ASEN 3728: Part 3 Notes

Contents

1 Introduction

This document provides a comprehensive overview of aircraft lateral and directional dynamics, covering the fundamental equations, stability derivatives, dynamic modes, and control system design principles. The document progresses from the complete nonlinear rigid body equations of motion to linearized models, and finally to control system design for lateral-directional stability.

2 Overview of Lateral/Directional Dynamics

Aircraft motion can be divided into two principal components:

- **Longitudinal dynamics**: Motion in the vertical plane (pitch, altitude)
- **Lateral-directional dynamics**: Motion outside the vertical plane (roll, yaw, sideslip)

The complexity of analysis increases as we make fewer assumptions. With fewer assumptions, the analysis becomes more accurate but also more difficult to perform analytically.

2.1 Big Picture

The complete lateral-directional dynamics begin with the nonlinear rigid body equations of motion (EOM). These equations include:

$$\dot{p} = f_p(p, r, v_s, \phi, \ldots) \tag{1}$$
$$\dot{r} = f_r(p, r, v_s, \phi, \ldots) \tag{2}$$
$$\dot{v}_s = f_v(p, r, v_s, \phi, \ldots) \tag{3}$$
$$\dot{\phi} = f_\phi(p, r, v_s, \phi, \ldots) \tag{4}$$

2.2 Linearization Process

Through a process of linearization around trim conditions, we obtain the linearized equations of motion:

$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0 \quad \text{(Lateral kinematic equation)} \tag{5}$$

$$\Delta\dot{\theta} = \Delta q \quad \text{(Longitudinal kinematic equation)} \tag{6}$$

$$\Delta\dot{u} = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m} \quad \text{(Longitudinal force equation)} \tag{7}$$

$$\Delta\dot{v} = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m} \quad \text{(Lateral force equation)} \tag{8}$$

$$\Delta\dot{w} = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m} \quad \text{(Vertical force equation)} \tag{9}$$

$$\Delta\dot{p} = \Gamma_3\Delta L + \Gamma_4\Delta N \quad \text{(Roll moment equation)} \tag{10}$$

$$\Delta\dot{q} = \frac{\Delta M}{I_y} \quad \text{(Pitch moment equation)} \tag{11}$$

$$\Delta\dot{r} = \Gamma_1\Delta L + \Gamma_8\Delta N \quad \text{(Yaw moment equation)} \tag{12}$$

where the Γ coefficients are functions of the moments of inertia:

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \tag{13}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \tag{14}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \tag{15}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma} \tag{16}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y} \tag{17}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y} \tag{18}$$

$$\Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma} \tag{19}$$

$$\Gamma_8 = \frac{I_x}{\Gamma} \tag{20}$$

with $\Gamma = I_xI_z - I_{xz}^2$.

3 Stability Derivatives and Aerodynamic Forces

The aerodynamic forces and moments are expressed in terms of non-dimensional coefficients:

$$Y = \frac{1}{2}\rho V^2 S C_Y(\beta, p, r, \delta_a, \delta_r) \tag{21}$$

$$L = \frac{1}{2}\rho V^2 S b C_l(\beta, p, r, \delta_a, \delta_r) \tag{22}$$

$$N = \frac{1}{2}\rho V^2 S b C_n(\beta, p, r, \delta_a, \delta_r) \tag{23}$$

These coefficients can be expanded using stability derivatives:

$$C_Y \approx C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r \tag{24}$$

$$C_l \approx C_{l_\beta}\beta + C_{l_p}\hat{p} + C_{l_r}\hat{r} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r \tag{25}$$

$$C_n \approx C_{n_\beta}\beta + C_{n_p}\hat{p} + C_{n_r}\hat{r} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r \tag{26}$$

where $\hat{p} = \frac{pb}{2V}$ and $\hat{r} = \frac{rb}{2V}$ are non-dimensional roll and yaw rates.

3.1 Interpretation of Stability Derivatives

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$.

3.1.1 β Derivatives

- C_{n_β}: Weathervane derivative (yaw stiffness). Typically positive, indicating a stabilizing effect.
- C_{l_β}: Dihedral effect. Typically negative, representing the roll moment due to sideslip.
- C_{Y_β}: Side force due to sideslip. Usually small and similar in sign to C_{n_β}.

