

# Final Review



ASEN 3128 Aircraft Dynamics  
UNIVERSITY OF COLORADO BOULDER

# Announcements

- Final Exam: Saturday, May 4<sup>th</sup> 7:30-10pm  
Notes: 3 pages      Front/Back
- Last Reading Quiz due Tonight
- FCQs, TA feedback due tonight (see Piazza)



# Outline

- Learning Goals
- Summary of Material

# Aircraft Dynamics

Newton's Laws	(ASEN 2703)	<i>Inertial Reference Frame</i>
+ Aerodynamics	(ASEN 2702, ASEN 3711)	<i>Body Reference Frame</i>
<hr/>		
Aircraft Dynamics (ASEN 3128)		

# Learning Goals

- How aircraft dynamics models are constructed
- How dynamics are simulated numerically
- How dynamical behavior is understood and specified
- How control is designed to achieve behavioral objectives
- Specific understanding of “how aircraft work” for a multirotor drone and a conventional (fixed-wing) airplane

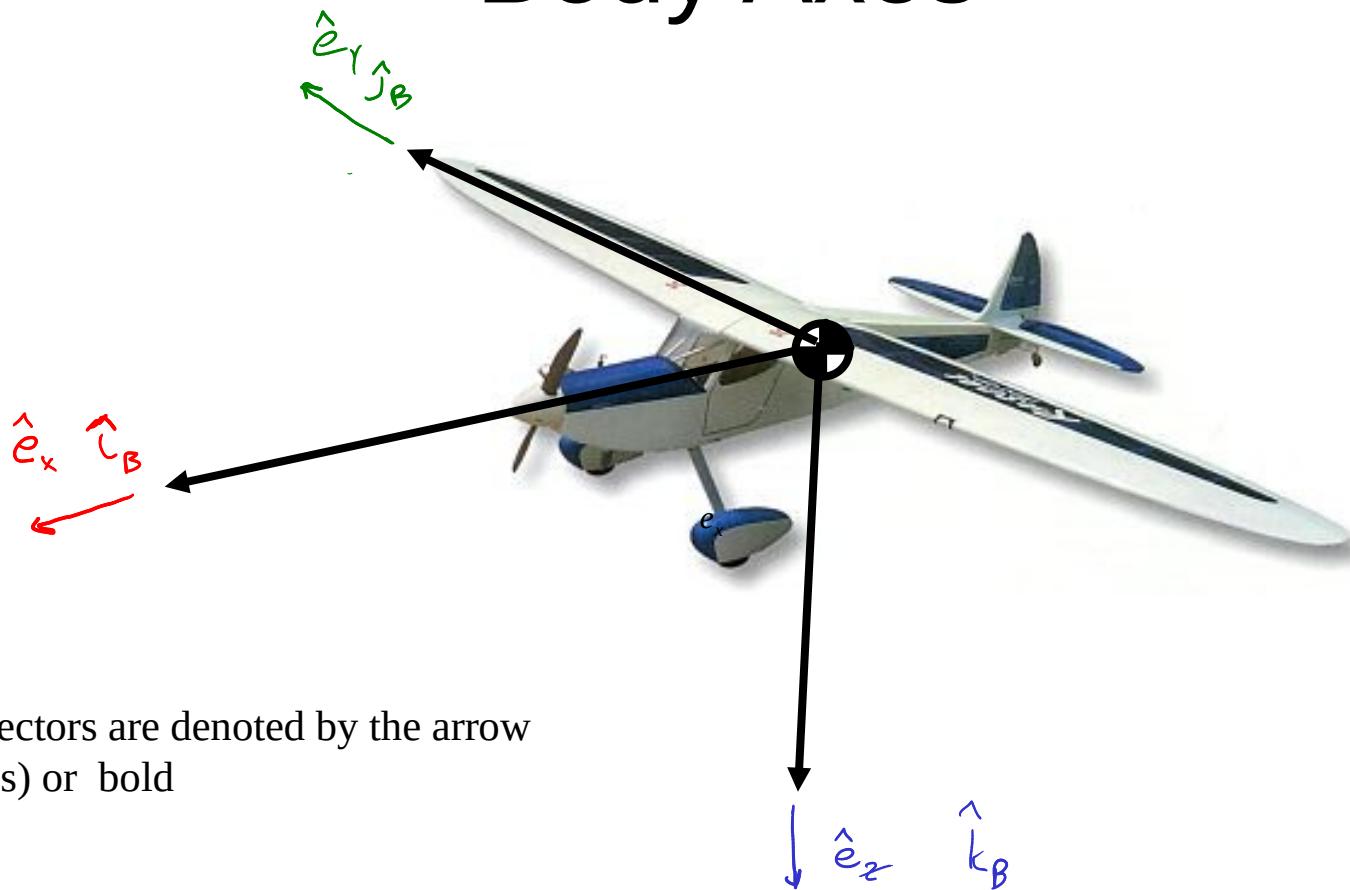


# Course Material

1. Nomenclature
2. Aircraft Static Stability
3. Euler Angles
4. Aircraft Equations of Motion (Kinematics + Dynamics)
5. Linearized Equations
6. Quadrotor Equations of Motion
7. Quadrotor Stability and Control
8. Longitudinal Dynamics of Conventional (Fixed-Wing) Aircraft
9. Lateral Dynamics of Conventional (Fixed Wing) Aircraft
10. State Space and Transfer Function-based Analysis
11. Mode Approximations
12. Conventional Aircraft Control



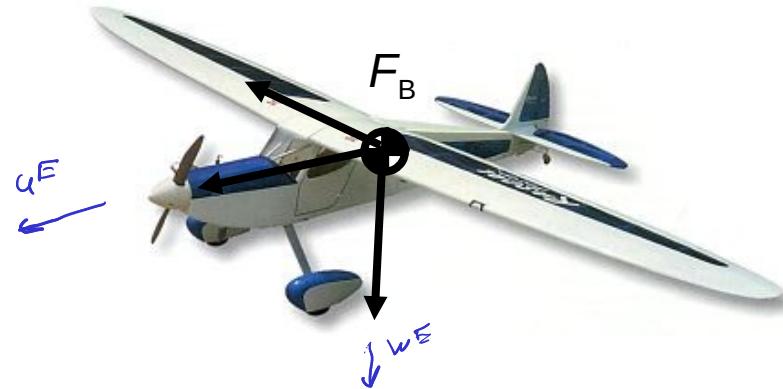
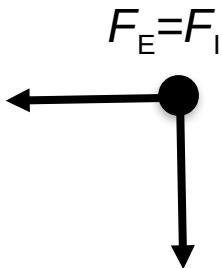
# Body Axes



Note, vectors are denoted by the arrow  
(lectures) or bold

- Aircraft dynamics and control are best described in a coordinate system defined by the aircraft body
  - Often called the “body coordinate system”
  - Defines positive sign convention for all vectors (velocity, acceleration, force)

# Coordinate Systems

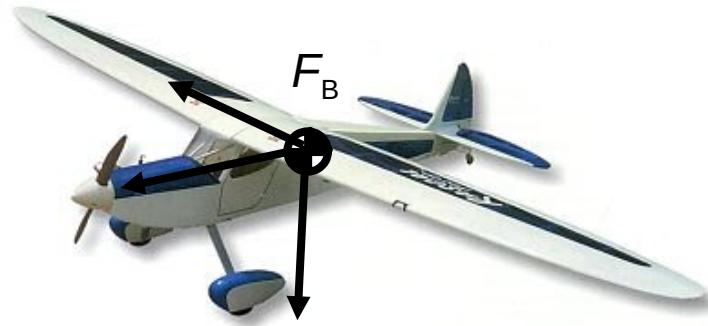
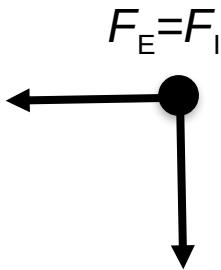


- Two main reference frames / coordinate systems
  - $F_B$  = Body coordinate system: Aerodynamic forces determined best
  - $F_E = F_I$  = Inertial coordinate system: Newton's Laws apply

- Superscripts denote what frame vector is relative to
  - $\vec{V}_B^E$  = inertial velocity written in body coordinate system
- Subscripts denote what coordinate system vector is written in

$$\vec{V}_B^E = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}$$

# Coordinate Systems



- Two main reference frames / coordinate systems
  - $F_B$  = Body coordinate system: Aerodynamic forces determined best
  - $F_E = F_I$  = Inertial coordinate system: Newton's Laws apply

