

# Final Review – Part 2: Conventional Aircraft



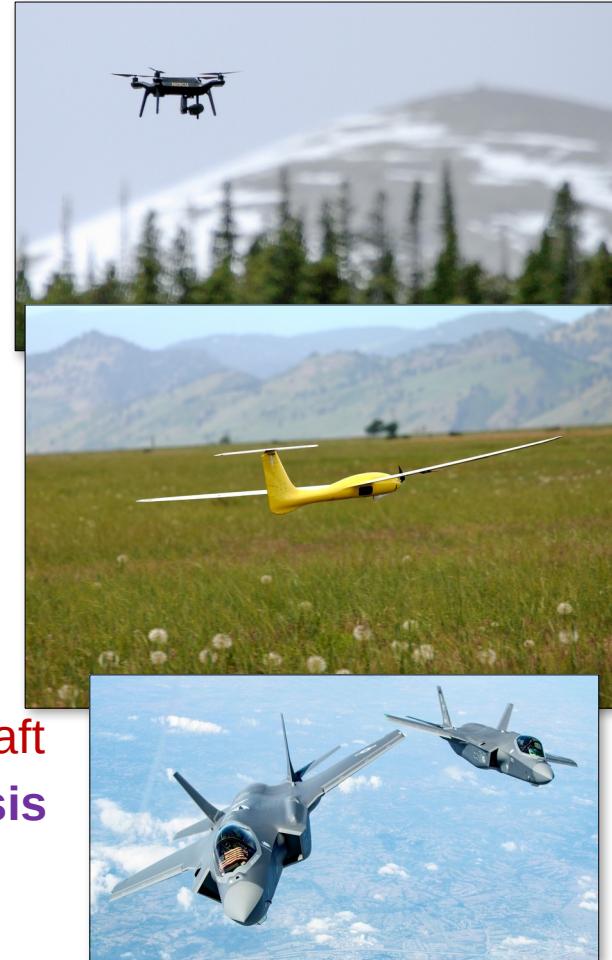
# Announcements

- Final Exam: Saturday, May 4<sup>th</sup> 7:30-10pm
- TAs will be at Office Hours and Study Hall to answer questions
- Sunberg Extra Office Hours: Friday 4-5 pm

# Course Material – Bold in Review 1

(See Syllabus for Reading Assignments)

- 1. Nomenclature**
2. Aircraft Static Stability
- 3. Euler Angles**
- 4. Aircraft Equations of Motion (Kinematics + Dynamics)**
- 5. Linearized Equations**
- 6. Quadrotor Equations of Motion**
- 7. Quadrotor Stability and Control**
8. Longitudinal Dynamics of Conventional (Fixed-Wing) Aircraft
9. Lateral Dynamics of Conventional (Fixed Wing) Aircraft
- 10. State Space and Transfer Function-based Analysis**
- 11. Mode Approximations**
- 12. Conventional Aircraft Control**



# Course Material – Bold in Review 2

(See Syllabus for Reading Assignments)

1. Nomenclature
2. Aircraft Static Stability
3. Euler Angles
4. Aircraft Equations of Motion (Kinematics + Dynamics)
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# Conventional Aircraft Dynamics

- Conventional Aircraft dynamics and aerodynamics are commonly separated into two groups:
  - Longitudinal
    - Up-down, pitch plane, pitching motions
  - Lateral-directional
    - Side-to-side, turning motions (roll and yaw)

# **LONGITUDINAL DYNAMICS**

# Longitudinal Forces and Moments

$$\begin{aligned} C_L &= C_{L_\alpha} \alpha + C_{L_q} \dot{q} + C_{L_{\delta_e}} \delta_e \\ C_D &= C_{D_{min}} + K (C_L - C_{L_{min}})^2 \\ C_m &= C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \dot{q} + C_{m_{\delta_e}} \delta_e \end{aligned}$$

$$C_{L_\alpha} = a = a_{wb} \left[ 1 + \frac{a_t S_t}{a_{wb} S} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m0_p} + a_t \bar{V}_H (\epsilon_0 + i_t) \left[ 1 - \frac{a_t S_t}{a S} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a} \bar{V}_H \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) - \frac{1}{a} \frac{\partial C_{m_p}}{\partial \alpha}$$

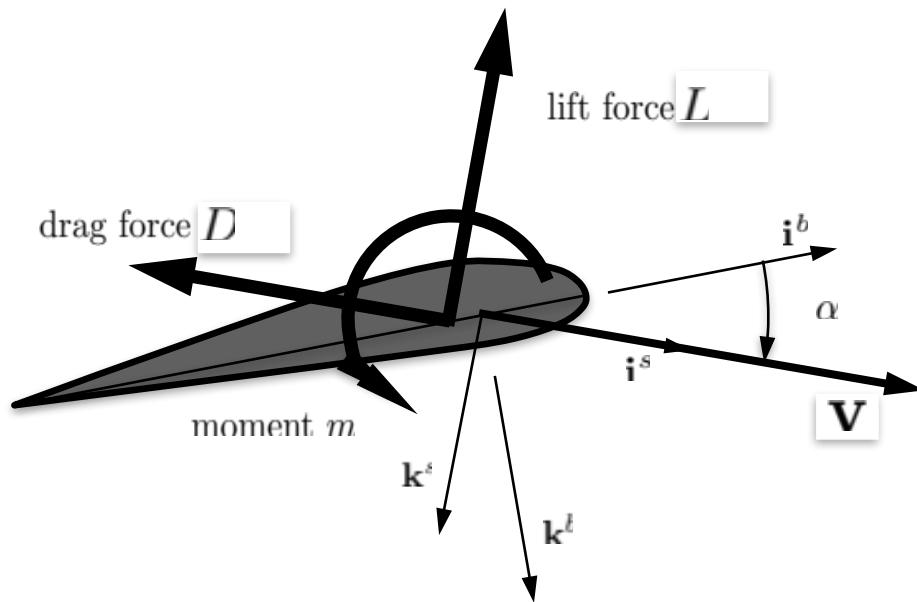
$$C_{m_\alpha} = C_{L_\alpha} (h - h_n)$$

*Direct dependence on CG location  $h$*

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e} \frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}} (h - h_{n_{wb}})$$

# Longitudinal Forces – Body Frame



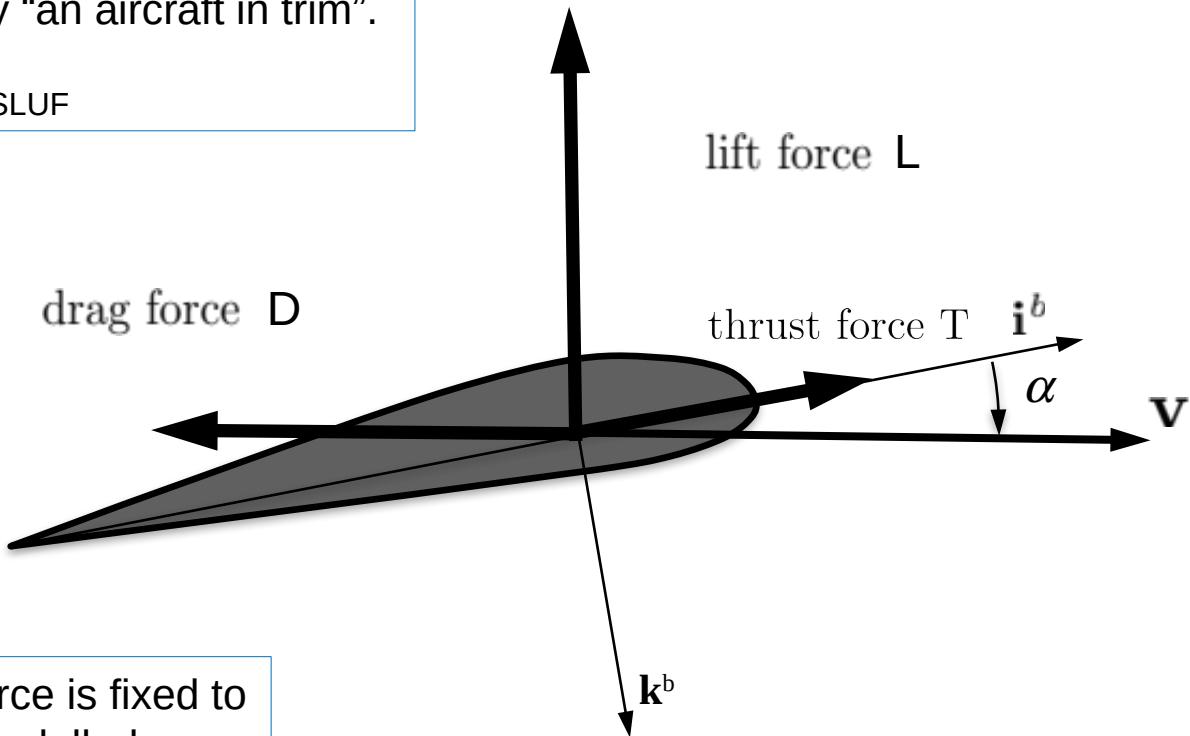
*Lift and drag defined in stability (wind) frame, so rotate into body frame*

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} -D \\ -L \end{pmatrix}$$

# Straight, level, unaccelerated flight (SLUF)

For a fixed-wing aircraft the main trim state will be straight, level, unaccelerated flight (SLUF)<sup>1</sup>. We will just say “an aircraft in trim”.

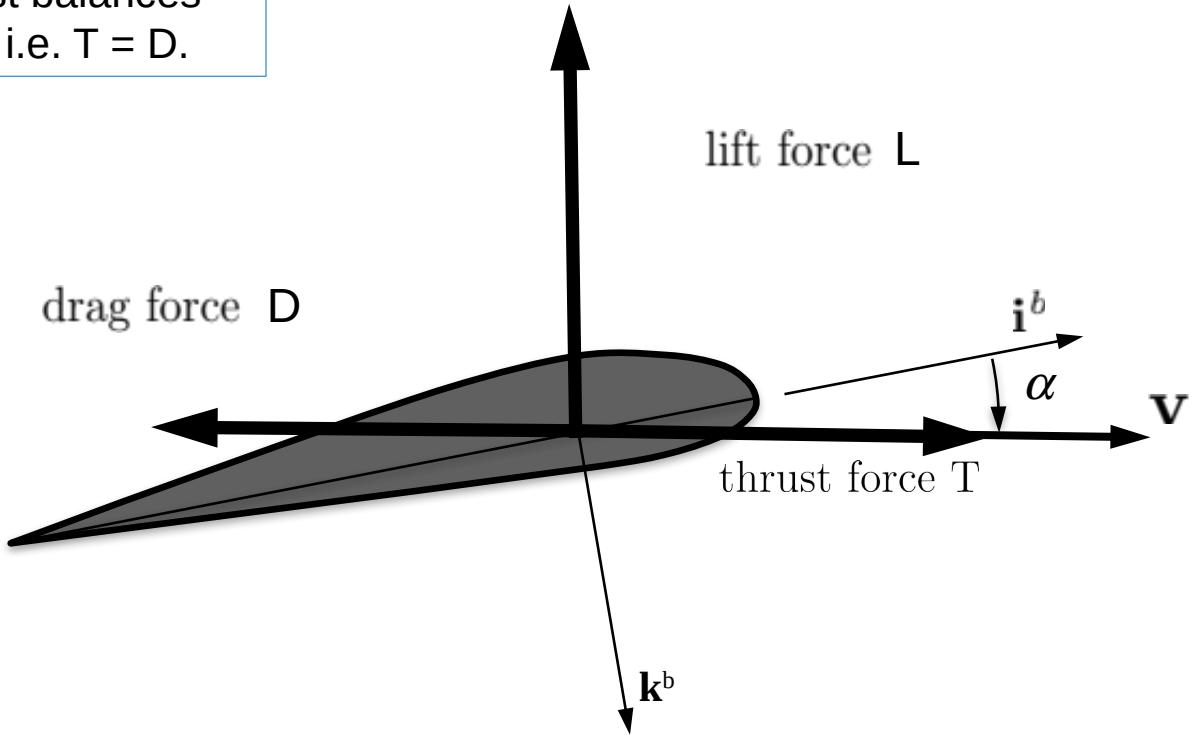
1. <https://en.wiktionary.org/wiki/SLUF>



In general, the thrust force is fixed to the body and is often modelled as pointing along the body x-axis.

