

Welcome to ASEN 3728

Aircraft Dynamics!



Aeronautical Engineering

1. Aerodynamics
2. Structures / Materials
3. Propulsion
4. Dynamics + Control

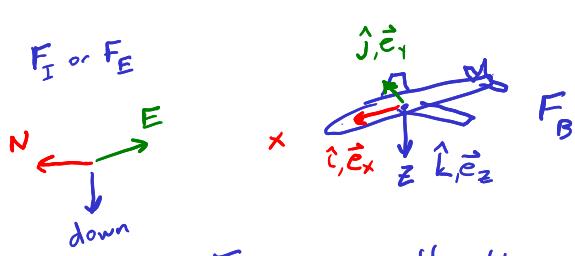
- Keep the pointy end forward
- Get the aircraft to where we want to go

+ Mathematical description of A/C behavior

+ computer simulation model

+ Design of A/C and Control Systems
to effect desirable dynamics

Notation + Conventions



Body Coordinate System

vectors: \vec{v} , \hat{v} or bold

Frame: collection of ≥ 3 points (distance between points is constant)

Inertial: translates with a constant velocity
↳ Newton's laws valid

Coordinate System: 3 unit vectors that allow measurement

\vec{v}_B^E if present, frame
"inertial velocity written in body coordinates"
 \mathbf{t} coordinate system

Forces and Moments

$$\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{F}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

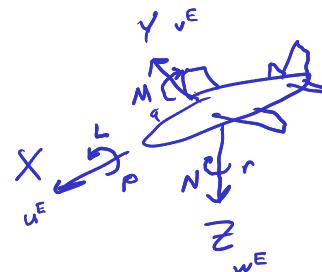
$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

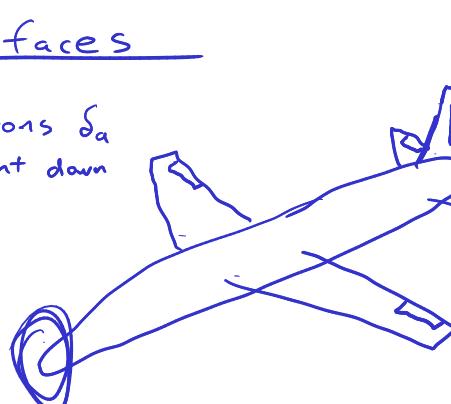
$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



$$V_g = |V_B^E| = \sqrt{u^E_{}^2 + v^E_{}^2 + w^E_{}^2}$$

Control Surfaces

Ailerons δ_a
+ δ_a = right down
+ $\delta_a \approx -L$



Rudder
+ δ_r = toward -Y direction
+ $\delta_r \approx -N$, Y
Elevator
+ δ_e = down
+ $\delta_e \approx -M$, -Z

Throttle: δ_t

+ $\delta_t \approx +X$ force
no moment

Wind

(not gravity)

Aerodynamic Forces + Moments
are functions of the A/C velocity wrt. the air

$$\vec{V}^E = \vec{V} + \vec{W}_{\text{wind}}$$

↑
air-relative

When no wind $\vec{V}^E = \vec{V}$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Wind Angles

angle of attack α

$$\alpha = \tan^{-1} \frac{w}{u}$$

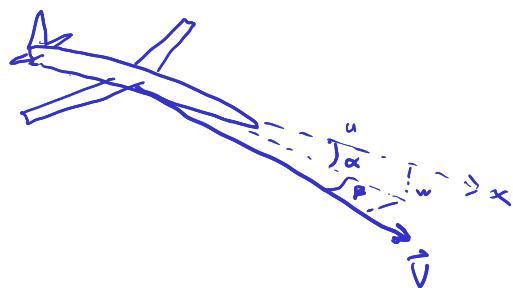
sideslip angle β

$$\beta = \sin^{-1} \frac{v}{V}$$

$$u = V \cos \beta \cos \alpha$$

$$v = V \sin \beta$$

$$w = V \cos \beta \sin \alpha$$

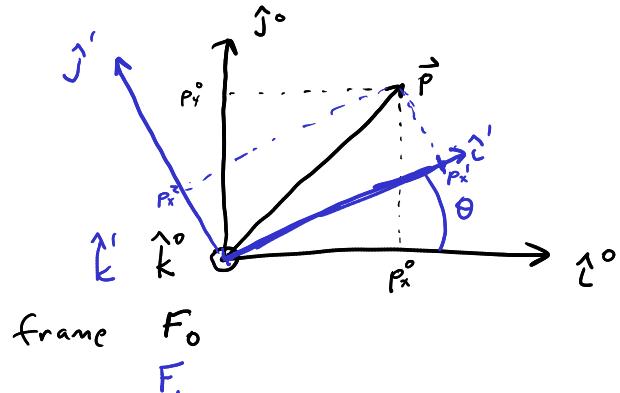


Orientation

"1" ϕ roll
 "2" θ pitch
 "3" ψ yaw

3-2-1

know \vec{v}_B^E
 want \vec{v}_E^E



$$\vec{p}_0 = \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix} \quad \vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix}$$

$$\vec{p} = p_x^0 \hat{i}^0 + p_y^0 \hat{j}^0 + p_z^0 \hat{k}^0 \leftarrow$$

$$\vec{p} = p_x^1 \hat{i}^1 + p_y^1 \hat{j}^1 + p_z^1 \hat{k}^1$$

→ want p_x^1 in terms of \vec{p}_0

$$p_x^1 = \vec{p} \cdot \hat{i}^1$$

$$= p_x^0 \hat{i}^0 \cdot \hat{i}^1 + p_y^0 \hat{j}^0 \cdot \hat{i}^1 + p_z^0 \hat{k}^0 \cdot \hat{i}^1$$

$$= \underbrace{\begin{bmatrix} \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \end{bmatrix}}_{\text{Direction Cosine Matrix}} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$$p_x^1 = [\cos \theta \quad \sin \theta \quad 0] \vec{p}_0$$

$$\vec{p}_1 = \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{bmatrix} = \begin{bmatrix} \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \\ \hat{i}^0 \cdot \hat{j}^1 & \hat{j}^0 \cdot \hat{j}^1 & \hat{k}^0 \cdot \hat{j}^1 \\ \hat{i}^0 \cdot \hat{k}^1 & \hat{j}^0 \cdot \hat{k}^1 & \hat{k}^0 \cdot \hat{k}^1 \end{bmatrix} \begin{bmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{bmatrix}$$

$$\vec{p}_1 = R_0^1 \vec{p}_0$$

Direction Cosine Matrix

Rotation Matrix

$$R_3(\theta)_{\substack{\text{t axis} \\ \text{t axis}}} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

Properties of DCMs

$$\vec{p}_2 = R_1^2 \vec{p}_1 \quad \vec{p}_1 = R_0^1 \vec{p}_0 \quad \vec{p}_2 = R_1^2 \vec{p}_1 = \vec{p}_2 = \underbrace{R_1^2 R_0^1}_{R_A^C} \vec{p}_0$$

Chaining

$$R_0^2 = R_1^2 [R_0^1]$$

$$R_A^C = R_B^C R_A^B$$

Inverse

$$\vec{p}_2 = R_1^2 \vec{p}_1$$

$$R_1^2 \vec{p}_2 = R_1^{2-1} R_1^2 \vec{p}_1 = R_2^1 \vec{p}_2$$

$$R_2^1 = (R_1^2)^{-1}$$

$$R_A^B = (R_B^A)^{-1}$$

since DCMs are orthonormal

$$(R_A^B)^{-1} = R_A^B{}^T$$

$$\begin{bmatrix} - \\ - \\ - \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_A^B = R_B^A{}^T$$

Euler Angles

Any rotation can be described by 3 Euler angles about 3 n-repeated axes

E.g. 3-1-3 rotation through α, β, γ

3-2-1 rotation through ψ, θ, ϕ

$$\vec{p}_B = R_E^B \vec{p}_E = \underbrace{R_1(\phi)}_{\uparrow \text{roll}} \underbrace{R_2(\theta)}_{\curvearrowright \text{pitch}} \underbrace{R_3(\psi)}_{\curvearrowleft \text{yaw}} \vec{p}_E$$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$\vec{p}_{\text{pilot}}^E = \vec{p}_{A/C}^E + \vec{p}_{\text{pilot}}^B$$

$$\vec{p}_{\text{pilot}}^E = \vec{p}_{A/C}^E + R_B^E \vec{p}_{\text{pilot}}^B$$

← abstract vectors

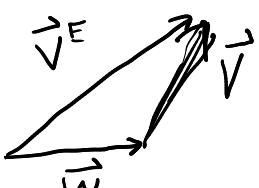
Vector Clarity

\vec{V}_B^E ↗ "frame of reference", "with respect to", "relative to"

↖ "coordinate frame", "expressed in", "written in"

↙ by convention

$$\vec{V}^E = \vec{V}^{(w)} + \vec{W}^{(E)}$$



True in Any coordinate system
as long as all in same coordinate system

$$\vec{V}_B^E = \vec{V}_B + \vec{W}_B$$

$\begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$ ↘ $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ + ↗

$$\vec{V}_E^E = \vec{V}_E + \vec{W}_E$$

Kinematics

Kinematics: "Geometry of Motion" (no forces)

Translational
Rotational

Dynamics: Effects of forces and moments on objects

Vector Derivatives

$\frac{d}{dt} \vec{p}$ = time rate of change of \vec{p}

$$\frac{d}{dt} \vec{p} = \vec{v}^E$$

$$\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$$

$$\dot{\vec{p}}_B \equiv \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Not Always
True

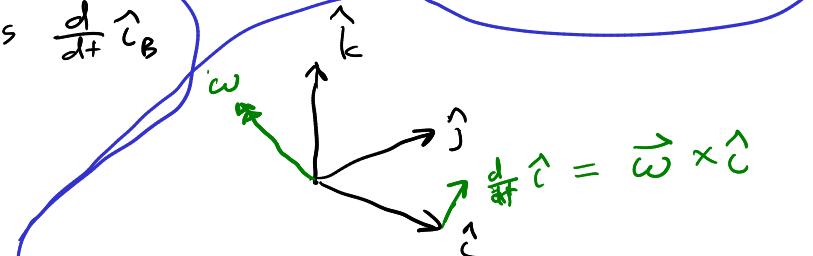
$$\begin{bmatrix} \hat{i}_B \\ \hat{j}_B \\ \hat{k}_B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \vec{p}_B &= p_x \hat{i}_B + p_y \hat{j}_B + p_z \hat{k}_B \\ \dot{\vec{p}}_B &= \dot{p}_x \hat{i}_B + \dot{p}_y \hat{j}_B + \dot{p}_z \hat{k}_B \end{aligned}$$

$$\frac{d}{dt}(uv) = u \frac{d}{dt}v + v \frac{d}{dt}u$$

$$\left(\frac{d}{dt} \vec{p} \right)_B = \dot{p}_x \hat{i}_B + p_x \frac{d}{dt} \hat{i}_B + \dot{p}_y \hat{j}_B + p_y \frac{d}{dt} \hat{j}_B + \dot{p}_z \hat{k}_B + p_z \frac{d}{dt} \hat{k}_B$$

What is $\frac{d}{dt} \hat{i}_B$



$$\tilde{\omega}_B = \tilde{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{aligned} &= p_x (\tilde{\omega}_B \times \hat{i}_B) + p_y (\tilde{\omega}_B \times \hat{j}_B) + p_z (\tilde{\omega}_B \times \hat{k}_B) \\ &= \tilde{\omega}_B \times \vec{p}_B \end{aligned}$$

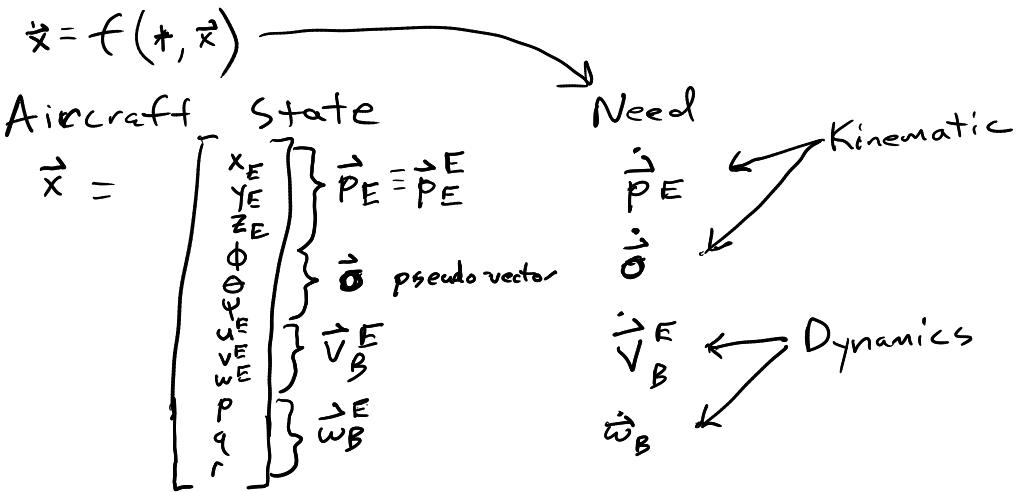
$$\frac{d}{dt} \vec{p}_B = \dot{\vec{p}}_B + \tilde{\omega}_B \times \vec{p}_B$$

$$= \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Kinematic Transport Theorem

Equations of motion for a 3D body

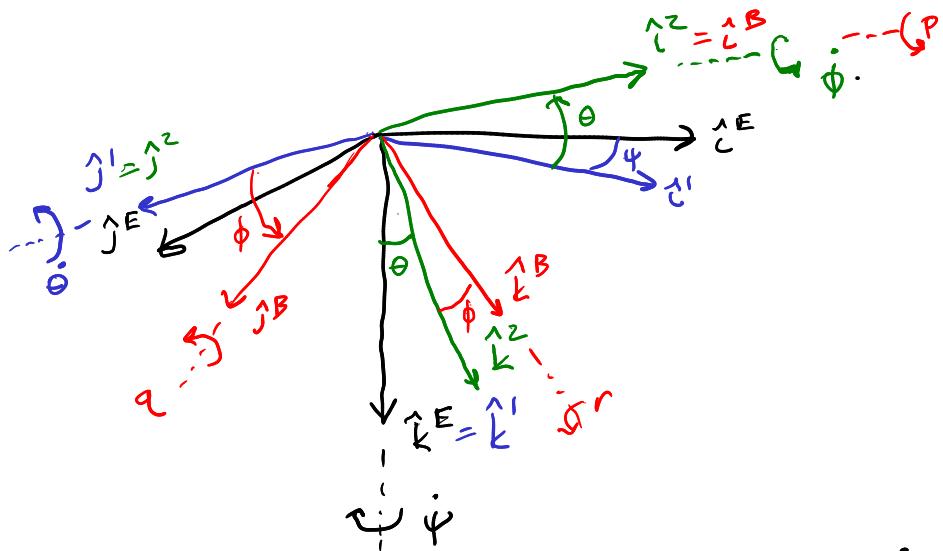


Translational Kinematics

$$\dot{\vec{p}_E} = \frac{d}{dt} \vec{p}_E = \vec{v}_E^E \times \vec{p}_E^E = \vec{v}_E^E = R_B^E \vec{v}_B^E$$

$$\boxed{\dot{\vec{p}_E} = R_B^E \vec{v}_B^E}$$

Rotational Kinematics



$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\vec{\omega} = \dot{\psi} \hat{k}_E + \dot{\theta} \hat{j}_I + \dot{\phi} \hat{i}_I$$

$$\vec{\omega}_B = \dot{\psi} \hat{k}_B + \dot{\theta} \hat{j}_B + \dot{\phi} \hat{i}_B$$

$$R_E^B = R_1(\phi) R_2(\theta) R_3(\psi)$$

$$R_I^B = R_1(\phi) R_2(\theta)$$

$$= \dot{\psi} R_E^B \hat{k}_E + \dot{\theta} R_E^B \hat{j}_I + \dot{\phi} \hat{i}_B$$

$$= R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix} + R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \text{invert} & & \\ 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\sec x = \frac{1}{\cos x}$$

$\dot{\vec{\alpha}} = T \vec{\omega}_B$

C attitude,
influence matrix

Dynamics

$$\vec{F} \quad \vec{G}$$

Translational Dynamics

Newton's 2nd Law

$$\vec{F} = m \vec{a}$$

$$\vec{F} = m \frac{d}{dt} \vec{V}^E$$

$$\frac{d}{dt} \vec{V}_B^E = \vec{V}_B^E + \vec{\omega}_B \times \vec{V}_B^E$$

$$m(\vec{V}_B^E + \vec{\omega}_B \vec{V}_B^E) = \vec{F}_B$$

$$\vec{V}_B^E = \frac{\vec{F}_B}{m} - \vec{\omega}_B \vec{V}_B^E$$

*

Rotational Dynamics

Euler's 2nd Law

$$\frac{d}{dt} \vec{h} = \vec{G} \quad \begin{matrix} \text{angular momentum} \\ \text{moments} \end{matrix}$$

$$\vec{h} = I \vec{\omega} \quad \begin{matrix} \text{moment of inertia} \end{matrix}$$

$$I = \begin{pmatrix} \int(y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int(x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int(x^2 + y^2) dm \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

$\dot{\vec{\omega}}_B$

$$\frac{d}{dt} \vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \vec{h}_B = \vec{G}_B$$

$$I \ddot{\vec{\omega}}_B + \vec{\omega}_B I \vec{\omega}_B = \vec{G}_B$$

$$\dot{\vec{\omega}}_B = I^{-1} [\vec{G}_B - \vec{\omega}_B I \vec{\omega}_B]$$

$$\dot{x} = \begin{bmatrix} \dot{P}_E \\ \dot{\theta} \\ \dot{V}_B^E \\ \dot{\omega}_B^E \end{bmatrix}$$

$$\dot{\vec{P}}_E = R_B^E \vec{V}_B$$

$$\dot{\theta} = T \vec{\omega}_B$$

*

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ x_B \\ y_B \\ w_B \\ p \\ q \\ r \end{bmatrix}$$

$$\dot{\vec{p}}_E^E = \vec{p}_E^E$$

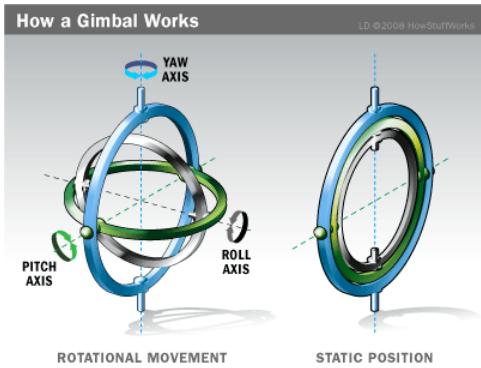
attitude influence

$$\dot{\vec{o}} = T \vec{\omega}_B$$

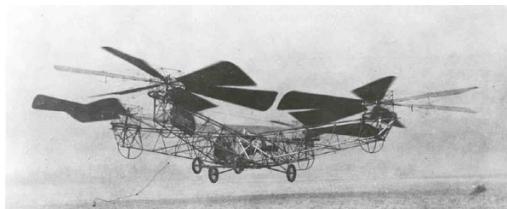
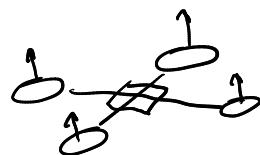
$$\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{\omega}}_B = I_B^{-1} [\vec{G}_B - \vec{\omega}_B \times I_B \vec{\omega}_B]$$

$$T = \begin{bmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{bmatrix}$$



Multi-copter
Quadrrotor



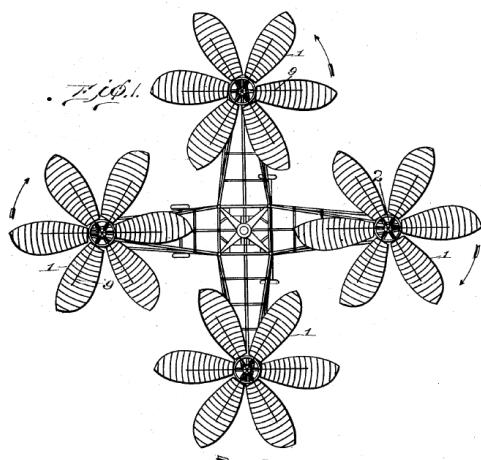
March 4, 1930.

