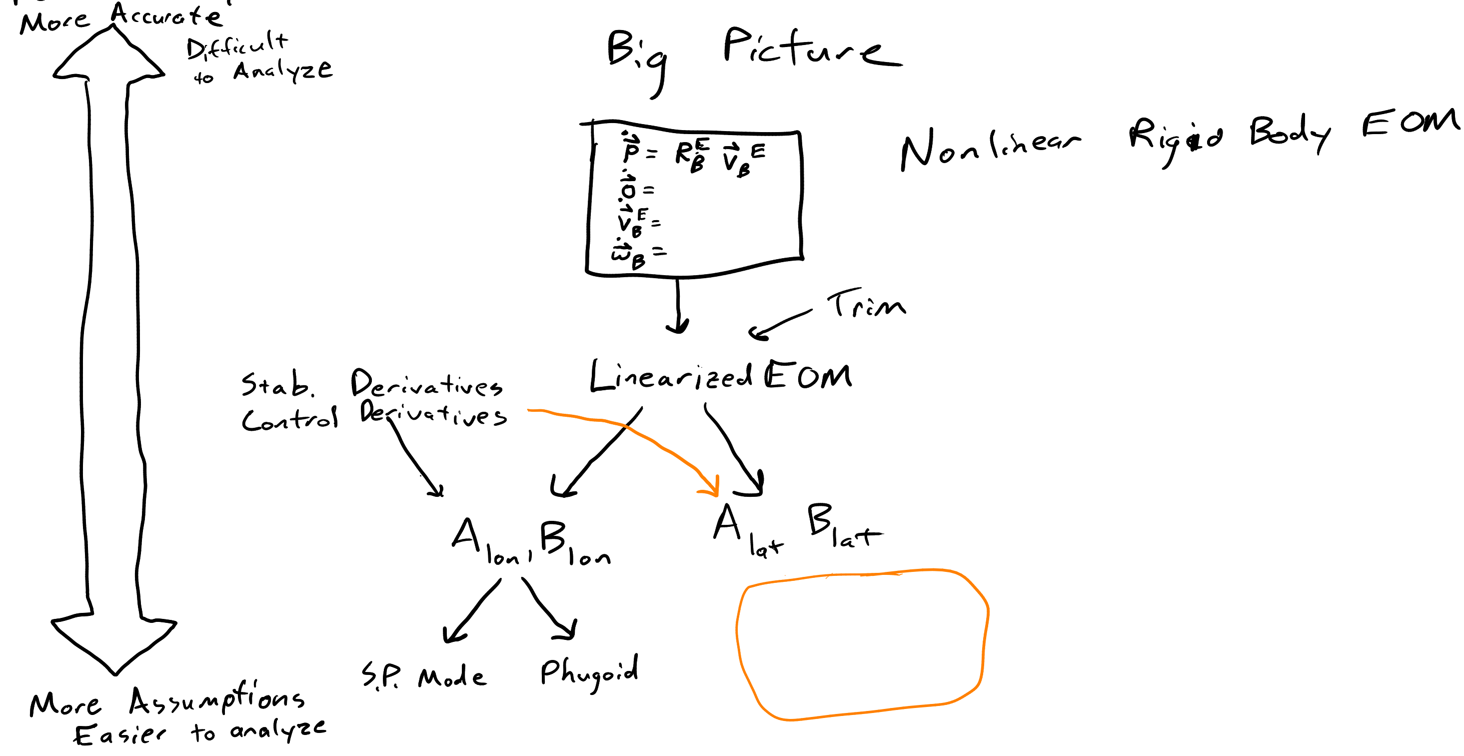
Lateral / Directional Dynamics



$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$ Lat

$$\Delta\dot{\theta} = \Delta q$$ long

$$\Delta\dot{u} = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3\Delta L + \Gamma_4\Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4\Delta L + \Gamma_8\Delta N$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$Y = \tfrac{1}{2}\rho V^2 S\, C_y(\beta, p, r, \delta_a, \delta_r)$$

$$L = \tfrac{1}{2}\rho V^2 S \underbrace{b}_{\text{span}} C_l(\beta, p, r, \delta_a, \delta_r)$$

$$N = \tfrac{1}{2}\rho V^2 S b\, C_n(\beta, p, r, \delta_a, \delta_r)$$

$$\hat{p} = \frac{pb}{2V_a}$$

$$
\begin{aligned}
Y &\approx \frac{1}{2}\rho V_a^2 S \left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r \right] \\
L &\approx \frac{1}{2}\rho V_a^2 Sb \left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r \right] \\
N &\approx \frac{1}{2}\rho V_a^2 Sb \left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r \right]
\end{aligned}
$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$
\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad
\mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}
$$

$$
\mathbf{A}_{lat} = \begin{pmatrix}
\dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\
\Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\
\Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\
0 & 1 & \tan\theta_0 & 0
\end{pmatrix}
$$

+β: wind coming from right

$$\beta = \sin^{-1}\left(\frac{\Delta v}{V}\right)$$

$$\beta \approx \frac{\Delta v}{u_0} = \hat{v}$$

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$

$$L = \frac{1}{2}\rho V^2 S b C_\ell$$

nondimensional
use lower-case for moments

$$
\begin{aligned}
L_v \equiv \left.\frac{\partial L}{\partial v}\right|_0 &= \frac{1}{2}\rho u_0^2 S b \left.\frac{\partial C_\ell}{\partial v}\right|_0 \\
&= \frac{1}{2}\rho u_0^2 S b \left.\frac{\partial C_\ell}{\partial \beta}\right|_0 \left.\frac{\partial \beta}{\partial v}\right|_0 \\
&= \boxed{\frac{1}{2}\rho u_0 S b C_{\ell_\beta}}
\end{aligned}
$$

$$C_{\ell_\beta} \equiv \left.\frac{\partial C_\ell}{\partial \beta}\right|_0 = \left.\frac{\partial C_\ell}{\partial \hat{v}}\right|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0} \Rightarrow \frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

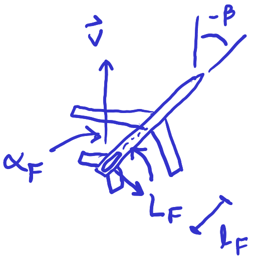
Table 5.2
Summary—Lateral Derivatives

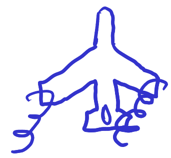
	C_y	C_l	C_n
β	$*-a_F\dfrac{S_F}{S}\left(1-\dfrac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V\left(1-\dfrac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F\dfrac{S_F}{S}\left(2\dfrac{z_F}{b}-\dfrac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V\left(2\dfrac{z_F}{b}-\dfrac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F\dfrac{S_F}{S}\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F\dfrac{S_F}{S}\dfrac{z_F}{b}\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V\left(2\dfrac{l_F}{b}+\dfrac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

β derivatives

C_{n_β}: weathervane derivatime / yaw stiffness Sign? +





side wash

$$\alpha_F = -\beta + \sigma$$

$$C_{L_F} = a_F(-\beta + \sigma)$$

$$N_F = -\underbrace{C_{L_F}\tfrac{1}{2}\rho V_F^2 S_F}_{L_F}\, l_F$$

$$C_{n_F} = -C_{L_F}\frac{S_F\, l_F}{S\, b}\left(\frac{V_F}{V}\right)^2 \qquad V_V = \frac{S_F l_F}{S b}$$

$$= -V_V C_{L_F}\left(\frac{V_F}{V}\right)^2$$

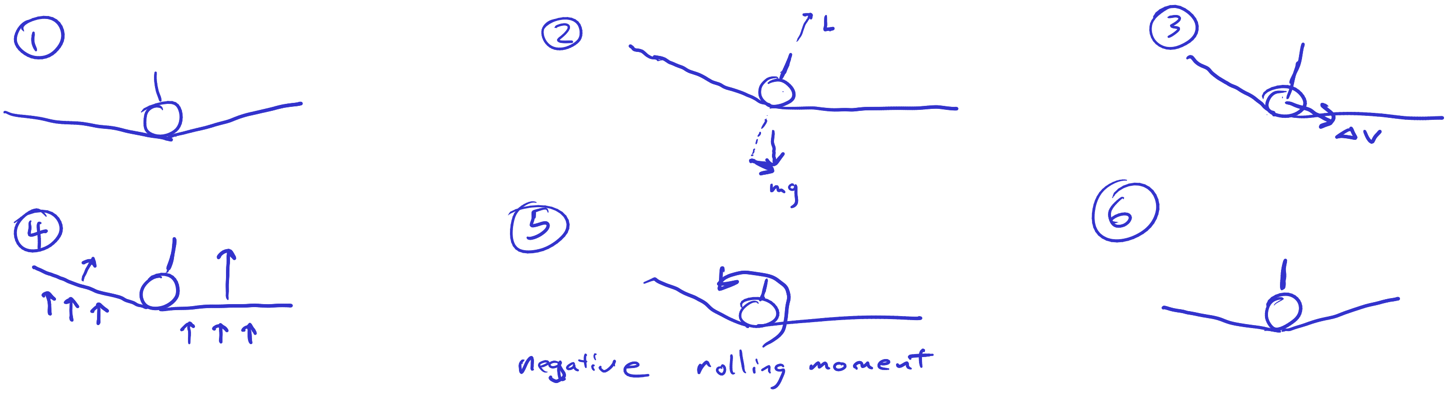
$$\left(C_{n_\beta}\right)_{Tail} = \left.\frac{\partial C_{n_F}}{\partial\beta}\right|_0 = -V_V\left(\frac{V_F}{V}\right)^2 \underbrace{\left.\frac{\partial C_{L_F}}{\partial\beta}\right|_0}_{a_F\left(-1+\frac{\partial\sigma}{\partial\beta}\right)}$$

$$\boxed{\left(C_{n_\beta}\right)_{Tail} = V_V a_F\left(\frac{V_F}{V}\right)^2\left(1-\frac{\partial\sigma}{\partial\beta}\right)}$$