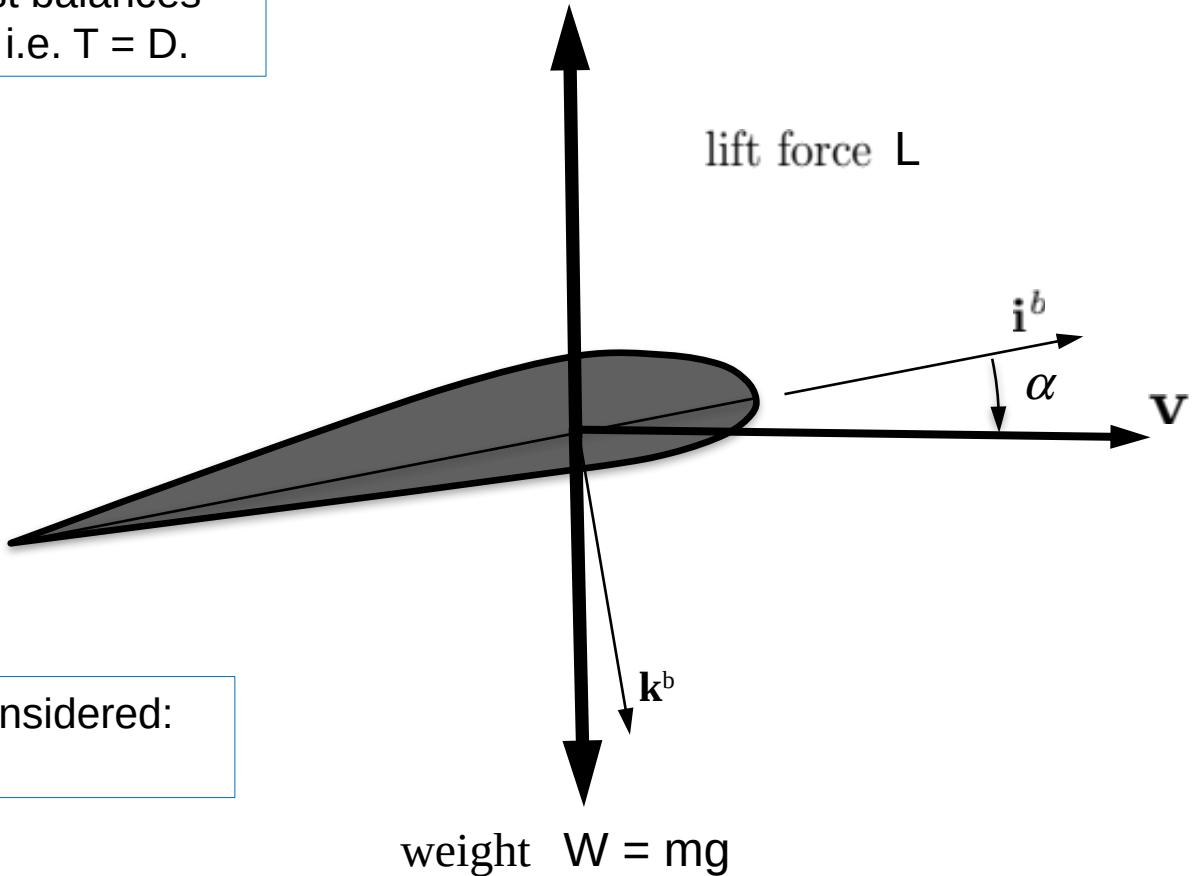
# Assumption: Thrust

For simple analysis of static stability (trim), we **assume** thrust balances drag and cancels it out, i.e.  $T = D$ .



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For simple analysis of static stability (trim), we **assume** thrust balances drag and cancels it out, i.e.  $T = D$ .



Only two forces are considered:  
Lift and Weight

# Equilibrium – Speed and Altitude

- At equilibrium we need

$$C_{m_{trim}} = 0$$

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V_a^2 S}$$

Density a function of  
(desired) altitude

Desired (air)speed

$$C_{L_{trim}} = C_{L_\alpha} \alpha_{trim} + C_{L_{\delta_e}} \delta_{e_{trim}}$$

$$C_{m_{trim}} = 0 = C_{m_0} + C_{m_\alpha} \alpha_{trim} + C_{m_{\delta_e}} \delta_{e_{trim}}$$

# Trim Angle of Attack and Elevator

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta_e}} \\ C_{m_\alpha} & C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

↓  
solving

$$\alpha_{trim} = \frac{C_{m_0} C_{L_{\delta_e}} + C_{m_{\delta_e}} C_{L_{trim}}}{\Delta}$$

$$\delta_{e_{trim}} = -\frac{C_{m_0} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_\alpha}$$

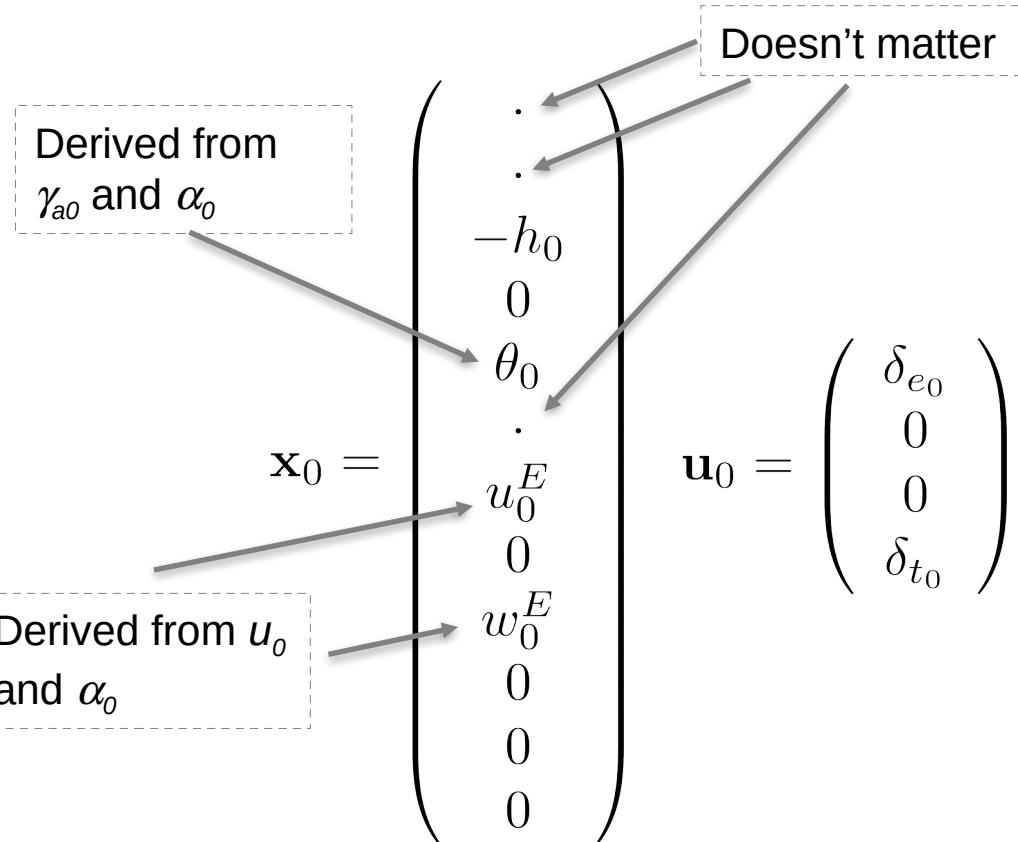
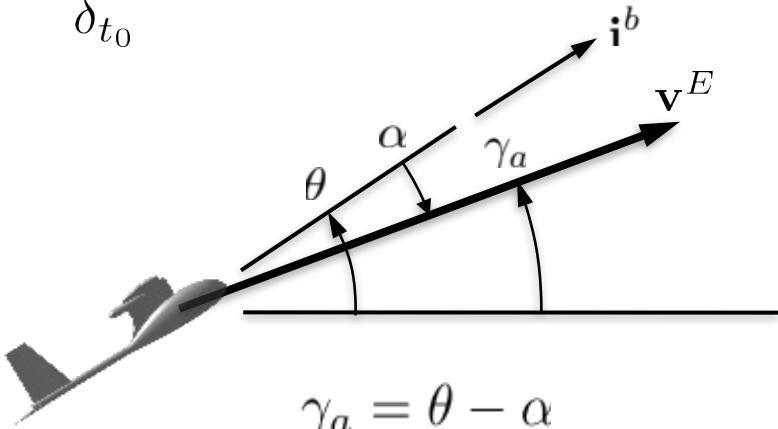
# General Trim Initial Condition

Straight, wings-level trim is defined by

- $u_0$  • speed
- $\gamma_{a0}$  • air-relative flight path angle
- $h_0$  • altitude (height)

which is used to determine

$$\begin{aligned}\alpha_0 \\ \delta_{e0} \\ \delta_{t0}\end{aligned}$$



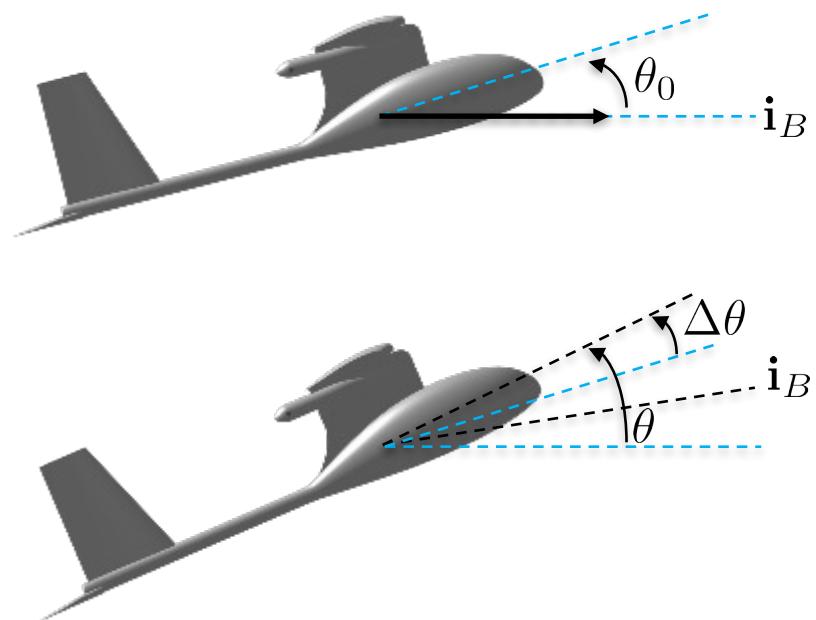
This further assumes no background wind.  
What would be different...?



# Variations

Can describe variables as reference condition (“trim condition”) plus small variation, e.g  $u^E = u = u_0 + \Delta u$

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x_{E_0} + \Delta x_E \\ y_{E_0} + \Delta y_E \\ z_{E_0} + \Delta z_E \\ \Delta\phi \\ \theta_0 + \Delta\theta \\ \Delta\psi \\ u_0 + \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$



# Recall Nonlinear EOM

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

# Linearized EOM

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

$$\Delta\dot{\theta} = \Delta q$$

$$\Delta\dot{u} = -g \cos \theta_0 \Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\Delta\dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

$$\Delta\dot{x}^E = \Delta u$$

$$\Delta\dot{y}^E = u_0 \cos \theta_0 \Delta\psi + \Delta v$$

$$\Delta\dot{z}^E = -u_0 \Delta\theta + \Delta w$$

$$\Delta\psi = \Delta r \sec \theta_0$$

These variables depend on each other.

They do not depend on these variables.



# Linearized Aero Forces and Moments

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

Force due to control  
variables (more later)

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

Combine these equations with the linearized dynamic equations, group terms and put into matrix form

# Longitudinal Equations

*Dynamics of Flight*, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

# 6x6 (a.k.a “flight path”) Dynamics Matrix

# Non-dimensionalization

Variable	Divisor	Non-dim Variable
$X, Y, Z$	$\frac{1}{2}\rho V^2 S$	$C_x, C_y, C_z$
$W$	$\frac{1}{2}\rho V^2 S$	$C_W$
$M$	$\frac{1}{2}\rho V^2 S \bar{c}$	$C_m$
$L, N$	$\frac{1}{2}\rho V^2 S \bar{b}$	$C_l, C_n$
$u, v, w$	$V$	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
$m$	$\rho S \bar{c}/2$	$\mu$
$I_y$	$\rho S (\bar{c}/2)^3$	$\hat{I}_y$
$I_x, I_z, I_{xz}$	$\rho S (b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

# Why Dimensionless?

Dimensional derivatives change with

Trim condition



Size



Dimensionless derivatives stay the same for an aircraft (with fixed CG)!