3.1.2 p Derivatives

- C_{l_p}: Roll damping. Always negative, representing resistance to roll.
- C_{n_p}: Yaw due to roll rate. Can be either positive or negative.

3.1.3 r Derivatives

- C_{n_r}: Yaw damping. Always negative, representing resistance to yaw.
- C_{l_r}: Roll due to yaw rate. Usually positive, contributing to spiral stability.

3.1.4 Control Derivatives

- $C_{l_{\delta_a}}$: Roll control power. Usually negative, indicating right roll for positive aileron.
- $C_{n_{\delta_a}}$: Adverse yaw. Usually positive, can cause aileron reversal.
- $C_{n_{\delta_r}}$: Rudder control power. Negative, indicating response to rudder input.

4 Factors Affecting Lateral Stability

Four significant factors affect lateral stability:

4.1 Dihedral Angle

The dihedral angle contributes to roll stability by generating a restoring roll moment when the aircraft encounters sideslip. As the sideslip angle increases, the windward wing experiences higher lift, creating a roll moment that tends to level the wings.

For a wing with dihedral angle Γ, the effective angle of attack on the left and right wings differs during sideslip:

$$\alpha^l \approx \frac{w^l}{u_0} = \alpha - \beta\Gamma \tag{27}$$

$$\alpha^r \approx \frac{w^r}{u_0} = \alpha + \beta\Gamma \tag{28}$$

This leads to a roll moment that is proportional to the negative of the dihedral angle:

$$C_{l_\beta} \propto -\Gamma \tag{29}$$

4.2 Wing Height

The vertical position of the wing relative to the fuselage affects lateral stability:

- High wing: Negative contribution to C_{l_β} (stabilizing)
- Low wing: Positive contribution to C_{l_β} (destabilizing)

4.3 Wing Sweep

Wing sweep contributes to lateral stability through an increase in the effective dihedral effect:

$$C_{l_\beta}^{\Lambda} \propto C_L V \sin(2\Lambda) \tag{30}$$

where Λ is the sweep angle. Positive sweep (swept back) increases dihedral effect.

4.4 Vertical Tail

The vertical tail contributes to both directional and lateral stability:

$$\Delta C_l^F = C_{L_F} S_F \frac{z_F}{Sb} (\frac{V_F}{V})^2 \tag{31}$$

$$C_{l_\beta}^F = -a_F (1 - \frac{\partial \sigma}{\partial \beta}) \frac{S_F z_F}{Sb} (\frac{V_F}{V})^2 \tag{32}$$

where S_F, z_F, and V_F represent the tail surface area, lever arm, and local velocity, respectively.

5 State-Space Representation of Lateral Dynamics

The linearized lateral-directional dynamics can be represented in state-space form:

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{c}_{lat} \tag{33}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix}, \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix} \tag{34}$$

The system matrix $\mathbf{A}_{lat}$ is:

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & (\frac{Y_r}{m} - u_0) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix} \tag{35}$$

For control inputs, the equation becomes:

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{B}_{lat} \mathbf{u}_{lat} \tag{36}$$

where $\mathbf{u}_{lat} = \begin{pmatrix} \delta_r \\ \delta_a \end{pmatrix}$ represents the rudder and aileron deflections.

6 Lateral Dynamic Modes

The lateral-directional dynamics of an aircraft typically exhibit three distinct modes:

6.1 Spiral Mode

The spiral mode is characterized by:

- Very slow convergence or divergence ($\lambda \approx -0.0073$)
- Time constant of approximately 137 seconds
- Primarily motion in roll and yaw angles
- Small sideslip and roll rate
- Large heading angle changes

The normalized eigenvector for the spiral mode:

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0.0068 \\ -0.0074 \\ 0.04 \\ 1.0 \\ -5.66 \end{pmatrix} \tag{37}$$

where the last component represents the heading angle ψ.

