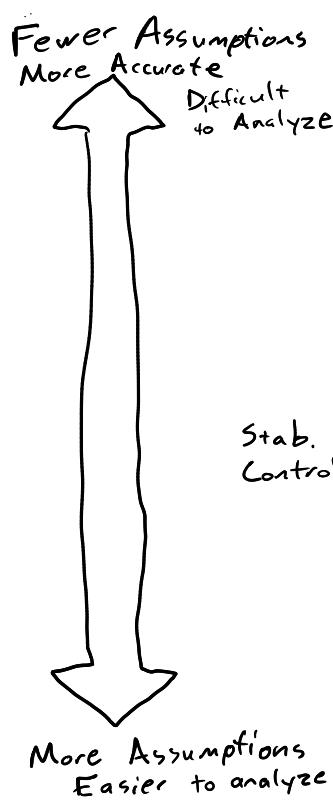
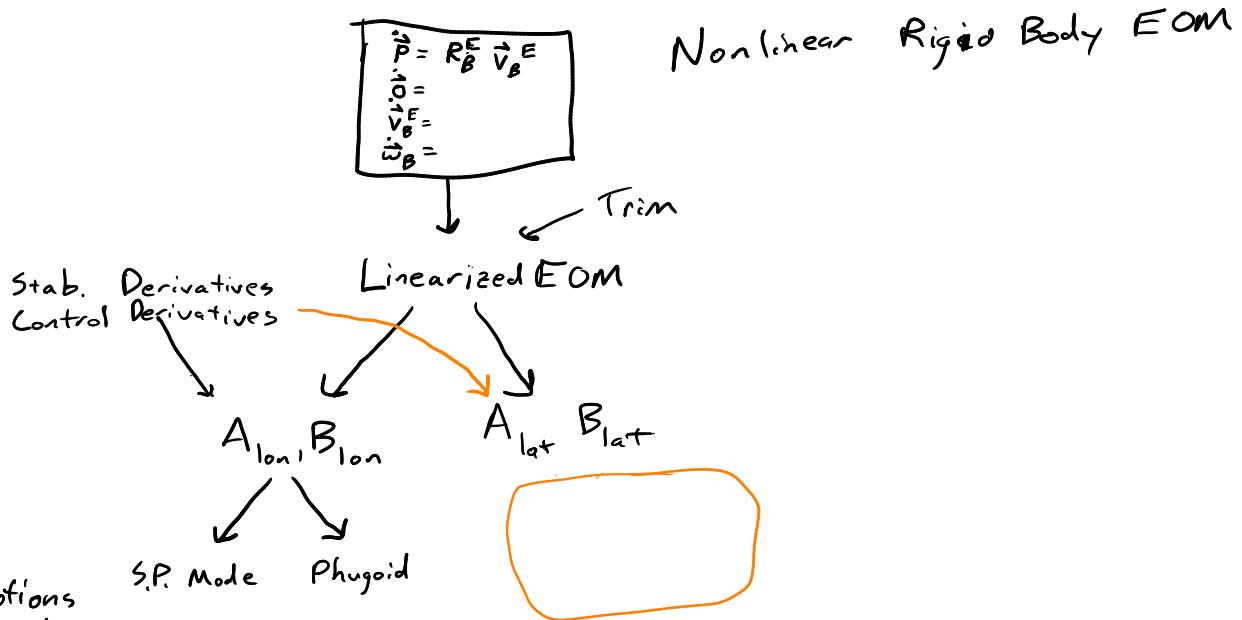


Lateral / Directional Dynamics



Big Picture



$$\Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0 \quad \leftarrow \text{Lat}$$

$$\Delta \dot{\theta} = \Delta q \quad \leftarrow \text{long}$$

$$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m} \quad \leftarrow$$

$$\Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m} \quad \leftarrow$$

$$\Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m} \quad \leftarrow$$

$$\Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N \quad \leftarrow$$

$$\Delta \dot{q} = \frac{\Delta M}{I_y} \quad \leftarrow$$

$$\Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N \quad \leftarrow$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y}$$

$$\Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$Y = \frac{1}{2} \rho V^2 S C_Y(\beta, \rho, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2} \rho V^2 S_b C_L(\beta, \rho, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2} \rho V^2 S_b C_n(\beta, \rho, r, \delta_a, \delta_r)$$

$$Y \approx \frac{1}{2} \rho V_a^2 S \left[C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \left(\frac{b}{2V_a} p \right) + C_{Y_r} \frac{b}{2V_a} r + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right]$$

$$L \approx \frac{1}{2} \rho V_a^2 S b \left[C_{l_0} + C_{l_\beta} \beta + C_{l_p} \frac{b}{2V_a} p + C_{l_r} \frac{b}{2V_a} r + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \right]$$

$$N \approx \frac{1}{2} \rho V_a^2 S b \left[C_{n_0} + C_{n_\beta} \beta + C_{n_p} \frac{b}{2V_a} p + C_{n_r} \frac{b}{2V_a} r + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \right]$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ . & 1 & \tan \theta_0 & 0 \end{pmatrix}$$



+ β : wind coming from right

$$\beta = \sin^{-1} \left(\frac{\Delta v}{V} \right)$$

$$\beta \approx \frac{\Delta v}{u_0} = \hat{v}$$

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2} \rho u_0 S C_{y_\beta}$	$\frac{1}{2} \rho u_0 b S C_{l_\beta}$	$\frac{1}{2} \rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4} \rho u_0 b S C_{y_p}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_p}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4} \rho u_0 b S C_{y_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_r}$

$$L = \frac{1}{2} \rho V^2 S b C_L \checkmark \text{ nondimensional use lower-case for moments}$$

$$\rightarrow L_v \equiv \frac{\partial L}{\partial v} \Big|_0 = \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial \beta} \Big|_0 \frac{\partial \beta}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0 S b C_{l_\beta}$$

$$C_{l_\beta} \equiv \frac{\partial C_L}{\partial \beta} \Big|_0 = \frac{\partial C_L}{\partial \hat{v}} \Big|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0} \Rightarrow \frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

