

# Conventional A/C Dynamics

## Longitudinal

Altitude  
Speed  
Pitch

## Lateral/Directional

Roll  
Yaw  
Sideslip

### Long. Forces and Moments

Lift

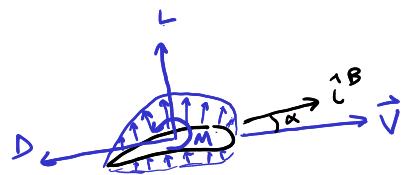
$$L = \frac{1}{2} \rho V_a^2 S \overset{\text{density}}{\cancel{C_L}} \overset{\text{Wing Area}}{\cancel{C_L}}$$

Drag

$$D = \frac{1}{2} \rho V_a^2 S \overset{\text{airspeed}}{\cancel{C_D}}$$

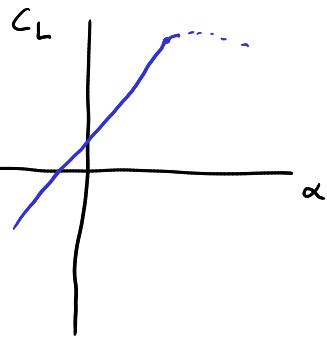
Pitch Moment

$$M = \frac{1}{2} \rho V_a^2 S \bar{z} C_m$$



Variable	Divisor	Non-dim Variable
$X, Y, Z$	$\frac{1}{2} \rho V^2 S$	$C_x, C_y, C_z$
$W$	$\frac{1}{2} \rho V^2 S$	$C_W$
$M$	$\frac{1}{2} \rho V^2 S \bar{c}$	$C_m$
$L, N$	$\frac{1}{2} \rho V^2 S \bar{b}$	$C_l, C_n$
$u, v, w$	$V$	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\dot{\hat{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\dot{\hat{\beta}}, \hat{p}, \hat{r}$
$m$	$\rho S \bar{c}/2$	$\mu$
$I_y$	$\rho S (\bar{c}/2)^3$	$\hat{I}_y$
$I_x, I_z, I_{xz}$	$\rho S (b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

### Lift



$$C_L(\alpha, q, \delta_e)$$

1st order Taylor series

$$L = \frac{1}{2} \rho V_a^2 S \left( C_{L_{\text{zero}}} + \frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e} \delta_e \right)$$

### Stability Derivatives

$$C_{a,b} = \left. \frac{\partial \text{nondimensionalized } a}{\partial \text{nondimensionalized } b} \right|_{\substack{\text{condition} \\ (\text{usually trim})}}$$

- based on linear assumptions
- main tool for connecting aerodynamics + dynamics
- determined by A/C geometry
- only accurate in a linear region (e.g. small  $\alpha$ )

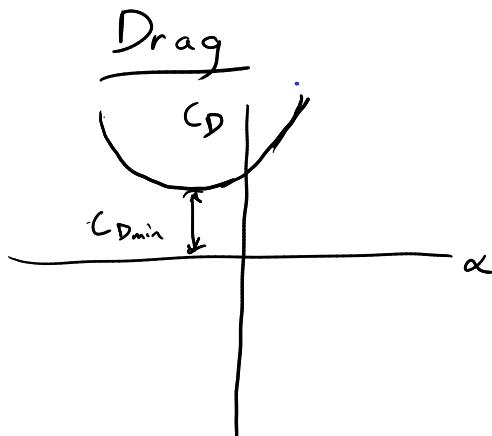


### Estimated

- Geometric Data  $\rightarrow C_{L\alpha} \approx \frac{\pi AR}{1 + \sqrt{1 + (\frac{AR}{2})^2}}$
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$L = \frac{1}{2} \rho V_a^2 S (C_{L_{zero}} + C_{L\alpha} \alpha + C_{Lq} \hat{q} + C_{L\delta_e} \delta_e)$$

$\hat{q} = q \frac{C}{2V_a}$



parasitic + induced  
 $\propto C_L^2$

$$C_D = C_{D_{min}} + K (C_L(\alpha, q, \delta_e) - C_{L_{min}})^2$$

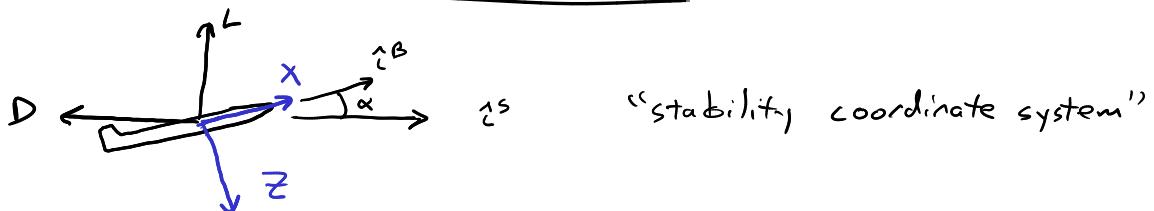
$$K = \frac{1}{\pi e AR}$$

Oswald's  
Efficiency

### Pitch Moment

$$M \approx \frac{1}{2} \rho V_a^2 S C (C_{m_{zero}} + C_{m\alpha} \alpha + C_{mq} \hat{q} + C_{m\delta_e} \delta_e)$$

### Lift + Drag $\rightarrow$ Body coordinates



$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} -D \\ -L \end{bmatrix}$$

# Longitudinal Stability

So far :  $C_L$ ,  $C_D$ ,  $C_m$

Now :  $C_{L\alpha}$ ,  $C_{m\alpha}$ ,  $C_{L_{se}}$ ,  $C_{m_{se}}$

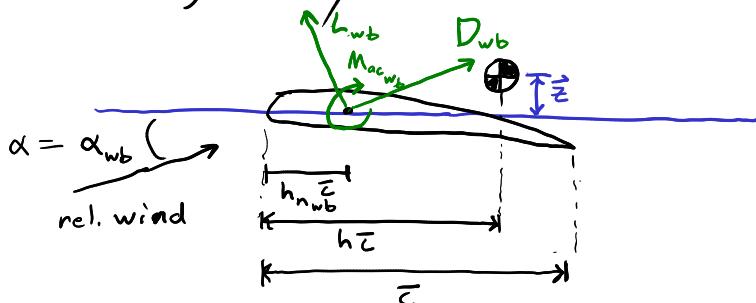
static margin :  $b_n - h$

Steady state forces  
trim, static stability

Three contributors 1. Wing / body

2. Propulsion (often small)  
3. Tail

Wing / body



Aerodynamic Center

$$\frac{\partial \text{Moment}}{\partial \alpha} = 0$$

Location stays constant  
Moment usually  $< 0$

Center of Pressure

$$\text{Moment} = 0$$

Changes w/  $\alpha$

Neutral Point

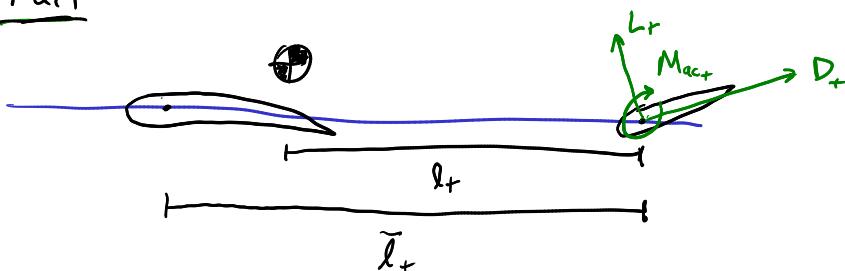
C.G. location that yields  $C_{m\alpha} = 0$   
A.C. of entire A/C

$$M_{wb} = M_{ac_{wb}} + (\underline{L \cos \alpha} + \underline{D \sin \alpha})(h - h_{nwb}) \bar{z} + (\underline{L \sin \alpha} - \underline{D \cos \alpha}) \bar{z}$$