C_{y_β}: usually small, similar derivation to C_{n_β} Sign? −

C_{l_β}: Dihedral Effect Sign? −

(looking from behind)



4 Significant Factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

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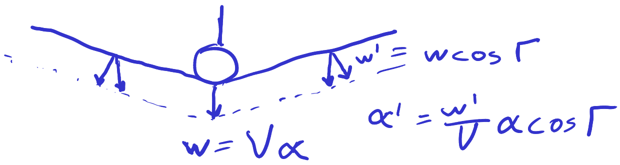
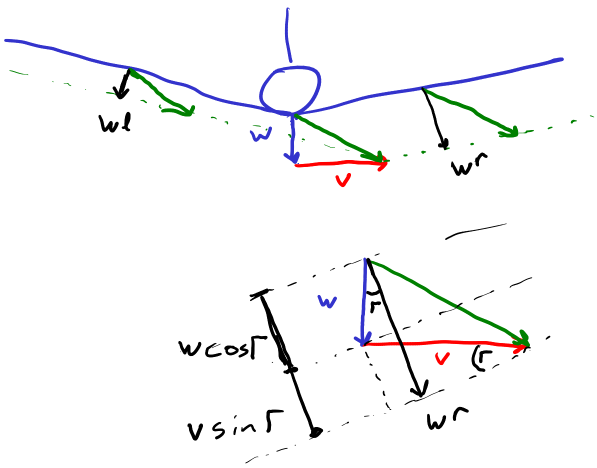
1 Dihedral Angle





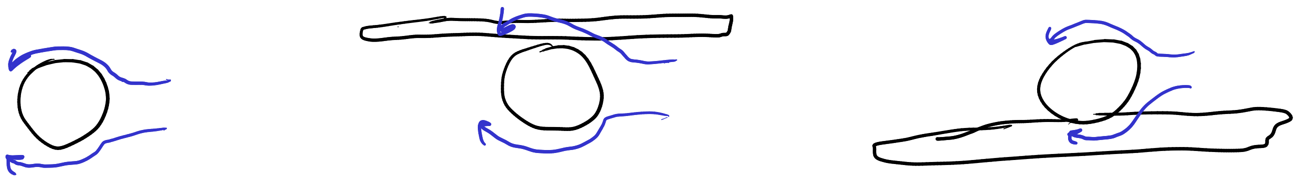
$$w^r = w\cos\Gamma + v\sin\Gamma \approx w + v\Gamma$$

$$\alpha^l \approx \frac{w^l}{u_0} = \alpha - \beta\Gamma \qquad \alpha^r \approx \frac{w^r}{u_0} = \alpha + \beta\Gamma$$

$$L \propto (\alpha^l - \alpha^r)$$

$$\boxed{C_{l_\beta}^{\Gamma} \propto -\Gamma}$$

2. Wing Height



High Wing: -ve C_{l_β} contribution

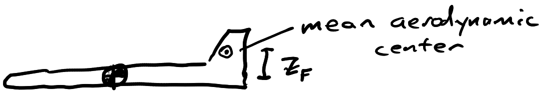
Low Wing: +ve C_{l_β} contribution

3. Wing Sweep

$$C_{\ell_\beta}^{\Lambda} \propto 2 C_L V^2 \sin(2\Lambda)$$

$$+\Lambda \Rightarrow +C_{\ell_\beta}$$

4. Tail



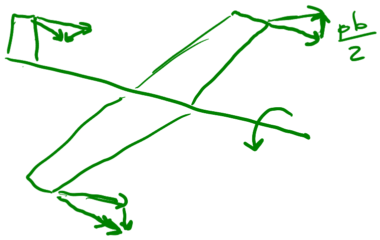
$$\Delta C_\ell^F = C_{L_F}\frac{S_F z_F}{S b} = a_F(-\beta+\sigma)\frac{S_F z_F}{S b}$$

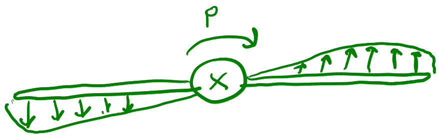
$$C_{\ell_\beta}^F = -a_F\left(1-\frac{\partial\sigma}{\partial\beta}\right)\frac{S_F z_F}{S b}\left(\frac{V_F}{V}\right)^2$$

p-derivatives

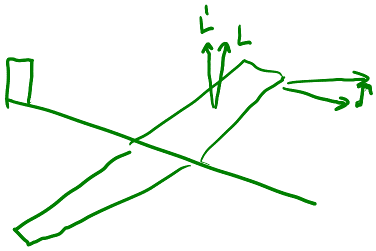
$\hat{p} = \frac{pb}{2V}$

C_{l_p}: roll damping Sign? $-$





C_{n_p} Wing Effect



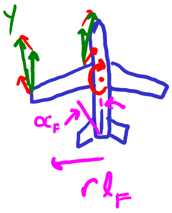
$$\left(C_{n_p}\right)_{tail} = a_F V_V \left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

C_{y_p} (usually small)

(similar to derivation for $\left(C_{n_\beta}\right)_{tail}$)

r-derivatives

Wing-Body



$\vec{\omega} \times \vec{p}^{F \to B}$

$r l_F$

$$\Delta\alpha_F = \frac{r l_F}{u_0} + r\frac{\partial\sigma}{\partial r}$$

$$= \hat{r}\left(\frac{2 l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$\frac{\partial\alpha_F}{\partial\hat{r}} = \left(\frac{2 l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$C_y = \frac{Y}{\frac{1}{2}\rho u_0^2 S} \qquad \left(C_y\right)_{tail} = \frac{\frac{1}{2}\rho u_0^2 S_F a_F \alpha_F}{\frac{1}{2}\rho u_0^2 S}$$

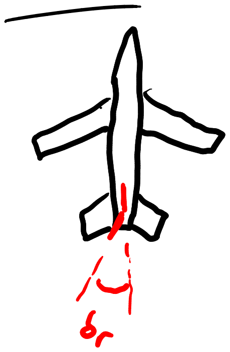
$$\boxed{\left(C_{y_{\hat{r}}}\right)_{tail} = \left.\frac{\partial (C_y)_{tail}}{\partial\hat{r}}\right|_0 = a_F\frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$

$$\boxed{\left(C_{l_r}\right)_{tail} = a_F\frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$

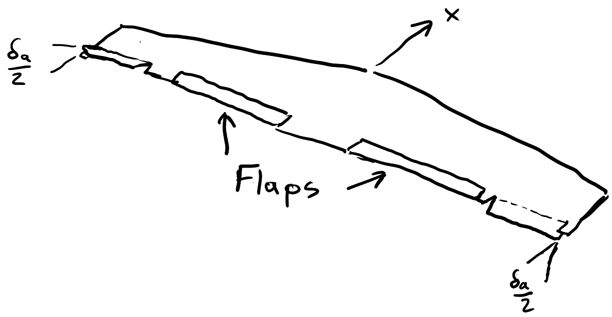
$$\boxed{\left(C_{n_r}\right)_{tail} = -a_F \underbrace{V_V}_{\frac{S_F}{S}\frac{l_F}{b}}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$ ← yaw damping

Lateral Control and Coordinated Turn

Rudder

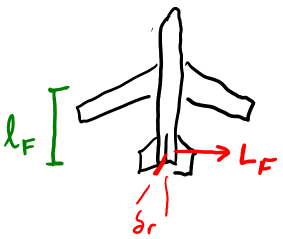


Aileron



Rudder Power

$C_{n_{\delta_r}}$



$$N_F = -l_F L_F = -l_F \tfrac{1}{2}\rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$

$$C_{n_F} = \frac{N_F}{\frac{1}{2}\rho V^2 S b} = -\underbrace{\frac{l_F S_F}{S b}}_{V_V}\left(\frac{V_F^2}{V^2}\right) C_{L_F} = -V_V\left(\frac{V_F^2}{V^2}\right) C_{L_F}$$

$$C_{n_{\delta_r}} = \left.\frac{\partial C_{n_F}}{\partial \delta_r}\right|_0 = -V_V\left(\frac{V_F^2}{V^2}\right)\underbrace{\left.\frac{\partial C_{L_F}}{\partial \delta_r}\right|_0}_{a_r} = \boxed{-a_r V_V\left(\frac{V_F^2}{V^2}\right)}$$