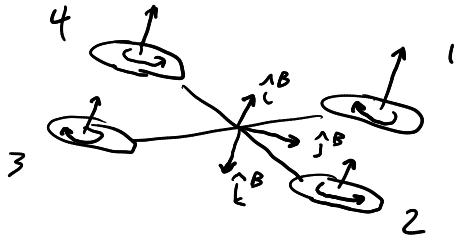
G. DE BOTHEZAT
HELICOPTER

1,749,471

Filed March 29, 1924

5 Sheets-Sheet 1





$$\vec{f} = {}^g\vec{f} + {}^a\vec{f} + {}^c\vec{f}$$

$$\vec{G} = {}^g\vec{G} + {}^c\vec{G}$$

$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$${}^c\vec{G} = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since symmetric about $i^B - k^B$ plane and $i^B - j^B$ plane

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I_B = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \underbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}_{\vec{\omega}_B \times \vec{v}_B^E} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$I_B^{-1} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \quad \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Aerodynamic Forces and Moments

$${}^a\vec{f} = -D \frac{\vec{v}}{|V|}$$



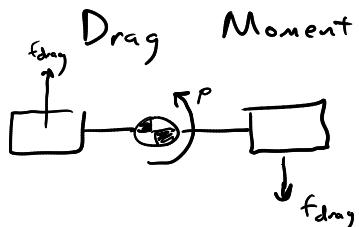
$$V_a = |\vec{V}| \quad \text{airspeed}$$

$$D = \frac{1}{2} \rho C_D A \quad \text{drag coefficient}$$

$$V_a^2 = \nu V_a^2 \quad \text{cross-sectional area}$$

$${}^a\vec{f}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = -\nu V_a^2 \frac{\vec{v}_B}{V_a}$$

$$= -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$



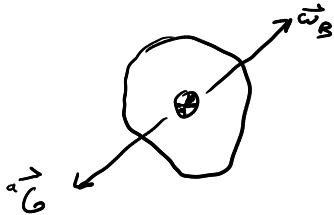
$$L_{drag} = -2l f_{drag}$$

$$= -2l \left(\frac{1}{2} \rho C_D A (\ell_p)^2 \right) \text{sign}(p)$$

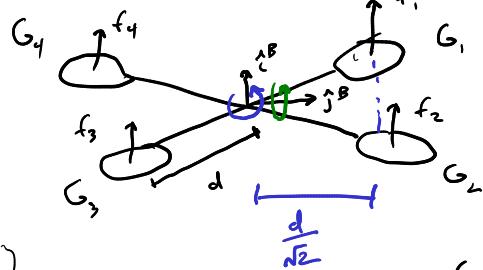
$\underline{-\rho^2 \text{sign}(p)} = -|p|p$

$$= -M |p| p$$

$$\vec{a}_G^c = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = -M \sqrt{p^2 + q^2 + r^2} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$



Control Forces + Moments



$$G_i = k_g C_D \omega_r^2$$

$$f_i = k_f C_L \omega_r^2$$

$$k_m = \frac{k_g C_D}{k_f C_L}$$

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ k_m & -k_m & k_m & -k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} \text{invert} \\ \text{ } \\ \text{ } \\ \text{ } \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

Multicopter

vs

Conv. Helicopter

Aerodynamic
Efficient



Mechanical
Control
Complexity



Power System
Complexity
Turbine



Electric



- differential
- 1st order
- ordinary
- nonlinear
- coupled

general form

$$\begin{cases} \dot{\vec{x}}_E = R_B^E \vec{v}_B \\ \vec{o} = T \vec{\omega}_B \\ \dot{\vec{v}}_B = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B \\ \dot{\vec{\omega}}_B = I_B^{-1} (\vec{g}_B - \vec{\omega}_B \times I_B \vec{\omega}_B) \end{cases}$$

$$T = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix}$$

$$\begin{array}{l} \phi = 90^\circ \\ \theta = 0 \\ \psi = 10^\circ \end{array} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ q = b \\ r \end{bmatrix}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ z_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pu^E - ru^E \\ qu^E - pv^E \end{pmatrix} + q \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_x - I_z}{I_y} pr \\ \frac{I_y - I_x}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L \\ \frac{1}{I_y} M \\ \frac{1}{I_z} N \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

$$O = \frac{1}{m} X + \frac{Q}{m} \quad \therefore X = 0 \quad Y = 0$$

$$O = g + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$Z_{c,0} = mg$$

"Never" trim condition

$$\begin{array}{l} \dot{x}_{E,0} = 0 \\ \dot{y}_{E,0} = 0 \\ \dot{z}_{E,0} = 0 \end{array} \quad \begin{array}{l} u_o^E = 0 \\ v_o^E = 0 \\ w_o^E = 0 \end{array} \quad \begin{array}{l} \dot{u}_o^E = 0 \\ \dot{v}_o^E = 0 \\ \dot{w}_o^E = 0 \end{array}$$

$$\begin{array}{l} p_o = 0 \\ q_o = 0 \\ r_o = 0 \end{array}$$

$$\begin{array}{l} \dot{p}_o = 0 \\ \dot{q}_o = 0 \\ \dot{r}_o = 0 \end{array}$$

$$\begin{array}{l} \phi_o = 0 \\ \theta_o = 0 \\ \psi_o = \text{any constant value} \end{array}$$

$$\begin{array}{l} f_{1,o} = mg \\ f_{2,o} = 0 \\ f_{3,o} = 0 \\ f_{4,o} = 0 \\ L_{c,o} = 0 \\ M_{c,o} = 0 \\ N_{c,o} = 0 \end{array} \boxed{Z_{c,o} = mg}$$

Linearization

$$\dot{\vec{x}} = \vec{x}_o + \Delta \vec{x}$$

$$\vec{u} = \vec{u}_o + \Delta \vec{u}$$

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \begin{bmatrix} x_{E,0} + \Delta x_E \\ y_{E,0} + \Delta y_E \\ z_{E,0} + \Delta z_E \\ \Delta \phi \\ \Delta \theta \\ \Delta \psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix}$$

$$\Delta \dot{\vec{x}} = A \Delta \vec{x} + B \Delta \vec{u}$$

Taylor Series

$$y = f(x, u)$$

$$x_0 + \Delta x$$

$$y_0 = f(x_0, u_0)$$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$= f(x_0, u_0) + \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}$$

$$\Delta y = \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}$$

$$x + y = x_0 + \Delta x + y_0 + \Delta y$$

$$\rightarrow x u \approx x_0 u_0 + \left. \frac{\partial x u}{\partial x} \right|_0 \Delta x + \left. \frac{\partial x u}{\partial u} \right|_0 \Delta u$$

$$= x_0 u_0 + u_0 \Delta x + x_0 \Delta u$$

alternative

$$x u = (x_0 + \Delta x)(u_0 + \Delta u) = x_0 u_0 + u_0 \Delta x + x_0 \Delta u + \Delta x \Delta u^0$$

$$\sin(\theta) \approx \sin(\theta_0) + \left. \frac{\partial \sin \theta}{\partial \theta} \right|_0 \Delta \theta$$

$$= \sin(\theta_0) + \cos(\theta_0) \Delta \theta$$

alternatively

$$\sin(\theta) = \sin(\theta_0 + \Delta \theta) = \sin(\theta_0) \cos(\Delta \theta)^1 + \cos(\theta_0) \sin(\Delta \theta)^2 \approx \sin(\theta_0) + \cos(\theta_0) \Delta \theta$$

$$\cos(\theta) \approx \cos(\theta_0) - \sin(\theta_0) \Delta \theta$$

Linearize QR equations of motion

Examples

$$\dot{\theta} = \cos \phi q - \sin \phi r \leftarrow$$

$$\dot{\theta}_0 + \Delta \dot{\theta} = \cos(\theta_0 + \Delta \theta) (q_0 + \Delta q) - \sin(\theta_0 + \Delta \theta) (r_0 + \Delta r)$$

$$\Delta \dot{\theta} \approx \cancel{\cos(\Delta \theta)} \Delta q - \cancel{\sin(\Delta \theta)} \Delta r$$

$$\approx \Delta q - \Delta \theta \Delta r^0$$

$\Delta \dot{\theta} = \Delta q$

$$\dot{w}^E = q u^E - p v^E + g \cos \theta \cos \phi + \frac{1}{m} Z + \frac{1}{m} Z_c$$

$$\Delta \dot{w}^E = \cancel{\Delta q} \cancel{\Delta u^0} - \cancel{\Delta p} \cancel{\Delta v^0} + g + \frac{1}{m} \Delta Z + \frac{1}{m} (Z_{c,0} + \Delta Z_c)$$

cancel

$$\Delta \dot{w}^E = \frac{1}{m} \Delta Z + \frac{1}{m} \Delta Z_c$$

Linearized Aerodynamics

$$Z = -\gamma w \sqrt{u^2 + v^2 + w^2}$$

$$= Z_0 + \left. \frac{\partial Z}{\partial u} \right|_0 \Delta u + \left. \frac{\partial Z}{\partial v} \right|_0 \Delta v + \left. \frac{\partial Z}{\partial w} \right|_0 \Delta w$$

$$\Delta Z =$$

" " " "

$$\left. \frac{\partial Z}{\partial u} \right|_0 = -Z \gamma w u \left(u^2 + v^2 + w^2 \right)^{-1/2}$$

$\left. \frac{\partial Z}{\partial u} \right|_0 = 0$

$\left. \frac{\partial Z}{\partial v} \right|_0 = 0$

$$\frac{\partial Z}{\partial w} = -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$\left. \frac{\partial Z}{\partial w} \right|_0 = 0$

$\begin{cases} \Delta L = 0 \\ \Delta M = 0 \\ \Delta N = 0 \end{cases}$

~~In real life~~
 ~~$Z(p, q, r)$~~

γ constant

Looks like $\frac{0}{0}$ take limit as
 $u, v, w \rightarrow 0$

$\Delta Z = 0$

$\begin{cases} Z = -\gamma w^2 \\ \frac{\partial Z}{\partial w} = -\gamma 2w^0 \end{cases}$

Review : State space representation of a Linear Dynamical System

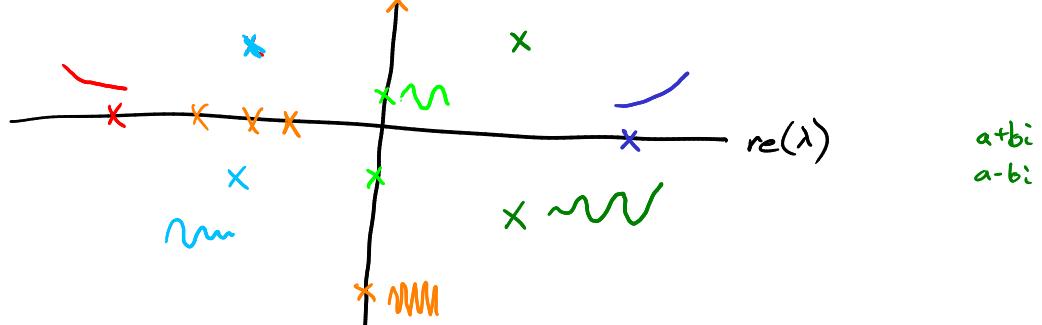
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\dot{\vec{y}} = C\vec{x} + D\vec{u}$$

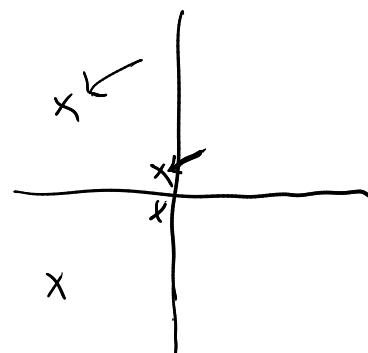
Eigenvectors Eigenvectors

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$m(\lambda) \vec{x}(+) = e^{A+} \vec{x}(0) = \sum_{i=1}^n k_i e^{\lambda_i t} \vec{v}_i$$



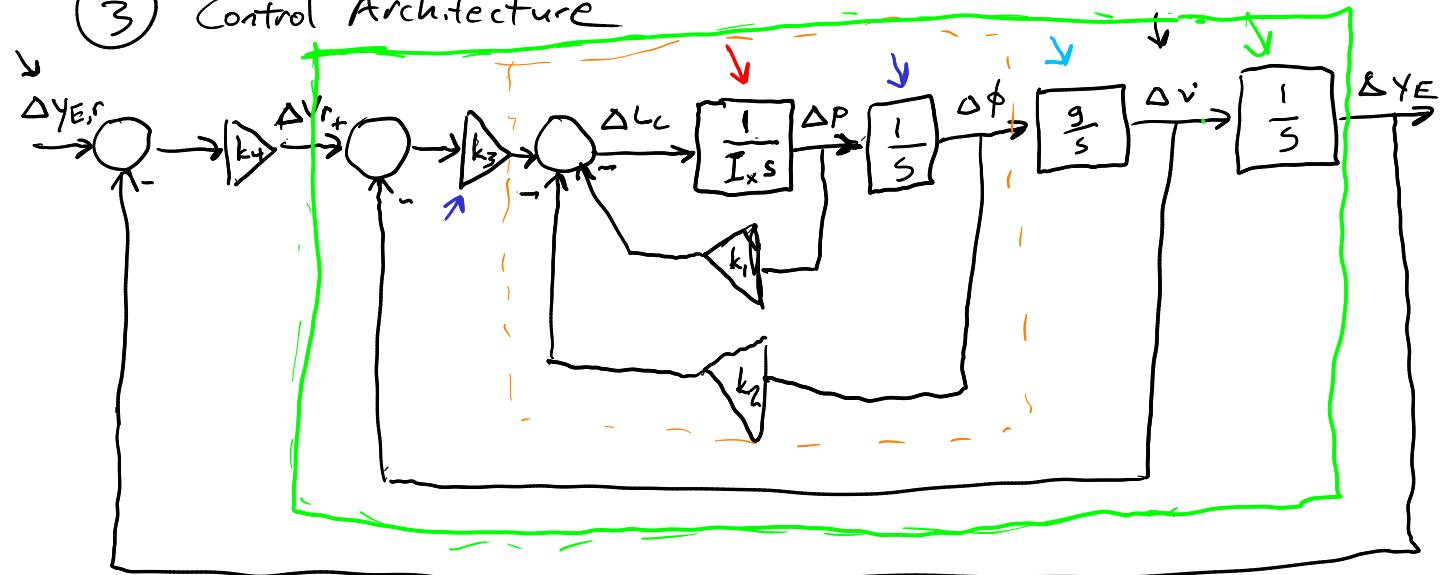
$$\begin{aligned}\vec{u} &= -K\vec{x} \\ \dot{\vec{x}} &= (A - BK)\vec{x} \\ &\quad A^{-1}\end{aligned}$$



(1)
(2)

$$\begin{bmatrix} \Delta Y_E \\ \Delta Y \\ \Delta \phi \\ \Delta P \end{bmatrix} = \begin{bmatrix} \Delta V \\ g \Delta \phi \\ \Delta P \\ \frac{1}{I_x} \Delta L_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta Y_E \\ \Delta V \\ \Delta \phi \\ \Delta P \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_x} \end{bmatrix} \Delta L_c$$

③ Control Architecture



(4) Choosing Gains

1) $E_1 + K_2$

2) K_4

3) K_3 Root Locus

$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi - k_3 \Delta v - k_3 k_4 \Delta y_E + k_3 k_4 \Delta y_{E,r}$$

$$\begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3 k_4}{I_x} \end{bmatrix} \Delta y_{E,r}$$

A^{cl}

$$\rightarrow \begin{bmatrix} \dot{\Delta \phi} \\ \dot{\Delta p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

Desired:

$$f = 0.7$$

$$\omega_n = 16 \text{ rad/s} \quad T = \frac{2\pi}{\omega_n}$$

$$\frac{T}{2} = 0.2s$$

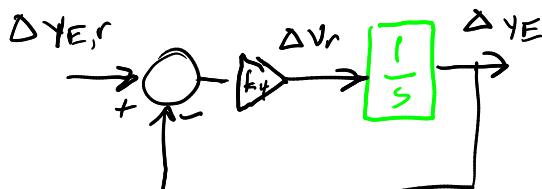
$$\lambda = -f\omega_n \pm i\omega_n \sqrt{1-f^2}$$

$$\lambda = -11.2 + 11.4i$$

$$k_1 = 0.0016$$

$$k_2 = 0.0179$$

$$\cancel{\lambda = -\frac{k_1}{2I_x} + \sqrt{\frac{k_1^2}{4I_x^2} - \frac{k_2}{I_x}}}$$



$$\dot{\Delta y}_E = k_4 (y_{E,r} - \Delta y_E)$$

$$\Delta y_E^{(+)} = \Delta y_{E,r} (1 - e^{-\frac{k_4}{T} t})$$

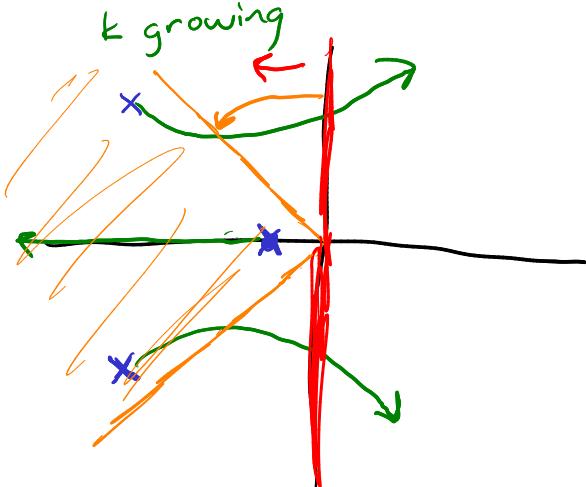
choose k_4 1 order of magnitude slower

$$T = \frac{1}{k_4} > 10 \cdot \frac{2\pi}{f\omega_n} = 5.6$$

$$k_4 = 0.17$$

$x: k=0$

Choose k_3 via Root Locus



Root Locus: plot of the poles/eigen values of a system as a parameter (usually a gain) changes

Conventional A/C Dynamics

Longitudinal

Altitude
Airspeed
Pitch



Lift
Drag
(Pitch Moment)

$$L = \frac{1}{2} \rho V_a^2 S C_L$$

$$D = \frac{1}{2} \rho V_a^2 S C_D$$

$$M = \frac{1}{2} \rho V_a^2 S_c C_m$$

↑ chord

Lateral-Directional

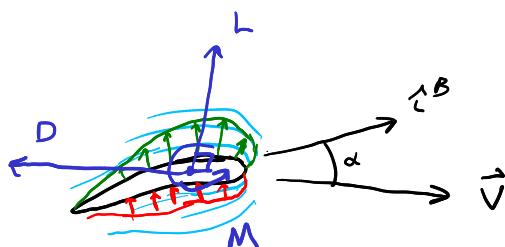
Roll
Yaw
Sideslip
Turning

$$C_L(\alpha, q, \delta_e) \quad \text{nonlinear}$$

$$C_D(\alpha, q, \delta_e)$$

$$C_m(\alpha, q, \delta_e)$$

Lift



1st order Taylor Series of $C_L(\alpha, q, \delta_e)$

$$L = \frac{1}{2} \rho V_a^2 S \left(C_{L_0} + \frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e} \delta_e \right)$$

$$= \frac{1}{2} \rho V_a^2 S \left(C_{L_0} + C_{L_\alpha} \alpha + C_{L_q} \frac{c}{2V_a} q + C_{L_{\delta_e}} \delta_e \right)$$

$\uparrow \quad \uparrow \quad \uparrow$

Nondimensional stability Derivative Nondim. control derivative

Stability Derivatives

- based linear assumptions / linearization
- Main tool connecting aerodynamics to dynamics
- Functions of A/C geometry

Example

Pitch Stiffness C_{m_α}

$$\ddot{\alpha} \approx \frac{k}{I} C_{m_\alpha} \alpha$$

$$\ddot{x} = -\frac{k_{spring}}{m} x$$

- Only valid in a linear region e.g. small α



$$AR = \frac{b^2}{S}$$

Estimated Using

- Geometric Data
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$\text{eg. } C_{L\alpha} = \frac{\pi AR}{1 + \sqrt{1 + (AR)^2}}$$

Drag

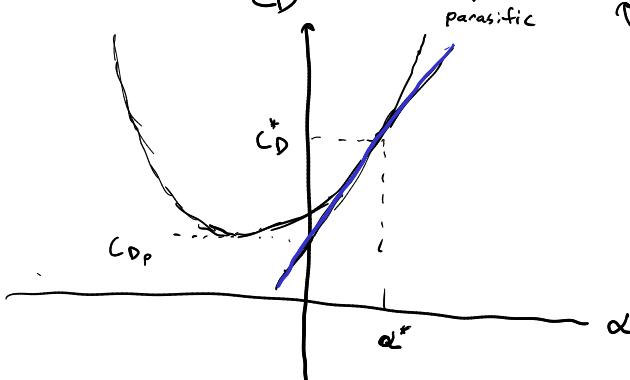
nonlinear

α
only

$$C_D(\alpha) = C_{D_p} + \frac{C_L(\alpha)^2}{\pi e AR}$$

↑
parasitic

$\alpha \in AR$
Oswald's efficiency
induced
drag



linearize

$$\underline{C_{D_0} + C_{D_\alpha} \alpha + C_{D_{\dot{\alpha}}} \dot{\alpha} + C_{D_{\ddot{\alpha}}} \ddot{\alpha}}$$

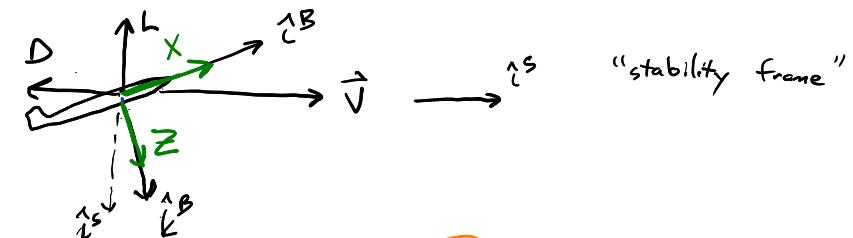
In this class:

$$K = \frac{1}{\pi e AR}$$

$$C_D = C_{D_{min}} + K(C_L(\alpha, q, \delta_e) - C_{L_{min}})^2$$

aerodynamic
forces

in \hat{i}^B and \hat{k}^B



Pitching Moment

$$M = \frac{1}{2} \rho V_a^2 S_c [C_{m_0} + C_{m_\alpha} \alpha + C_{m_q} \dot{\alpha} + C_{m_{\dot{\alpha}}} \ddot{\alpha}]$$

trim coefficient
of moment

pitch
stiffness
coefficient

pitch
damping
coefficient

T

Longitudinal Trim

Last time: Long. Forces + Moments

$$L = \frac{1}{2} \rho V_a^2 S (C_{L_0} + C_{L\alpha} \overset{\text{coordinate def.}}{\underbrace{\alpha}} + C_{Lq} \overset{\text{coordinate def.}}{\underbrace{q}} + C_{L\delta_e} \overset{\text{coordinate def.}}{\underbrace{\delta_e}})$$

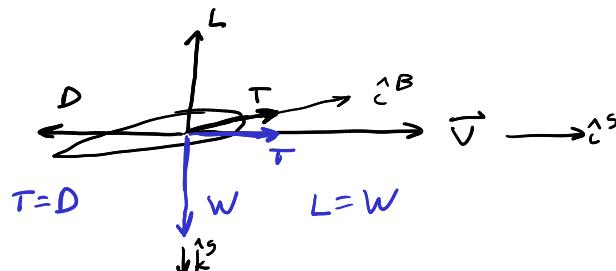
$$D = \frac{1}{2} \rho V_a^2 S (C_{D_{min}} + K (C_L(\alpha, q, \delta_e) - C_{L_{min}})^2)$$

$$M = \frac{1}{2} \rho V_a^2 S \bar{c} (C_{m_0} + C_{m\alpha} \overset{\text{coordinate def.}}{\underbrace{\alpha}} + C_{mq} \overset{\text{coordinate def.}}{\underbrace{q}} + C_{m\delta_e} \overset{\text{coordinate def.}}{\underbrace{\delta_e}})$$

$$C_{a_b} \equiv \frac{\partial C_a}{\partial b}$$

Trim: SLUF: Steady Level Unaccelerated Flight
 ↗ just "in trim"