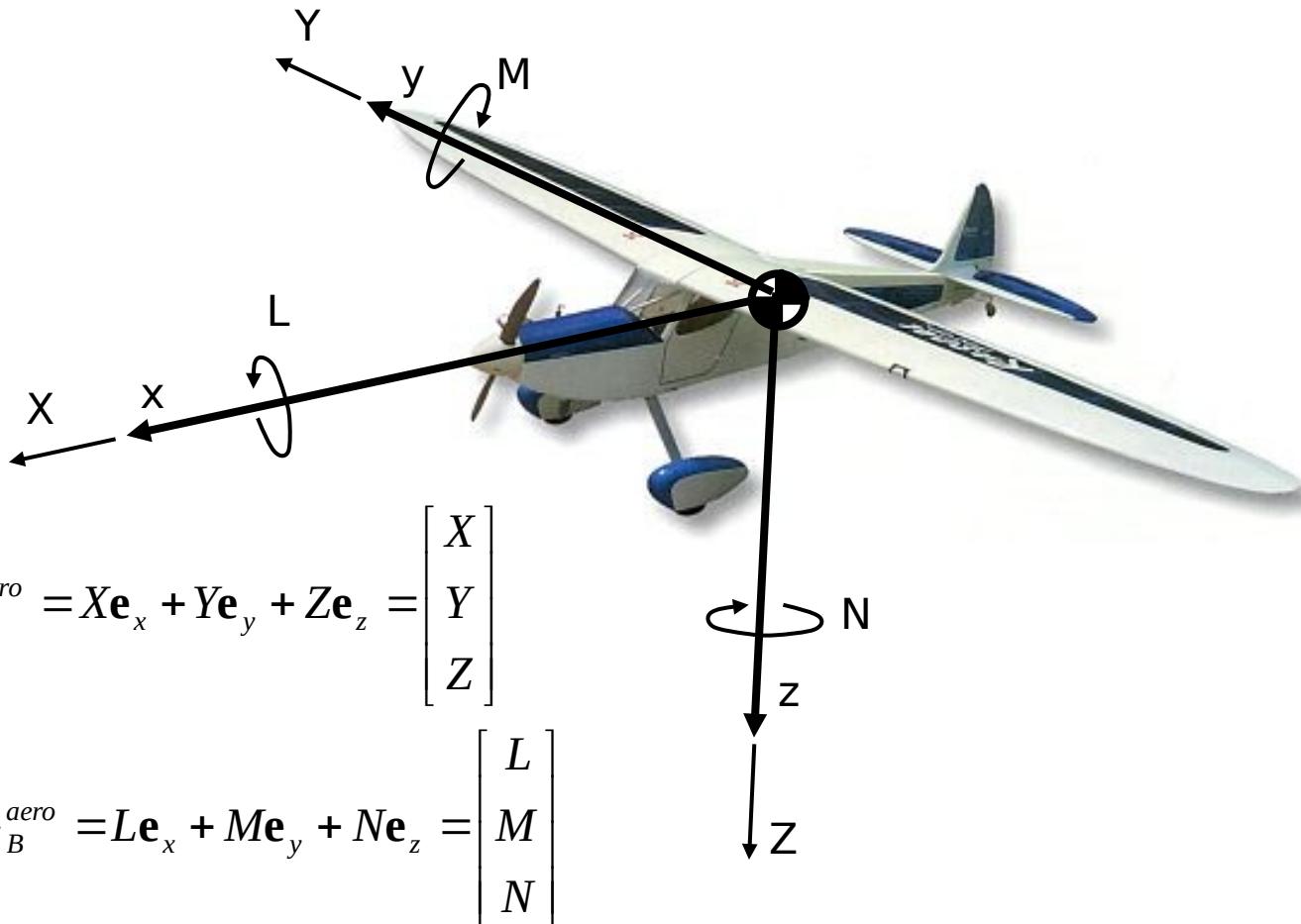
Source of most confusion deriving A/C equations of motion is representing inertial vectors in body coordinate systems

- Superscripts denote what frame vector is relative to

$\overset{\searrow}{V_B^E}$  = inertial velocity written in body coordinate system

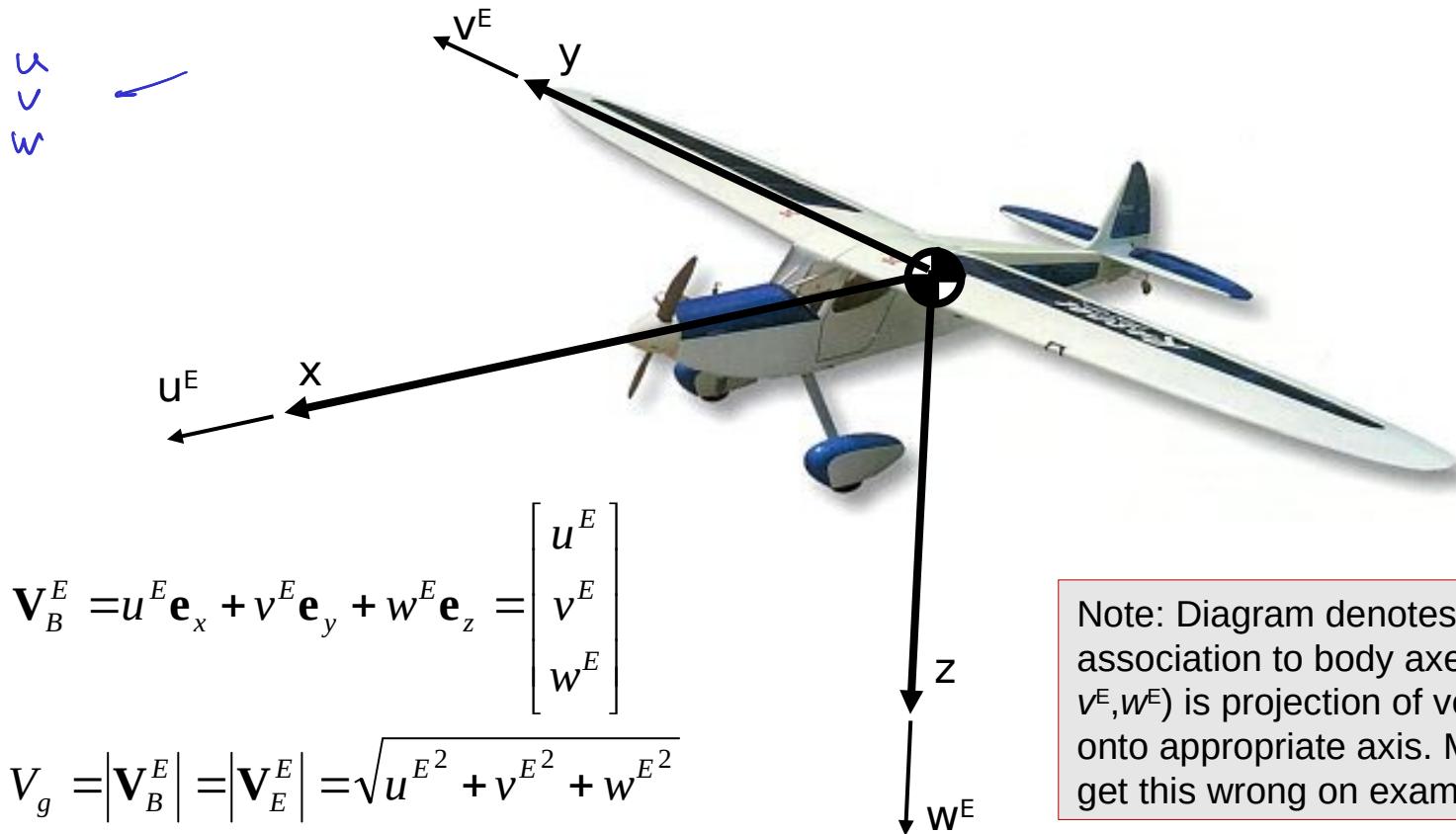
- Subscripts denote what coordinate system vector is written in

# Aerodynamic Forces and Moments



- Aerodynamic Force and moment components are represented by capital letters

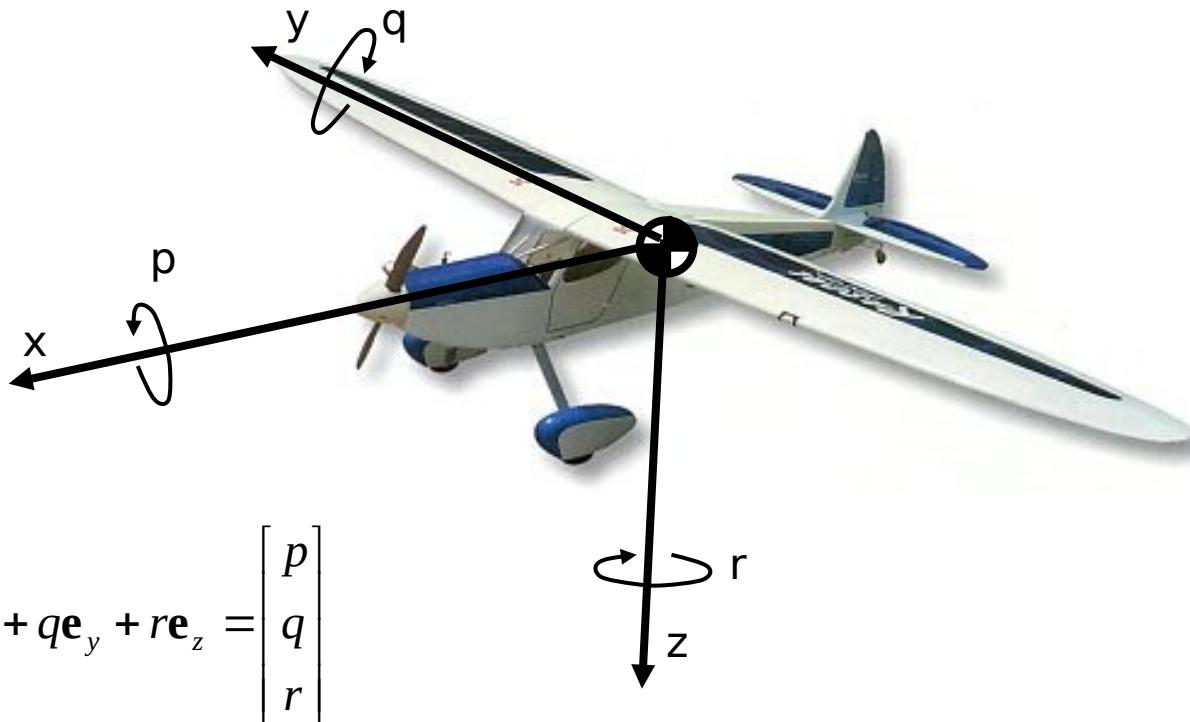
# Inertial Velocity Components



Note: Diagram denotes variable association to body axes. Value of  $(u^E, v^E, w^E)$  is projection of velocity vector onto appropriate axis. Many students get this wrong on exams!