Other nondim. rudder control derivatives

$C_{Y_{\delta_r}}$ sign? +

$C_{l_{\delta_r}}$ sign? +

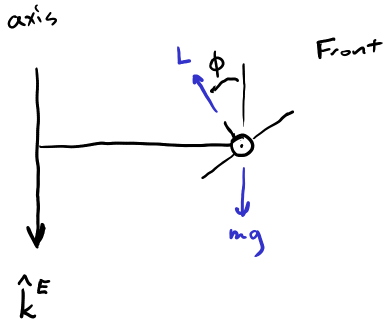
Aileron nondim. control derivatives

$C_{l_{\delta_a}}$ sign? − aileron reversal can occur due to wing twist

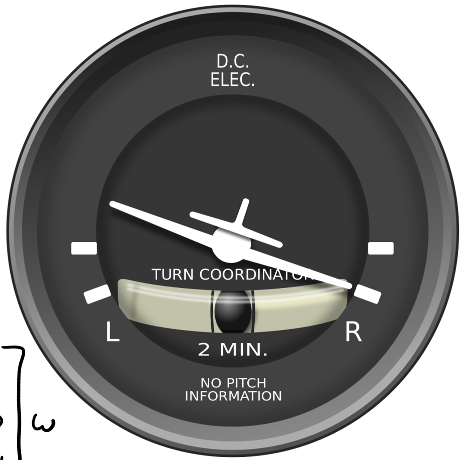
$C_{n_{\delta_a}}$ sign? can be either $C_{n_{\delta_a}} > 0$, called adverse yaw

$C_{Y_{\delta_a}}$ usually small

Coordinated Turn



- angular velocity vector is constant and aligned with inertial z
- No aerodynamic forces in a/c y direction



$$\omega = \frac{u_0}{R} \qquad \vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_B = R^B_E \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix}\omega = \begin{bmatrix} -\theta \\ \sin\phi \\ \cos\phi \end{bmatrix}\omega$$

assume θ small but ϕ is not

$$L\cos\phi = mg$$

$$L\sin\phi = ma_n = m\frac{u_0^2}{R} = m\omega u_0$$

$$\boxed{\tan\phi} = \frac{L\sin\phi}{L\cos\phi} = \frac{m\omega u_0}{mg} = \boxed{\frac{\omega u_0}{g}}$$

From EOM

Assumed, no wind, $v \ll u$, $V = u = u_0$

$$Z = -mg\cos\phi - mqu$$

load factor "g's"

$$n = -\frac{Z}{mg} = \cos\phi + \frac{qu_0}{g}$$

$$= \cos\phi + \frac{\omega u_0 \sin\phi}{g}$$

$$= \cos\phi + \tan\phi\sin\phi$$

$$\boxed{n = \sec\phi}$$

$$n = \frac{L}{W}$$

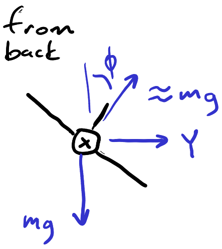
$$\Delta C_L = \frac{L - mg}{\frac{1}{2}\rho V^2 S} = (n-1)C_W$$

determined by ω, θ, ϕ

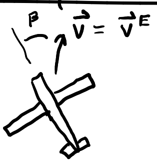
Coordinated Turn	$\beta, p, r, \delta_r, \delta_a$	α, q, δ_e
$C_Y = 0$	$= C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta_r}}\delta_r + C_{Y_{\delta_a}}\delta_a$	
$C_\ell = 0$	$=$ ⋮	
$C_n = 0$	$=$ ⋮	
$C_m = 0$	$=$ ⋮	
$C_L = (n-1)C_W$	$=$	

$$\begin{array}{c} Y \\ l \\ n \end{array} \begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_\beta} & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix} \begin{bmatrix}\beta\\ \delta_r \\ \delta_a\end{bmatrix} = \begin{bmatrix} C_{Y_p} & C_{Y_r} \\ C_{l_p} & C_{l_r} \\ C_{n_p} & C_{n_r}\end{bmatrix} \overbrace{\begin{bmatrix}\theta \\ -\cos\phi\end{bmatrix}\frac{\omega b}{2u_0}}^{\begin{bmatrix}\hat p\\ \hat r\end{bmatrix}}$$

$$\begin{bmatrix} C_{m_\alpha} & C_{m_{\delta e}} \\ C_{L_\alpha} & C_{L_{\delta e}}\end{bmatrix}\begin{bmatrix}\Delta\alpha \\ \Delta\delta e\end{bmatrix} = -\begin{bmatrix}C_{m_q}\\ C_{L_q}\end{bmatrix}\overbrace{\frac{\omega \bar c \sin\phi}{2u_0}}^{\hat q} + \begin{bmatrix}0 \\ (n-1)C_W\end{bmatrix}$$



Steady Sideslip



$$Y + mg\sin\phi = 0$$
$$Y + mg\phi = 0$$
$$L = 0$$
$$N = 0$$

$$-mg\phi = Y = Y_v v + Y_p \cancel{p}^{\,0} + Y_r \cancel{r}^{\,0} + Y_{\delta a}\delta_a + Y_{\delta_r}\delta_r$$
$$L = \dots$$
$$N =$$

$$mg\phi + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r = -Y_v v$$

$$\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix}\begin{bmatrix}\delta_r \\ \delta_a \\ \phi\end{bmatrix} = -\begin{bmatrix} Y_v \\ L_v \\ N_v\end{bmatrix} v \quad \beta u_0$$

Steady Sideslip

For Piper Cherokee

$$\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix}\begin{bmatrix}\delta_r \\ \delta_a \\ \phi\end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394\end{bmatrix} v \qquad (7.8,4)$$

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta \doteq v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\delta_r/\beta = .303$$
$$\delta_a/\beta = -2.96$$
$$\phi/\beta = .104$$

We see that a positive sideslip (to the right) of say 10° would entail left rudder of 3° and right aileron of 29.6°. Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only 1° to the right so the sideslip is almost flat.

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

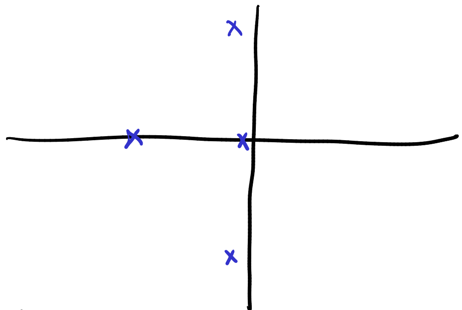
$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\dot{\vec{x}} = A_{lat}\,\vec{x}$$

$$\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02 + 9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02 - 9.47e-01i$	$3.49e-02$	$9.47e-01$



$$\mathbf{v}_1 = \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \quad \mathbf{v}_{3/4} = \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

Augmented State Space Dynamics Matrix

$$\begin{bmatrix} \Delta\dot{v} \\ \vdots \\ \\ \\ \\ \\ \end{bmatrix} = \left[\begin{array}{cccc|cc} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{array}\right] \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta y_E \end{bmatrix}$$

1 if $\theta_0 = 0$

$$\Delta\dot{\psi} = \Delta r \sec\theta_0$$

$$\Delta\dot{y}_E = u_0 \cos\theta_0 \Delta\psi + \Delta v$$

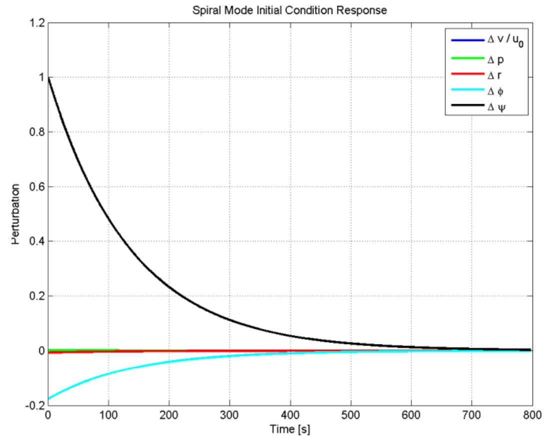
Spiral Mode

$\lambda = -0.0073$

$\tau = 137\,s$

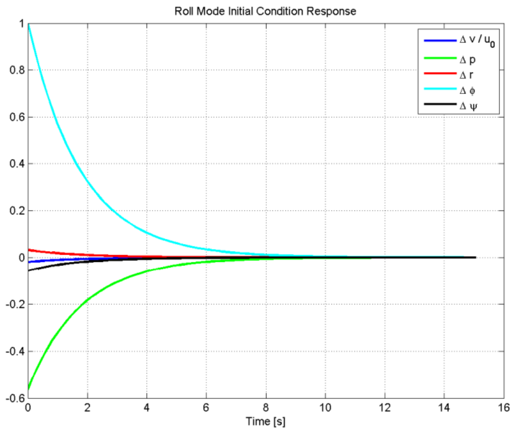
$$\hat{v}_1 = \begin{bmatrix} 0.0068 \\ -0.0074 \\ 0.04 \\ 1.0 \\ -5.66 \end{bmatrix} \begin{array}{l} \hat{v} = \beta \leftarrow \text{small} \\ p \leftarrow \text{small} \\ r \leftarrow \text{a little} \\ \phi \leftarrow \text{large} \\ \psi \leftarrow \text{large} \end{array}$$

normalize to $\Delta\phi = 1$
+ nondimensionalize velocity



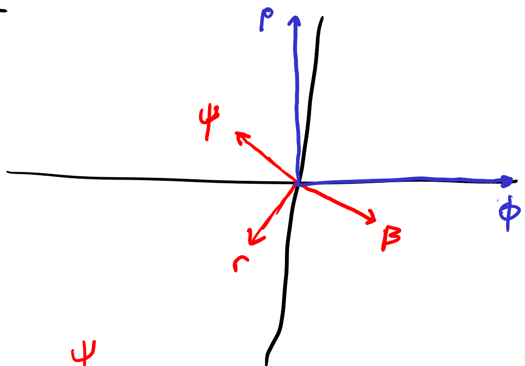
Roll Mode

$$\hat{v}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \begin{array}{l} \hat{v} = \beta \leftarrow \text{small} \\ p \leftarrow \text{large} \\ r \leftarrow \text{small} \\ \phi \leftarrow \text{large} \\ \psi \leftarrow \text{small} \end{array}$$