# Longitudinal Dimensional Derivatives

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u} \quad X_u = \rho u_0 S C_{W_0} \sin \theta_0 + \frac{1}{2} \rho u_0 S C_{x_u}$$

$$Z_w = \frac{1}{2} \rho u_0 S C_{z_\alpha} \quad X_w = \frac{1}{2} \rho u_0 S C_{x_\alpha}$$

$$Z_{\dot{w}} = \frac{1}{4} \rho \bar{c} S C_{z_{\dot{\alpha}}} \quad X_{\dot{w}} = \frac{1}{4} \rho \bar{c} S C_{x_{\dot{\alpha}}}$$

$$Z_q = \frac{1}{4} \rho u_0 \bar{c} S C_{z_q} \quad X_q = \frac{1}{4} \rho u_0 \bar{c} S C_{x_q}$$

$$M_u = \frac{1}{2} \rho u_0 S \bar{c} C_{m_u}$$

$$M_w = \frac{1}{2} \rho u_0 S \bar{c} C_{m_\alpha}$$

$$M_{\dot{w}} = \frac{1}{4} \rho \bar{c}^2 S C_{m_{\dot{\alpha}}}$$

$$M_q = \frac{1}{4} \rho u_0 \bar{c}^2 S C_{m_q}$$

# Example Derivative Derivations:

$$\left( \frac{\partial V}{\partial w} \right)_0 = 0 \quad \text{so} \quad Z_w = \left( \frac{\partial Z}{\partial w} \right)_0 = \frac{1}{2} \rho u_0 S \left( \frac{\partial C_z}{\partial \hat{w}} \right)_0$$
$$Z_w = \frac{1}{2} \rho u_0 S C_{z_\alpha} \quad \boxed{\hat{w} = \Delta \alpha}$$

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$$Z_{\dot{w}} = \left( \frac{\partial Z}{\partial \dot{w}} \right)_0 = \frac{1}{2} \rho u_0 \left( \frac{\partial C_z}{\partial \dot{\alpha}} \right)_0 \quad Z_q = \left( \frac{\partial Z}{\partial q} \right)_0 = \frac{1}{2} \rho u_0^2 \left( \frac{\partial C_z}{\partial q} \right)_0$$
$$= \frac{1}{2} \rho u_0 (\bar{c}/2u_0) \left( \frac{\partial C_z}{\partial \dot{\alpha}} \right)_0 \quad = \frac{1}{2} \rho u_0^2 (\bar{c}/2u_0) \left( \frac{\partial C_z}{\partial \hat{q}} \right)_0$$
$$= \frac{1}{4} \rho \bar{c} S C_{z_{\dot{\alpha}}} \quad = \frac{1}{4} \rho u_0 \bar{c} S C_{z_q}$$

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# Example: Pitch Rate Derivatives

$$Z_q = \frac{1}{4} \rho u_0 \bar{c} S C_{z_q}$$

$$X_q = \frac{1}{4} \rho u_0 \bar{c} S C_{x_q}$$

$$M_q = \frac{1}{4} \rho u_0 \bar{c}^2 S C_{m_q}$$

$$C_{Z_q} = -C_{L_q}$$

$$C_{m_q} = C_{m_q}$$

Stability frame (lift/drag)  
to body frame

$$= -2a_t V_H$$

$$= -2a_t V_H \frac{l_t}{c}$$

Shape/geometry and mass  
to aerodynamic coefficients



# Example: Pitch Rate Derivatives

$$Z_q = \frac{1}{4} \rho u_0 \bar{c} S C_{z_q}$$

$$X_q = \frac{1}{4} \rho u_0 \bar{c} S C_{x_q} \xrightarrow{0} \text{by assumption}$$

$$M_q = \frac{1}{4} \rho u_0 \bar{c}^2 S C_{m_q}$$

$$\Delta C_m = -V_H \Delta C_{L_t} = -a_t V_H \frac{q l_t}{u_0}$$

$$C_{m_q} = \left( \frac{\partial C_m}{\partial \hat{q}} \right)_0$$

Remember: derivatives are taken w.r.t. nondimensional values ()

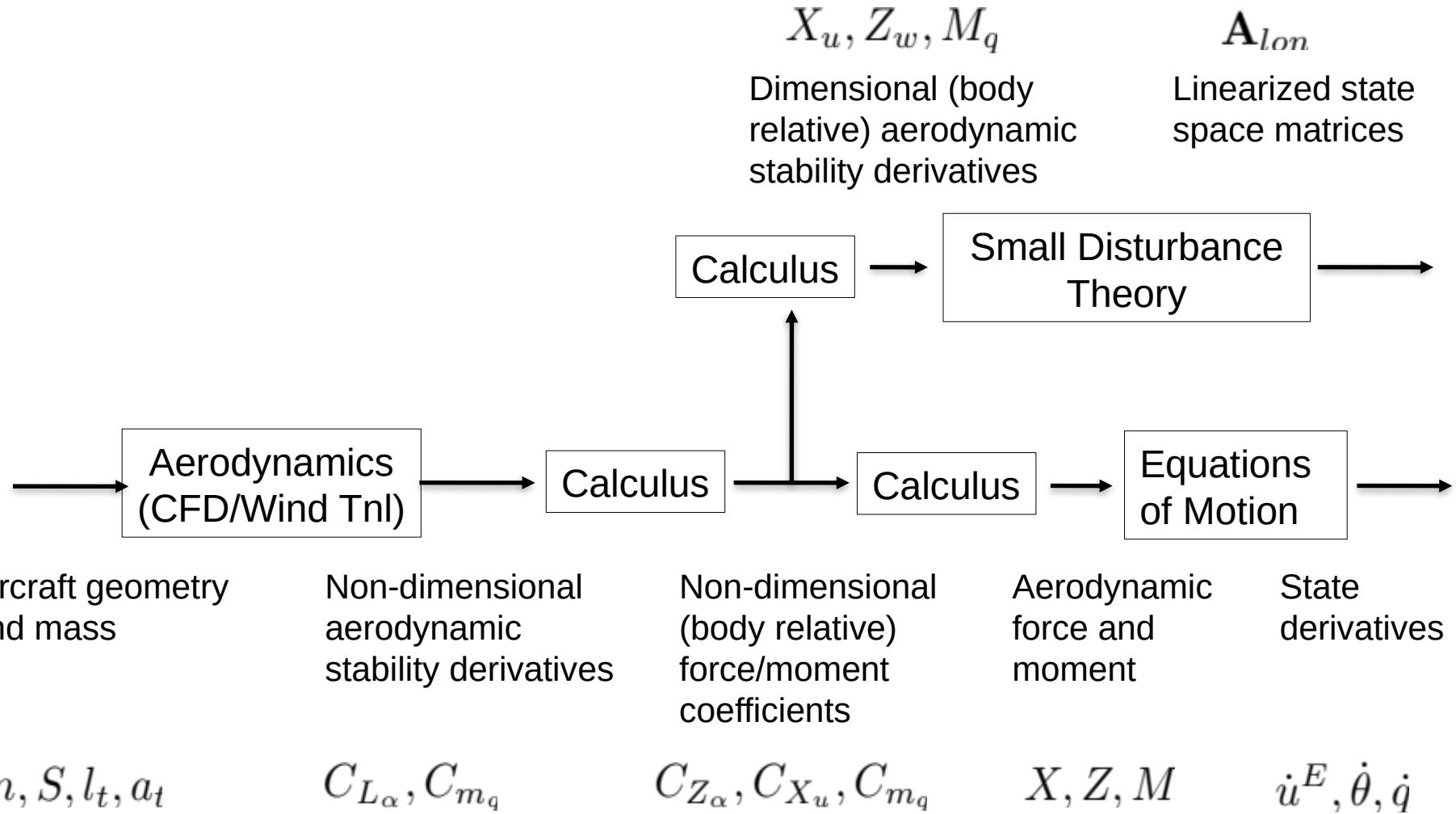
$$= \frac{2u_0}{c} \left( \frac{\partial C_m}{\partial q} \right)_0$$

$$= -\frac{2u_0}{c} a_t V_H \frac{l_t}{u_0}$$

$$= -2a_t V_H \frac{l_t}{c}$$



# Big Picture



# Longitudinal Modes

Boeing 747 Jet Transport (Etkin 6.2)



$$\begin{aligned} h_0 &= 40000 \text{ ft} \\ V_0 &= 774 \text{ ft/s} \\ \gamma_0 = \theta_0 = \alpha_0 &= 0^\circ \end{aligned}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

# Longitudinal Modes

$\lambda_i$	$\zeta$	$\omega_n$	Eigenvalues With their damping ratio and natural frequency
$-3.72e - 01 + 8.88e - 01i$ $-3.72e - 01 - 8.88e - 01i$	$3.87e - 01$ $3.87e - 01$	$9.62e - 01$ $9.62e - 01$	
$-3.29e - 03 + 6.72e - 02i$ $-3.29e - 03 - 6.72e - 02i$	$4.89e - 02$ $4.89e - 02$	$6.73e - 02$ $6.73e - 02$	

**Short period** →

**Phugoid** →

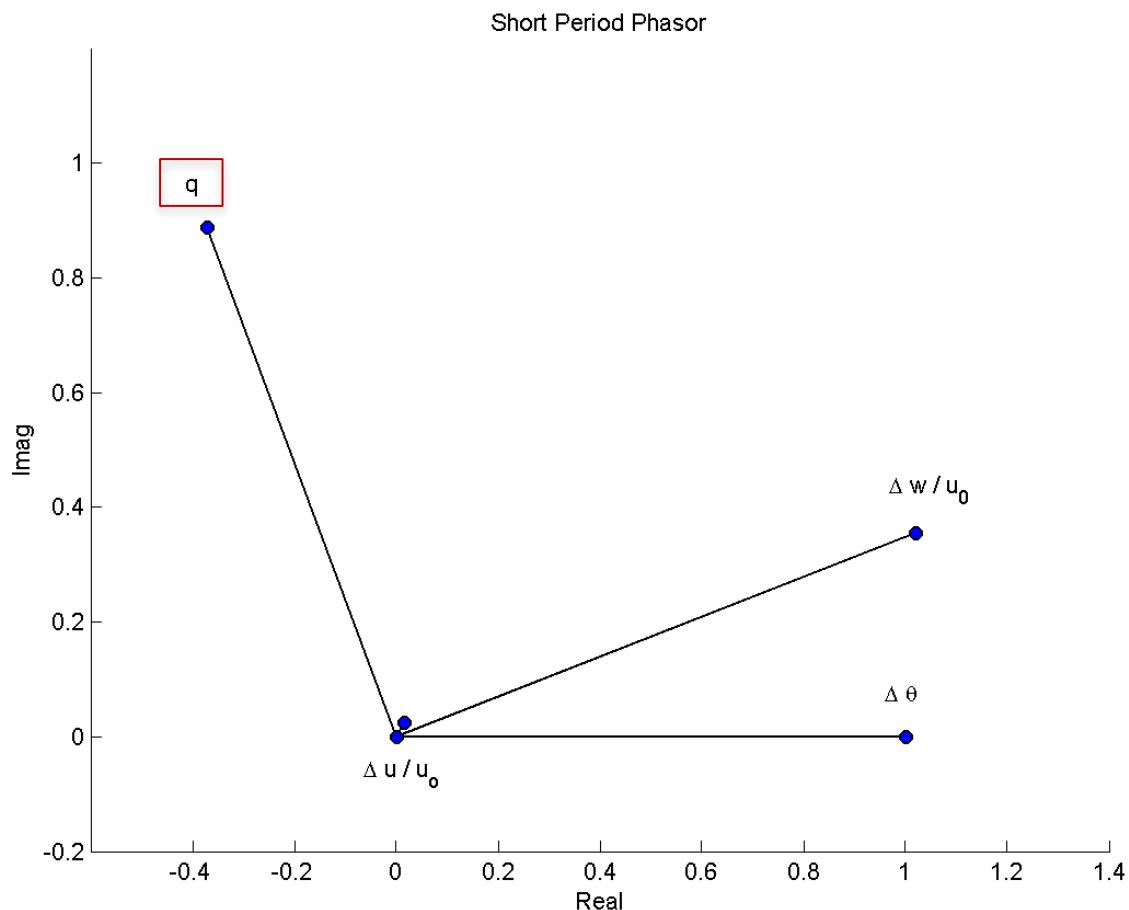
Associated Eigenvectors

$$\mathbf{v}_{1/2} = \begin{pmatrix} 0.0211 \pm 0.0166i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \mp 0.0004i \end{pmatrix}$$
$$\mathbf{v}_{3/4} = \begin{pmatrix} -0.9983 \\ -0.0573 \pm 0.0097i \\ -0.0001 \mp 0.0000i \\ 0.0001 \pm 0.0021i \end{pmatrix}$$

# Short Period Mode

$$\begin{aligned}\lambda_{1/2} &= -.372 + .888i \\ \zeta &= 0.387 \leftarrow \text{well damped} \\ \omega_n &= 0.962 \leftarrow \text{fast response}\end{aligned}$$

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ \boxed{0.9623\angle 112.74^\circ} \\ 1.0000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{matrix}$$



The short period mode is characterized by a fast oscillatory response in pitch. The airspeed perturbations are small and the flight path angle stays close to zero since pitch and angle of attack are almost equivalent.