only keep most important terms + nondimensionalize

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}} (h - h_{nwb})$$

Tail



$$L = L_{wb} + L_+$$

$$= C_{L_{wb}} \left( \frac{1}{2} \rho V^2 S \right) + C_{L_+} \left( \frac{1}{2} \rho V^2 S_+ \right) \Rightarrow C_L = C_{L_{wb}} + \underbrace{\frac{S_+}{S} C_{L_+}}$$

$$M_+ = -l_+ L_+ = -l_+ C_{L_+} \left( \frac{1}{2} \rho V^2 S_+ \right) \Rightarrow C_{m_+} = \underbrace{-l_+ \frac{S_+}{S} C_{L_+}}$$

more convenient  
b/c C.G. can change

$$= -V_H C_{L_+}$$

("volume ratio")

$$\bar{V}_H = \frac{l_+}{\bar{c}} \frac{S_+}{S} \Rightarrow V_H = \bar{V}_H - \frac{S_+}{S} (h - h_{nwb})$$

$$C_{m+} = -\bar{V}_H C_{L+} + C_{L+} \frac{S_+}{S} (h - h_{nwb})$$

Wing/body

$$\rightarrow C_m = \underbrace{C_{m_{acwb}}}_{\text{Tail}} + \underbrace{C_L (h - h_{nwb})}_{\text{Tail}} - \bar{V}_H C_{L+} + C_{mp}$$

propulsion

$$C_{m\alpha} = \frac{\partial C_{m_{acwb}}}{\partial \alpha} + C_{L\alpha} (h - h_{nwb}) - \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} + \frac{\partial C_{mp}}{\partial \alpha}$$

Want  $h_n \equiv cg$  location where  $C_{m\alpha} = 0$

$$0 = C_{L\alpha} (h_n - h_{nwb}) - \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} + \frac{\partial C_{mp}}{\partial \alpha}$$

$$h_n = h_{nwb} + \frac{1}{C_{L\alpha}} \left( \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} - \frac{\partial C_{mp}}{\partial \alpha} \right)$$

tail correction

$$C_{m\alpha} = C_{L\alpha} (h - h_n)$$

$C_{m\alpha} < 0$  for stability

Static margin

$K_n \equiv h_n - h$  must be  $> 0$  for stability

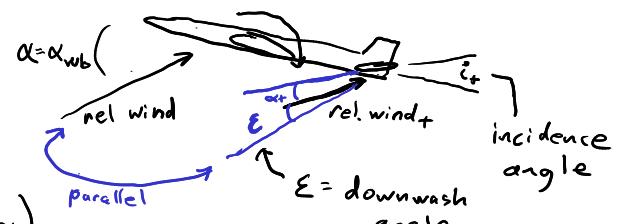
back to  $h_n$  eq.

Lift curve slope  $a \equiv C_{L\alpha}$

$$C_{L_{wb}} = a_{wb} \alpha_{wb} = a_{wb} \alpha$$

$$C_{L+} = a_+ \alpha_+$$

$$\alpha_+ = \alpha - i_+ - (\varepsilon_{zero} + \frac{\partial \varepsilon}{\partial \alpha} \alpha)$$



$$\frac{\partial C_{L+}}{\partial \alpha} = a_+ \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right)$$

$$h_n = h_{nwb} + \frac{a_+}{a} \bar{V}_H \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) - \frac{1}{a} \frac{\partial C_{mp}}{\partial \alpha}$$

$$C_{L\alpha} = a = a_{wb} \left[ 1 + \frac{a_+ S_+}{a_{wb} S} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right]$$

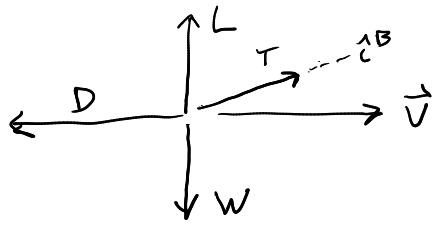
### Longitudinal Control

$\delta_e$  changes  $C_{L+}$

$$C_{L_{\delta e}} = \frac{\partial C_{L+}}{\partial \delta_e} \frac{S_+}{S} = a_e \frac{S_+}{S}$$

$$C_{m_{\delta e}} = -a_e \bar{V}_H + C_{L_{\delta e}} (h - h_{nwb})$$

## Linear Trim Estimation



For Linear trim,  $T=D$ ,  $L=W$

$$C_{L_{\text{trim}}} = \frac{W}{\frac{1}{2} \rho V^2 S} = C_{L_{\text{zero}}} + C_{L_{\alpha}} \alpha_{\text{trim}} + C_{L_{\delta_e}} \delta_{\text{trim}}$$

$$C_{m_{\text{trim}}} = C_{m_{\text{zero}}} + C_{m_{\alpha}} \alpha_{\text{trim}} + C_{m_{\delta_e}} \delta_{\text{trim}} = 0$$

$$\begin{bmatrix} C_{L_{\alpha}} & C_{L_{\delta_e}} \\ C_{m_{\alpha}} & C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \alpha_{\text{trim}} \\ \delta_{\text{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{\text{trim}}} \\ -C_{m_{\text{zero}}} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{\text{trim}} \\ \delta_{\text{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{\alpha}} & C_{L_{\delta_e}} \\ C_{m_{\alpha}} & C_{m_{\delta_e}} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{\text{trim}}} \\ -C_{m_{\text{zero}}} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Cramer's rule)

$$\alpha_{\text{trim}} = \frac{C_{m_{\text{zero}}} C_{L_{\delta_e}} + C_{m_{\delta_e}} C_{L_{\text{trim}}}}{\Delta}$$

$$\Delta = C_{L_{\alpha}} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_{\alpha}}$$

$$\delta_{\text{trim}} = -\frac{C_{m_{\text{zero}}} C_{L_{\alpha}} + C_{m_{\alpha}} C_{L_{\text{trim}}}}{\Delta}$$

# Longitudinal Linear Model

$$\dot{\vec{p}}_E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\vec{v}_B^E = \frac{\vec{f}_e}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{\omega}} = I^{-1} [\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B]$$

2 differences

1. Aerodynamic Forces

2. I more complex

Symmetry about x-z axis  $\Rightarrow I_{xy} = I_{yz} = 0$

$$I_B^{-1} = \begin{bmatrix} I_z & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \quad I_{xz} \neq 0$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \quad \Gamma_4 = \frac{I_{xz}}{\Gamma} \quad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \quad \Gamma_5 = \frac{I_z - I_x}{I_y} \quad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \quad \Gamma_6 = \frac{I_{xz}}{I_y} \quad \Gamma = I_x I_z - I_{xz}^2$$



$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi c_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_8 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Linearize about trim state

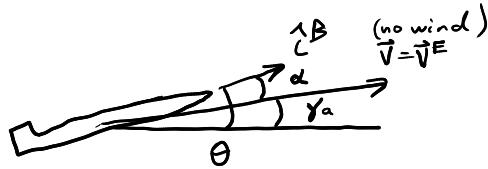
Inputs:  $U_0, h_0, \gamma_{a_0}$

airspeed  $= V = V_a$

altitude

air-relative flight-path angle

$\alpha_0, \delta_{e0}, \delta_{r0}$



$$\vec{x} = \begin{bmatrix} X_E \\ Y_E \\ Z_E \\ \phi \\ \theta \\ \psi \\ q \\ r \\ w \end{bmatrix} = \vec{x}_0 + \vec{\Delta x}$$

$$\vec{x}_0 = \begin{bmatrix} \cdot \\ -h_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For linearization:  $\vec{V} = \vec{V}^E$  (no wind)  
assume  $w_0^E = 0$

$$\vec{u} = \begin{bmatrix} \delta_e \\ \delta_\alpha \\ \delta_r \\ \delta_\tau \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} \delta_{e0} \\ 0 \\ 0 \\ \delta_{r0} \end{bmatrix}$$