Table 5.2
Summary—Lateral Derivatives

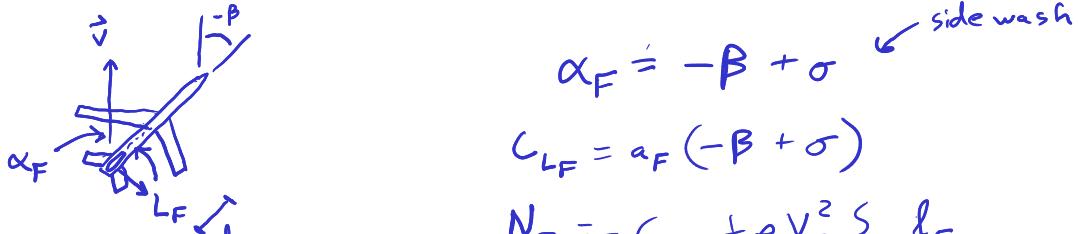
	C_y	C_l	C_n
β	$* -a_F \frac{S_F}{S} \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$
\hat{p}	$* -a_F \frac{S_F}{S} \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left(2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
\hat{r}	$* a_F \frac{S_F}{S} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.

N.A. means no formula available.

β derivatives

C_{n_β} : weathervane derivative / yaw stiffness Sign? +



$$\alpha_F = -\beta + \sigma \quad \text{side wash}$$

$$C_{L_F} = a_F (-\beta + \sigma)$$

$$N_F = - C_{L_F} \frac{1}{2} \rho V_F^2 S_F l_F$$

$$C_{n_F} = - C_{L_F} \frac{S_F l_F}{S b} \left(\frac{V_F}{V} \right)^2$$

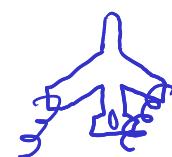
$$V_V = \frac{S_F l_F}{S b}$$

$$= - V_V C_{L_F} \left(\frac{V_F}{V} \right)^2$$

$$(C_{n_\beta})_{\text{Tail}} = \frac{\partial C_{n_F}}{\partial \beta} \Big|_0 = - V_V \left(\frac{V_F}{V} \right)^2 \frac{\partial C_{L_F}}{\partial \beta} \Big|_0$$

$$a_F \left(-1 + \frac{\partial \sigma}{\partial \beta} \right)$$

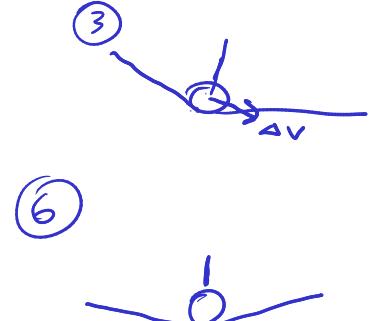
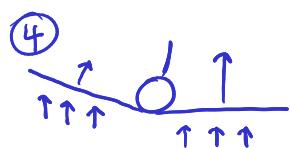
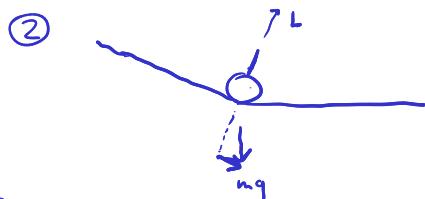
$$(C_{n_\beta})_{\text{Tail}} = V_V a_F \left(\frac{V_F}{V} \right)^2 \left(1 - \frac{\partial \sigma}{\partial \beta} \right)$$



C_{y_β} : usually small, similar derivation to C_{n_β} Sign? -

C_{l_β} : Dihedral Effect Sign? -

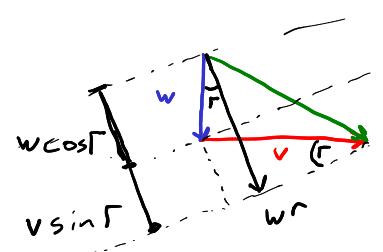
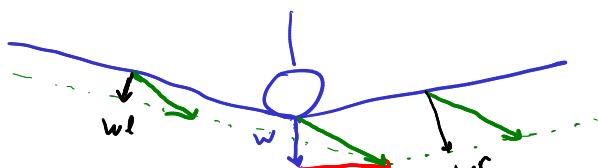
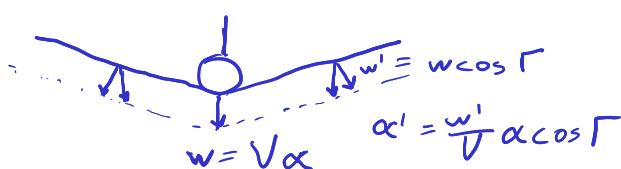
(looking from behind)



4 Significant Factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

1 Dihedral Angle



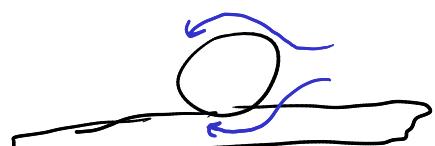
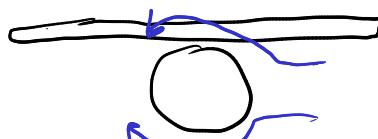
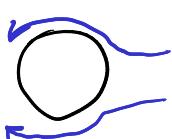
$$w^r = w \cos \Gamma + v \sin \Gamma \approx w + v \Gamma$$

$$\alpha^l \approx \frac{w^l}{u_0} = \alpha - \beta \Gamma$$

$$\alpha^r \approx \frac{w^r}{u_0} = \alpha + \beta \Gamma$$

$$\boxed{\begin{aligned} L \alpha (\alpha^l - \alpha^r) \\ C_{L\beta} \alpha - \Gamma \end{aligned}}$$

2. Wing Height



High Wing: -ve $C_{L\beta}$ contribution

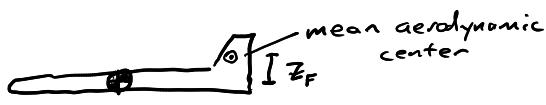
Low Wing: +ve $C_{L\beta}$ contribution

3. Wing Sweep



$$C_{\ell_B}^A \propto 2 C_L V^2 \sin(\alpha) \\ + \lambda \Rightarrow + C_{\lambda_B}$$