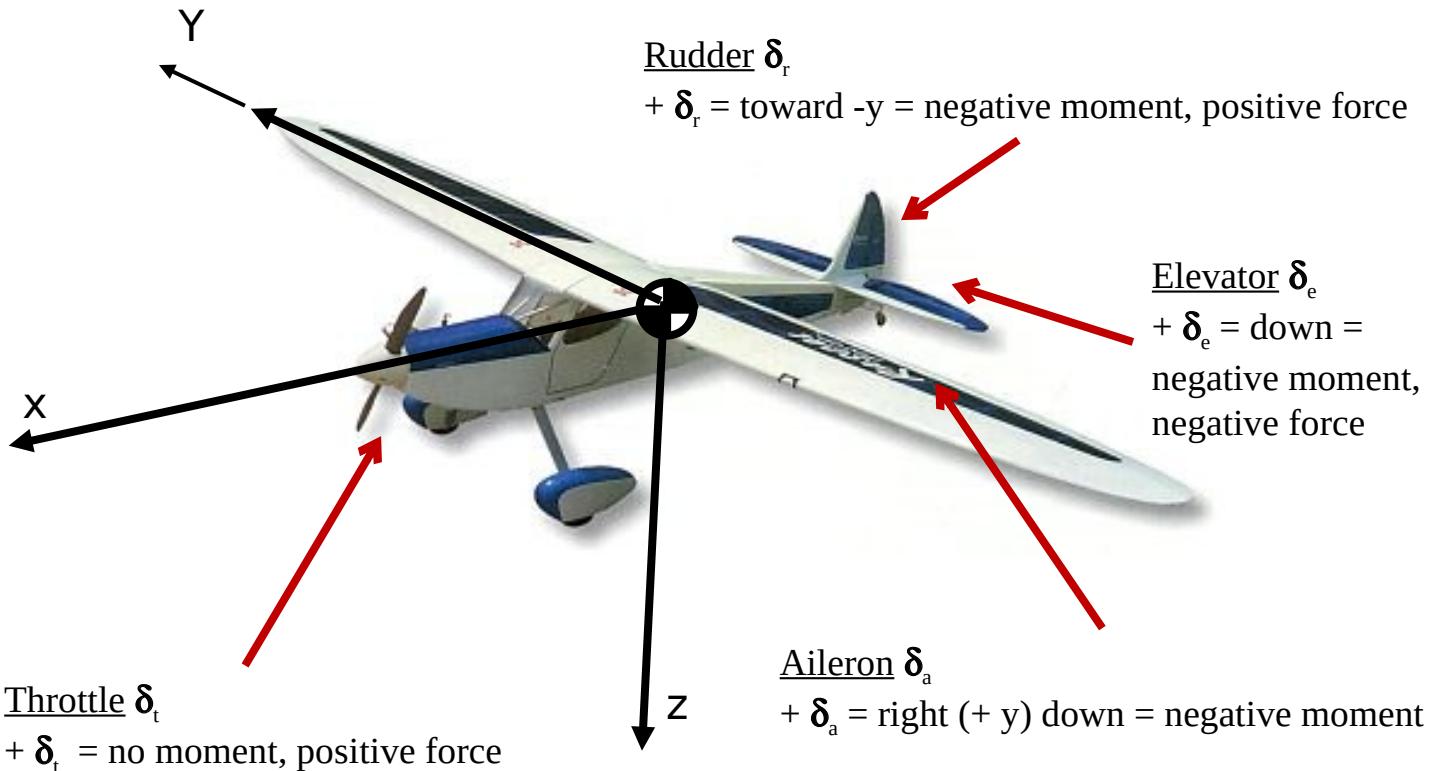
- The inertial (linear) velocity is express in body coordinates
  - Aircraft ground speed  $V_g$  can be derived from GPS measurements

# Inertial Angular Velocity



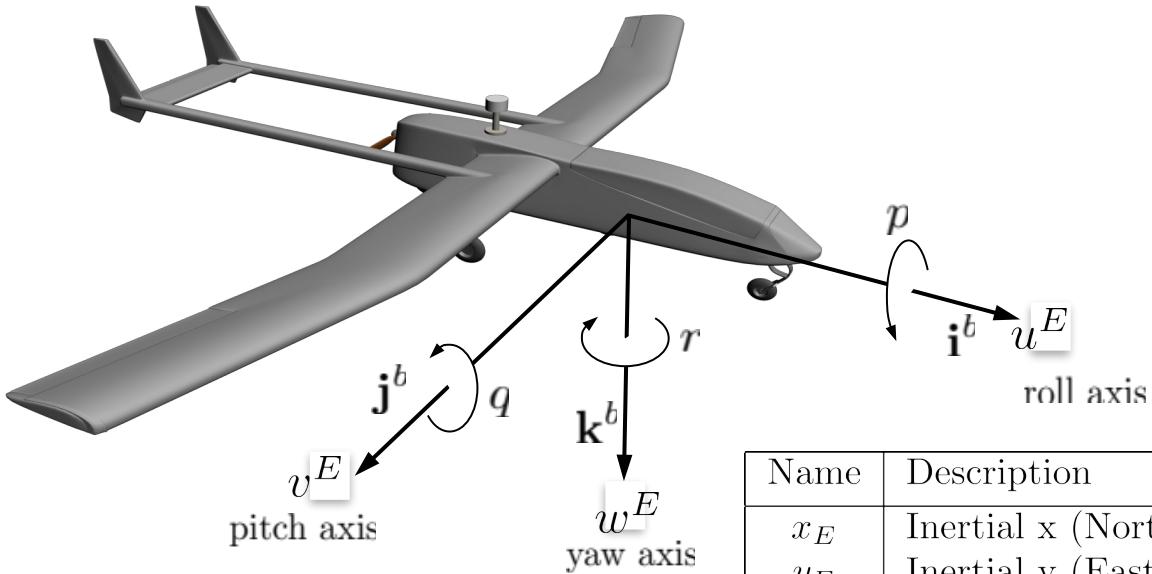
- The inertial angular velocity is also expressed in body coordinates
  - Components can be measured by rate gyros

# Four Control Surfaces



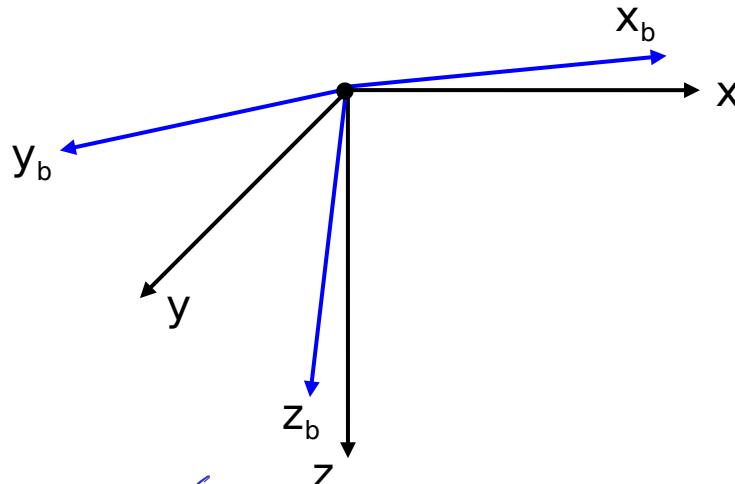
- We will focus on a standard aircraft with four control surfaces
  - 3 actual surfaces plus throttle

# Aircraft State Variables



Name	Description
$x_E$	Inertial x (North) position
$y_E$	Inertial y (East) position
$z_E$	Inertial z (Down) position
$\phi$	Roll angle
$\theta$	Pitch angle
$\psi$	Yaw angle
$u^E$	Inertial velocity vector component along $\mathbf{i}_B$
$v^E$	Inertial velocity vector component along $\mathbf{j}_B$
$w^E$	Inertial velocity vector component along $\mathbf{k}_B$
$p$	Angular velocity vector component along $\mathbf{i}_B$ (Roll rate)
$q$	Angular velocity vector component along $\mathbf{j}_B$ (Pitch rate)
$r$	Angular velocity vector component along $\mathbf{k}_B$ (Yaw rate)

# Inertial Frame to Body Frame Transformation



$$\vec{v}_B^E = R_E^B \vec{v}_E^E$$

$$R_E^B = R_E^B{}^T$$

$$\vec{v}_E^E = R_E^{B^T} \vec{v}_B^E$$

~~$$R_E^B(\phi, \theta, \psi) = R_{v2}^B(\phi) R_{v1}^E(\theta) R_E^{v1}(\psi)$$~~

$$\begin{aligned}
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}
 \end{aligned}$$

$c_\theta = \cos \theta$   
 $s_\theta = \sin \theta$   
etc.



# Vector Derivatives

Consider the vector  $\mathbf{p}$

$\frac{d\mathbf{p}}{dt}$  is the time rate of change of the vector  $\mathbf{p}$

If  $\mathbf{p}$  was the inertial position of an aircraft, then  $\frac{d\mathbf{p}}{dt}$  would be the inertial velocity.