Dutch Roll

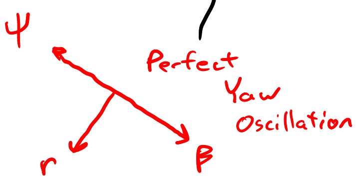
$$\hat{v}_3 = \begin{bmatrix} 0.321\angle -28^\circ \\ 0.9471\angle 92^\circ \\ 0.2915\angle -112^\circ \\ 1.0 \\ 0.3078\angle 155^\circ \end{bmatrix} \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$

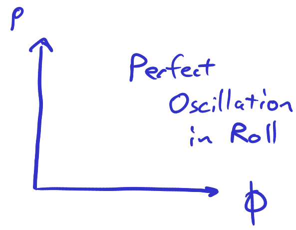


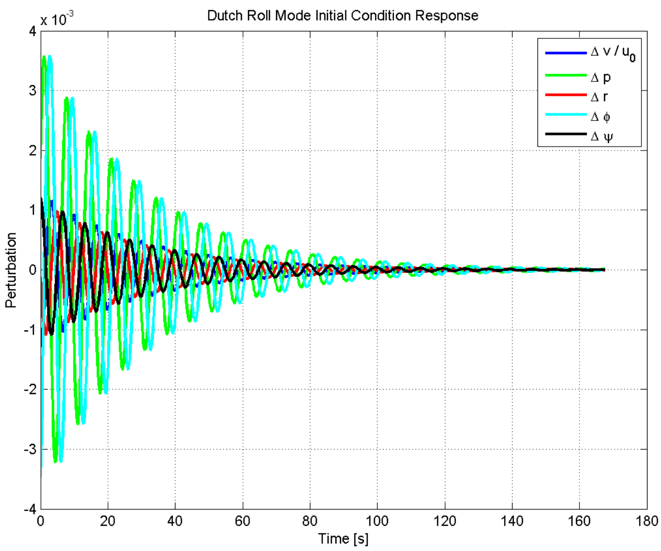
$\lambda_{3,4} = -0.033 \pm 0.947i$

$\zeta = 0.0349$ Poor Damping

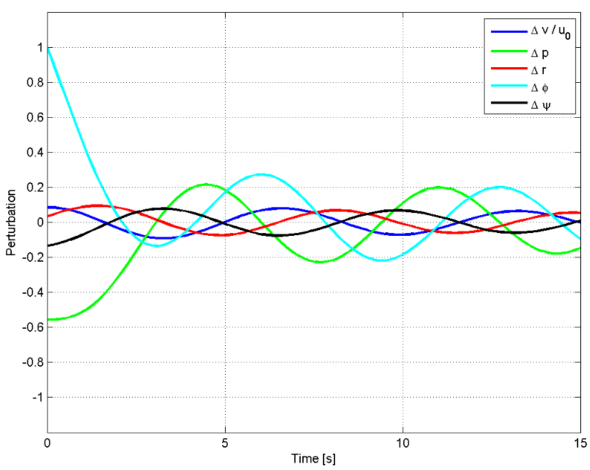
$\omega_n = 0.947$ relatively fast







$$\mathbf{x}(0) = 0.4 \cdot Re(\mathbf{v}_r) + 0.4 \cdot Re(\mathbf{v}_{dr}) + 0.2 \cdot Re(\mathbf{v}_{spi})$$



Lateral Mode Approximations

$$A_{lat} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

e.g. $\mathcal{Y}_v = \frac{Y_v}{m}$

Roll Approximation

$r = 0 \quad v = 0$

$$\dot{p} = \mathcal{L}_p p$$

$|A - \lambda I|$ if A is a scalar $\lambda = A$

$$\lambda_{r,approx} = \mathcal{L}_p$$
$$= -0.434$$

$\lambda_r = -0.562$

23% difference

2x2 Spiral Approximation

$\rightarrow p = 0$

$\rightarrow \dot{p} = 0$

Ignore side force

$$\begin{bmatrix} \dot{v} \\ \cancel{\dot{p}} \to 0 \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cancel{\mathcal{Y}_v} & \cancel{\mathcal{Y}_p} & \cancel{\mathcal{Y}_r} & \cancel{g\cos\theta_0} \\ \mathcal{L}_v & \cancel{\mathcal{L}_p} & \mathcal{L}_r & \cancel{0} \\ \mathcal{N}_v & \cancel{\mathcal{N}_p} & \mathcal{N}_r & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = \mathcal{L}_v v + \mathcal{L}_r r \Rightarrow v = -\frac{\mathcal{L}_r}{\mathcal{L}_v} r$$

$$\dot{r} = -\mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} r + \mathcal{N}_r r = \underbrace{\left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right)}_{\lambda_{s,approx}} r$$

$\lambda_{s,approx} = -0.0296$

$\lambda_s = -0.0073$

not great

Characteristic-Eqn.-based spiral approx

$$|A_{lat} - \lambda I| = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

since $\lambda_s << 1$

$$D\lambda + E = 0$$

$$\lambda_{s,approx} = -\frac{E}{D}$$

$$E = g\left[(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r)\cos\theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p)\sin\theta_0\right]$$

$$D = -g(\mathcal{L}_v \cos\theta_0 + \mathcal{N}_v \sin\theta_0) + u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

for 747 $\quad \lambda_{s,approx} = -0.00725 \quad$ very close!

In Book

Characteristic-based spiral + roll eigen value approximation

Dutch Roll Approx

Assume $\phi = p = 0$ $\qquad \frac{Y_r}{m} << u_0$

$$\begin{bmatrix} \mathcal{Y}_v & \cancel{\mathcal{Y}_p} & \cancel{\mathcal{Y}_r} \to u_0 & \cancel{g\cos\theta_0} \\ \cancel{\mathcal{L}_v} & \cancel{\mathcal{L}_p} & \cancel{\mathcal{L}_r} & \cancel{0} \\ \mathcal{N}_v & \cancel{\mathcal{N}_p} & \mathcal{N}_r & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\lambda + (\mathcal{Y}_v \mathcal{N}_r + u_0 \mathcal{N}_v) = 0$$

$$\lambda_{dr,\,approx} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Laplace Transforms and Transfer Functions

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 u(t)$$

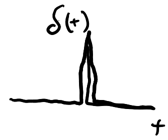
$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} + Du \end{aligned} \quad \Longleftrightarrow \quad G_{yu}(s)$$

Review: Properties of Laplace transforms

$$\mathcal{L}[x(t)](s) = \int_0^\infty e^{-st} x(t)\, dt$$

$$x(t) \Longleftrightarrow X(s)$$

Appendix A1



$$\dot{x}(t) \Longleftrightarrow sX(s) - x(t=0)$$

$$\int_0^t x(\tau)\,d\tau \Longleftrightarrow \frac{1}{s} X(s)$$

$4\sin(2t) + 2\cos(t)$

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha x(s) + \beta y(s)$$

$$\frac{4\cdot 2}{s^2+2^2} + \frac{2s}{s^2+1}$$

Review: Transfer Function

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 u(t)$$

Assume 0 initial conditions

$$s^2 x(s) + 2\zeta\omega_n s x(s) + \omega_n^2 x(s) = \omega_n^2 u(s)$$

$$\underline{(s^2 + 2\zeta\omega_n s + \omega_n^2)}\, x(s) = \omega_n^2 \underline{u(s)}$$

Transfer Function

$$\boxed{G_{xu}(s) \equiv \frac{x(s)}{u(s)}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

0 initial conditions

Can you find a TF for any differential equation?

$$\ddot{x}(t) = -\dot{x}(t)^2 + u(t)$$

nonlinear

✗ no TF for nonlinear diff eq.