4. Tail



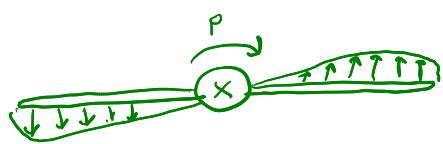
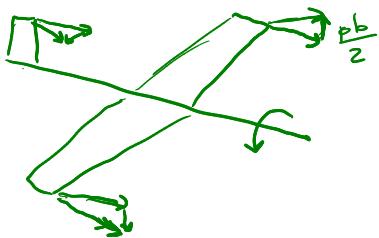
$$\Delta C_L^F = C_{L_F} \frac{S_F z_F}{S_b} = \alpha_F \left(-\beta + \sigma \right) \frac{S_F z_F}{S_b} \\ C_{\ell_B}^F = -\alpha_F \left(1 - \frac{\partial \sigma}{\partial \beta} \right) \frac{S_F z_F}{S_b} \left(\frac{V_F}{V} \right)^2$$



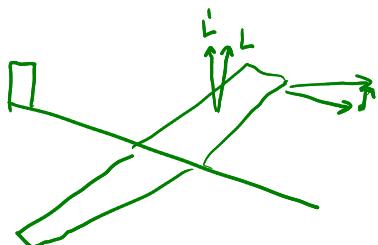
p-derivatives

$$\hat{p} = \frac{\rho b}{2V}$$

$C_{\ell p}$: roll damping Sign? -



C_{n_p} Wing Effect



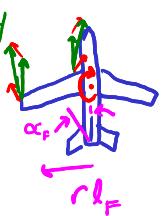
$$(C_{n_p})_{tail} = \alpha_F V_v \left(2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{p}} \right)$$

C_{y_p} (usually small)

(similar to derivation for $(C_{n_B})_{tail}$)

r-derivatives

Wing-Body



$$\Delta \alpha_F = \frac{r l_F}{u_0} + r \frac{\partial \sigma}{\partial r}$$

$$= \hat{r} \left(\frac{2 l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\frac{\partial \alpha_F}{\partial \hat{r}} = \left(\frac{2 l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\vec{\omega} \times \vec{v}_P$$

$$C_y = \frac{Y}{\frac{1}{2} \rho u_0^2 S}$$

$$(C_y)_{tail} = \frac{\frac{1}{2} \rho u_0^2 S_F \alpha_F \hat{r}}{\frac{1}{2} \rho u_0^2 S}$$

$$(C_y)_{tail} = \frac{\partial (C_y)_{tail}}{\partial \hat{r}} \Big|_0 = \alpha_F \frac{S_F}{S} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{\ell r})_{tail} = \alpha_F \frac{S_F}{S} \frac{z_F}{b} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial z} \right)$$

$$(C_{nr})_{tail} = -\alpha_F \frac{V_v}{u_0} \left(2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\frac{S_F}{S} \frac{l_F}{b}$$

yaw damping

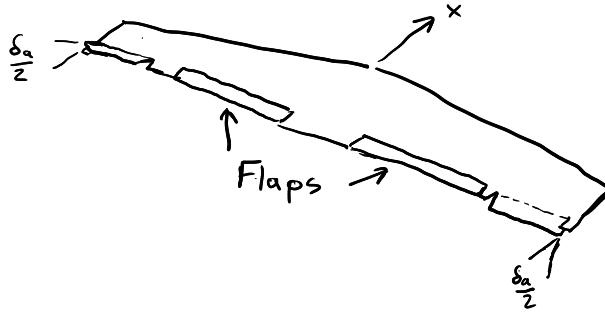


Lateral Control and Coordinated Turn

Rudder



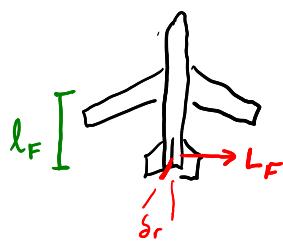
Aileron



Rudder Power

$$C_{n_{\delta_r}}$$

$$N_F = -l_F L_F = -l_F \frac{1}{2} \rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$



$$C_{n_F} = \frac{N_F}{\frac{1}{2} \rho V^2 S_b} = - \underbrace{\frac{l_F S_F}{S_b}}_{V_v} \left(\frac{V_F^2}{V^2} \right) C_{L_F} = -V_v \left(\frac{V_F^2}{V^2} \right) C_{L_F}$$

$$C_{n_{\delta_r}} \equiv \left. \frac{\partial C_{n_F}}{\partial \delta_r} \right|_0 = -V_v \left(\frac{V_F^2}{V^2} \right) \left. \frac{\partial C_{L_F}}{\partial r} \right|_0 = \boxed{-a_r V_v \left(\frac{V_F^2}{V^2} \right)}$$

Other nondim. rudder control derivatives

$$C_{\gamma_{\delta_r}} \text{ sign? } +$$

$$C_{q_{\delta_r}} \text{ sign? } +$$

Aileron nondim. control derivatives

$$C_{l_{\delta_a}} \text{ sign? } -$$

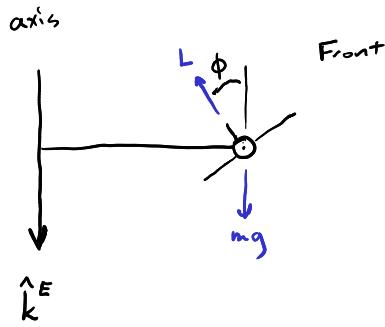
aileron reversal can occur due to wing twist

$$C_{n_{\delta_a}} \text{ sign? can be either}$$

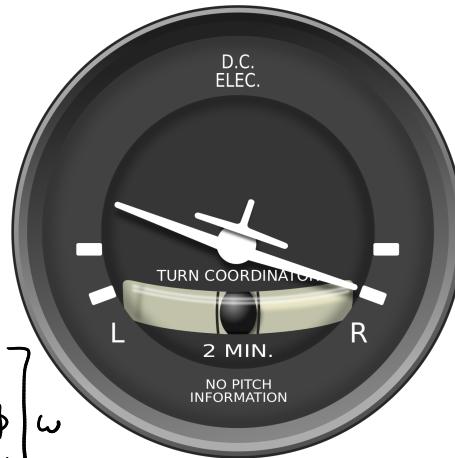
$C_{n_{\delta_a}} > 0$, called adverse yaw

$$C_{\gamma_{\delta_a}} \text{ usually small}$$

Coordinated Turn



- angular velocity vector is constant and aligned with inertial \hat{z}
- No aerodynamic forces in a/c γ direction