Examples to Linearize

$$\dot{\theta} = \cos \phi q - \sin \phi r$$

$$\dot{\phi}_0 + \Delta \dot{\phi} = \cos \phi_0 q_0 - \sin \phi_0 r_0 + \left. \frac{\partial}{\partial \phi} (\cos \phi q - \sin \phi r) \right|_0 \Delta \phi$$

$$+ \left. \frac{\partial}{\partial q} (\cos \phi q - \sin \phi r) \right|_0 \Delta q$$

$$+ \left. \frac{\partial}{\partial r} (\cos \phi q - \sin \phi r) \right|_0 \Delta r$$

$$= -\sin \phi_0 q_0 - \cos \phi_0 r_0 + \cos \phi_0 \Delta q - \sin \phi_0 \Delta r$$

$\Delta \dot{\phi} = \Delta q$

$$\dot{u} = r v - q w - g \sin \theta + \frac{X}{m}$$

$$\dot{u}_0 + \Delta \dot{u} = \cancel{r_0 v_0} - \cancel{q_0 w_0} - g \sin \theta_0 + \cancel{\frac{X_0}{m}} + \left. \frac{\partial}{\partial r} rv \right|_0 \Delta r + \left. \frac{\partial}{\partial v} rv \right|_0 \Delta v + \left. \frac{\partial}{\partial q} (-qw) \right|_0 \Delta q$$

$$+ \left. \frac{\partial}{\partial w} (-qw) \right|_0 \Delta w + \cancel{\left. \frac{\partial}{\partial \theta} (g \sin \theta) \right|_0 \Delta \theta} + \left. \frac{\partial}{\partial X} \left( \frac{X}{m} \right) \right|_0 \Delta X$$

$$= \cancel{v_0 \Delta r} + \cancel{p_0 \Delta v} - \cancel{w_0 \Delta q} - \cancel{q_0 \Delta w} - g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta X$$

$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta X$

## Lateral

$$\rightarrow \Delta\dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

$$\rightarrow \Delta\dot{\theta} = \Delta q$$

## Long.

$$\rightarrow \Delta\dot{u} = -g \cos \theta_0 \Delta\theta + \frac{\Delta X}{m}$$

$$\rightarrow \Delta\dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta\phi + \frac{\Delta Y}{m}$$

$$\rightarrow \Delta\dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta\theta + \frac{\Delta Z}{m}$$

$$\rightarrow \Delta\dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\rightarrow \Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\rightarrow \Delta\dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

Dimensional stab derivs.

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

$$X_u \equiv \left. \frac{\partial X}{\partial u} \right|_0$$

$$C_{X_u} \equiv \left. \frac{\partial C_x}{\partial u} \right|_0$$

$$\hat{\omega} = \frac{w}{u_0} \approx \alpha$$

$$\alpha = \tan(\frac{\Delta w}{u_0 + \Delta u}) \approx \frac{w}{u_0}$$

Table 4.4

Longitudinal Dimensional Derivatives

	X	$X_u$	Z	M
u	$\rho u_0 S C_{w_0} \sin \theta_0 + \frac{1}{2} \rho u_0 S C_{x_u}$		$-\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$	$\frac{1}{2} \rho u_0 \bar{c} S C_{m_u}$
w		$\frac{1}{2} \rho u_0 S C_{x_\alpha}$	$\frac{1}{2} \rho u_0 S C_{z_\alpha}$	$\frac{1}{2} \rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4} \rho u_0 \bar{c} S C_{x_q}$		$\frac{1}{4} \rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4} \rho u_0 \bar{c}^2 S C_{m_q}$
w	$\frac{1}{4} \rho c S C_{x_{\dot{w}}}$		$\frac{1}{4} \rho c S C_{z_{\dot{w}}}$	$\frac{1}{4} \rho c^2 S C_{m_{\dot{w}}}$

$$Z_u \equiv \left. \frac{\partial Z}{\partial u} \right|_0$$

$$Z = \frac{1}{2} \rho V^2 S C_Z$$

$$\begin{aligned} \left. \frac{\partial Z}{\partial u} \right|_0 &= \frac{1}{2} \rho S \left( \left. \frac{\partial V^2}{\partial u} \right|_0 C_Z + V^2 \left. \frac{\partial C_Z}{\partial u} \right|_0 \right) \\ &= \frac{1}{2} \rho S \left( Z_{u_0} C_{Z_0} + u_0^2 \left. \frac{\partial C_Z}{\partial u} \right|_0 \right) \end{aligned}$$

$$Z_u = -\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$$

$$\hat{u} = \frac{u}{V} = \frac{u}{u_0} \quad u = \hat{u} u_0$$

$$\left. \frac{\partial C_Z}{\partial u} \right|_0 = \left. \frac{\partial C_Z}{u_0 \hat{u}} \right|_0 = \frac{1}{u_0} C_{Z_u}$$

$$C_{Z_0} = -C_{w_0} \cos \theta_0$$

Table 5.1

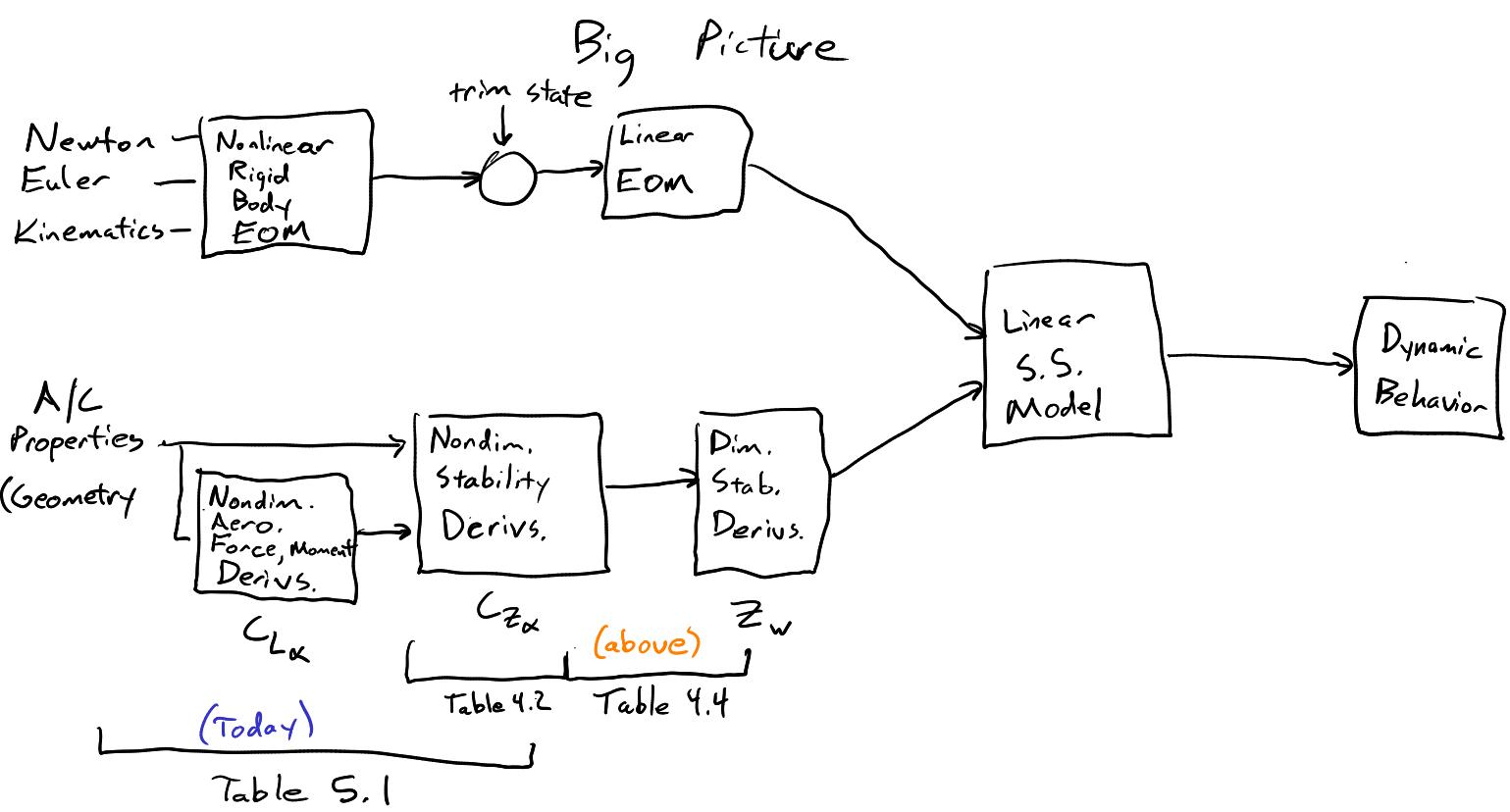
Summary—Longitudinal Derivatives

	$C_x$	$C_z$	$C_m$
$\hat{u}^\dagger$	$\mathbf{M}_0 \left( \frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_i V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_i V_H \frac{l_i}{c} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_i V_H$	$*-2a_i V_H \frac{l_i}{c}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\dagger C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$



