

Welcome to ASEN 3728

Aircraft Dynamics!



## Aeronautical Engineering

1. Aerodynamics
2. Structures / Materials
3. Propulsion
4. Dynamics + Control

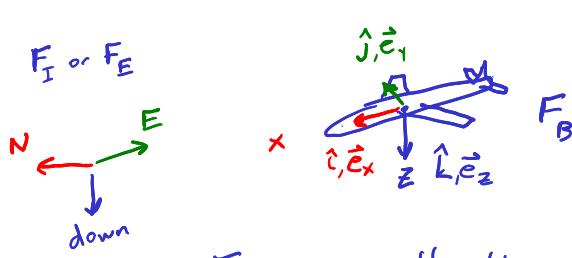
- Keep the pointy end forward
- Get the aircraft to where we want to go

+ Mathematical description of A/C behavior

+ computer simulation model

+ Design of A/C and Control Systems  
to effect desirable dynamics

## Notation + Conventions



Body Coordinate System

Vectors:  $\vec{\cdot}$ ,  $\hat{\cdot}$  <sup>unit</sup> or bold

Frame: collection of  $\geq 3$  points (distance between points is constant)

Inertial: translates with a constant velocity  
↳ Newton's laws valid

Coordinate System: 3 unit vectors that allow measurement

$\vec{v}_B^E$  if present, frame  
"inertial velocity written in body coordinates"  
Coordinate system

Forces and Moments

$$\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{F}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

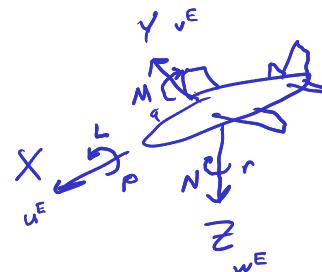
$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

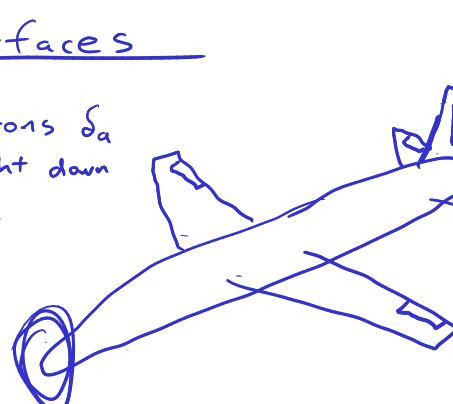
$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



$$V_g = |V_B^E| = \sqrt{u^E_{}^2 + v^E_{}^2 + w^E_{}^2}$$

## Control Surfaces

Ailerons  $\delta_a$   
 $+\delta_a$  = right down  
 $+\delta_a \approx -L$



Rudder  
 $+\delta_r$  = toward  $-Y$  direction  
 $+\delta_r \approx -N$ ,  $Y$

Elevator  
 $+\delta_e$  = down  
 $+\delta_e \approx -M$ ,  $-Z$

Throttle:  $\delta_t$

$+\delta_t \approx +X$  force  
no moment

Wind

(not gravity)

Aerodynamic Forces + Moments  
are functions of the A/C velocity wrt. the air

$$\vec{V}^E = \vec{V} + \vec{W}_{\text{wind}}$$

↑  
air-relative

When no wind  $\vec{V}^E = \vec{V}$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Wind Angles

angle of attack  $\alpha$

$$\alpha = \tan^{-1} \frac{w}{u}$$

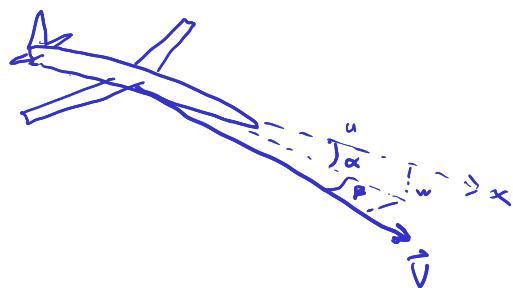
sideslip angle  $\beta$

$$\beta = \sin^{-1} \frac{v}{V}$$

$$u = V \cos \beta \cos \alpha$$

$$v = V \sin \beta$$

$$w = V \cos \beta \sin \alpha$$

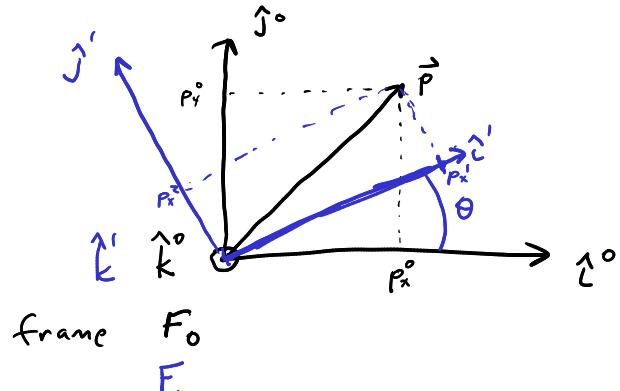


# Orientation

"1"  $\phi$  roll  
 "2"  $\theta$  pitch  
 "3"  $\psi$  yaw

3-2-1

know  $\vec{v}_B^E$   
 want  $\vec{v}_E^E$



$$\vec{p}_0 = \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix} \quad \vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix}$$

$$\vec{p} = p_x^0 \hat{i} + p_y^0 \hat{j} + p_z^0 \hat{k} \leftarrow$$

$$\vec{p} = p_x^1 \hat{i}' + p_y^1 \hat{j}' + p_z^1 \hat{k}'$$

→ want  $p_x^1$  in terms of  $\vec{p}_0$

$$p_x^1 = \vec{p} \cdot \hat{i}'$$

$$= p_x^0 \hat{i} \cdot \hat{i}' + p_y^0 \hat{j} \cdot \hat{i}' + p_z^0 \hat{k} \cdot \hat{i}'$$

$$= [\hat{i} \cdot \hat{i}' \quad \hat{j} \cdot \hat{i}' \quad \hat{k} \cdot \hat{i}'] \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$$p_x^1 = [\cos \theta \quad \sin \theta \quad 0] \vec{p}_0$$

$$\vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix} = \begin{bmatrix} \hat{i} \cdot \hat{i}' & \hat{j} \cdot \hat{i}' & \hat{k} \cdot \hat{i}' \\ \hat{i} \cdot \hat{j}' & \hat{j} \cdot \hat{j}' & \hat{k} \cdot \hat{j}' \\ \hat{i} \cdot \hat{k}' & \hat{j} \cdot \hat{k}' & \hat{k} \cdot \hat{k}' \end{bmatrix} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$$\vec{p}_1 = R_0^T \vec{p}_0$$

Direction Cosine Matrix

Rotation Matrix

$$R_3(\theta)_{\substack{\text{t axis} \\ \text{t axis}}} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

## Properties of DCMs

$$\vec{p}_2 = R_1^2 \vec{p}_1 \quad \vec{p}_1 = R_0^1 \vec{p}_0 \quad \vec{p}_2 = R_1^2 \vec{p}_1 = \vec{p}_2 = \underbrace{R_1^2 R_0^1}_{R_A^C} \vec{p}_0$$

Chaining

$$R_0^2 = R_1^2 [R_0^1]$$

$$R_A^C = R_B^C R_A^B$$

Inverse

$$\vec{p}_2 = R_1^2 \vec{p}_1$$

$$R_1^2 \vec{p}_2 = R_1^{2-1} R_1^2 \vec{p}_1 = R_2^1 \vec{p}_2$$

$$R_2^1 = (R_1^2)^{-1}$$

$$R_A^B = (R_B^A)^{-1}$$

since DCMs are orthonormal

$$(R_A^B)^{-1} = R_A^B{}^T$$

$$\begin{bmatrix} - \\ - \\ - \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_A^B = R_B^A{}^T$$

## Euler Angles

Any rotation can be described by 3 Euler angles about 3 n-repeated axes

E.g. 3-1-3 rotation through  $\alpha, \beta, \gamma$

3-2-1 rotation through  $\psi, \theta, \phi$

$$\vec{p}_B = R_E^B \vec{p}_E = \underbrace{R_1(\phi)}_{\uparrow \text{roll}} \underbrace{R_2(\theta)}_{\curvearrowright \text{pitch}} \underbrace{R_3(\psi)}_{\curvearrowleft \text{yaw}} \vec{p}_E$$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$\vec{p}_{\text{pilot}}^E = \vec{p}_{A/C}^E + \vec{p}_{\text{pilot}}^B$$

$$\vec{p}_{\text{pilot}}^E = \vec{p}_{A/C}^E + R_B^E \vec{p}_{\text{pilot}}^B$$

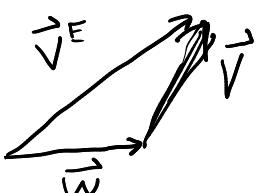
← abstract vectors

# Vector Clarity

$\vec{V}_B^E$  ← "frame of reference", "with respect to", "relative to"

← "coordinate frame", "expressed in", "written in"

→  $\vec{V}^E = \vec{V}^{(w)} + \vec{W}^{(E)}$  ← by convention



True in Any coordinate system  
as long as all in same coordinate system

$$\vec{V}_B^E = \vec{V}_B + \vec{W}_B$$

$\begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$      $\begin{bmatrix} u \\ v \\ w \end{bmatrix} +$

$$\vec{V}_E^E = \vec{V}_E + \vec{W}_E$$

# Kinematics

Kinematics: "Geometry of Motion" (no forces)

Translational  
Rotational

Dynamics: Effects of forces and moments on objects

## Vector Derivatives

$\frac{d}{dt} \vec{p}$  = time rate of change of  $\vec{p}$

$$\frac{d}{dt} \vec{p} = \vec{v}^E$$

$$\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$$

$$\dot{\vec{p}}_B \equiv \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Not Always  
True

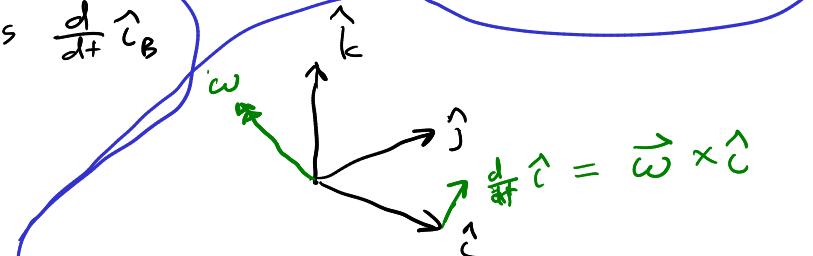
$$\begin{bmatrix} \hat{i}_B \\ \hat{j}_B \\ \hat{k}_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \vec{p}_B &= p_x \hat{i}_B + p_y \hat{j}_B + p_z \hat{k}_B \\ \dot{\vec{p}}_B &= \dot{p}_x \hat{i}_B + \dot{p}_y \hat{j}_B + \dot{p}_z \hat{k}_B \end{aligned}$$

$$\frac{d}{dt}(uv) = u \frac{d}{dt}v + v \frac{d}{dt}u$$

$$\left( \frac{d}{dt} \vec{p} \right)_B = \dot{p}_x \hat{i}_B + p_x \frac{d}{dt} \hat{i}_B + \dot{p}_y \hat{j}_B + p_y \frac{d}{dt} \hat{j}_B + \dot{p}_z \hat{k}_B + p_z \frac{d}{dt} \hat{k}_B$$

What is  $\frac{d}{dt} \hat{i}_B$



$$\tilde{\omega}_B = \tilde{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{aligned} &= p_x (\tilde{\omega}_B \times \hat{i}_B) + p_y (\tilde{\omega}_B \times \hat{j}_B) + p_z (\tilde{\omega}_B \times \hat{k}_B) \\ &= \tilde{\omega}_B \times \vec{p}_B \end{aligned}$$

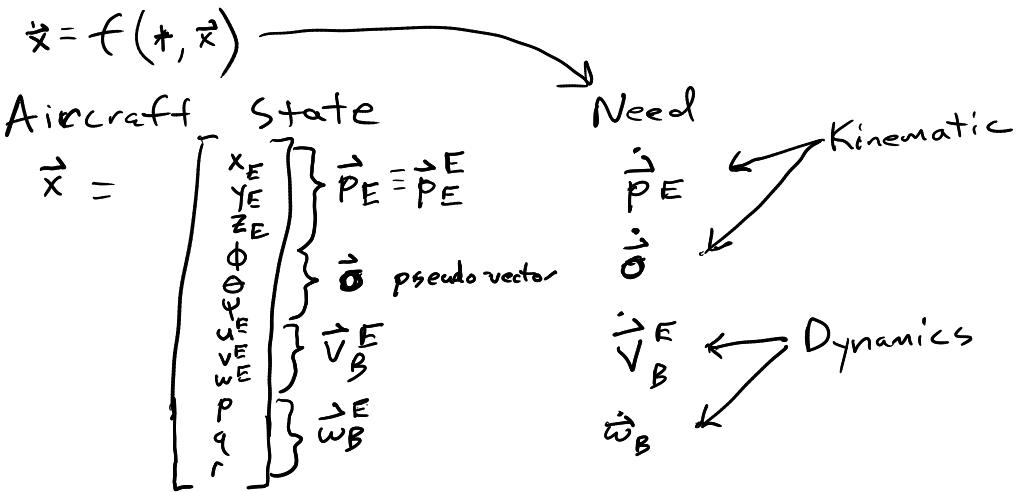
$$\frac{d}{dt} \vec{p}_B = \dot{\vec{p}}_B + \tilde{\omega}_B \times \vec{p}_B$$

$$= \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Kinematic Transport Theorem

# Equations of motion for a 3D body

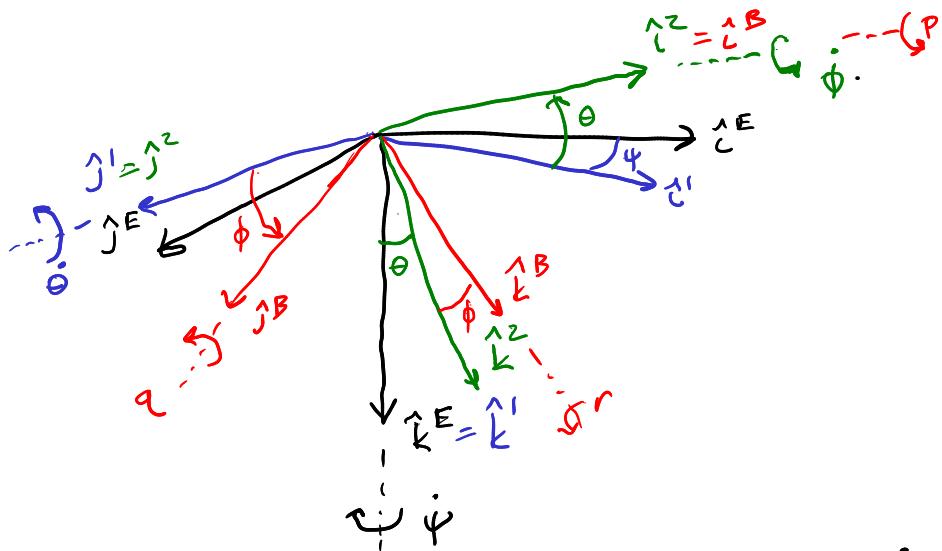


## Translational Kinematics

$$\dot{\vec{p}_E} = \frac{d}{dt} \vec{p}_E = \vec{v}_E^E \times \vec{p}_E^E = \vec{v}_E^E = R_B^E \vec{v}_B^E$$

$$\boxed{\dot{\vec{p}_E} = R_B^E \vec{v}_B^E}$$

## Rotational Kinematics



$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega} = \dot{\psi} \hat{i}^E + \dot{\theta} \hat{j}^I + \dot{\phi} \hat{k}^B$$

$$\vec{\omega}_B = \dot{\psi} \hat{i}_B^E + \dot{\theta} \hat{j}_B^I + \dot{\phi} \hat{k}_B^B$$

$$= \dot{\psi} R_E^B \hat{i}_E^E + \dot{\theta} R_E^B \hat{j}_I^I + \dot{\phi} \hat{k}_B^B$$

$$= R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix} + R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$R_E^B = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$R_1 = R_1(\phi) R_2(\theta)$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \text{invert} & & \\ 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\sec x = \frac{1}{\cos x}$$