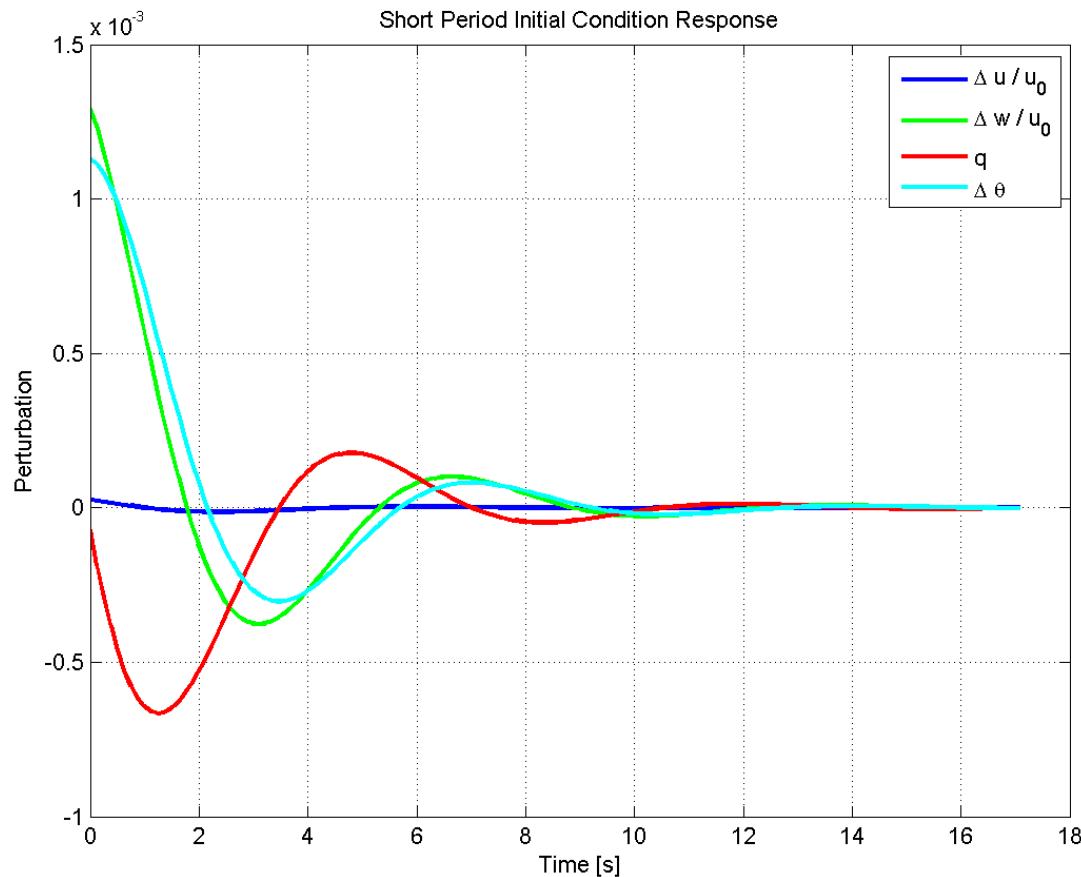
# Short Period Mode

$$\lambda_{1/2} = -0.372 + 0.888i$$

$$\zeta = 0.387$$

$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



The short period mode is characterized by a fast oscillatory response in pitch. The airspeed perturbations are small and the flight path angle stays close to zero since pitch and angle of attack are almost equivalent.

# Phugoid Mode

Phugoid Phasor

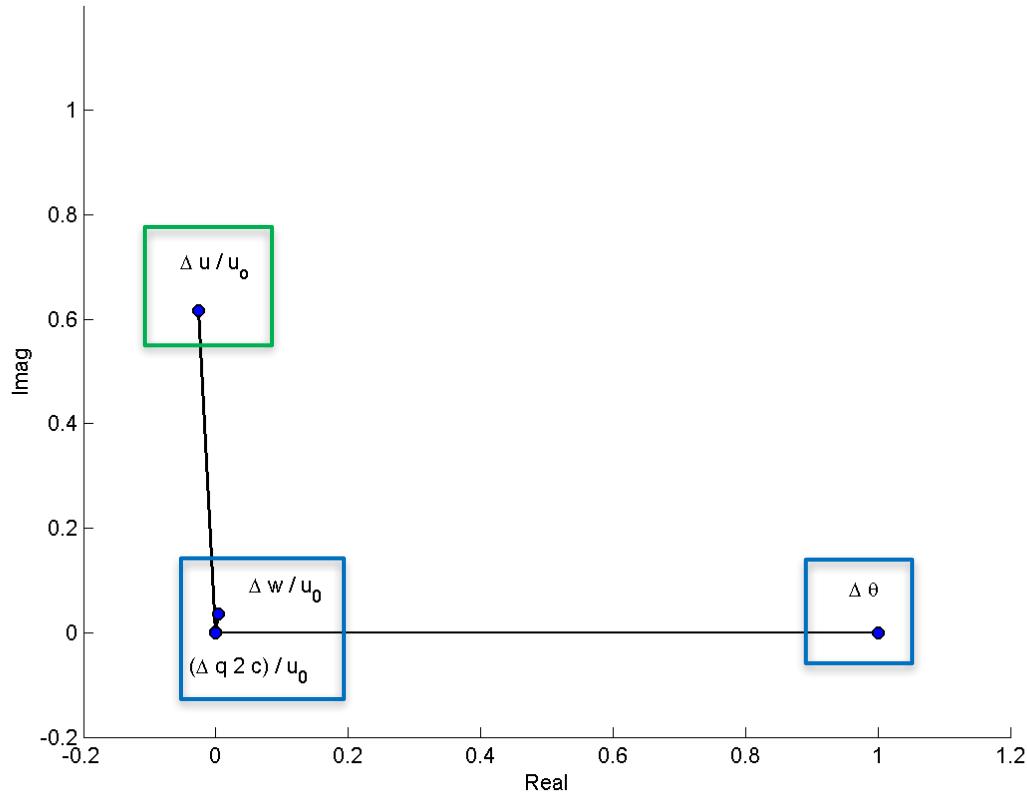
$$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$$

$$\zeta = 0.0489 \quad \text{poorly damped}$$

$$\omega_n = 0.0673 \quad \text{slow response}$$

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0012\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ \hat{q} \\ \Delta\theta \end{matrix}$$

Non-dimensionalize and scale so  $\Delta\theta$  term is 1



The phugoid mode is characterized by damped vertical oscillations of the aircraft's trajectory about the nominal flight path with speed increasing (decreasing) as the height decreases (increases). It can be interpreted as an exchange of potential and kinetic energy

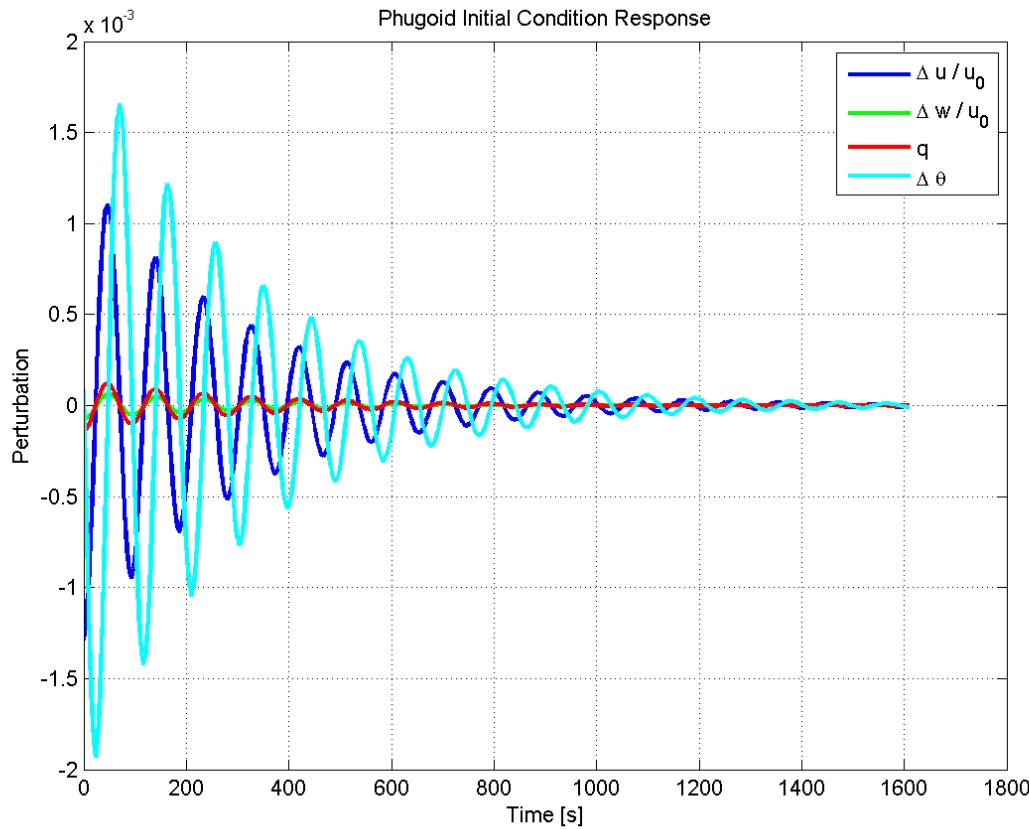
# Phugoid Mode

$$\lambda_{3/4} = -3.29e - 03 + 6.72e - 02i$$

$$\zeta = 0.0489 \quad \text{poorly damped}$$

$$\omega_n = 0.0673 \quad \text{slow response}$$

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



The phugoid mode is characterized by damped vertical oscillations of the aircraft's trajectory about the nominal flight path with speed increasing (decreasing) as the height decreases (increases). It can be interpreted as an exchange of potential and kinetic energy

# Short Period Approximation

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ 0.9623\angle 112.74^\circ \\ 1.0000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{matrix} \longrightarrow \begin{matrix} \Delta u = 0 \\ \theta_0 = \alpha_0 = 0 \end{matrix}$$

$$\begin{pmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{pmatrix} = \begin{pmatrix} \frac{Z_w}{m} & u_0 \\ \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m} \right] & \frac{1}{I_y} [M_q + M_{\dot{w}} u_0] \end{pmatrix} \begin{pmatrix} \Delta w \\ \Delta q \end{pmatrix}$$

Analytical solution for eigenvalues of 2<sup>nd</sup> order system is straightforward.

$$\lambda^2 - \lambda \left[ \frac{Z_w}{m} + \frac{1}{I_y} (M_q + M_{\dot{w}} u_0) \right] - \frac{1}{I_y} \left( u_0 M_w - \frac{M_q Z_w}{m} \right) = 0$$

$$-2\zeta\omega_n$$

$$-\omega_n^2$$



# Short Period Approximation

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0290\angle 57.38^\circ \\ 1.0803\angle 19.20^\circ \\ 0.9623\angle 112.74^\circ \\ 1.0000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{matrix}$$

## Full longitudinal matrix

$$\lambda_{1/2} = -.372 + .888i$$

$$\zeta = 0.387$$

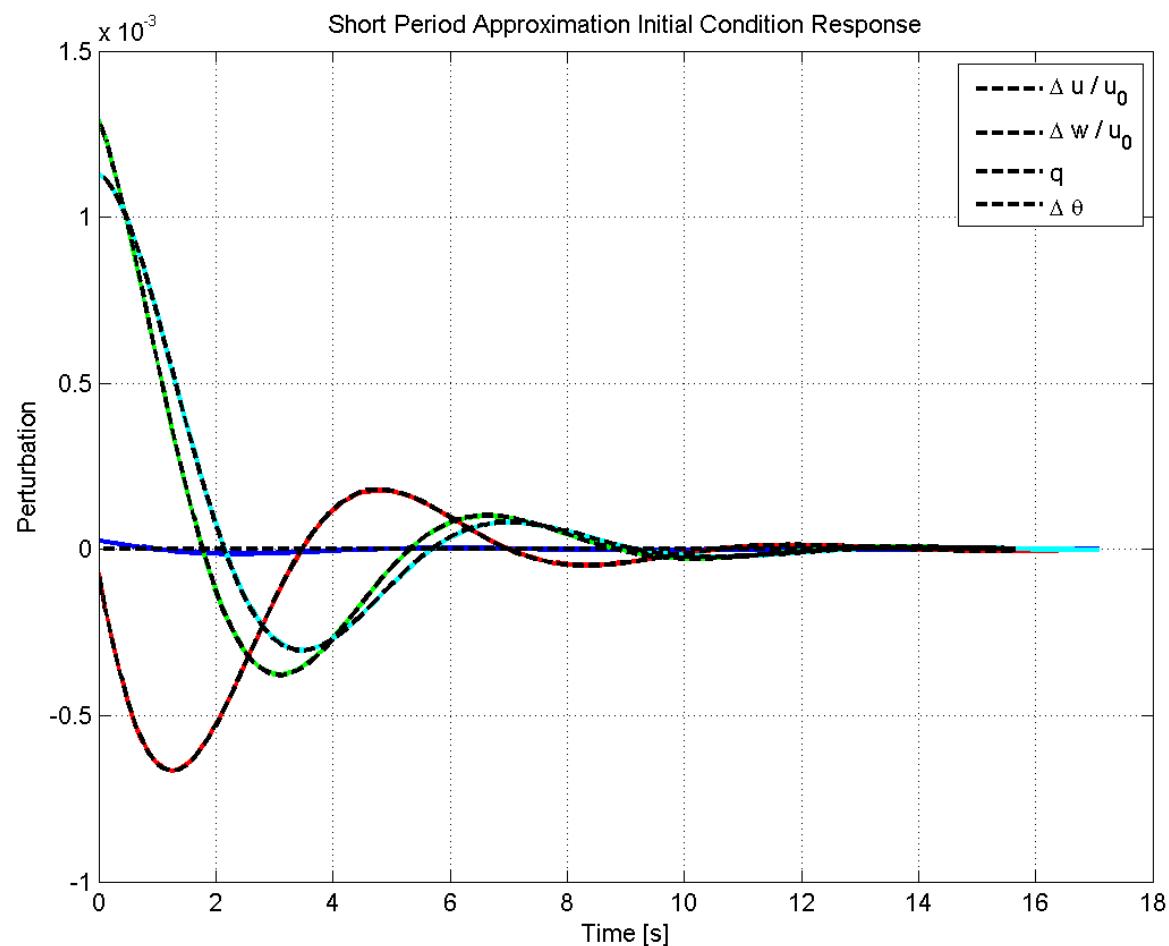
$$\omega_n = 0.962$$

## Short period approximation

$$\lambda_{sp} = -.371 + .889i$$

$$\zeta = 0.385$$

$$\omega_n = 0.963$$



# Phugoid Approximation

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0673\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{matrix} \longrightarrow \begin{array}{l} \text{Lanchester (1908)} \\ \Delta\alpha = 0 \\ \theta_0 = \alpha_0 = 0 \\ q = 0 \\ T = D \end{array}$$