# Nondimensional Stability Derivatives

Table 5.1  
Summary—Longitudinal Derivatives

	$C_x$	$C_z$	$C_m$
$\hat{u}^+$	$\mathbf{M}_0 \left( \frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$* -2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$* -2a_t V_H \frac{l_t}{c} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$* -2a_t V_H$	$* -2a_t V_H \frac{l_t}{c}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\dagger C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

$\alpha$ -derivatives

$$\boxed{C_{m_\alpha} = C_{L_\alpha} (h - h_n)}$$

$$C_{z_\alpha}$$

$$Z = -L \cos \alpha - D \sin \alpha$$

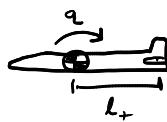
$$C_z = -(C_L \cos \alpha + C_D \sin \alpha) \\ \approx -(C_L + C_D \alpha)$$

$$C_{z_\alpha} = \frac{\partial C_z}{\partial \alpha} \Big|_0 = -(C_{L_\alpha} + C_{D_0} + \cancel{C_{D_\alpha}})$$

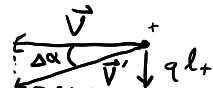
$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

$q$ -derivatives

Wing-body | Tail



velocity observed by tail



$$\Delta C_{L_t} = \alpha_t \Delta \alpha \xrightarrow{\text{Lift curve slope of tail}} = \alpha_t + \tan^{-1} \left( \frac{q l_t}{u_0} \right) \approx \alpha_t + \frac{q l_t}{u_0}$$

$$\Delta C_L = \frac{S_t}{S} \Delta C_{L_t}$$

$$= \frac{S_t}{S} \alpha_t + \frac{q l_t}{u_0}$$

$$(C_{z_q})_{tail}$$

$$C_{z_q} = \frac{\partial C_z}{\partial q} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_z}{\partial q} \Big|_0 = -\frac{2u_0}{c} \frac{\partial C_L}{\partial q} \Big|_0$$

$$(C_{z_q})_{tail} = -\frac{2u_0}{c} \alpha_t \frac{S_t}{S} \frac{l_t}{u_0} = \boxed{-2\alpha_t V_H}$$

$$V_H = \frac{S_t l_t}{S c}$$

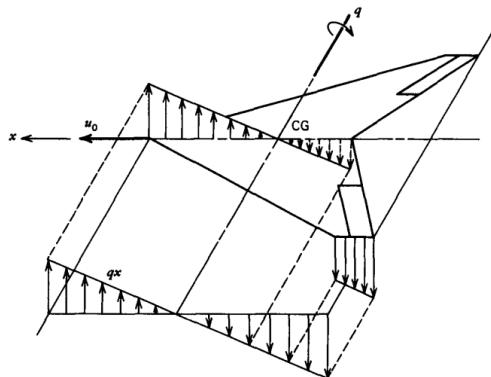
$(C_{m\dot{\alpha}})_{tail}$ 

$$\Delta C_m = -V_H \Delta C_{L+} = \alpha_+ V_H \frac{q l_+}{u_0}$$

$$C_{m\dot{\alpha}} \equiv \frac{\partial C_m}{\partial \dot{\alpha}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_m}{\partial q} \Big|_0$$

$$(C_{m\dot{\alpha}})_{tail} = -2\alpha_+ V_H \frac{l_+}{c}$$

## Wing - Body



measure in  
wind tunnel  
or CFD

Figure 5.4 Wing velocity distribution due to pitching.

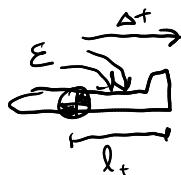
## alpha derivatives

### Unsteady effects

Wing-body → determined by initial response  
or oscillation of wing in wind tunnel or flight test

## Tail

### Downwash lag



$$\begin{aligned} -\Delta\alpha_+ &= \Delta\varepsilon = -\frac{\partial\varepsilon}{\partial\alpha} \dot{\alpha} \Delta t \\ &= -\frac{\partial\varepsilon}{\partial\alpha} \dot{\alpha} \frac{l_+}{u_0} \end{aligned}$$

$$\Delta C_{L+} = \alpha_+ \Delta\alpha_+ = \alpha_+ \dot{\alpha} \frac{l_+}{u_0} \frac{\partial\varepsilon}{\partial\alpha}$$

$$\Delta C_L = \alpha_+ \dot{\alpha} \frac{l_+ S_+}{u_0 S} \frac{\partial\varepsilon}{\partial\alpha}$$

$$(C_{Z\dot{\alpha}})_{tail} \equiv \frac{\partial C_Z}{\partial \dot{\alpha} c} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_Z}{\partial \dot{\alpha}} \Big|_0 = \boxed{-2\alpha_+ \frac{l_+ S_+}{c S} \frac{\partial\varepsilon}{\partial\alpha}}$$

$$(C_{m\dot{\alpha}})_{tail} = -2\alpha_+ V_H \frac{l_+}{c} \frac{\partial\varepsilon}{\partial\alpha}$$

## u derivatives

3 important factors:

- Compressibility: Mach Number

- Dynamic Pressure:  $p_d = \frac{1}{2} \rho V^2$

- Thrust

- Different from the dynamic pressure in nondimensionalization.

Changes in  $C_L$ ,  $C_D$  etc. due to changes in dynamic pressure.

$$C_{X_u} = \left. \frac{\partial C_x}{\partial u} \right|_0$$

$$C_{*u} = \underbrace{\left. \frac{\partial C_*}{\partial M} \right|_0 \left. \frac{\partial M}{\partial u} \right|_0}_{+} + \underbrace{\left. \frac{\partial C_*}{\partial p_d} \right|_0 \left. \frac{\partial p_d}{\partial u} \right|_0}_{+} + \underbrace{\left. \frac{\partial C_*}{\partial C_T} \right|_0 \left. \frac{\partial C_T}{\partial u} \right|_0}_{+}$$

$$M = \frac{V}{a}$$

$$\left. \frac{\partial M}{\partial u} \right|_0 = u_0 \left. \frac{\partial M}{\partial u} \right|_0 = \frac{u_0}{a} \left. \frac{\partial V}{\partial u} \right|_0 = M_0 \quad \text{Mach number at trim}$$

$$\left. \frac{\partial p_d}{\partial u} \right|_0 = u_0 \left. \frac{\partial p_d}{\partial u} \right|_0 = u_0 \frac{1}{2} \rho \left. \frac{\partial V^2}{\partial u} \right|_0 = u_0 \rho u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2} \rho V^2 S}$$

$$\begin{aligned} \left. \frac{\partial C_T}{\partial u} \right|_0 &= u_0 \left. \frac{\partial C_T}{\partial u} \right|_0 = u_0 \left( \frac{\partial T / \partial u}{\frac{1}{2} \rho V^2 S} - \frac{Z T}{\frac{1}{2} \rho V^3 S} \right) \Big|_0 \\ &= \frac{\partial T / \partial u}{\frac{1}{2} \rho u_0 S} \Big|_0 - Z C_{T_0} \end{aligned}$$