$\dot{\vec{\alpha}} = T \vec{\omega}_B$

C attitude,  
influence matrix

## Dynamics

$$\vec{F} \quad \vec{G}$$

### Translational Dynamics

Newton's 2nd Law

$$\vec{F} = m \vec{a}$$

$$\vec{F} = m \frac{d}{dt} \vec{V}^E$$

$$\frac{d}{dt} \vec{V}_B^E = \vec{V}_B^E + \vec{\omega}_B \times \vec{V}_B^E$$

$$m(\vec{V}_B^E + \vec{\omega}_B \vec{V}_B^E) = \vec{F}_B$$

$$\vec{V}_B^E = \frac{\vec{F}_B}{m} - \vec{\omega}_B \vec{V}_B^E$$

\*

### Rotational Dynamics

Euler's 2nd Law

$$\frac{d}{dt} \vec{h} = \vec{G} \quad \begin{matrix} \text{angular momentum} \\ \downarrow \text{moments} \end{matrix}$$

$$\vec{h} = I \vec{\omega} \quad \begin{matrix} \downarrow \text{moment of inertia} \end{matrix}$$

$$I = \begin{pmatrix} \int(y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int(x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int(x^2 + y^2) dm \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

$\dot{\vec{\omega}}_B$

$$\frac{d}{dt} \vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \vec{h}_B = \vec{G}_B$$

$$I \ddot{\vec{\omega}}_B + \vec{\omega}_B I \vec{\omega}_B = \vec{G}_B$$

$$\dot{\vec{\omega}}_B = I^{-1} [\vec{G}_B - \vec{\omega}_B I \vec{\omega}_B]$$

$$\dot{x} = \begin{bmatrix} \dot{P}_E \\ \dot{\theta} \\ \dot{V}_B^E \\ \dot{\omega}_B^E \end{bmatrix}$$

$$\dot{\vec{P}}_E = R_B^E \dot{\vec{V}}_B$$

$$\dot{\theta} = T \vec{\omega}_B$$

\*

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ x_B \\ y_B \\ w_B \\ p \\ q \\ r \end{bmatrix}$$

$$\dot{\vec{p}}_E^E = \vec{p}_E^E$$

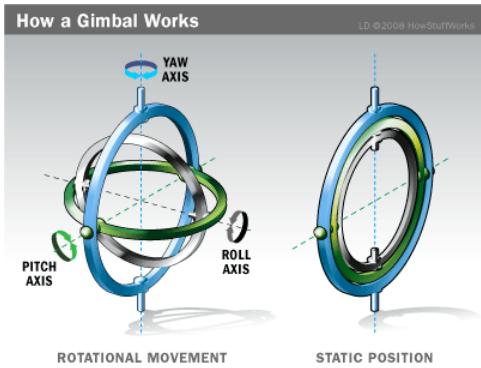
$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

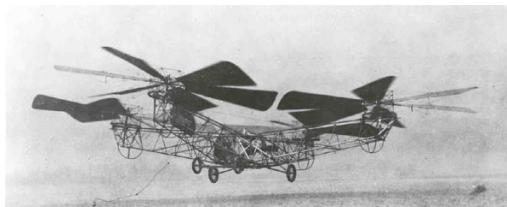
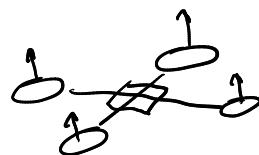
$$\dot{\vec{\omega}}_B = I_B^{-1} [\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B]$$

attitude influence

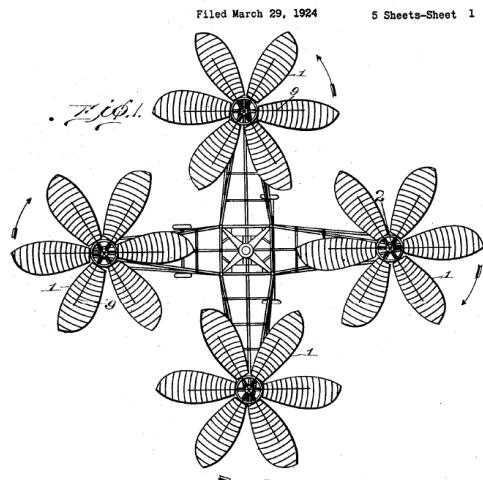
$$T = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix}$$

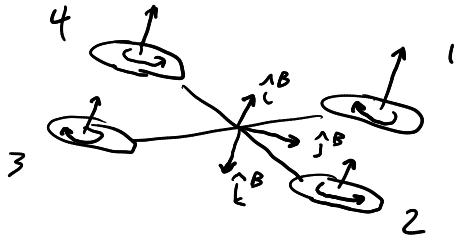


Multi-copter  
Quadrrotor



March 4, 1930. G. DE BOTHEZAT HELICOPTER Filed March 29, 1924 1,749,471 5 Sheets-Sheet 1





$$\vec{f} = {}^a\vec{f} + {}^c\vec{f} + {}^e\vec{f}$$

$$\vec{G} = {}^a\vec{G} + {}^c\vec{G}$$

$${}^e\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^e\vec{G} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since symmetric about  $i^B - k^B$  plane and  $i^B - j^B$  plane

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \underbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}_{\vec{\omega}_B \times \vec{v}_B^E} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$I_B^{-1} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad \vec{I}_B^{-1} = \begin{bmatrix} I_y - I_z & qr \\ I_z - I_x & pr \\ I_x - I_y & pq \end{bmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix}}_{\vec{I}_B^{-1} (\vec{\omega}_B \times \vec{I}_B)} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

### Aerodynamic Forces and Moments

$${}^a\vec{f} = -D \frac{\vec{v}}{|V|}$$



$$V_a = |\vec{V}| \quad \text{airspeed}$$

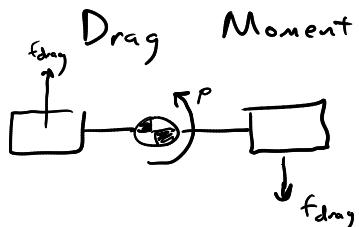
$$D = \frac{1}{2} \rho C_D A \quad \text{drag force}$$

$\vec{V}$   
 density  
 coefficient  
 of drag  
 (shape)

$$V_a^2 = \nu V_a^2 \quad \text{cross-sectional area}$$

$${}^a\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = -\nu V_a^2 \frac{\vec{v}_B}{V_a}$$

$$= -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$



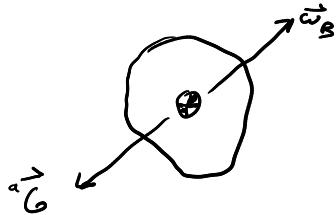
$$L_{drag} = -2l f_{drag}$$

$$= -2l \left( \frac{1}{2} \rho C_D A (\ell_p)^2 \right) \text{sign}(p)$$

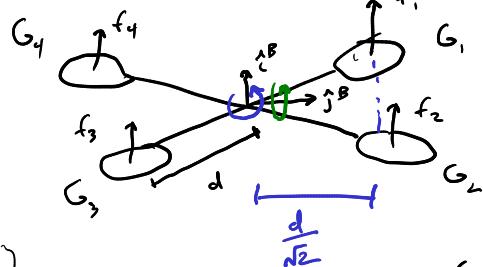
$\underline{-\rho^2 \text{sign}(p)} = -|p|p$

$$= -M |p| p$$

$$\vec{a}_{G_B} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = -M \sqrt{p^2 + q^2 + r^2} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



### Control Forces + Moments



$$G_i = k_g C_D \omega_r^2$$

$$f_i = k_f C_L \omega_r^2$$

$$k_m = \frac{k_g C_D}{k_f C_L}$$

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ k_m & -k_m & k_m & -k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \text{invert} \\ \text{ } \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

Multicopter

vs

Conv. Helicopter

Aerodynamic  
Efficient

Mechanical  
Control  
Complexity

Power System  
Complexity  
Turbine

Electric



- differential
- 1st order
- ordinary
- nonlinear
- coupled

general form

$$\begin{cases} \dot{\vec{x}}^E = R_B^E \vec{v}_B^E \\ \dot{\vec{o}} = T \vec{\omega}_B \\ \dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ \dot{\vec{\omega}}_B = I_B^{-1} (\vec{g}_B - \vec{\omega}_B \times I_B \vec{\omega}_B) \end{cases}$$

$$T = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix}$$

$$\begin{array}{l} \phi = 90^\circ \\ \theta = 0 \\ \psi = 10^\circ \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q = b \\ r \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pu^E - ru^E \\ qu^E - pv^E \end{pmatrix} + q \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_x - I_z}{I_y} pr \\ \frac{I_y - I_x}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

$$O = \frac{1}{m} X + \frac{Q}{m} \quad \therefore X = 0$$

$$O = g + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$Z_{c,0} = mg$$

Quadrupoles

"Never" trim condition

$$\begin{array}{l} \dot{x}_{E,0} = 0 \\ \dot{y}_{E,0} = 0 \\ \dot{z}_{E,0} = 0 \end{array} \quad \begin{array}{l} u_{o,0}^E = 0 \\ v_{o,0}^E = 0 \\ w_{o,0}^E = 0 \end{array} \quad \begin{array}{l} \dot{u}_{o,0}^E = 0 \\ \dot{v}_{o,0}^E = 0 \\ \dot{w}_{o,0}^E = 0 \end{array}$$

$$\begin{array}{l} p_o = 0 \\ q_o = 0 \\ r_o = 0 \end{array}$$

$$\begin{array}{l} \dot{p}_o = 0 \\ \dot{q}_o = 0 \\ \dot{r}_o = 0 \end{array}$$

$$\begin{array}{l} \phi_o = 0 \\ \theta_o = 0 \\ \psi_o = \text{any constant value} \end{array}$$

$$\begin{array}{l} f_{1,o} = mg \\ f_{2,o} = 0 \\ f_{3,o} = 0 \\ f_{4,o} = 0 \\ L_{c,o} = 0 \\ M_{c,o} = 0 \\ N_{c,o} = 0 \end{array} \boxed{Z_{c,0} = mg}$$

Linearization

$$\dot{\vec{x}} = \vec{x}_o + \Delta \vec{x}$$

$$\vec{u} = \vec{u}_o + \Delta \vec{u}$$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta \phi \\ \Delta \theta \\ \Delta \psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$$\Delta \dot{\vec{x}} = A \Delta \vec{x} + B \Delta \vec{u}$$

## Taylor Series

$$y = f(x, u)$$

$$x_0 + \Delta x$$

$$y_0 = f(x_0, u_0)$$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$= f(x_0, u_0) + \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}$$

$$\Delta y = \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}$$

$$x + y = x_0 + \Delta x + y_0 + \Delta y$$

$$\rightarrow x u \approx x_0 u_0 + \left. \frac{\partial x u}{\partial x} \right|_0 \Delta x + \left. \frac{\partial x u}{\partial u} \right|_0 \Delta u$$

$$= x_0 u_0 + u_0 \Delta x + x_0 \Delta u$$

alternative

$$x u = (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + u_0 \Delta x + x_0 \Delta u + \Delta x \Delta u^0$$

$$\sin(\theta) \approx \sin(\theta_0) + \left. \frac{\partial \sin \theta}{\partial \theta} \right|_0 \Delta \theta$$

$$= \sin(\theta_0) + \cos(\theta_0) \Delta \theta$$

alternatively

$$\sin(\theta) = \sin(\theta_0 + \Delta \theta) = \sin(\theta_0) \cos(\Delta \theta)^1 + \cos(\theta_0) \sin(\Delta \theta)^2 \approx \sin(\theta_0) + \cos(\theta_0) \Delta \theta$$

$$\cos(\theta) \approx \cos(\theta_0) - \sin(\theta_0) \Delta \theta$$

Linearize QR equations of motion

Examples

$$\dot{\theta} = \cos \phi q - \sin \phi r \leftarrow$$

$$\dot{\theta}_0 + \Delta \dot{\theta} = \cos(\theta_0 + \Delta \theta) (q_0 + \Delta q) - \sin(\theta_0 + \Delta \theta) (r_0 + \Delta r)$$

$$\Delta \dot{\theta} \approx \cancel{\cos(\Delta \theta)} \Delta q - \cancel{\sin(\Delta \theta)} \Delta r$$

$$\approx \Delta q - \Delta \theta \Delta r^0$$

$\Delta \dot{\theta} = \Delta q$

$$\dot{w}^E = q u^E - p v^E + g \cos \theta \cos \phi + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$\Delta \dot{w}^E = \cancel{\Delta q} \cancel{\Delta u^0} - \cancel{\Delta p} \cancel{\Delta v^0} + g + \frac{1}{m} \Delta Z + \frac{1}{m} (Z_{c,0} + \Delta Z_c)$$

cancel

$$\Delta \dot{w}^E = \frac{1}{m} \Delta Z + \frac{1}{m} \Delta Z_c$$

Linearized Aerodynamics

$$Z = -\gamma w \sqrt{u^2 + v^2 + w^2}$$

$$= Z_0 + \left. \frac{\partial Z}{\partial u} \right|_0 \Delta u + \left. \frac{\partial Z}{\partial v} \right|_0 \Delta v + \left. \frac{\partial Z}{\partial w} \right|_0 \Delta w$$

$$\Delta Z =$$

" " " "

$$\left. \frac{\partial Z}{\partial u} \right|_0 = -Z \gamma w u \left( u^2 + v^2 + w^2 \right)^{-1/2}$$

$\left. \frac{\partial Z}{\partial u} \right|_0 = 0$

$\left. \frac{\partial Z}{\partial v} \right|_0 = 0$

$$\frac{\partial Z}{\partial w} = -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$\left. \frac{\partial Z}{\partial w} \right|_0 = 0$

$\begin{cases} \Delta L = 0 \\ \Delta M = 0 \\ \Delta N = 0 \end{cases}$

~~In real life~~  
 ~~$Z(p, q, r)$~~

$\gamma$  constant

Looks like  $\frac{0}{0}$  take limit as  
 $u, v, w \rightarrow 0$

$\Delta Z = 0$

$\begin{cases} Z = -\gamma w^2 \\ \frac{\partial Z}{\partial w} = -\gamma 2w^0 \end{cases}$

