{m} - \frac{\gamma u |u|}{m}$

Trim condition

$$u_0 = 30 \text{ m/s steady}$$

$$\dot{u}_0 = 0 = \frac{e_f}{m} - \frac{\gamma u_0 |u_0|}{m}$$

$$\Delta \dot{u} = \frac{\partial g}{\partial u} \Big|_0 \Delta u = \frac{\partial (-\gamma u^2 \text{sign}(u))}{\partial u} \Big|_0 \Delta u = -\gamma (2u \text{sign}(u) + u^2 \frac{\partial \text{sign}(u)}{\partial u}) \Big|_0 \Delta u = -\gamma 2u_0 \Delta u$$

$$\boxed{\Delta \dot{u} = -\gamma 2u_0 \Delta u}$$

$$\frac{\partial Z_d}{\partial w} \Big|_0 = -\gamma \left( \sqrt{u^2 + v^2 + w^2} + w \frac{1}{2} \frac{1}{\sqrt{u^2 + v^2 + w^2}} 2w \right) \Big|_0 = -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}} \Big|_0$$

looks like  $\frac{\partial}{\partial}$

$$\lim_{u,v,w \rightarrow 0} -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$$\lim_{r \rightarrow 0} \frac{r^2 \cdot \text{stuff}}{r} = 0$$

can solve with spherical coordinates

$$u = r \cos \theta \sin \phi$$

$$v = r \sin \theta \sin \phi$$

$$w = r \cos \phi$$

$$\frac{\partial Z_d}{\partial u} \Big|_0 = 0$$

$$\frac{\partial Z_d}{\partial v} \Big|_0 = 0$$

$$\boxed{\Delta \dot{w}^E = \frac{\Delta Z_d}{m}}$$

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{y}_E \\ \Delta \dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \Delta \dot{\phi} \\ \Delta \dot{\theta} \\ \Delta \dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

Drag force would show up here

$$\rightarrow \begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta \theta \\ \Delta \phi \\ 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x} \Delta L_c \\ \frac{1}{I_y} \Delta M_c \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{y}_E \\ \Delta \dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{\phi} \\ \Delta \dot{\theta} \\ \Delta \dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta \theta \\ \Delta \phi \\ 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x} \Delta L_c \\ -\frac{1}{I_y} \Delta M_c \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

Longitudinal

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{u} \\ \Delta \dot{\theta} \\ \Delta \dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g \Delta \theta \\ \Delta q \\ \frac{1}{I_u} \Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ q \Delta \phi \\ \Delta p \\ \frac{1}{I_r} \Delta L_c \end{pmatrix}$$

Vertical

$$\begin{pmatrix} \Delta \dot{z}_E \\ \Delta \dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m} \Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta \dot{\psi} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

State space model

$$\begin{aligned} \dot{\vec{x}} &= A \vec{x} + B \vec{u} \\ \vec{y} &= C \vec{x} + D \vec{u} \end{aligned}$$

$$\ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = \omega_n^2 u$$

$$\begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \frac{1}{I_x} \Delta L_c \end{bmatrix}$$

$$\begin{aligned} \Delta p &= \Delta \dot{\phi} \\ \Delta \dot{p} &= \Delta \ddot{\phi} \end{aligned}$$

$$\Delta \ddot{\phi} = \frac{1}{I_x} \Delta L_c$$

$$\text{Solutions? } \Delta \phi(+)$$

Assume  $\Delta L_c$  is constant

$$\Delta \dot{\phi}(+) = \Delta \dot{\phi}_o + \int_{\tau=0}^{+} \Delta \dot{\phi}(t) dt = \Delta \dot{\phi}_o + \frac{1}{I_x} \Delta L_c +$$

$$\Delta \phi(+) = \Delta \phi_o + \int_{\tau=0}^{+} \Delta \dot{\phi}(+) dt = \Delta \phi_o + \Delta \dot{\phi}_o t + \frac{1}{2} \frac{1}{I_x} \Delta L_c t^2$$

