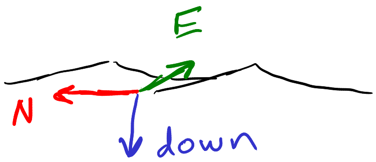
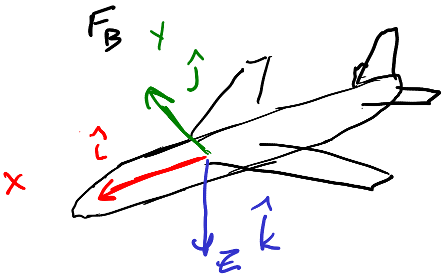
Notation and Conventions

F_I or F_E





- Frame of reference : Collection of ≥ 3 points w/ constant distance between each other

- Inertial frame : A frame that translates with constant (possibly 0) velocity and does not rotate
  - Newton's second law is valid in an inertial frame

- Coordinate System : Three orthogonal unit vectors that allow measurement and vector representation

- Vector Notation

$\rightarrow$ or **bold** : vector

$\wedge$: unit vector

object that we are referring to

often implied

$^{a/c}\vec{V}^{E}_{B}$

frame that this vector is relative to
often implied

"with respect to"
"relative to"

coordinate system

"written in"
"expressed in"
"coordinate frame"

$$\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

"inertial velocity written in body coordinates"

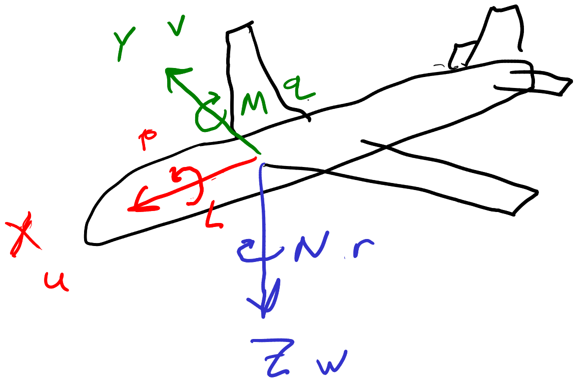
$\vec{V}^{W}_{B}$ $\qquad$ $\vec{V}^{E}_{E}$

Forces, Moments, and Velocities

$$\vec{A} = \underbrace{X\hat{i} + Y\hat{j} + Z\hat{k}}$$

Aerodynamic (incl. propulsion NOT gravity)

$$\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

$$\vec{V}^E_B = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{\omega}^E_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$V_g = |\vec{V}^E| = \sqrt{u^{E2} + v^{E2} + w^{E2}}$$

Anatomy of an Airplane aka. "Conventional A/C"

Ailerons

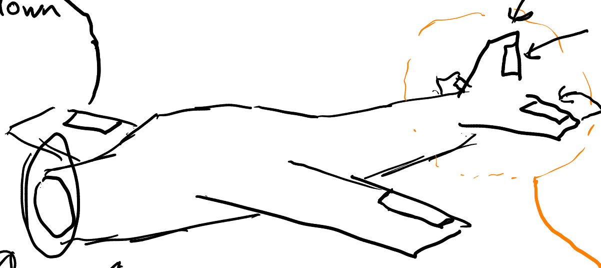
$+\delta_a$ = right down

$\delta_a \rightsquigarrow -L$

Propulsion

$\delta_p \rightsquigarrow +X$

Sometimes δ_t throttle



Vertical Stabilizer "Vert. Fin" "Tail Fin"

Rudder $+\delta_r$: toward $-y$

$+\delta_r \rightsquigarrow +N, Y$

Horizontal Stabilizer

Elevator $+\delta_e$: down

$+\delta_e \rightsquigarrow -M, -Z$

Empennage "Tail"

Propeller

Fuselage

Wing

Wind

Aerodynamic Forces and Moment (i.e. not gravity) are functions of the A/C velocity w.r.t. the Air

$\vec{V}^{W}$

by convention $\vec{V} \equiv \vec{V}^{W}$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$V = |\vec{V}|$

$$\boxed{\vec{V}^{E} = \vec{V}^{(W)} + \vec{W}^{(E)}}$$

When $\vec{W} = \vec{0}$, $\vec{V} = \vec{V}^{E}$

Example

Wind blowing east @ 8 m/s

A/C pointing north and traveling northward with respect to the wind at 60 m/s

What is the A/c velocity vector relative to earth in NED

$$\vec{V}^{E}_{E} = \vec{V}_E + \vec{W}_E$$

$$= \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 8 \\ 0 \end{bmatrix}$$

$\cancel{\vec{V}_B + \vec{W}_E}$

Illegal!

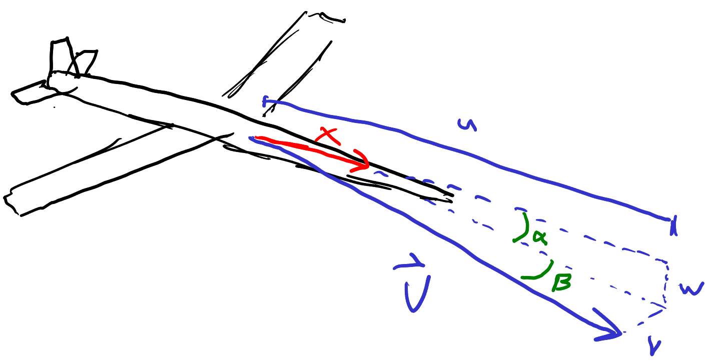
Wind Angles

Angle of Attack

$\alpha \equiv \operatorname{atan}\left(\frac{w}{u}\right)$

Sideslip Angle

$\beta \equiv \sin^{-1} \frac{v}{V}$



$u = V \cos\beta \cos\alpha$

$v = V \sin\beta$

$w = V \cos\beta \sin\alpha$

Orientation

axis		E.G.
"1"	ϕ : roll	45°
"2"	θ : pitch	90°
"3"	ψ : yaw	45°

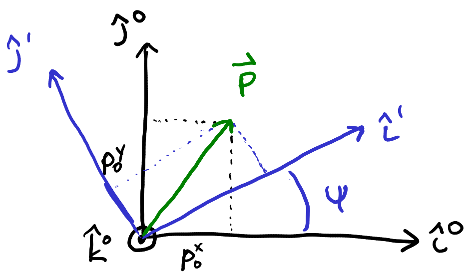
By convention A/C orientation is defined by a 3-2-1 sequence of rotations throug $\psi - \theta - \phi$

Key task: changing the coordinate sys. that a vector is written in

Ex.

know: $\vec{V}_B^E$ want: $\vec{V}_E^E$

F_0 F_1



$$\vec{p}_0 = \begin{array}{c} \text{know} \\ \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix} \end{array} \qquad \vec{p}_1 = \begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix}$$

$$\vec{p} = p_0^x \hat{i}^0 + p_0^y \hat{j}^0 + p_0^z \hat{k}^0$$

F_1 is related to F_0 by a 3-rotation through ψ

want p_1^x in terms of $\vec{p}$

$$p_1^x = \vec{p} \cdot \hat{i}'$$
$$= p_0^x \hat{i}^0 \cdot \hat{i}' + p_0^y \hat{j}^0 \cdot \hat{i}' + p_0^z \hat{k}^0 \cdot \hat{i}'$$
$$= p_0^x \cos\psi + p_0^y \sin\psi + p_0^z \cdot 0$$
$$p_1^x = [\cos\psi \quad \sin\psi \quad 0] \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