Review : State space representation of a Linear Dynamical System

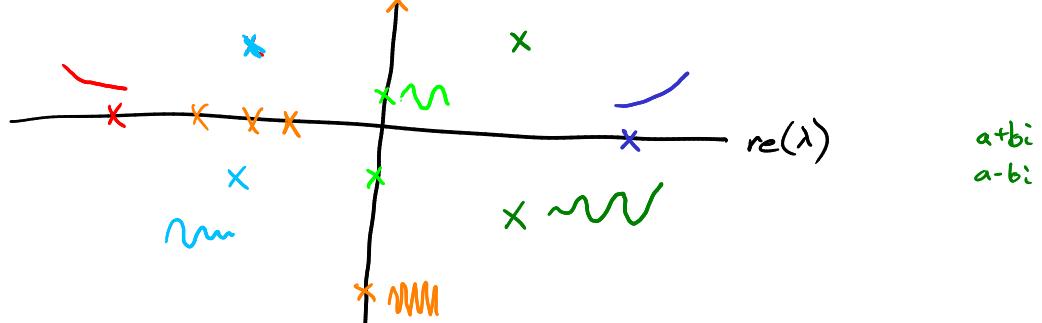
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\dot{\vec{y}} = C\vec{x} + D\vec{u}$$

Eigenvectors Eigenvectors

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

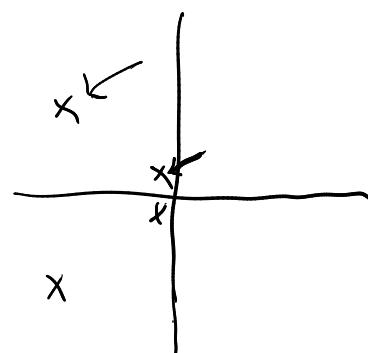
$$m(\lambda) \vec{x}(+) = e^{A+} \vec{x}(0) = \sum_{i=1}^n k_i e^{\lambda_i t} \vec{v}_i$$



$$\vec{u} = -K\vec{x}$$

$$\dot{\vec{x}} = (A - BK)\vec{x}$$

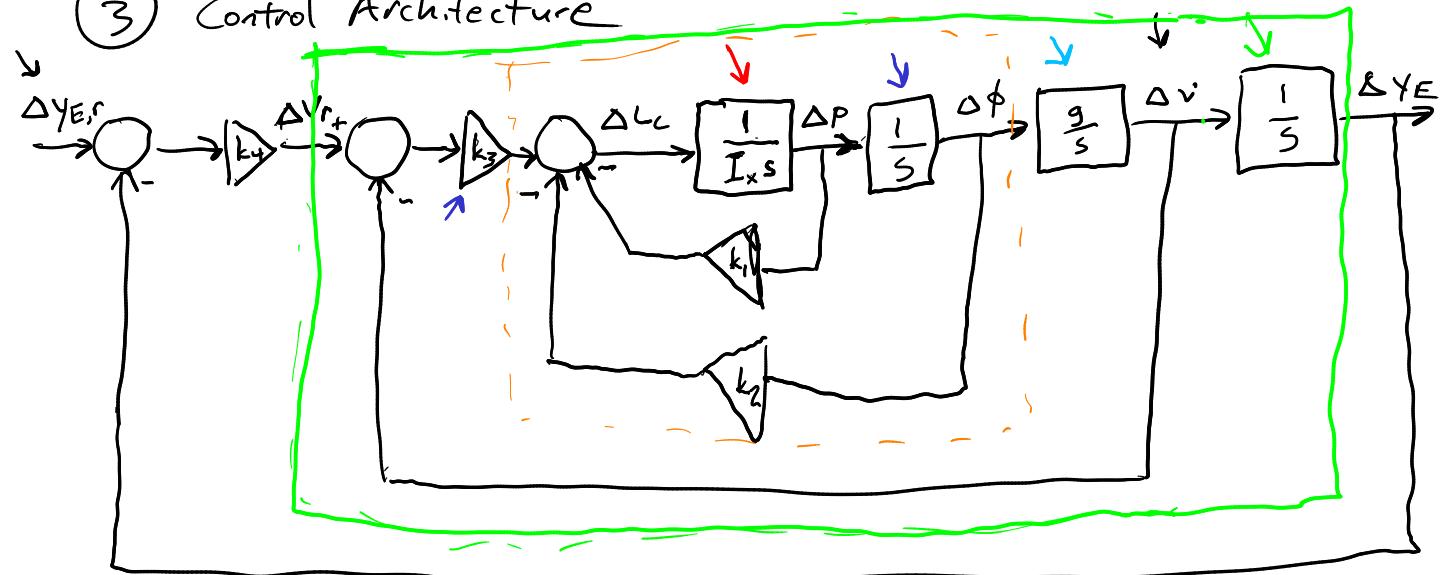
$$A^{cl}$$



(1)  
(2)

$$\begin{bmatrix} \Delta Y_E \\ \Delta Y \\ \Delta \phi \\ \Delta P \end{bmatrix} = g \begin{bmatrix} \Delta V \\ \Delta L_C \\ \frac{1}{I_x} \Delta L_C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta Y_E \\ \Delta V \\ \Delta \phi \\ \Delta P \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_x} \end{bmatrix} \Delta L_C$$

③ Control Architecture



(4) Choosing Gains

1)  $E_1 + k_2$

2)  $k_4$

3)  $k_3$  Root Locus

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi - k_3 \Delta v - k_3 k_4 \Delta y_E + k_3 k_4 \Delta y_{E,r}$$

$$\begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3 k_4}{I_x} \end{bmatrix} \Delta y_{E,r}$$

$A^{cl}$

$$\rightarrow \begin{bmatrix} \dot{\Delta \phi} \\ \dot{\Delta p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

$$\rightarrow |A - \lambda I| = 0$$

$$\lambda^2 + \frac{k_2}{I_x} \lambda + \frac{k_1}{I_x} = 0$$

$$\lambda = -\frac{k_1}{2I_x} + \sqrt{\frac{k_1^2}{4I_x^2} - \frac{k_2}{I_x}}$$

Desired:

$$f = 0.7$$

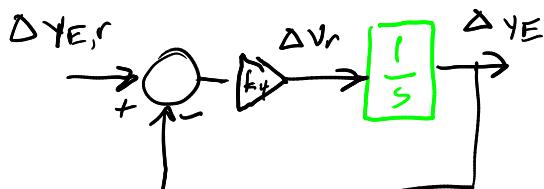
$$\omega_n = 16 \text{ rad/s} \quad T = \frac{2\pi}{\omega_n}$$

$$\frac{T}{2} = 0.2s$$

$$\lambda = -f\omega_n \pm i\omega_n \sqrt{1-f^2}$$

$$\lambda = -11.2 + 11.4i$$

$$\boxed{k_1 = 0.0016} \\ \boxed{k_2 = 0.0179}$$



$$\dot{\Delta y}_E = k_4 (y_{E,r} - \Delta y_E)$$

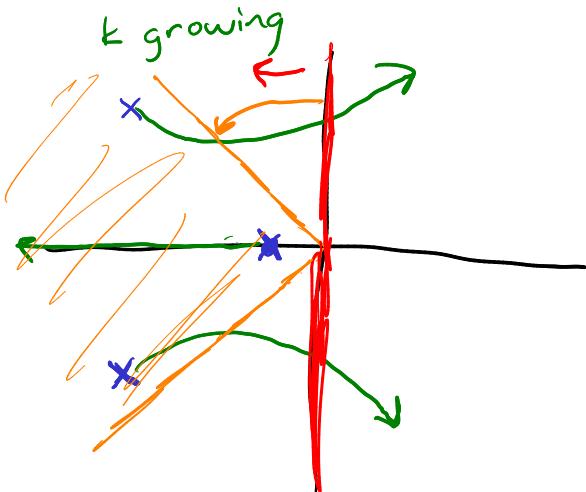
$$\Delta y_E^{(+)} = \Delta y_{E,r} (1 - e^{-\frac{k_4}{T}})$$

choose  $k_4$  1 order of magnitude slower

$$T = \frac{1}{k_4} > 10 \cdot \frac{2\pi}{f\omega_n} = 5.6$$

$$\boxed{k_4 = 0.17}$$

$x: k=0$  Choose  $k_3$  via Root Locus



Root Locus: plot of the poles/eigen values of a system as a parameter (usually a gain) changes

# Conventional A/C Dynamics

## Longitudinal

Altitude  
Airspeed  
Pitch



Lift  
Drag  
(Pitch Moment)

$$L = \frac{1}{2} \rho V_a^2 S C_L$$

$$D = \frac{1}{2} \rho V_a^2 S C_D$$

$$M = \frac{1}{2} \rho V_a^2 S_c C_m$$

↑ chord

## Lateral-Directional

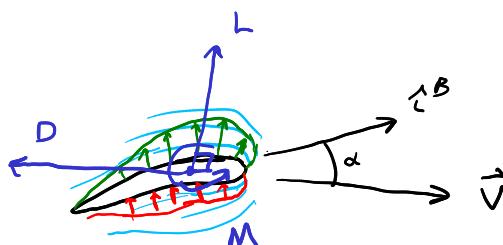
Roll  
Yaw  
Sideslip  
Turning

$$C_L(\alpha, q, \delta_e) \quad \text{nonlinear}$$

$$C_D(\alpha, q, \delta_e)$$

$$C_m(\alpha, q, \underline{\delta_e})$$

## Lift



1st order Taylor Series of  $C_L(\alpha, q, \delta_e)$

$$L = \frac{1}{2} \rho V_a^2 S \left( C_{L_0} + \frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e} \delta_e \right)$$

$$= \frac{1}{2} \rho V_a^2 S \left( C_{L_0} + C_{L_\alpha} \alpha + C_{L_q} \frac{c}{2V_a} q + C_{L_{\delta_e}} \delta_e \right)$$

$\uparrow \quad \uparrow \quad \uparrow$   
Nondimensional stability Derivative      Nondim. control derivative

## Stability Derivatives

- based linear assumptions / linearization
- Main tool connecting aerodynamics to dynamics
- Functions of A/C geometry

### Example

Pitch Stiffness  $C_{m_\alpha}$

$$\ddot{\alpha} \approx \frac{k}{I} C_{m_\alpha} \alpha$$

$$\ddot{x} = -\frac{k_{spring}}{m} x$$

- Only valid in a linear region e.g. small  $\alpha$



$$AR = \frac{b^2}{S}$$

### Estimated Using

- Geometric Data
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$\text{eg. } C_{L_\alpha} = \frac{\pi AR}{1 + \sqrt{1 + (AR)^2}}$$

## Drag

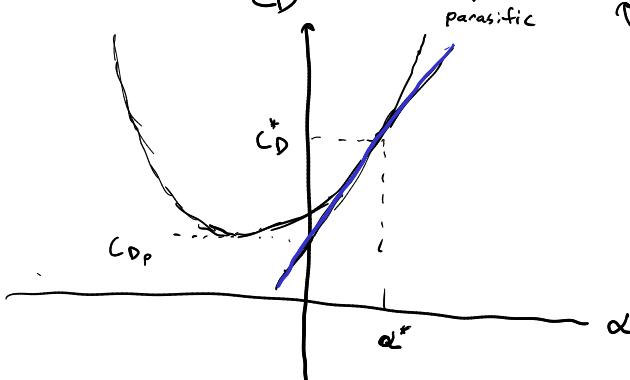
nonlinear

$\alpha$   
only

$$C_D(\alpha) = C_{D_p} + \frac{C_L(\alpha)^2}{\pi e AR}$$

parasitic

$\nearrow \alpha \in AR$   
Oswalds efficiency  
induced drag



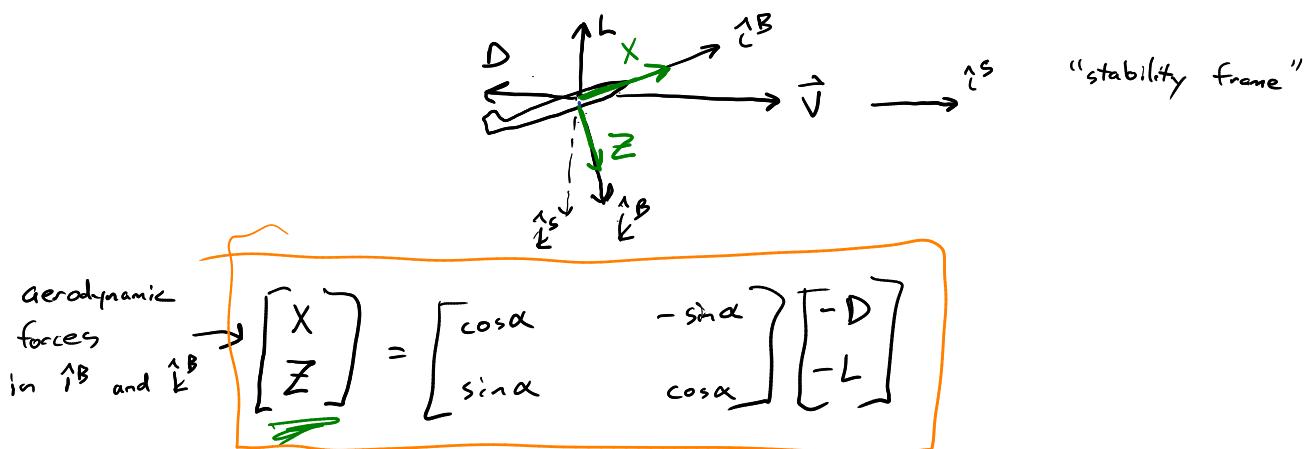
linearize

$$\underline{C_{D_0} + C_{D_\alpha} \alpha + C_{D_{\dot{\alpha}}} \dot{\alpha} + C_{D_{\ddot{\alpha}}} \ddot{\alpha}}$$

In this class:

$$K = \frac{1}{\pi e AR}$$

$$C_D = C_{D_{min}} + K(C_L(\alpha, q, \delta_e) - C_{L_{min}})^2$$



## Pitching Moment

$$M = \frac{1}{2} \rho V_a^2 S_c [C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \dot{\alpha} + C_{m_{\dot{\alpha}}} \ddot{\alpha}]$$

trim coefficient of moment

pitch stiffness coefficient

pitch damping coefficient

$\ddot{\alpha}$

# Longitudinal Trim

Last time: Long. Forces + Moments

$$L = \frac{1}{2} \rho V_a^2 S (C_{L_0} + C_{L\alpha} \overset{\text{coordinate def.}}{\underbrace{\alpha}} + C_{Lq} \overset{\text{coordinate def.}}{\underbrace{q}} + C_{L\delta_e} \overset{\text{coordinate def.}}{\underbrace{\delta_e}})$$

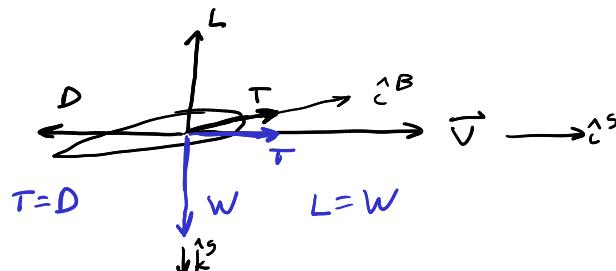
$$D = \frac{1}{2} \rho V_a^2 S (C_{D_{min}} + K (C_L(\alpha, q, \delta_e) - C_{L_{min}})^2)$$

$$M = \frac{1}{2} \rho V_a^2 S \bar{c} (C_{m_0} + C_{m\alpha} \overset{\text{coordinate def.}}{\underbrace{\alpha}} + C_{mq} \overset{\text{coordinate def.}}{\underbrace{q}} + C_{m\delta_e} \overset{\text{coordinate def.}}{\underbrace{\delta_e}})$$

$$C_{a_b} \equiv \frac{\partial C_a}{\partial b}$$

Trim: SLUF: Steady Level Unaccelerated Flight  
 ↗ just "in trim"