For linear trim calc



At trim, forces + moments about C.G. sum to 0

$$\overset{\text{S dir:}}{T_{trim}} = D_{trim}$$

$$\overset{\text{S dir:}}{L_{trim}} = W$$

$$M_{trim} = 0$$

V_a fixed

$$\cancel{C_{L_{trim}}} = C_{L\alpha_{trim}} = \frac{W}{\frac{1}{2} \rho V_a^2 S}$$

$$\cancel{C_{m_{trim}}} = C_{m_0} + C_{m\alpha_{trim}} = 0$$

$$C_L = C_{L\alpha} + C_{L\delta_e}$$

$$C_m = C_{m_0} + C_{m\alpha} + C_{m\delta_e} = 0$$

$$\begin{bmatrix} C_{L\alpha} & C_{L\delta_e} \\ C_{m\alpha} & C_{m\delta_e} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{trim} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{trim} \end{bmatrix} = \begin{bmatrix} C_{L\alpha} & C_{L\delta_e} \\ C_{m\alpha} & C_{m\delta_e} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{trim}} \\ -C_{m_0} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(special case of
Cramer's Rule)

$$\alpha_{trim} = \frac{C_{m_0} C_{L\delta_e} + C_{m\delta_e} C_{L_{trim}}}{\Delta}$$

$$\delta_{trim} = - \frac{C_{m_0} C_{L\alpha} + C_{m\alpha} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L\alpha} C_{m\delta_e} - C_{L\delta_e} C_{m\alpha}$$

$$C_{m\alpha} = C_{L\alpha} (h - h_n)$$

(from slides)

$$C_{L_\alpha} = a = a_{wb} \left[1 + \frac{a_t S_t}{a_{wb} S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$C_{m_0} = C_{m_{ac_{wb}}} + C_{m_{0_p}} + a_t \bar{V}_H (\epsilon_0 + i_t) \left[1 - \frac{a_t S_t}{a S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$$

$$h_n = h_{n_{wb}} + \frac{a_t}{a} \bar{V}_H \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) - \frac{1}{a} \frac{\partial C_{m_p}}{\partial \alpha}$$

$$C_{m_\alpha} = C_{L_\alpha} (h - h_n)$$

Direct dependence on CG location h

$$C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e} \frac{S_t}{S} = a_e \frac{S_t}{S}$$

$$C_{m_{\delta_e}} = -a_e \bar{V}_H + C_{L_{\delta_e}} (h - h_{n_{wb}})$$

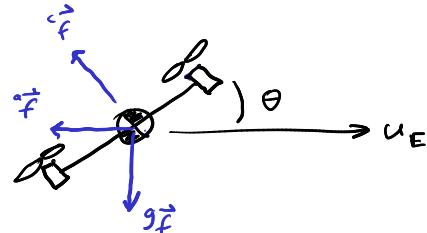
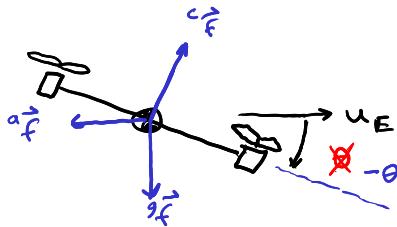
Some mistakes on hw

if $\dot{x}(0) = \sum_i k_i \vec{v}_i$ $\dot{x}(t) = \sum_i k_i e^{\lambda_i t} \vec{v}_i$ not gains

Rate of climb $= \dot{z}_E = -\dot{z}_E$

$\dot{h} = -\dot{z}_E$

Diagram



Linearization

Method 1: fo. Taylor Series

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} \Delta x + \frac{\partial f}{\partial y} \Big|_{(x_0, y_0)} \Delta y$$

$$xy = x_0 y_0 + y_0 \Delta x + x_0 \Delta y$$

$$\begin{aligned} \sin(\theta) &= \sin(\theta_0) + \frac{\partial \sin \theta}{\partial \theta} \Big|_{\theta_0} \Delta \theta \\ &= \sin(\theta_0) + \cos(\theta_0) \Delta \theta \end{aligned}$$

$$\dot{x} = \underbrace{a \Delta x + b}_{\text{may include } x_0}$$

Method 2

$$f(x, y) \rightarrow f(x_0 + \Delta x, y_0 + \Delta y) \rightarrow \text{expand out/cancel out} \rightarrow \text{apply small disturbance approx}$$

$$xy \rightarrow (x_0 + \Delta x)(y_0 + \Delta y) \rightarrow x_0 y_0 + y_0 \Delta x + x_0 \Delta y + \Delta x \Delta y$$

$$\begin{aligned} \sin \theta &\rightarrow \sin(\theta_0 + \Delta \theta) \rightarrow \sin(\theta_0) \cos(\Delta \theta) + \cos(\theta_0) \sin(\Delta \theta) \\ &\rightarrow \sin(\theta_0) + \cos(\theta_0) \Delta \theta \end{aligned}$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{w}_2 \\ \dot{q}_2 \end{bmatrix}$$

$$\dot{\vec{V}}_B^E$$

$$\dot{\vec{W}}_E$$

Longitudinal Dynamics

Last Time

$$C_L = C_{L\alpha} \alpha + C_{L\delta e} \delta e = C_{L\text{trim}}$$

$$C_m = C_{m\alpha} \alpha + C_{m\delta e} \delta e \rightarrow = 0$$

$$C_{m\alpha} = C_{L\alpha} (h - h_n) \leftarrow$$

Today: Linear Longitudinal EOM
Stability Derivatives

$$\dot{\vec{r}}_E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{\omega}}_B = I^{-1} [\vec{G}_B - \vec{\omega}_x \times I \vec{\omega}_B]$$

Symmetry about x-z axis

$$I_{xy} = I_{yz} = 0$$



$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{I_x}{\Gamma} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz} (I_x - I_y + I_z)}{\Gamma}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma}$$

$$\Gamma_7 = \frac{I_x (I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z (I_z - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y}$$

$$\Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

→ $\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

→ $\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

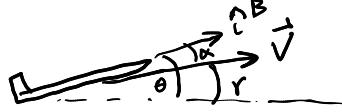
$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_8 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

$$V = u_0$$

Trim State
Inputs u_0, h_0, γ_{ao}
speed altitude γ_{air} relative flight path angle

determine →

$$\alpha_0, \delta_{e0}, \delta_{t0}$$



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \begin{bmatrix} \Delta x_E \\ \Delta y_E \\ \Delta z_E \\ \Delta \phi \\ \Delta \theta \\ \Delta \psi \\ \Delta u^E \\ \Delta v^E \\ \Delta w^E \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} \quad \vec{x}_0 = \begin{bmatrix} \cdot \\ -h_0 \\ 0 \\ 0 \\ u_0^E \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} \delta_e \\ 0 \\ 0 \\ 0 \\ \delta_t \end{bmatrix}$$

$$\dot{\theta}_0 + \Delta \dot{\theta} = \cos \theta_0 \dot{\theta}_0 + \frac{\partial \cos \theta_0}{\partial \phi} \Delta \phi + \frac{\partial \cos \theta_0}{\partial \theta} \Delta \theta - \sin \theta_0 - \frac{\partial \sin \theta_0}{\partial \phi} \Delta \phi - \frac{\partial \sin \theta_0}{\partial \theta} \Delta \theta$$

$$\Delta \dot{\theta} = \Delta \dot{\theta}$$

$$\dot{x}_0 + \Delta \dot{x} = \dot{x}_0 + \Delta \dot{x} + v_0 \Delta r + g \Delta v - g \Delta v - g \Delta w - u_0 \Delta q - \Delta \dot{q} \cancel{- g \sin \theta_0 - g \cos \theta_0 \Delta \theta + \frac{1}{m} (\dot{x}_0 + \Delta x)}$$

$$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta x \quad \text{cancel}$$

→ $\Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0$

Longitudinal

$$X_u \equiv \frac{\partial X}{\partial u}$$

→ $\Delta \dot{\theta} = \Delta q$

Lat-d

→ $\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m}$

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

→ $\Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m}$

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

→ $\Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m}$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

→ $\Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

→ $\Delta \dot{q} = \frac{\Delta M}{I_y}$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

→ $\Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$

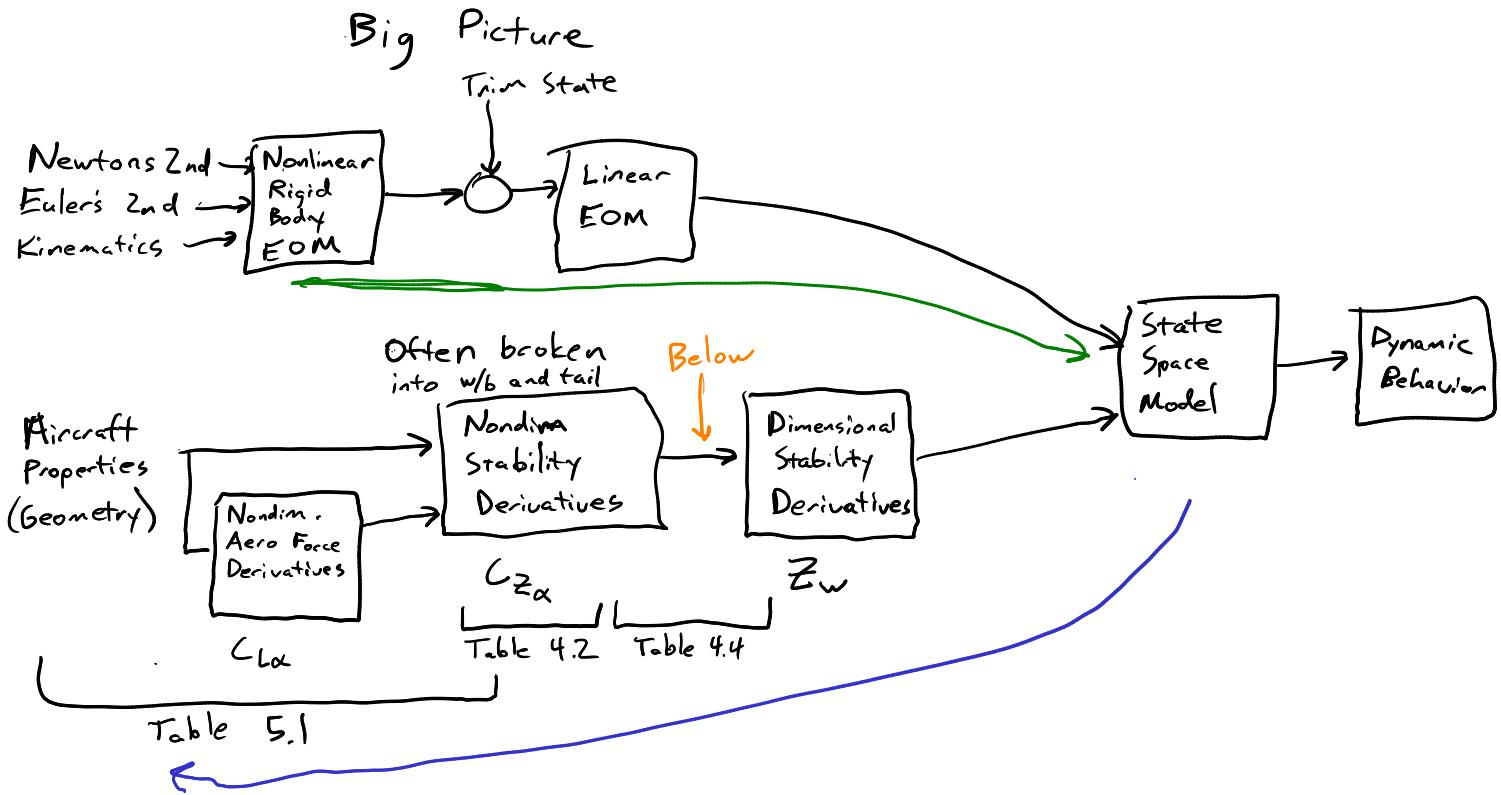
$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_w}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m} & \frac{Z_w}{m - Z_w} & \frac{Z_q + mu_0}{m - Z_w} & \frac{-mg \sin \theta_0}{m - Z_w} \\ \frac{1}{I_y} \left[M_u + \frac{M_w Z_u}{m - Z_w} \right] & \frac{1}{I_y} \left[M_w + \frac{M_w Z_w}{m - Z_w} \right] & \frac{1}{I_y} \left[M_q + \frac{M_w (Z_q + mu_0)}{m - Z_w} \right] & \frac{-M_w mg \sin \theta_0}{I_y (m - Z_w)} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2} \rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2} \rho V^2 S$	C_W
M	$\frac{1}{2} \rho V^2 S \bar{c}$	C_m
L, N	$\frac{1}{2} \rho V^2 S \bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\dot{\hat{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\dot{\hat{\beta}}, \hat{p}, \hat{r}$
m	$\rho S \bar{c}/2$	μ
I_y	$\rho S (\bar{c}/2)^3$	\hat{I}_y
I_x, I_z, I_{xz}	$\rho S (b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

$$C_{z_u} = \frac{\partial C_z}{\partial \hat{u}}$$

Wind - Angle Approximations

$$\begin{aligned} \Delta \alpha &= \tan^{-1} \frac{\Delta w}{V} \\ &\approx \hat{w} \\ \Delta \beta &= \sin^{-1} \frac{\Delta v}{V} \\ &\approx \hat{v} \end{aligned}$$

Table 4.4
Longitudinal Dimensional Derivatives

	X	Z	M
u	$\rho u_0 S C_{w_0} \sin \theta_0 + \frac{1}{2} \rho u_0 S C_{x_u}$	$-\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$	$\frac{1}{2} \rho u_0 \bar{C} S C_{m_u}$
w	$\frac{1}{2} \rho u_0 S C_{x_\alpha}$	$\frac{1}{2} \rho u_0 S C_{z_\alpha}$	$\frac{1}{2} \rho u_0 \bar{C} S C_{m_\alpha}$
q	$\frac{1}{4} \rho u_0 \bar{C} S C_{x_q}$	$\frac{1}{4} \rho u_0 \bar{C} S C_{z_q}$	$\frac{1}{4} \rho u_0 \bar{C}^2 S C_{m_q}$
\dot{w}	$\frac{1}{4} \rho \bar{C} S C_{x_\alpha}$	$\frac{1}{4} \rho \bar{C} S C_{z_\alpha}$	$\frac{1}{4} \rho \bar{C}^2 S C_{m_\alpha}$

$$\begin{aligned}
 Z_u &\equiv \frac{\partial Z}{\partial u} \Big|_o \quad Z = \frac{1}{2} \rho V^2 S C_Z \\
 \frac{\partial Z}{\partial u} \Big|_o &= \frac{1}{2} \rho S \left(\frac{\partial V^2}{\partial u} \Big|_o C_Z + \frac{\partial C_Z}{\partial u} \Big|_o V^2 \right) \\
 &= \frac{1}{2} \rho S Z_u C_{Z_0} + \frac{1}{2} \rho u_0^2 S \frac{\partial C_Z}{\partial u} \Big|_o \\
 Z_u &= -\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}
 \end{aligned}$$

$\frac{\partial f(x)g(x)}{\partial x} = f(x) \frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x} g(x)$
 $C_{z_u} \equiv \frac{\partial C_Z}{\partial \alpha}$
 $\frac{\partial C_Z}{\partial u} = \frac{\partial C_Z}{\partial \hat{u}_{u_0}} = \frac{1}{u_0} \frac{\partial C_Z}{\partial \alpha}$
 $C_{Z_0} = -C_{W_0} \cos \theta_0$

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
\hat{u}^\dagger	$\mathbf{M}_0 \left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left(1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$* -2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$* -2a_t V_H \frac{l_t}{\bar{C}} \frac{\partial \epsilon}{\partial \alpha}$
\hat{q}	Neg.	$* -2a_t V_H$	$* -2a_t V_H \frac{l_t}{\bar{C}}$

Neg. means usually negligible.

*means contribution of the tail only, formula for wing-body not available.

$$\ddot{C}_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

Nondimensional Longitudinal Stability Derivatives

Table 5.1

Summary—Longitudinal Derivatives

	C_x	C_z	C_m
\hat{u}^+	$M_0 \left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left(1 - \frac{\partial C_D}{\partial C_T} \right)$	$-M_0 \frac{\partial C_L}{\partial M} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$M_0 \frac{\partial C_m}{\partial M} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_t V_H \frac{l_t}{c} \frac{\partial \epsilon}{\partial \alpha}$
\hat{q}	Neg.	$*-2a_t V_H$	$*-2a_t V_H \frac{l_t}{c}$

Neg. means usually negligible.

*means contribution of the tail only, formula for wing-body not available.

$$\hat{t} C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

α derivatives

$$C_{m_\alpha} = C_{L_\alpha} (h - h_N)$$

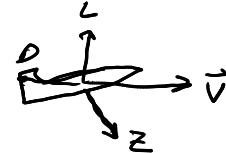
C_{z_α}

$$Z = -L \cos \alpha - D \sin \alpha$$

$$C_Z = -(C_L \cos \alpha + C_D \sin \alpha) \\ = -(C_L + C_D \alpha)$$

$$C_{Z_\alpha} = \frac{\partial C_Z}{\partial \alpha} \Big|_0 = -(C_{L_\alpha} + C_{D_0} + \alpha \frac{\partial C_D}{\partial \alpha})$$

$$\boxed{C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$



u derivatives

3 important factors:

- Compressibility : Mach Number
- Dynamic Pressure: $p_d = \frac{1}{2} \rho V^2$
- Thrust

$$C_{x_u} \equiv \frac{\partial C_x}{\partial \hat{u}} \Big|_0$$

$$M \equiv \frac{V}{a} \leftarrow \text{speed of sound}$$

- Different from the dynamic pressure in nondimensionalization
Changes in C_L, C_D , etc., due to changes in dynamic pressure

$$C_{*u} = \frac{\partial C_*}{\partial M} \Big|_0 \frac{\partial M}{\partial \hat{u}} \Big|_0 + \frac{\partial C_*}{\partial p_d} \Big|_0 \frac{\partial p_d}{\partial \hat{u}} \Big|_0 + \frac{C_*}{\partial C_T} \Big|_0 \frac{\partial C_T}{\partial \hat{u}} \Big|_0$$

$$\rightarrow \frac{\partial M}{\partial \hat{u}} \Big|_0 = u_0 \frac{\partial M}{\partial u} \Big|_0 = \frac{u_0}{a} \frac{\partial V}{\partial u} \Big|_0 = M_0$$

$* \in \{x, z, m\}$

$$\frac{\partial p_d}{\partial u} \Big|_0 = u_0 \frac{\partial p_d}{\partial u} \Big|_0 = u_0 \frac{1}{2} \rho \frac{\partial V^2}{\partial u} \Big|_0 = u_0 \rho \cdot u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2} \rho V^2 S}$$

$$\frac{\partial C_T}{\partial u} \Big|_0 = u_0 \frac{\partial C_T}{\partial u} \Big|_0 = u_0 \left(\frac{\partial T}{\partial u} \Big|_0 - \frac{2T}{\frac{1}{2} \rho V^2 S} \right) \Big|_0 = \frac{\partial T}{\partial u} \Big|_0 - 2C_{T_0}$$

$\frac{\partial (f(x))}{\partial x} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

3 cases

Gliding case : $C_{Tu} = 0$

Constant Thrust (Jet) : $C_{Tu} = -2C_{T_0}$

Constant Power (Prop) : $C_{Tu} = -3C_{T_0}$
 $TV = \text{constant}$

$$\frac{\partial T}{\partial u} \Big|_0 = -\frac{T_0}{u_0}$$

C_{Xu}

$$C_X \approx C_T - C_D$$

$$\frac{\partial C_X}{\partial M} \Big|_0 = \frac{\partial C_T}{\partial M} \Big|_0 - \frac{\partial C_D}{\partial M} \Big|_0$$

$$\frac{\partial C_X}{\partial p_d} \Big|_0 = \frac{\partial C_T}{\partial p_d} \Big|_0 - \frac{\partial C_D}{\partial p_d} \Big|_0$$

$$\frac{\partial C_X}{\partial C_T} \Big|_0 = 1 - \frac{\partial C_D}{\partial C_T} \Big|_0$$

$$C_{Xu} = M_0 \left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) \Big|_0 - \rho u_0^2 \frac{\partial C_D}{\partial p_d} \Big|_0 + C_{Tu} \left(1 - \frac{\partial C_D}{\partial C_T} \Big|_0 \right)$$

C_{Zu}

Assume $C_Z = -C_L$

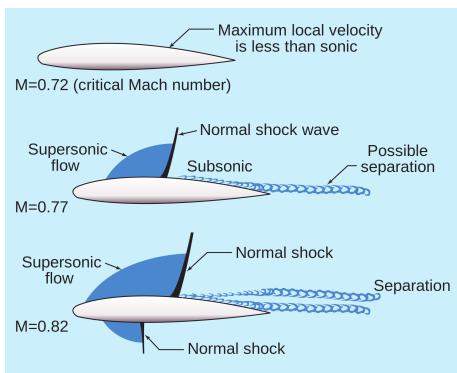
$$C_{Zu} = -M_0 \underbrace{\frac{\partial C_L}{\partial M} \Big|_0}_{\text{Small except transonic}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} \Big|_0 - C_{Tu} \frac{\partial C_L}{\partial C_T} \Big|_0$$

Small except transonic

C_{mu}

$$C_{mu} = M_0 \underbrace{\frac{\partial C_m}{\partial M} \Big|_0}_{\text{Mach Tuck}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} \Big|_0 + C_{Tu} \frac{\partial C_m}{\partial C_T} \Big|_0$$

Mach Tuck



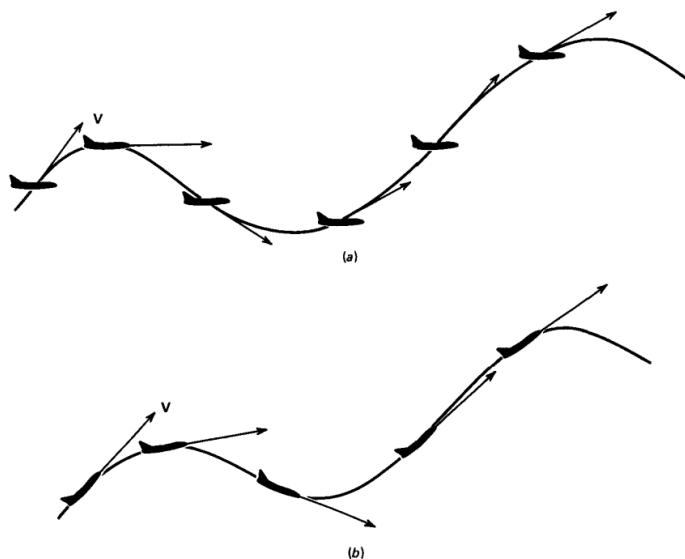


Figure 5.2 (a) Motion with zero q , but varying α_x . (b) Motion with zero α_x but varying q .