$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \sin\phi \cos\theta \\ \cos\phi \cos\theta \end{bmatrix} \omega = \begin{bmatrix} -\theta \\ \sin\phi \\ \cos\phi \end{bmatrix} \omega$$

assume θ small
but ϕ is not

$$L \cos\phi = mg$$

$$L \sin\phi = m a_n = m \frac{u_0^2}{R} = m \omega u_0$$

$$\boxed{\tan\phi} = \frac{L \sin\phi}{L \cos\phi} = \frac{m \omega u_0}{mg} = \boxed{\frac{\omega u_0}{g}}$$

From EOM

$$\boxed{Z = -mg \cos\phi - mqu} \quad \leftarrow \text{Assumed, no wind, } V \ll u, \quad V = u = u_0$$

$$\text{load factor "g's"} \quad n = -\frac{Z}{mg} = \cos\phi + \frac{qu_0}{g}$$

$$= \cos\phi + \frac{\omega u_0 \sin\phi}{g}$$

$$= \cos\phi + \tan\phi \sin\phi$$

$$\boxed{n = \sec\phi}$$

$$n = \frac{L}{w}$$

$$\Delta C_L = \frac{L - mg}{\frac{1}{2} \rho V^2 S} = (n - 1) C_w$$

Coordinated Turn

$$C_x = 0$$

$$= C_{y_p} \beta + C_{y_p} \hat{p} + C_{y_r} \hat{r} + C_{y_s} \delta_r + C_{y_a} \delta_a$$

$$C_R = 0$$

$$= \vdots$$

$$C_n = 0$$

$$= \vdots$$

$$C_m = 0$$

$$= \vdots$$

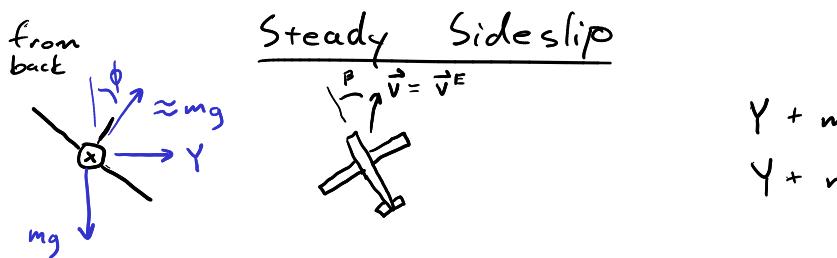
$$C_L = (n - 1) C_w =$$

$$\boxed{\beta, p, r, \delta_r, \delta_a \quad \alpha, q, \delta_e}$$

determined by ω, θ, ϕ

$$\begin{array}{l}
 Y \\
 l \\
 n \\
 \hline
 \end{array} \quad
 \begin{bmatrix}
 C_{Y\beta} & C_{Y\delta_r} & 0 \\
 C_{l\beta} & C_{l\delta_r} & C_{l\delta_a} \\
 C_{n\beta} & C_{n\delta_r} & C_{n\delta_a}
 \end{bmatrix} \begin{bmatrix}
 \beta \\
 \delta_r \\
 \delta_a
 \end{bmatrix} = \begin{bmatrix}
 C_{Y\beta} & C_{Y\delta_r} \\
 C_{l\beta} & C_{l\delta_r} \\
 C_{n\beta} & C_{n\delta_r}
 \end{bmatrix} \begin{bmatrix}
 \theta \\
 -\cos\phi \\
 \frac{\omega b}{2u_0}
 \end{bmatrix}$$

$$\begin{bmatrix}
 C_{m\alpha} & C_{m\delta_e} \\
 C_{L\alpha} & C_{L\delta_e}
 \end{bmatrix} \begin{bmatrix}
 \Delta\alpha \\
 \Delta\delta_e
 \end{bmatrix} = - \begin{bmatrix}
 C_{m\alpha} \\
 C_{L\alpha}
 \end{bmatrix} \frac{\omega \bar{c} \sin\phi}{2u_0} + \begin{bmatrix}
 0 \\
 (n-1)C_w
 \end{bmatrix}$$



$$\begin{aligned}
 Y + mg \sin\phi &= 0 \\
 Y + mg\phi &= 0 \\
 L &= 0 \\
 N &= 0
 \end{aligned}$$

$$\begin{aligned}
 -mg\phi &= Y = Y_v v + Y_p \overset{\circ}{\beta} + Y_r \overset{\circ}{\alpha} + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r \\
 L &= \dots \\
 N &=
 \end{aligned}$$

$$mg\phi + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r = -Y_v v$$

$$\begin{bmatrix}
 Y_{\delta_r} & 0 & mg \\
 L_{\delta_r} & L_{\delta_a} & 0 \\
 N_{\delta_r} & N_{\delta_a} & 0
 \end{bmatrix} \begin{bmatrix}
 \delta_r \\
 \delta_a \\
 \phi
 \end{bmatrix} = - \begin{bmatrix}
 Y_v \\
 L_v \\
 N_v
 \end{bmatrix} v \quad \beta = \frac{\phi}{u_0}$$

Steady Sideslip

For Piper Cherokee



$$\begin{bmatrix}
 280.7 & 0 & 2400 \\
 755.7 & -3821.9 & 0 \\
 -3663.5 & 359 & 0
 \end{bmatrix} \begin{bmatrix}
 \delta_r \\
 \delta_a \\
 \phi
 \end{bmatrix} = \begin{bmatrix}
 2.991 \\
 102.93 \\
 -19.394
 \end{bmatrix} v \quad (7.8, 4)$$

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta = v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\begin{aligned}
 \delta_r/\beta &= .303 \\
 \delta_a/\beta &= -2.96 \\
 \phi/\beta &= .104
 \end{aligned}$$

We see that a positive sideslip (to the right) of say 10° would entail left rudder of 3° and right aileron of 29.6° . Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only 1° to the right so the sideslip is almost flat.