For linear trim calc



At trim, forces + moments about C.G. sum to 0

$$\overset{\text{S dir:}}{T_{trim}} = D_{trim}$$

$$\overset{\text{S dir:}}{L_{trim}} = W$$

$$M_{trim} = 0$$

$V_a$  fixed

$$\cancel{C_{L_{trim}}} = C_{L\alpha_{trim}} = \frac{W}{\frac{1}{2} \rho V_a^2 S}$$

$$\cancel{C_{m_{trim}}} = C_{m_0} + C_{m\alpha_{trim}} = 0$$

$$C_L = C_{L\alpha} + C_{L\delta_e}$$

$$C_m = C_{m_0} + C_{m\alpha} + C_{m\delta_e} = 0$$

$$\begin{bmatrix} C_{L\alpha} & C_{L\delta_e} \\ C_{m\alpha} & C_{m\delta_e} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{trim} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{trim} \end{bmatrix} = \begin{bmatrix} C_{L\alpha} & C_{L\delta_e} \\ C_{m\alpha} & C_{m\delta_e} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(special case of  
Cramer's Rule)

$$\alpha_{trim} = \frac{C_{m_0} C_{L\delta_e} + C_{m\delta_e} C_{L_{trim}}}{\Delta}$$

$$\delta_{trim} = - \frac{C_{m_0} C_{L\alpha} + C_{m\alpha} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L\alpha} C_{m\delta_e} - C_{L\delta_e} C_{m\alpha}$$

$$C_{m\alpha} = C_{L\alpha} (h - h_n)$$

(from slides)

$$C_{L_\alpha} = a = a_{wb} \left[ 1 + \frac{a_t S_t}{a_{wb} S} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m_{0_p}} + a_t \bar{V}_H (\epsilon_0 + i_t) \left[ 1 - \frac{a_t S_t}{a S} \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a} \bar{V}_H \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) - \frac{1}{a} \frac{\partial C_{m_p}}{\partial \alpha}$$

$$C_{m_\alpha} = C_{L_\alpha} (h - h_n)$$

*Direct dependence on CG location  $h$*

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e} \frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}} (h - h_{n_{wb}})$$

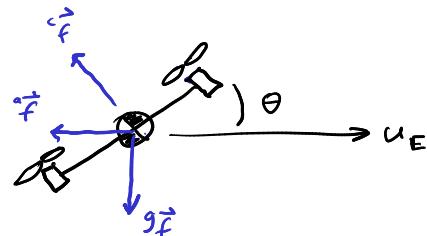
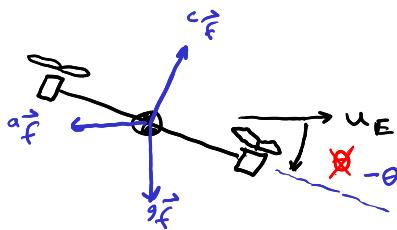
## Some mistakes on hw

if  $\dot{x}(0) = \sum_i k_i \vec{v}_i$        $\dot{x}(t) = \sum_i k_i e^{\lambda_i t} \vec{v}_i$        $\downarrow$  not gains

Rate of climb  $= \dot{z}_E = -\dot{z}_E$

$\dot{h} = -\dot{z}_E$

## Diagram



## Linearization

Method 1: fo. Taylor Series

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} \Delta x + \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} \Delta y$$

$$xy = x_0 y_0 + y_0 \Delta x + x_0 \Delta y$$

$$\begin{aligned} \sin(\theta) &= \sin(\theta_0) + \frac{\partial \sin \theta}{\partial \theta} \Big|_{\theta_0} \Delta \theta \\ &= \sin(\theta_0) + \cos(\theta_0) \Delta \theta \end{aligned}$$

$$\dot{x} = \underbrace{a \Delta x + b}_{\text{may include } x_0}$$

2 methods

Method 2

$$f(x, y) \rightarrow f(x_0 + \Delta x, y_0 + \Delta y) \rightarrow \text{expand out/cancel out} \rightarrow \text{apply small disturbance approx}$$

$$xy \rightarrow (x_0 + \Delta x)(y_0 + \Delta y) \rightarrow x_0 y_0 + y_0 \Delta x + x_0 \Delta y + \Delta x \Delta y$$

$$\begin{aligned} \sin \theta &\rightarrow \sin(\theta_0 + \Delta \theta) \rightarrow \sin(\theta_0) \cos(\Delta \theta) + \cos(\theta_0) \sin(\Delta \theta) \\ &\rightarrow \sin(\theta_0) + \cos(\theta_0) \Delta \theta \end{aligned}$$

$$\dot{x} = \begin{bmatrix} W_2 \\ Q_2 \end{bmatrix}$$

$$\dot{\vec{V}}_B^E$$

$$\vec{W}_E$$

# Longitudinal Dynamics

# Last Time

$$C_L = C_{L\alpha} \alpha + C_{L\delta e} \delta e = C_{L\text{trim}}$$

$$C_m = C_{m\alpha} \alpha + C_{m\delta e} \delta e \rightarrow = 0$$

$$C_{m\alpha} = C_{L\alpha} (h - h_n) \leftarrow$$

Today: Linear Longitudinal EOM  
Stability Derivatives

$$\dot{\vec{r}}_E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{\omega}}_B = I^{-1} [\vec{G}_B - \vec{\omega}_x \times I \vec{\omega}_B]$$

Symmetry about x-z axis

$$I_{xy} = I_{yz} = 0$$



$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{I_x}{\Gamma} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz} (I_x - I_y + I_z)}{\Gamma}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma}$$

$$\Gamma_7 = \frac{I_x (I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z (I_z - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y}$$

$$\Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

→  $\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

→  $\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

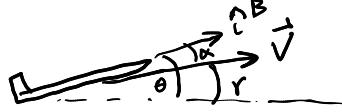
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_8 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

$$V = u_0$$

Trim State  
Inputs  $u_0, h_0, \gamma_{ao}$   
speed altitude  $\gamma_{air}$  relative flight path angle

determine →

$$\alpha_0, \delta_{e0}, \delta_{t0}$$



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \begin{bmatrix} \Delta x_E \\ \Delta y_E \\ \Delta z_E \\ \Delta \phi \\ \Delta \theta \\ \Delta \psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} \quad \vec{x}_0 = \begin{bmatrix} \cdot \\ -h_0 \\ 0 \\ 0 \\ u_0^E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} \delta_e \\ 0 \\ 0 \\ 0 \\ \delta_t \end{bmatrix}$$

$$\dot{\theta}_0 + \Delta \dot{\theta} = \cos \theta_0 \dot{\theta}_0 + \frac{\partial \cos \theta_0}{\partial \phi} \Delta \phi + \frac{\partial \cos \theta_0}{\partial \theta} \Delta \theta - \sin \theta_0 - \frac{\partial \sin \theta_0}{\partial \phi} \Delta \phi - \frac{\partial \sin \theta_0}{\partial \theta} \Delta \theta$$

$$\Delta \dot{\theta} = \Delta \dot{\theta}$$

$$\dot{x}_0 + \Delta \dot{x} = \dot{x}_0 + \Delta \dot{x} + v_0 \Delta r + g \Delta v - g \Delta v - g \Delta w - u_0 \Delta q - \Delta \dot{q} \cancel{- g \sin \theta_0 - g \cos \theta_0 \Delta \theta + \frac{1}{m} (\dot{x}_0 + \Delta x)}$$

$$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta x \quad \text{cancel}$$

→  $\Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0$

Longitudinal

$$X_u \equiv \frac{\partial X}{\partial u}$$

→  $\Delta \dot{\theta} = \Delta q$

Lat-d

→  $\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m}$

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

→  $\Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m}$

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

→  $\Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m}$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

→  $\Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

→  $\Delta \dot{q} = \frac{\Delta M}{I_y}$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

→  $\Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$

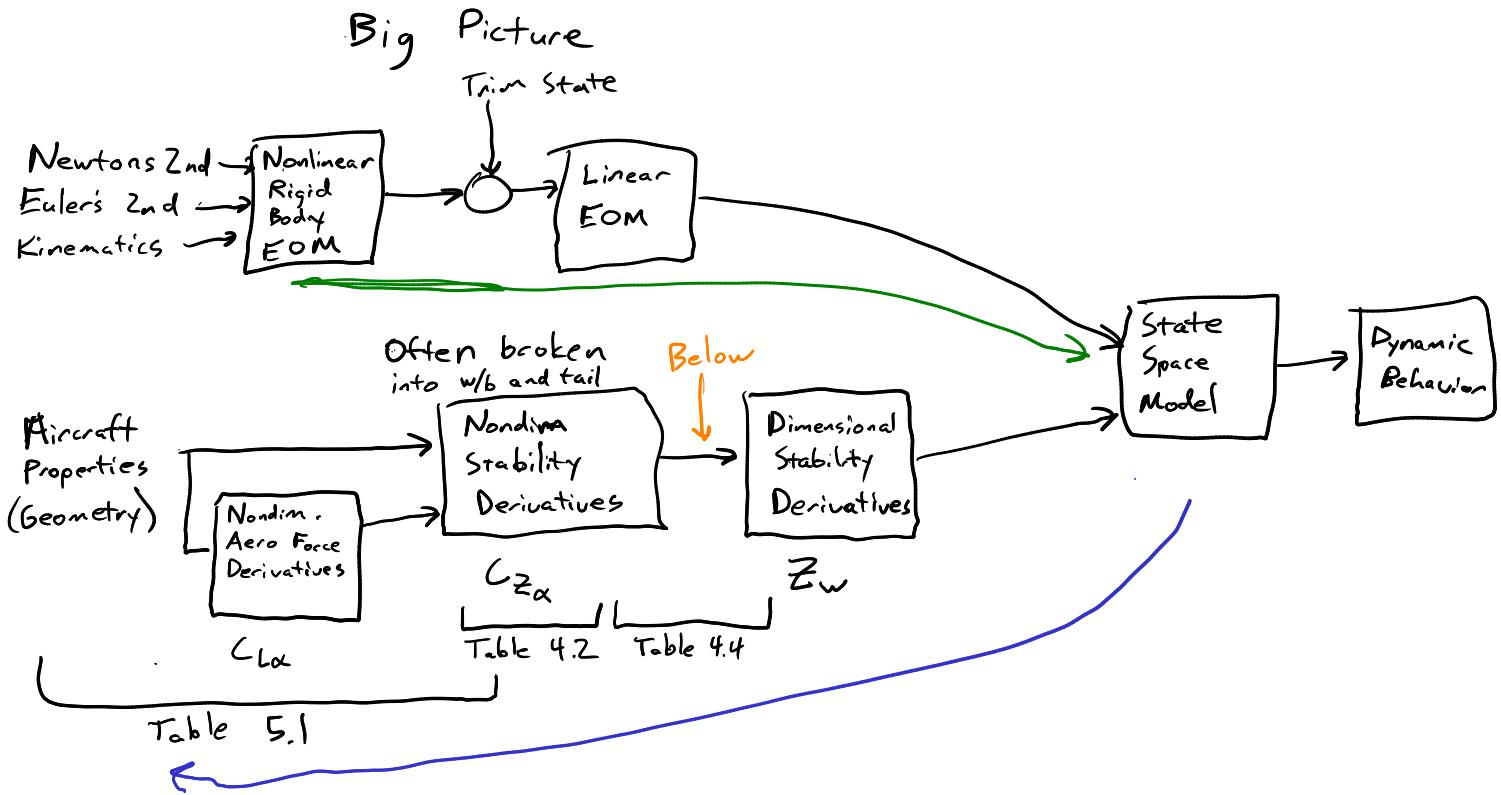
$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_w}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m} & \frac{Z_w}{m - Z_w} & \frac{Z_q + mu_0}{m - Z_w} & \frac{-mg \sin \theta_0}{m - Z_w} \\ \frac{1}{I_y} \left[ M_u + \frac{M_w Z_u}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_w Z_w}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_w (Z_q + mu_0)}{m - Z_w} \right] & \frac{-M_w mg \sin \theta_0}{I_y (m - Z_w)} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Variable	Divisor	Non-dim Variable
$X, Y, Z$	$\frac{1}{2} \rho V^2 S$	$C_x, C_y, C_z$
$W$	$\frac{1}{2} \rho V^2 S$	$C_W$
$M$	$\frac{1}{2} \rho V^2 S \bar{c}$	$C_m$
$L, N$	$\frac{1}{2} \rho V^2 S \bar{b}$	$C_l, C_n$
$u, v, w$	$V$	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\dot{\hat{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\dot{\hat{\beta}}, \hat{p}, \hat{r}$
$m$	$\rho S \bar{c}/2$	$\mu$
$I_y$	$\rho S (\bar{c}/2)^3$	$\hat{I}_y$
$I_x, I_z, I_{xz}$	$\rho S (b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

$$C_{z_u} = \frac{\partial C_z}{\partial \hat{u}}$$

Wind - Angle Approximations

$$\begin{aligned} \Delta \alpha &= \tan^{-1} \frac{\Delta w}{V} \\ &\approx \hat{w} \\ \Delta \beta &= \sin^{-1} \frac{\Delta v}{V} \\ &\approx \hat{v} \end{aligned}$$

**Table 4.4**  
**Longitudinal Dimensional Derivatives**

	X	Z	M
$u$	$\rho u_0 S C_{w_0} \sin \theta_0 + \frac{1}{2} \rho u_0 S C_{x_u}$	$-\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$	$\frac{1}{2} \rho u_0 \bar{C} S C_{m_u}$
$w$	$\frac{1}{2} \rho u_0 S C_{x_\alpha}$	$\frac{1}{2} \rho u_0 S C_{z_\alpha}$	$\frac{1}{2} \rho u_0 \bar{C} S C_{m_\alpha}$
$q$	$\frac{1}{4} \rho u_0 \bar{C} S C_{x_q}$	$\frac{1}{4} \rho u_0 \bar{C} S C_{z_q}$	$\frac{1}{4} \rho u_0 \bar{C}^2 S C_{m_q}$
$\dot{w}$	$\frac{1}{4} \rho \bar{C} S C_{x_\alpha}$	$\frac{1}{4} \rho \bar{C} S C_{z_\alpha}$	$\frac{1}{4} \rho \bar{C}^2 S C_{m_\alpha}$

$$\begin{aligned} Z_u &\equiv \frac{\partial Z}{\partial u} \Big|_o \quad Z = \frac{1}{2} \rho V^2 S C_Z \\ \frac{\partial Z}{\partial u} \Big|_o &= \frac{1}{2} \rho S \left( \frac{\partial V^2}{\partial u} \Big|_o C_Z + \frac{\partial C_Z}{\partial u} \Big|_o V^2 \right) \\ &= \frac{1}{2} \rho S Z_u C_{Z_o} + \frac{1}{2} \rho u_o^2 S \frac{\partial C_Z}{\partial u} \Big|_o \\ Z_u &= -\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u} \end{aligned}$$

$\frac{\partial f(x)g(x)}{\partial x} = f(x) \frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x} g(x)$   
 $C_{z_u} \equiv \frac{\partial C_Z}{\partial \alpha}$   
 $\frac{\partial C_Z}{\partial u} = \frac{\partial C_Z}{\partial \hat{u}_{u_0}} = \frac{1}{u_0} \frac{\partial C_Z}{\partial \alpha}$   
 $C_{Z_o} = -C_{W_o} \cos \theta_0$