General

$$R_0^1 = \begin{array}{ccc} \hat{i}^0 & \hat{j}^0 & \hat{k}^0 \end{array} \atop \begin{bmatrix} \hat{i}^0\cdot\hat{i}' & \hat{j}^0\cdot\hat{i}' & \hat{k}^0\cdot\hat{i}' \\ \hat{i}^0\cdot\hat{j}' & \hat{j}^0\cdot\hat{j}' & \hat{k}^0\cdot\hat{j}' \\ \hat{i}^0\cdot\hat{k}' & \hat{j}^0\cdot\hat{k}' & \hat{k}^0\cdot\hat{k}' \end{bmatrix}$$

$$\begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix} = \underbrace{\begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\psi)} \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

axis angle

R_E^B R_B^E

$$\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$$

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

Properties of Direction Cosine Matrices (DCMs)

$\vec{p}_B = R_A^B \vec{p}_A$

Book: $\vec{p}_B = L_{BA} \vec{p}_A$

1. Chaining

$$\vec{p}_C = R_B^C \vec{p}_B \qquad \vec{p}_B = R_A^B \vec{p}_A$$

$$= \underbrace{R_B^C R_A^B}_{R_A^C} \vec{p}_A$$

$$\boxed{R_A^C = R_B^C R_A^B}$$

2. Inverse

$$\vec{p}_B = R_A^B \vec{p}_A$$

$$(R_A^B)^{-1} \vec{p}_B = \cancel{(R_A^B)^{-1} R_A^B} \vec{p}_A$$

$$R_B^A \vec{p}_B = \vec{p}_A \qquad R_B^A = (R_A^B)^{-1}$$

Since DCMs are orthonormal (columns are orthogonal unit vectors)

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_A^B (R_A^B)^T = I$$

$$(R_A^B)^{-1} = (R_A^B)^T$$

$$\boxed{R_B^A = R_A^{B\,T}}$$

Earth-to-body DCM

3-2-1 rotation through ψ, θ, ϕ

$$\vec{p}_B = \underbrace{R_1(\phi) R_2(\theta) R_3(\psi)}\, \vec{p}_E$$

$$\boxed{\vec{p}_B = R_E^B \vec{p}_E}$$

$c_\theta = \cos\theta$
$s_\theta = \sin\theta$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

Example:

Want: ${}^{pilot}\vec{p}_E^{\,E}$ know ${}^{B}\vec{p}_E^{\,E}$, ${}^{pilot}\vec{p}_B^{\,B}$ ✗, R_E^B

$${}^{pilot}\vec{p}^{\,E} = {}^{B}\vec{p}^{\,E} + {}^{pilot}\vec{p}^{\,B}$$

$${}^{pilot}\vec{p}_E^{\,E} = {}^{B}\vec{p}_E^{\,E} + {}^{pilot}\vec{p}_E^{\,B} \text{✗} = {}^{B}\vec{p}_E^{\,E} + R_B^E\, {}^{pilot}\vec{p}_B^{\,B}$$

$(R_E^B)^T$

Kinematics

Kinematics: "Geometry of motion" (no forces)

Dynamics/Kinetics: Effects of forces and moments on an object

Vector derivatives

$\vec{p}$: shorthand $\vec{p}^E$

$\frac{d}{dt}\vec{p} \equiv$ time rate of change of $\vec{p}$ $\leftarrow$ "Vector derivative"

$\vec{V}^E \equiv \frac{d}{dt}\vec{p}$

$\dot{\vec{p}}_B \equiv$ time rate of change of elements "coordinates" of $\vec{p}_B$

$$\text{if} \quad \vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} \quad \text{then} \quad \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Question

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \overset{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix} \leftarrow$$

Not always true

$\hat{i}_B$: shorthand for $\hat{i}_B^B$

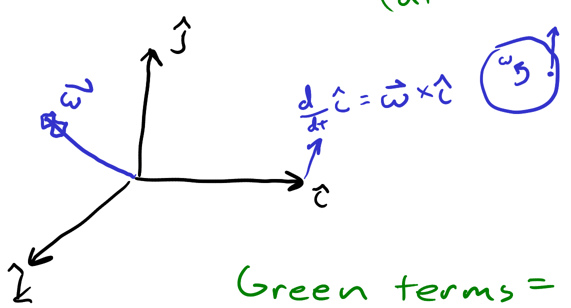
Recall: $\frac{d}{dt}(uv) = u\frac{d}{dt}v + v\frac{d}{dt}u$

$$\vec{p}_B = x_B\hat{i}_B + y_B\hat{j}_B + z_B\hat{k}_B$$

$$\rightarrow \dot{\vec{p}}_B = \dot{x}_B\hat{i}_B + \dot{y}_B\hat{j}_B + \dot{z}_B\hat{k}_B$$

$$\left(\frac{d}{dt}\vec{p}\right)_B = \boxed{\dot{x}_B\hat{i}_B} + \underline{x_B\frac{d}{dt}\hat{i}_B} + \boxed{\dot{y}_B\hat{j}_B} + \underline{y_B\frac{d}{dt}\hat{j}_B} + \boxed{\dot{z}_B\hat{k}_B} + \underline{z_B\frac{d}{dt}\hat{k}_B}$$

what is $\left(\frac{d}{dt}\hat{i}\right)_B$



$$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Green terms $= x_B(\vec{\omega}_B\times\hat{i}_B) + y_B(\vec{\omega}_B\times\hat{j}_B) + z_B(\vec{\omega}_B\times\hat{k}_B)$

$= \vec{\omega}_B\times\vec{p}_B$

$$\boxed{\left(\frac{d}{dt}\vec{p}\right)_B = \dot{\vec{p}}_B + \vec{\omega}_B\times\vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B\vec{p}_B \qquad \tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

"Kinematic Transport Theorem"

$$\left(\frac{d}{dt}\vec{p}\right)_E = \dot{\vec{p}}_E + \underbrace{\cancel{{}^E\vec{\omega}_E^E}}_{0}\times\vec{p}_E = \dot{\vec{p}}_E$$

Aircraft Equations of Motion (EOM)

$$\dot{\vec{x}} = f(t,\vec{x},\vec{u}) \longrightarrow \text{Need}$$

$$\begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{p}_E = \vec{p}_E^{\,E} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\Theta} \text{ "pseudo-vector" array of numbers} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{V}_B^E \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\omega}_B^E \end{matrix}$$

Need:

$\dot{\vec{p}}_E$ ← Kinematics

$\dot{\vec{\Theta}}$ ← Kinematics

$\dot{\vec{V}}_B^E$ ← Dynamics

$\dot{\vec{\omega}}_B$ ← Dynamics

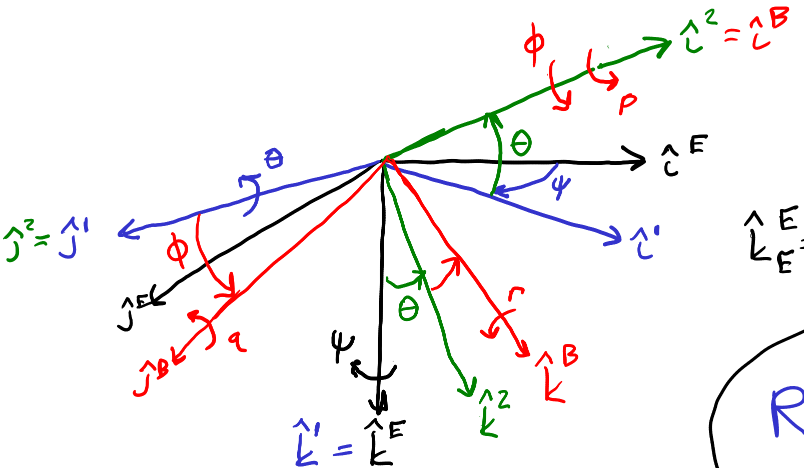
Translational Kinematics

$$\dot{\vec{p}}_E^{\,E} = \frac{d}{dt}\vec{p}_E - \underbrace{\cancel{{}^E\vec{\omega}_E^E}}_{0} \times \vec{p}_E^{\,E} = \frac{d}{dt}\vec{p}_E = \vec{V}_E^E = R_B^E \vec{V}_B^E$$