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left(\frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

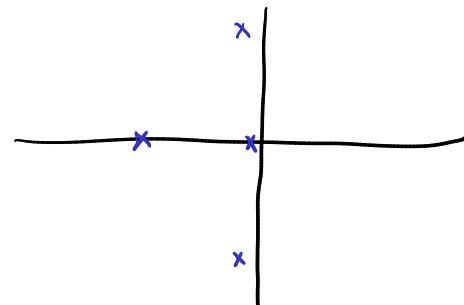


$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}_{lat} \mathbf{x}$$

$$\mathbf{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

	λ_i	ζ	ω_n
→	$-7.30e - 03$	$1.00e + 00$	$7.30e - 03$
→	$-5.62e - 01$	$1.00e + 00$	$5.62e - 01$
→	$-3.30e - 02 + 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$
→	$-3.30e - 02 - 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$



$$\begin{pmatrix} \mathbf{v}_1 \\ 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_2 \\ -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_{3/4} \\ -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

Augmented State Space Dynamics Matrix

$$\begin{bmatrix} \Delta v \\ \vdots \\ \Delta \phi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{lat} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & \sec \theta_0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & \begin{matrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \\ \Delta \psi \\ \Delta \gamma_E \end{matrix} \end{bmatrix}$$

$$\dot{\Delta \psi} = \Delta r \sec \theta_0 \quad \text{if } \theta_0 = 0$$

$$\dot{\Delta \gamma_E} = u_0 \cos \theta_0 \Delta \psi + \Delta v$$

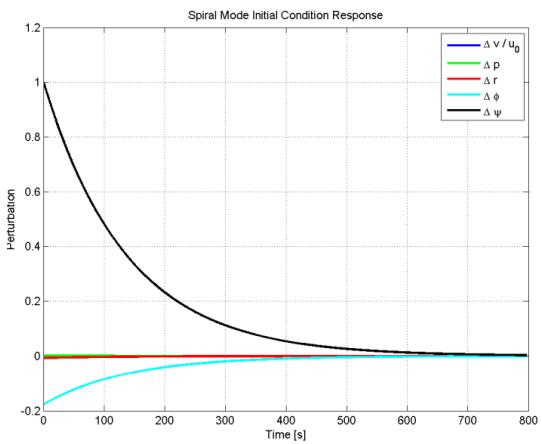
Spiral Mode

$$\lambda = -0.0073$$

$$\tau = 137 \text{ s}$$

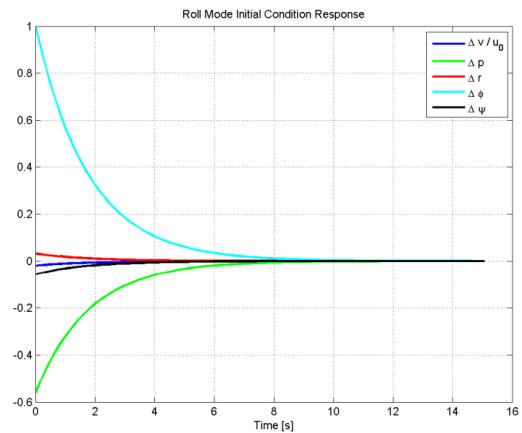
$$\hat{\vec{v}}_1 = \begin{bmatrix} 0.0068 \\ -0.074 \\ 0.04 \\ 1.0 \\ -5.66 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{small} \\ \leftarrow \text{a little} \\ \leftarrow \text{large} \\ \leftarrow \text{large} \end{array}$$

Normalize to $\Delta\phi = 1$
+ nondimensionalize velocity



Roll Mode

$$\hat{\vec{v}}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \end{array}$$



Dutch Roll

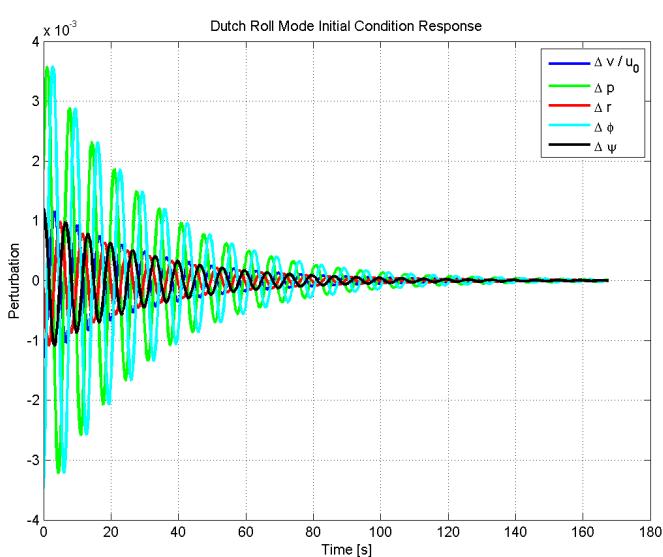
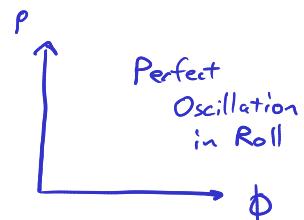
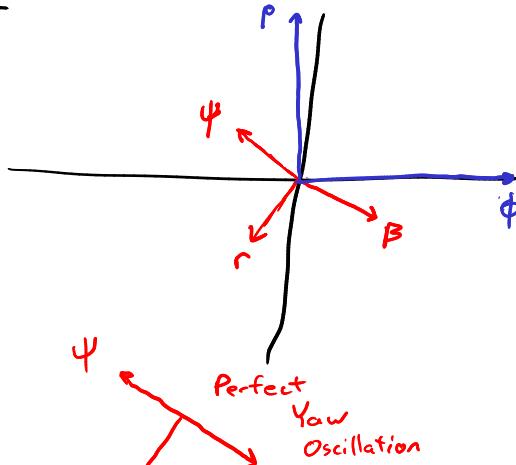
$$\hat{\vec{v}}_3 = \begin{bmatrix} 0.321 \angle -28^\circ \\ 0.9471 \angle 92^\circ \\ 0.2915 \angle -112^\circ \\ 1.0 \\ 0.3078 \angle 155^\circ \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$