Sum the forces in the Z-direction and apply Newton's Law

$$W - L = m\Delta\ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m\Delta\ddot{z}_E$$

$$m\Delta\ddot{z}_E + \rho g S C_{W_0} \Delta z_E = 0$$

$$\Delta\ddot{z}_E + \frac{\rho g S C_{W_0}}{m} \Delta z_E = 0$$

$$\omega_n = \sqrt{\frac{\rho g S C_{W_0}}{m}} \longrightarrow T = 2\pi/\omega_n = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = \pi\sqrt{2} \frac{u_0}{g} = 0.138 u_0 = 0.453 u_0$$

**Full longitudinal matrix**

$$T = 93 \text{ s}$$

**Lanchester phugoid approximation**

$$T = 107 \text{ s}$$

when  $u_0$  in feet per second

when  $u_0$  in meters per second



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# Phugoid Approximation

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.6170\angle 92.36^\circ \\ 0.0359\angle 82.78^\circ \\ 0.0673\angle 92.80^\circ \\ 1.000\angle 0.0^\circ \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{\alpha} \\ q \\ \Delta\theta \end{matrix} \quad \xrightarrow{\hspace{1cm}} \quad \begin{matrix} \Delta\alpha = 0 \\ \theta_0 = \alpha_0 = 0 \end{matrix}$$

$$\begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{pmatrix} = \begin{pmatrix} X_u/m & -g \\ -\frac{Z_u}{mu_0} & 0 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta\theta \end{pmatrix}$$

↓

Analytical solution for eigenvalues of 2<sup>nd</sup> order system is straightforward.

$$\lambda^2 - \lambda \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n} - \underbrace{\frac{Z_u g}{mu_0}}_{-\omega_n^2} = 0$$

# Pitch Hold Controller

# Break



Warning: Video shows Mild Injury

positive\_kp\_gain\_skiMeme.mp4

# Frew.1

Warning! Motion sickness possible.

<https://www.youtube.com/watch?v=pGFnz9ffjOI>

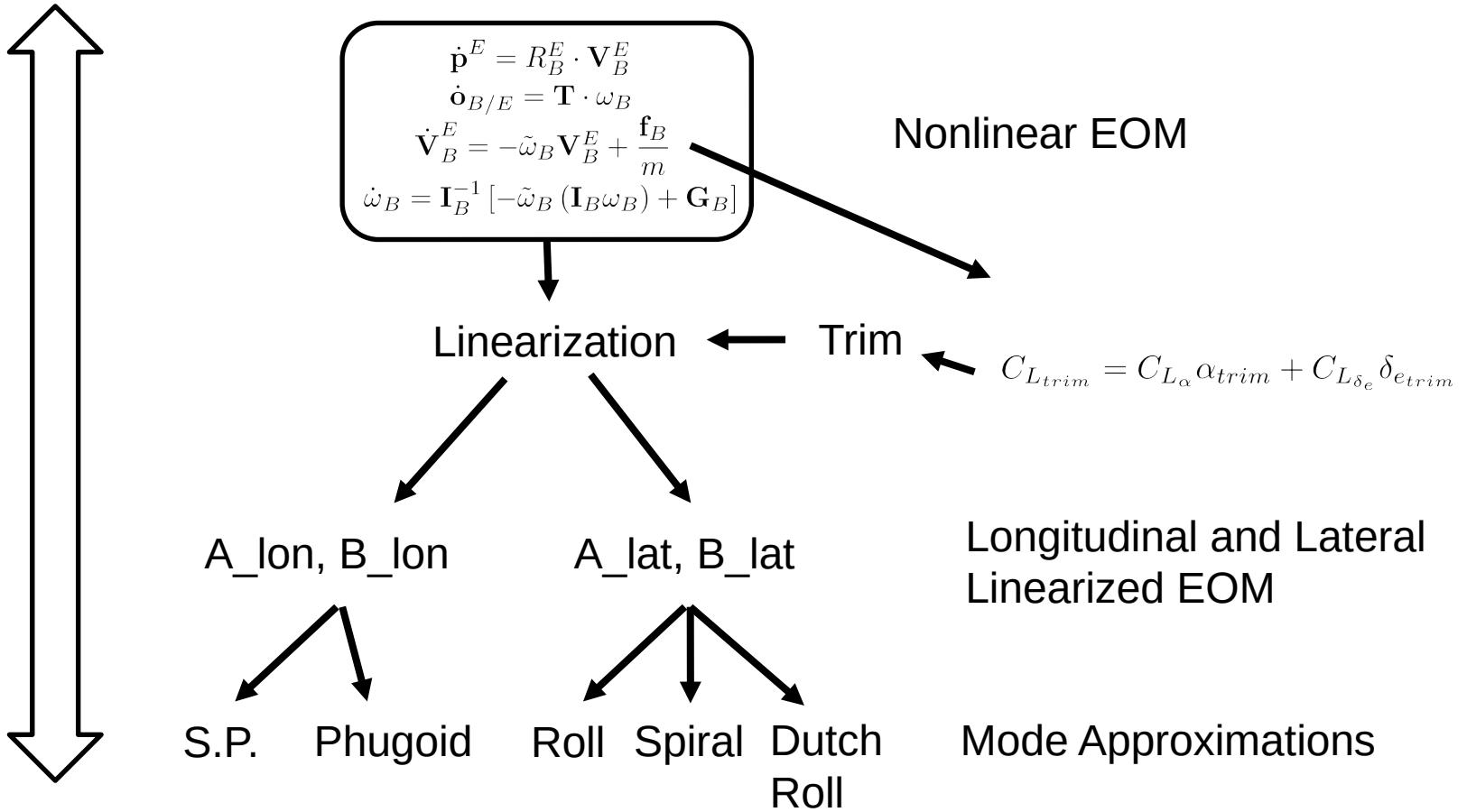
# Break



# LATERAL DYNAMICS

# Big Picture

More Accurate,  
Fewer Assumptions



More Assumptions  
Easier to Analyze

# Lateral Equations

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

# Lateral Matrix

Define matrix elements in order to simplify notation

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g \cos \theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix} \quad \mathbf{B}_{lat} = \begin{pmatrix} 0 & \mathcal{Y}_{\delta_r} \\ \mathcal{L}_{\delta_a} & \mathcal{L}_{\delta_r} \\ \mathcal{N}_{\delta_a} & \mathcal{N}_{\delta_r} \\ 0 & 0 \end{pmatrix}$$

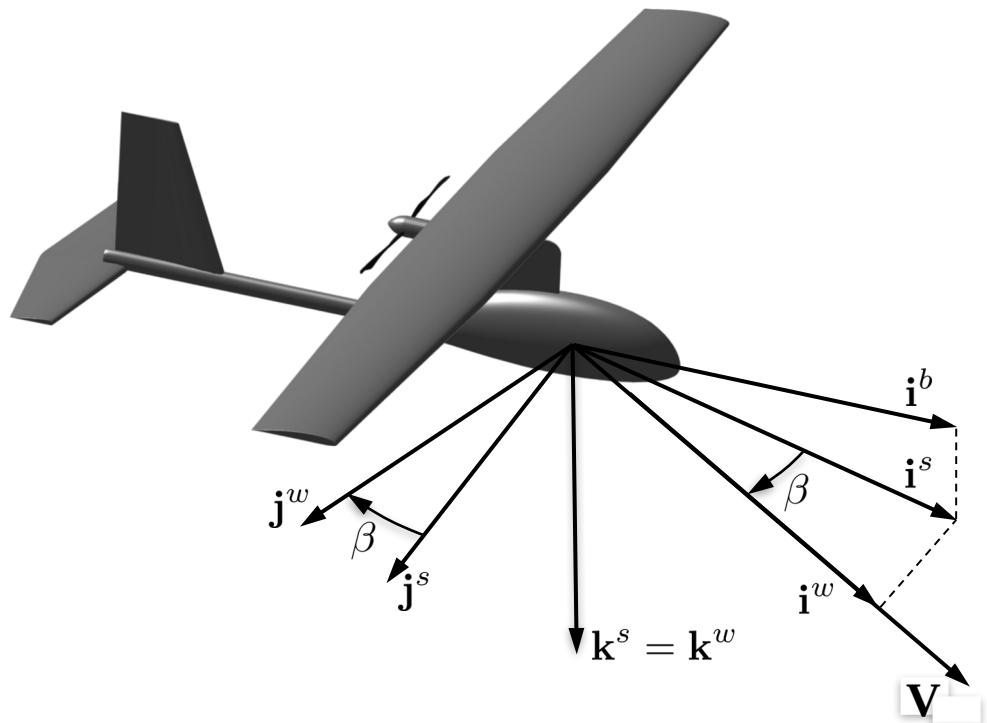
# 6x6 (a.k.a “flight path”) Dynamics Matrix

# Sideslip

Aerodynamics a function of the sideslip angle, not roll or yaw

$$\Delta\beta = \sin^{-1} \frac{\Delta v}{V}$$

$$\approx \frac{\Delta v}{u_0} = \hat{v}$$



# Dimensional Derivatives

$$Y_v = \frac{1}{2} \rho u_0 S C_{y_\beta}$$

$$L_v = \frac{1}{2} \rho u_0 b S C_{l_\beta}$$

$$N_v = \frac{1}{2} \rho u_0 b S C_{n_\beta}$$

$$Y_p = \frac{1}{4} \rho u_0 b S C_{y_p}$$

$$L_p = \frac{1}{4} \rho u_0 b^2 S C_{l_p}$$

$$N_p = \frac{1}{4} \rho u_0 b^2 S C_{n_p}$$

$$Y_r = \frac{1}{4} \rho u_0 b S C_{y_r}$$

$$L_r = \frac{1}{4} \rho u_0 b^2 S C_{l_r}$$

$$N_r = \frac{1}{4} \rho u_0 b^2 S C_{n_r}$$

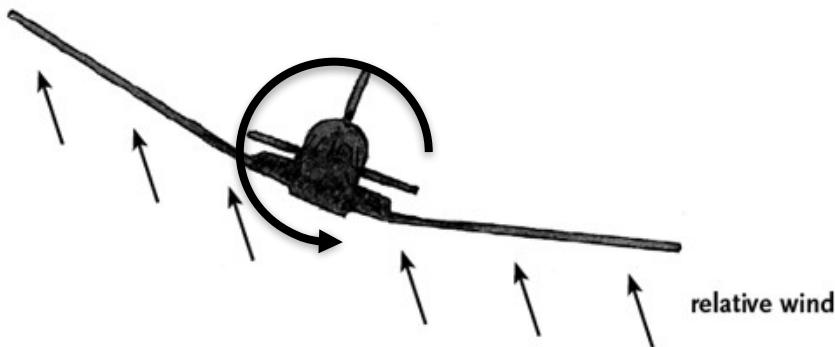
# Beta Derivatives

$$Y_v = \frac{1}{2} \rho u_0 S C_{y_\beta}$$

$$L_v = \frac{1}{2} \rho u_0 b S C_{l_\beta}$$

$$N_v = \frac{1}{2} \rho u_0 b S C_{n_\beta}$$

$C_{l_\beta}$  = dihedral effect



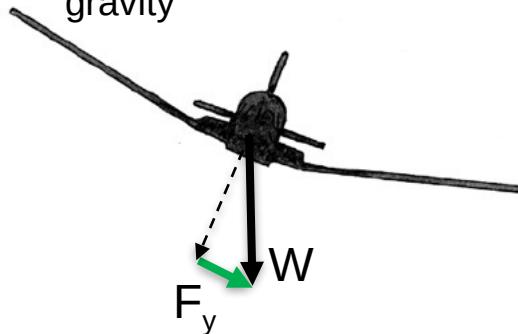
- Even though roll stiffness is small, aircraft are often still stable in roll
- A roll angle induces a side force due to gravity, which induces a sideslip
- Based on  $C_{l_\beta}$  a negative (restoring) moment is created
- The derivative  $C_{l_\beta}$  is called “dihedral effect” because it tends to depend heavily on wing dihedral