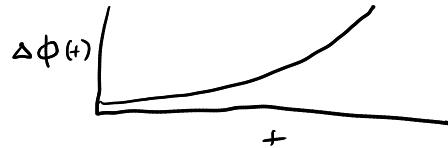
Exactly at hover  $\Delta \phi_o = \Delta \dot{\phi}_o = \Delta L_c = 0$

$$\Delta \phi(+) = 0$$

If  $\Delta \dot{\phi}_o > 0$ ,  $\Delta L_c = 0$

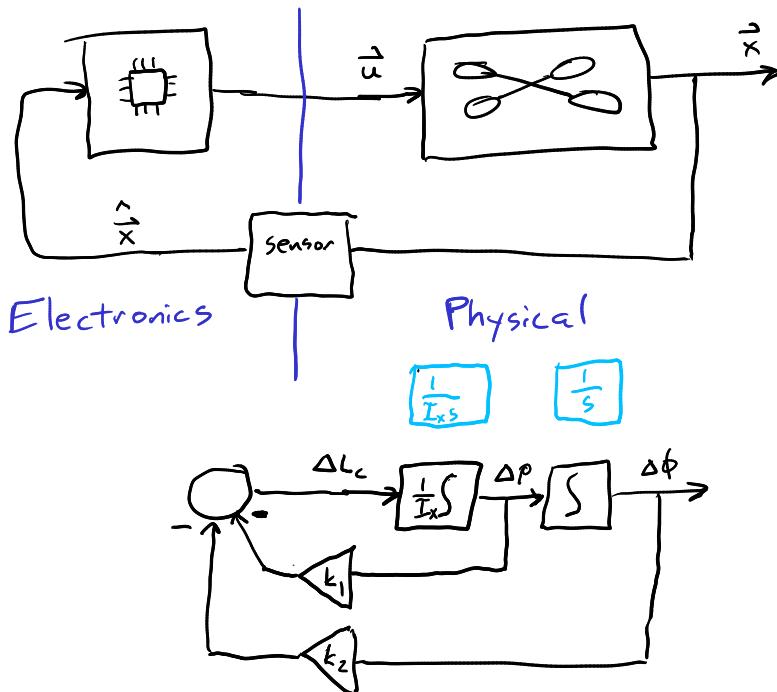


If  $\Delta L_c > 0$



If initial conditions are nonzero (always in real life), vehicle will crash

# Solution: Feedback Control



$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi$$

$k_1$  Deriv. gain       $k_2$  prop. gain

$$\dot{\Delta \phi} = \frac{1}{I_x} \Delta L_c$$

$$= \frac{1}{I_x} (-k_1 \Delta p - k_2 \Delta \phi)$$

$$\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0$$

$$\ddot{\Delta \phi} + \frac{k_1}{I_x} \dot{\Delta \phi} + \frac{k_2}{I_x} \Delta \phi = 0$$

$$\ddot{\Delta \phi} + 2\zeta\omega_n \dot{\Delta \phi} + \omega_n^2 \Delta \phi = 0$$

If  $\lambda$  are real and distinct

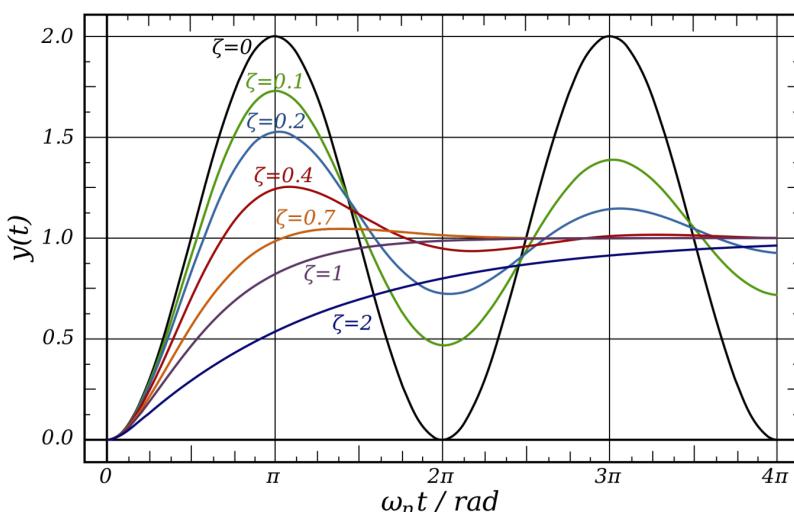
$$\Delta \phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