$$\lambda_{3,4} = -0.033 \pm 0.947i$$

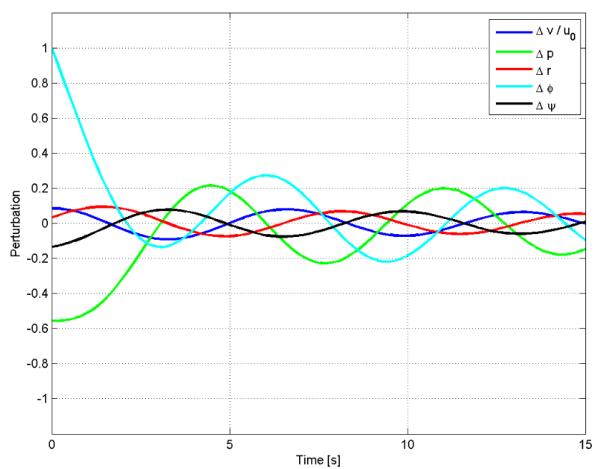
$$f = 0.0349$$

Poor Damping
relatively fast

$$\omega_n = 0.947$$



$$\mathbf{x}(0) = 0.4 \cdot \operatorname{Re}(\mathbf{v}_r) + 0.4 \cdot \operatorname{Re}(\mathbf{v}_{dr}) + 0.2 \cdot \operatorname{Re}(\mathbf{v}_{spi})$$



Lateral Mode Approximations

$$A_{lat} = \begin{bmatrix} Y_v & Y_p & Y_r & g \cos \theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{eg. } Y_v = \frac{Y_v}{m}$$

Roll Approximation

$$r=0 \quad v=0 \quad \dot{\rho} = L_p \rho \quad |A - \lambda I| \quad \text{if } A \text{ is a scalar}$$

$$\lambda_{r, \text{approx}} = L_p \\ = -0.434$$

$$\lambda_r = -0.562 \\ 23\% \text{ difference}$$

2x2 Spiral Approximation

$$\rightarrow p=0$$

$$\rightarrow \dot{p}=0$$

Ignore
side force

$$\begin{bmatrix} v \\ \dot{v} \\ r \\ \dot{r} \\ \phi \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r & g \cos \theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \rho \\ r \\ \phi \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ i \\ r \\ \dot{r} \\ \phi \end{bmatrix} = \begin{bmatrix} L_v & L_r \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = L_v v + L_r r \Rightarrow v = -\frac{L_r}{L_v} r$$

$$\dot{r} = -N_v \frac{L_r}{L_v} r + N_r r = \underbrace{\left(\frac{N_r L_v - N_v L_r}{L_v} \right)}_{\lambda_s, \text{approx}} r$$

$$\lambda_s, \text{approx} = -0.0296 \\ \lambda_s = -0.0073$$

not great

Characteristic-Eqn.-based spiral approx

$$|A_{lat} - \lambda I| = A \lambda^4 + B \lambda^3 + C \lambda^2 + D \lambda + E = 0$$

since $\lambda_s \ll 1$

$$D \lambda + E = 0$$

$$\lambda_{s, \text{approx}} = -\frac{E}{D}$$

$$E = g [(N_r L_v - N_v L_r) \cos \theta_0 + (N_v L_p - L_v N_p) \sin \theta_0]$$

$$D = -g (L_v \cos \theta_0 + N_v \sin \theta_0) + u_0 (L_v N_p - L_p N_v)$$

$$\text{for 747} \quad \lambda_{s, \text{approx}} = -0.00725 \quad \text{very close!}$$

In Book

Characteristic-based spiral + roll eigenvalue approximation

Dutch Roll Approx

Assume $\phi = p = 0$

$$\frac{Y_r}{m} \ll u_0$$

$$\begin{bmatrix} y_v & y_p & y_r^{\approx u_0} & g \cos \theta_0 \\ \dot{z}_v & \dot{z}_p & \dot{z}_r & 0 \\ N_v & N_p & N_r & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} y_v & -u_0 \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$\lambda^2 - (y_v + N_r) \lambda + (y_v N_r + u_0 N_v) = 0$$

$$\lambda_{dr, \text{approx}} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Laplace Transforms and Transfer Functions

$$\ddot{x}(+) + 2\zeta\omega_n \dot{x}(+) + \omega_n^2 x(+) = \omega_n^2 u(+)$$

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ \vec{y} &= C\vec{x} + Du \end{aligned} \quad \iff \quad G_{yu}(s)$$

Review: Properties of Laplace transforms

$$\mathcal{L}[x(+)](s) = \int_0^\infty e^{-st} x(+) dt$$

$$x(+) \iff x(s)$$

Appendix A1



$$\dot{x}(+) \iff s x(s) - x(+|_{t=0})$$

$$\int_0^+ x(\tau) d\tau \iff \frac{1}{s} x(s)$$

$$4 \sin(zt) + 2 \cos(zt)$$

$$\alpha x(+) + \beta y(+) \iff \alpha x(s) + \beta y(s)$$

$$\frac{4 \cdot z}{s^2 + z^2} + \frac{2s}{s^2 + 1}$$

Review: Transfer Function

$$\ddot{x}(+) + 2\zeta\omega_n \dot{x}(+) + \omega_n^2 x(+) = \omega_n^2 u(+)$$

Assume 0 initial conditions

$$s^2 x(s) + 2\zeta\omega_n s x(s) + \omega_n^2 x(s) = \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) x(s) = \omega_n^2 u(s)$$

Transfer Function

$$G_{xu}(s) \equiv \frac{x(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

↑
0 initial conditions

Can you find a TF for any differential equation?