# Dihedral effect

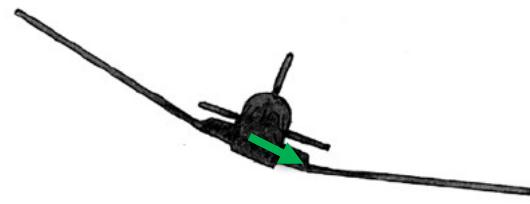
Aircraft flying into the page, straight and level



Positive roll induces positive side-force due to gravity



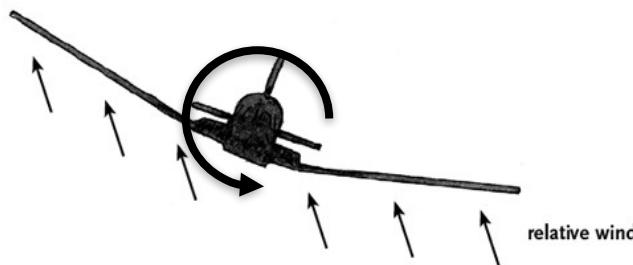
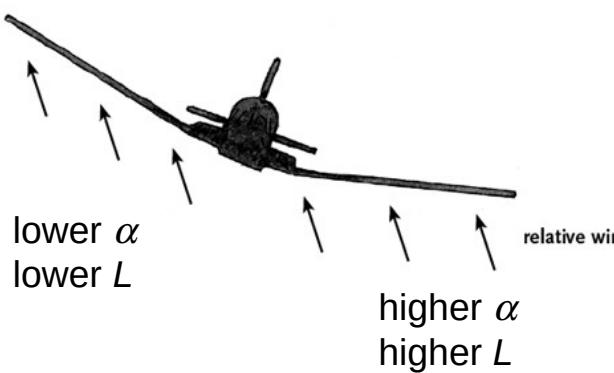
Side-force induces positive sideslip



Sideslip creates imbalance of angle of attack, lift

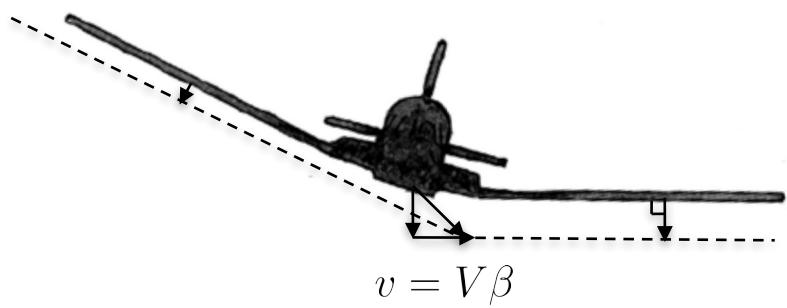
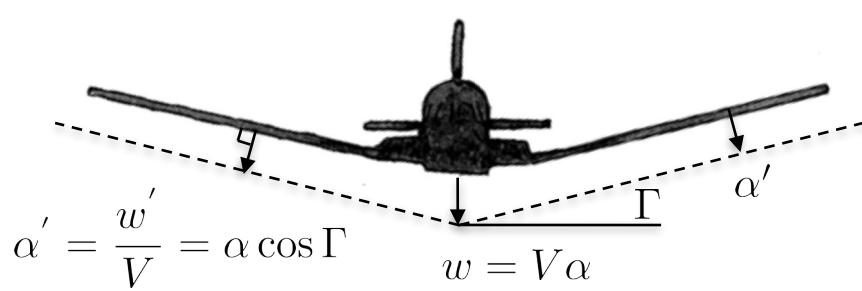
Imbalance in lift creates negative roll moment

Aircraft returns to straight and level flight



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# Dihedral Effect – Dihedral Angle



$$\Delta C_l^\Gamma \propto \Delta \alpha C_L$$

Asymmetric lift cause a negative (restoring) roll moment

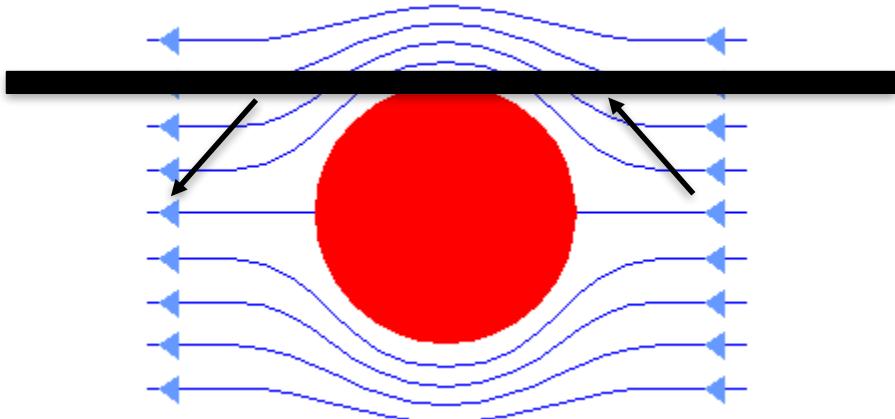
$$C_{l_\beta}^\Gamma \propto \Gamma$$

$$C_{l_\beta}^\Gamma \propto C_L$$

# Dihedral Effect – Wing Location

Down-wash on  
left wing  
decreases  $\alpha$ ,  
decreases  $L$

Up-wash on  
right wing  
increases  $\alpha$ ,  
increases  $L$



high wing

Asymmetric lift cause a negative  
(restoring) roll moment

Aircraft flying into the page

# Dihedral Effect – Wing Sweep

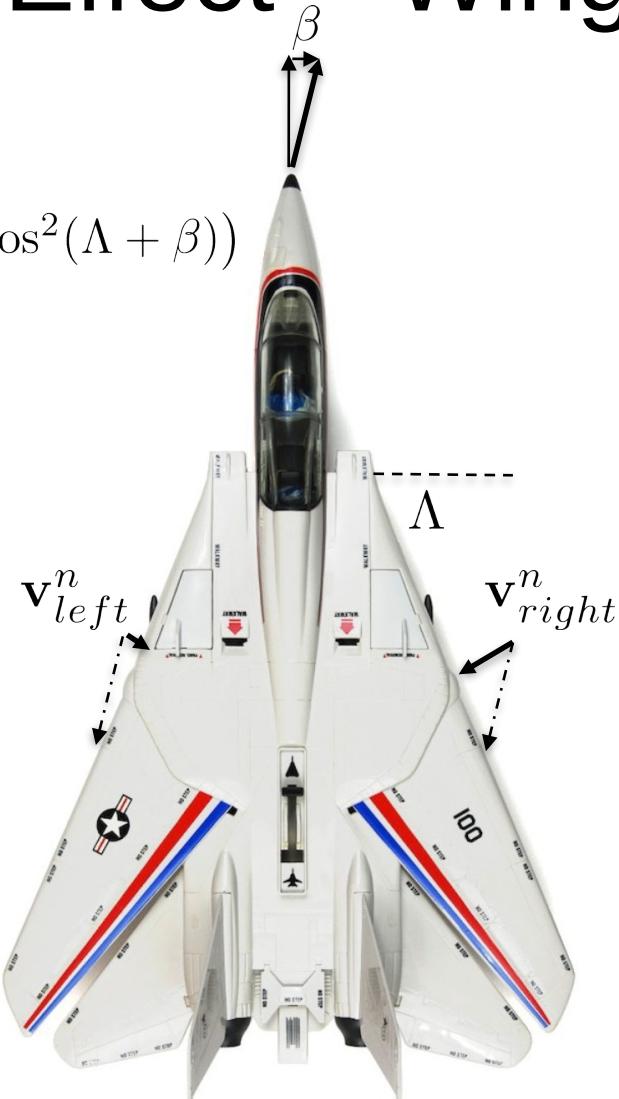
$$\Delta C_l^\Lambda \propto C_L \left( V_{left}^2 - V_{right}^2 \right)$$

$$\Delta C_l^\Lambda \propto C_L V^2 \left( \cos^2(\Lambda - \beta) - \cos^2(\Lambda + \beta) \right)$$

$$\Delta C_l^\Lambda \propto 2\beta C_L V^2 \sin 2\Lambda$$

$$C_{l_\beta}^\Lambda \propto 2C_L V^2 \sin 2\Lambda$$

Increased lift on the right, decreased lift on the left leads to negative (restoring) roll moment

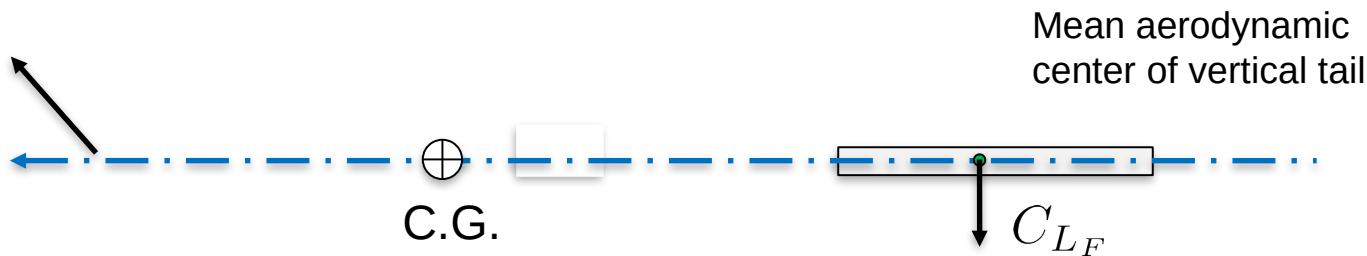


Lift generated on a swept wing is a function of the normal component of wind velocity

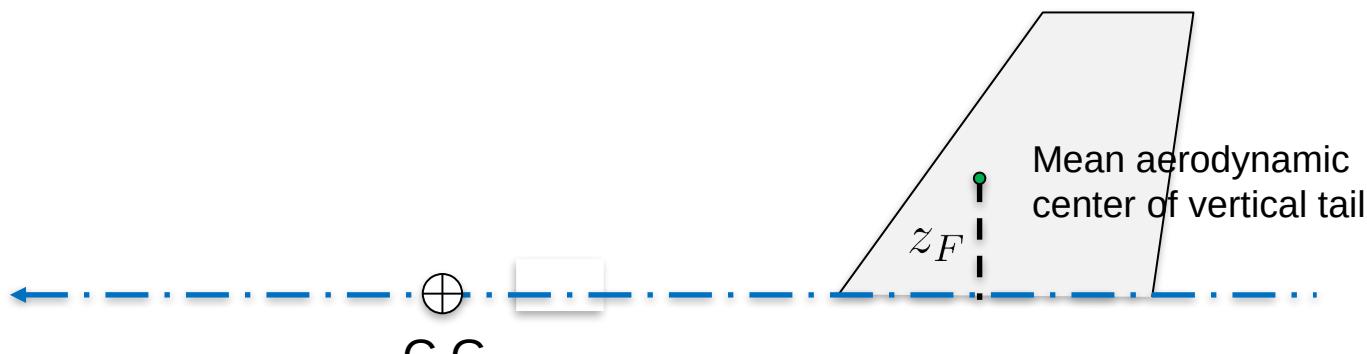
[http://www.yojoe.com/images/resize/w/MAX/vehicles/83/skystriker/skystriker\\_top\\_swept.jpg](http://www.yojoe.com/images/resize/w/MAX/vehicles/83/skystriker/skystriker_top_swept.jpg)

# Dihedral Effect – Vertical Tail

Top view



Side view



$$\Delta C_l^F = C_{L_F} \frac{S_F z_F}{Sb} = a_F (-\beta + \sigma) \frac{S_F z_F}{Sb}$$

$$C_{l_\beta}^F = -a_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \frac{S_F z_F}{Sb}$$