# Vector Derivatives

Now consider the vector  $\mathbf{p}_B$  written in a particular coordinate frame, e.g. the body coordinate system

$$\mathbf{p}_B = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$$

The time rates of change of the individual components of the vector is defined to be

$$\dot{\mathbf{p}}_B = \begin{pmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{pmatrix}$$



If the reference frame  $F_B$  is rotating then

$$\frac{d\mathbf{p}_B}{dt} \neq \dot{\mathbf{p}}_B$$

# Differentiation of a Vector

Kinematic Transport Theorem

$$\frac{d}{dt} \mathbf{p}_B = \dot{\mathbf{p}}_B + (\boldsymbol{\omega}_B \times \mathbf{p}_B) = \dot{\mathbf{p}}_B + \tilde{\boldsymbol{\omega}}_B \mathbf{p}_B$$

$$\tilde{\boldsymbol{\omega}}_B = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$$

# Rotational Kinematics

$$\begin{aligned}
 \begin{pmatrix} p \\ q \\ r \end{pmatrix} &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}_{v2}^b(\phi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathcal{R}_{v2}^b(\phi) \mathcal{R}_{v1}^{v2}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
 &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}
 \end{aligned}$$

Inverting gives:

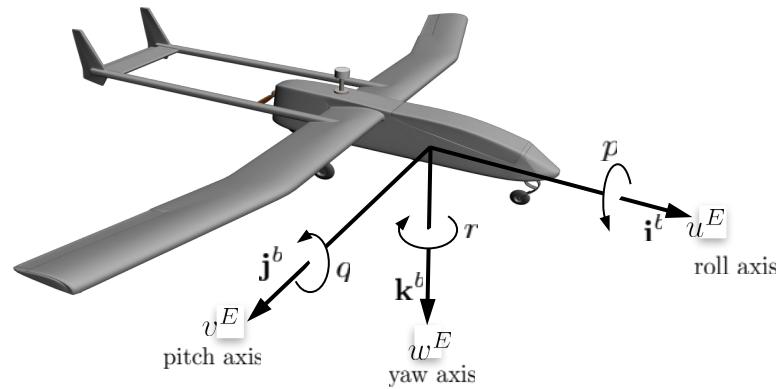
Not rotation matrices  
(not orthonormal)

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Vector form:  $\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \boldsymbol{\omega}_B$

↑ attitude      ↓ influence

My notation



# Kinematic Equations

Six of the 12 state equations for the aircraft come from the kinematic equations relating positions and velocities:

$$\frac{d\mathbf{p}_E^E}{dt} = R_B^E \cdot \mathbf{v}_B^E$$

“Orientation vector” not in a coordinate system  
(Also referred to as “attitude” or just “Euler angles”)

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \boldsymbol{\omega}_B$$

The remaining six equations will come from applying Newton’s 2<sup>nd</sup> law to the translational and rotational motion of the aircraft.

# Dynamic Equations

The remaining six equations for the aircraft come from the dynamic equations relating derivatives of velocities and forces and moments:

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} [-\tilde{\omega}_B (\mathbf{I}_B \omega_B) + \mathbf{G}_B]$$

# Rotational Dynamics

$$\dot{\omega}_B = \mathbf{I}_B^{-1} [-\tilde{\omega}_B (\mathbf{I}_B \omega_B) + \mathbf{G}_B]$$
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \underbrace{\begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix}}_{\text{underbrace}} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y}$$

$$\Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma = I_x I_z - I_{xz}^2$$



University of Colorado  
Boulder

Same as Eq. 4.5,9, though book does not give the equations in this form

# Equation of Motion in Vector Form

$$\dot{\mathbf{p}}_E^E = R_B^E \mathbf{v}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \boldsymbol{\omega}_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\boldsymbol{\omega}}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\boldsymbol{\omega}}_B = \mathbf{I}_B^{-1} [-\tilde{\boldsymbol{\omega}}_B (\mathbf{I}_B \boldsymbol{\omega}_B) + \mathbf{G}_B]$$

# Equation of Motion Summary

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

# Quadrotor Equations of Motion

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

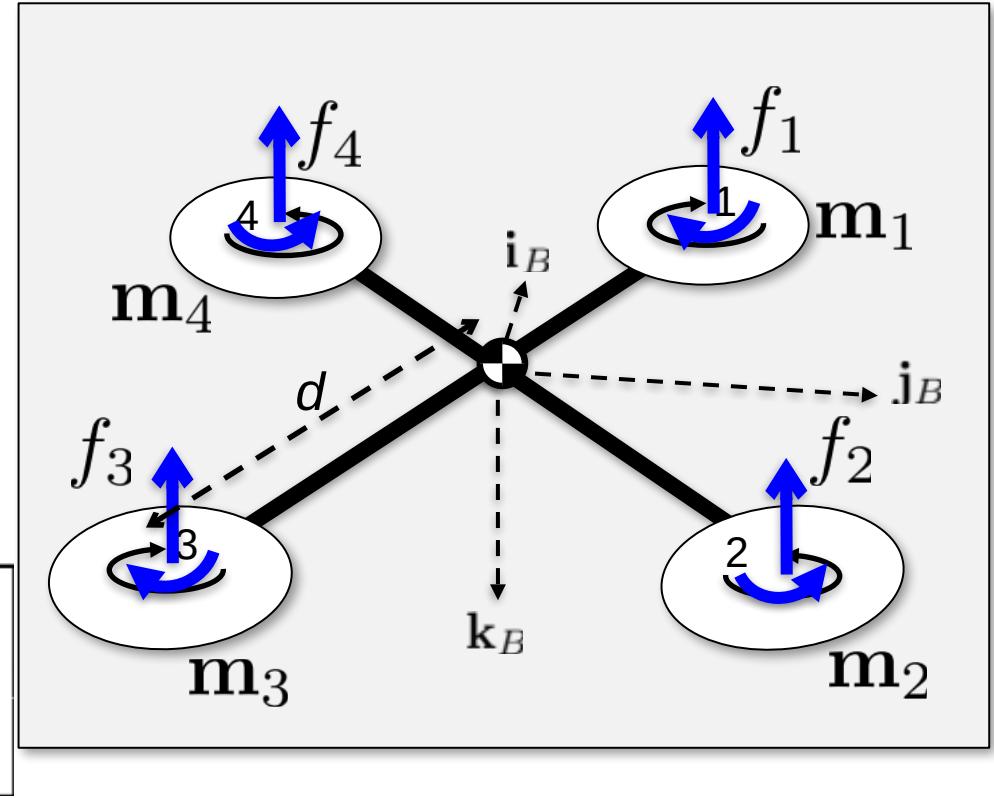
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

# Control Forces and Moments

$${}^c\mathbf{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^c\mathbf{m}_B = \begin{bmatrix} \frac{d}{\sqrt{2}} (-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{\sqrt{2}} (f_1 - f_2 - f_3 + f_4) \\ m_1 - m_2 + m_3 - m_4 \end{bmatrix}$$



# Equilibrium and Static Stability

- Equilibrium = The states do not change without a disturbance or control action
  - Derivative of the state vector is zero (almost)
- Static stability = Consider the equilibrium condition and whether aircraft returns to condition after small perturbation
- Dynamic Stability = Details of how a system returns to equilibrium
- For aircraft we call the equilibrium condition the “**trim condition**”
- For fixed-wing aircraft the trim condition is
  - “Straight, level, unaccelerated flight (SLUF)”
- For a quadrotor the trim state is
  - “Hovering”
  - “Steady hover”

# Trim Condition

In a “**steady hover**” equilibrium condition

$$x_{E,0} = \cdot$$

$$\gamma_{E,0} = \cdot$$

$$z_{E,0} = \cdot$$

$$\dot{x}_{E,0} = 0$$

$$\dot{y}_{E,0} = 0$$

$$\dot{z}_{E,0} = 0$$

$$u_0^E = 0$$

$$v_0^E = 0$$

$$w_0^E = 0$$

$$\dot{u}_0^E = 0$$

$$\dot{v}_0^E = 0$$

$$\dot{w}_0^E = 0$$

$$p_0 = 0$$

$$q_0 = 0$$

$$r_0 = 0$$

$$\dot{p}_0 = 0$$

$$\dot{q}_0 = 0$$

$$\dot{r}_0 = 0$$

$$\phi_0 = 0$$

$$\theta_0 = 0$$

$$\psi_0 = \cdot$$

Subscript 0 denotes  
trim value

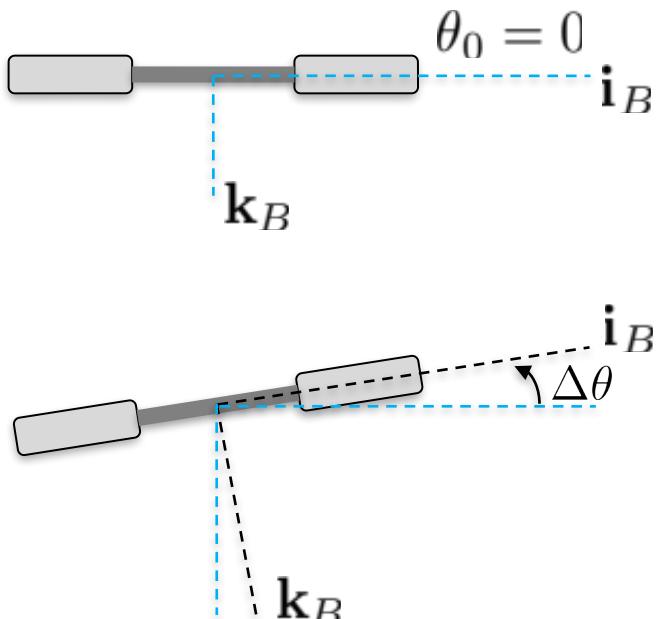
Any constant value



# Small Variations

Can describe variables as reference condition (“trim condition”) plus small variation, e.g  $u^E = u = u_0 + \Delta u$