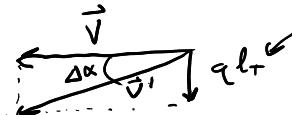
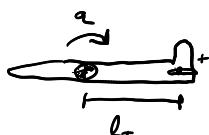
q - derivatives

Wing-body

Tail

Tail

velocity observed by tail



$(C_{Zq})_{tail}$

$$C_{Zq} = \frac{\partial C_Z}{\partial \dot{q}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_Z}{\partial q} \Big|_0 = - \frac{2u_0}{c} \frac{\partial C_L}{\partial q} \Big|_0$$

$$(C_{Zq})_{tail} = - \frac{2u_0}{c} a_+ \frac{S+L+}{S u_0} = \boxed{-2a_+ V_H}$$

$$\begin{aligned} \Delta C_L &= a_+ \Delta \alpha = a_+ \tan^{-1} \frac{q l+}{u_0} \approx a_+ \frac{q l+}{u_0} \\ \Delta C_L &= \frac{S+}{S} \Delta C_L \\ &= \frac{S+}{S} a_+ \frac{q l+}{u_0} \end{aligned}$$

$$V_H = \frac{S+ l+}{S c}$$

$(C_{m_q})_{tail}$

$$\Delta C_m = - V_H \Delta C_L = a_+ V_H \frac{q l+}{u_0}$$

$$C_{m_q} = \frac{\partial C_m}{\partial \dot{q}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_m}{\partial q} \Big|_0$$

$$(C_{m_q})_{tail} = -2a_+ V_H \frac{l+}{c}$$

Wing-Body

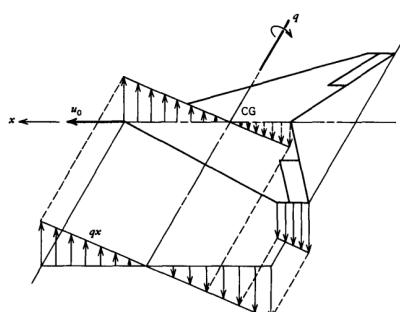


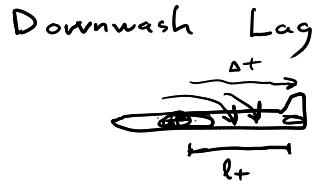
Figure 5.4 Wing velocity distribution due to pitching.

$\dot{\alpha}$ derivatives

Unsteady effects

Wing-Body → Determined by initial response
or oscillation of wing in wind tunnel
or flight test

Tail



$$\Delta \varepsilon = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \Delta t = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \frac{l_+}{u_0}$$

$$= -\Delta \alpha_+$$

$(C_{Z\dot{\alpha}})_{tail}$

$$\Delta C_L = a_+ \Delta \alpha_+ = a_+ \dot{\alpha} \frac{l_+}{u_0} \frac{\partial \varepsilon}{\partial \alpha}$$

$$\rightarrow \Delta C_L = a_+ \dot{\alpha} \frac{l_+ + S_r}{u_0 S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$C_{Z\dot{\alpha}} = \frac{\partial C_Z}{\partial \frac{\alpha \varepsilon}{S u_0}} \Big|_0 = \frac{2 u_0}{c} \frac{\partial C_Z}{\partial \alpha} = -2 a_+ \frac{l_+ + S_r}{c S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{Z\dot{\alpha}})_{tail} = -2 a_+ V_H \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{m\dot{\alpha}})_{tail} = -2 a_+ V_H \frac{l_+}{c} \frac{\partial \varepsilon}{\partial \alpha}$$

Longitudinal Modes

$$\mathbf{x}(t) = \sum_i k_i \vec{v}_i e^{\lambda_i t}$$

Dynamics of Flight, Eq. (4.9,18) $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$



$$h_0 = 40k \text{ ft}$$

$$V_0 = 774 \text{ ft/s}$$

$$\gamma_0 = \theta_0 = \alpha_0 = 0$$

$$\omega_n = \sqrt{a^2 + b^2}$$

$$\lambda_{1,2} = -0.37 \pm 0.89i$$

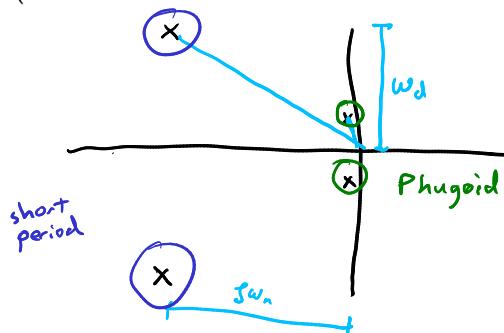
$a = -0.37$ $b = 0.89$
 $\omega_n = 0.96$, $\zeta = 0.38$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$a = -0.0033$ $b = 0.067$
 $\omega_n = 0.067$, $\zeta = 0.049$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & -mg \sin \theta_0 \\ \frac{1}{I_y} \left[M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Eigenvectors

$$A \vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I) v_i = 0$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0 - \lambda & 1 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \lambda_1 = -1, \lambda_2 = -2$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$(A - \lambda_1 I) v_1 = 0$$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

let $\vec{v}_1[1] = 1$
 $\vec{v}_1[2] = -1$

$$\vec{v}_1[1] + \vec{v}_1[2] = 0$$

$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.001i \\ 0.0011 \mp 0.0004i \end{bmatrix} \quad \begin{array}{l} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{array}$$

$$\vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097i \\ -0.0001 \mp 0.0006i \\ 0.0001 \pm 0.002i \end{bmatrix}$$

$$x(t) = \sum_i L_i \vec{v}_i e^{\lambda_i t}$$

$$x(0) = \text{Re}(\vec{v}_1) = \sum_i L_i \vec{v}_i e^{\lambda_i 0} = 0.5 \vec{v}_1 + 0.5 \vec{v}_2$$

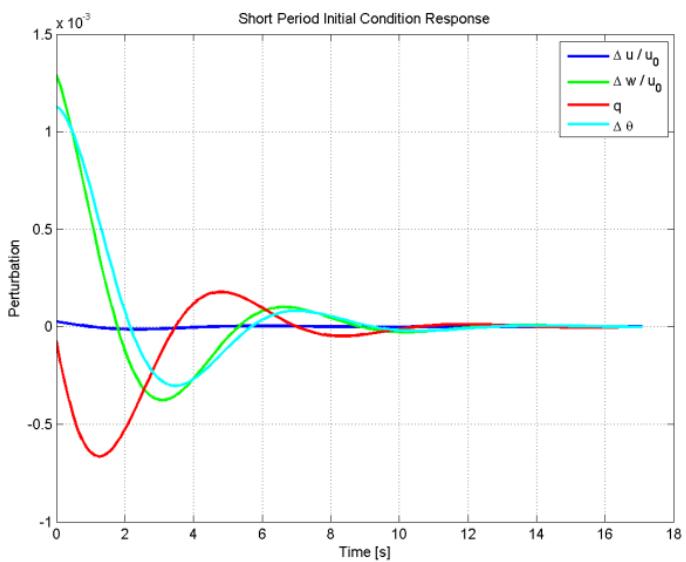
$$x(t) = 0.5 \vec{v}_1 e^{\lambda_1 t} + 0.5 \vec{v}_2 e^{\lambda_2 t}$$

$$\lambda_{1/2} = -0.372 + 0.888i$$

$$\zeta = 0.387$$

$$\omega_n = 0.962$$

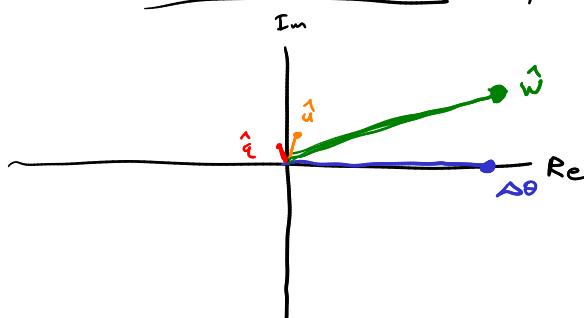
$$x(0) = \text{Re}(\vec{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



$$\vec{v}_{1,2} / \vec{v}_{1,2}[u] = \vec{v}'_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i / & & & \\ 0.9996 / & & & \\ -0.0001 \pm 0.001i / 0.0011 - 0.0004i \\ 0.0011 \mp 0.0004i / 0.0011 - 0.0004i \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ 1.0 & & & \end{bmatrix}$$

$$\hat{v}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.0066 \pm 0.016i \\ 1.0 \end{bmatrix} \quad \begin{array}{l} \hat{u} = \frac{\Delta u}{u_0} \\ \hat{w} = \frac{\Delta w}{u_0} \approx \alpha \\ \hat{q} = \frac{\Delta q}{2u_0} \\ \Delta \theta \end{array}$$

Phasor Plot (\hat{v}_1 , Short Period)



- \hat{v} {
 α and $\Delta \theta$ change in-phase
 \hat{u} relatively constant
- λ {
 well-damped
 high frequency

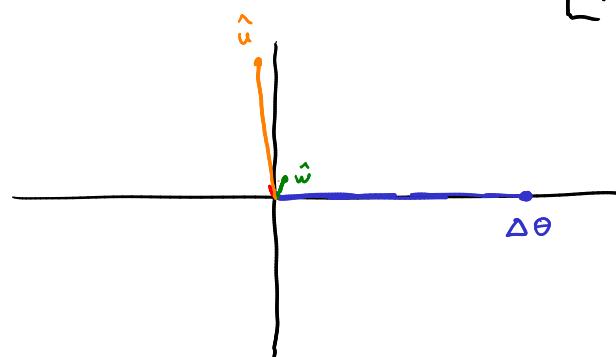
Phugoid Mode

phugoid = "flight"



side note:

$$\begin{aligned} z &= a + bi \\ &= r \angle \phi \\ &= r e^{i\phi} \\ r &= \sqrt{a^2 + b^2}; \quad \phi = \text{atan} 2(b, a) \end{aligned}$$

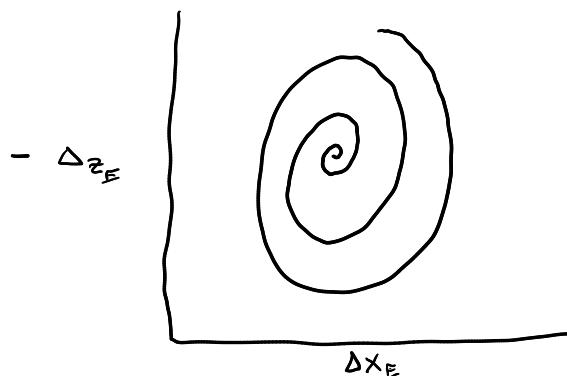
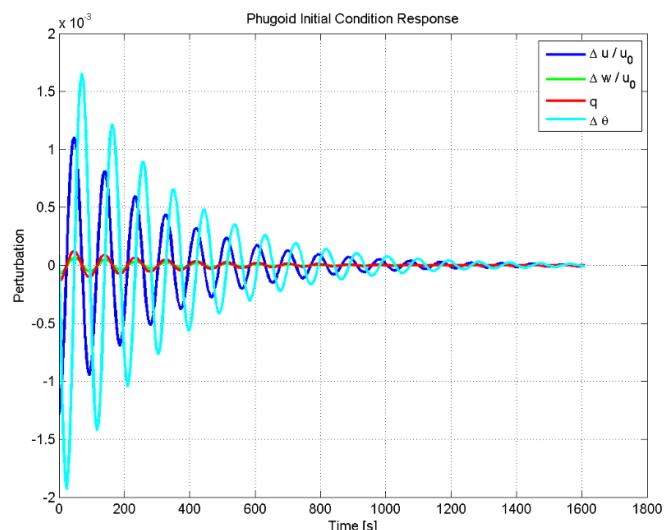


$$\hat{v}_{3,4} = \begin{bmatrix} 0.62 \angle 92^\circ \\ 0.036 \angle 83^\circ \\ \underline{0.0012 \angle 93^\circ} \\ 1.0 \angle 0 \end{bmatrix} \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ \hat{q} \\ \Delta\theta \end{array}$$

- { Large $\hat{u}, \Delta\theta$ out-of-phase oscillations (offset $\sim 90^\circ$)
small α }
- λ { low frequency
low damping }

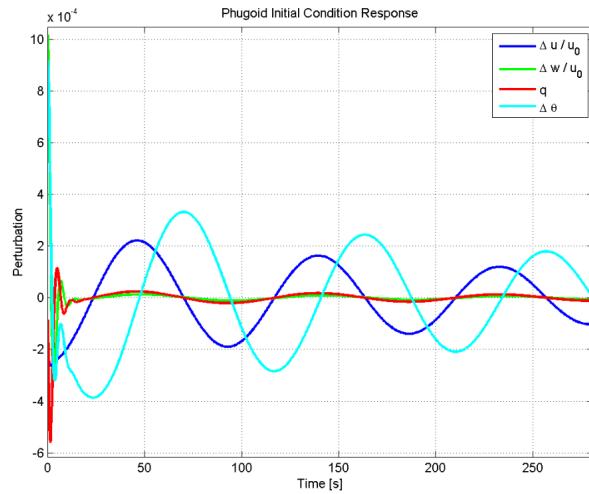
$$\begin{aligned} \lambda_{3/4} &= -3.29e-03 + 6.72e-02i \\ \zeta &= 0.0489 \quad \leftarrow \text{poorly damped} \\ \omega_n &= 0.0673 \quad \leftarrow \text{slow response} \end{aligned}$$

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



General Response

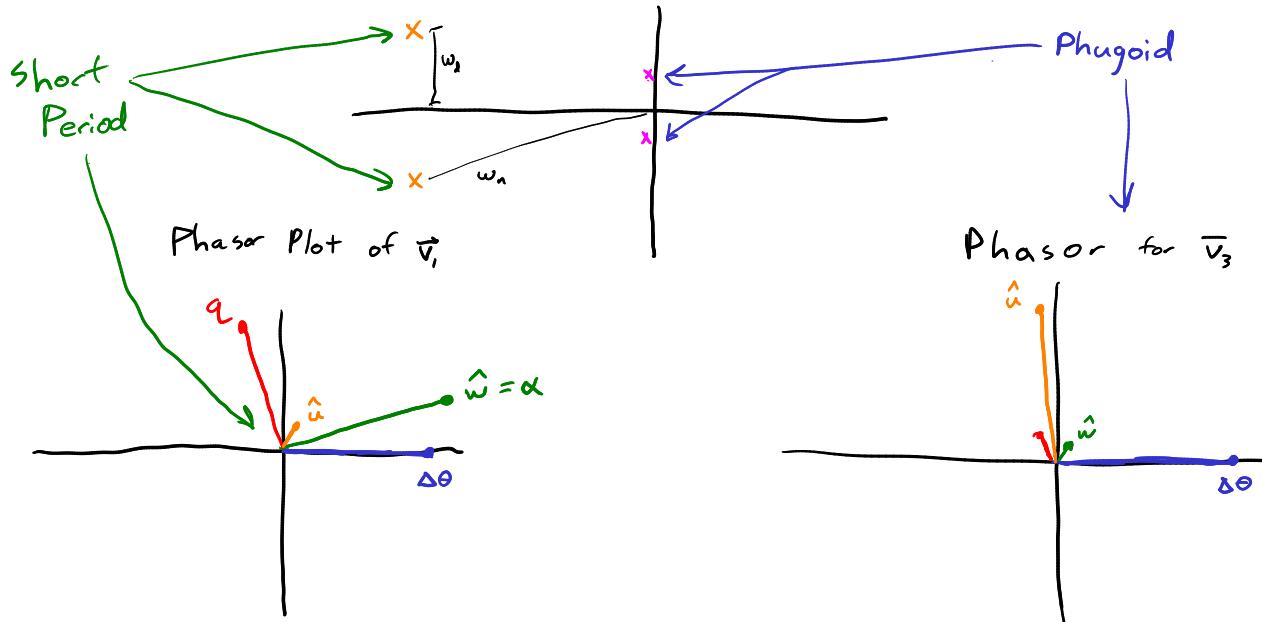
$$\mathbf{x}_0 = 0.8 \cdot \text{Re}(\mathbf{v}_{sp}) + 0.2 \cdot \text{Re}(\mathbf{v}_{ph})$$



Longitudinal Modal Approximations

$$\dot{\vec{x}}_{lon} = \underbrace{A_{lon} \vec{x}_{lon}}_{\text{Eigen}} + \vec{c}_{lon}$$

$$\vec{x}_h = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$



Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\dot{\vec{x}}_{lon} = \vec{A}_{lon} \vec{x}_{lon} + \vec{c}_{lon}$

$$\vec{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \vec{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\vec{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -a \cos \theta_0 \\ \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & -m q \sin \theta_0 \\ \frac{1}{I_y} \left[M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_q + \frac{M_{\dot{w}} (Z_w + m u_0)}{m - Z_{\dot{w}}} \right] & -M_{\dot{w}} m g \sin \theta_0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume: $\Delta u = 0$

$$\theta_0 = 0$$

$$Z_{\dot{w}} \ll m$$

$$Z_q \ll m u_0$$

If we also assume no vertical motion,

$$\theta_0 = 0 \text{ implies } \Delta \theta = \alpha \approx \frac{\Delta w}{u_0}$$

$$\begin{bmatrix} \Delta \dot{w} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m} \\ \frac{1}{I_y} \left[M_w + \frac{M_{\dot{w}} Z_w}{m} \right] \end{bmatrix} \underbrace{\frac{1}{I_y} \left[M_q + M_{\dot{w}} u_0 \right]}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y} \left[M_q + M_{\dot{w}} u_0 \right] \right]}_{-2 \zeta \omega_n} \lambda - \underbrace{\frac{1}{I_y} \left(u_0 M_w - \frac{M_{\dot{w}} Z_w}{m} \right)}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

<u>Dimensional Stab. Deriv.</u>	<u>Nondim. Stab. Deriv.</u>	<u>A/C Params</u>
$Z_w = \frac{\partial Z}{\partial w} \Big _0 = \frac{1}{2} \rho u_0 S C_{Z\alpha}$	$C_{Z\alpha}$	$C_{Z\alpha} = -C_{D_0} - C_{L\alpha}$
M_w	$C_{m\alpha}$	$= C_{L\alpha}(h - h_n)$
M_w	$C_{m\alpha}$	\dots
M_q	$C_{m\alpha}$	$-2a_n V_H \frac{L}{C}$

How accurate is this approximation?

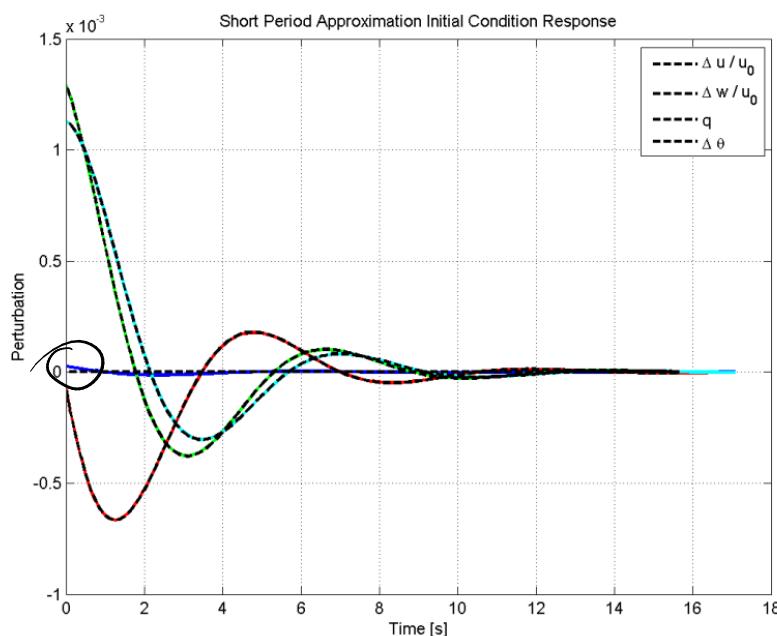
For 747
@ cruise

Full A_{bn}

$$\lambda_{1,2} = -0.372 \pm 0.88j; \quad g = 0.387 \quad \omega_n = 0.962$$

S.P. approx

$$\lambda_{sp} = -0.371 \pm .889; \quad g = 0.385 \quad \omega_n = 0.963$$



(note: here since $\Delta\theta \neq \frac{\Delta w}{u_0}$, there is some vertical motion)

Phugoid Mode

Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - mg \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g \Delta z_E + u_0^2$$

$$C_L = C_{L0} = C_{W0}$$

$$L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \rho u_0^2 S C_{W0} + \rho g S C_{W0} \Delta z_E = W + \rho g S C_{W0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W0}}{m} \Delta z_E}_{\omega_n^2} = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = \boxed{\pi \sqrt{2} \frac{u_0}{g}} = 0.138 u_0 \leftarrow \text{if } u_0 \text{ in ft/s}\right. \\ \left. 0.453 u_0 \leftarrow \text{if } u_0 \text{ in m/s}\right.$$

for 747

$$\frac{\text{Full } A_{lon}}{T = 93s}$$

$$\frac{\text{Lanchester}}{T = 107s}$$

"ZxZ"

Phugoid Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} - & \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ & \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & -mg \sin \theta_0 \\ I_y \begin{bmatrix} M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \end{bmatrix} & 0 & I_y \begin{bmatrix} M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \end{bmatrix} & I_y \begin{bmatrix} M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \end{bmatrix} & -\frac{M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