**Table 5.1**  
**Summary—Longitudinal Derivatives**

	$C_x$	$C_z$	$C_m$
$\hat{u}^\dagger$	$\mathbf{M}_0 \left( \frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$* -2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$* -2a_t V_H \frac{l_t}{\bar{C}} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$* -2a_t V_H$	$* -2a_t V_H \frac{l_t}{\bar{C}}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\ddot{\alpha} C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

# Nondimensional Longitudinal Stability Derivatives

Table 5.1

Summary—Longitudinal Derivatives

	$C_x$	$C_z$	$C_m$
$\hat{u}^+$	$M_0 \left( \frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-M_0 \frac{\partial C_L}{\partial M} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$M_0 \frac{\partial C_m}{\partial M} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{c} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{c}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\hat{t} C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

$\alpha$  derivatives

$$C_{m_\alpha} = C_{L_\alpha} (h - h_N)$$

$C_{z_\alpha}$

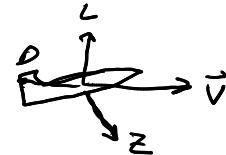
$$Z = -L \cos \alpha - D \sin \alpha$$

$$C_Z = -(C_L \cos \alpha + C_D \sin \alpha)$$

$$= -(C_L + C_D \alpha)$$

$$C_{Z_\alpha} = \frac{\partial C_Z}{\partial \alpha} \Big|_0 = -(C_{L_\alpha} + C_{D_0} + \alpha \frac{\partial C_D}{\partial \alpha} \Big|_0)$$

$$\boxed{C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$



$u$  derivatives

3 important factors:

- Compressibility : Mach Number

- Dynamic Pressure:  $p_d = \frac{1}{2} \rho V^2$

- Thrust

$$C_{x_u} \equiv \frac{\partial C_x}{\partial \hat{u}} \Big|_0$$

$$M \equiv \frac{V}{a} \quad \text{speed of sound}$$

- Different from the dynamic pressure in nondimensionalization  
Changes in  $C_L, C_D$ , etc., due to changes in dynamic pressure

$$C_{*u} = \frac{\partial C_*}{\partial M} \Big|_0 \frac{\partial M}{\partial \hat{u}} \Big|_0 + \frac{\partial C_*}{\partial p_d} \Big|_0 \frac{\partial p_d}{\partial \hat{u}} \Big|_0 + \frac{C_*}{\partial C_T} \Big|_0 \frac{\partial C_T}{\partial \hat{u}} \Big|_0$$

$$\rightarrow \frac{\partial M}{\partial \hat{u}} \Big|_0 = u_0 \frac{\partial M}{\partial u} \Big|_0 = \frac{u_0}{a} \frac{\partial V}{\partial u} \Big|_0 = M_0$$

$$* \in \{x, z, m\}$$

$$\frac{\partial p_d}{\partial u} \Big|_0 = u_0 \frac{\partial p_d}{\partial u} \Big|_0 = u_0 \frac{1}{2} \rho \frac{\partial V^2}{\partial u} \Big|_0 = u_0 \rho \cdot u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2} \rho V^2 S}$$

$$\frac{\partial C_T}{\partial u} \Big|_0 = u_0 \frac{\partial C_T}{\partial u} \Big|_0 = u_0 \left( \frac{\partial T}{\partial u} \Big|_0 - \frac{2T}{\frac{1}{2} \rho V^2 S} \right) \Big|_0 = \frac{\partial T}{\partial u} \Big|_0 - 2C_{T_0}$$

$\frac{\partial (f(x))}{\partial x} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

3 cases

Gliding case :  $C_{Tu} = 0$

Constant Thrust (Jet) :  $C_{Tu} = -2C_{T_0}$

Constant Power (Prop) :  $C_{Tu} = -3C_{T_0}$

$TV = \text{constant}$

$$\frac{\partial T}{\partial u} \Big|_0 = -\frac{T_0}{u_0}$$

$C_{Xu}$

$$C_X \approx C_T - C_D$$

$$\frac{\partial C_X}{\partial M} \Big|_0 = \frac{\partial C_T}{\partial M} \Big|_0 - \frac{\partial C_D}{\partial M} \Big|_0$$

$$\frac{\partial C_X}{\partial p_d} \Big|_0 = \cancel{\frac{\partial C_T}{\partial p_d} \Big|_0} - \frac{\partial C_D}{\partial p_d} \Big|_0$$

$$\frac{\partial C_X}{\partial C_T} \Big|_0 = 1 - \frac{\partial C_D}{\partial C_T} \Big|_0$$

$$C_{Xu} = M_0 \left( \frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) \Big|_0 - \rho u_0^2 \frac{\partial C_D}{\partial p_d} \Big|_0 + C_{Tu} \left( 1 - \frac{\partial C_D}{\partial C_T} \Big|_0 \right)$$

$C_{Zu}$

Assume  $C_Z = -C_L$

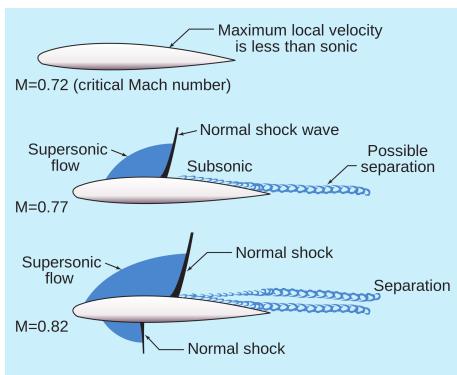
$$C_{Zu} = -M_0 \underbrace{\frac{\partial C_L}{\partial M} \Big|_0}_{\text{Small except transonic}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} \Big|_0 - C_{Tu} \frac{\partial C_L}{\partial C_T} \Big|_0$$

Small except transonic

$C_{mu}$

$$C_{mu} = M_0 \underbrace{\frac{\partial C_m}{\partial M} \Big|_0}_{\text{Mach Tuck}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} \Big|_0 + C_{Tu} \frac{\partial C_m}{\partial C_T} \Big|_0$$

Mach Tuck



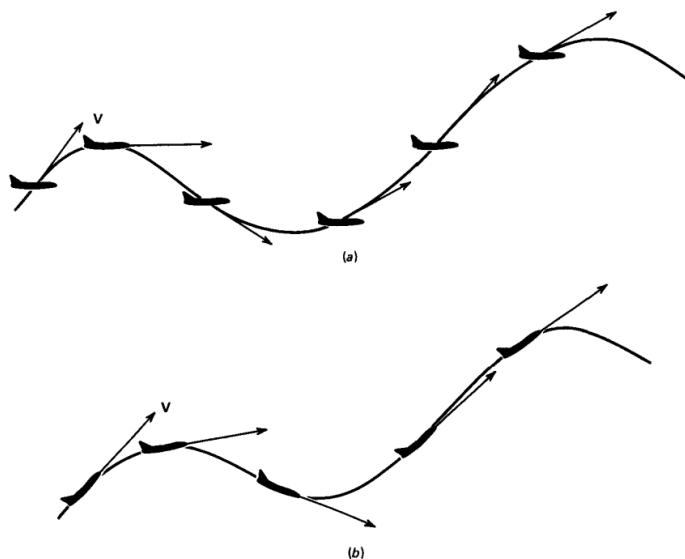


Figure 5.2 (a) Motion with zero  $q$ , but varying  $\alpha_x$ . (b) Motion with zero  $\alpha_x$  but varying  $q$ .

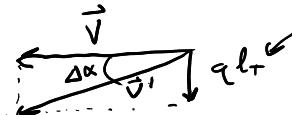
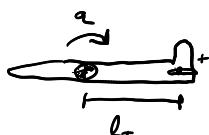
$q$ - derivatives

Wing-body

Tail

Tail

velocity observed by tail



$(C_{Zq})_{tail}$

$$C_{Zq} = \frac{\partial C_Z}{\partial \dot{q}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_Z}{\partial q} \Big|_0 = - \frac{2u_0}{c} \frac{\partial C_L}{\partial q} \Big|_0$$

$$(C_{Zq})_{tail} = - \frac{2u_0}{c} a_+ \frac{S+L+}{S u_0} = \boxed{-2a_+ V_H}$$

$$\begin{aligned} \Delta C_L &= a_+ \Delta \alpha = a_+ \tan^{-1} \frac{qL+}{u_0} \approx a_+ \frac{qL+}{u_0} \\ \Delta C_L &= \frac{S+}{S} \Delta C_L \\ &= \frac{S+}{S} a_+ \frac{qL+}{u_0} \end{aligned}$$

$$V_H = \frac{S+L+}{Sc}$$

$(C_{m_q})_{tail}$

$$\Delta C_m = -V_H \Delta C_L = a_+ V_H \frac{qL+}{u_0}$$

$$C_{m_q} = \frac{\partial C_m}{\partial \dot{q}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_m}{\partial q} \Big|_0$$

$$(C_{m_q})_{tail} = -2a_+ V_H \frac{l+}{c}$$

Wing-Body

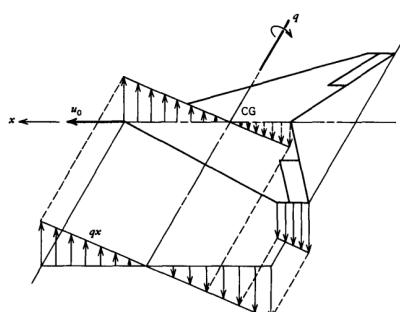


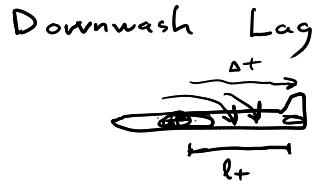
Figure 5.4 Wing velocity distribution due to pitching.

## $\dot{\alpha}$ derivatives

Unsteady effects

**Wing-Body** → Determined by initial response  
or oscillation of wing in wind tunnel  
or flight test

Tail



$$\Delta \varepsilon = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \Delta t = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \frac{l_+}{u_0}$$

$$= -\Delta \alpha_+$$

$(C_{Z\dot{\alpha}})_{tail}$

$$\Delta C_L = a_+ \Delta \alpha_+ = a_+ \dot{\alpha} \frac{l_+}{u_0} \frac{\partial \varepsilon}{\partial \alpha}$$

$$\rightarrow \Delta C_L = a_+ \dot{\alpha} \frac{l_+ + S_r}{u_0 S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$C_{Z\dot{\alpha}} = \frac{\partial C_Z}{\partial \frac{\alpha \varepsilon}{2 u_0}} \Big|_0 = \frac{2 u_0}{c} \frac{\partial C_Z}{\partial \alpha} = -2 a_+ \frac{l_+ + S_r}{c S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{Z\dot{\alpha}})_{tail} = -2 a_+ V_H \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{m\dot{\alpha}})_{tail} = -2 a_+ V_H \frac{l_+}{c} \frac{\partial \varepsilon}{\partial \alpha}$$

# Longitudinal Modes

$$\mathbf{x}(t) = \sum_i k_i \vec{v}_i e^{\lambda_i t}$$

Dynamics of Flight, Eq. (4.9,18)  $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$



$$h_0 = 40k \text{ ft}$$

$$V_0 = 774 \text{ ft/s}$$

$$\gamma_0 = \theta_0 = \alpha_0 = 0$$

$$\omega_n = \sqrt{a^2 + b^2}$$

$$\lambda_{1,2} = -0.37 \pm 0.89i$$

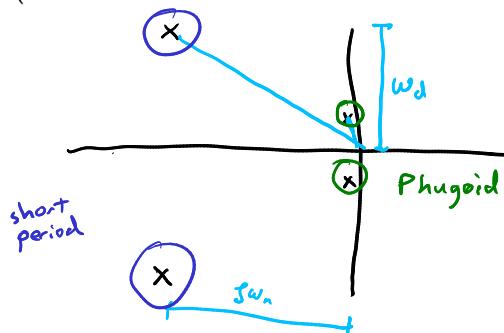
$a = -0.37$     $b = 0.89$   
 $\omega_n = 0.96$ ,  $\zeta = 0.38$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$a = -0.0033$     $b = 0.067$   
 $\omega_n = 0.067$ ,  $\zeta = 0.049$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & -mg \sin \theta_0 \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



## Eigenvectors

$$A \vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I) v_i = 0$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0 - \lambda & 1 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \lambda_1 = -1, \lambda_2 = -2$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$(A - \lambda_1 I) v_1 = 0$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

let  $\vec{v}_1[1] = 1$   
 $\vec{v}_1[2] = -1$

$$\vec{v}_1[1] + \vec{v}_1[2] = 0$$

$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.001i \\ 0.0011 \mp 0.0004i \end{bmatrix} \quad \begin{array}{l} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{array}$$

$$\vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097i \\ -0.0001 \mp 0.0006i \\ 0.0001 \pm 0.002i \end{bmatrix}$$

$$x(t) = \sum_i L_i \vec{v}_i e^{\lambda_i t}$$

$$x(0) = \text{Re}(\vec{v}_1) = \sum_i L_i \vec{v}_i e^{\lambda_i 0} = 0.5 \vec{v}_1 + 0.5 \vec{v}_2$$

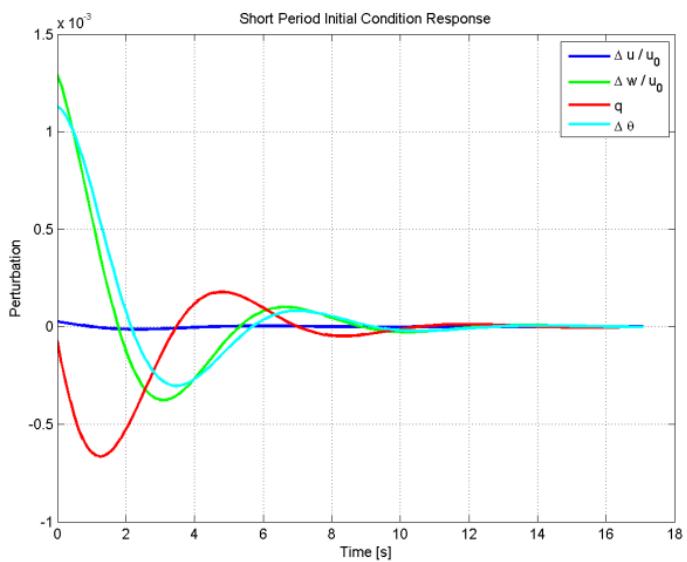
$$x(t) = 0.5 \vec{v}_1 e^{\lambda_1 t} + 0.5 \vec{v}_2 e^{\lambda_2 t}$$

$$\lambda_{1/2} = -0.372 + .888i$$

$$\zeta = 0.387$$

$$\omega_n = 0.962$$

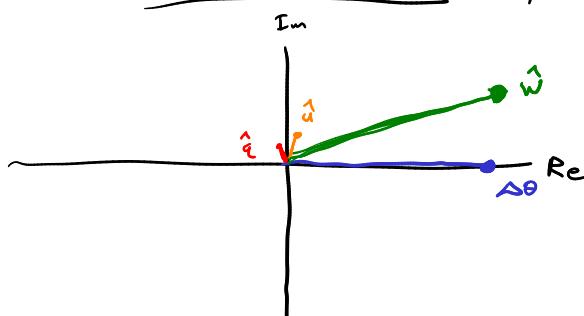
$$x(0) = \text{Re}(\vec{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



$$\vec{v}_{1,2} / \vec{v}_{1,2}[u] = \vec{v}'_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i / & " & " \\ 0.9996 / & " & " \\ -0.0001 \pm 0.001i / 0.0011 - 0.001i \\ 0.0011 \mp 0.0004i / 0.0011 - 0.0004i \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & 1.0 \end{bmatrix}$$

$$\hat{v}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.0066 \pm 0.016i \\ 1.0 \end{bmatrix} \quad \begin{array}{l} \hat{u} = \frac{\Delta u}{u_0} \\ \hat{w} = \frac{\Delta w}{u_0} \approx \alpha \\ \hat{q} = \frac{\Delta q}{2u_0} \\ \Delta \theta \end{array}$$