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta\phi \\ \Delta\theta \\ \psi_0 + \Delta\psi \\ \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$



# Linearization

$$f(x, u)$$

Option 1

$$f(x_0 + \Delta x, u_0 + \Delta u)$$

expand, assume  $\Delta$  small

Ex -

$$y = xu$$
$$x_0 + \Delta y \approx (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + x_0 \Delta u + u_0 \Delta x + \cancel{\Delta x \Delta u}^0$$

$$\Delta y \approx x_0 \Delta u + u_0 \Delta x$$

Option 2

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x} \Big|_{x_0} \Delta x + \frac{\partial f}{\partial u} \Big|_{u_0} \Delta u$$



# Linearized Quadrotor EOM

## Longitudinal

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{u} \\ \Delta\dot{\theta} \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g\Delta\theta \\ \Delta q \\ \frac{1}{I_y}\Delta M_c \end{pmatrix}$$

## Lateral

$$\begin{pmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_r}\Delta L_c \end{pmatrix}$$

## Vertical

$$\begin{pmatrix} \Delta\dot{z}_E \\ \Delta\dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m}\Delta Z_c \end{pmatrix}$$

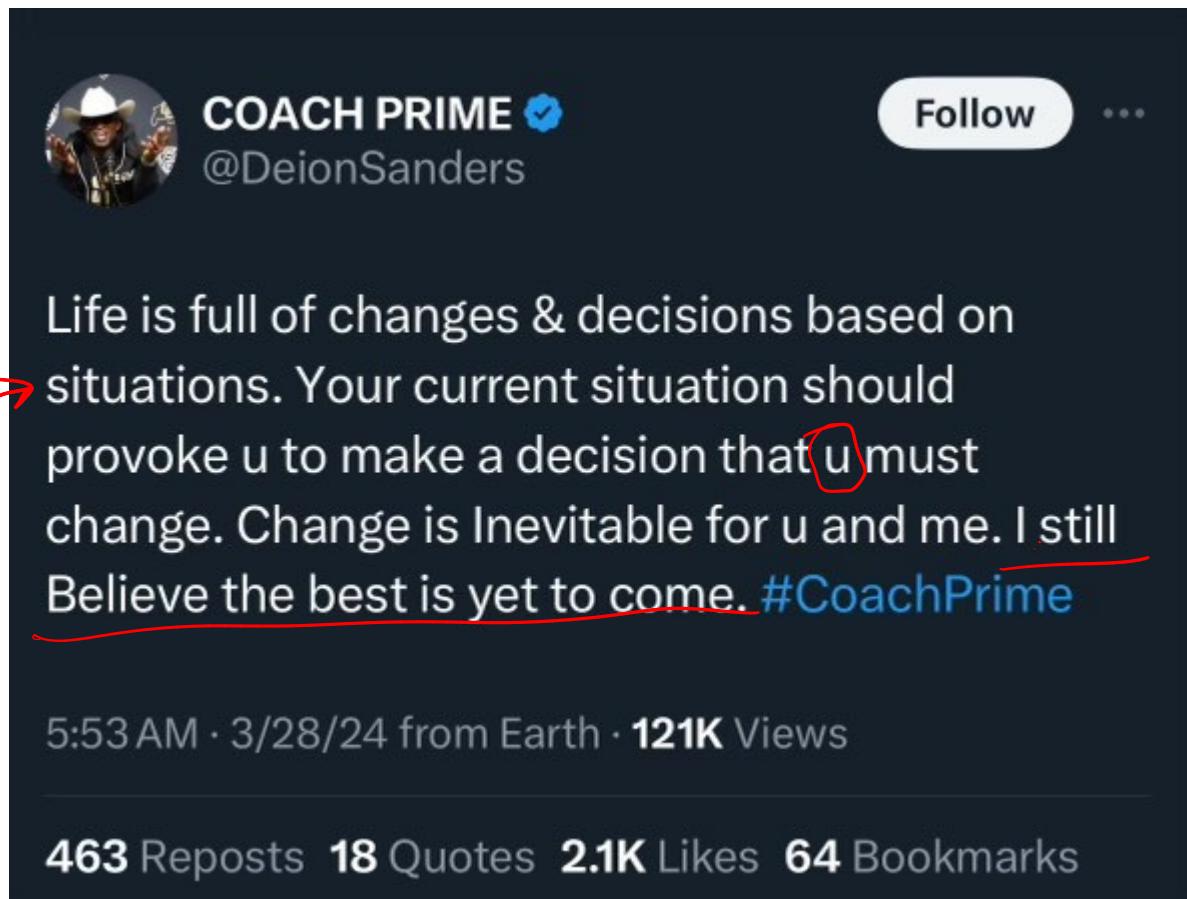
## Spin

$$\begin{pmatrix} \Delta\dot{\psi} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

# Unleash the Memes!



# My understanding of control systems:



COACH PRIME   
@DeionSanders

Follow ...

Life is full of changes & decisions based on situations. Your current situation should provoke u to make a decision that u must change. Change is Inevitable for u and me. I still Believe the best is yet to come. #CoachPrime

5:53 AM · 3/28/24 from Earth · 121K Views

463 Reposts 18 Quotes 2.1K Likes 64 Bookmarks

Optimal

Annotations: A red arrow points from the word 'Optimal' to the first sentence. Another red arrow points from the word 'Optimal' to the word 'must'. A red circle highlights the word 'u' in the sentence 'provoke u to make a decision that u must change'. A red underline is placed under the phrase 'Believe the best is yet to come.'

# Longest Flight Ever?

# Longest Flight Ever?

64 days, 22 hours, 19 minutes

# Longest Flight Ever?

64 days, 22 hours, 19 minutes



- Oil changes
- Several hours when both pilots were asleep

# Break



# Course Material

1. Nomenclature
2. Aircraft Static Stability
3. Euler Angles
4. Aircraft Equations of Motion (Kinematics + Dynamics)
5. Linearized Equations
6. Quadrotor Equations of Motion
7. Quadrotor Stability and Control
8. Longitudinal Dynamics of Conventional (Fixed-Wing) Aircraft
9. Lateral Dynamics of Conventional (Fixed Wing) Aircraft
10. State Space and Transfer Function-based Analysis
11. Mode Approximations
12. Conventional Aircraft Control



# Linear ODEs

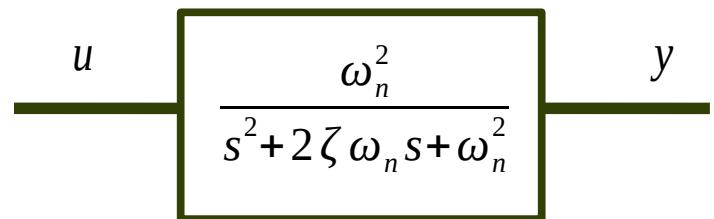
$$\begin{aligned}\dot{x} &= A x + B u \\ y &= C x + D u\end{aligned}$$

Aircraft dynamics (mostly) uses this one  
It is called a “state space” model.

These contain (roughly) the same information!

Always keep this in mind!

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$$

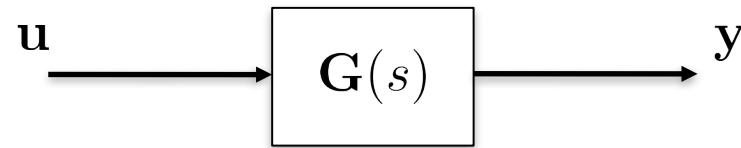


# MIMO Linear Systems

- Two representations

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

State Space Model



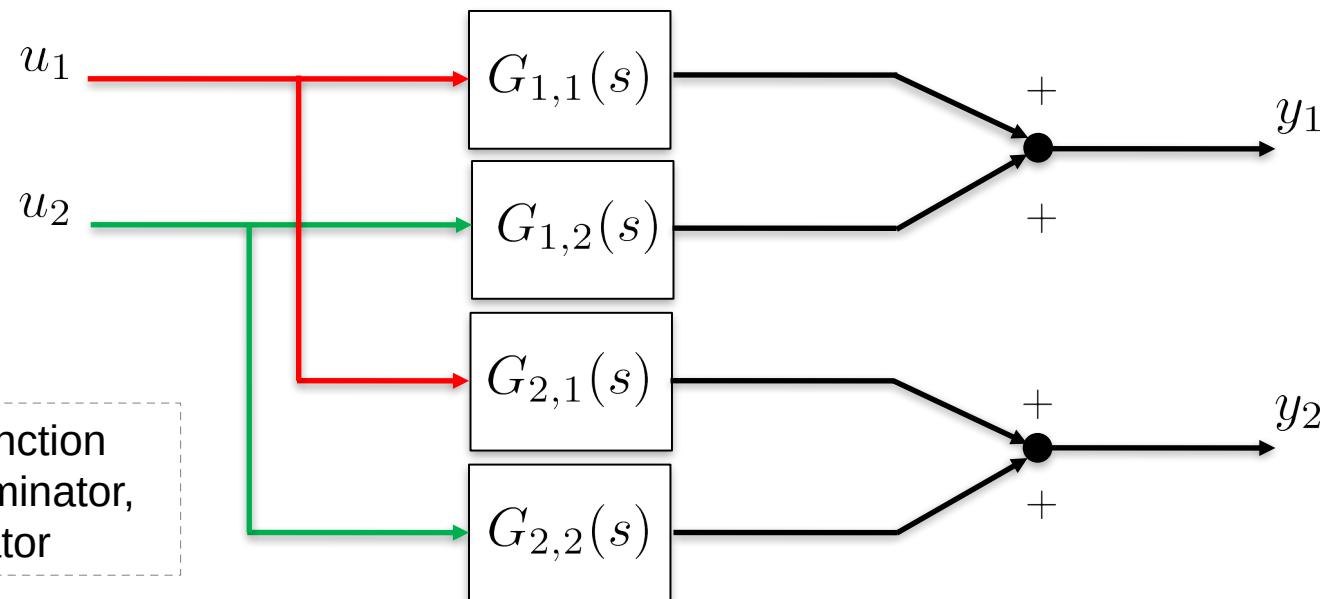
Transfer Function  
Matrix

# MIMO Linear Systems

- Two representations

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$



# Translating between representations

SS  $\rightarrow$  TF

$$G_{yu}(s) = C(sI - A)^{-1}B = \frac{C \text{adj}(sI - A) B}{|sI - A|}$$

adj = transpose of cofactor

TF  $\rightarrow$  SS

$$G_{yu}(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$a_0 = 1$  →

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & 0 & \cdots & 0 \\ & & 0 & \ddots & 1 & 0 \\ & & & \ddots & 0 & 1 \\ & & & & 0 & 1 \\ a_n & a_{n-1} & \cdots & a_1 & & \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [b_n - a_n b_0, \dots, b_1 - a_1 b_0]$$