3 Cases

$$C_{T_0} = C_{D_0} + C_{W_0} \sin \theta_0$$

Gliding:  $C_{T_u} = 0$

Constant Thrust (Jet):  $C_{T_u} = -2 C_{T_0}$

Constant Power (Prop):

$$T V = \text{constant}$$

$$\left. \frac{\partial T}{\partial u} \right|_0 = - \frac{T_0}{u_0}$$

$$C_{T_u} = -3 C_{T_0}$$

$C_{X_u}$ 

$$C_x \approx C_T - C_D$$

$$\frac{\partial C_x}{\partial M} \Big|_o = \frac{\partial C_T}{\partial M} \Big|_o - \frac{\partial C_D}{\partial M} \Big|_o$$

$$\frac{\partial C_x}{\partial p_d} \Big|_o = \frac{\partial C_T}{\partial p_d} \Big|_o - \frac{\partial C_D}{\partial p_d} \Big|_o$$

$$\frac{\partial C_x}{\partial C_T} \Big|_o = 1 - \frac{\partial C_D}{\partial C_T} \Big|_o$$

$$C_{X_u} = M_o \left( \frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) \Big|_o - \rho u_o^2 \frac{\partial C_D}{\partial p_d} \Big|_o + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right) \Big|_o$$

 $C_{Z_u}$ 

Assume  $C_z = -C_L$

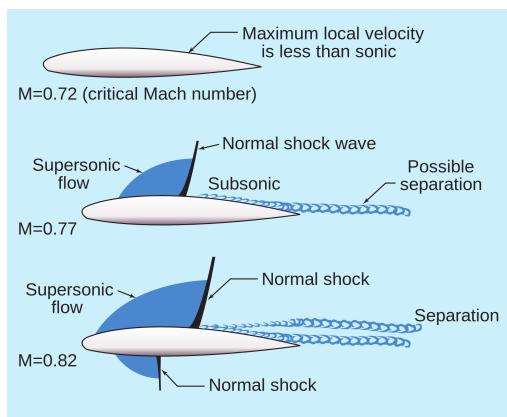
$$C_{Z_u} = -M_o \frac{\partial C_L}{\partial M} \Big|_o - \rho u_o^2 \frac{\partial C_L}{\partial p_d} \Big|_o - C_{T_u} \frac{\partial C_L}{\partial C_T} \Big|_o$$

small except  
for transonic

 $C_{m_u}$ 

$$C_{m_u} = M_o \frac{\partial C_m}{\partial M} \Big|_o + \rho u_o^2 \frac{\partial C_m}{\partial p_d} \Big|_o + C_{T_u} \frac{\partial C_m}{\partial C_T} \Big|_o$$

Mach Tuck



## Longitudinal Modes

$$\dot{\vec{x}} = A\vec{x}$$

$n \times n$

(assume that  $A$  has  $n$  distinct non zero eigenvalues)

$$\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

Full State Space

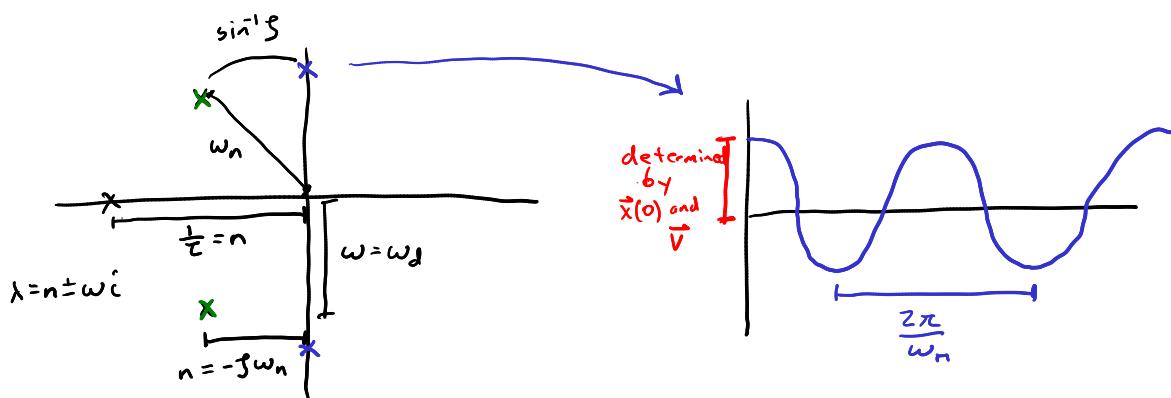
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

What does this mean?

A single real-valued  $(\lambda, \vec{v})$  pair, or a pair of complex-valued  $((\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2))$  describes a mode.

$\lambda$ : "speed" of mode  
 $\vec{v}$ : "shape" of mode



### Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I)\vec{v}_i = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \lambda_1 = -1, \quad \lambda_2 = -2$$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \propto$$

$$\text{let } (\vec{v}_1)_1 = 1$$

$$(\vec{v}_1)_2 = -1$$

$$(\vec{v}_1)_1 + (\vec{v}_1)_2 = 0$$

$$(A - \lambda_2 I)\vec{v}_2 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \vec{v}_2 = 0 \quad \therefore \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \propto$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i e^{\lambda_i t} = \sum_i q_i \vec{v}_i = q_1 \vec{v}_1 + q_2 \vec{v}_2 + \dots + q_n \vec{v}_n = \sqrt{\vec{q}}$$

$$\vec{x}(0) = \sqrt{\vec{q}}$$

$$\vec{q} = V^{-1} \vec{x}(0)$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

## Conventional A/C Long. Modes

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$h_0 = 40k \text{ ft}$$

$$u_0 = 774 \text{ ft/s}$$

$$\gamma_0 = \theta_0 = \alpha_0 = 0$$

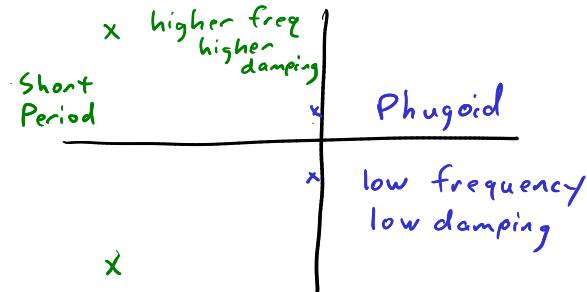
$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = -0.37 \pm 0.89i$$

$$\omega_n = 0.96 \quad \zeta = 0.38$$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$$\omega_n = 0.067 \quad \zeta = 0.049$$



$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \mp 0.0004i \end{bmatrix}$$

$$\begin{aligned} \Delta U & \\ \Delta W & \\ \Delta q & \\ \Delta \theta & \end{aligned}$$

$$\vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097i \\ -0.0001 \mp 0i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

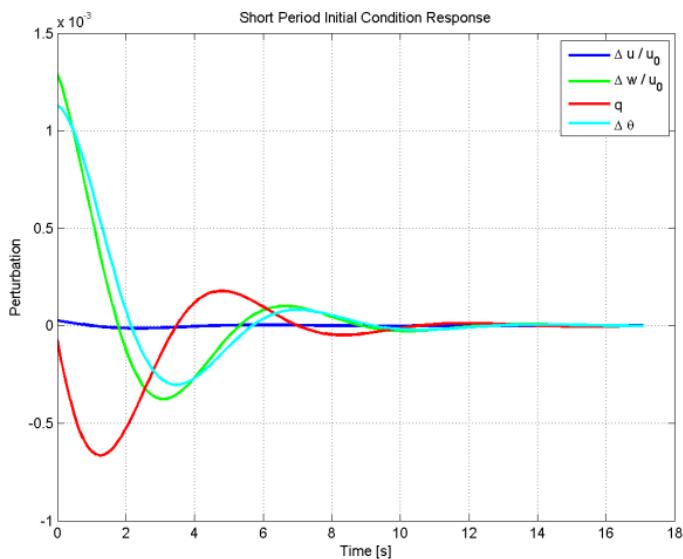
$$\vec{x}(0) = 0.5 \vec{v}_1 + 0.5 \vec{v}_2 = \operatorname{Re}(\vec{v}_{1,2})$$

$$\lambda_{1/2} = -0.372 + 0.888i$$

$$\zeta = 0.387$$

$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



Phasor Plot: plot of eigenvector in complex plane

Aside: Polar Coordinates of a complex number

$$z = a + bi$$

$$z = r e^{i\phi}$$

$$z = r \angle \phi$$

$$r = \sqrt{a^2 + b^2} \quad \phi = \text{atan} z(b, a)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