Review: How to use a TF

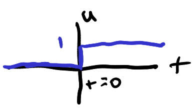
1. Input → Output

$$u(t) \to u(s) \longrightarrow x(s) = G_{xu}(s)\,u(s) \longrightarrow x(t)$$

$$\frac{s+1}{s^3+2s^2+3s+4} \to \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \cdots \xrightarrow{\text{table}} x(t)$$

partial frac decomposition

Ex. $u(t)$ = step function



$$u(s) = \frac{1}{s}$$

$$x(s) = G_{xu}(s)\,u(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{1}{s}$$

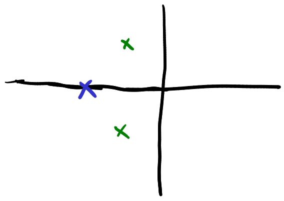
$$x(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

2. Stability

$\underbrace{\text{Roots of TF denominator}}_{\text{a.k.a. poles}}$ are eigenvalues of A matrix

↰ $G(s) \to \infty$ if s is at a pole

$\frac{1}{s+1}$, pole $= -1$



If all poles are on LHP system is stable

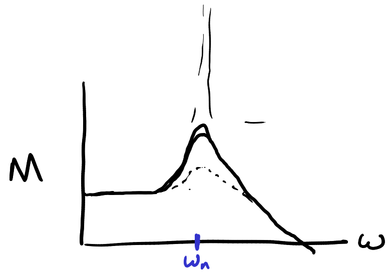
$$G_{xu}(s) = \frac{1}{(s+2)\underline{(s^2+2s+2)}}$$

$s=-2$ $\quad$ $s=-1\pm i$

3. Steady-State Behavior

a) Final Value Theorem

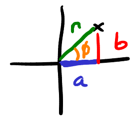
$$\boxed{\text{If } sx(s) \text{ is stable} \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)}$$



b) Frequency Response / Harmonic Response

$$\boxed{\text{If } u(t) = A\cos(\omega t) \text{ the steady-state } x(t) \text{ is } AM\cos(\omega t+\phi) \quad M = |G_{xu}(i\omega)| \text{ and } \phi = \angle G_{xu}(i\omega)}$$

$z = a + b_i$ or $re^{\phi i}$ or $r\angle\phi$



$$G(s) = \frac{1}{s^2+2s+2}$$

$$G(i\omega) = \frac{1}{-\omega^2+2i\omega+2}$$

$$= \frac{1}{\underbrace{-\omega^2+2}_{a} + \underbrace{2\omega i}_{b}} = \frac{1}{\underbrace{\sqrt{(-\omega^2+2)^2+(2\omega)^2}}_{r}\, e^{i\,\underbrace{\operatorname{atan}\left(\frac{2\omega}{-\omega^2+2}\right)}_{\phi}}}$$

$$= \frac{1}{r} e^{-i\phi}$$

$$\rightarrow \quad \begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} + Du \end{aligned} \quad \Longleftrightarrow \quad G_{yu}(s)$$

State Space to TF $\qquad$ ss2tf $\qquad$ tf(sys)

$$s\vec{x}(s) = A\vec{x}(s) + Bu(s)$$

$$(sI-A)\vec{x}(s) = Bu(s)$$

$$\vec{x}(s) = (sI-A)^{-1}Bu(s)$$

$$y(s) = C\vec{x}(s) + Du(s)$$

$$y(s) = C(sI-A)^{-1}Bu(s) + Du(s)$$

assume $D=0$ from here on

$$M^{-1} = \frac{\operatorname{adj}(M)}{|M|} \qquad (sI-A)^{-1} = \frac{\operatorname{adj}(sI-A)}{|sI-A|}$$

Adjugate: Transpose of Cofactor Matrix F

$$F_{ij} = (-1)^{i+j} |M_{-i-j}|$$

$-i$ = all except i

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \cdots \\ \vdots & & \end{bmatrix}$

$$\operatorname{adj}(M) = F^T$$

$$G_{yu}(s) = \frac{Y(s)}{u(s)} = \boxed{C(sI-A)^{-1}B} = \boxed{\frac{C \operatorname{adj}(sI-A) B}{|sI-A|}} = \frac{N(s)}{D(s)}$$

Roots of $D(s)$ are eigenvalues of A

TF to state space

tf2ss

$\leftarrow G_{ode}$

$$G_{yu}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots b_m}{a_0 s^n + \dots \dots a_{n-1} s + a_n}$$

Ex: $G_{yu}(s) = \dfrac{b_0 s + b_1}{s^2 + a_1 s + a_2}$

multiply by $\dfrac{x(s)}{x(s)}$

$$\frac{Y(s)}{u(s)} = G_{yu}(s) = \frac{b_0 s x(s) + b_1 x(s)}{s^2 x(s) + a_1 s x(s) + a_2 x(s)}$$

$\rightarrow$ $y(t) = b_0 \dot{x}(t) + b_1 x(t)$ $\leftarrow$

$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$

$\rightarrow$ $\ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u(t)$ $\leftarrow$

$\dot{\vec{x}} = A\vec{x} + Bu$

$y = C\vec{x} + Du$

$$\rightarrow \overset{\dot{\vec{x}}}{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}} = \underset{A}{\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}} + \underset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u \leftarrow$$

$$y = \underset{C}{\begin{bmatrix} b_1 & b_0 \end{bmatrix}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underset{D}{[0]} u \leftarrow$$

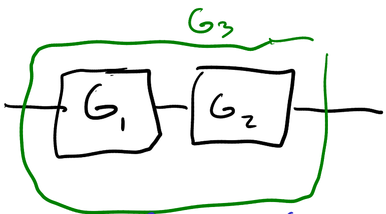
$$G_{y_u}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots b_m}{a_0 s^n + \quad \dots \quad a_{n-1} s + a_n}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & & & & 0 \\ & & & & & 1 \\ -a_n & \cdots & & \cdots & & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

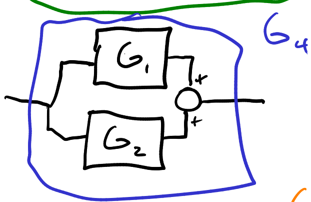
$$C = [b_m \dots \quad b_0 \underbrace{0\ 0\ 0}_{\text{if } n > m+1}] \qquad D = [0]$$

$\vec{x}$ may not correspond to physical states

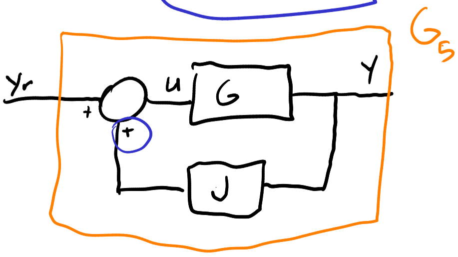
Review: Block Diagrams



$G_3 = G_1 G_2$



$G_4 = G_1 + G_2$



$G_5 = \frac{y}{y_r}$

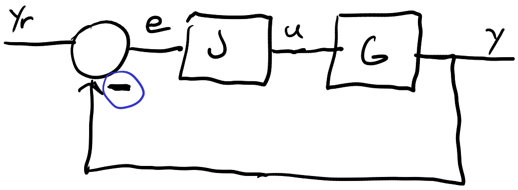
$y = Gu$

$u = y_r + Jy$

$y = G(y_r + Jy)$

$(1 - JG)\,y = G\,y_r$

$\frac{y}{y_r} = \boxed{\frac{G}{1 - JG}}$



$y = JGe$

$e = y_r - y$

$y = JG(y_r - y)$

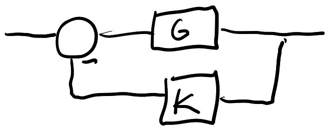
$(1 + JG)\,y = JG\,y_r$

$\frac{y}{y_r} = \boxed{\frac{JG}{1 + JG}}$

$\frac{G}{1 + JG}$ if −'ve feedback

$\frac{JG}{1 - JG}$ for +'ve feedback

TFs and Root Loci



$G \leftarrow$ open loop TF

$$G^{cl} = \frac{G}{1+KG}$$

$$k=0 \Rightarrow G^{cl} = G$$

$\boxed{\text{Root locus starts at poles of the OL systems}}$

~~$G^{cl} = \frac{G}{1+KG} = \frac{\frac{1}{K}G}{\frac{1}{K}+G}$~~

~~as $k \to \infty$ CL poles approach the zeros of G~~

$J = K$

$$G^{cl} = \frac{G}{1+KG} \quad \text{or} \quad \frac{KG}{1+KG}$$

$$1 + KG = 0$$

$$\frac{1}{K} + G = 0$$

$$G(s) = \frac{N(s)}{D(s)}$$

As K gets large $G(s) = -\frac{1}{K}$

When K is large, two possibilities for CL poles

1. N(s) is close to zero

   $\boxed{\text{CL poles close to O.L. zeros}}$

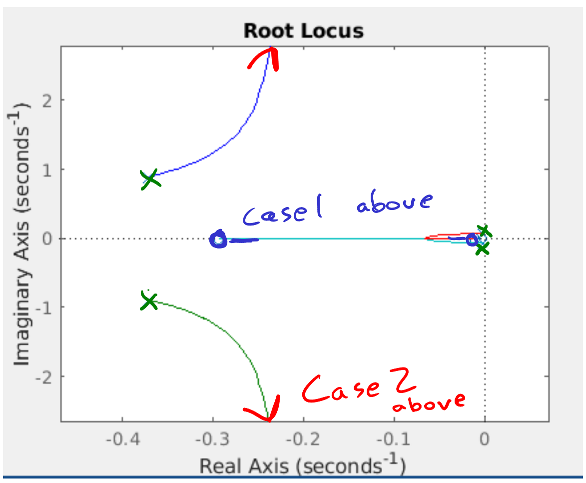
2. D(s) is very large

   $\boxed{\text{Magnitude of CL poles is very large}}$

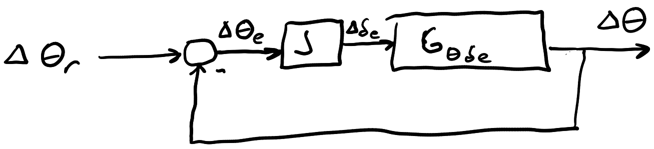
B 744 Long dynamics with $\Delta\delta_e = -k\,\Delta\theta$