$$D = [0]$$



# Laplace Transforms

$$\frac{dx(t)}{dt} \Leftrightarrow s x(s) \quad (\text{assuming zero initial})$$

$$\int_0^t x(\tau) d\tau \Leftrightarrow \frac{1}{s} x(s)$$

$$\alpha x(t) + \beta y(t) \Leftrightarrow \alpha x(s) + \beta y(s)$$

$$\text{step : } u(s) = \frac{1}{s}$$

$$\text{impulse : } u(s) = 1$$

$$G_{xu}(s) = \frac{x(s)}{u(s)} \qquad x(s) = G_{xu}(s) u(s)$$

# Modal Analysis

When initial condition is

$$\mathbf{x}(0) = \sum_{i=1}^n k_i \mathbf{v}_i$$

The solution is

$$\mathbf{x}(t) = \sum_{i=1}^n k_i e^{\lambda_i \cdot t} \mathbf{v}_i$$

eigenvalue

eigenvector

Eigenvalues are solutions to the characteristic equation,

For second order system,

but the characteristic equation could be higher order!

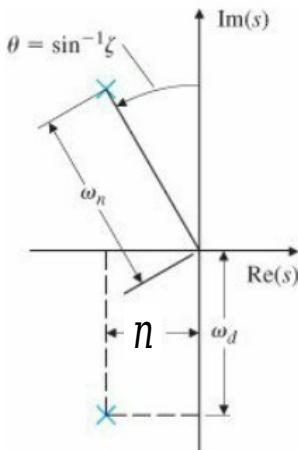
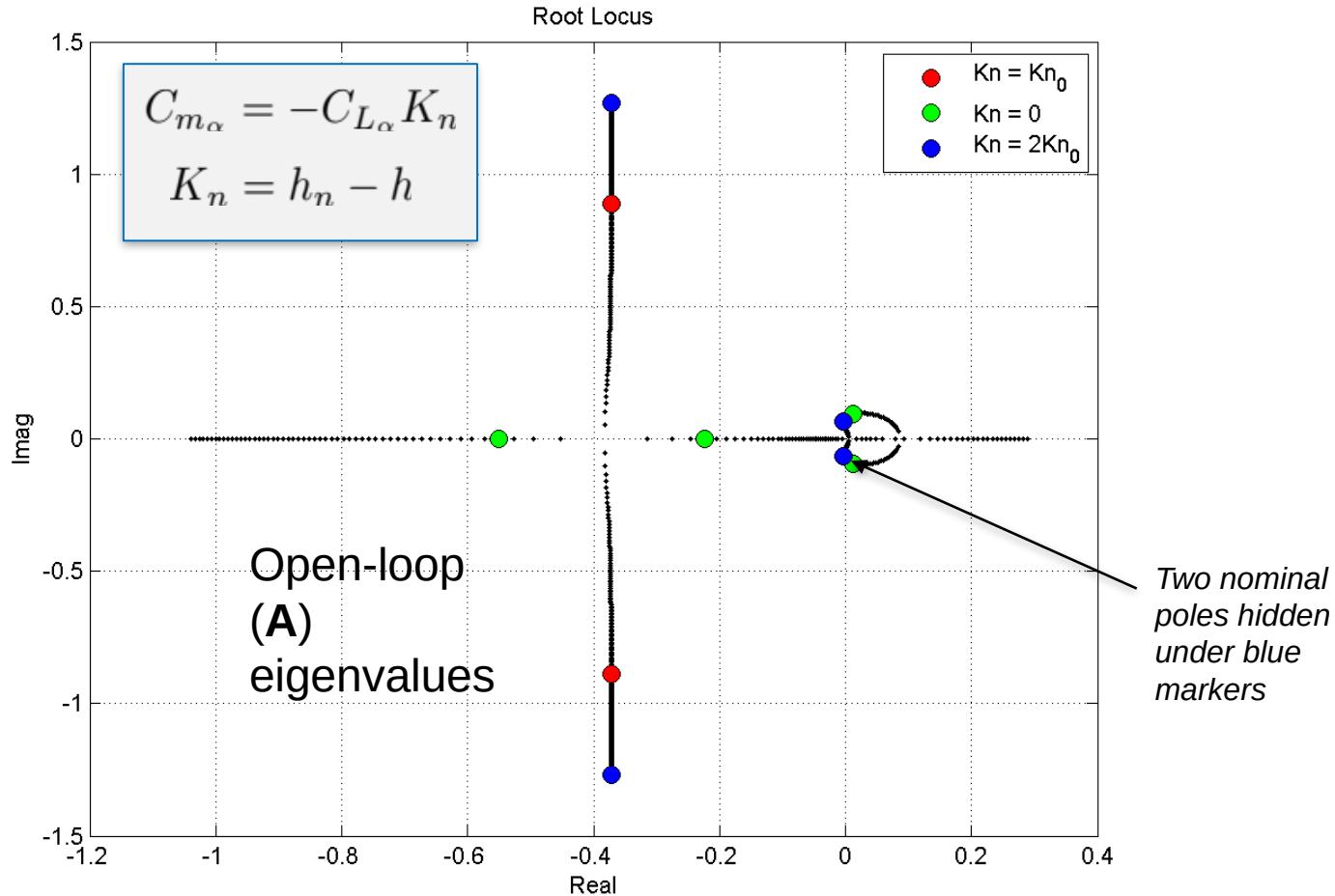


Image from Franklin, Powell, and Emami-Naeini

Roots of the transfer function denominator are the eigenvalues of A!