$$\boxed{\dot{\vec{p}}_E = \left(R_E^B\right)^T \vec{V}_B^E}$$

Rotational Kinematics

want: $\dot{\vec{\Theta}} = \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$ have: $\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$



$$\vec{\omega} = p\,\hat{i}^B + q\,\hat{j}^B + r\,\hat{k}$$

$$\vec{\omega} = \dot\psi\,\hat{k}^E + \dot\theta\,\hat{j}^1 + \dot\phi\,\hat{i}^B$$

$$\vec{\omega}_B = \dot\psi\,\hat{k}^E_B + \dot\theta\,\hat{j}^1_B + \dot\phi\,\hat{i}^B_B$$

$$\hat{k}^E_E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{\omega}_B = \dot\psi\,\underbrace{R_E^B}\,\underbrace{\hat{k}^E_E} + \dot\theta\,\underbrace{R_1^B}\,\hat{j}^1_1 + \dot\phi\,\hat{i}^B_B$$

$$R_E^B = \underbrace{R_1(\phi)R_2(\theta)R_3(\psi)}_{B \quad 2 \quad 1 \quad E} \qquad R_1^B = R_1(\phi)R_2(\theta)$$

$$\vec{\omega}_B = \dot\psi\,R_1(\phi)R_2(\theta)R_3(\psi)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot\theta\,R_1(\phi)R_2(\theta)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot\phi\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix}}_{\text{invert}} \underbrace{\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}}_{\dot{\vec{\Theta}}}$$

$t\theta = \tan\theta$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi / c\theta & c\phi / c\theta \end{bmatrix}}_{} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Eqn. 4.4, 7 in book

$$\boxed{\dot{\vec{\Theta}} = T \vec{\omega}_B}$$

T = "attitude influence matrix"

Dynamics $\quad \vec{f} \quad \vec{G}$

Translational Dynamics

Newton's 2nd Law

$$\vec{f} = m\vec{a}$$

$$\vec{f} = m \frac{d}{dt}\vec{V}^E$$

want $\dot{\vec{V}}^E_B$

$$\left(\frac{d}{dt}\vec{V}^E\right)_B = \dot{\vec{V}}^E_B + {}^B\vec{\omega}^E_B \times \vec{V}^E_B$$

$$\dot{\vec{V}}^E_B = \frac{d}{dt}\vec{V}^E_B - \vec{\omega}_B \times \vec{V}^E_B$$

$$\boxed{\dot{\vec{V}}^E_B = \frac{\vec{f}_B}{m} - \underbrace{\vec{\omega}_B \times \vec{V}^E_B}_{\tilde{\omega}_B \vec{V}^E_B}}$$

Rotational Dynamics

"Euler's 2nd Law"

$$\frac{d}{dt}\vec{h} = \vec{G}$$

angular momentum

moment

$I = I_B$ book

$$\vec{h} = I\vec{\omega}$$

$$\mathbf{I} = \begin{pmatrix} \int (y^2+z^2)\,\mathrm{dm} & -\int xy\,\mathrm{dm} & -\int xz\,\mathrm{dm} \\ -\int xy\,\mathrm{dm} & \int (x^2+z^2)\,\mathrm{dm} & -\int yz\,\mathrm{dm} \\ -\int xz\,\mathrm{dm} & -\int yz\,\mathrm{dm} & \int (x^2+y^2)\,\mathrm{dm} \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

Want $\dot{\vec{\omega}}_B$

$$\frac{d}{dt}\vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \times \vec{h}_B = \vec{G}_B$$

$$I\dot{\vec{\omega}}_B + \vec{\omega}_B \times I\vec{\omega}_B = \vec{G}_B$$

$$\boxed{\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \dot{\vec{\omega}}_B = I^{-1}\left(\vec{G}_B - \vec{\omega}_B \times I\vec{\omega}_B\right)}$$

A/C EOM

$$\boxed{\dot{\vec{X}} = \begin{bmatrix} \dot{\vec{p}}^{E}_{E} \\ \dot{\vec{o}} \\ \dot{\vec{v}}^{E}_{B} \\ \dot{\vec{\omega}}^{E}_{B} \end{bmatrix} = \begin{bmatrix} \left(R^{B}_{E}\right)^{T} \vec{v}^{E}_{B} \\ T\vec{\omega}_{B} \\ \frac{\vec{f}_{B}}{m} - \vec{\omega}_{B} \times \vec{v}^{E}_{B} \\ I^{-1}(\vec{\tau}_{B} - \vec{\omega}_{B} \times I\vec{\omega}_{B} \end{bmatrix}}$$

Quadrotors

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{p}_E \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\Theta} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{v}^E_B \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\omega}_B \end{matrix}$$

A/C EOM

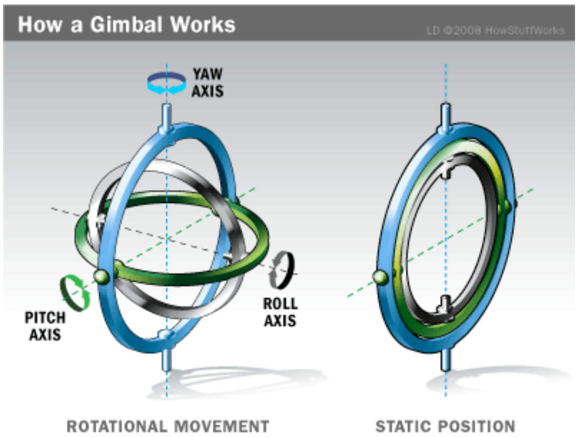
$$\boxed{\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}^E_E \\ \dot{\vec{\Theta}} \\ \dot{\vec{v}}^E_B \\ \dot{\vec{\omega}}^E_B \end{bmatrix} = \begin{bmatrix} (R^B_E)^T \vec{v}^E_B \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}^E_B \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}}$$

$$T = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi / c\theta & c\phi / c\theta \end{bmatrix}$$

From Quiz MC2

$$T = \begin{bmatrix} & 0 & \\ & 0 & \\ & 1 & \end{bmatrix} \quad \vec{\omega}_B = \begin{bmatrix} 0 \\ 10°/s \\ 0 \end{bmatrix}$$

$$\vec{\Theta} = \begin{bmatrix} 0 \\ 0 \\ 10°/s \end{bmatrix}$$



Homework P1 Monospinner

$\vec{F}$: sum of all forces acting on A/C

$\vec{F} = \underbrace{\vec{A}}_{\text{aerodynamic forces}} + {}^{g}\vec{F}$ ← gravity

$\vec{G}$: sum of all moments acting about the C.G. of A/C

Multirotor Case

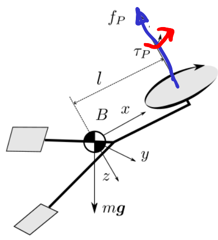
$$\vec{f} = \underbrace{{}^d\vec{f} + {}^c\vec{f}}_{\vec{A}} + {}^g\vec{f}$$

drag control

$$\vec{G} = {}^d\vec{G} + {}^c\vec{G}$$

drag control

Monospinner Assignment



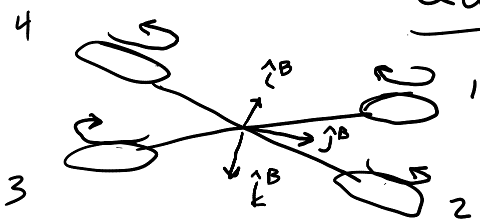
$${}^d\vec{f} = 0 \quad \text{(by assumption)}$$