# Beta Derivatives

$$Y_v = \frac{1}{2} \rho u_0 S C_{y_\beta}$$

$$L_v = \frac{1}{2} \rho u_0 b S C_{l_\beta}$$

$$N_v = \frac{1}{2} \rho u_0 b S C_{n_\beta}$$

$$C_{Y_\beta} = C_{Y_\beta}$$

$$C_{l_\beta} = C_{l_\beta}$$

$$C_{n_\beta} = C_{n_\beta}$$

$$= -a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \frac{S_F}{S}$$

Function of i.) dihedral; ii.) sweep;  
iii.) high/low wing; iv.) high/low tail

$$= V_V a_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$$

# Lateral Modes

Boeing 747 Jet Transport (Etkin 6.2)



$$\begin{aligned} h_0 &= 40000 \text{ ft} \\ V_0 &= 774 \text{ ft/s} \\ \gamma_0 = \theta_0 = \alpha_0 &= 0^\circ \end{aligned}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

# Lateral Modes

$\lambda_i$	$\zeta$	$\omega_n$
$-7.30e - 03$	$1.00e + 00$	$7.30e - 03$
$-5.62e - 01$	$1.00e + 00$	$5.62e - 01$
$-3.30e - 02 + 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$
$-3.30e - 02 - 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_1 \\ 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_2 \\ -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_{3/4} \\ -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

**Spiral**

**Roll**

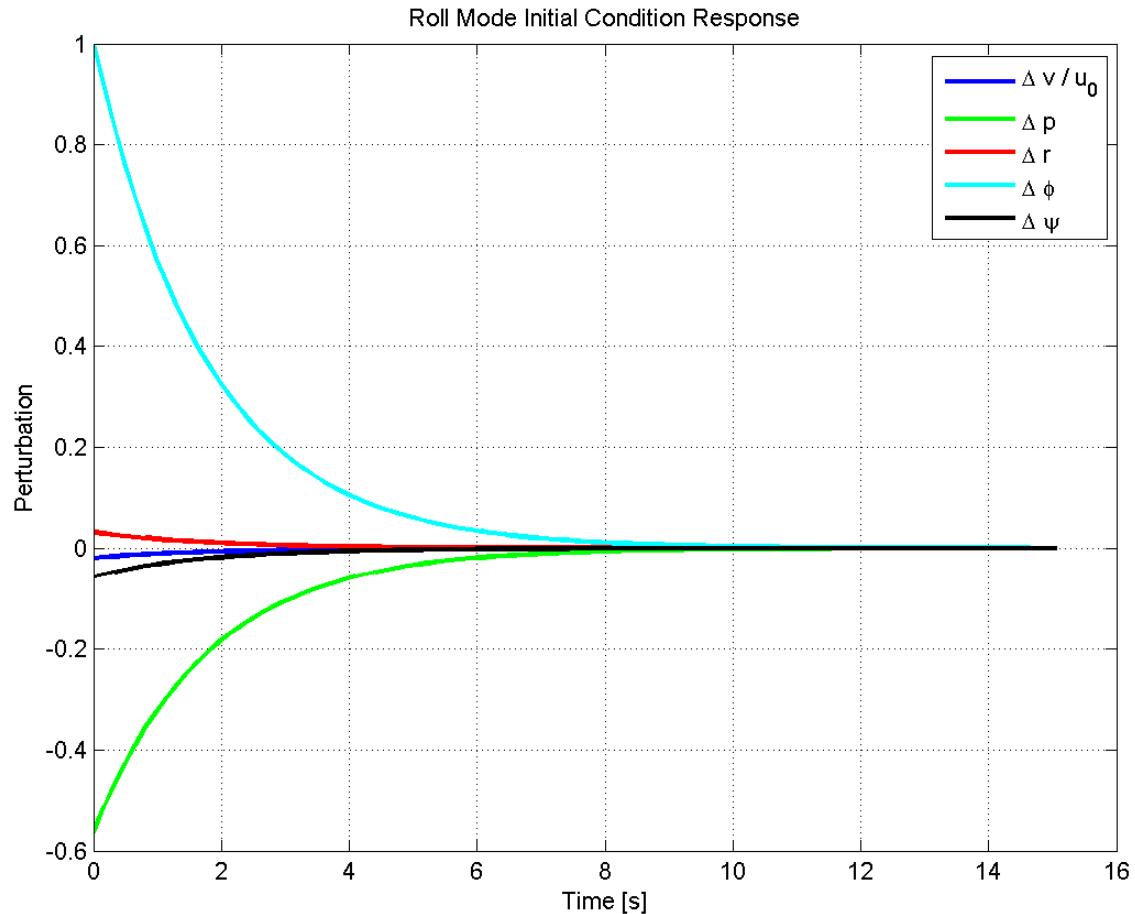
**Dutch Roll**

# Roll Mode

$$\lambda_2 = -0.56248$$
$$t_{half} = 1.23 \text{ s}$$

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0000 \\ -0.0562 \end{pmatrix} \quad \begin{matrix} \hat{v} = \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{matrix}$$

Non-dimensionalize and scale so  $\Delta\phi$  term is 1



The roll mode is characterized by a **fast roll rate** with very small sideslip, yaw rate, and roll angle.

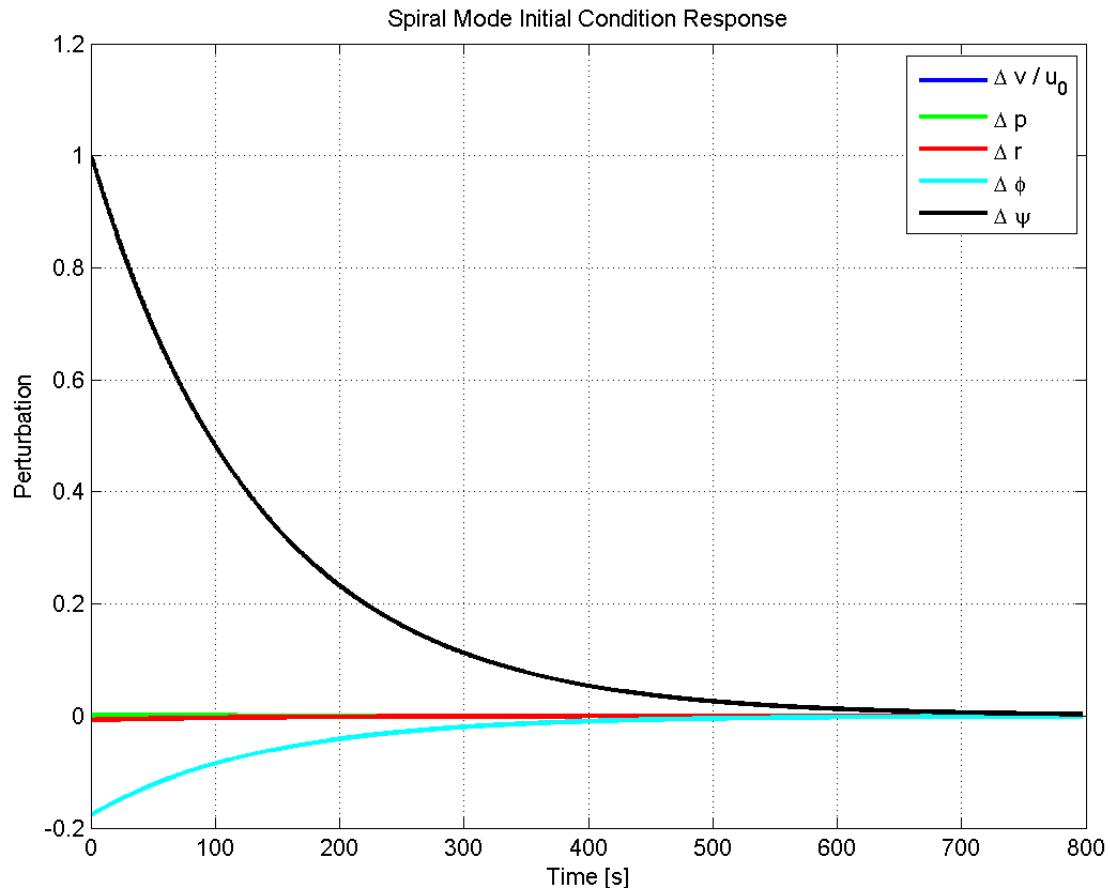
# Spiral Mode

$$\lambda_1 = -0.0073$$

$$t_{half} = 95 \text{ s}$$

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0000 \end{pmatrix} \quad \begin{matrix} \hat{v} = \Delta\beta \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \end{matrix}$$

Non-dimensionalize and scale so  $\Delta\psi$  term is 1



The spiral mode is characterized by strong coupling in roll and yaw, with sideslip remaining relatively fixed. The spiral mode is **often unstable**.

# Dutch Roll Mode

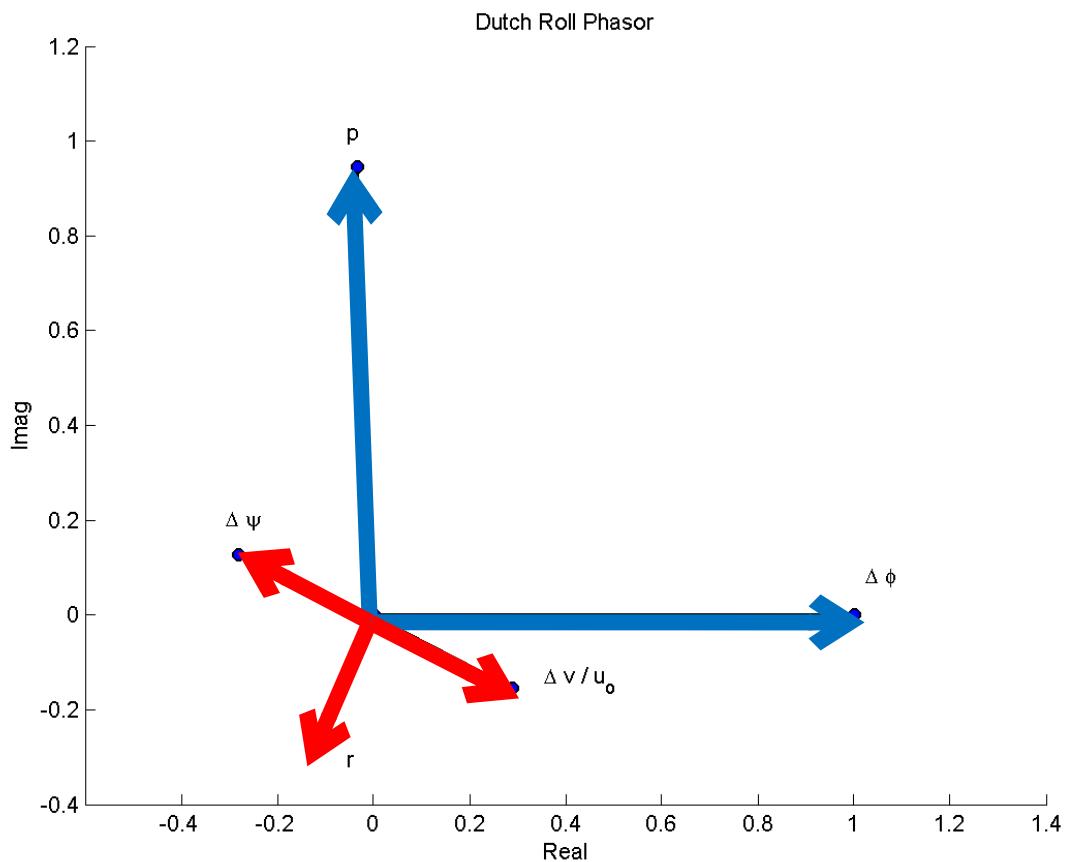
$$\lambda_{3/4} = -0.033 + 0.947i$$

$$\zeta = 0.0349 \leftarrow \text{Not well damped}$$

$$\omega_n = 0.947 \leftarrow \text{fast response}$$

Phugoid (for comparison)

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.3271\angle -28.0^\circ \\ 0.9471\angle 92.0^\circ \\ 0.2915\angle -112.3^\circ \\ 1.0000\angle 0.0^\circ \\ 0.3078\angle 155.7^\circ \end{pmatrix} \begin{matrix} \hat{\beta} \\ p \\ r \\ \Delta\phi \\ \Delta\psi \end{matrix}$$



The Dutch roll mode is characterized by a “tail-wagging” motion whereby the horizontal flight path stays relatively straight (sideslip and yaw tend oscillate with opposite sign and balance each other out).