$$\ddot{x}(+) = -\dot{x}(z^2) + u(+) \quad \text{X no TF for nonlinear diff eq.}$$

↑ nonlinear

Review: How to use a TF

1. Input → Output

$$u(+) \rightarrow u(s) \rightarrow x(s) = G_{xu}(s) u(s) \rightarrow x(+)$$

$$\frac{s+1}{s^3 + 2s^2 + 3s + 4} \rightarrow \frac{C_1}{s+a_1} + \frac{C_2}{s+a_2} \dots \xrightarrow{\text{table}}$$

↑
partial frac
decomposition

Ex. $u(+) = \text{step function}$



$$u(s) = \frac{1}{s}$$

$$x(s) = G_{xu}(s) u(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{s}$$

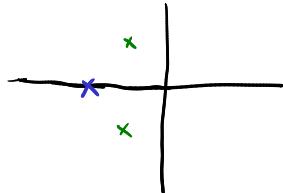
$$x(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

2. Stability

Roots of TF denominator are eigenvalues of A matrix
a.k.a. poles

$\curvearrowleft G(s) \rightarrow \infty$ if s is at a pole

$$\frac{1}{s+1}, \text{ pole} = -1$$



If all poles are on LHP system is stable

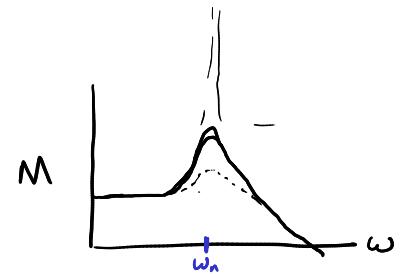
$$G_{xu}(s) = \frac{1}{(s+2)(s^2+2s+2)}$$

$$s=-2 \quad s=-1 \pm i$$

3. Steady-State Behavior

a) Final Value Theorem

If $sx(s)$ is stable

$$\lim_{s \rightarrow \infty} s x(s) = \lim_{s \rightarrow 0} s x(s)$$


b) Frequency Response / Harmonic Response

If $u(t) = A \cos(\omega t)$ the steady-state $x(t)$ is $AM \cos(\omega t + \phi)$

$$M = |G_{xu}(i\omega)| \quad \text{and} \quad \phi = \angle G_{xu}(i\omega)$$

$$z = a + bi \quad \text{or} \quad re^{j\phi} \quad \text{or} \quad r \angle \phi$$

$$G(s) = \frac{1}{s^2 + 2s + 2}$$

$$G(i\omega) = \frac{1}{-\omega^2 + 2i\omega + 2}$$

$$= \frac{1}{\underline{-\omega^2 + 2} + \frac{2i\omega}{b}}$$

$$= \frac{1}{\sqrt{(-\omega^2 + 2)^2 + (2\omega)^2}} e^{i \tan^{-1} \frac{2\omega}{-\omega^2 + 2} \phi}$$

$$= \frac{1}{r} e^{-i\phi}$$

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ \vec{y} &= C\vec{x} + Du \end{aligned} \quad \iff \quad G_{yu}(s)$$

State Space to TF

$$ssZ + f$$

$$tf(sys)$$

$$s\vec{x}(s) = A\vec{x}(s) + Bu(s)$$

$$(sI - A)\vec{x}(s) = Bu(s)$$

$$\vec{x}(s) = (sI - A)^{-1}Bu(s)$$

$$y(s) = C\vec{x}(s) + Du(s) \quad \text{assume } D=0 \text{ from here on}$$

$$y(s) = C(sI - A)^{-1}Bu(s) + Du(s)$$

$$M^{-1} = \frac{\text{adj}(M)}{|M|} \quad (sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

Adjugate: Transpose of Cofactor Matrix F

$$F_{ij} = (-1)^{i+j} |M_{-i-j}|$$

$\neg i = \text{all except } i$

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} |ef| & -|df| & \dots \\ |hi| & -|gi| & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

$$\text{adj}(M) = F^T$$

$$G_{yu}(s) = \frac{Y(s)}{U(s)} = \boxed{C(sI - A)^{-1}B} = \boxed{\frac{C \text{adj}(sI - A)B}{|sI - A|}} = \frac{N(s)}{D(s)}$$

Roots of $D(s)$ are eigenvalues of A

TF to state space

+ 2ss

$$G_{yu}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + \dots + a_{n-1} s + a_n}$$

$$\text{Ex: } G_{yu}(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$$

multiply by $\frac{x(s)}{x(s)}$

$$\frac{Y(s)}{U(s)} = G_{yu}(s) = \frac{b_0 s x(s) + b_1 x(s)}{s^2 x(s) + a_1 s x(s) + a_2 x(s)}$$

$$\rightarrow y(t) = b_0 \dot{x}(t) + b_1 x(t) \quad \leftarrow$$

$$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$$

$$\rightarrow \ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u(t) \quad \leftarrow$$

$$\dot{\vec{x}} = A\vec{x} + Bu \rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & & \\ -a_2 & -a_1 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} u \quad \leftarrow$$

$$y = C\vec{x} + Du \quad \leftarrow$$

$$C = \begin{bmatrix} b_1 & b_0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$G_{Y_u}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + \dots + a_{n-1} s + a_n}$$

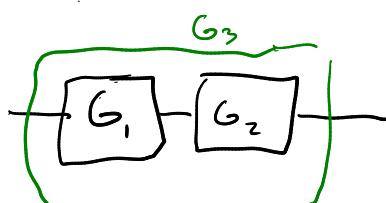
$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & & \\ & & & & 0 & \\ -a_n & \dots & \dots & \dots & -a_1 & \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

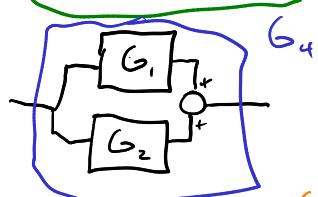
$$C = [b_m \dots b_0 \underbrace{0 \ 0 \ 0}_{\text{if } n > m+1}] \quad D = [0]$$