$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_P \end{bmatrix}$$

$${}^d\vec{G} = 0 \quad \text{(by assumption)}$$

$${}^c\vec{G}_B = \begin{bmatrix} 0 \\ 0 \\ -\tau_P \end{bmatrix} + {}^P\vec{p}_B^{\,B} \times {}^c\vec{f}_B$$

Quadrotor Case



$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} \qquad {}^c\vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since the quadrotor is symmetric about the $\hat{i}^B$-$\hat{k}^B$ and $\hat{j}^B$-$\hat{k}^B$ planes

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad I^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

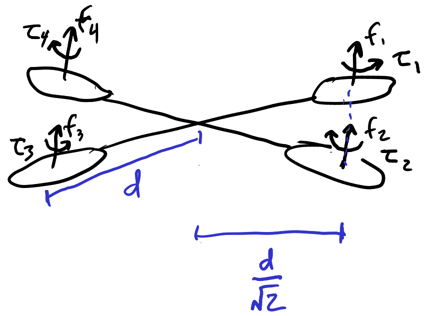
$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + \underline{g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix}} + \frac{1}{m} \overbrace{\begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix}}^{{}^d\vec{f}_B} + \frac{1}{m} \overbrace{\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}}^{{}^c\vec{f}_B}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Control Forces and Moments



$$ {}^{c}\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix} $$

$$ {}^{c}\vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} \frac{d}{\sqrt{2}}(-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{\sqrt{2}}(f_1 - f_2 - f_3 + f_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{bmatrix} $$

ω_i

$$ f_i = k_f C_L (\omega_i)^2 $$

$$ \tau_i = k_\tau C_D (\omega_i)^2 $$

$$ \boxed{\tau_i = k_m f_i} $$

$$ k_m = \frac{k_\tau C_D}{k_f C_L} $$

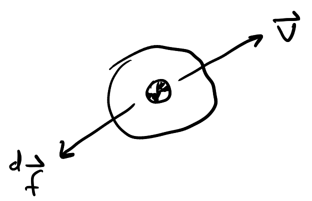
control forces + Moments $\Longleftrightarrow$ individual rotor forces

$$ \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} $$

$\Downarrow$ invert

$$ \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} $$

Drag Forces

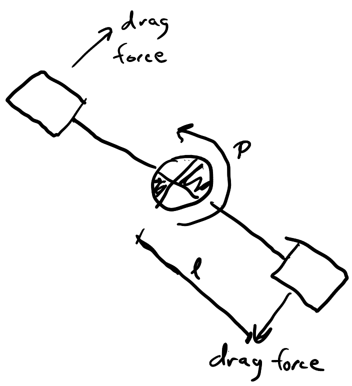


$$ d\vec{f} = -D \frac{\vec{v}}{V_a} \qquad V_a = |\vec{v}| $$

$$ D = \frac{1}{2}\rho V_a^2 C_D A = \nu V_a^2 $$

$$ d\vec{f}_B = \begin{bmatrix} X_d \\ Y_d \\ Z_d \end{bmatrix} = -\nu V_a^2 \frac{\vec{v}_B}{V_a} = \boxed{-\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}} $$

Drag Moments



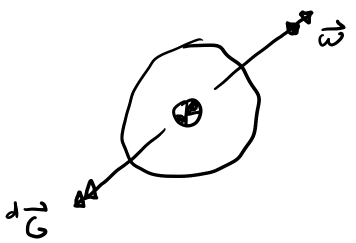
$$ \mathrm{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} $$

$$ L_{drag} = -2 l f_{drag} $$

$$ = \underbrace{-2 l \tfrac{1}{2}\rho C_D A}_{\mu} (l p)^2 \,\mathrm{sign}(p) $$

$$ = -\mu p |p| $$

General case



$$ \boxed{ d\vec{G}_B = \begin{bmatrix} L_d \\ M_d \\ N_d \end{bmatrix} = -\mu |\vec{\omega}| \begin{bmatrix} p \\ q \\ r \end{bmatrix} } $$

Quadrotor Linear Model

$\dot{\vec{x}} = f(t, \vec{x}, \vec{u})$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E^E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} (R_E^B)^T \vec{v}_B^E \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

- differential
- first order
- ordinary (not partial)
- coupled
- nonlinear

simulate

Linearization

$\vec{x} \approx \vec{x}_0 + \Delta\vec{x}$ $\qquad \dot{\vec{x}} \approx \Delta\dot{\vec{x}}$ $\qquad \vec{u} \approx \vec{u}_0 + \Delta\vec{u}$

trim condition

For a quadrotor, "Hover" trim condition

dot means any value

$$\vec{x}_0 = \begin{bmatrix} x_{E,0} \\ y_{E,0} \\ z_{E,0} \\ \phi_0 \\ \theta_0 \\ \psi_0 \\ u_0^E \\ v_0^E \\ w_0^E \\ p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix} Z_{c,0} \\ L_{c,0} \\ M_{c,0} \\ N_{c,0} \end{bmatrix} = \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Want Linear EOM

$\Delta\dot{\vec{x}} = A\Delta x + B\Delta u$

Approach: Use first-order Taylor Series approx

$y = f(x,u)$

$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$

$$y_0 + \Delta y \approx f(x_0, u_0) + \left.\frac{\partial f}{\partial x}\right|_0 \Delta x + \left.\frac{\partial f}{\partial u}\right|_0 \Delta u + \cancel{H.O.T.}$$

ignore

cancel

2 Approaches for finding Taylor series

1. Calculate partial derivatives (always work)
2. Substitute $x = x_0 + \Delta x$ use small number approximations (sometimes faster)

$\sin(\Delta x) \approx \Delta x$
$\cos(\Delta x) \approx 1$
$\Delta x \Delta u \approx 0$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot u^E \\ \dot v^E \\ \dot w^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m}\begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot p \\ \dot q \\ \dot r \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Example

$$\dot\theta = \overbrace{\cos(\phi) q - \sin(\phi) r}^{f(\phi,q,r)}$$

Approach 1: $\cancel{\dot\theta_0}^{0} + \Delta\dot\theta = \cancel{\cos(\phi_0) q_0}^{0} - \cancel{\sin(\phi_0) r_0}^{0} + \frac{\partial f}{\partial \phi}\Big|_0 \Delta\phi + \frac{\partial f}{\partial q}\Big|_0 \Delta q + \frac{\partial f}{\partial r}\Big|_0 \Delta r$

$$= \left(\cancel{\frac{\partial \cos(\phi)}{\partial\phi} q_0 - \frac{\partial \sin\phi}{\partial\phi} r_0}^{0}\right)\Delta\phi + \cos(\phi_0)\Delta q - \cancel{\sin(\phi_0)}^{0}\Delta r$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

Approach 2: $\dot\theta_0 + \Delta\dot\theta = \cos(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{q_0}^{0} + \Delta q) - \sin(\cancel{\phi_0}^{0} + \Delta\phi)(\cancel{r_0}^{0} + \Delta r)$

$$= 1\Delta q - \cancel{\Delta\phi\Delta r}^{0}$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

Harder

$$\dot w^E = \underbrace{qu^E - pv^E + g\cos\theta\cos\phi + \frac{1}{m}Z_d + \frac{1}{m}Z_c}_{f(\phi,\theta,u,v,w,p,q,Z_c)} \qquad Z_d(u,v,w)$$