If  $\lambda$  are complex

$$\zeta = \frac{k_1}{2\sqrt{k_2 I_x}} \quad \omega_n = \sqrt{\frac{k_2}{I_x}}$$

$$\lambda = -\zeta\omega_n \pm i\omega_n \sqrt{1-\zeta^2}$$

$$\Delta \phi(t) = e^{-\zeta\omega_n t} (C_3 \sin(\omega_d t) + C_4 \cos(\omega_d t))$$



$$\begin{bmatrix} \dot{\Delta\phi} \\ \dot{\Delta p} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \frac{1}{I_x} \Delta L_c \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\vec{x}} \\ \vec{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta L_c \end{bmatrix}$$

$$\begin{array}{c} \dot{\vec{y}} \\ \vec{y} \end{array} = \begin{array}{c} C \\ \vec{x} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Delta L_c \end{bmatrix} \quad D \quad \vec{u}$$

$$\vec{u} = \begin{bmatrix} \Delta L_c \end{bmatrix} = -K \vec{x} = -[k_2 \ k_1] \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix}$$

$$\dot{\vec{x}} = \underbrace{A \vec{x} - BK \vec{x}}_{A^{cl}} = \underbrace{(A - BK)}_{A^{cl}} \vec{x}$$

$$A^{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{I_x} \end{bmatrix} [k_2 \ k_1] = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} \end{bmatrix}$$

$$\dot{\vec{x}} = A^{cl} \vec{x} \quad \text{what are solutions of } \vec{x} = A^{cl} \vec{x}?$$

scalar case

$$\dot{x} = ax \Rightarrow x(t) = x(0)e^{at}$$

analogously

$$\dot{\vec{x}} = A \vec{x} \Rightarrow \vec{x}(+) = e^{A+} \vec{x}(0)$$

$$e^{At} = I^+ + A + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 \dots \quad (\text{Taylor Series})$$

## Modal Analysis

Eigenvalues and Eigen vectors

$$A \vec{v}_i = \lambda_i \vec{v}_i$$

Suppose that  $\vec{x}_0 = \vec{v}_i$

$$\begin{aligned} \vec{x}(+) &= (I^+ + A + \frac{t^2}{2!} A^2 + \dots) \vec{v}_i \\ &= \vec{v}_i + \lambda_i \vec{v}_i + \frac{t^2}{2!} \lambda_i^2 \vec{v}_i \dots \end{aligned}$$

$$\vec{x}(+) = \vec{v}_i e^{\lambda_i t}$$

$$\text{If } \vec{x}_0 = \sum_i q_i \vec{v}_i$$

$$\text{then } \vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$$\vec{q} = V^{-1} \vec{x}$$

matrix of eigenvector columns

# Finding Eigenvalues

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$(A - \lambda_i I) \vec{v}_i = 0$$

only has nontrivial solutions if  
 $|A - \lambda_i I| = 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

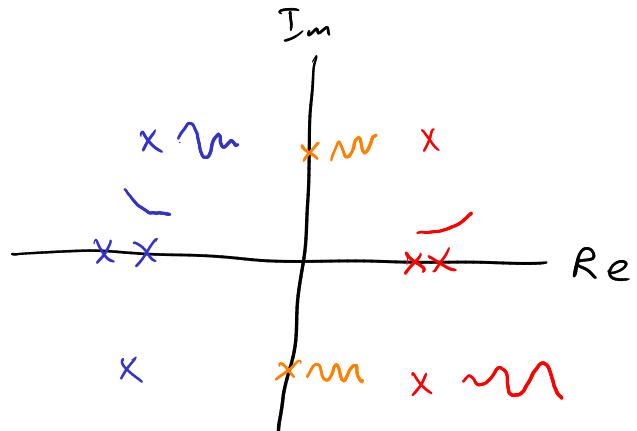
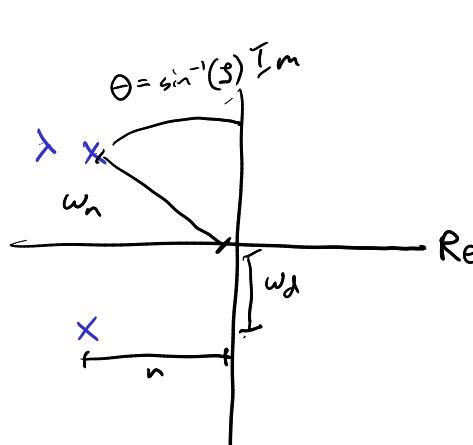
solve with quadratic formula