6.2 Roll Mode

The roll mode is characterized by:

- Rapid convergence ($\lambda \approx -0.5622$)
- Primary motion in roll rate
- Small sideslip and yaw rate
- Large changes in roll angle

The normalized eigenvector for the roll mode:

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{pmatrix} \tag{38}$$

6.3 Dutch Roll Mode

The Dutch roll mode is characterized by:

- Oscillatory motion ($\lambda_{3,4} = -0.033 \pm 0.947i$)
- Damping ratio $\zeta = 0.0349$ (poorly damped)

- Natural frequency $\omega_n = 0.947$ rad/s
- Coupled motion in sideslip, roll, and yaw
- Manifests as a combined rolling and yawing oscillation

The normalized eigenvector for the Dutch roll mode:

$$\hat{\mathbf{v}}_{3,4} = \begin{pmatrix} 0.321\angle-28^\circ \\ 0.407\angle 182^\circ \\ 0.294\angle 285^\circ \\ 1.0 \\ 0.307\angle 185^\circ \end{pmatrix} \tag{39}$$

7 Modal Approximations

Simplified models can provide insight into the dynamic modes:

7.1 Roll Approximation

Assuming $r = v = 0$:

$$\dot{p} = \mathcal{L}_p p \tag{40}$$

This gives an approximate eigenvalue:

$$\lambda_{r,approx} = \mathcal{L}_p \approx -0.434 \tag{41}$$

compared to the actual value $\lambda_r = -0.562$ (23% difference).

7.2 Spiral Approximation

The spiral mode can be approximated using a 2×2 system by assuming $p = \dot{p} = 0$ and neglecting side force:

$$\begin{pmatrix} \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \mathcal{Y}_v & \mathcal{Y}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{pmatrix} \begin{pmatrix} v \\ r \end{pmatrix} \tag{42}$$

This yields:

$$v = -\frac{\mathcal{L}_r}{\mathcal{L}_v} r \tag{43}$$

$$\dot{r} = \left(\mathcal{N}_r - \mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} \right) r \tag{44}$$

The approximate eigenvalue is:

$$\lambda_{s,approx} = \mathcal{N}_r - \mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} \approx -0.0096 \tag{45}$$

compared to the actual value $\lambda_s = -0.0073$.

7.3 Dutch Roll Approximation

For the Dutch roll mode, a characteristic-equation based approach yields:

$$\lambda_{s,approx} = -\frac{E}{D} \tag{46}$$

where:

$$E = g[(\mathcal{N}_r\mathcal{L}_v - \mathcal{N}_v\mathcal{L}_r)\cos\theta_0 + (\mathcal{N}_v\mathcal{L}_r - \mathcal{L}_v\mathcal{N}_p)\sin\theta_0] \tag{47}$$

$$D = -g(\mathcal{L}_v\cos\theta_0 + \mathcal{N}_v\sin\theta_0) + u_0(\mathcal{L}_v\mathcal{N}_p - \mathcal{L}_p\mathcal{N}_v) \tag{48}$$

For a Boeing 747, this gives $\lambda_{dr,approx} = -0.00725$, which is very close to the actual value.

8 Transfer Functions and Laplace Transforms

8.1 Review of Laplace Transforms

The Laplace transform converts time-domain functions to the s-domain:

$$\mathcal{L}[x(t)](s) = \int_0^\infty e^{-st}x(t)dt \tag{49}$$

Key properties include:

$$x(t) \Leftrightarrow X(s) \tag{50}$$

$$\dot{x}(t) \Leftrightarrow sX(s) - x(t=0) \tag{51}$$

$$\int_0^t x(\tau)d\tau \Leftrightarrow \frac{1}{s}X(s) \tag{52}$$

8.2 Transfer Functions

For linear systems:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{53}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{54}$$

The transfer function is defined as:

$$G_{yu}(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{N(s)}{D(s)} \tag{55}$$

The roots of $D(s)$ are the eigenvalues of $\mathbf{A}$.

8.3 Using Transfer Functions

Transfer functions are useful for:

1. Input-output analysis
2. Stability analysis (poles in left half-plane indicate stability)
3. Steady-state behavior via the Final Value Theorem
4. Frequency response analysis

9 Coordinated Turn and Steady Sideslip

9.1 Coordinated Turn

A coordinated turn is characterized by:

- Angular velocity vector aligned with inertial z-axis
- No aerodynamic forces in aircraft y-direction
- Bank angle related to turn rate: $\tan\phi = \frac{\omega u_0}{g}$
- Load factor: $n = \sec\phi$