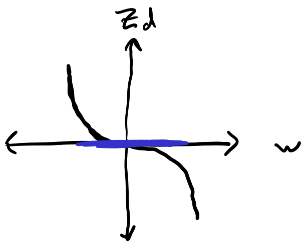
Assume no wind
$u = u^E,\ v = v^E,\ w = w^E$

$$\Delta\dot w = \cancel{\frac{\partial f}{\partial\phi}\Big|_0 \Delta\phi}^{0} + \cancel{\frac{\partial f}{\partial\theta}\Big|_0 \Delta\theta}^{0} + \underbrace{\frac{\partial f}{\partial u}\Big|_0 \Delta u + \frac{\partial f}{\partial v}\Big|_0 \Delta v + \frac{\partial f}{\partial w}\Big|_0 \Delta w}_{0} + \cancel{\frac{\partial f}{\partial p}\Big|_0 \Delta p}^{0} + \cancel{\frac{\partial f}{\partial q}\Big|_0 \Delta q}^{0} + \underbrace{\frac{\partial f}{\partial Z_c}\Big|_0}_{\frac{1}{m}} \Delta Z_c$$

$$Z_d = -\nu w \sqrt{u^2 + v^2 + w^2}$$

Simple case: assume $u, v = 0$

$$Z_d = -\nu w|w| = -\nu w^2 \operatorname{sign}(w)$$

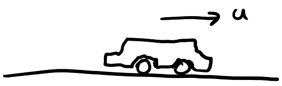


$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\nu\left(2w\operatorname{sign}(w) + w^2 \frac{\partial}{\partial w}\operatorname{sign}(w)\right)\Big|_0$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = 0$$

No drag force in Linear model!

Simple "Non hover" example



EOM $\dot{u} = \underbrace{\frac{e_f}{m} - \frac{\nu u|u|}{m}}_{g}$

Trim condition

$u_0 = 30\,m/s$ steady

$\dot{u}_0 = 0 = \frac{e_{f_0}}{m} - \frac{\nu u_0|u_0|}{m}$

$$\Delta \dot{u} = \frac{\partial g}{\partial u}\Big|_0 \Delta u = \frac{\partial(-\nu u^2 \,\text{sign}(u))}{\partial u}\Big|_0 \Delta u = -\nu\left(2u\,\text{sign}(u) + u^2 \cancel{\frac{\partial\, \text{sign}(u)}{\partial u}}^{\,0}\right)\Big|_0 \Delta u$$
$$= -\nu 2 u_0 \Delta u$$

$$\boxed{\Delta \dot{u} = -\nu 2 u_0 \Delta u}$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\nu\left(\sqrt{u^2+v^2+w^2} + w\frac{1}{2}\frac{1}{\sqrt{u^2+v^2+w^2}}2w\right)\Big|_0 = -\nu \frac{u^2+v^2+2w^2}{\sqrt{u^2+v^2+w^2}}\Big|_0$$

looks like $\frac{0}{0}$

can solve with spherical coordinates
$u = r\cos\theta\sin\phi$
$v = r\sin\theta\sin\phi$
$w = r\cos\phi$

$$\lim_{u,v,w\to 0} -\nu \frac{u^2+v^2+2w^2}{\sqrt{u^2+v^2+w^2}}$$

$$\lim_{r\to 0} \frac{r^2\cdot \text{stuff}}{r} = 0$$

$$\frac{\partial Z_d}{\partial u}\Big|_0 = 0$$

$$\frac{\partial Z_d}{\partial v}\Big|_0 = 0$$

$$\boxed{\Delta \dot{w}^E = \frac{\Delta Z_c}{m}}$$

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{y}_E \\ \Delta \dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \Delta \dot{\phi} \\ \Delta \dot{\theta} \\ \Delta \dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

Drag force would show up here

$$\rightarrow \begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta\theta \\ \Delta\phi \\ 0 \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x}\Delta L_c \\ \frac{1}{I_y}\Delta M_c \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

$$\begin{pmatrix}\Delta\dot{x}_E\\ \Delta\dot{y}_E\\ \Delta\dot{z}_E\end{pmatrix} = \begin{pmatrix}\Delta u\\ \Delta v\\ \Delta w\end{pmatrix}$$

$$\begin{pmatrix}\Delta\dot{\phi}\\ \Delta\dot{\theta}\\ \Delta\dot{\psi}\end{pmatrix} = \begin{pmatrix}\Delta p\\ \Delta q\\ \Delta r\end{pmatrix}$$

$$\begin{pmatrix}\Delta\dot{u}\\ \Delta\dot{v}\\ \Delta\dot{w}\end{pmatrix} = g\begin{pmatrix}-\Delta\theta\\ \Delta\phi\\ 0\end{pmatrix} + \frac{1}{m}\begin{pmatrix}0\\ 0\\ \Delta Z_c\end{pmatrix}$$

$$\begin{pmatrix}\Delta\dot{p}\\ \Delta\dot{q}\\ \Delta\dot{r}\end{pmatrix} = \begin{pmatrix}\frac{1}{I_x}\Delta L_c\\ \frac{1}{I_y}\Delta M_c\\ \frac{1}{I_z}\Delta N_c\end{pmatrix}$$

Longitudinal

$$\begin{pmatrix}\Delta\dot{x}_E\\ \Delta\dot{u}\\ \Delta\dot{\theta}\\ \Delta\dot{q}\end{pmatrix} = \begin{pmatrix}\Delta u\\ -g\Delta\theta\\ \Delta q\\ \frac{1}{I_y}\Delta M_c\end{pmatrix}$$

Lateral

$$\begin{pmatrix}\Delta\dot{y}_E\\ \Delta\dot{v}\\ \Delta\dot{\phi}\\ \Delta\dot{p}\end{pmatrix} = \begin{pmatrix}\Delta v\\ g\Delta\phi\\ \Delta p\\ \frac{1}{I_x}\Delta L_c\end{pmatrix}$$

Vertical

$$\begin{pmatrix}\Delta\dot{z}_E\\ \Delta\dot{w}\end{pmatrix} = \begin{pmatrix}\Delta w\\ \frac{1}{m}\Delta Z_c\end{pmatrix}$$

Spin

$$\begin{pmatrix}\Delta\dot{\psi}\\ \Delta\dot{r}\end{pmatrix} = \begin{pmatrix}\Delta r\\ \frac{1}{I_z}\Delta N_c\end{pmatrix}$$

state space model

$$\boxed{\begin{aligned}\dot{\vec{x}} &= A\vec{x} + B\vec{u}\\ \vec{y} &= C\vec{x} + D\vec{u}\end{aligned}}$$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$$

$$u \longrightarrow \boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n + \omega_n^2}} \longrightarrow y$$

$$\begin{bmatrix}\Delta\dot{\phi}\\ \Delta\dot{p}\end{bmatrix} = \begin{bmatrix}\Delta p\\ \frac{1}{I_x}\Delta L_c\end{bmatrix} \qquad \begin{aligned}\Delta p &= \Delta\dot{\phi}\\ \Delta\dot{p} &= \Delta\ddot{\phi}\end{aligned}$$

$$\Delta\ddot{\phi} = \frac{1}{I_x}\Delta L_c$$

Solutions? $\Delta\phi(t)$

Assume ΔL_c is constant

$$\Delta\dot{\phi}(t) = \Delta\dot{\phi}_0 + \int_{\tau=0}^{t}\Delta\ddot{\phi}(\tau)\,d\tau = \Delta\dot{\phi}_0 + \frac{1}{I_x}\Delta L_c t$$

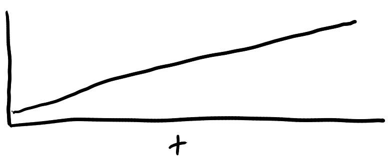
$$\Delta\phi(t) = \Delta\phi_0 + \int_{\tau=0}^{t}\Delta\dot{\phi}(\tau)\,d\tau = \Delta\phi_0 + \Delta\dot{\phi}_0 t + \frac{1}{2}\frac{1}{I_x}\Delta L_c t^2$$