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_2}{I_x} & \frac{k_1}{I_x} - \lambda \end{vmatrix} = \lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{I_x} \pm \sqrt{\frac{k_1^2}{4I_x} - \frac{k_2}{I_x}}$$

$$= n \pm i\omega_d$$

$$= -\zeta\omega_n \pm i\omega_n \sqrt{1 - \zeta^2}$$



# Solutions to linear ODEs

$$\dot{x} = ax \Rightarrow x(0)e^{at}$$

$$\ddot{x} + \frac{2\zeta\omega_n}{a}\dot{x} + \frac{\omega_n^2}{b}x = 0 \Rightarrow \text{characteristic eqn} \quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

quad. form

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \text{if } \lambda \text{ real and distinct}$$

$$x(t) = e^{-\zeta\omega_n t} (C_3 \sin(\omega_d t) + C_4 \cos(\omega_d t))$$

$$\dot{\vec{x}} = A\vec{x} \Rightarrow e^{At}\vec{x}(0)$$

$$\sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$\vec{v}_i$  are eigenvectors

$\lambda_i$  are eigenvalues

$$\text{if } \vec{x}(0) = a\vec{v}_1 + b\vec{v}_2$$

$$\vec{x}(t) = a\vec{v}_1 e^{\lambda_1 t} + b\vec{v}_2 e^{\lambda_2 t}$$

$\vec{v}_i$  Eigenvectors: "shape" of the mode

which state variables are actively changing

$\lambda_i$  Eigenvalues : "speed" of the mode

how fast does it oscillate, decay, or diverge

## Linear Control Design Process

1. Derive EOM

2. Linearize and Separate EOM

3. Design Control Architecture

4. Choose Gain Values

5. Testing in Linear Simulation

6. Test in Nonlinear Sim

1. PID tuning  
2. Pole Assignment  
3. Root Locus

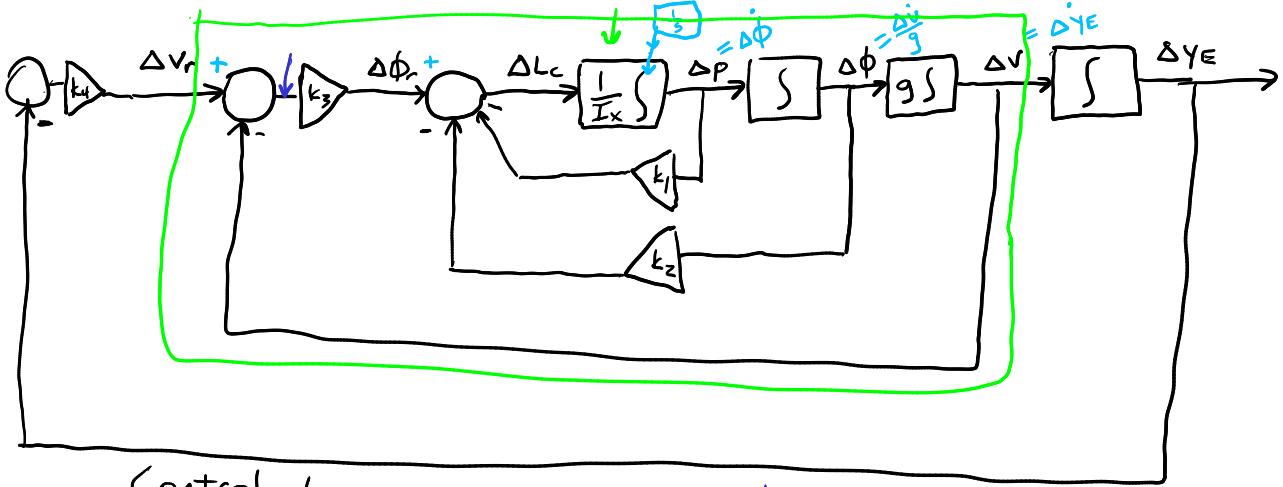
4. Optimal Control (LQR)

$$\dot{\vec{x}} = \begin{bmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} \Delta v \\ g \Delta \phi \\ \Delta p \\ \frac{1}{I_x} \Delta L_C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta L_C \end{bmatrix}$$