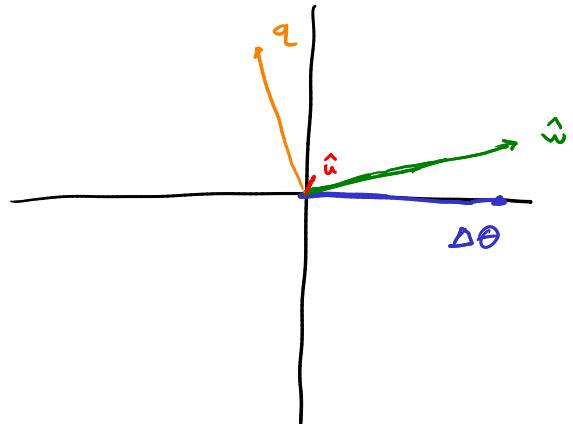
For consistent phasor plots,

- normalize so that  $\Delta\theta = 1$
- nondimensionalize  $u$  and  $w$

Short Period

$$\vec{V}'_{1,2} = \vec{V}_{1,2} / (\vec{V}_{1,2})_4 = \begin{bmatrix} & \\ & \\ & \\ & \\ & 1.0 \end{bmatrix}$$

$$\hat{V}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.37 \pm 0.99i \\ 1.0 \end{bmatrix} \quad \begin{aligned} \hat{u} &= \frac{\Delta u}{u_0} \\ \hat{w} &= \frac{\Delta w}{w_0} \approx \alpha \\ q & \\ \Delta\theta & \end{aligned} \Rightarrow$$

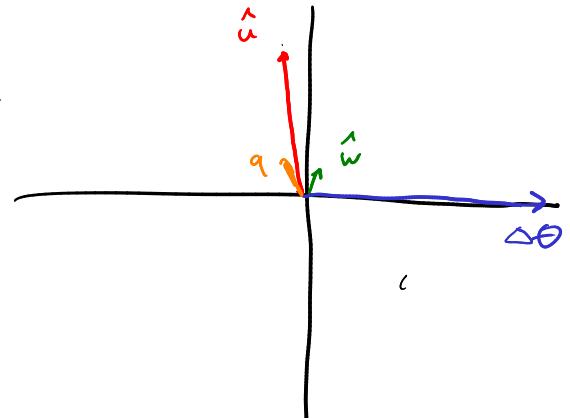


# Phugoid Mode

phugoid = "flight"



$$\vec{v}_{3,9} = \begin{bmatrix} 0.62 < 92^\circ \\ 0.036 < 83^\circ \\ 0.067 < 93^\circ \\ 1.0 < 0 \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{\omega} = \alpha \\ q \\ \Delta\theta \end{matrix}$$



$\lambda_{3,4} \Rightarrow \begin{cases} \text{low frequency} \\ \text{low damping} \end{cases}$

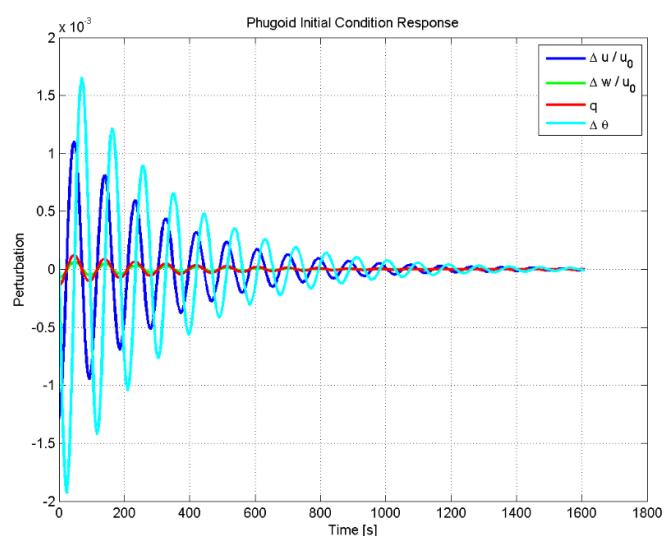
$\vec{v}_{3,4} \Rightarrow \begin{cases} \text{Large } \hat{u} \text{ and } \theta \text{ oscillations out of phase (90° offset)} \\ \text{small } \alpha \end{cases}$

$$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$$

$$\zeta = 0.0489 \leftarrow \text{poorly damped}$$

$$\omega_n = 0.0673 \leftarrow \text{slow response}$$

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



# Longitudinal Mode Approximations

$$\vec{v} = \begin{bmatrix} a+bi \\ b \end{bmatrix}$$

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + B_{lon} \vec{u}_{lon}$$

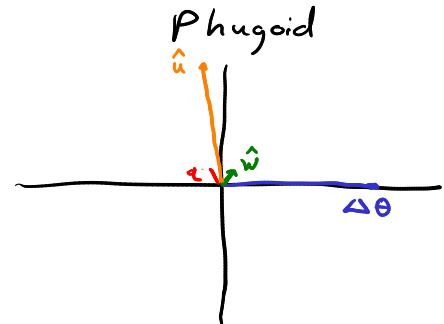
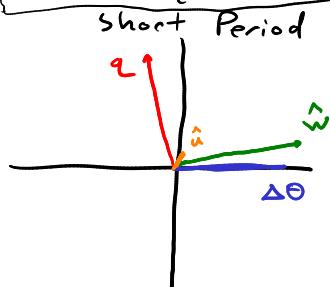
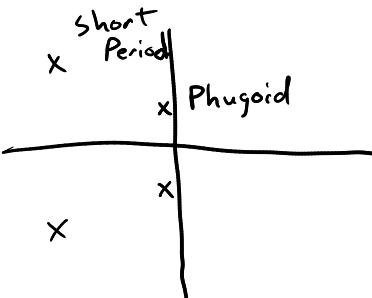
$$\dot{\vec{y}} = C \vec{x}_{lon} + D \vec{u}_{lon}$$

$\downarrow I \quad \downarrow O$

$$\vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$\vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$$\hat{w} = \frac{\Delta w}{u_0}$$



## Short period Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + \vec{c}_{lon}$$

Assume:  $\Delta u = 0$

$$Z_w \ll m$$

$$Z_q \ll m u_0$$

$$\theta_0 = 0$$

No vertical motion

$$\Delta \theta = \Delta \alpha = \frac{\Delta w}{u_0}$$

$$\vec{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \vec{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$A_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_w} & \frac{Z_w}{m - Z_w} & \frac{Z_q + mu_0}{m - Z_w} & \frac{-mg \sin \theta_0}{m - Z_w} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_w} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_w)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$\begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} \\ \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_w} \right] \end{bmatrix}}_{A_{sp}} \underbrace{\begin{bmatrix} u_0 \\ \frac{1}{I_y} \left[ M_q + M_{\dot{w}} u_0 \right] \end{bmatrix}}_{\lambda} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[ \frac{Z_w}{m} + \frac{1}{I_y} (M_q + M_{\dot{w}} u_0) \right]}_{-2 \zeta \omega_n} \lambda - \underbrace{\frac{1}{I_y} (u_0 M_w - \frac{M_q Z_w}{m})}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

### Dimensional Stab. Deriv.

$$Z_w = \frac{\partial Z}{\partial w} \Big|_0 = \frac{1}{2} \rho u_0 S C_{Z_\alpha}$$

Table 4.4

$$M_w$$

$$M_w$$

$$M_q$$

### Nondim. Stab. Deriv.

$$C_{Z_\alpha}$$

$$C_m \alpha$$

$$C_m \alpha$$

$$C_m q$$

### A/C Properties

$$= -C_{L_\alpha} - C_{D_0}$$

Table 5.1

$$= C_{L_\alpha} (h - h_n)$$

....

$$(C_m)_\text{tail} = -2 \alpha_+ V_H \frac{l + c}{c}$$

How accurate is this approximation?