Exactly at hover $\Delta\phi_0 = \Delta\dot{\phi}_0 = \Delta L_c = 0$

$$\Delta\phi(t) = 0$$

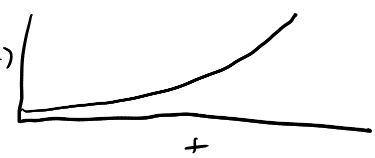
If $\Delta\dot{\phi}_0 > 0$, $\Delta L_c = 0$ If $\Delta L_c > 0$

$\Delta\phi(t)$



t

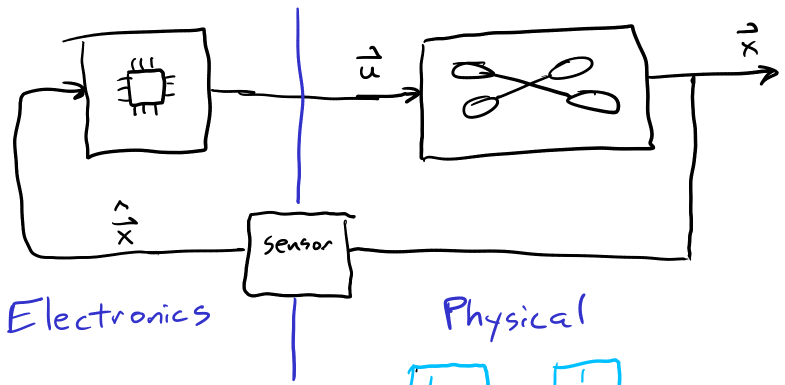
$\Delta\phi(t)$

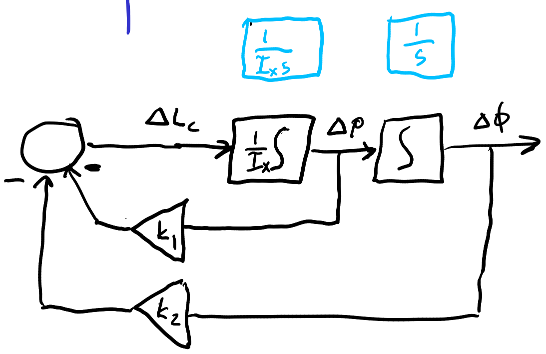


t

If initial conditions are nonzero (always in real life), vehicle will crash

Solution: Feedback Control





$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi$$

Deriv. gain — prop. gain

$$\Delta\ddot{\phi} = \frac{1}{I_x}\Delta L_c$$

$$= \frac{1}{I_x}\left(-k_1\Delta p - k_2 \Delta\phi\right)$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\boxed{\Delta\ddot{\phi} + \frac{k_1}{I_x}\Delta\dot{\phi} + \frac{k_2}{I_x}\Delta\phi = 0} \Longleftrightarrow \boxed{\Delta\ddot{\phi} + 2\zeta\omega_n\Delta\dot{\phi} + \omega_n^2\Delta\phi = 0}$$

$$\zeta = \frac{k_1}{2\sqrt{k_2 I_x}} \qquad \omega_n = \sqrt{\frac{k_2}{I_x}}$$

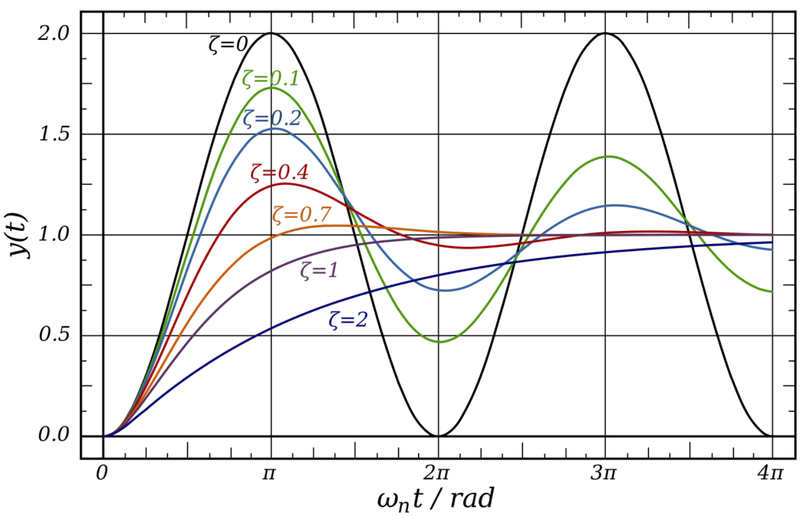
$$\lambda = -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$$

If λ are real and distinct

$$\boxed{\Delta\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}}$$

If λ are complex

$$\boxed{\Delta\phi(t) = e^{-\zeta\omega_n t}\left(C_3\sin(\omega_d t) + C_4\cos(\omega_d t)\right)}$$



$$\begin{bmatrix}\Delta\dot\phi\\ \Delta\dot p\end{bmatrix} = \begin{bmatrix}\Delta p\\ \frac{1}{I_x}\Delta L_c\end{bmatrix} \Longrightarrow \overset{\dot{\vec x}}{\begin{bmatrix}\Delta\dot\phi\\ \Delta\dot p\end{bmatrix}} = \overset{A}{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}} \overset{\vec x}{\begin{bmatrix}\Delta\phi\\ \Delta p\end{bmatrix}} + \overset{B}{\begin{bmatrix}0\\ \frac{1}{I_x}\end{bmatrix}} \overset{\vec u}{\begin{bmatrix}\Delta L_c\end{bmatrix}}$$

$$\underset{\vec Y}{\begin{bmatrix}\Delta\phi\end{bmatrix}} \underset{=}{=} \underset{C}{\begin{bmatrix}1 & 0\end{bmatrix}} \underset{\vec x}{\begin{bmatrix}\Delta\phi\\ \Delta p\end{bmatrix}} + \underset{D}{\begin{bmatrix}0\end{bmatrix}} \underset{\vec u}{\begin{bmatrix}\Delta L_c\end{bmatrix}}$$

$$\vec u = \begin{bmatrix}\Delta L_c\end{bmatrix} = -K\vec x = -\begin{bmatrix}k_2 & k_1\end{bmatrix}\begin{bmatrix}\Delta\phi\\ \Delta p\end{bmatrix}$$

$$\dot{\vec x} = \underbrace{A\vec x - BK\vec x} = \underbrace{(A - BK)}_{A^{cl}}\vec x$$

$$A^{cl} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0\\ \frac{1}{I_x}\end{bmatrix}\begin{bmatrix}k_2 & k_1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x}\end{bmatrix}$$

$\dot{\vec x} = A^{cl}\vec x$ what are solutions of $\dot{\vec x} = A^{cl}\vec x$?

scalar case

$\dot x = ax \Longrightarrow x(t) = x(0)e^{at}$

analogously

$$\boxed{\dot{\vec x} = A\vec x \Longrightarrow \vec x(t) = e^{At}\vec x(0)}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 \dots \quad \text{(Taylor Series)}$$

Modal Analysis

Eigenvalues and Eigenvectors

$$A\vec v_i = \lambda_i \vec v_i$$

Suppose that $\vec x_0 = \vec v_i$

$$\vec x(t) = \left(I + tA + \frac{t^2}{2!}A^2 \dots\right)\vec v_i$$

$$= \vec v_i + t\lambda_i v_i + \frac{t^2}{2!}\lambda_i^2 \vec v_i \dots$$

$$\vec x(t) = \vec v_i e^{\lambda_i t}$$

$$\boxed{\begin{aligned}&\text{If} \quad \vec x_0 = \sum_i q_i \vec v_i\\ &\text{then} \quad \vec x(t) = \sum_i q_i \vec v_i e^{\lambda_i t}\end{aligned}}$$