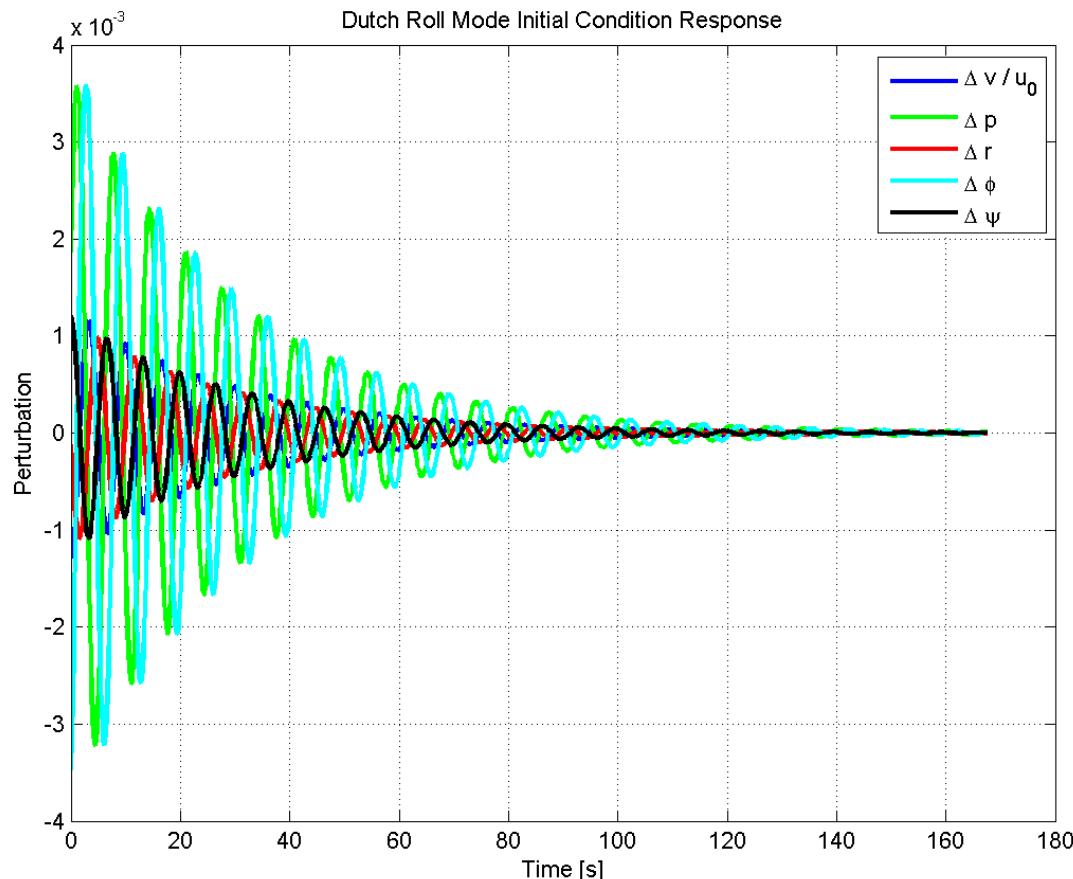
# Dutch Roll Mode

$$\lambda_{3/4} = -0.033 + 0.947i$$

$$\zeta = 0.0349$$

$$\omega_n = 0.947$$

$$\mathbf{x}(0) = Re(\mathbf{v}_{3/4}) = \begin{pmatrix} -0.9958 \\ 0.0019 \\ -0.0001 \\ -0.0035 \\ 0.0012 \end{pmatrix}$$



The Dutch roll mode is characterized by a “tail-wagging” motion whereby the horizontal flight path stays relatively straight (sideslip and yaw tend oscillate with opposite sign and balance each other out).

# Review: All the modes

Name	<i>Primary variables</i>	Fast/Slow	Damping
Short Period	Angle of attack	Fast	Well-damped
Phugoid	Speed, Altitude	Slow	Poorly-Damped
Roll	Roll rate	Fast	Over-Damped
Spiral	Roll, Yaw	Slow	Over-Damped OR Unstable
Dutch Roll	All lateral Degrees of Freedom	Fast	Poorly-Damped

Trust the mathematical properties of the matrix over your intuition!

# Spiral Approximation

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0000 \end{pmatrix} \longrightarrow \begin{array}{l} \Delta\dot{p} = \Delta p = 0 \\ \text{Assume no side-force} \\ (\text{ignore Y-force equation}) \end{array}$$

$$\begin{pmatrix} 0 \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta r \end{pmatrix}$$

$$\Delta\dot{r} = \left( \frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right) \Delta r \quad \lambda_{spi} = \frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v}$$

**Full lateral matrix**

$$\begin{aligned} \lambda_1 &= -.0073 \\ t_{half} &= 95 \text{ s} \end{aligned}$$

**Spiral mode approximation**

$$\begin{aligned} \lambda_{spi} &= -0.0296 \\ t_{half} &= 23.4 \text{ s} \end{aligned}$$

# Spiral Approximation

$$\begin{aligned}\lambda_1 &= -.0073 \\ t_{half} &= 95 \text{ s}\end{aligned}$$

Since spiral eigenvalue is so small, characteristic equation can be approximated by last two terms

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$



$$D\lambda + E = 0$$

$$\lambda = -\frac{E}{D}$$

$$E = g [(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r) \cos \theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p) \sin \theta_0]$$

$$D = -g (\mathcal{L}_v \cos \theta_0 + \mathcal{N}_v \sin \theta_0) + u_0 (\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

# Spiral Approximation

$$\begin{aligned}\lambda_1 &= -.0073 \\ t_{half} &= 95 \text{ s}\end{aligned}$$

Since spiral eigenvalue is so small, characteristic equation can be approximated by last two terms

$$D\lambda + E = 0$$

$$\lambda = -\frac{E}{D}$$

$$E = g [(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r) \cos \theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p) \sin \theta_0]$$

$$D = -g (\mathcal{L}_v \cos \theta_0 + \mathcal{N}_v \sin \theta_0) + u_0 (\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

## Full lateral matrix

$$\begin{aligned}\lambda_1 &= -.0073 \\ t_{half} &= 95 \text{ s}\end{aligned}$$

## Spiral mode approximation

$$\begin{aligned}\lambda_{spi} &= -0.00725 \\ t_{half} &= 96 \text{ s}\end{aligned}$$

# Roll Approximation

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0000 \\ -0.0562 \end{pmatrix}$$

Ignore everything except roll rate

$$\Delta \dot{p} = \mathcal{L}_p \Delta p$$

$$(\Delta \dot{\phi} = \Delta p)$$

$$\lambda_r = \mathcal{L}_p$$

**Full lateral matrix**

$$\lambda_2 = -.56248$$

$$t_{half} = 1.23 \text{ s}$$

**Roll mode approximation**

$$\lambda_r = -.4342$$

$$t_{half} = 1.60 \text{ s}$$

# Roll and Spiral Approximation

Assume side force produces same yaw rate as  $\beta = 0$ .  
Also assume  $Y_p = Y_r = 0$ .

$$\begin{pmatrix} 0 \\ \Delta\dot{p} \\ \Delta\dot{r} \\ \Delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -u_0 & g \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix}$$



$$C\lambda^2 + D\lambda + E = 0$$

$$E = g(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r)$$

**Full lateral matrix**

**Approximation**

$$D = u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v) - g \mathcal{L}_v$$

$$\lambda_1 = -.0073$$

$$\lambda_{spi} = -0.00734$$

$$C = u_o \mathcal{N}_v$$

$$\lambda_2 = -.56248$$

$$\lambda_r = -0.597$$

# Dutch Roll Approximation

$$\mathbf{v}_{3/4} = \begin{pmatrix} 0.3271\angle -28.0^\circ \\ 0.9471\angle 92.0^\circ \\ 0.2915\angle -112.3^\circ \\ 1.0000\angle 0.0^\circ \\ 0.3078\angle 155.7^\circ \end{pmatrix} \quad \xrightarrow{\text{Assume a flat, snaking motion}} \quad \Delta p = \Delta \phi = 0 \\ Y_r = 0$$

$$\begin{pmatrix} \Delta \dot{v} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta r \end{pmatrix}$$

$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r) \lambda + (\mathcal{Y}_v \mathcal{N}_r + u_0 \mathcal{N}_v) = 0$$

**Full lateral matrix**

$$\lambda_{3/4} = -0.033 + 0.947i$$

**Dutch roll mode approximation**

$$\lambda_{dr} = -0.1008 \pm 0.9157i$$



# Eigenvalues of Mode Approximations

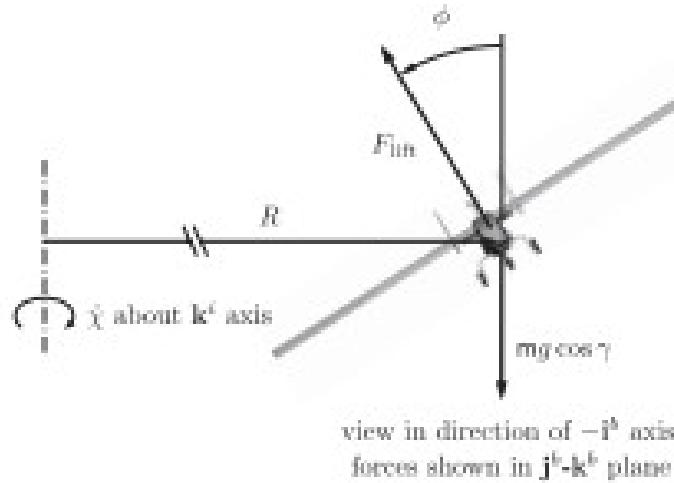
$$\dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta_a} \delta_a$$
$$\dot{x} = A \dot{x} + B u$$

$$|A - \lambda I| = 0$$

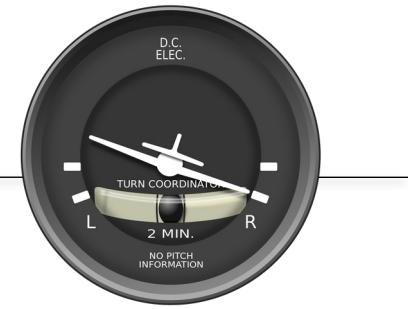
$\lambda = \text{A}$  if A is scalar

$$\dot{p} = \underbrace{\left( \mathcal{L}_p - \mathcal{L}_{\delta_a} k \right)}_{\delta_a = -k p} p$$

# Coordinated Turn



- A “banked” or “coordinated” turn is one in which
  - Angular velocity vector is a constant and aligned with inertial z direction
  - No Aerodynamic forces in the y direction ()
    - Pilot “flying the turn on the turn-and-bank indicator”



# Coordinated Turn

$$\begin{pmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_\beta} & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{pmatrix} \begin{pmatrix} \beta \\ \delta_r \\ \delta_a \end{pmatrix} = \begin{pmatrix} C_{Y_p} & C_{Y_r} \\ C_{l_p} & C_{l_r} \\ C_{n_p} & C_{n_r} \end{pmatrix} \begin{pmatrix} \theta \\ -\cos \phi \end{pmatrix} \frac{\omega b}{2u_0}$$

$$\begin{pmatrix} C_{m_\alpha} & C_{m_{\delta_e}} \\ C_{L_\alpha} & C_{L_{\delta_e}} \end{pmatrix} \begin{pmatrix} \Delta \alpha \\ \Delta \delta_e \end{pmatrix} = - \begin{pmatrix} C_{m_q} \\ C_{L_q} \end{pmatrix} \frac{\omega \bar{c} \sin \phi}{2u_0} + \begin{pmatrix} 0 \\ (n-1)C_W \end{pmatrix}$$



All these variables are perturbations, but only these two are added to non-zero trim values

These are based on small climb rate

May not work perfectly in nonlinear simulation – why?



# Roll Controller



# Outline

- Example Problems
- Questions

# From 2019 Exam

TRUE or FALSE?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

# From 2019 Exam

TRUE or **FALSE?**

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

The spiral mode has a real eigenvalue, so it does not have a damping ratio.

# From 2019 Exam

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For the augmented lateral dynamical system with  $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$ , the control law  $\mathbf{u} = -\mathbf{K}^* \mathbf{x}_{lat}$  may have no effect on certain eigenvalues of the system.

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Feedback of the first four terms does not change the eigenvalues associated with the last two terms, ie the zero eigenvalues.

# From 2019 Exam

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Bank angle not in the linear equation for time rate of change of bank angle

$$\dot{\Delta\phi} = \Delta p + \Delta r \tan \theta_0$$

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$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \boxed{\Gamma_3 L_v + \Gamma_4 N_v} & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

# From 2019 Exam

TRUE or FALSE?

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# Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ( $\lambda_5 = 0$ ,  $\lambda_6 = 0$ ). Further, they both have the same eigenvector  $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^T$  associated with one of these eigenvalues. Why are the other eigenvectors associated with  $\lambda_5 = 0$  different?

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$$\Delta \dot{x}^E = \Delta u$$

$$\Delta \psi = \Delta r \sec \theta_0$$

$$\Delta \dot{z}^E = -u_0 \Delta \theta + \Delta w$$

$$\Delta \dot{y}^E = u_0 \cos \theta_0 \Delta \psi + \Delta v$$

# Questions?