Assume $Z_{\dot{w}} \ll m$

$Z_u \ll mu_0$

Δq small

$\Delta \alpha = 0$

$\theta_0 = 0$

$$\begin{bmatrix} \Delta u \\ \Delta \dot{w} \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m} \Delta u + u_0 \Delta q \rightarrow \Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$\lambda^2 - \frac{X_u}{m} \lambda - \frac{Z_u g}{mu_0} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$$

$$C_{z_u} = \underbrace{-M_0 \frac{\partial C_L}{\partial M}_0}_{\text{small}} - \underbrace{\rho u_0^2 \frac{\partial C_L}{\partial p}_0}_{\text{small}} - \underbrace{C_{T_u} \frac{\partial C_L}{\partial C_T}_0}_{\text{small}} - C_{T_u} \frac{\partial C_X}{\partial C_T}_0$$

$$C_{z_u} = 0$$

$$Z_u \approx -\rho u_0 S C_{W_0}$$

$$X_u = \rho u_0 S C_{W_0} \sin \theta^0 + \frac{1}{2} \rho u_0 S C_{X_u}$$

$$C_{X_u} = -2 C_{T_0} \quad (\text{constant thrust})$$

$$C_{T_0} = C_{D_0} + C_{W_0} \sin \theta^0$$

$$X_u \approx -\rho u_0 S C_{D_0}$$

$$\omega_n = \sqrt{-\frac{Z_{ug}}{m u_0}} = \sqrt{\frac{\rho S C_{W_0} g}{m}} \quad \text{Same as Lanchester}$$

$$\zeta = -\frac{X_u}{2} \sqrt{\frac{u_0}{m Z_{ug}}} = \frac{\rho u_0 S C_{D_0}}{2} \sqrt{\frac{u_0}{\frac{1}{2} mg \rho u_0 S C_{L_0}}}$$

$$= C_{D_0} \sqrt{\frac{\frac{1}{2} \rho u_0^2 S}{\frac{1}{2} mg} \frac{1}{C_{L_0}}}$$

$$\zeta = \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}} \quad \text{High } \zeta_D = \begin{matrix} \text{less energy} \\ \text{loss} \end{matrix} \\ = \text{less damping}$$

747

Full Δ_{10n}

$$\lambda_{3,4} = -3.29 \times 10^{-3} \pm 6.72 \times 10^{-2} i$$

$$\zeta = 0.0489$$

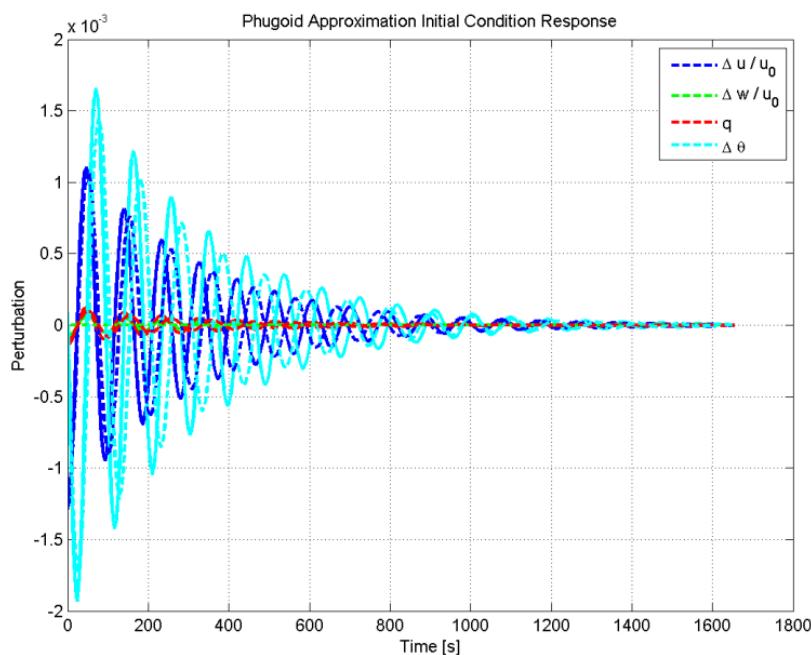
$$\omega_n = 0.0673$$

Ph Appr ox

$$\lambda_{ph} = -3.43 \times 10^{-3} \pm 6.11 \times 10^{-2} i$$

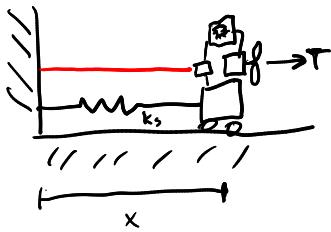
$$\zeta = 0.0561$$

$$\omega_n = 0.0612$$



Longitudinal Control

State Space Review



$$s_f = -k_s(x - x_0)$$

$$m\ddot{x} = -k_s(x - x_0)$$

$$m\Delta\ddot{x} = -k_s(\Delta x + \Delta T)$$

$$T_0 = 0$$

$$T = \Delta T$$

$$\vec{u} = [\Delta T]$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\Delta x} \\ \dot{\Delta \dot{x}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{k_s}{m} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \vec{u}$$

$$\Delta \dot{x} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \vec{u}$$

$$|A - \lambda I| = 0 = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_s}{m} & -\lambda \end{vmatrix} = \lambda^2 + \frac{k_s}{m} = 0$$

$$\lambda^2 + \underbrace{2\zeta\omega_n\lambda}_{0} + \underbrace{\omega_n^2}_{\frac{k_s}{m}} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \sqrt{\frac{k_s}{m}}$$

$$\Delta T = -k_p \Delta x - k_d \Delta \dot{x}$$

$$\vec{u} = -K \vec{x} \quad K = [k_p \quad k_d] \quad \vec{u} = [-k_p \quad -k_d] \vec{x}$$

$$\dot{\vec{x}} = A \vec{x} - B K \vec{x}$$

$$\dot{\vec{x}} = \underbrace{(A - BK)}_{A^{cl}} \vec{x}$$

$$A^{cl} = \begin{bmatrix} 0 & 1 \\ \frac{(k_s + k_p)}{m} & -\frac{k_d}{m} \end{bmatrix}$$

Longitudinal Control

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

δ_e = elevator + downward deflection

δ_t = throttle + more power

δ_p in book

dimensional control derivatives

$$\Delta X_c = X_{de} \delta_e + X_{dt} \delta_t$$

$$\Delta Z_c = Z_{de} \delta_e + Z_{dt} \delta_t \quad \text{often zero}$$

$$\Delta M_c = M_{de} \delta_e + M_{dt} \delta_t$$

747

$$B_{lon} = \begin{bmatrix} -0.000187 & 9.66 \\ -1.785 & 0 \\ -1.158 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + B_{lon} \vec{u}_{lon}$$

$$\begin{bmatrix} \dot{x}_{de} \\ \dot{z}_{de} \\ \dot{m} \\ \dot{M}_{de} \end{bmatrix} = \begin{bmatrix} \frac{X_{de}}{m} & \frac{X_{dt}}{m} \\ \frac{Z_{de}}{m-Z_w} & \frac{Z_{dt}}{m-Z_w} \\ \frac{M_{de}}{I_y} + \frac{M_w Z_{de}}{I_y(m-Z_w)} & \frac{M_{dt}}{I_y} + \frac{M_w Z_{de}}{I_y(m-Z_w)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$



$$Y = [0 \ 0 \ 1] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} + [0 \ 0] \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

$$C_Y = Y \rightarrow C_Y$$

$$\delta_e = 10^\circ$$

$$\delta_t = \frac{1}{6} \approx 0.05 \text{ rad}$$

Longitudinal Stability Augmentation

B 747

$$A_{sp} = \begin{bmatrix} -0.3151 & 773.98 \\ -0.0010 & -0.4285 \end{bmatrix}$$

$$|A - \lambda I| = (-0.3151 - \lambda)(-0.4285 - \lambda) - (-0.0010)(773.98)$$

$$\lambda = -0.372 \pm 0.889i$$

$$\omega_n = 0.964$$

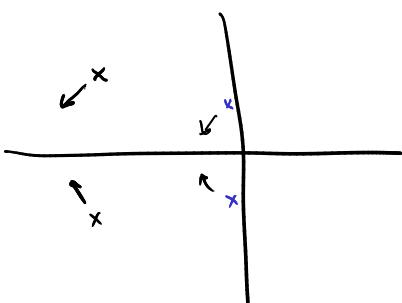
$$\zeta = 0.386 \times 0.7$$

$$\delta_e = -k_s \Delta q \quad \text{measured by rate gyro}$$

$$B_{sp} = \begin{bmatrix} -1.785 & 0 \\ -1.158 & 0 \end{bmatrix}$$

$$K_{sp} \vec{x}_{sp}$$

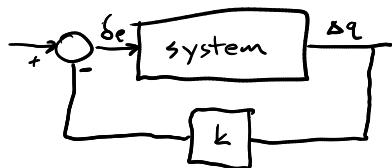
$$\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} = - \begin{bmatrix} 0 & k_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$



$$A^{cl} = A_{sp} - B_{sp}K_{sp} = A_{sp} - \begin{bmatrix} 0 & -17.85k_s \\ 0 & -1.158k_s \end{bmatrix}$$

$$= \begin{bmatrix} -0.3151 & 773.98 + 17.85k_s \\ -0.0010 & -0.4285 + 1.158k_s \end{bmatrix}$$

Matlab rlocus

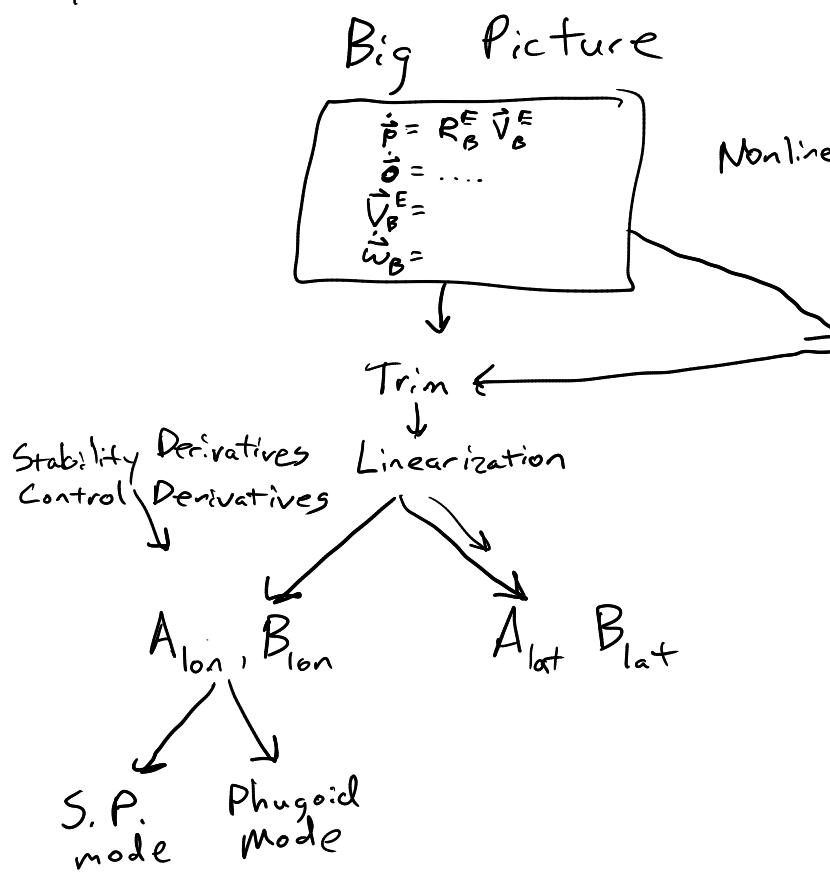


$$C_{\alpha_2} = [0 \ 1] \quad D_{\alpha_2} = [0]$$

$$B_{\delta e} = \begin{bmatrix} -17.85 \\ -1.158 \end{bmatrix}$$

More Accurate
Fewer Assumption

Lateral / Directional Dynamics



$$C_L = C_{L_{\text{atm}}} \alpha_{\text{trim}} + C_{L_{\text{se}}} \delta_{\text{trim}}$$

$$C_m = C_{m_{\text{atm}}} + C_{m_{\text{se}}} \alpha_{\text{trim}} + C_{m_{\text{se}}} \delta_{\text{trim}}$$

More Assumptions
Easier to Analyze

$$\rightarrow \Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

$$\rightarrow \Delta \dot{\theta} = \Delta q$$

$$\rightarrow \Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m}$$

$$\rightarrow \Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m}$$

$$\rightarrow \Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m}$$

$$\rightarrow \Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\rightarrow \Delta \dot{q} = \frac{\Delta M}{I_y}$$

$$\rightarrow \Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

$$Y = \frac{1}{2} \rho V^2 S C_Y (\beta, \rho, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2} \rho V^2 S b C_l (\beta, \rho, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2} \rho V^2 S b C_n (\beta, \rho, r, \delta_a, \delta_r)$$

$$\hat{P} = \frac{\rho b}{2 u_0}$$

$$Y \approx \frac{1}{2} \rho V_a^2 S \left[C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \frac{b}{2V_a} p + C_{Y_r} \frac{b}{2V_a} r + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right]$$

$$L \approx \frac{1}{2} \rho V_a^2 S b \left[C_{l_0} + C_{l_\beta} \beta + C_{l_p} \frac{b}{2V_a} p + C_{l_r} \frac{b}{2V_a} r + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \right]$$

$$N \approx \frac{1}{2} \rho V_a^2 S b \left[C_{n_0} + C_{n_\beta} \beta + C_{n_p} \frac{b}{2V_a} p + C_{n_r} \frac{b}{2V_a} r + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \right]$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 \underline{L_v} + \Gamma_4 N_v & \Gamma_3 \underline{L_p} + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

$$\begin{aligned} \Gamma_1 &= \frac{I_{xz} (I_x - I_y + I_z)}{\Gamma} & \Gamma_4 &= \frac{I_{xz}}{\Gamma} & \Gamma_7 &= \frac{I_x (I_x - I_y) + I_{xz}^2}{\Gamma} \\ \Gamma_2 &= \frac{I_z (I_z - I_y) + I_{xz}^2}{\Gamma} & \Gamma_5 &= \frac{I_z - I_x}{I_y} & \Gamma_8 &= \frac{I_x}{\Gamma} \\ \Gamma_3 &= \frac{I_z}{\Gamma} & \Gamma_6 &= \frac{I_{xz}}{I_y} & \Gamma &= I_x I_z - I_{xz}^2 \end{aligned}$$

Sideslip: β + $\beta \Rightarrow$ wind coming from right



$$\beta = \sin^{-1} \frac{\Delta V}{V}$$

$$\beta \approx \frac{\Delta V}{u_0} = \hat{V}$$

$$L = \frac{1}{2} \rho V^2 S b C_L$$

$$C_{l\beta} \equiv \frac{\partial C_L}{\partial \beta} = \frac{\partial C_L}{\partial \dot{V}}$$

$$L_v = \frac{\partial L}{\partial v} \Big|_0 = \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial \beta} \Big|_0 \frac{\partial \beta}{\partial v} \Big|_0$$

$$L_v = \frac{1}{2} \rho u_0 b S C_{l\beta}$$

$$\beta = \dot{V} = \frac{\Delta V}{u_0}$$

$$\frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

$$L_p = \frac{\partial L}{\partial p} \Big|_0 = \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial p} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 b S \frac{\partial C_L}{\partial \hat{p}} \Big|_0 \frac{\partial \hat{p}}{\partial p} \Big|_0$$

$$L_p = \frac{1}{4} \rho u_0 b^2 S C_{l\hat{p}}$$

$$C_{l\hat{p}} \equiv \frac{\partial C_L}{\partial \hat{p}}$$

$$\hat{p} = \frac{b p}{2 u_0}$$

Table 4.5
Lateral Dimensional Derivatives

	<i>Y</i>	<i>L</i>	<i>N</i>
<i>v</i>	$\frac{1}{2} \rho u_0 S C_{y\beta}$	$\frac{1}{2} \rho u_0 b S C_{l\beta}$	$\frac{1}{2} \rho u_0 b S C_{n\beta}$
<i>p</i>	$\frac{1}{4} \rho u_0 b S C_{y\hat{p}}$	$\frac{1}{4} \rho u_0 b^2 S C_{l\hat{p}}$	$\frac{1}{4} \rho u_0 b^2 S C_{n\hat{p}}$
<i>r</i>	$\frac{1}{4} \rho u_0 b S C_{y_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_r}$

Table 5.2
Summary—Lateral Derivatives

	<i>C_y</i>	<i>C_l</i>	<i>C_n</i>
<i>β</i>	$* -a_F \frac{S_F}{S} \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$
<i>p</i>	$* -a_F \frac{S_F}{S} \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
<i>r</i>	$* a_F \frac{S_F}{S} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

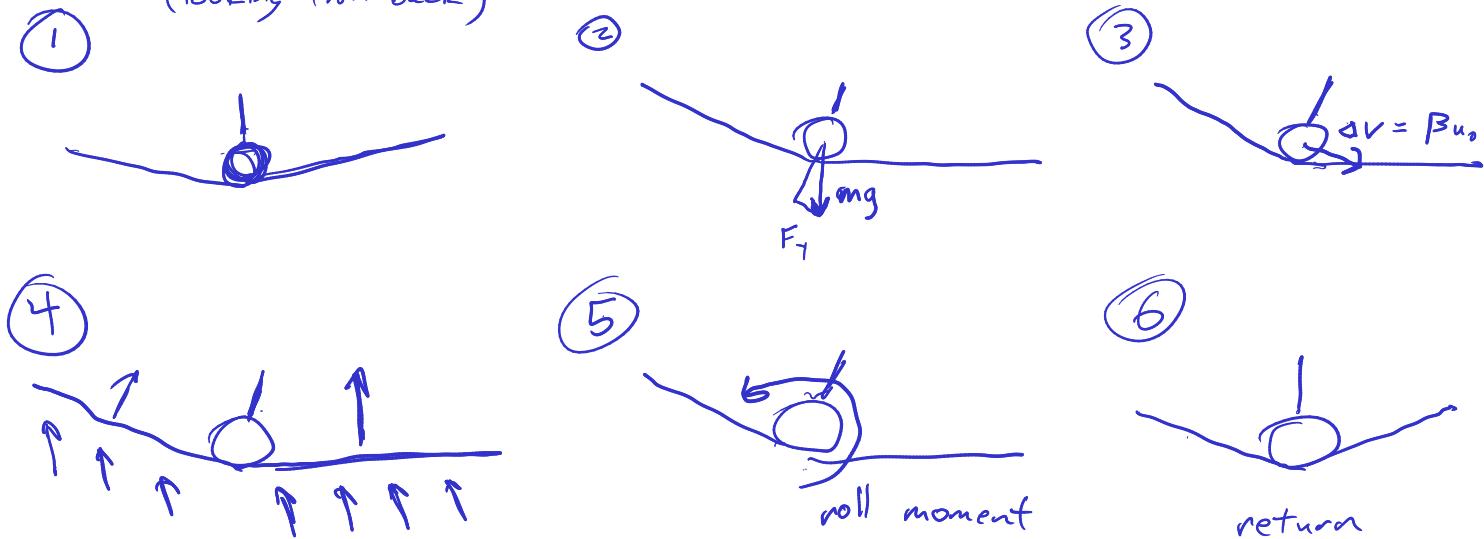
*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.

N.A. means no formula available.

$C_{l\beta}$: Dihedral Effect

sign: negative

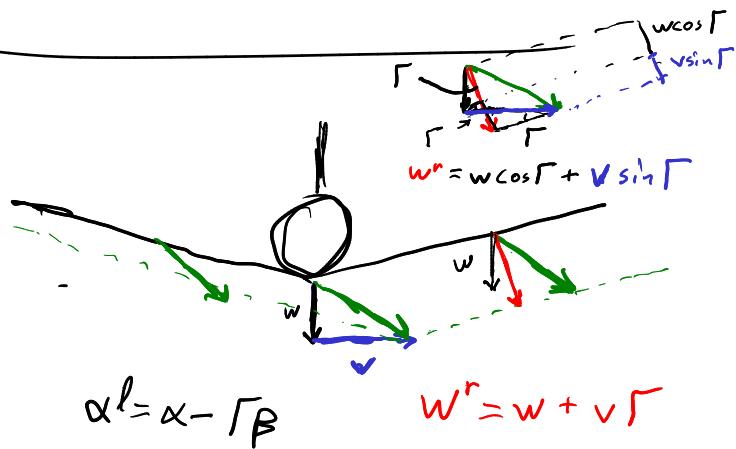
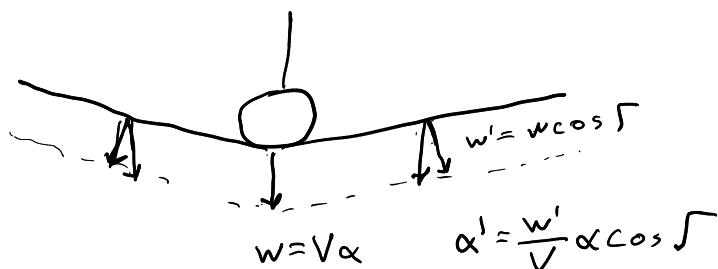
(looking from back)



4 significant factors

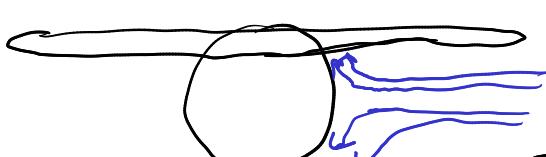
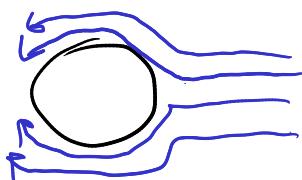
1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

1. Dihedral Angle

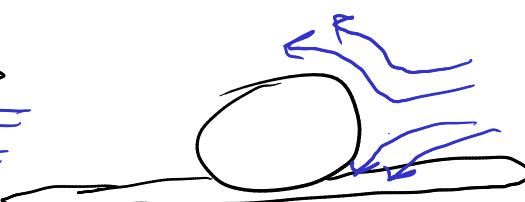


$$C_{l\beta} \propto \Gamma$$

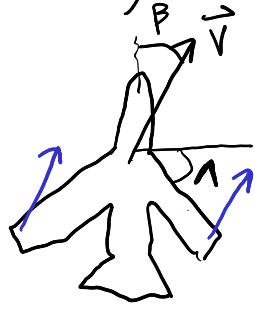
2. Wing Height



high wing $\Rightarrow -C_{l\beta}$
low wing $\Rightarrow +C_{l\beta}$

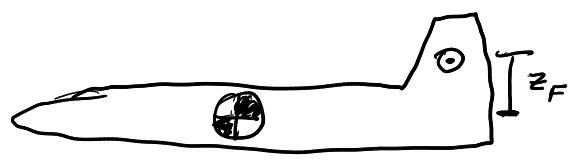


3. Wing Sweep



$$C_{L\beta}^1 \propto 2 C_L V^2 \sin 2\Lambda$$

4. Tail



$$\Delta C_L^F = C_{L_F} \frac{S_F z_F}{S_b} = \alpha_F (-\beta + \sigma) \frac{S_F z_F}{S_b}$$

$$C_{L\beta}^F = -\alpha_F \left(1 - \frac{\partial \sigma}{\partial \beta}\right) \frac{S_F z_F}{S_b} \left(\frac{V_F}{V}\right)^2$$



Lateral Stability Derivatives

Sideslip , Coordinated Turn

Table 5.2
Summary—Lateral Derivatives

	C_y	C_l	C_n
β	$* -a_F \frac{S_F}{S} \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$
\hat{p}	$* -a_F \frac{S_F}{S} \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
\hat{r}	$* a_F \frac{S_F}{S} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

*means contribution of the tail only, formula for wing-body not available; $V_F/V = 1$.