$\vec q = V^{-1}\vec x$

matrix of eigenvector columns

Finding Eigenvalues

$A\vec{v}_i = \lambda_i \vec{v}_i$

$(A - \lambda_i I)\vec{v}_i = 0$ only has nontrivial solutions if $|A - \lambda_i I| = 0$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad |A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$

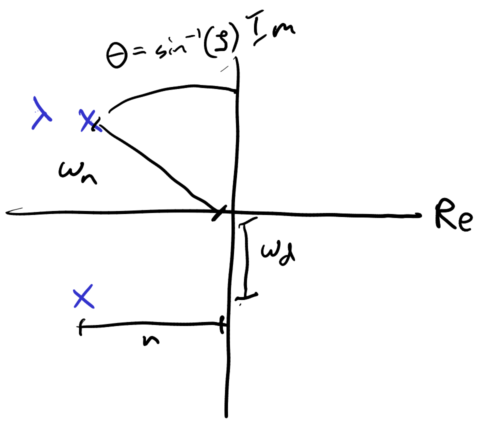
solve with quadratic formula

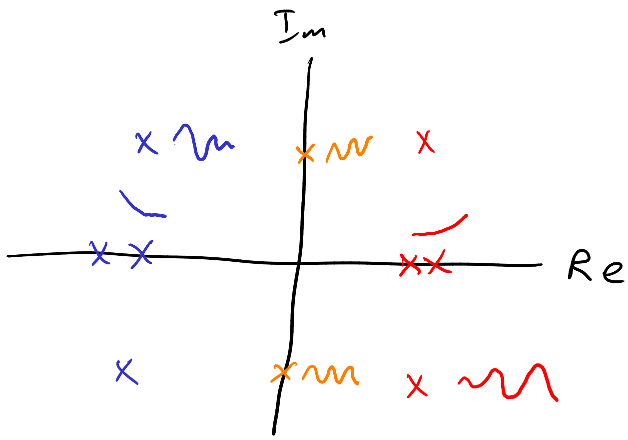
$$|A^{cl} - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} - \lambda \end{vmatrix} = \lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{I_x} \pm \sqrt{\frac{k_1^2}{4I_x} - \frac{k_2}{I_x}}$$

$$= n \pm i\omega_d$$

$$= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$





Solutions to linear ODEs

$$\dot{x} = ax \Rightarrow x(0)e^{at}$$

$$\ddot{x} + \underbrace{2\zeta\omega_n}_{a}\dot{x} + \underbrace{\omega_n^2}_{b}x = 0 \Longrightarrow$$

characteristic eqn $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

quad. form

$$\lambda = -\zeta\omega_n \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$ if λ real and distinct

$$x(t) = e^{-\zeta\omega_n t}\left(C_3\sin(\omega_d t) + C_4\cos(\omega_d t)\right)$$

$$\dot{\vec{x}} = A\vec{x} \Rightarrow e^{At}\vec{x}(0)$$

or

$$\sum_i q_i\vec{v}_i e^{\lambda_i t}$$

$\vec{v}_i$ are eigenvectors

λ_i are eigenvalues

if $\vec{x}(0) = a\vec{v}_1 + b\vec{v}_2$

$$\vec{x}(t) = a\vec{v}_1e^{\lambda_1 t} + b\vec{v}_2e^{\lambda_2 t}$$

$\vec{v}_i$ Eigenvectors: "shape" of the mode
which state variables are actively changing

λ_i Eigenvalues : "speed" of the mode
how fast does it oscillate, decay, or diverge

Linear Control Design Process

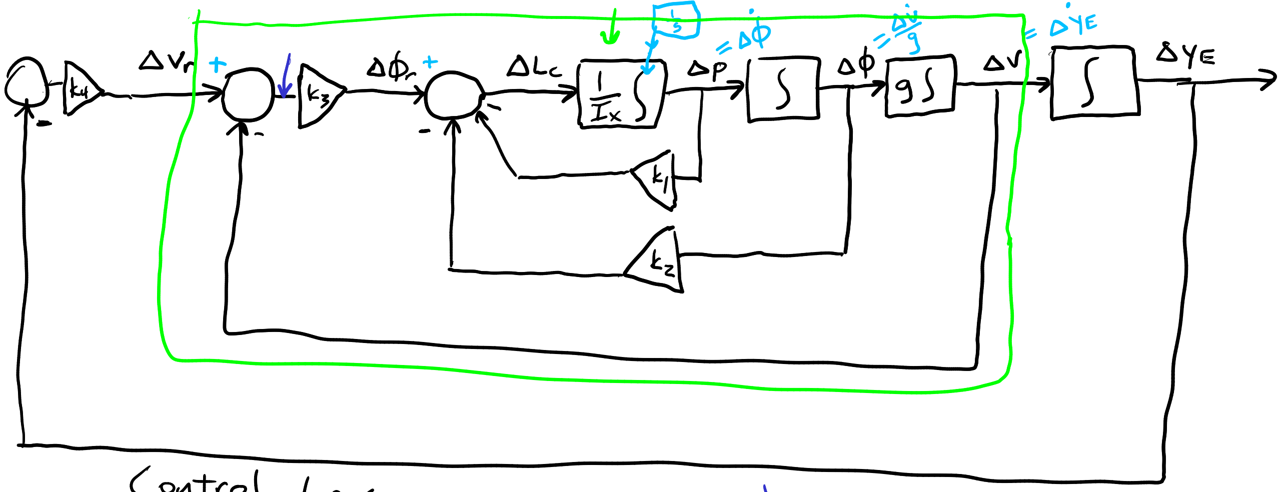
1. Derive EOM
2. Linearize and Separate EOM
3. Design Control Architecture
4. Choose Gain Values
5. Testing in Linear Simulation
6. Test in Nonlinear Sim

1. PID tuning
2. Pole Assignment
3. Root Locus
4. Optimal Control (LQR)

$$\underbrace{\begin{bmatrix}\Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p}\end{bmatrix}}_{\dot{\vec{x}}} = \begin{bmatrix}\Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\Delta y_E \\ \Delta v \\ \Delta\phi \\ \Delta p\end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix}0 \\ 0 \\ 0 \\ \frac{1}{I_x}\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\Delta L_c\end{bmatrix}}_{\vec{u}}$$

$$\vec{y} = \underset{I}{C}\vec{x} + \underset{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}{D}\vec{u}$$

$$I_x = 7\times10^{-5}\ \text{kgm}^2$$



Control Law

$$\longrightarrow \quad \Delta L_c = -k_1\Delta p - k_2\Delta\phi + k_3(-k_4\Delta Y_E - \Delta V)$$

$$= -k_1\Delta p - k_2\Delta\phi - k_3k_4\Delta Y_E - k_3\Delta V$$

$$[\Delta L_c] = \vec{u} = -K\vec{x} = -\underbrace{\left[k_3k_4 \;\vdots\; k_3 \;\vdots\; k_2 \;\vdots\; k_1\right]}\begin{bmatrix}\Delta Y_E\\ \Delta V\\ \Delta\phi\\ \Delta p\end{bmatrix}$$

$$A^{cl} = A - BK = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & g & 0\\ 0 & 0 & 0 & 1\\ \frac{-k_3k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x}\end{bmatrix} \qquad \dot{\vec{x}} = A^{cl}\vec{x}$$

Choosing Gains

1. Choose k_1 and k_2 with the "pole placement" strategy

Pole placement

- Decide where we want eigenvalues (poles) to be
- solve for k_1 and k_2

For 2nd order system

$$\begin{bmatrix}\Delta\dot\phi\\ \Delta\dot p\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ \frac{-k_2}{I_x} & -\frac{k_1}{I_x}\end{bmatrix}}_{A^{cl}}\begin{bmatrix}\Delta\phi\\ \Delta p\end{bmatrix}$$