The equations for a coordinated turn are:

$$C_Y = 0 \tag{56}$$
$$C_L = 0 \tag{57}$$
$$C_n = 0 \tag{58}$$
$$C_m = 0 \tag{59}$$
$$C_L = (n-1)C_W \tag{60}$$

9.2 Steady Sideslip

In steady sideslip:

$$Y + mg\sin\phi = 0 \tag{61}$$
$$Y + mg\phi = 0 \tag{62}$$
$$L = 0 \tag{63}$$
$$N = 0 \tag{64}$$

This leads to:

$$\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} v\beta \tag{65}$$

For a Piper Cherokee, analysis shows that positive sideslip requires positive rudder (10°) and negative aileron (-2.96°).

10 Control System Design

10.1 Pitch-Hold Controller

A pitch-hold controller aims to:

1. Improve damping (open-loop phugoid has low damping)
2. Reduce steady-state error

Three controller types are analyzed:

10.1.1 Proportional Controller

$$\Delta\delta_e = K(\Delta\theta_r - \Delta\theta) \tag{66}$$

With $K = -0.5$, this improves damping but results in steady-state error.

10.1.2 PI Controller

$$J = K(1 + \frac{1}{s}) \tag{67}$$

This eliminates steady-state error but oscillations persist.

10.1.3 PID Controller

$$J = K(1 + \frac{1}{s} + s) \tag{68}$$

With $K = -0.5$, this provides good damping for all modes and zero steady-state error.

10.2 Roll Controller

A roll controller typically includes:

1. Yaw damper: $\delta_r = -k_r r$
2. Proportional roll control: $\delta_a = k_p(\phi_r - \phi)$

For individual tuning, multiple feedback loops may be implemented:

$$\delta_a = -K_p p - K_\phi(\phi_r - \phi) \tag{69}$$
$$\delta_r = -K_r r \tag{70}$$

11 Root Locus and Feedback Control

The root locus technique traces the movement of closed-loop poles as a function of feedback gain:

$$G^{cl} = \frac{G}{1 + KG} \tag{71}$$

Key properties:

- Root locus starts at open-loop poles
- As gain increases, poles approach zeros of the open-loop system
- System stability depends on all poles being in the left half-plane

For a Boeing 747 with a yaw damper implementing $\Delta\delta_r = -k\Delta\theta$:

- Open-loop poles: $-0.37 \pm 0.89i$ and $-0.0033 \pm 0.067i$
- Open-loop zeros: -0.013 and -0.2948

Two scenarios for closed-loop poles:

1. Closed-loop poles approach open-loop zeros
2. Magnitudes of closed-loop poles become very large

12 List of Variables and Symbols

Symbol	Description
ϕ	Roll angle
θ	Pitch angle
ψ	Yaw angle (heading)
p	Roll rate
q	Pitch rate
r	Yaw rate
u	Forward velocity component
v	Lateral velocity component
w	Vertical velocity component
β	Sideslip angle
α	Angle of attack
δ_a	Aileron deflection
δ_r	Rudder deflection
δ_e	Elevator deflection
L	Rolling moment
M	Pitching moment
N	Yawing moment
X	Longitudinal force
Y	Lateral force
Z	Vertical force
I_x	Moment of inertia about x-axis
I_y	Moment of inertia about y-axis
I_z	Moment of inertia about z-axis
I_{xz}	Product of inertia
Γ	Denominator in inertia equations ($I_xI_z - I_{xz}^2$)
$\Gamma_1 - \Gamma_8$	Inertia coupling parameters
ρ	Air density
S	Wing reference area
b	Wing span
V	Airspeed
m	Aircraft mass
g	Gravitational acceleration
C_Y	Side force coefficient
C_l	Rolling moment coefficient
C_n	Yawing moment coefficient
C_{Y_β}	Side force derivative with respect to sideslip
C_{l_β}	Rolling moment derivative with respect to sideslip (dihedral effect)
C_{n_β}	Yawing moment derivative with respect to sideslip (weathervane effect)
C_{l_p}	Rolling moment derivative with respect to roll rate (roll damping)
C_{n_r}	Yawing moment derivative with respect to yaw rate (yaw damping)
λ	Eigenvalue
ζ	Damping ratio
ω_n	Natural frequency
Λ	Wing sweep angle
Γ	Dihedral angle