Phasor Plot ( $\hat{v}_1$ , Short Period)



- ↪ { $\alpha$  and  $\Delta \theta$  change in-phase  
 $\hat{u}$  relatively constant}
- ↪ {well-damped  
high frequency}

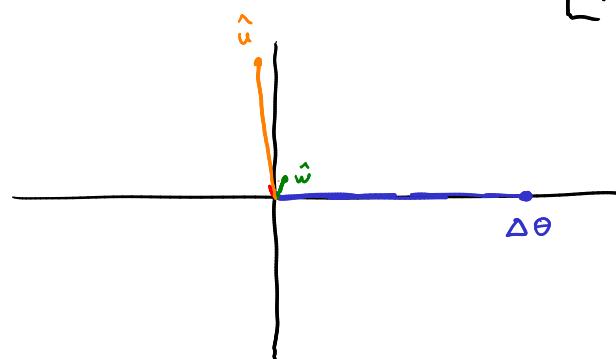
# Phugoid Mode

phugoid = “flight”



side note:

$$\begin{aligned} z &= a + bi \\ &= r \angle \phi \\ &= r e^{i\phi} \\ r &= \sqrt{a^2 + b^2}; \quad \phi = \text{atan} 2(b, a) \end{aligned}$$

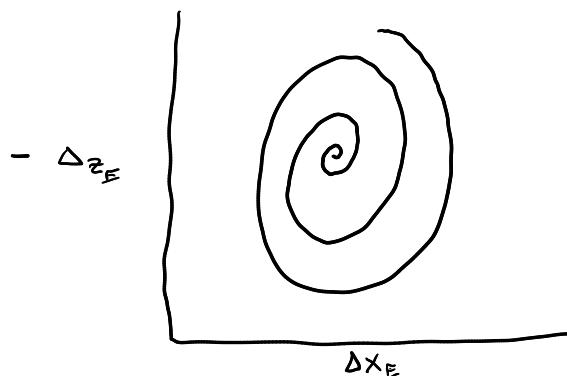
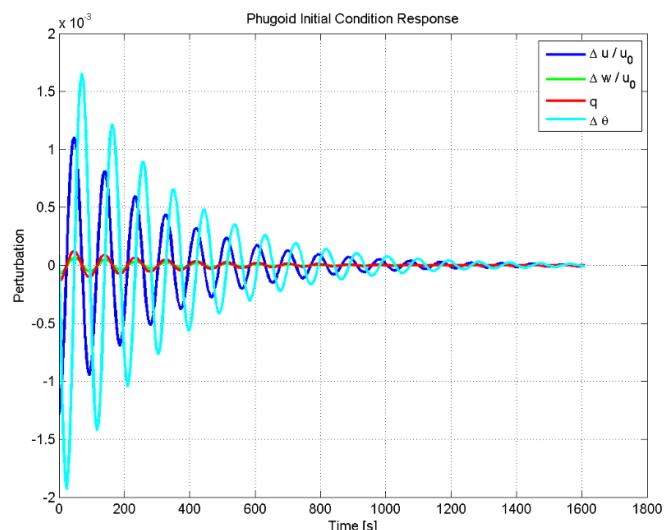


$$\hat{v}_{3,4} = \begin{bmatrix} 0.62 \angle 92^\circ \\ 0.036 \angle 83^\circ \\ \underline{0.0012 \angle 93^\circ} \\ 1.0 \angle 0 \end{bmatrix} \quad \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ \hat{q} \\ \Delta \theta \end{array}$$

- $\hat{v}$  { Large  $\hat{u}, \Delta \theta$  out-of-phase oscillations (offset  $\sim 90^\circ$ )  
small  $\alpha$
- $\lambda$  { low frequency  
low damping

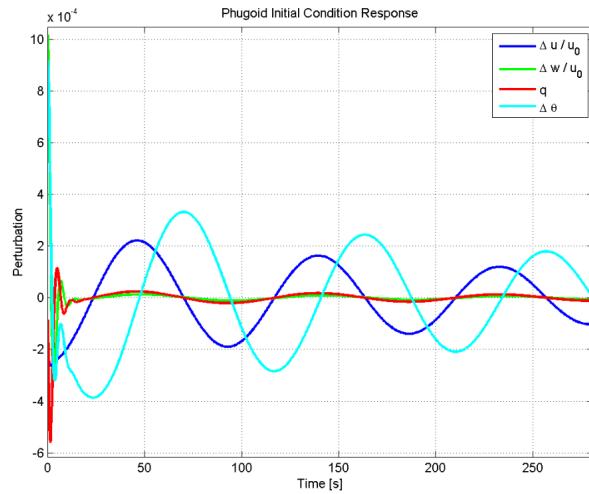
$$\begin{aligned} \lambda_{3/4} &= -3.29e-03 + 6.72e-02 i \\ \zeta &= 0.0489 \quad \leftarrow \text{poorly damped} \\ \omega_n &= 0.0673 \quad \leftarrow \text{slow response} \end{aligned}$$

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



## General Response

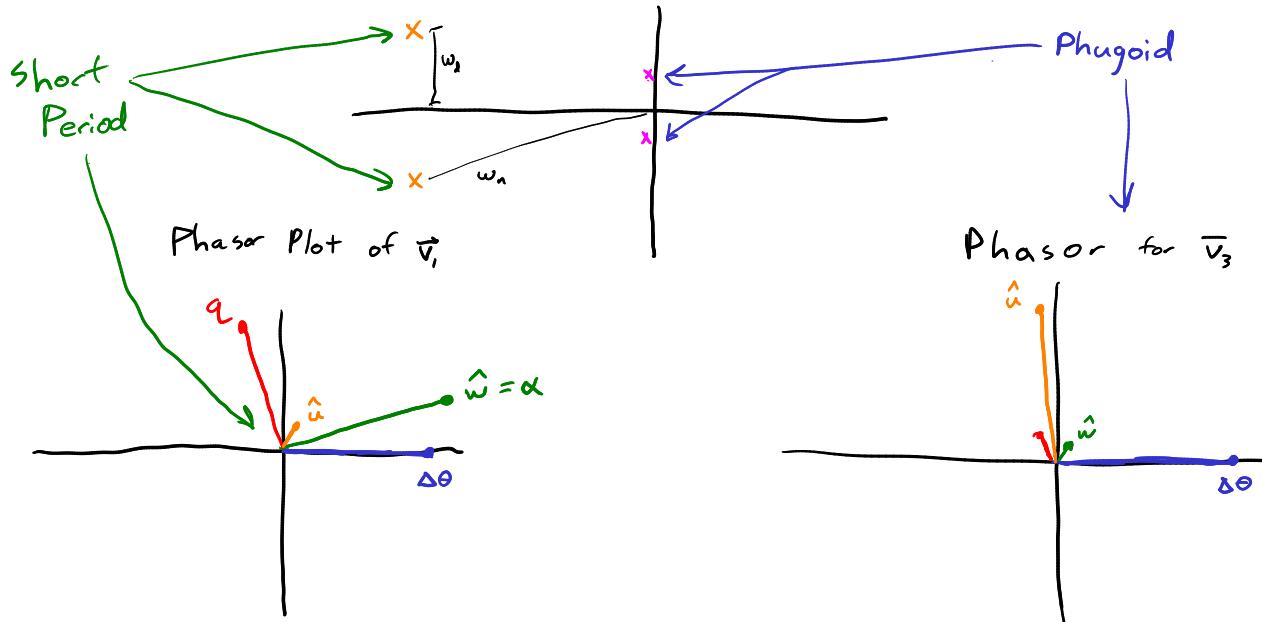
$$\mathbf{x}_0 = 0.8 \cdot \text{Re}(\mathbf{v}_{sp}) + 0.2 \cdot \text{Re}(\mathbf{v}_{ph})$$



# Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon} \vec{x}_{lon}}_{\text{Eigen}} + \vec{c}_{lon}$$

$$\vec{x}_h = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$



## Short Period Approx

*Dynamics of Flight, Eq. (4.9,18)*       $\dot{\vec{x}}_{lon} = \vec{A}_{lon} \vec{x}_{lon} + \vec{c}_{lon}$

$$\vec{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \vec{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\vec{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -a \cos \theta_0 \\ \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & -m q \sin \theta_0 \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_w + m u_0)}{m - Z_{\dot{w}}} \right] & -M_{\dot{w}} m g \sin \theta_0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume:  $\Delta u = 0$

$$\theta_0 = 0$$

$$Z_{\dot{w}} \ll m$$

$$Z_q \ll m u_0$$

If we also assume no vertical motion,

$$\theta_0 = 0 \text{ implies } \Delta \theta = \alpha \approx \frac{\Delta w}{u_0}$$

$$\begin{bmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m} \\ \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m} \right] \end{bmatrix} \underbrace{\frac{1}{I_y} \left[ M_q + M_{\dot{w}} u_0 \right]}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[ \frac{Z_w}{m} + \frac{1}{I_y} \left[ M_q + M_{\dot{w}} u_0 \right] \right]}_{-2 \zeta \omega_n} \lambda - \underbrace{\frac{1}{I_y} \left( u_0 M_w - \frac{M_{\dot{w}} Z_w}{m} \right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

<u>Dimensional Stab. Deriv.</u>	<u>Nondim. Stab. Deriv.</u>	<u>A/C Params</u>
$Z_w = \frac{\partial Z}{\partial w} \Big _0 = \frac{1}{2} \rho u_0 S C_{Z\alpha}$	$C_{Z\alpha}$	$C_{Z\alpha} = -C_{D_0} - C_{L\alpha}$
$M_w$	$C_{m\alpha}$	$= C_{L\alpha} (h - h_n)$
$M_w$	$C_{m\alpha}$	$\dots$
$M_q$	$C_{m\alpha}$	$-2a_n V_H \frac{L}{C}$

How accurate is this approximation?

For 747  
@ cruise

Full A<sub>bn</sub>

$$\lambda_{1,2} = -0.372 \pm 0.88j;$$

$$g = 0.387$$

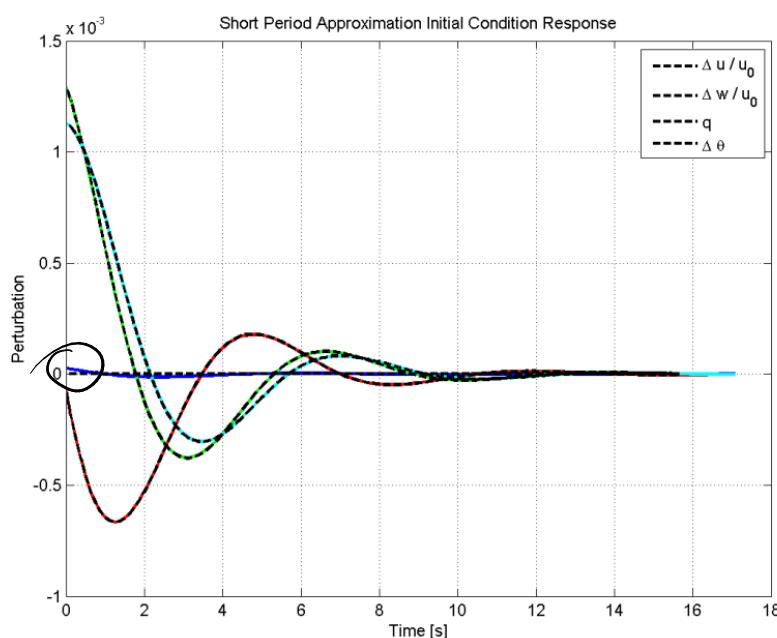
$$w_n = 0.962$$

S.P. approx

$$\lambda_{sp} = -0.371 \pm .88j;$$

$$g = 0.385$$

$$w_n = 0.963$$



(note: here since  $\Delta\theta \neq \frac{\Delta w}{u_0}$ , there is some vertical motion)

### Phugoid Mode

Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - mg \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g \Delta z_E + u_0^2$$

$$C_L = C_{L0} = C_{W0}$$

$$L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \rho u_0^2 S C_{W0} + \rho g S C_{W0} \Delta z_E = W + \rho g S C_{W0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W0}}{m} \Delta z_E}_{w_n^2} = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi \sqrt{2} \frac{u_0}{g}} = 0.138 u_0 \leftarrow \text{if } u_0 \text{ in ft/s}\right. \\ \left. 0.453 u_0 \leftarrow \text{if } u_0 \text{ in m/s}\right.$$

for 747

$$\frac{\text{Full } A_{lon}}{T = 93s}$$

$$\frac{\text{Lanchester}}{T = 107s}$$

"ZxZ"

Phugoid Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} - & \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ & \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & -mg \sin \theta_0 \\ I_y \begin{bmatrix} M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \end{bmatrix} & 0 & I_y \begin{bmatrix} M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \end{bmatrix} & I_y \begin{bmatrix} M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \end{bmatrix} & -\frac{M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume  $Z_{\dot{w}} \ll m$

$Z_u \ll mu_0$

$\Delta q$  small

$\Delta \alpha = 0$

$\theta_0 = 0$

$$\begin{bmatrix} \Delta u \\ \Delta \dot{w} \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m} \Delta u + u_0 \Delta q \rightarrow \Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$\lambda^2 - \frac{X_u}{m} \lambda - \frac{Z_u g}{mu_0} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$$

$$C_{z_u} = \underbrace{-M_0 \frac{\partial C_L}{\partial M}_0}_{\text{small}} - \underbrace{\rho u_0^2 \frac{\partial C_L}{\partial p}_0}_{\text{small}} - \underbrace{C_{T_u} \frac{\partial C_L}{\partial C_T}_0}_{\text{small}} - C_{T_u} \frac{\partial C_X}{\partial C_T}_0$$

$$C_{z_u} = 0$$

$$Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{W_0} \sin \theta^0 + \frac{1}{2} \rho u_0 S C_{X_u}$$

$$C_{X_u} = -2 C_{T_0} \quad (\text{constant thrust})$$

$$C_{T_0} = C_{D_0} + C_{W_0} \sin \theta^0$$

$$X_u \approx -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{-\frac{Z_{ug}}{m u_0}} = \sqrt{\frac{\rho S C_{W_0} g}{m}} \quad \text{Same as Lanchester}$$

$$\zeta = -\frac{X_u}{2} \sqrt{\frac{u_0}{m Z_{ug}}} = \frac{\rho u_0 S C_{D_0}}{2} \sqrt{\frac{u_0}{\frac{1}{2} mg \rho u_0 S C_{L_0}}}$$

$$= C_{D_0} \sqrt{\frac{\frac{1}{2} \rho u_0^2 S}{\frac{1}{2} mg} \frac{1}{C_{L_0}}}$$

$$\zeta = \frac{C_{D_0}}{C_{L_0}} \quad \text{High } \zeta_D = \begin{matrix} \text{less energy} \\ \text{loss} \end{matrix} \\ = \text{less damping}$$