# Root Locus: Effect of CG Location



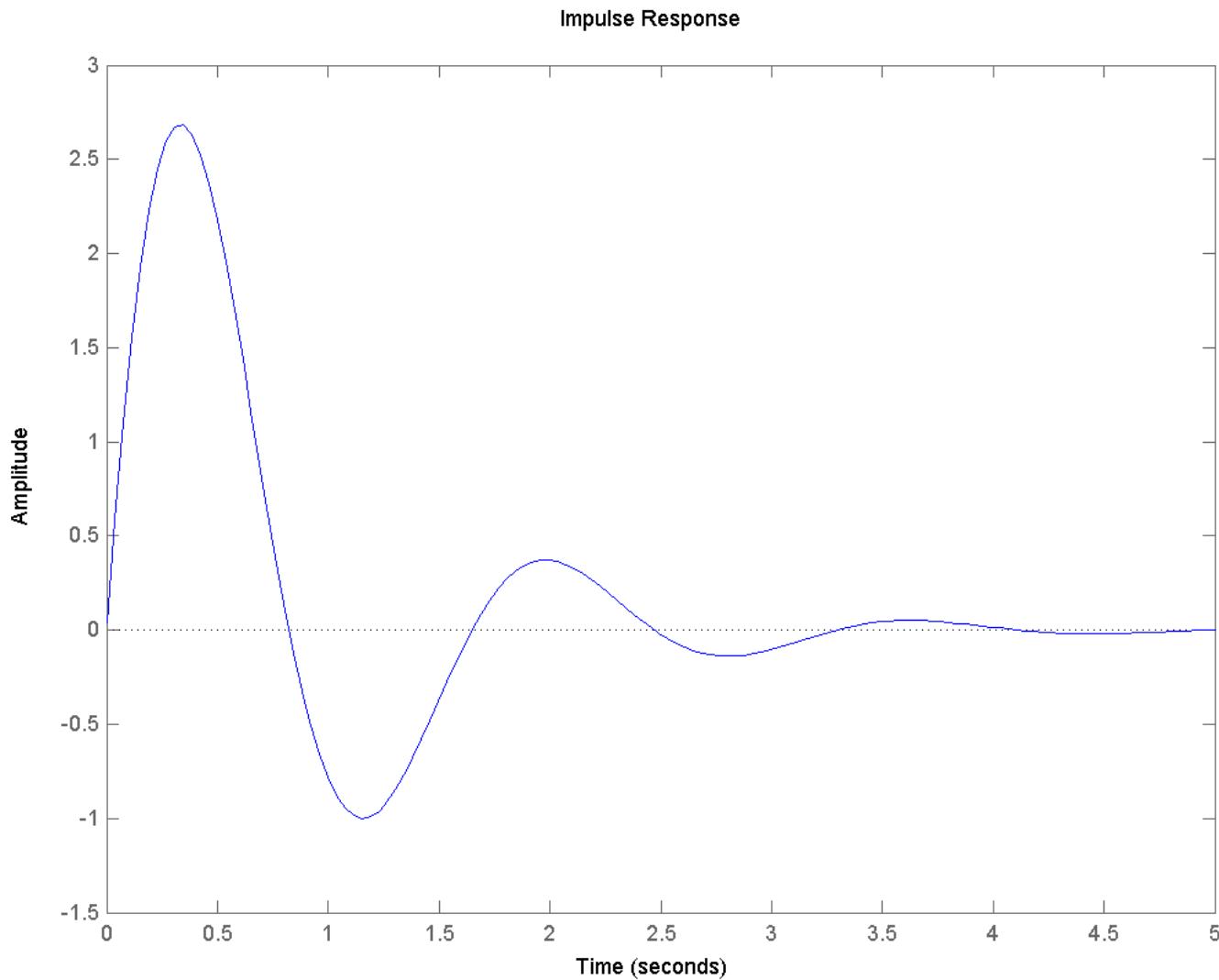
# Impulse Response Example

**Second order system**

$$G(s) = \frac{16}{s^2 + 2.4s + 16}$$

$$\zeta = 0.3$$

$$\omega_n = 4$$



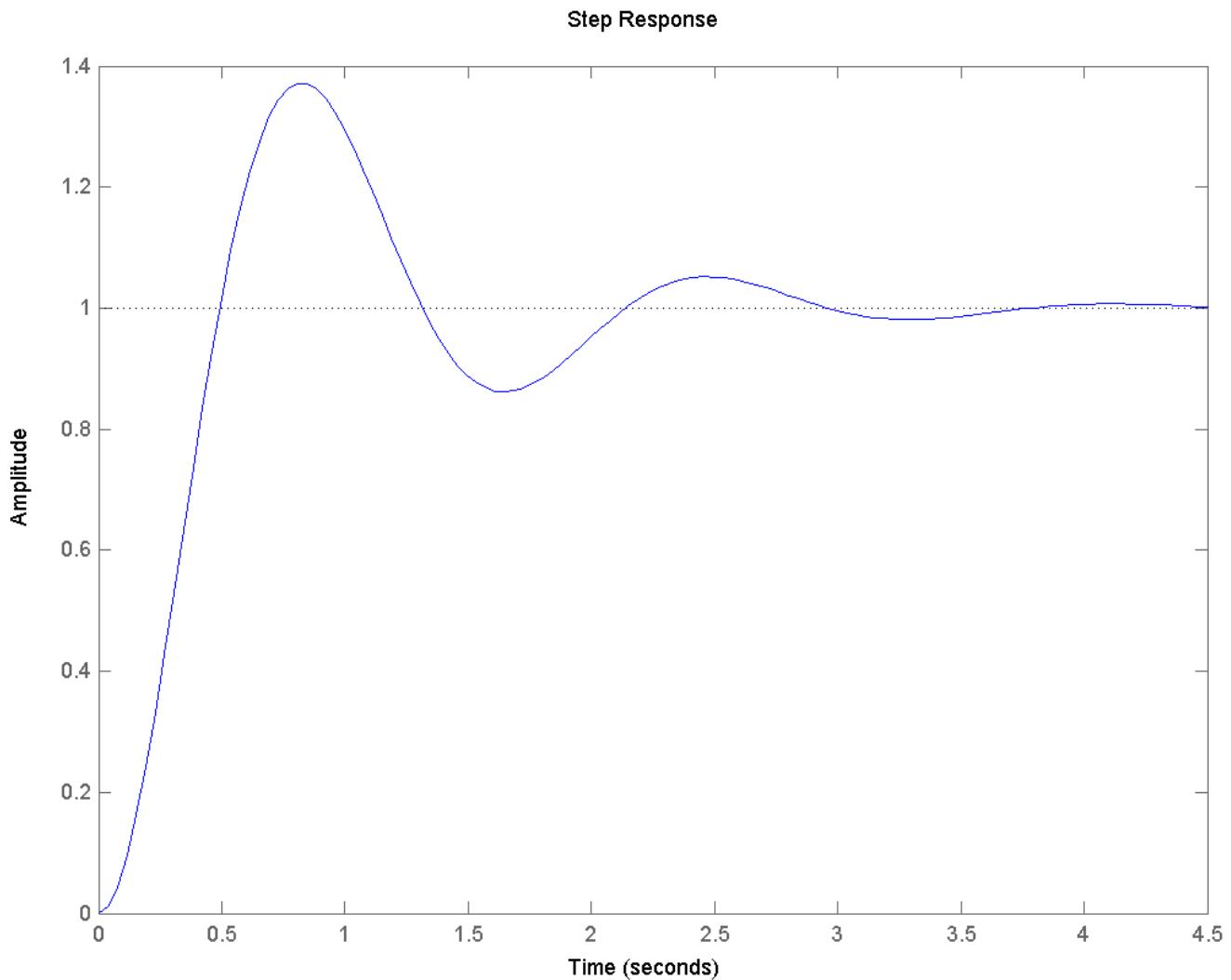
# Step Response Example

**Second order system**

$$G(s) = \frac{16}{s^2 + 2.4s + 16}$$

$$\zeta = 0.3$$

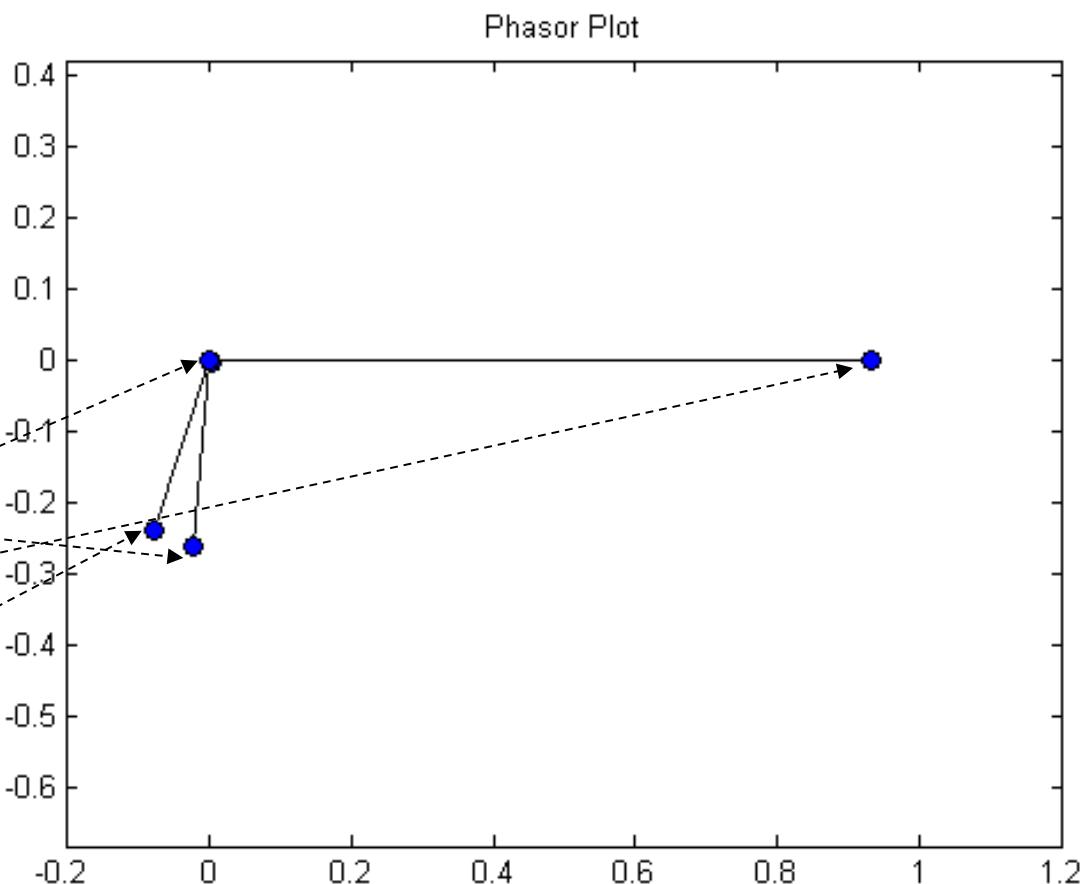
$$\omega_n = 4$$



# Phasor Plot

$$\mathbf{v}_1 = \begin{bmatrix} 0.0032 - 0.0021i \\ -0.0222 - 0.2608i \\ 0.9321 \\ -0.0755 - 0.2389i \end{bmatrix}$$

$$= \begin{bmatrix} 0.0038\angle -33.5239^\circ \\ 0.2617\angle -94.8632^\circ \\ 0.9321\angle 0.0^\circ \\ 0.2505\angle -107.5398^\circ \end{bmatrix}$$



# Feedback control

Let's consider a **feedback control law**

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi$$

which we can implement by measuring the roll rate and roll angle and "feeding them back" into the command for the roll moment.