O.L Poles : $-0.37 \pm 0.89i$
$-0.0033 \pm 0.067i$

O.L. Zeros : -0.0113, -0.2948



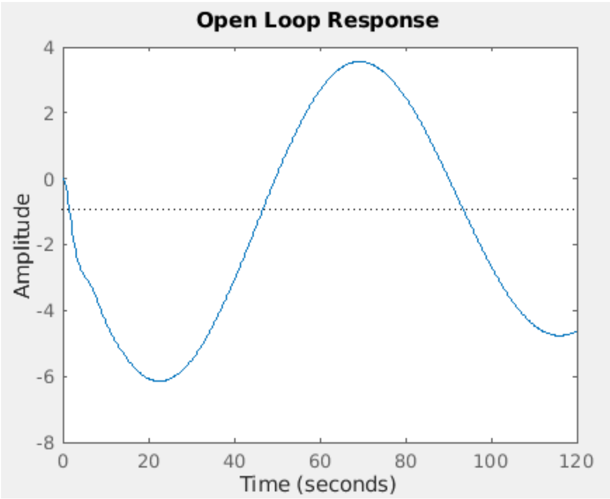
Designing a Pitch-Hold Controller



$\Delta\theta_e = \Delta\theta_r - \Delta\theta$

Goals

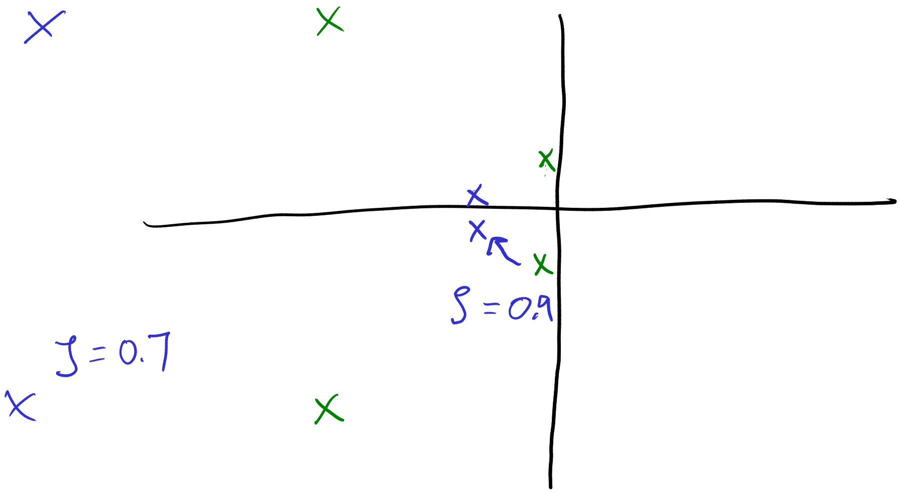
1. Improve Damping (O.L. Phugoid has low damping)
2. Low steady-state error $|\theta - \theta_r| \to 0$



Previously

$$\Delta\delta_e = \underbrace{-k_\theta}_{\text{proportional}} \Delta\theta - \underbrace{k_q}_{\text{derivative}} \Delta q$$