N.A. means no formula available.

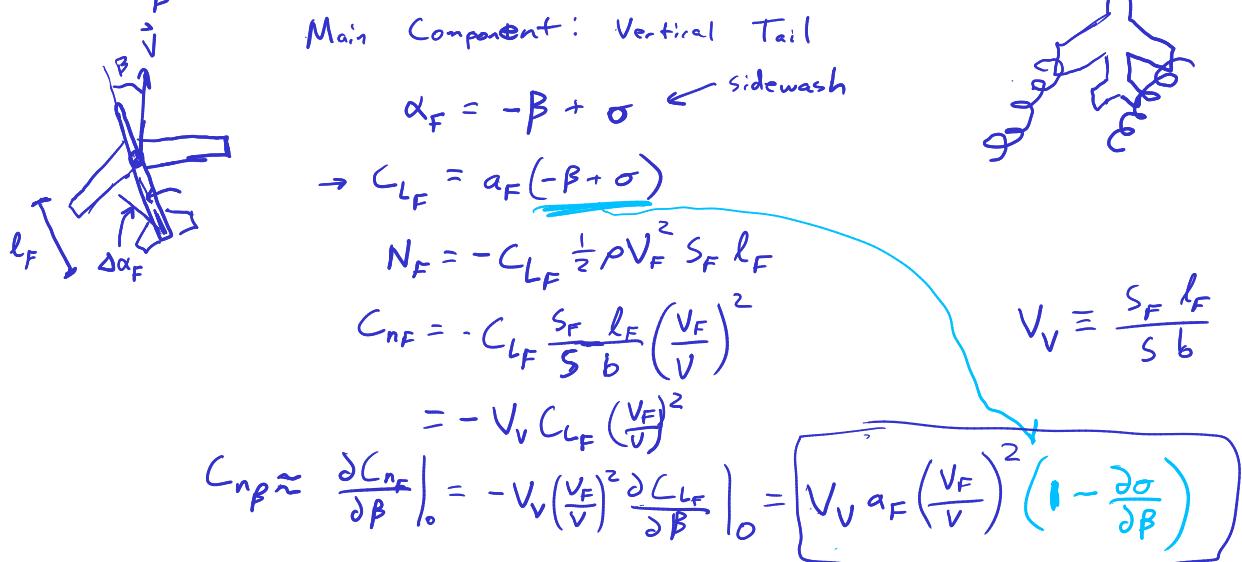
β derivatives

$C_{l\beta}$: Dihedral Effect

- 1) Wing Height
- 2) Dihedral Angle
- 3) Vertical Tail
- 4) Wing Sweep

$C_{n\beta}$: Weathervane Derivative Sign? (+)

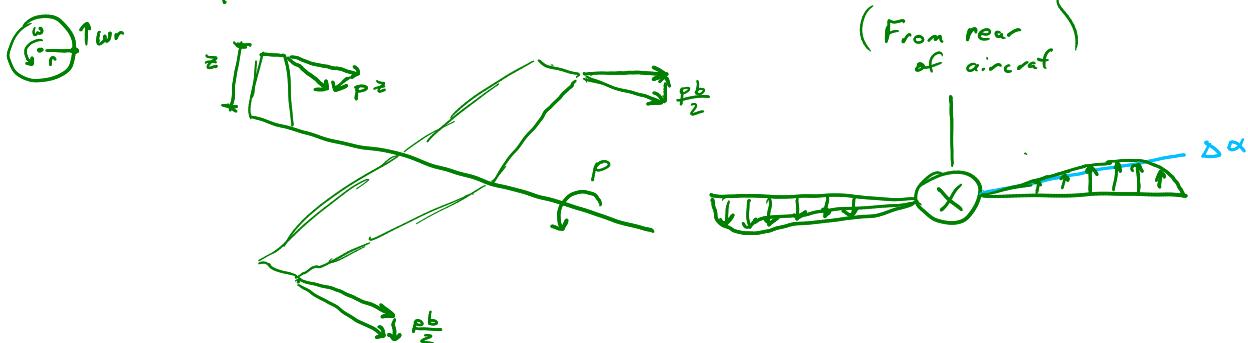
Main Component: Vertical Tail



$C_{y\beta}$ (usually small) (similar derivation to $C_{n\beta}$)

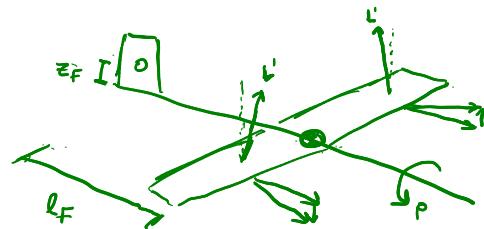
p derivatives

C_{l_p} roll damping derivative (-)



C_{np}

Wing Effects

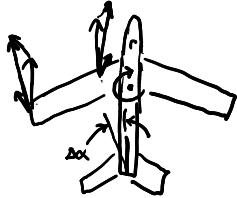


$$\text{Tail Effect } \Delta \alpha_F = -\frac{P z_F}{u_0} + P \frac{\partial \sigma}{\partial p} = -\hat{P} \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{P}} \right)$$

$$(\Delta C_n)_{tail} = -\Delta C_{Y_F} \frac{S_F}{S} \frac{l_F}{b} = \alpha_F V_V \hat{P} \left(2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{P}} \right)$$

$$(C_{np})_{tail} = \alpha_F V_V \left(2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{P}} \right)$$

C_{Yp} (usually small) (Similar derivation to $(C_{np})_{tail}$)

r-derivatives

$$\Delta \alpha_F = \frac{r l_F}{u_0} + r \frac{\partial \sigma}{\partial r}$$

$$= \hat{r} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{Yr})_{tail} = \alpha_F \frac{S_F}{S} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

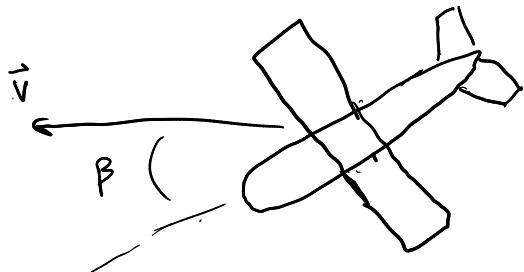
$$(C_{lr})_{tail} = \alpha_F \frac{S_F}{S} \frac{z_F}{b} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{nr})_{tail} = -\alpha_F V_V \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$
← yaw damping derivative



2 steady flight conditions (not "level")

Sideslip



$$\Delta Y + mg \Delta \phi = 0$$

$$\Delta L = 0$$

$$\Delta N = 0$$

can incorporate
1-engine out

$$\Delta Y = Y_v \Delta v + Y_p \delta_r + Y_a \delta_a + Y_s \delta_r$$

$$\begin{bmatrix} Y_{\delta_r} & 0 & mg \\ L_{\delta_r} & L_{\delta_a} & 0 \\ N_{\delta_r} & N_{\delta_a} & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \Delta \phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} \Delta v$$

invert

βu_0

Steady Sideslip

For Piper Cherokee



$$\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix}$$

βu_0 (7.8,4)

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta \doteq v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\delta_r/\beta = .303$$

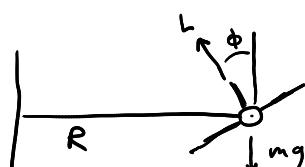
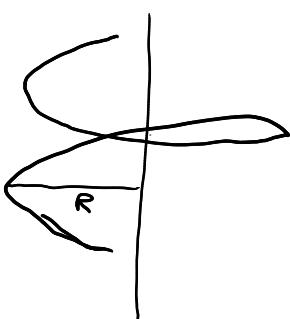
$$\delta_a/\beta = -2.96$$

$$\phi/\beta = .104$$

We see that a positive sideslip (to the right) of say 10° would entail left rudder of 3° and right aileron of 29.6° . Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only 1° to the right so the sideslip is almost flat.

Coordinated Turn

- angular velocity vector is constant and aligned with inertial \hat{z}
- No aerodynamic forces in Y direction



$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \sin \phi \cos \theta \\ \cos \phi \cos \theta \end{bmatrix} \omega$$



Since there is increased need for lift due to bank angle, involves both lat and lon dynamics

$$L \cos \phi = mg \cos \theta$$

$$L \sin \phi = m \frac{u_0^2}{R}$$

$$= m \omega u_0$$

$$\rightarrow \tan \phi = \frac{L \sin \phi}{L \cos \phi} = \frac{m \omega u_0}{m g \cos \theta} = \boxed{\frac{\omega u_0}{g \cos \theta}}$$

Assume: no wind, $\theta = \gamma$, $v \ll u$, $\cos \theta = 1$, $\sin \theta = \Theta$, $V = u = u_0$

From EOM:

$$Z = -mg \cos \phi - mqu$$

$$\text{load factor } n = -\frac{Z}{mg} = \cos \phi + \frac{qu}{g}$$

"g's"

$$= \cos \phi + \frac{\omega u_0 \sin \phi}{g}$$

$$= \cos \phi + \tan \phi \sin \phi$$

$$\boxed{n = \sec \phi}$$

$$\rightarrow n = \frac{L}{w}$$

$$\Delta C_L = \frac{L - mg}{\frac{1}{2} \rho V^2 S} = (n-1) C_w$$

Coordinated Turn

$$C_y = 0$$

$$C_l = 0$$

$$C_n = 0$$

$$C_m = 0$$

$$C_L = (n-1) C_w$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\Theta \omega \\ \sin \phi \omega \\ \cos \phi \omega \end{bmatrix}$$

$$= C_{y_B} \beta + C_{yP} \dot{\beta} + C_{y\delta_r} \delta_r + C_{y\delta_a} \delta_a$$

... ...

... ...

... ...

... .

$$\begin{bmatrix} C_{y_B} & C_{y\delta_r} & 0 \\ C_{l_P} & C_{l\delta_r} & C_{l\delta_a} \\ C_{n_P} & C_{n\delta_r} & C_{n\delta_a} \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} C_{y_P} & C_{y_n} \\ C_{l_P} & C_{l_n} \\ C_{n_P} & C_{n_n} \end{bmatrix} \begin{bmatrix} \Theta \\ -\cos \phi \end{bmatrix} \frac{w_b}{Z u_0}$$

$$\begin{bmatrix} C_{m_\alpha} & C_{m_{\delta e}} \\ C_{L_\alpha} & C_{L_{\delta e}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \delta e \end{bmatrix} = - \begin{bmatrix} C_{m_q} \\ C_{L_q} \end{bmatrix} \frac{w \bar{C} \sin \phi}{Z u_0} + \begin{bmatrix} 0 \\ (n-1) C_w \end{bmatrix}$$

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$



$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x}(t) = \sum_i c_i e^{\lambda_i t} \vec{v}_i$$

determined by initial condition

	λ_i	ζ
	-7.30e - 03	1.00e + 00
	-5.62e - 01	5.62e - 01
	$-3.30e - 02 + 9.47e - 01i$	$3.49e - 02$
	$-3.30e - 02 - 9.47e - 01i$	$9.47e - 01$

$\frac{\Delta v}{\Delta p}$	$\frac{\mathbf{v}_1}{\mathbf{v}_2}$	$\frac{\mathbf{v}_{3/4}}{-1.0000}$
$\frac{\Delta r}{\Delta \phi}$ $\frac{\Delta \psi}{\Delta y_E}$	$\begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix}$ $\begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix}$	$\begin{pmatrix} 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$

Flight Path State Space Dynamics Matrix

(also exists for lon)

$$\rightarrow \Delta \dot{\psi} = \Delta r \sec \theta_0$$

$$\Delta \dot{y}_E = u_0 \cos \theta_0 \Delta \Psi + \Delta v$$

$$\frac{\Delta z_E}{\Delta x_E}$$

$$\begin{bmatrix} \Delta v \\ \vdots \\ \Delta r \\ \Delta \phi \end{bmatrix} = A_{lat} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 & \sec \theta_0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \\ \Delta \psi \\ \Delta y_E \end{bmatrix}$$

Spiral Mode

$$\tilde{V}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{small} \\ \leftarrow \text{some} \\ \leftarrow \text{large} \\ \leftarrow \text{large} \end{array}$$

nondimensionalize

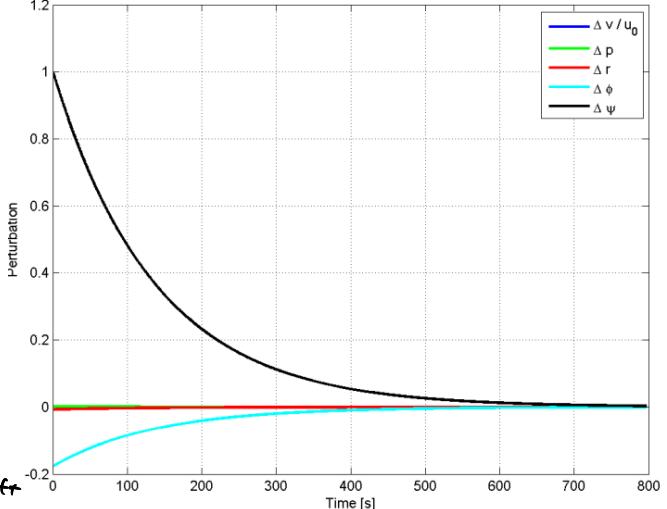
velocities, normalize

so that $\Delta\psi = 1$

$\lambda_1 = -0.0073 \leftarrow \text{stable for } T/47$
 $\tau = 137 \text{ s}$

unstable for many aircraft

Spiral Mode Initial Condition Response



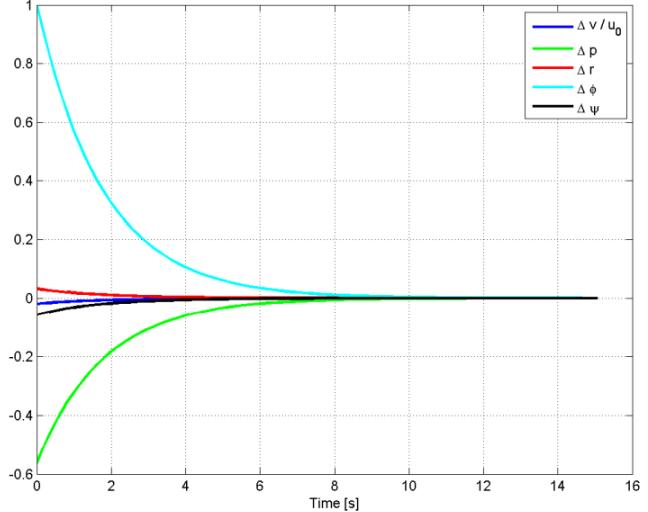
Roll

$$\tilde{V}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \end{array}$$

$$\lambda_2 = -0.5625$$

$$\tau = 1.78 \text{ s}$$

Roll Mode Initial Condition Response



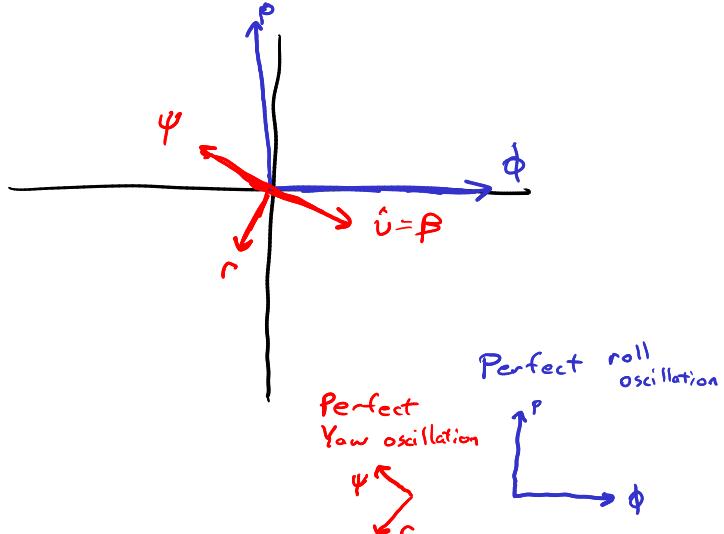
Dutch Roll

$$\tilde{V}_3 = \begin{bmatrix} 0.3271 \angle -28^\circ \\ 0.9471 \angle 92^\circ \\ 0.2915 \angle -112.3^\circ \\ 1.0 \\ 0.3078 \angle 155^\circ \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$

$$\lambda_{3/4} = -0.033 \pm 0.947i$$

$$\zeta = 0.0349 \leftarrow \text{not well-damped}$$

$$\omega_n = 0.947 \leftarrow \text{fast}$$



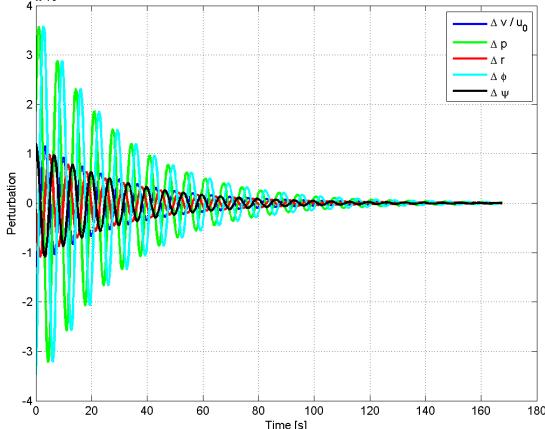
Phugoid (for comparison)

$$\lambda = -0.00329 \pm 0.0672i$$

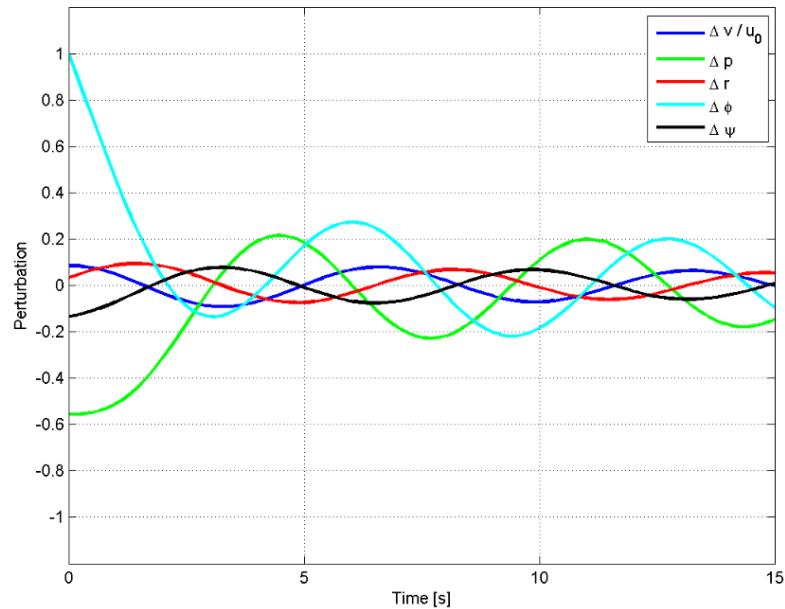
$$\zeta = 0.0981$$

$$\omega_n = 0.0673$$

Dutch Roll Mode Initial Condition Response



$$\mathbf{x}(0) = 0.4 \cdot Re(\mathbf{v}_r) + 0.4 \cdot Re(\mathbf{v}_{dr}) + 0.2 \cdot Re(\mathbf{v}_{spi})$$



Review: All modes

Name	Primary Variables	Fast / Slow	Damping
Short Period	α	Fast	Well-damped
Phugoid	Speed/altitude	Slow	Poorly
Roll	Roll rate	Fast	(over)
Spiral	Yaw, Roll	Slow	(over)/unstable
Dutch Roll	β , Roll	Fast	Poorly

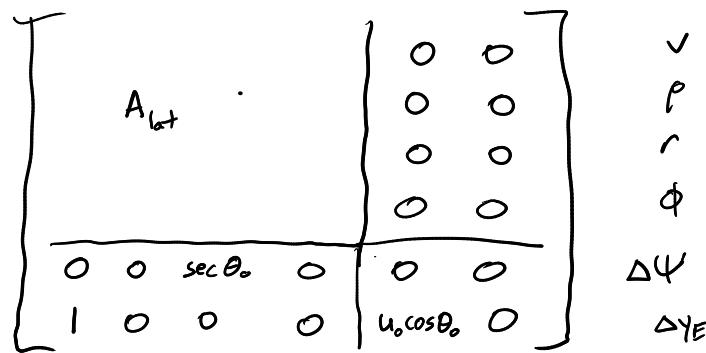
Trust the mathematical properties of A
over your intuition!