$\dot{\vec{x}}$        $A$        $\vec{x}$        $B$        $\vec{u}$

$$\vec{y} = \begin{matrix} \text{"} \\ \text{I} \\ \text{"} \\ \vec{x} \\ \text{I} \\ \text{"} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

$$I_x = 7 \times 10^{-5} \text{ kg m}^2$$



Control Law

$$\rightarrow \Delta L_c = -k_1 \Delta P - k_2 \Delta \phi + k_3 (-k_4 \Delta y_E - \Delta V)$$

$$= -k_1 \Delta P - k_2 \Delta \phi - k_3 k_4 \Delta y_E - k_3 \Delta V$$

$$[\Delta L_c] = \vec{u} = -K \vec{x} = -\underbrace{[k_3 k_4 | k_3 | k_2 | k_1]}_{\text{Matrix}} \begin{bmatrix} \Delta y_E \\ \Delta V \\ \Delta \phi \\ \Delta P \end{bmatrix}$$

$$A^{cl} = A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \quad \dot{\vec{x}} = A^{cl} \vec{x}$$

Choosing Gains

1. Choose  $k_1$  and  $k_2$  with the "pole placement" strategy

Pole placement

- Decide where we want eigenvalues (poles) to be
- Solve for  $k_1$  and  $k_2$

For 2nd order system

$$\lambda = -j\omega_n \pm i\omega_n \sqrt{1 - S^2}$$

$$|A^{cl} - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x} \lambda + \frac{k_2}{I_x} = 0$$

$$\begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{P} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta P \end{bmatrix}$$