For 747  
@ cruise

### Full A<sub>lon</sub>

$$\lambda_{1,2} = -0.372 \pm 0.888i$$

$$g = 0.387$$

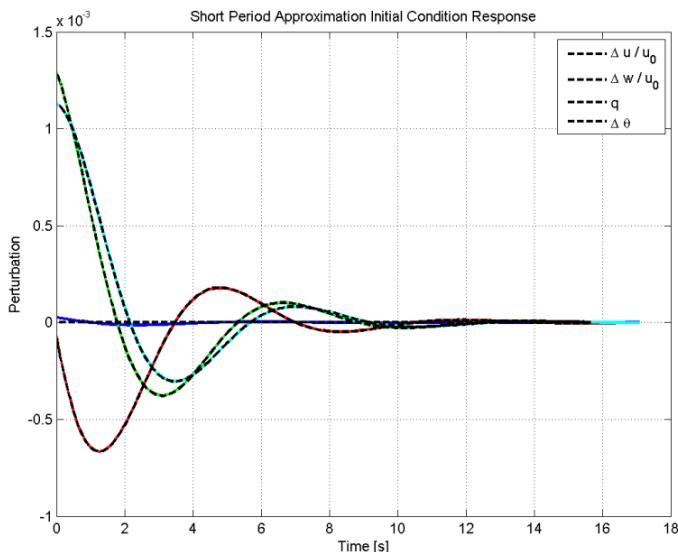
$$\omega_n = 0.962$$

### S.P. Approx

$$\lambda_{sp} = -0.371 \pm 0.889i$$

$$g = 0.385$$

$$\omega_n = 0.963$$



### Phugoid Mode

### Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - mg \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g \Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \rho u_0^2 S C_{W_0} + \rho g S C_{W_0} \Delta z_E = W + \rho g S C_{W_0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m} \Delta z_E}_{\omega_n^2} = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = 2\pi \sqrt{\frac{\frac{1}{2} u_0^2 m g}{g^2 \frac{1}{2} \rho u_0^2 S C_{W_0}}} = \boxed{\pi \sqrt{2} \frac{u_0}{g}}$$

$$\boxed{T = 0.138 u_0 \text{ if } u_0 \text{ in f/s} \\ = 0.453 u_0 \text{ if } u_0 \text{ in m/s}}$$

for 747

$$\underline{\text{Full Aer}}$$
  
 $T = 93s$

$$\underline{\text{Lanchester}}$$
  
 $T = 107s$

### "2x2" Phugoid Approximation

Dynamics of Flight, Eq. (4.9,18)  $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_w}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & 0 & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_w} & 0 & 0 & -mg \sin \theta_0 \\ \frac{1}{I_y} \left[ M_u + \frac{M_w Z_u}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_w \right] & \frac{1}{I_y} \left[ M_q + \frac{M_w (Z_q + mu_0)}{m - Z_w} \right] & -M_w mg \sin \theta_0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \Delta \alpha &= 0 \\ \Delta q &\text{ small} \\ Z_w &\ll m \\ Z_q &\ll mu_0 \\ \theta_0 &= 0 \end{aligned}$$

$$\rightarrow \begin{bmatrix} \Delta u \\ \Delta w = \Delta \alpha u_0 \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m} \Delta u + u_0 \Delta q$$

$$\Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\Delta \dot{\theta} = \Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\boxed{\begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}}$$

$A_{ph}$

$$|A_{ph} - \lambda I| = \lambda^2 - \underbrace{\frac{X_u}{m} \lambda}_{-2\beta \omega_n} - \underbrace{\frac{Z_u g}{mu_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{Z_u}$$

$$C_{Z_u} = -M_o \frac{\partial C_L}{\partial M} \Big|_0 - \rho u_0^2 \frac{\partial C_L}{\partial p_d} \Big|_0 - C_{T_u} \frac{\partial C_L}{\partial C_T} \Big|_0$$

assume  $\frac{\text{small}}{\text{small}} \quad \frac{\text{small}}{\text{small}} \quad \frac{\text{small}}{\text{small}}$

$$C_{Z_u} = 0 \quad \therefore \boxed{Z_u = -\rho u_0 S C_{W_0}}$$

$$X_u = \rho u_0 S C_{w_0} \sin \theta_0^0 + \frac{1}{2} \rho u_0 S C_{x_u}$$

$$C_{x_u} = -2 C_{T_0}$$

$$C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0^0$$

$$\underline{X_u = -\rho u_0 S C_{D_0}}$$

$$\boxed{\omega_u = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}}$$

Same as Lanchester

$$\zeta = \frac{-X_u}{2} \sqrt{\frac{-u_0}{m Z_u g}} = \frac{\rho u_0 S C_{D_0}}{2} \sqrt{\frac{u_0}{\frac{1}{2} \rho u_0 S C_{L_0}}}$$

substitute  $m g = \frac{1}{2} \rho u_0^2 S C_{L_0}$  and cancel

$$\boxed{\zeta = \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}}}$$

High  $\frac{C_D}{C_L} \Rightarrow$  less damping

(b/c less energy loss)

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Full  $A_{6x1}$

$$\lambda_{3,4} = -3.29 \times 10^{-3} \pm 6.72 \times 10^{-2} i$$

$$\zeta = 0.0489$$

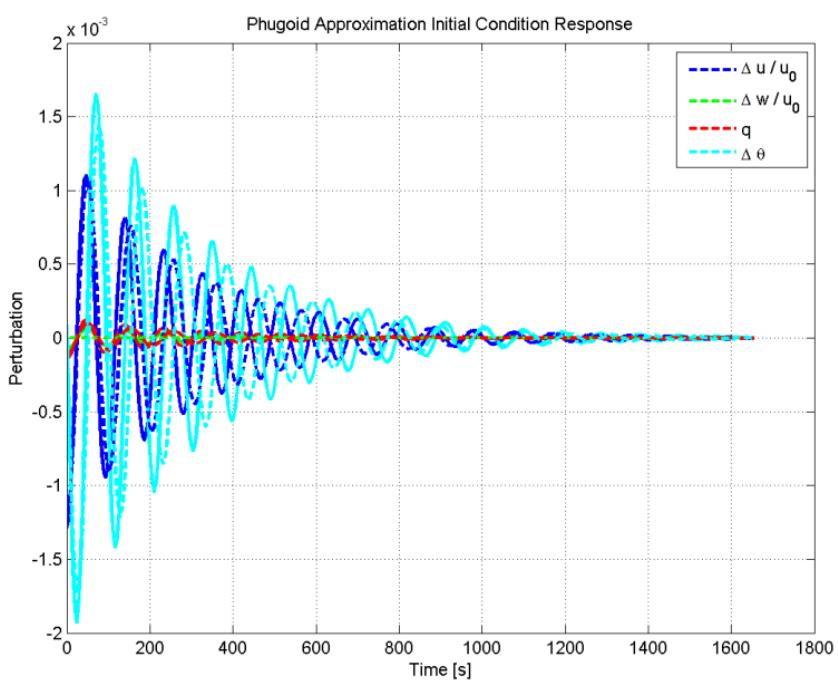
$$\omega_n = 0.0673$$

$2 \times 2$  Ph. Approx

$$\lambda_{ph} = -3.43 \times 10^{-3} \pm 6.11 \times 10^{-2} i$$

$$\zeta = 0.0561$$

$$\omega_n = 0.0612$$



# Longitudinal Control

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\vec{c}_{lon} = B_{lon} \vec{u}_{lon}$$

$$\vec{u}_{lon} = \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_t \end{bmatrix}$$