747

Full  $\Delta_{10n}$

$$\lambda_{3,4} = -3.29 \times 10^{-3} \pm 6.72 \times 10^{-2} i$$

$$\zeta = 0.0489$$

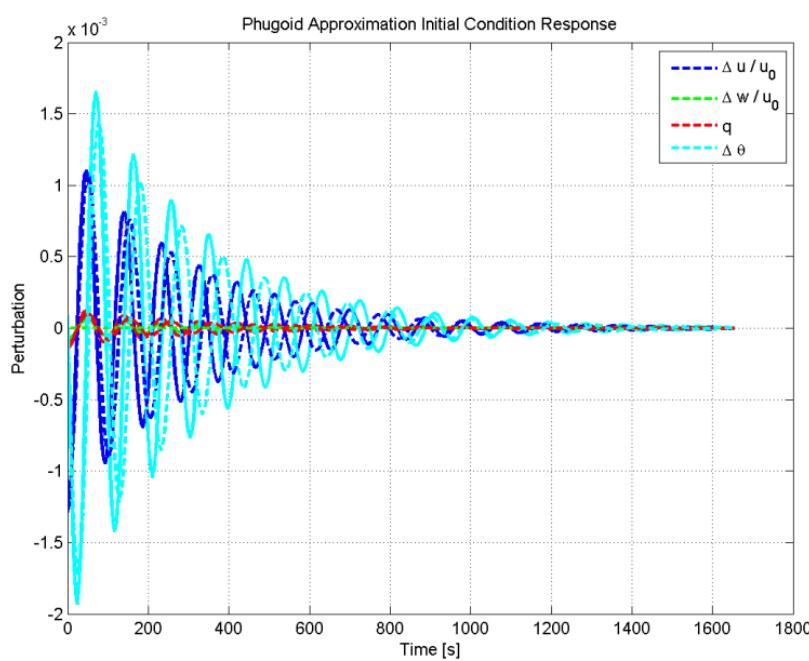
$$\omega_n = 0.0673$$

Ph Appr ox

$$\lambda_{ph} = -3.43 \times 10^{-3} + 6.11 \times 10^{-2} i$$

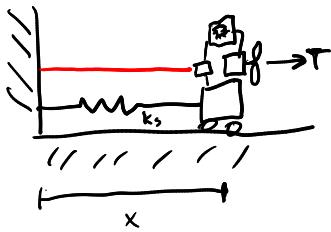
$$\zeta = 0.0561$$

$$\omega_n = 0.0612$$



## Longitudinal Control

### State Space Review



$$s_f = -k_s(x - x_0)$$

$$m\ddot{x} = -k_s(x - x_0)$$

$$m\Delta\ddot{x} = -k_s(\Delta x + \Delta T)$$

$$T_0 = 0$$

$$T = \Delta T$$

$$\vec{u} = [\Delta T]$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta \dot{x}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{k_s}{m} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \vec{u}$$

$$\Delta \dot{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \vec{u}$$

$$|A - \lambda I| = 0 = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_s}{m} & -\lambda \end{vmatrix} = \lambda^2 + \frac{k_s}{m} = 0$$

$$\lambda^2 + \underbrace{2\zeta\omega_n\lambda}_{0} + \underbrace{\omega_n^2}_{\frac{k_s}{m}} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \sqrt{\frac{k_s}{m}}$$

$$\Delta T = -k_p \Delta x - k_d \Delta \dot{x}$$

$$\vec{u} = -K \vec{x} \quad K = [k_p \quad k_d] \quad \vec{u} = [-k_p \quad -k_d] \vec{x}$$

$$\dot{\vec{x}} = A \vec{x} - B K \vec{x}$$

$$\dot{\vec{x}} = \underbrace{(A - BK)}_{A^{cl}} \vec{x}$$

$$A^{cl} = \begin{bmatrix} 0 & 1 \\ \frac{(k_s + k_p)}{m} & -\frac{k_d}{m} \end{bmatrix}$$

## Longitudinal Control

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\delta_e$  = elevator + downward deflection

$\delta_t$  = throttle + more power

$\delta_p$  in book

dimensional control derivatives

$$\Delta X_c = X_{de} \delta_e + X_{dt} \delta_t$$

$$\Delta Z_c = Z_{de} \delta_e + Z_{dt} \delta_t \quad \text{often zero}$$

$$\Delta M_c = M_{de} \delta_e + M_{dt} \delta_t$$

747

$$B_{lon} = \begin{bmatrix} -0.000187 & 9.66 \\ -1.785 & 0 \\ -1.158 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + B_{lon} \vec{u}_{lon}$$

$$\begin{bmatrix} \dot{x}_{de} \\ \dot{z}_{de} \\ \dot{m} \\ \dot{M}_{de} \end{bmatrix} = \begin{bmatrix} \frac{X_{de}}{m} & \frac{X_{dt}}{m} \\ \frac{Z_{de}}{m-Z_w} & \frac{Z_{dt}}{m-Z_w} \\ \frac{M_{de}}{I_y} + \frac{M_w Z_{de}}{I_y(m-Z_w)} & \frac{M_{dt}}{I_y} + \frac{M_w Z_{de}}{I_y(m-Z_w)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$



$$Y = [0 \ 0 \ 1] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} + [0 \ 0] \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

$$C_Y = Y \rightarrow C_Y$$

$$\delta_e = 10^\circ$$

$$\delta_t = \frac{1}{6} \approx 0.05 \text{ rad}$$

### Longitudinal Stability Augmentation

B747

$$A_{sp} = \begin{bmatrix} -0.3151 & 773.98 \\ -0.0010 & -0.4285 \end{bmatrix}$$

$$|A - \lambda I| = (-0.3151 - \lambda)(-0.4285 - \lambda) - (-0.0010)(773.98)$$

$$\lambda = -0.372 \pm 0.889i$$

$$\omega_n = 0.964$$

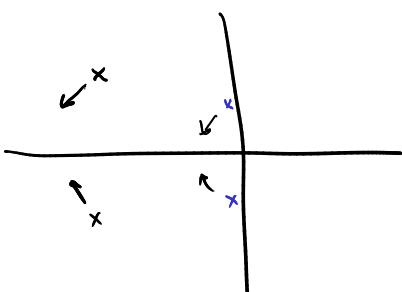
$$\zeta = 0.386 \times 0.7$$

$$\delta_e = -k_s \Delta q \quad \text{measured by rate gyro}$$

$$B_{sp} = \begin{bmatrix} -1.785 & 0 \\ -1.158 & 0 \end{bmatrix}$$

$$K_{sp} \vec{x}_{sp}$$

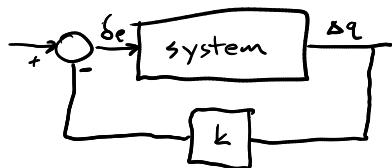
$$\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} = - \begin{bmatrix} 0 & k_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$



$$A^{cl} = A_{sp} - B_{sp}K_{sp} = A_{sp} - \begin{bmatrix} 0 & -17.85k_s \\ 0 & -1.158k_s \end{bmatrix}$$

$$= \begin{bmatrix} -0.3151 & 773.98 + 17.85k_s \\ -0.0010 & -0.4285 + 1.158k_s \end{bmatrix}$$

Matlab rlocus

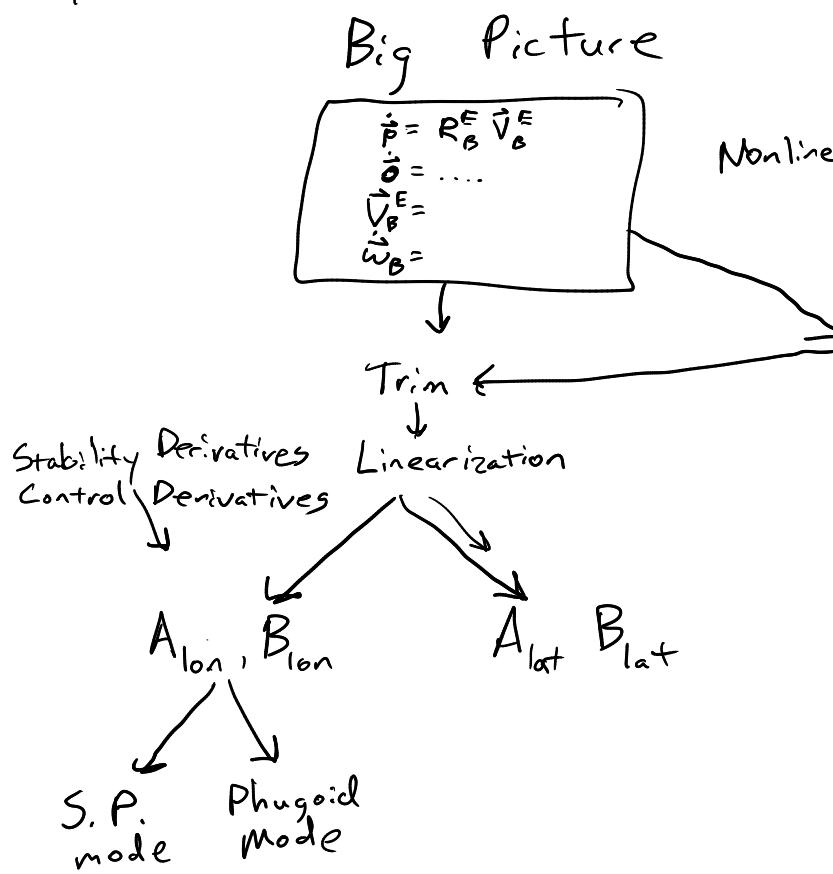


$$C_{\alpha_2} = [0 \ 1] \quad D_{\alpha_2} = [0]$$

$$B_{\delta e} = \begin{bmatrix} -17.85 \\ -1.158 \end{bmatrix}$$

More Accurate  
Fewer Assumption

## Lateral / Directional Dynamics



$$C_L = C_{L_{\text{atm}}} \alpha_{\text{trim}} + C_{L_{\text{se}}} \delta_{\text{trim}}$$

$$C_m = C_{m_{\text{atm}}} + C_{m_{\text{se}}} \alpha_{\text{trim}} + C_{m_{\text{se}}} \delta_{\text{trim}}$$

More Assumptions  
Easier to Analyze

$$\rightarrow \Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

$$\rightarrow \Delta \dot{\theta} = \Delta q$$

$$\rightarrow \Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m}$$

$$\rightarrow \Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m}$$

$$\rightarrow \Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m}$$

$$\rightarrow \Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\rightarrow \Delta \dot{q} = \frac{\Delta M}{I_y}$$

$$\rightarrow \Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

$$Y = \frac{1}{2} \rho V^2 S C_Y (\beta, \rho, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2} \rho V^2 S b C_l (\beta, \rho, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2} \rho V^2 S b C_n (\beta, \rho, r, \delta_a, \delta_r)$$

$$\hat{P} = \frac{\rho b}{2 u_0}$$

$$Y \approx \frac{1}{2} \rho V_a^2 S \left[ C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \frac{b}{2V_a} p + C_{Y_r} \frac{b}{2V_a} r + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right]$$

$$L \approx \frac{1}{2} \rho V_a^2 S b \left[ C_{l_0} + C_{l_\beta} \beta + C_{l_p} \frac{b}{2V_a} p + C_{l_r} \frac{b}{2V_a} r + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \right]$$

$$N \approx \frac{1}{2} \rho V_a^2 S b \left[ C_{n_0} + C_{n_\beta} \beta + C_{n_p} \frac{b}{2V_a} p + C_{n_r} \frac{b}{2V_a} r + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \right]$$

For symmetric aircraft,  $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 \underline{L_v} + \Gamma_4 N_v & \Gamma_3 \underline{L_p} + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

$$\begin{aligned} \Gamma_1 &= \frac{I_{xz} (I_x - I_y + I_z)}{\Gamma} & \Gamma_4 &= \frac{I_{xz}}{\Gamma} & \Gamma_7 &= \frac{I_x (I_x - I_y) + I_{xz}^2}{\Gamma} \\ \Gamma_2 &= \frac{I_z (I_z - I_y) + I_{xz}^2}{\Gamma} & \Gamma_5 &= \frac{I_z - I_x}{I_y} & \Gamma_8 &= \frac{I_x}{\Gamma} \\ \Gamma_3 &= \frac{I_z}{\Gamma} & \Gamma_6 &= \frac{I_{xz}}{I_y} & \Gamma &= I_x I_z - I_{xz}^2 \end{aligned}$$

Sideslip:  $\beta$  +  $\beta \Rightarrow$  wind coming from right



$$\beta = \sin^{-1} \frac{\Delta V}{V}$$

$$\beta \approx \frac{\Delta V}{u_0} = \hat{V}$$

$$L = \frac{1}{2} \rho V^2 S b C_L$$

$$C_{l\beta} \equiv \frac{\partial C_L}{\partial \beta} = \frac{\partial C_L}{\partial \dot{V}}$$

$$L_v = \frac{\partial L}{\partial v} \Big|_0 = \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial \beta} \Big|_0 \frac{\partial \beta}{\partial v} \Big|_0$$

$$L_v = \frac{1}{2} \rho u_0 b S C_{l\beta}$$

$$\beta = \dot{V} = \frac{\Delta V}{u_0}$$

$$\frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

$$L_p = \frac{\partial L}{\partial p} \Big|_0 = \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial p} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial \hat{p}} \Big|_0 \frac{\partial \hat{p}}{\partial p} \Big|_0$$

$$L_p = \frac{1}{4} \rho u_0 b^2 S C_{l\hat{p}}$$

$$C_{l\hat{p}} \equiv \frac{\partial C_L}{\partial \hat{p}}$$

$$\hat{p} = \frac{b p}{2 u_0}$$

**Table 4.5**  
**Lateral Dimensional Derivatives**

	<i>Y</i>	<i>L</i>	<i>N</i>
<i>v</i>	$\frac{1}{2} \rho u_0 S C_{y\beta}$	$\frac{1}{2} \rho u_0 b S C_{l\beta}$	$\frac{1}{2} \rho u_0 b S C_{n\beta}$
<i>p</i>	$\frac{1}{4} \rho u_0 b S C_{y\hat{p}}$	$\frac{1}{4} \rho u_0 b^2 S C_{l\hat{p}}$	$\frac{1}{4} \rho u_0 b^2 S C_{n\hat{p}}$
<i>r</i>	$\frac{1}{4} \rho u_0 b S C_{y_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_r}$

**Table 5.2**  
**Summary—Lateral Derivatives**

	<i>C<sub>y</sub></i>	<i>C<sub>l</sub></i>	<i>C<sub>n</sub></i>
$\beta$	$* -a_F \frac{S_F}{S} \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$
$\hat{p}$	$* -a_F \frac{S_F}{S} \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
$\hat{r}$	$* a_F \frac{S_F}{S} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