Lateral Mode Approximations and Control Surfaces

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix} = \mathbf{B} \mathbf{u}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \underline{\Gamma_3 L_v + \Gamma_4 N_v} & \underline{\Gamma_3 L_p + \Gamma_4 N_p} & \underline{\Gamma_3 L_r + \Gamma_4 N_r} & 0 \\ \underline{\Gamma_4 L_v + \Gamma_8 N_v} & \underline{\Gamma_4 L_p + \Gamma_8 N_p} & \underline{\Gamma_4 L_r + \Gamma_8 N_r} & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

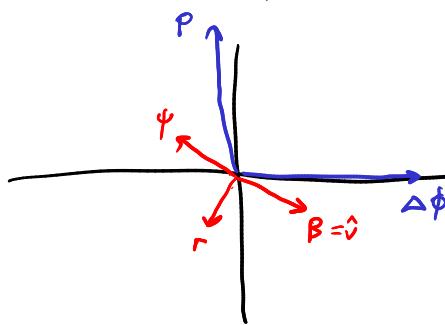
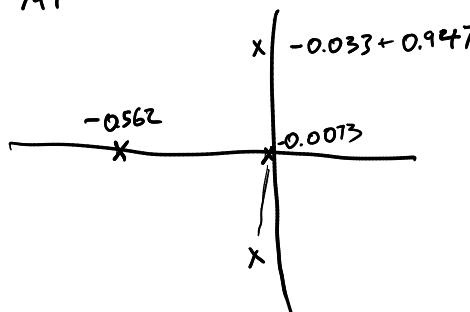


$$\lambda_i \quad \zeta \quad \omega_n$$

$$\begin{array}{lll}
 -7.30e-03 & 1.00e+00 & 7.30e-03 \\
 -5.62e-01 & 1.00e+00 & 5.62e-01 \\
 -3.30e-02 + 9.47e-01i & 3.49e-02 & 9.47e-01 \\
 -3.30e-02 - 9.47e-01i & 3.49e-02 & 9.47e-01
 \end{array}$$

$$\begin{array}{ccc}
 \begin{pmatrix} \mathbf{v}_1 \\ 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} &
 \begin{pmatrix} \mathbf{v}_2 \\ -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} &
 \begin{pmatrix} \mathbf{v}_{3/4} \\ -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix} \\
 \text{Spiral} & \text{Roll} & \text{Dutch Roll}
 \end{array}$$

747



$$A_{lat} = \begin{bmatrix} y_v & y_p & y_r & g \cos \theta_0 \\ l_v & l_p & l_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

2×2 Spiral Approx.

$$\tilde{v}_1 = \begin{bmatrix} -0.0012 \\ 0.0013 \\ -0.0073 \\ -0.1768 \\ 1.0 \end{bmatrix} \quad \begin{array}{l} p=0 \\ \dot{p}=0 \\ \text{Ignore side force} \end{array} \quad \begin{bmatrix} v \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} y_v & y_p & y_r & g \cos \theta_0 \\ l_v & l_p & l_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ r \\ \phi \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} l_v & l_r \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = l_v v + l_r r \quad \therefore \quad v = -\frac{l_r}{l_v} r$$

$$\dot{r} = -N_v \frac{l_r}{l_v} r + N_r r = \left(\frac{N_r l_v - N_v l_r}{l_v} \right) r$$

\downarrow $\leftarrow x = Ax$

$$|A - \lambda I| = 0 \quad \lambda = A \quad \text{if } A \text{ is a scalar}$$

$$\boxed{\lambda_{s, \text{approx}} = \left(\frac{N_r l_v - N_v l_r}{l_v} \right)} \quad = -0.0296 \quad \text{for B747}$$

$T = \frac{1}{\lambda} = 33.8 \text{ s}$

$$-0.0073 \quad T = 137 \text{ s}$$

Not a great approximation

Characteristic Eqn-Based Spiral Approx

$$|A_{lat} - \lambda I| = A \underline{\lambda^4} + B \underline{\lambda^3} + C \underline{\lambda^2} + D \underline{\lambda} + E = 0$$

since $\lambda_s \ll 1$

$$D \underline{\lambda} + E = 0$$

$$\boxed{\lambda_{s, \text{approx}} = -\frac{E}{D}}$$

$$\rightarrow E = g \left[(N_r l_v - N_v l_r) \cos \theta_0 + (N_v l_p - l_v N_p) \sin \theta_0 \right]$$

$$D = -g (l_v \cos \theta_0 + N_v \sin \theta_0) + u_0 (l_v N_p - l_p N_v)$$

$$\text{for B747} \quad \lambda_{s, \text{approx}} = -0.00725 \quad < 1\% \text{ error}$$

One necessary condition for stability is $E > 0$

$$(C_{l_p} C_{n_r} - C_{l_r} C_{n_p}) \cos \theta_0 + (C_{l_p} C_{n_p} - C_{l_r} C_{n_r}) \sin \theta_0 > 0$$

Roll Approximation

$$\dot{p} = L_p p$$

$$\lambda_{r, \text{approx}} = L_p = -0.434$$

$$\lambda_r = -0.562$$

23% difference

Roll + Spiral Approximation

Assume side force due to gravity produces same yaw rate that would exist with $\beta = 0$

$$0 = -u_0 r + g \phi \leftarrow$$

Also assume $Y_p = Y_r = 0$

$$\begin{bmatrix} 0 \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -u_0 & g \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\left| \tilde{A} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix} \right| = C \lambda^2 + D \lambda + E = 0$$

$$C = u_0 N_v$$

$$D = u_0 (L_v N_p - L_p N_v) - g L_v$$

$$E = g (L_v N_r - L_r N_v)$$

$$B747 \quad \lambda_{s, \text{approx}} = -0.00734 \quad \lambda_{r, \text{approx}} = -0.597$$

$$\text{"true"} \quad -0.0073 \quad -0.562$$

Dutch Roll Approx

Assume $\phi = p = 0 \quad Y_r = 0$

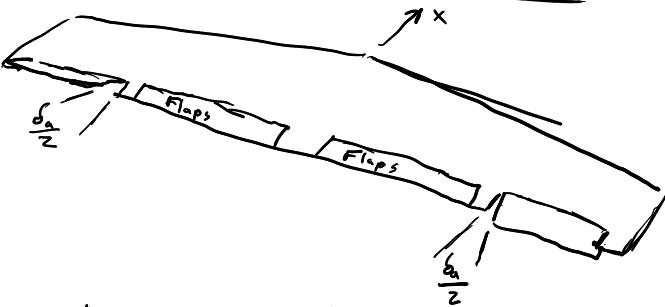
$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} Y_v & -u_0 \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$\lambda^2 - (Y_v + N_r) \lambda - (Y_v N_r + u_0 N_v) = 0$$

$$\lambda_{dr, \text{approx}} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Lateral Control Surfaces



$$\dot{\vec{x}}_{lat} = A_{lat} \vec{x}_{lat} + B_{lat} \vec{u}_{lat}$$

\vec{x}_{lat}

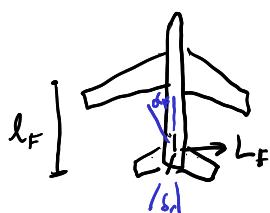
$$\vec{u}_{lat} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

$$B_{lat} = \begin{bmatrix} 0 & \frac{Y_{sr}}{m} \\ \Gamma_3 L_{\delta_a} + \Gamma_4 N_{\delta_a} & \Gamma_3 L_{\delta_r} + \Gamma_4 N_{\delta_r} \\ \Gamma_4 L_{\delta_a} + \Gamma_3 N_{\delta_a} & \Gamma_4 L_{\delta_r} + \Gamma_3 N_{\delta_r} \\ 0 & 0 \end{bmatrix}$$

Table 7.1
Dimensional Control Derivatives

	X	Z	M
δ_e	$C_{x_{\delta_e}} \frac{1}{2} \rho u_0^2 S$	$C_{z_{\delta_e}} \frac{1}{2} \rho u_0^2 S$	$C_{m_{\delta_e}} \frac{1}{2} \rho u_0^2 S \bar{c}$
δ_p	$C_{x_{\delta_p}} \frac{1}{2} \rho u_0^2 S$	$C_{z_{\delta_p}} \frac{1}{2} \rho u_0^2 S$	$C_{m_{\delta_p}} \frac{1}{2} \rho u_0^2 S \bar{c}$

	Y	L	N
δ_a	$C_{y_{\delta_a}} \frac{1}{2} \rho u_0^2 S$	$C_{l_{\delta_a}} \frac{1}{2} \rho u_0^2 S b$	$C_{n_{\delta_a}} \frac{1}{2} \rho u_0^2 S b$
δ_r	$C_{y_{\delta_r}} \frac{1}{2} \rho u_0^2 S$	$C_{l_{\delta_r}} \frac{1}{2} \rho u_0^2 S b$	$C_{n_{\delta_r}} \frac{1}{2} \rho u_0^2 S b$



$$N_F = -l_F L_F = -l_F \frac{1}{2} \rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$

$$C_{n_F} = \frac{N_F}{\frac{1}{2} \rho V^2 S b} = - \underbrace{\frac{l_F S_F}{S b} \left(\frac{V_F^2}{V^2} \right)}_{V_V} C_{L_F} = - V_V C_{L_F} \left(\frac{V_F^2}{V^2} \right)$$

$$C_{n_{\delta_r}} = \left. \frac{\partial C_{n_F}}{\partial \delta_r} \right|_0 = - V_V \left(\frac{V_F^2}{V^2} \right) \left. \frac{\partial C_{L_F}}{\partial \delta_r} \right|_0 = \boxed{- a_n V_V \left(\frac{V_F^2}{V^2} \right)}$$

a_n

$\gamma_{\delta_r}, N_{\delta_r}, L_{\delta_r}$

Ailerons

L_{δ_a} - Negative

Aileron reversal can occur due to twist in the wings

N_{δ_a} - Can be either sign

if $N_{\delta_a} > 0$ this is called adverse yaw

γ_{δ_a} - usually small

State Space \leftrightarrow Laplace

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

$$G_{yu}(s)$$

Review Properties of Laplace Transforms



$$\mathcal{L}[x(t)](s) = \int_0^\infty e^{-st} x(t) dt$$

$$x(t) \iff x(s)$$

Appendix A1

$$\dot{x}(t) \iff s x(s) - x(0)$$

$$\int_0^t x(\tau) d\tau \iff \frac{1}{s} x(s)$$

$$\alpha x(t) + \beta y(t) \iff \alpha x(s) + \beta y(s)$$

Transfer Function

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 u$$

$$\ddot{x} = -2\zeta\omega_n \dot{x} - \omega_n^2 x + \omega_n^2 u$$

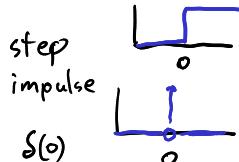
$$s^2 x(s) = -2\zeta\omega_n s x(s) - \omega_n^2 x(s) + \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) x(s) = \omega_n^2 u(s)$$

$$G_{xu}(s) = \frac{X(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{X } \ddot{x} = -\dot{x}^2 + u$$

No transfer function for nonlinear systems



$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$u(t) \rightarrow u(s)$$

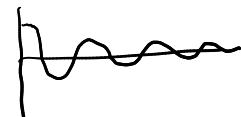
$$\begin{array}{ll} \text{step} & u(s) = \frac{1}{s} \\ \text{impulse} & u(s) = 1 \end{array}$$

$$X(s) = G_{xu}(s) u(s)$$

$$X(s) \rightarrow x(t)$$

table, method of partial fractions

$$\frac{b+s}{s^3 + a_3 s^2 + a_2 s + a_1} \rightarrow \frac{c_1}{s^2 + d_1} + \frac{c_2}{s^2 + d_2}$$



Final Value Theorem

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s x(s)$$

Only if $s x(s)$ is stable

Two representations of a Linear System

$$\begin{array}{l} \xrightarrow{\quad} \dot{x} = Ax + Bu \\ \xrightarrow{\quad} y = Cx + Du \end{array} \quad G_{yu}(s)$$

$$|A - \lambda I|$$

State space to TF
given SS want G_y

$$s x(s) = A x(s) + B u(s)$$

$$(sI - A)x(s) = B u(s)$$

$$x(s) = (sI - A)^{-1} B u(s)$$

$$y(s) = \underbrace{(C(sI - A)^{-1} B + D)}_{\text{from here on assume } D=0} u(s)$$

(from here on assume $D=0$)

$$M^{-1} = \frac{\text{adj}(M)}{|M|}$$

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

Adjugate: transpose of cofactor matrix F

$$F_{ij} = (-1)^{i+j} \left| M_{-i,-j} \right|$$

\nwarrow
 $-i \in \text{all rows except } i$

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & ; & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \vdots & ; & \vdots \\ \begin{vmatrix} a & b \\ d & e \end{vmatrix} & ; & -\begin{vmatrix} a & c \\ g & i \end{vmatrix} \end{bmatrix}$

$$G_{yu}(s) = \frac{y(s)}{u(s)} = C(sI - A)^{-1} B = \boxed{\frac{C \text{adj}(sI - A) B}{|sI - A|}} = \frac{N(s)}{D(s)}$$

$(sI - A) = 0$

Roots of $D(s)$ are the eigenvalues of A

stability \iff all roots of $D(s)$ in LHP
all eigenvalues of A in LAP

TF to SS

tf2ss

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

e.g. $\frac{Y(s)}{U(s)} = G(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$

multiply by $\frac{x(s)}{x(s)}$

$$\frac{y}{u} = \frac{b_0 s x + b_1 x}{s^2 x + a_1 s x + a_2 x}$$

$$\begin{aligned} y(+) &= b_0 \dot{x}(+) + b_1 x(+) \\ u(+) &= \dot{x}(+) + a_1 \dot{x}(+) + a_2 x(+) \\ \dot{x}(+) &= -a_1 \dot{x}(+) - a_2 x(+) + u \end{aligned}$$

$$\dot{\vec{x}} = A \vec{x} + B u$$

$$\rightarrow \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = C \vec{x} + D u$$

$$\rightarrow y = [b_1 \ b_0] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [0] u$$

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & & 1 \\ a_n & \cdots & & & a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [b_n \ a_n \ b_{n-1} \ \cdots \ b_1 \ -a_1 \ b_0] \quad D = [0]$$

\vec{x} may not correspond to interpretable physical variables

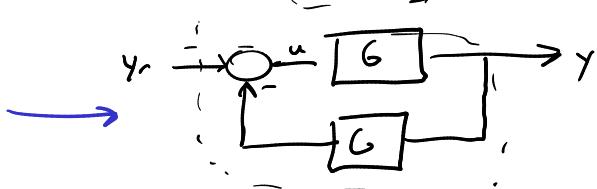
Why transfer function



$$G_3 = G_1 G_2$$

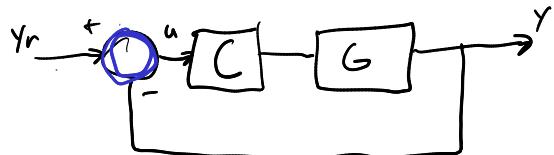


$$G_6 = G_4 + G_5$$



$$\begin{aligned} y &= G u \\ u &= y_r - C y \\ y &= G(y_r - C y) \\ (1 + G C) y &= G y_r \\ \frac{y}{y_r} &= \frac{G}{1 + G C} \end{aligned}$$

poles



$$y = G_C u$$

$$u = y_r - y$$

$$y = G C (y_r - y)$$

$$y(1+GC) = GC y_r$$

$$\frac{y}{y_r} = \frac{GC}{1+GC}$$

Example

$$A_{ph} = \begin{bmatrix} -0.0025 & -30 \\ 0.0001 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$B_{ph, \delta_t} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$C_{ph, \theta} = [0 \ 1]$$

$$D = [0]$$

1) find $G_{\theta \delta_t}(s)$

$$G_{\theta \delta_t}(s) = \frac{C \text{ adj}(sI - A) B}{|sI - A|}$$

$$G_{\theta \delta_t}(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0.0001 \end{bmatrix} \frac{10}{(s+0.002s)s - 0.0001 \cdot (-30)}$$

$$= \frac{0.001}{s^2 + 0.002s + 0.003}$$

$$sI - A = \begin{bmatrix} s+0.002s & -30 \\ -0.0001 & s \end{bmatrix}$$

$$F = \begin{bmatrix} & \\ & 0.0001 \end{bmatrix}$$

2) $\lim_{t \rightarrow \infty} \Delta \theta(t)$ in response to a δ_t step function with magnitude ε

$$\delta_t(+)=1(+)\varepsilon$$

$$\delta_t(s) = \frac{1}{s}\varepsilon$$

$$\Delta \theta(s) = G_{\theta \delta_t}(s) \delta_t(s) = \frac{0.001\varepsilon}{s(s^2 + 0.002s + 0.003)}$$

$$\lim_{s \rightarrow 0} s \Delta \theta(s) = \lim_{s \rightarrow 0} \frac{0.001\varepsilon}{s(s^2 + 0.002s + 0.003)} = \frac{0.001\varepsilon}{0.003} = \frac{\varepsilon}{3}$$

$$\begin{bmatrix} G_{y_1 u_1}(s) & G_{y_1 u_2}(s) \\ G_{y_2 u_1}(s) & G_{y_2 u_2}(s) \end{bmatrix} = \frac{C \text{ adj}(sI - A) B}{|sI - A|}$$

$$= \frac{\begin{bmatrix} N_{y_1 u_1}(s) & N_{y_1 u_2}(s) \\ N_{y_2 u_1}(s) & N_{y_2 u_2}(s) \end{bmatrix}}{|sI - A| = D(s)}$$

$$G_{y_1 u_2}(s) = \frac{N_{y_1 u_2}(s)}{D(s)}$$

Automatic Flight Control Systems

- Autopilot
 - Controls a/c trajectory w/o requiring input from pilot
- Stability Augmentation System (SAS) ← adjusting stability derivatives
 - Designed to improve dynamic stability
 - Relies on sensors, not on pilot inputs
- Control Augmentation System (CAS) ← adjusting control derivatives
 - Pilot input has two paths to control surfaces
 - Mechanical system
 - through the CAS electrical path
- Fly by Wire (FBW) ← choose your own derivatives
 - No mechanical link from pilot to actuators
 - (a CAS with complete authority)



Autopilot: 1910s
(A/C: Lockheed Vega (1930s))



SAS (F-104) 1950s



CAS (F-15) Early '70s



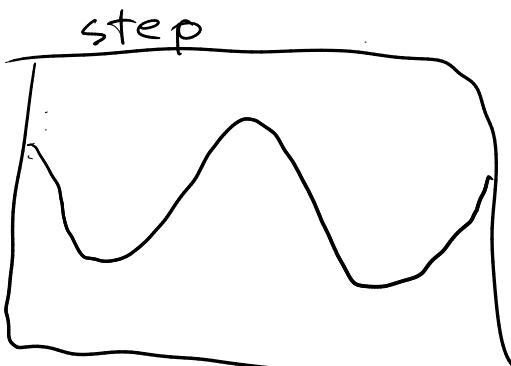
FBW (F-16) Late '70s

Control Design Task 1: Pitch Attitude Controller

Input: θ_r Goal: $\theta \rightarrow \theta_r$ quickly (minimal Phugoid)

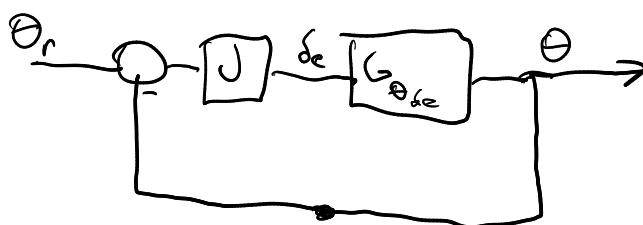


$J=1$

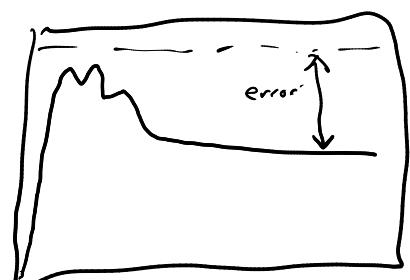
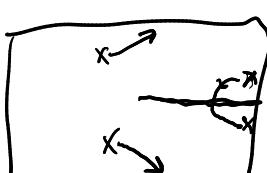


$J=K$

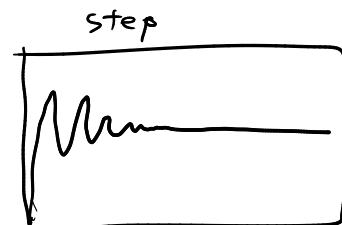
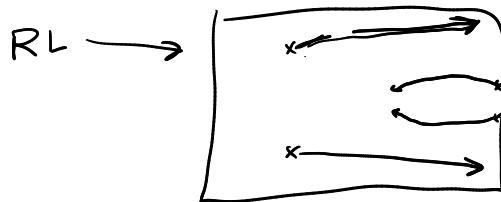
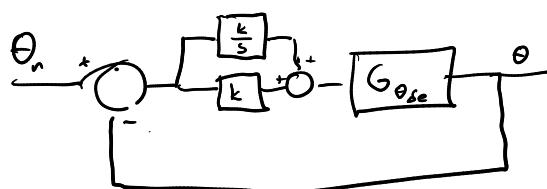
$K = -0.5$



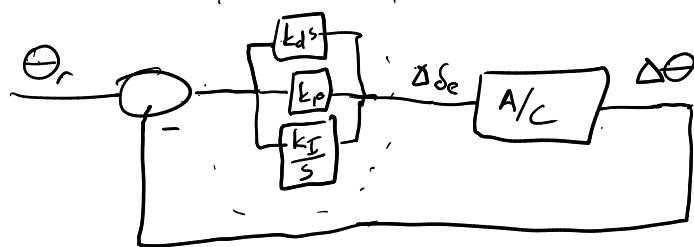
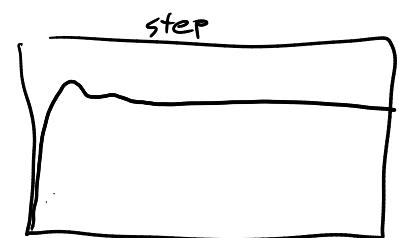
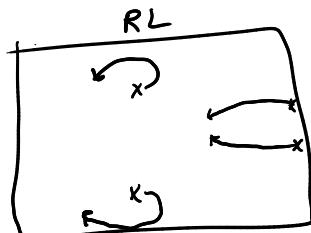
RL