← elevator  
 + = down  
 ← throttle  
 + = more thrust

dimensional control derivatives

$$\Delta X_c = X_{\delta e} \Delta \delta_e + X_{\delta t} \Delta \delta_t$$

$$\Delta Z_c = Z_{\delta e} \Delta \delta_e + Z_{\delta t} \Delta \delta_t \quad \text{often } 0$$

$$\Delta M_c = M_{\delta e} \Delta \delta_e + M_{\delta t} \Delta \delta_t$$

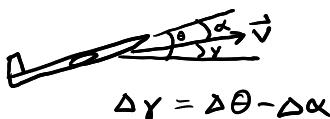
$$B_{lon} = \begin{bmatrix} \frac{X_{\delta e}}{m} & \frac{X_{\delta t}}{m} \\ \frac{Z_{\delta e}}{m - Z_{\dot{w}}} & \frac{Z_{\delta t}}{m - Z_{\dot{w}}} \\ \frac{M_{\delta e} + M_{\dot{w}} Z_{\delta e}}{I_y} & \frac{M_{\delta t} + M_{\dot{w}} Z_{\delta t}}{I_y} \\ 0 & 0 \end{bmatrix}$$

	$\delta_e$	$\delta_t$
$\Delta u$	-0.000187	9.66
$\Delta w$	-17.85	0
$\Delta q$	-1.158	0
$\Delta \theta$	0	0

$$\dot{\vec{x}}_{lon} = \mathbf{A}_{lon} \vec{x}_{lon} + \mathbf{B}_{lon} \vec{u}_{lon}$$

C and D depend on output

E.g. flight path angle  $\gamma$



$$\Delta \gamma = \Delta \theta - \Delta \alpha$$

$$\Delta \gamma = \underbrace{\begin{bmatrix} 0 & -\frac{1}{u_0} & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}}_D$$

$$\vec{\gamma} = [\Delta \gamma] = C_{\gamma} \vec{x}_{lon} + \underbrace{[0]}_D \vec{u}_{lon}$$

Open-loop step response to control inputs

$$B_{\delta e} = B_{lon}(:, 1)$$

$$B_{\delta t} = B_{lon}(:, 2)$$

$$\delta_e: 1^\circ$$

$$\delta_t: \frac{1}{6} = 0.05 \text{ rad}$$

$\delta_e \Rightarrow \gamma$  little change

$\delta_t \Rightarrow \gamma$  significant increase

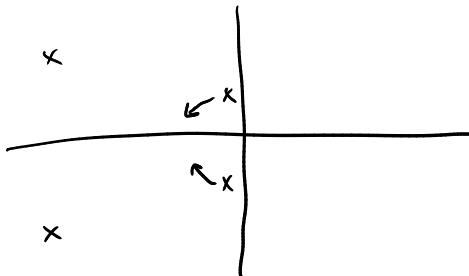
## Longitudinal stability augmentation

Realistic main goal:

- increase phugoid damping

On homework: change s.p. damping ratio

- not usually a real-life goal
- easy to do by hand



Increase Phugoid damping with

$$\Delta \delta_e = -k_\theta \Delta \theta$$

$$\dot{\vec{x}}_{ph} = \begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta \theta} \end{bmatrix} = \underbrace{A_{ph} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}}_{\text{previous}} + B_{ph, \delta e} \Delta \delta_e$$

$$B_{\text{lon}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}$$

for 747

Derivation of  $A_{ph}$  and  $B_{ph, \delta e}$

Assume  $\dot{q}$  only depends on  $\alpha$ ,  $q$ , and  $\delta e$   
solve for  $\dot{q}=0$  (after short period has damped out)

$$\dot{\vec{q}} = \frac{M_w}{I_Y} \Delta w + \frac{M_{\delta e}}{I_Y} \Delta \delta e + \frac{M_q}{I_Y} \Delta q$$

$$\Rightarrow \Delta w = -\frac{M_{\delta e}}{M_w} \Delta \delta e$$

solve for  $\dot{\Delta w} = 0$  (after s.p. oscillations damp out)

$$\dot{\Delta w} = 0 = \frac{Z_u}{m} \Delta u + \frac{Z_w}{m} \Delta w + \frac{m u_o}{m} \Delta \dot{q} + \frac{Z_{\delta e}}{m} \Delta \delta e$$

solve for  $\dot{\Delta \theta}$ , substitute  $\Delta w$  above

$$\frac{m u_o}{m} \dot{\Delta \theta} = -\frac{Z_u}{m} \Delta u - \frac{Z_w}{m} \left( -\frac{M_{\delta e}}{M_w} \Delta \delta e \right) - \frac{Z_{\delta e}}{m} \Delta \delta e$$

$$\dot{\Delta \theta} = -\frac{Z_u}{m u_o} \Delta u + \left( \frac{Z_w}{m u_o} \frac{M_{\delta e}}{M_w} - \frac{Z_{\delta e}}{m u_o} \right) \Delta \delta e$$

$$\dot{\Delta u} = \frac{X_u}{m} \Delta u + \frac{X_{\delta e}}{m} \Delta \delta e$$

$$\begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta \theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_o} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} \frac{X_{\delta e}}{m} \\ \frac{Z_w M_{\delta e}}{m u_o} - \frac{Z_{\delta e}}{m u_o} \end{bmatrix} \Delta \delta e$$

$$A_{ph} = \begin{bmatrix} -0.0069 & -32.2 \\ 0.0001 & 0 \end{bmatrix}$$

$$B_{ph, \delta e} = \begin{bmatrix} 0 \\ -0.0002 \\ -0.44 \end{bmatrix}$$

$$\Delta \delta_e = -k_\theta \Delta \theta$$

$$\vec{u} = -K_{ph, \theta} \vec{x}_{ph}$$

$$= -[0 \ k_\theta] \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$\begin{aligned}
 A^{cl} &= A_{ph} - B_{ph\theta_e} K_{ph\theta} \\
 &= A_{ph} - \begin{bmatrix} 0 \\ -0.44 \end{bmatrix} [0 \quad k_\theta] \\
 &= A_{ph} - \begin{bmatrix} 0 & 0 \\ 0 & -0.44k_\theta \end{bmatrix} \\
 &= \begin{bmatrix} -0.0069 & -32.2 \\ -0.0001 & 0.44k_\theta \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 |A^{cl} - \lambda I| &= (-0.0069 - \lambda)(0.44k_\theta - \lambda) - (-0.0001)(-32.2) = 0 \\
 &= \lambda^2 + \underbrace{(-0.44k_\theta + 0.0069)\lambda}_{2\zeta\omega_n} - \underbrace{0.003k_\theta - 0.0032}_{-\omega_n^2} = 0
 \end{aligned}$$

$$\zeta = 0.7 \Rightarrow k_\theta = -0.2$$

Plug back into  $4 \times 4$

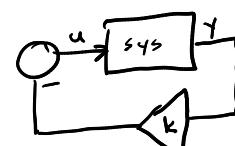
$$\begin{aligned}
 \Delta \delta_e &= -K_\theta \vec{x}_{ion} \\
 &= -[0 \ 0 \ 0 \ k_\theta] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}
 \end{aligned}$$

$$A^{cl} = A_{ion} - \underline{B_{\delta e}} K_\theta$$


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nlocus assumes  $\underline{u} = -k_y$

$$\Delta \delta_e = -k_\theta \Delta \theta$$



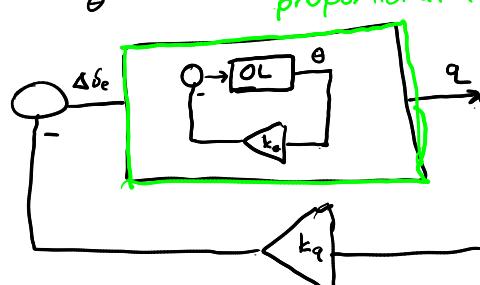
$$\begin{aligned}
 y = \Delta \theta &= [0 \ 0 \ 0 \ 1] \begin{bmatrix} \vec{x}_{ion} \\ C_\theta \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} \\
 D &= 0
 \end{aligned}$$


---

derivative gain

$$\Delta \delta_e = \underbrace{-k_\theta \Delta \theta}_{\text{proportional}} - \underbrace{k_q \Delta q}_{\text{derivative}}$$

$k_\theta = -0.5$  proportional feedback



$$\Delta q = [0 \ 0 \ 1 \ 0] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

from root locus choose  $k_q = -1$