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$|A^{cl} - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0 \qquad \text{analogous to} \qquad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

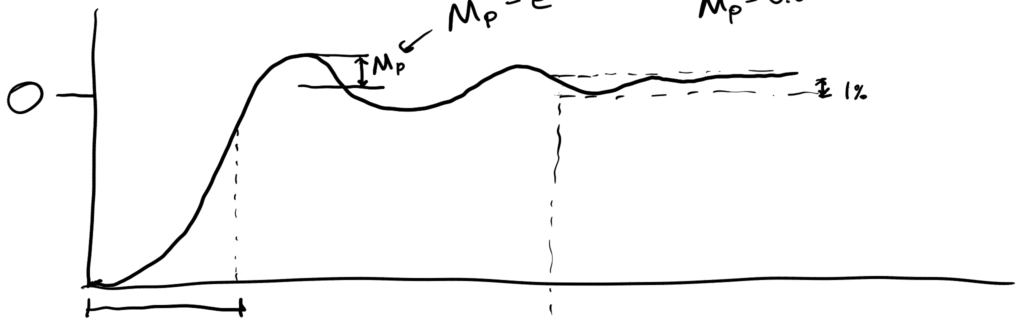
$$k_1 = 2\zeta\omega_n I_x \qquad k_2 = \omega_n^2 I_x$$

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ $\quad 0 \le \zeta \le 1$

$M_p = 0.05$ for 0.7

ω_n

ζ



rise time

$t_r = \frac{1.8}{\omega_n}$

settling time

$t_s = \frac{4.6}{\zeta\omega_n}$

try

$t_r = 0.2 \Rightarrow \omega_n = 9 \text{ rad/s}$

$\zeta = 0.7$

$\lambda = -\omega_n\zeta \pm i\,\omega_n\sqrt{1-\zeta^2}$

$\lambda = -6.3 \pm 6.4i$

$k_1 = 8.8 \times 10^{-4}$ $\qquad k_2 = 5.7 \times 10^{-3}$

$\dot{\vec{x}} = \underbrace{A^{cl}}_{2\times 2}\vec{x}$ $\leftarrow \begin{bmatrix}\Delta\phi \\ \Delta p\end{bmatrix}$

solutions look like

$\vec{x}(0) = \sum_i q_i \vec{v}_i$

$\vec{x}(t) = \sum_i q_i \vec{v}_i \underline{e^{\lambda_i t}}$ $\Longleftrightarrow$ $\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

λ_i are solutions to

$|A^{cl} - \lambda I| = 0$

if A^{cl} is 2×2

$\lambda^2 + \underbrace{\quad}_{\boxed{\frac{k_1}{I_x}}}\lambda + \underbrace{\quad}_{\boxed{\frac{k_2}{I_x}}} = 0$

$\Delta\ddot{\phi} + \underbrace{\quad}_{2\zeta\omega_n}\Delta\dot{\phi} + \underbrace{\quad}_{\omega_n^2} = 0$

solutions look like

$\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

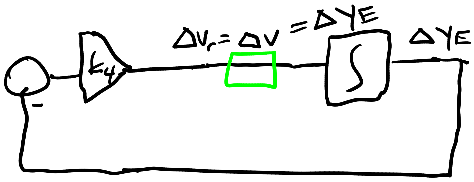
(C, λ might be complex $\Rightarrow$ sin/cos)

λ_i are solutions to

$\lambda^2 + \boxed{2\zeta\omega_n}\lambda + \boxed{\omega_n^2} = 0$

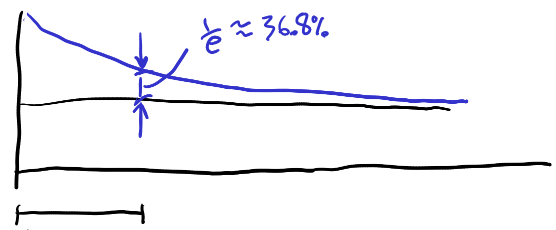
$k_1 = 2\zeta\omega_n I_x$ $\qquad k_2 = \omega_n^2 I_x$

Choose k_4 with pole placement



$\Delta\dot{Y}_E = -k_4 \Delta Y_E \Longrightarrow \Delta Y_E(t) = \Delta Y_E(0)\, e^{-k_4 t}$

e^{at}



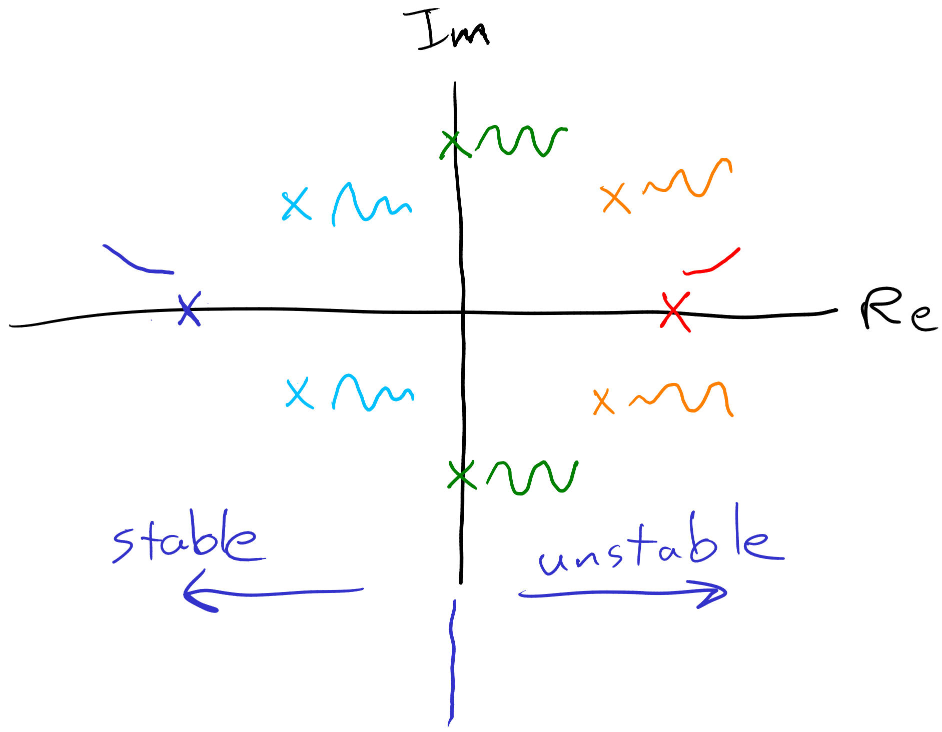
$\tau = -\frac{1}{a}$ if $a<0$

$\tau_y = \frac{1}{k_4}$

inner loop (ϕ, p) has settling time of $\frac{4.6}{\zeta\omega_n} \approx 0.7$

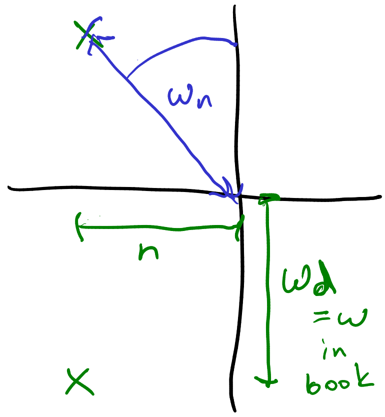
choose τ_y 10x larger ~ 7 sec, ≈ 5

$\tau_y = 5 \Rightarrow k_4 = 0.2$



$n \pm i\omega_d$

$\theta = \sin^{-1}(\zeta)$



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$