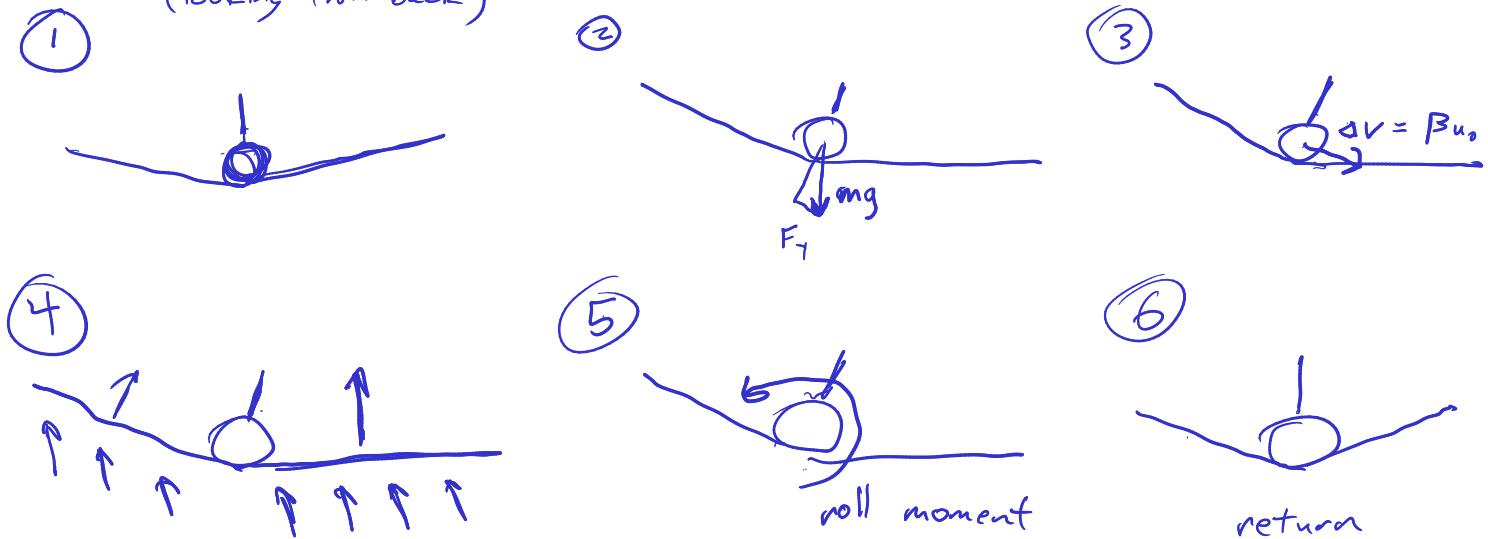
\*means contribution of the *tail only*, formula for wing-body not available;  $V_F/V = 1$ .

N.A. means no formula available.

# $C_{l\beta}$ : Dihedral Effect

sign: negative

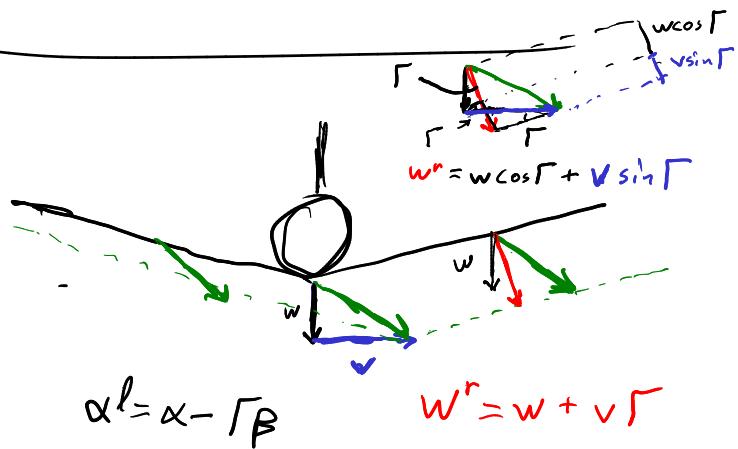
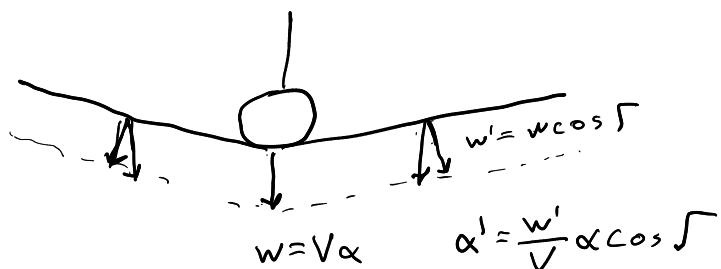
(looking from back)



4 significant factors

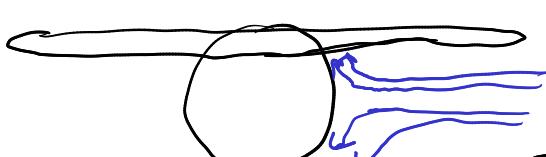
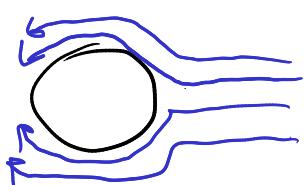
1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

## 1. Dihedral Angle

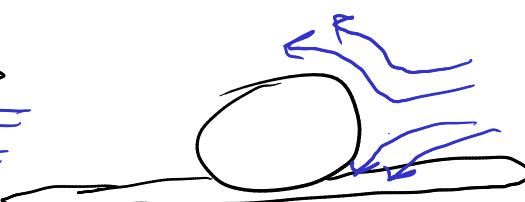


$$C_{l\beta} \propto \Gamma$$

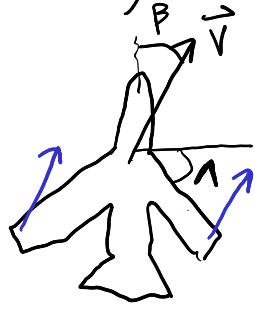
## 2. Wing Height



high wing  $\Rightarrow -C_{l\beta}$   
low wing  $\Rightarrow +C_{l\beta}$

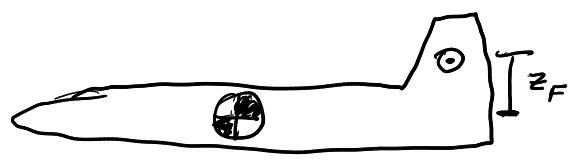


### 3. Wing Sweep



$$C_{L\beta}^1 \propto 2 C_L V^2 \sin 2\Lambda$$

### 4. Tail



$$\Delta C_L^F = C_{L_F} \frac{S_F z_F}{S_b} = \alpha_F (-\beta + \sigma) \frac{S_F z_F}{S_b}$$

$$C_{L\beta}^F = -\alpha_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \frac{S_F z_F}{S_b} \left(\frac{V_F}{V}\right)^2$$



# Lateral Stability Derivatives

## Sideslip , Coordinated Turn

**Table 5.2**  
Summary—Lateral Derivatives

	$C_y$	$C_l$	$C_n$
$\beta$	$* -a_F \frac{S_F}{S} \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$
$\hat{p}$	$* -a_F \frac{S_F}{S} \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
$\hat{r}$	$* a_F \frac{S_F}{S} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

\*means contribution of the tail only, formula for wing-body not available;  $V_F/V = 1$ .

N.A. means no formula available.

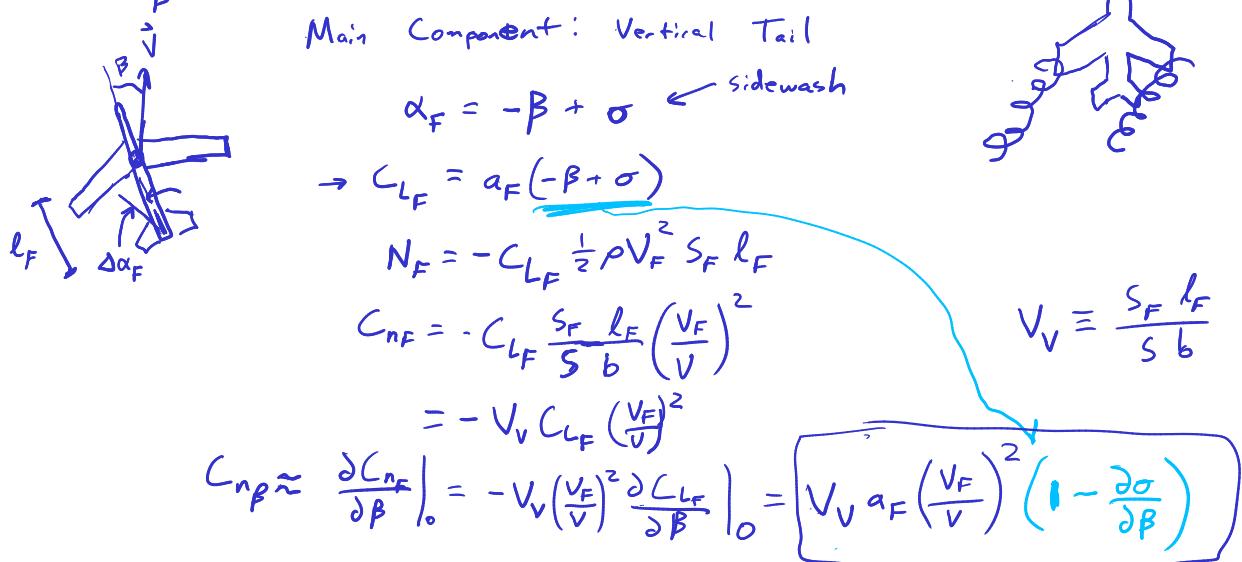
### $\beta$ derivatives

$C_{l\beta}$ : Dihedral Effect

- 1) Wing Height
- 2) Dihedral Angle
- 3) Vertical Tail
- 4) Wing Sweep

$C_{n\beta}$ : Weathervane Derivative Sign? (+)

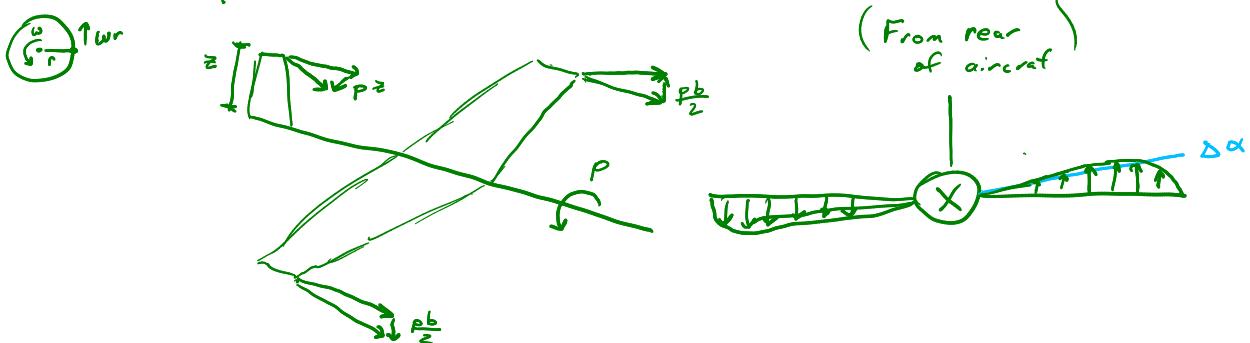
Main Component: Vertical Tail



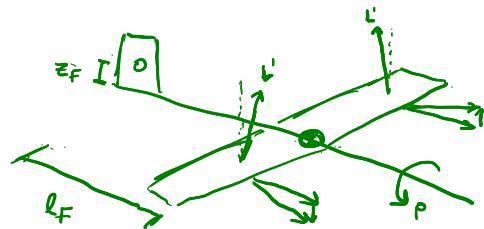
$C_{y\beta}$  (usually small) (similar derivation to  $C_{n\beta}$ )

### $p$ derivatives

$C_{l_p}$  roll damping derivative (-)



$C_{np}$  Wing Effects



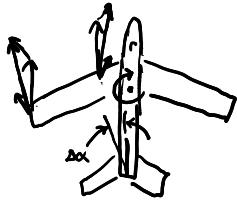
Tail Effect  $\Delta \alpha_F = -\frac{P z_F}{u_0} + P \frac{\partial \sigma}{\partial p} = -\hat{P} \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{P}} \right)$

$$(\Delta C_n)_{tail} = -\Delta C_{Y_F} \frac{S_F}{S} \frac{l_F}{b} = \alpha_F V_V \hat{P} \left( 2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{P}} \right)$$

$$(C_{np})_{tail} = \alpha_F V_V \left( 2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{P}} \right)$$

$C_{Yp}$  (usually small) (Similar derivation to  $(C_{np})_{tail}$ )

r-derivatives



$$\Delta \alpha_F = \frac{r l_F}{u_0} + r \frac{\partial \sigma}{\partial r}$$

$$= \hat{r} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{Yr})_{tail} = \alpha_F \frac{S_F}{S} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

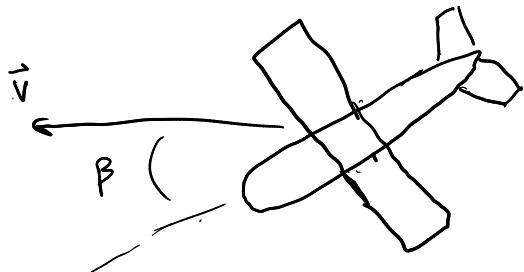
$$(C_{lr})_{tail} = \alpha_F \frac{S_F}{S} \frac{z_F}{b} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{nr})_{tail} = -\alpha_F V_V \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$
← yaw damping derivative



## 2 steady flight conditions (not "level")

### Sideslip



$$\Delta Y + mg \Delta \phi = 0$$

$$\Delta L = 0$$

$$\Delta N = 0$$

can incorporate  
1-engine out

$$\Delta Y = Y_v \Delta v + Y_p \delta_r + Y_a \delta_a + Y_s \delta_r$$

$$\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \Delta \phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} \Delta v$$

invert

$\beta u_0$

## Steady Sideslip

For Piper Cherokee



$$\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix}$$

$\beta u_0$  (7.8,4)

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that  $\beta \doteq v/u_0$ , with  $u_0$  given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\delta_r/\beta = .303$$

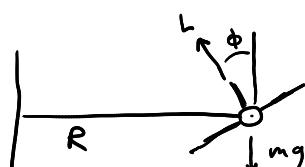
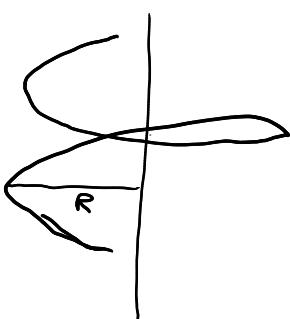
$$\delta_a/\beta = -2.96$$

$$\phi/\beta = .104$$

We see that a positive sideslip (to the right) of say  $10^\circ$  would entail left rudder of  $3^\circ$  and right aileron of  $29.6^\circ$ . Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only  $1^\circ$  to the right so the sideslip is almost flat.

### Coordinated Turn

- angular velocity vector is constant and aligned with inertial  $\hat{z}$
- No aerodynamic forces in  $Y$  direction



$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \sin \phi \cos \theta \\ \cos \phi \cos \theta \end{bmatrix} \omega$$



Since there is increased need for lift due to bank angle, involves both lat and lon dynamics

$$L \cos \phi = mg \cos \theta$$

$$L \sin \phi = m \frac{u_0^2}{R}$$

$$= m \omega u_0$$

$$\tan \phi = \frac{L \sin \phi}{L \cos \phi} = \frac{m \omega u_0}{mg \cos \theta} = \boxed{\frac{\omega u_0}{g \cos \theta}}$$