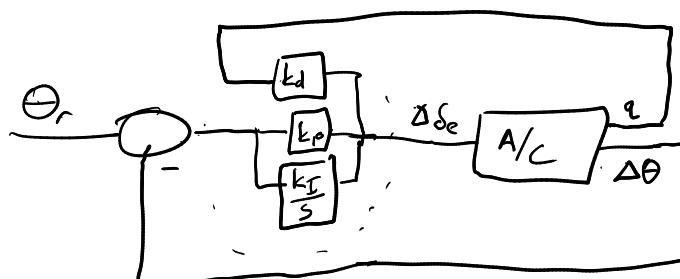
$$J = K \left(1 + \frac{1}{s}\right)$$



$$J = K \left(1 + \frac{1}{s} + s\right)$$



$\Delta \theta$ $\int \frac{d\Delta \theta}{dt}$ very large



$$\Delta \theta_e = -k_p \Delta \theta$$

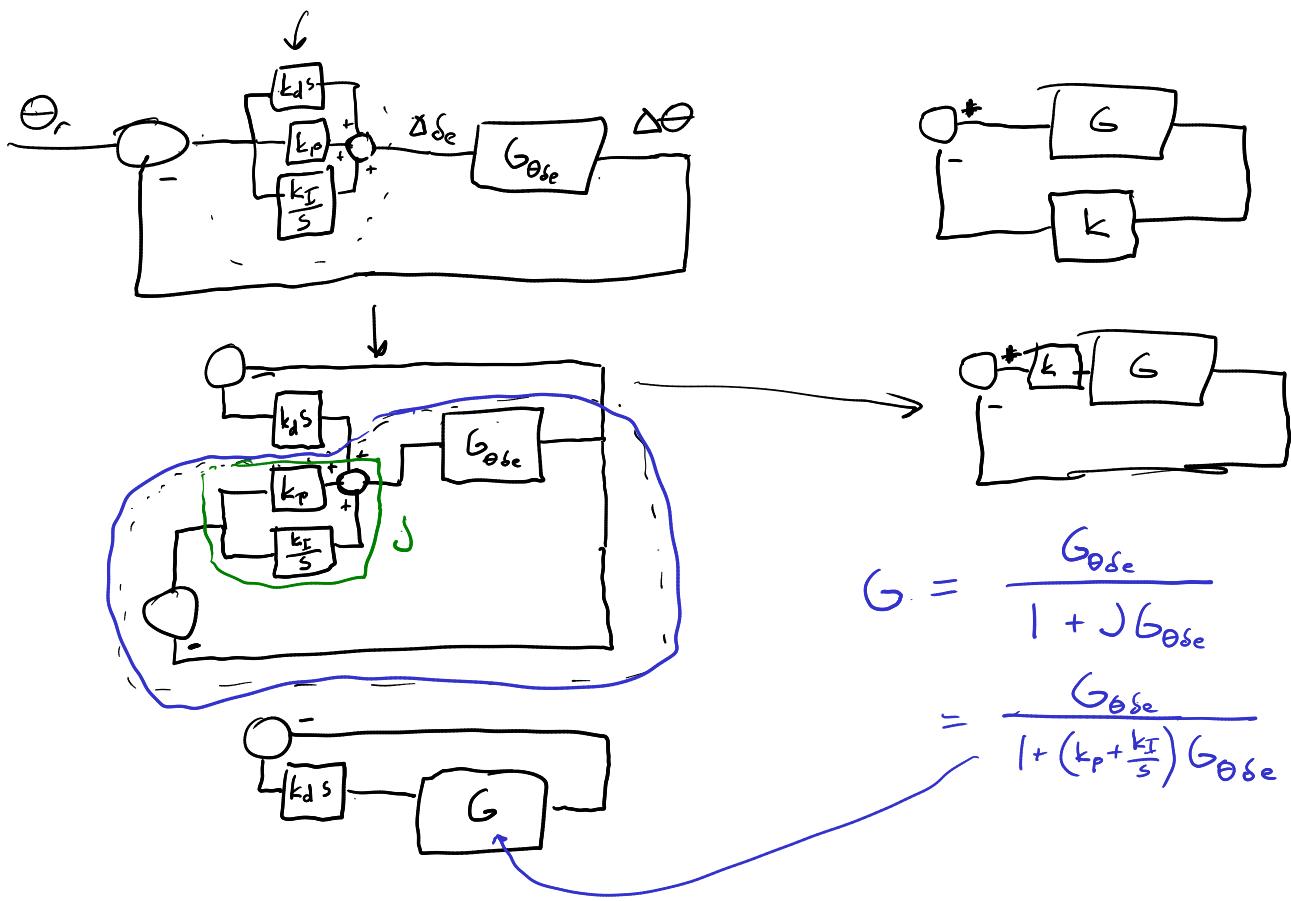
$$\bar{u} = -K \bar{x}$$

$$\begin{bmatrix} \dot{\theta}_e \\ \dot{\theta}_r \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & k_p \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{w} \\ q \\ \Delta \theta \end{bmatrix}$$

A-BK

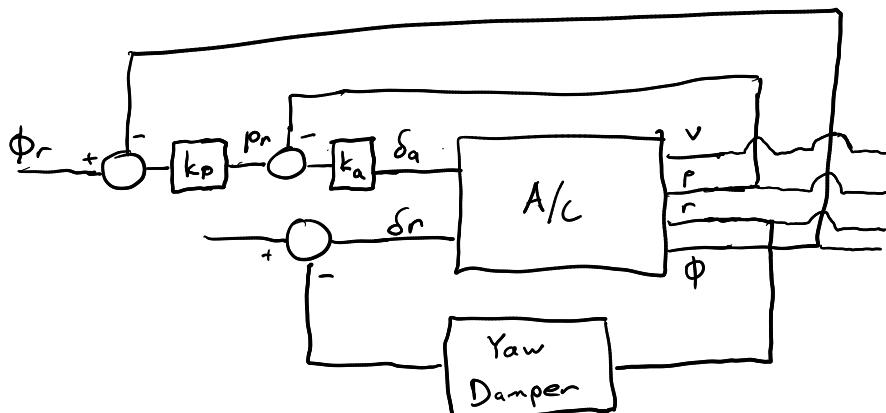
$$A_{lon} - \begin{bmatrix} -0.0002k_p \\ -17.85 \\ -1.158 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & k_p \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{lon} - \begin{bmatrix} 0 & 0 & 0 & -0.0002k_p \\ 0 & 0 & 0 & -17.85k_p \\ 0 & 0 & 0 & -1.158k_p \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Control Task 2: Roll Controller

Input: $\phi_r = \phi_c$ Goals: $\phi \rightarrow \phi_r$ quickly Alleviate Dutch Roll
 In Book



Part 1: Yaw Damper

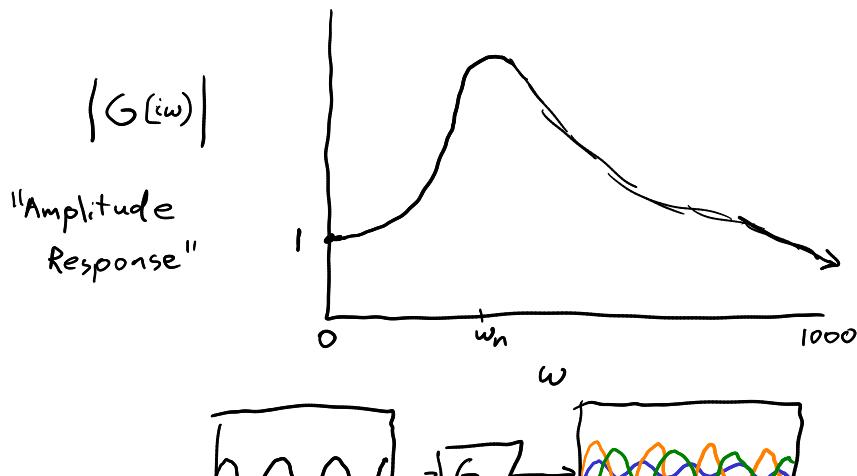
$$\delta_r = -k_r r \quad \text{from RL. choose } k_r = -1.9$$

Problem: In steady right turn for $k_r = -1.9$, rudder will be positive (wrong way for coordinated turn)

Want: $\delta_r \rightarrow 0$ at low frequency

$\delta_r = -k_r r$ near dutch roll freq.

Review: Frequency Response

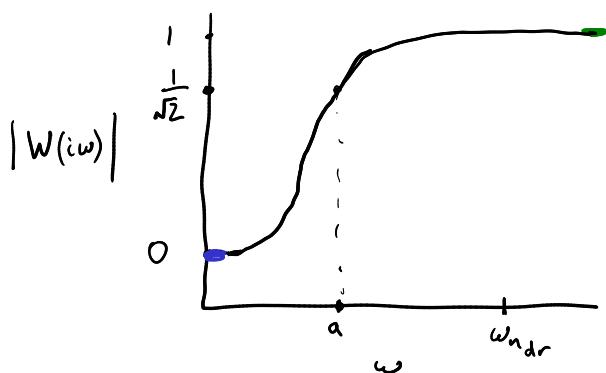


$$G(s) = \frac{\omega_n^2}{s^2 + 2s\omega_n s + \omega_n^2}$$

$$|G(0)| = 1$$

$$\lim_{\omega \rightarrow \infty} |G(i\omega)| = 0$$

"Washout" / high pass filter



$$W(s) = \frac{s}{s+a}$$

$$|W(0)| = 0$$

$$\lim_{\omega \rightarrow \infty} |W(\omega_i)| = \lim_{\omega \rightarrow \infty} \left| \frac{\omega_i}{\omega_i + a} \right| = 1$$

$$|W(a_i)| = \left| \frac{a_i}{a + a_i} \right| = \left| \frac{a e^{\frac{\pi i}{2}}}{\sqrt{2} a e^{\frac{\pi i}{2}}} \right| = \frac{1}{\sqrt{2}}$$

$$\frac{a_i}{a} \sqrt{\frac{\pi i}{a}}$$

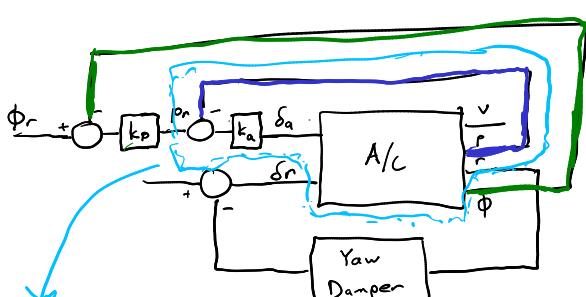
for 747 $\lambda_{dr} = -0.033 \pm 0.947i$

choose $a = 0.1$

$$\omega_d \quad \omega_{ndr} \approx 1$$

$$\delta_r = -k_r \frac{s}{s+0.1}$$

Part 2: Inner and Outer loop Aileron Controller



with $k_a = -1$
this looks like 0.25

$$\delta_a = (p_r - p) k_a$$

from A_{int}
 $\lambda_r = -0.5625$

$$\dot{p} = L_p p + L_{\delta_a} \delta_a$$

$$-0.4342 = \frac{1}{T} \quad 0.1431$$

$$\dot{p} = (L_p - L_{\delta_a} k_a) p + L_{\delta_a} k_a p_r$$

$$= \frac{1}{T} p$$

at steady state, $\dot{p} = 0$

choose $k_a = -1$

$$-\frac{L_{\delta_a} k_a}{L_p - L_{\delta_a} k_a} = 0.25$$

$$p_\infty = -\frac{L_{\delta_a} k_a}{L_p - L_{\delta_a} k_a} p_r$$

Not = 1

Outer Loop

Assume: $\dot{\phi} = p$, roll dynamics are much faster than ϕ response that we want

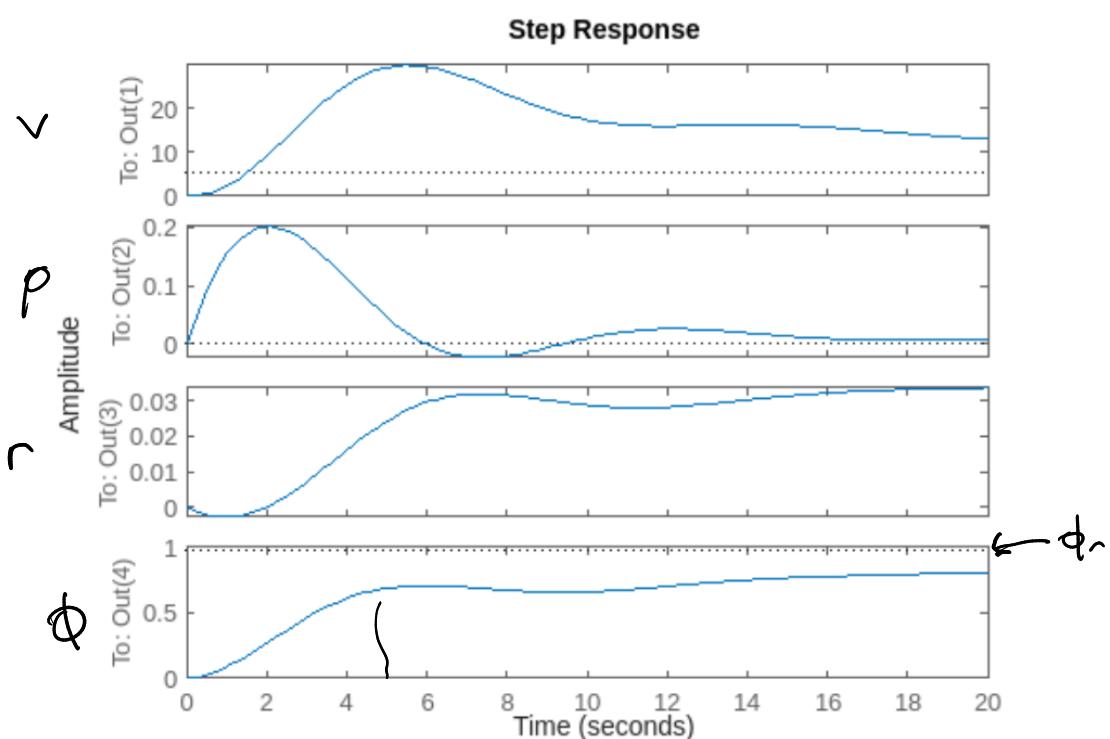


$$\dot{\phi} = 0.25 k_p (\phi_r - \phi)$$

$$\dot{\phi} = -0.25 k_p \phi + 0.25 k_p \phi_r$$

Based on Root Locus

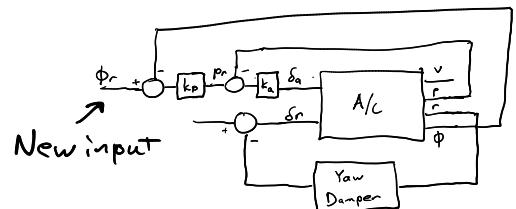
choose $k_p = 1.5$



State Space Gain Matrix for this Controller (without washout)

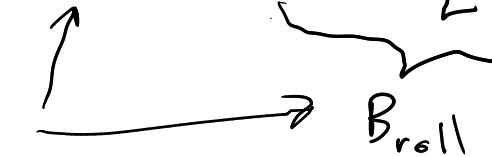
$$\dot{x} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = -K \dot{x} = -\underbrace{\begin{bmatrix} 0 & k_a & 0 & k_a k_p \\ 0 & 0 & k_r & 0 \end{bmatrix}}_K \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\begin{aligned} \delta_a &= -k_a p + k_a p_r \\ &= -k_a p - k_a k_p \phi \\ \delta_r &= -k_r r \end{aligned}$$



$$A_{roll} = A_{lat} - B_{lat} K$$

$$\dot{\vec{x}} = A_{roll} \vec{x} + B_{lat} \begin{bmatrix} k_p & k_a \\ 0 & 0 \end{bmatrix} \phi_c$$



Taste of Optimal Control

Techniques for choosing gains

1. Pole Assignment

$$A^{cl} = A - BK$$

solve for entries in K to achieve
desired A^{cl} eigenvalues

2. Hand Tuning (often PID)

3. Root Locus

4. Optimal Control



Optimization Problem

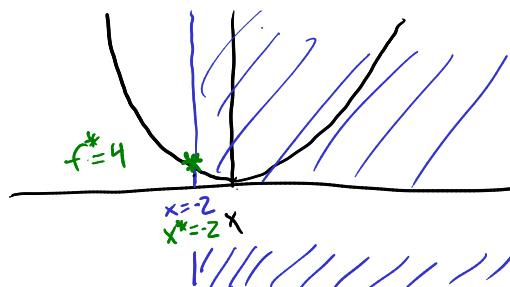
$$\begin{array}{l} \text{minimize } f(x) \leftarrow \text{objective} \\ \text{subject to } g(x) \leq 0 \leftarrow \text{constraints} \\ \text{optimization variables } x \end{array}$$

Engineer specifies f and g ; computer finds x^* that minimizes $f(x)$

Example

$$\underset{x}{\text{minimize}} \quad x^2$$

$$\text{subject to } x+2 \leq 0 \quad x \leq 2$$



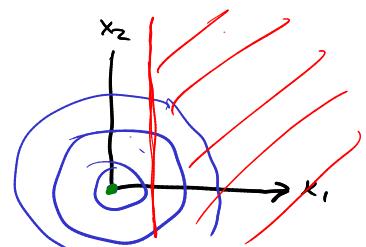
Example

$$\underset{\vec{x}}{\text{minimize}} \quad x_1^2 + x_2^2$$

$$\text{subject to } x_1 - 1 \leq 0$$

$$\nabla f = 0$$

$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Optimal Control

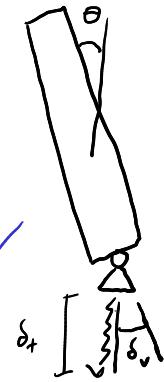
minimize $J(\vec{x}, \vec{u})$

$$\vec{x}(t), \vec{u}(t)$$

subject to $\dot{\vec{x}} = f(\vec{x}, \vec{u})$

$$\vec{x}_{\min} \leq \vec{x} \leq \vec{x}_{\max}$$

$$\vec{u}_{\min} \leq \vec{u} \leq \vec{u}_{\max}$$



minimize $\| \vec{x}(T) - \vec{x}_{\text{target}} \|$

subject to $\text{fuel}(t) \geq 0$

$$\theta_{\min} \leq \theta(t) \leq \theta_{\max}$$

Special Problem : LQR

"Linear Quadratic Regulator"

$$\underset{\vec{u}(t), \vec{x}(t)}{\text{minimize}} \quad J(\vec{x}, \vec{u}) = \int_0^{\infty} \vec{x}(t)^T Q \vec{x}(t) + \vec{u}(t)^T R \vec{u}(t) dt$$

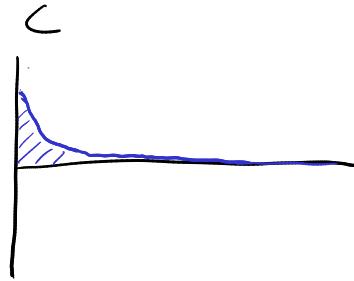
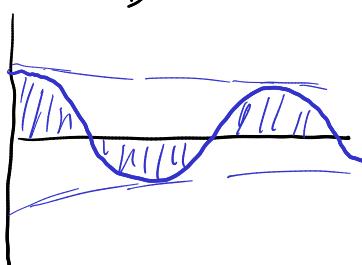
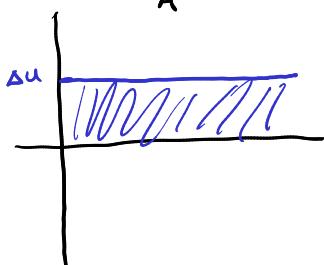
subject to $\dot{\vec{x}}(t) = A \vec{x}(t) + B \vec{u}(t)$

$$\vec{x}(0) = \vec{x}_0$$

Ex: Long Dynamics

$$Q = \begin{bmatrix} c_u & 0 & 0 & 0 \\ 0 & c_w & 0 & 0 \\ 0 & 0 & c_q & 0 \\ 0 & 0 & 0 & c_\theta \end{bmatrix} \Rightarrow \vec{x}^T Q \vec{x} = c_u \Delta u^2 + c_w \Delta w^2 + c_q \Delta q^2 + c_\theta \Delta \theta^2$$

$$R = \begin{bmatrix} c_{\delta_e} & 0 \\ 0 & c_{\delta_r} \end{bmatrix} \Rightarrow \vec{u}^T R \vec{u} = c_{\delta_e} \Delta \delta_e^2 + c_{\delta_r} \Delta \delta_r^2$$



$$J(\vec{x}, \vec{u}) = \int_0^{\infty} \vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u} dt$$

$$c_u \Delta u^2$$

Relatively Large $Q \Rightarrow$ smaller state deviations

Relatively Large $R \Rightarrow$ smaller control deviations

Linear Analytic Solution !

$$\underline{u(t) = -K x(t)}$$

$$\underset{x, u}{\text{minimize}} \quad J(x, u) = \int_0^\infty x^T Q x + u^T R u \, dt$$

↓
cost change
 $J_o(x, u) = x(0)^T Q x(0)$
+ $u(0)^T R u(0)$

$$V^*(x(\tau)) = \min \int_{\tau}^{\infty} x^T Q x + u^T R u \, dt$$

$$J^* = J^*(x^*, u^*) = V^*(x(0))$$

$$V^*(x(\tau)) = \min \left\{ \int_{\tau}^{\tau+d\tau} x^T Q x + u^T R u \, dt + \int_{\tau+d\tau}^{\infty} x^T Q x + u^T R u \, dt \right\}$$

immediate future

$$= \min \left\{ (x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)) d\tau + V^*(x(\tau+d\tau)) \right\}$$

$$\cancel{V^*(x(\tau))} = \min \left\{ \dots + \cancel{V^*(x(\tau))} + \frac{d}{dt} V^*(x(\tau)) + \frac{\partial V^*}{\partial x} |_{x(\tau)} \cdot \dot{x} \right\}$$

$\rightarrow A_x + B_u$

$$O = \min_u \left\{ x^T Q x + u^T R u + (A_x + B_u) \cdot \nabla V^* \right\}$$

assumption $V^*(x(\tau)) = x(\tau)^T S x(\tau)$

$$\nabla V^* = 2Sx$$

$$O = \min_{u(\tau)} \underbrace{(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) + (A x(\tau) + B u(\tau)) \cdot 2Sx(\tau))}_{f(u)}$$

minimized when $\nabla f = 0$

$$\nabla f = 2R_u + B^T 2Sx = 0$$

$$u = -R^{-1}B^T S x$$

K

$$\boxed{u = -Kx}$$

S is a solution to

$$O = Q + A^T S + S A + S B R^{-1} B^T S$$

In matlab

$$\boxed{K = lqr(A, B, Q, R)}$$

Traditional Methods:

Use RL, hand tuning, Pole assignment \rightarrow choose K directly

LQR

Choose $Q, R \rightarrow$ computer calculates K