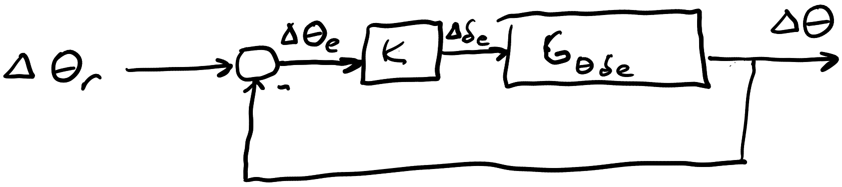
gain



$k_\theta = -0.5$

$k_q = -1$

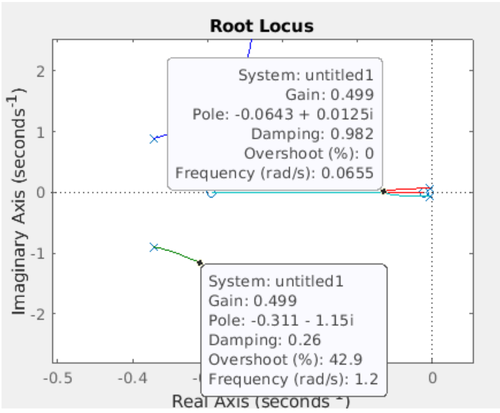
Controller 1 — Proportional Controller

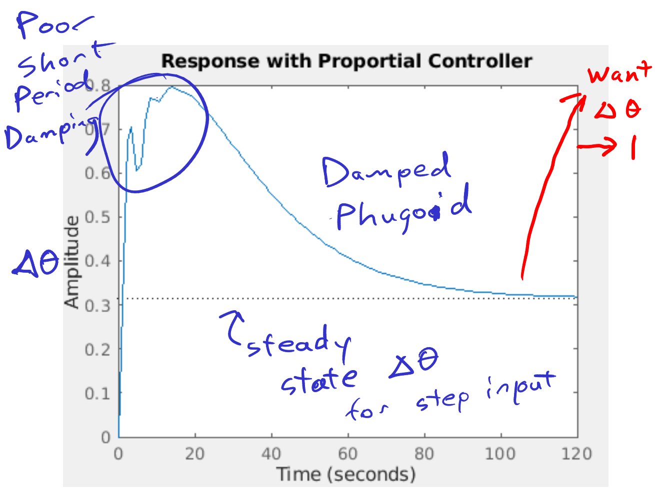


$J = K$

$\Delta\delta_e = K(\Delta\theta_r - \Delta\theta)$

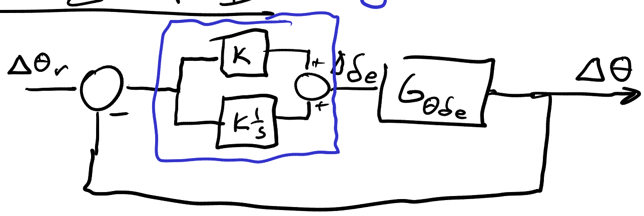
Choose $K = -0.5$

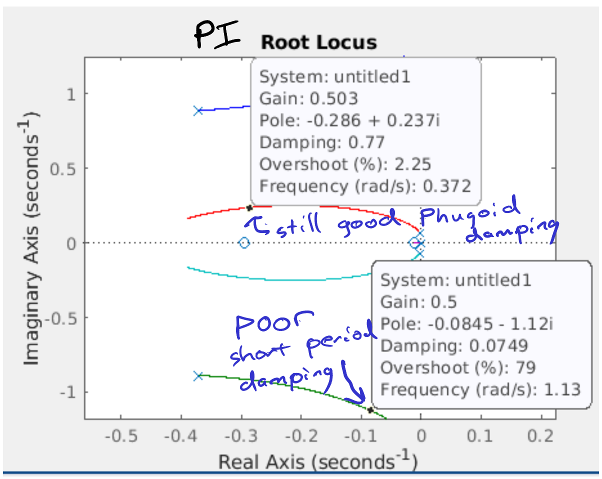


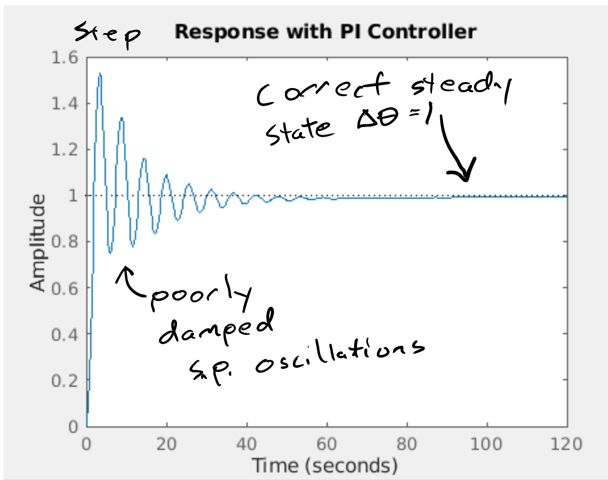


Controller 2: PI

$J = K\left(1 + \frac{1}{s}\right)$

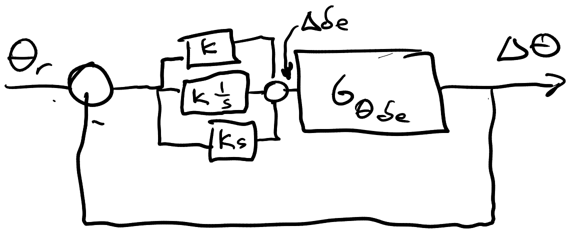


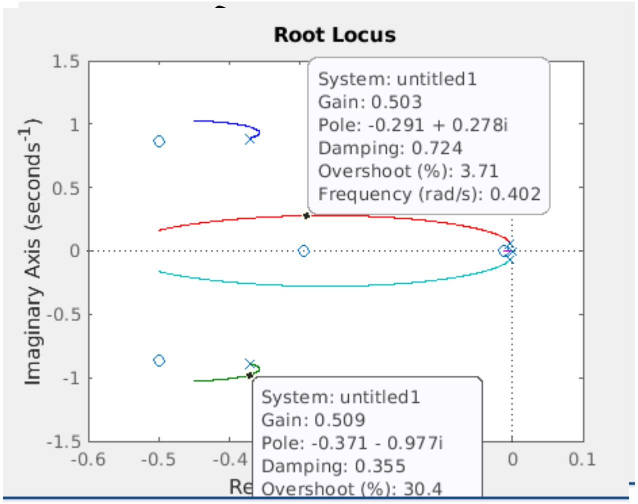




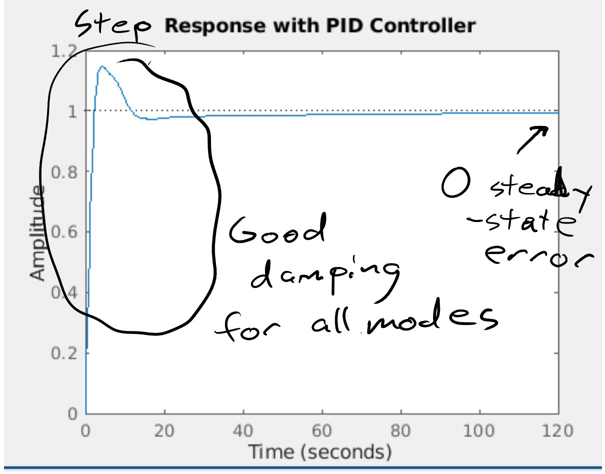
Controller 3: PID

$J = K\left(1 + \frac{1}{s} + s\right)$



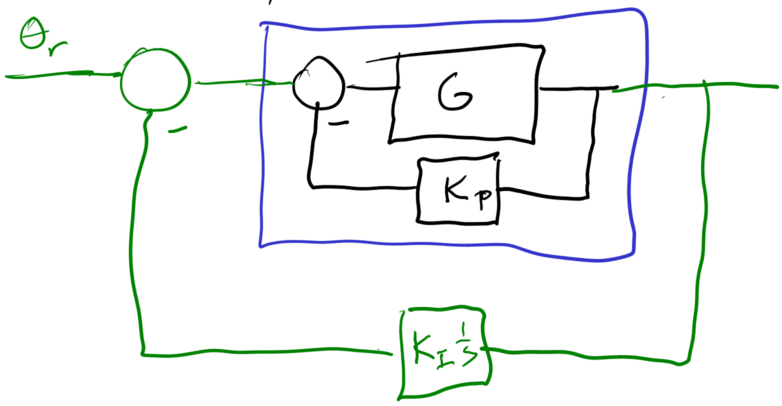


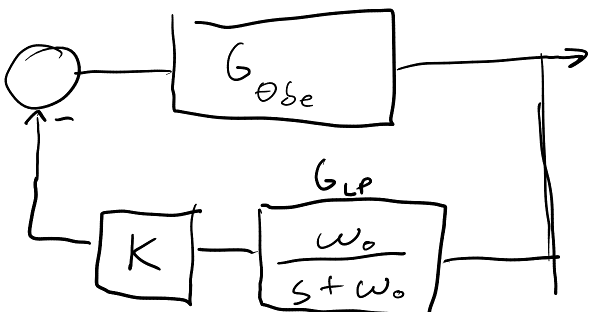
Choose $K = -0.5$

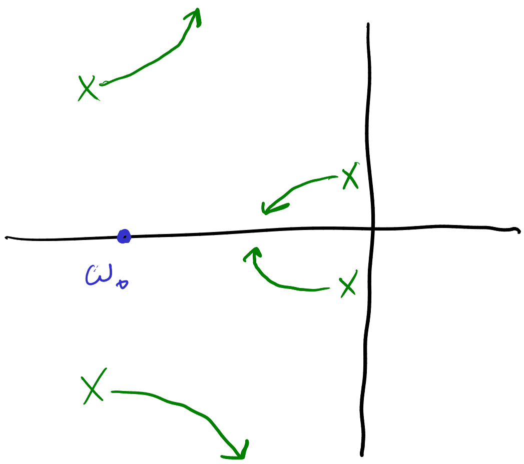


$$J = -0.5\left(1 + \frac{1}{s} + s\right)$$

If you wanted to tune individually

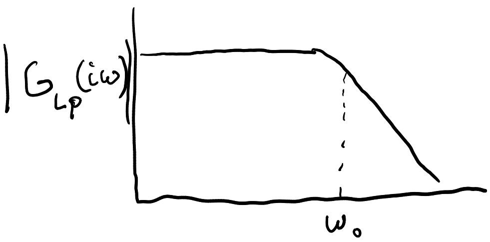




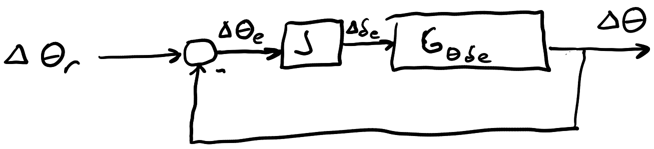


Frequency response

P controller



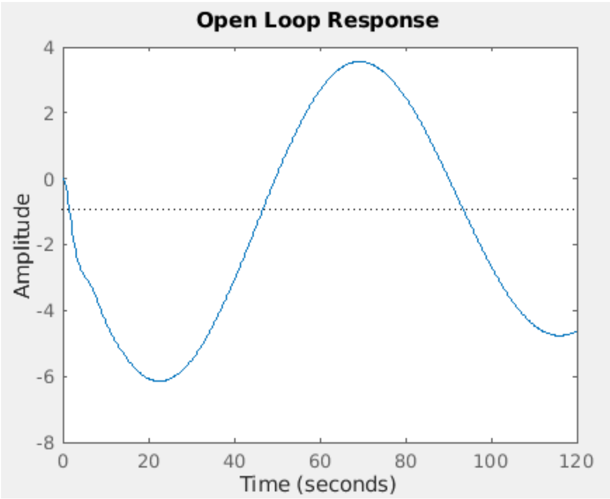
Designing a Pitch-Hold Controller



$\Delta\theta_e = \Delta\theta_r - \Delta\theta$

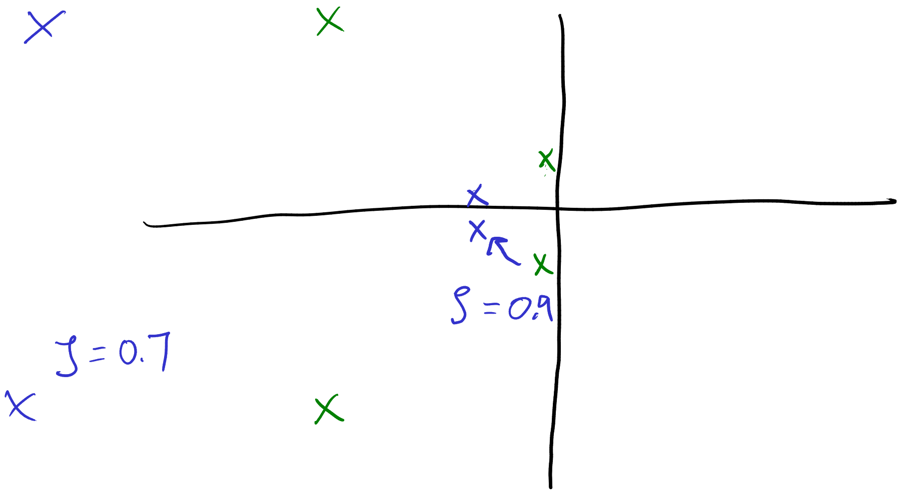
Goals

1. Improve Damping (O.L. Phugoid has low damping)
2. Low steady-state error $|\theta - \theta_r| \to 0$



Previously

$$\Delta\delta_e = \underbrace{-k_\theta}_{\text{proportional}} \Delta\theta - \underbrace{k_q}_{\text{derivative gain}} \Delta q$$