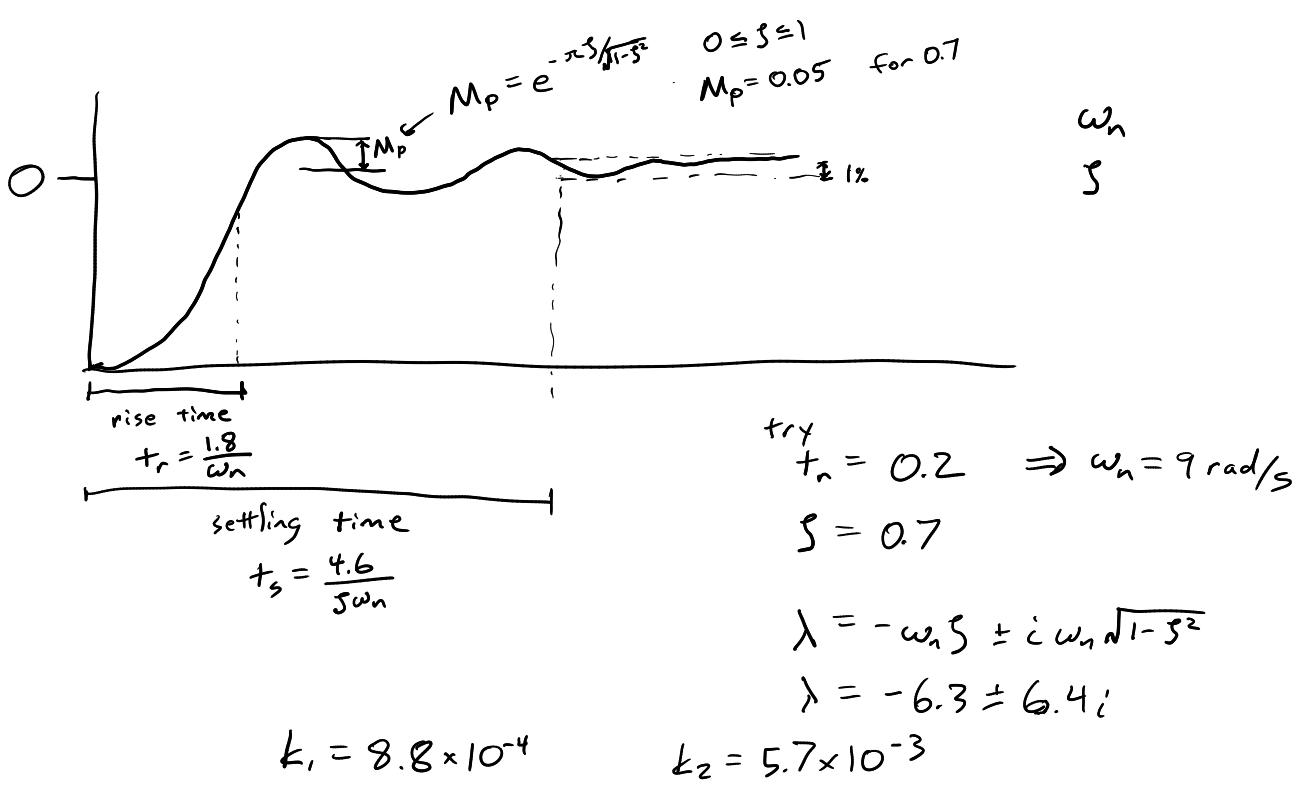
$$A^{cl}$$

analogous to

$$\lambda^2 + 2j\omega_n \lambda + \omega_n^2 = 0$$

$$k_1 = 2j\omega_n I_x$$

$$k_2 = \omega_n^2 I_x$$



$$\dot{\vec{x}} = A^{cl} \vec{x} \leftarrow \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

solutions look like

$$\ddot{\Delta \phi} + \frac{\Delta \phi}{2\zeta\omega_n} + \frac{\Delta \phi}{\omega_n^2} = 0$$

solutions look like

$$\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

( $C, \lambda$  might be complex  $\Rightarrow \sin/\cos$ )

$\lambda_i$  are solutions to

$$\lambda^2 + [2\zeta\omega_n]\lambda + [\omega_n^2] = 0$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i$$

$$\vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t} \iff$$

$\lambda_i$  are solutions to

$$|A^{cl} - \lambda I| = 0$$

if  $A^{cl}$  is  $2 \times 2$

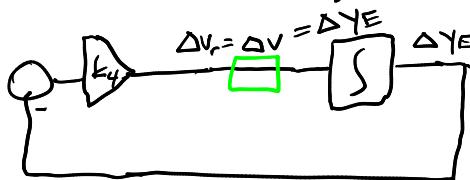
$$\lambda^2 + \frac{k_1}{I_x} \lambda + \frac{k_2}{I_x} = 0$$