# Closed Loop Behavior

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Original system, “plant”



$$\mathbf{u} = -\mathbf{Kx}$$

Control law

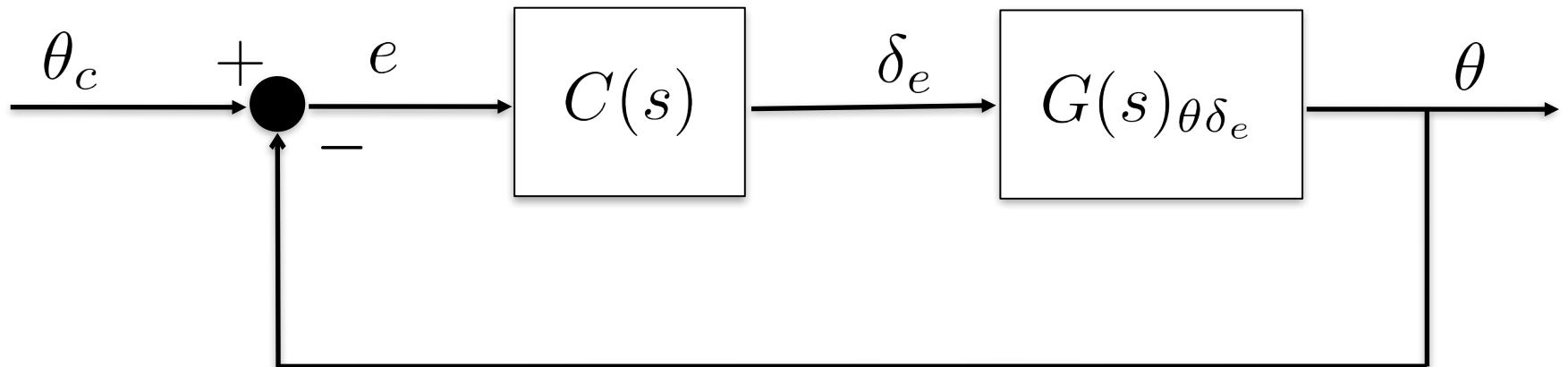
---

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK}) \mathbf{x}$$

Closed loop dynamics

The eigenvalues and eigenvectors of this closed loop state space matrix describe the “closed loop behavior”.

# Feedback Control: Pitch



$$G_{\theta \theta_c} = \frac{C G_{\theta \delta_e}}{1 + C G_{\theta \delta_e}}$$

Should be able to

- derive transfer functions for simple block diagrams
- pick gains for simple controllers

# General Approach to Control Design by Hand

1. Write down the control law
2. Determine the closed-loop representation of the system

$$A - BK \text{ or } \frac{CG}{1+CG}$$

3. Find the performance indicating variables (usually eigenvalues of closed-loop system)
4. Calculate gains to achieve goal

# Optimal Control: LQR

# Questions

# Question from Piazza

## Problem 4 (10 pts)

Consider a controller for a quadrotor aircraft that feeds back roll angle error  $\phi_c - \phi$ , where  $\phi_c$  is the commanded roll angle, and roll rate  $p$  to the roll control moment  $L_c$ . The mass and moments of inertia of the quadrotor are:  $m = 0.1$  kg,  $I_x = 2\text{e-}4$  kg m<sup>2</sup>,  $I_y = 1.6\text{e-}4$  kg m<sup>2</sup>,  $I_z = 8\text{e-}4$  kg m<sup>2</sup>.

- a. (4 pts) Use the linearized equations to design a feedback control law such that the closed loop behavior of the roll angle has natural frequency  $\omega_n = 3$  rad/sec and damping ratio  $\zeta = 0.5$ .
- b. (4 pts) Consider the outer loop feedback control law  $\phi_c = k(y_c - y_E)$  where  $\phi_c$  is the roll command passed to the controller designed for Problem 4.a and  $y_c$  is the lateral position command. What are the eigenvalues of the closed loop state matrix for the case when  $k = -2$ .

# Most Missed Quiz Questions

**Q4**

1 Point

If  $\phi = 90^\circ$  and  $\theta = 0^\circ$  and the aircraft is rotating at 10 degrees per second about the body y-axis, which of the following is true?

- $\dot{\theta} = -10^\circ/s$
- $\dot{\psi} = 10^\circ/s$
- $\dot{\theta} = 10^\circ/s$
- $\dot{\psi} = -10^\circ/s$

# Most Missed Quiz Questions

**Q4**

1 Point

A quadrotor is in steady hover. A small perturbation causes a small forward motion (+x inertial direction). The linearized equations give a resulting drag force on the aircraft that is

- a small value in the negative x direction.
- zero.
- a small value in the positive x direction.
- a small value upward.

# Most Missed Quiz Questions

**Q1**

1 Point

At trim, the quad copter is stable.

True

False



University of Colorado  
Boulder

# Most Missed Quiz Questions

**Q2**

**1 Point**

A student team designed a conventional aircraft with a horizontal tail behind the main wing. The team collected data in a wind tunnel and derived the stability and control derivatives for the aircraft. The team forgot the sign convention they used for the elevator angle but determined that  $C_{m_{\delta_e}} > 0$ . What is the sign convention used by the team for the elevator deflection?

- A positive elevator corresponds to an upward deflection.
- A positive elevator corresponds to a downward deflection.
- The angle of attack is needed to determine the sign convention.
- The area of the horizontal tail is needed to determine the sign convention.



University of Colorado  
Boulder

# Most Missed Quiz Questions

**Q2**

**1 Point**

An aircraft is flying along in trim at height  $h$  with speed  $V$ . At time  $t_0$  the pilot introduces a positive step input to the elevator angle. Which statement is TRUE?

- The natural frequency of the phugoid mode is increased by the step change in the elevator angle.
- The aircraft will oscillate, but it will eventually decay and the aircraft will return to trim at the same height and same speed.
- The short period mode, but not the phugoid mode, will be excited in the aircraft.
- The short period and phugoid modes will be excited in the aircraft.

# Most Missed Quiz Questions

**Q3**

**1 Point**

Which feature of an aircraft will increase (make more positive/less negative) the dihedral effect stability derivative?

- A positive dihedral angle on the main wings.
- The main wing mounted below the center of the fuselage.
- A positive sweep angle on the main wing.
- A vertical tail above the center of the fuselage.

# Most Missed Quiz Questions

**Q2**

**1 Point**

In addition to damping the yaw rate of an aircraft, the yaw damper has the effect of speeding up (i.e. making the eigenvalue more negative) which other aircraft mode?

- Spiral mode
- Roll mode
- Short period mode
- Phugoid mode

# Most Missed Quiz Questions

A control system designer wants to use LQR with the standard longitudinal linearized dynamics to reduce phugoid oscillations.

Which  $Q$  matrix would be most appropriate for the cost function?

$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$



# Outline

- Example Problems
- Questions

# From 2019 Exam

TRUE or FALSE?

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

# From 2019 Exam

TRUE or **FALSE?**

Improving the damping ratio of the spiral mode is a major design objective of lateral stability augmentation system design.

The spiral mode has a real eigenvalue, so it does not have a damping ratio.

# From 2019 Exam

TRUE or FALSE?

For the augmented lateral dynamical system with  $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$ , the control law  $\mathbf{u} = -\mathbf{K}^* \mathbf{x}_{lat}$  may have no effect on certain eigenvalues of the system.

# From 2019 Exam

**TRUE** or FALSE?

For the augmented lateral dynamical system with  $\mathbf{x}_{lat} = [\Delta v, \Delta p, \Delta r, \Delta \phi, \Delta \psi, \Delta y]^T$ , the control law  $\mathbf{u} = -\mathbf{K}^* \mathbf{x}_{lat}$  may have no effect on certain eigenvalues of the system.

Feedback of the first four terms does not change the eigenvalues associated with the last two terms, ie the zero eigenvalues.

# From 2019 Exam

TRUE or FALSE?

Near trim for straight and level airplane flight, the rate of change in bank angle at time  $t$  depends on the bank angle at time  $t$ .

# From 2019 Exam

TRUE or **FALSE?**

Near trim for straight and level airplane flight, the rate of change in bank angle at time  $t$  depends on the bank angle at time  $t$ .

Bank angle not in the linear equation for time rate of change of bank angle

$$\Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

# From 2019 Exam

TRUE or FALSE?

For straight and level flight, an airplane's change in roll rate near trim at time  $t$  generally depends on the change in sideslip angle near trim at time  $t$ .

# From 2019 Exam

**TRUE** or FALSE?

For straight and level flight, an airplane's change in roll rate near trim at time  $t$  generally depends on the change in sideslip angle near trim at time  $t$ .

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \boxed{\Gamma_3 L_v + \Gamma_4 N_v} & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

# From 2019 Exam

TRUE or FALSE?

Aircraft 1 and Aircraft 2 are identical except for one feature: the vertical tail for Aircraft 1 is above the centerline of the fuselage whereas the vertical tail for Aircraft 2 is below.  $C_{n_{\beta,1}} = C_{n_{\beta,2}}$ .

# From 2019 Exam

**TRUE** or FALSE?

Aircraft 1 and Aircraft 2 are identical except for one feature: the vertical tail for Aircraft 1 is above the centerline of the fuselage whereas the vertical tail for Aircraft 2 is below.  $C_{n_{\beta,1}} = C_{n_{\beta,2}}$ .

# Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ( $\lambda_5 = 0$ ,  $\lambda_6 = 0$ ). Further, they both have the same eigenvector  $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^T$  associated with one of these eigenvalues. Why are the other eigenvectors associated with  $\lambda_5 = 0$  different?

# Advanced Bonus Question

The augmented (six state) state space matrices for the lateral and longitudinal dynamics of an aircraft both have two eigenvalues of zero ( $\lambda_5 = 0$ ,  $\lambda_6 = 0$ ). Further, they both have the same eigenvector  $\mathbf{x}_6 = [0, 0, 0, 0, 0, 1]^\top$  associated with one of these eigenvalues. Why are the other eigenvectors associated with  $\lambda_5 = 0$  different?

$$\Delta \dot{x}^E = \Delta u$$

$$\Delta \psi = \Delta r \sec \theta_0$$

$$\Delta \dot{z}^E = -u_0 \Delta \theta + \Delta w$$

$$\Delta \dot{y}^E = u_0 \cos \theta_0 \Delta \psi + \Delta v$$

# Questions?