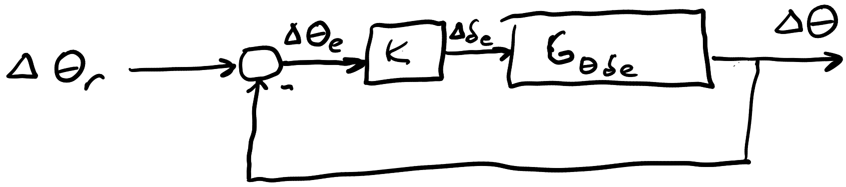


$k_\theta = -0.5$

$k_q = -1$

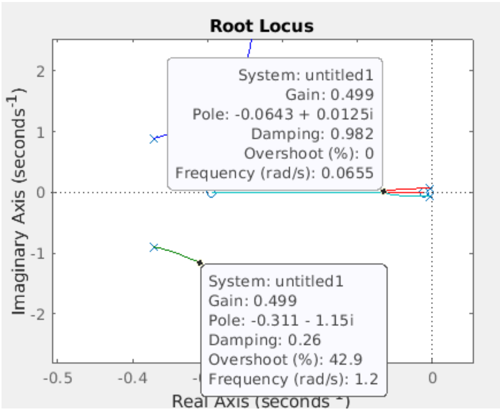
Controller 1

Proportional Controller

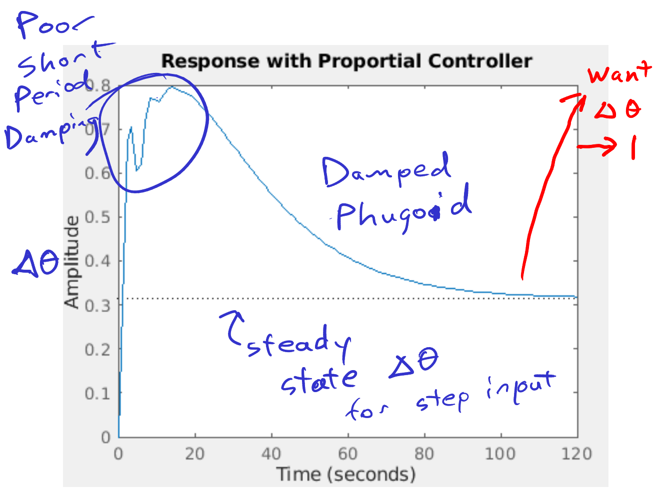


$$J = K$$

$$\Delta\delta_e = K(\Delta\theta_r - \Delta\theta)$$

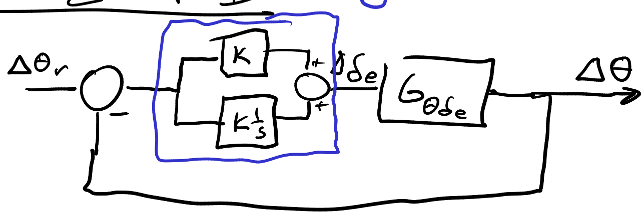


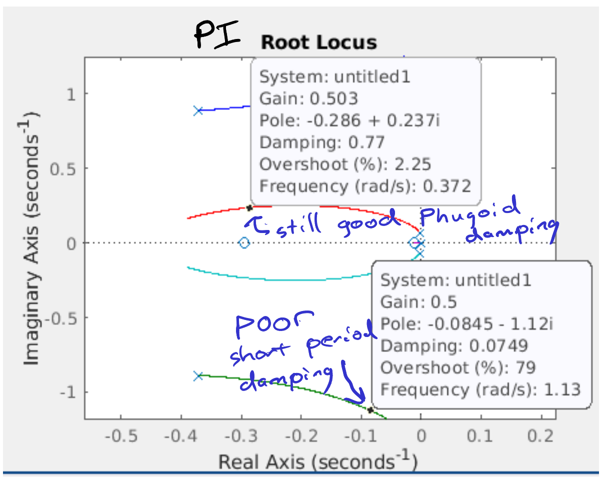
Choose $K = -0.5$

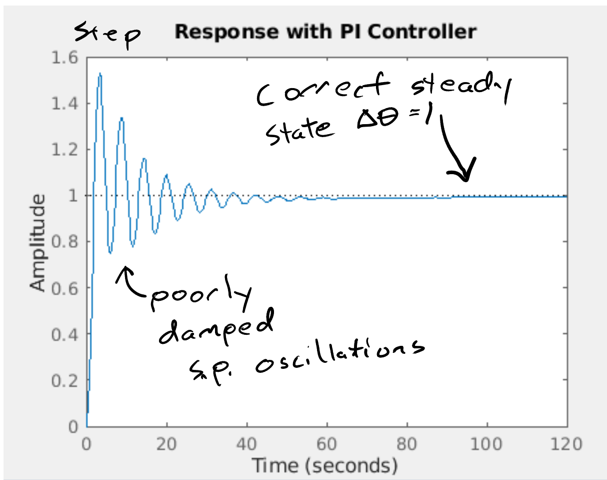


Controller 2: PI

$$J = K\left(1 + \frac{1}{s}\right)$$

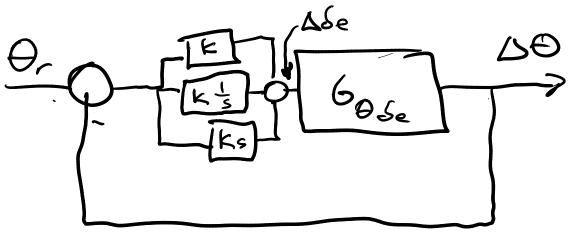


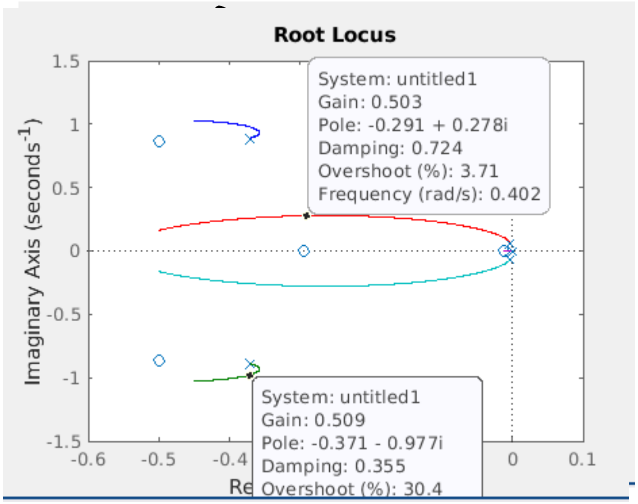




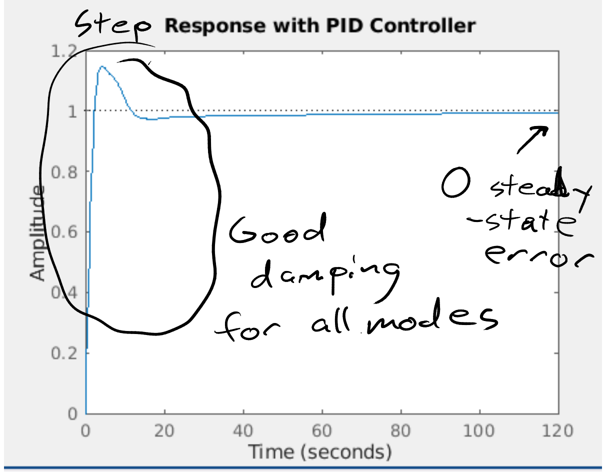
Controller 3: PID

$$J = K\left(1 + \frac{1}{s} + s\right)$$



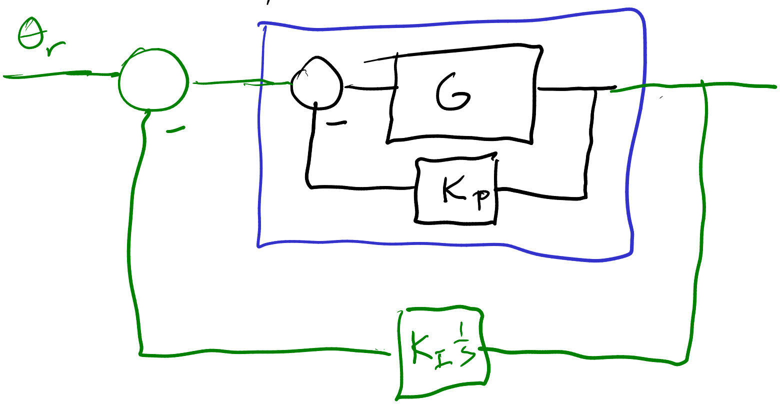


Choose $K = -0.5$

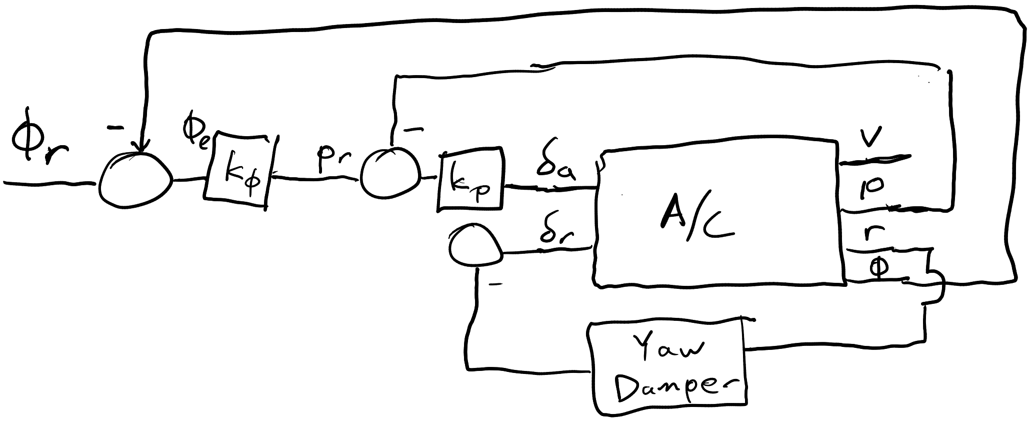


$$J = -0.5\left(1 + \frac{1}{s} + s\right)$$

If you wanted to tune individually



Designing a Roll Controller



Part 1: Yaw Damper

$\delta_r = -k_r r$