$$\boxed{\frac{k_1}{I_x}}$$

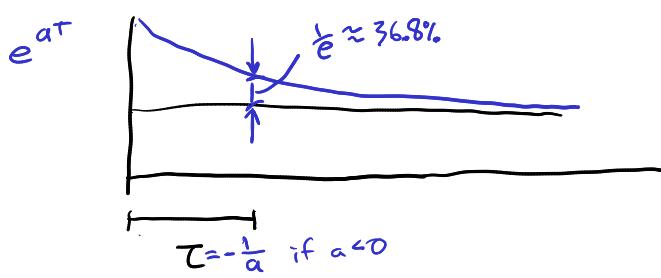
$$\boxed{\frac{k_2}{I_x}}$$

$$k_1 = 2\zeta\omega_n I_x \quad k_2 = \omega_n^2 I_x$$

Choose  $k_4$  with pole placement



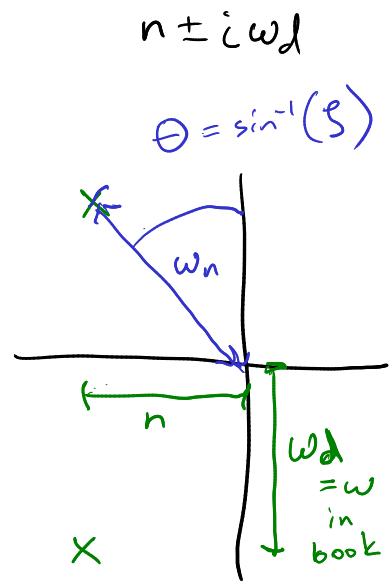
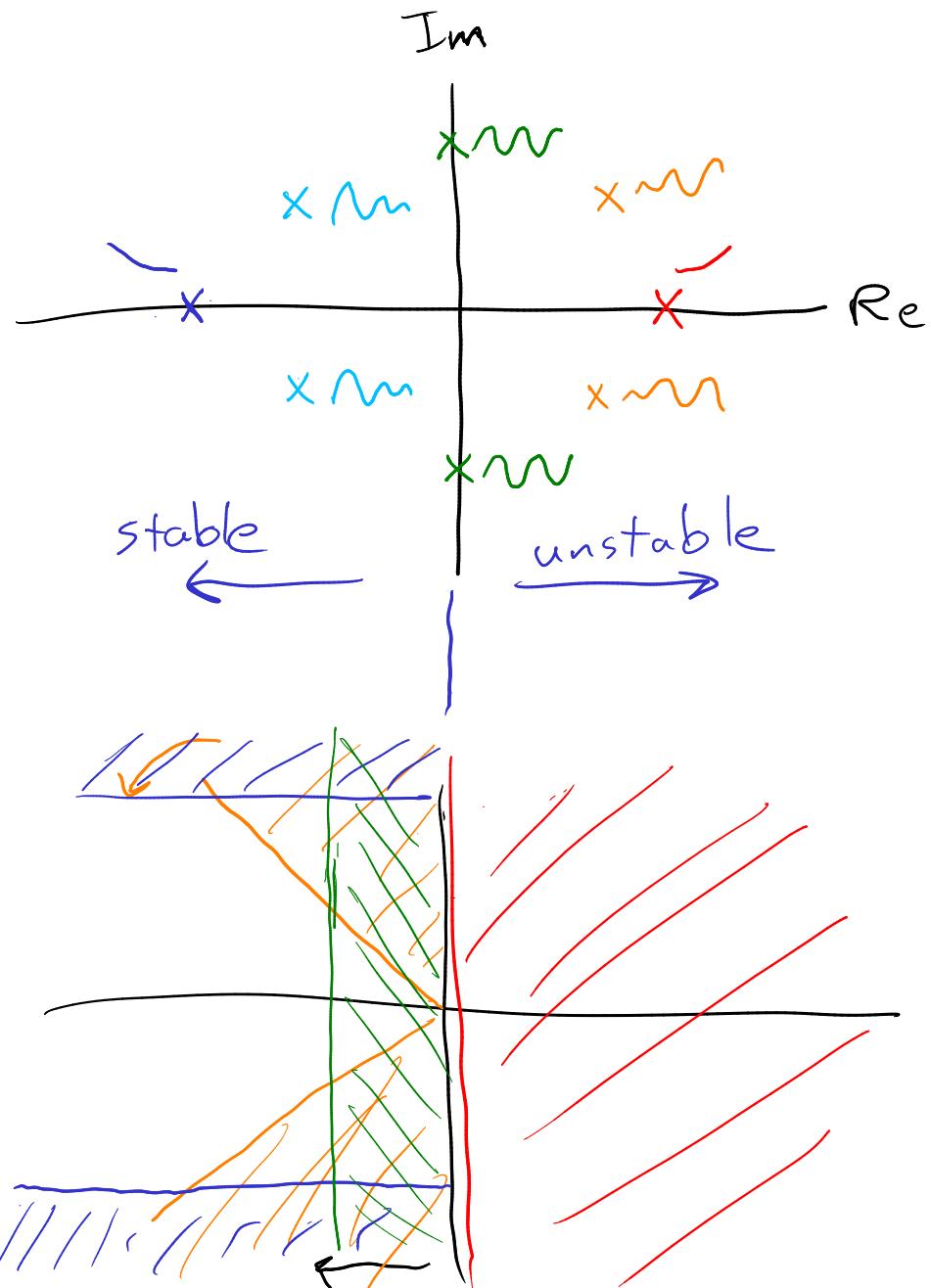
$$\dot{\Delta y_E} = -k_4 \Delta y_E \implies \Delta y_E(+) = \Delta y_E(0) e^{-k_4 t}$$



$$T_y = \frac{1}{k_4}$$

inner loop  $(\phi, p)$  has settling time of  $\frac{4.6}{\zeta\omega_n} \approx 0.7$   
choose  $T_y$  10x larger  $\approx 7 \text{ sec.} \approx 5$

$$T_y = 5 \Rightarrow k_4 = 0.2$$



$$\omega_d = \omega_n \sqrt{1 - \beta^2}$$