\vec{x} may not correspond to physical states

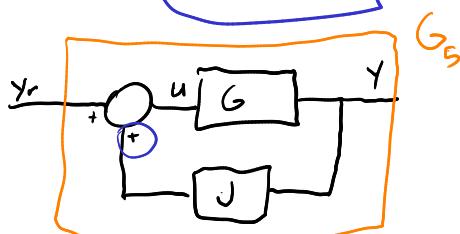
Review: Block Diagrams



$$G_3 = G_1 G_2$$



$$G_4 = G_1 + G_2$$



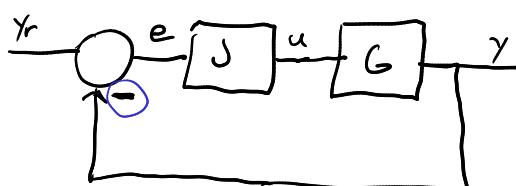
$$G_5 = \frac{Y}{Y_r}$$

$$\begin{aligned} Y &= Gu \\ u &= Y_r + Jy \end{aligned}$$

$$Y = G(Y_r + Jy)$$

$$(1-JG)Y = GY_r$$

$$\frac{Y}{Y_r} = \boxed{\frac{G}{1-JG}}$$



$$Y = JGe$$

$$e = Y_r - Y$$

$$Y = JG(Y_r - Y)$$

$$(1+JG)Y = JGY_r$$

$$\frac{Y}{Y_r} = \boxed{\frac{JG}{1+JG}}$$

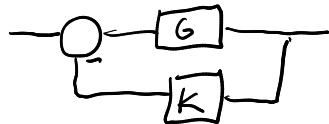
$$\frac{G}{1+JG}$$

if -ive feedback

$$\frac{JG}{1-JG}$$

for +ve feedback

TFs and Root Loci

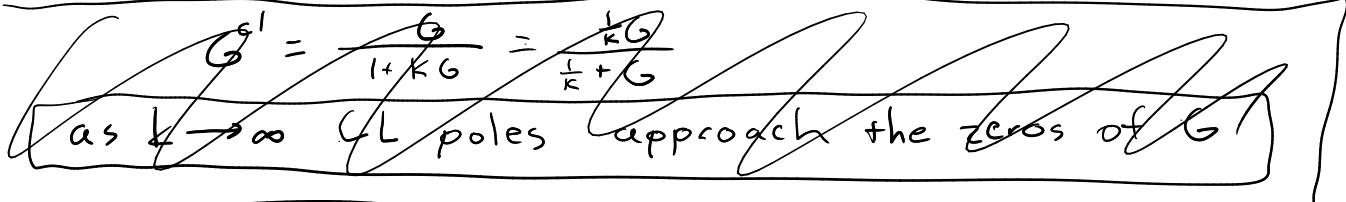


$G \leftarrow$ open loop TF

$$G^{cl} = \frac{G}{1+KG}$$

$$K=0 \Rightarrow G^{cl}=G$$

Root locus starts at poles of the OL systems



$$J=K$$

$$G^{cl} = \frac{G}{1+KG} \quad \text{or} \quad \frac{KG}{1+KG}$$

$$1+KG=0$$

$$\frac{1}{K} + G = 0$$

$$G(s) = \frac{N(s)}{D(s)}$$

$$\text{As } K \text{ gets large} \quad G(s) = -\frac{1}{K}$$

When K is large, two possibilities for CL poles

1. $N(s)$ is close to zero

CL poles close to O.L. zeros

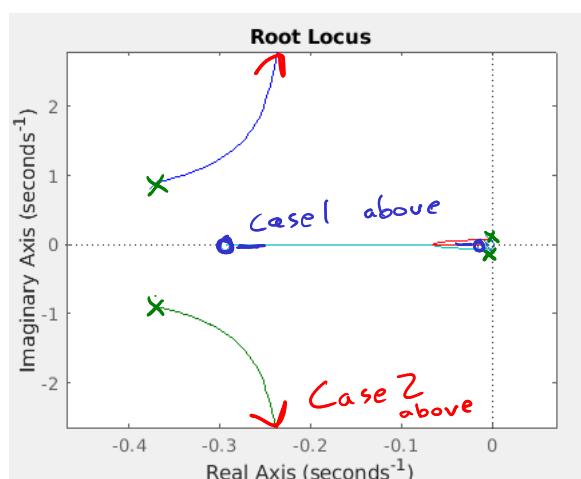
2. $D(s)$ is very large

Magnitude of CL poles is very large

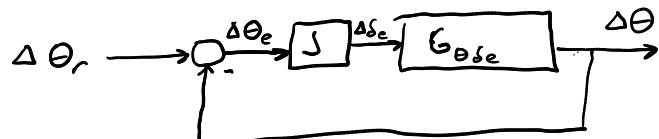
B744 Long dynamics with $\Delta\delta_e = -K\Delta\theta$

O.L. Poles : $-0.37 \pm 0.89i$
 $-0.0033 \pm 0.067i$

O.L. Zeros : $-0.0113, -0.2948$



Designing a Pitch - Hold Controller



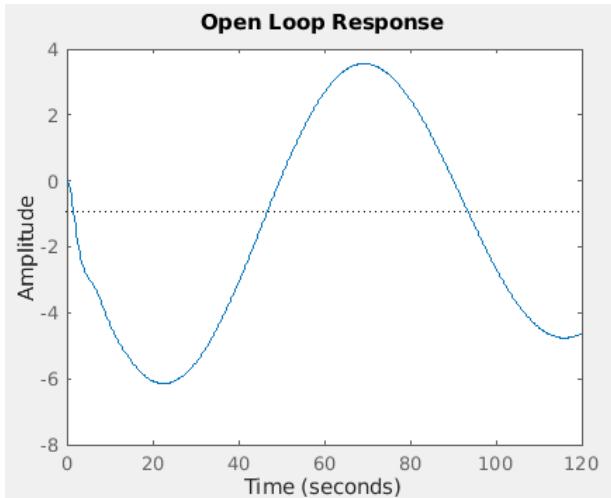
$$\Delta\theta_e = \Delta\theta_r - \Delta\theta$$

Goals

1. Improve Damping

(O.L. Phugoid
has low damping)

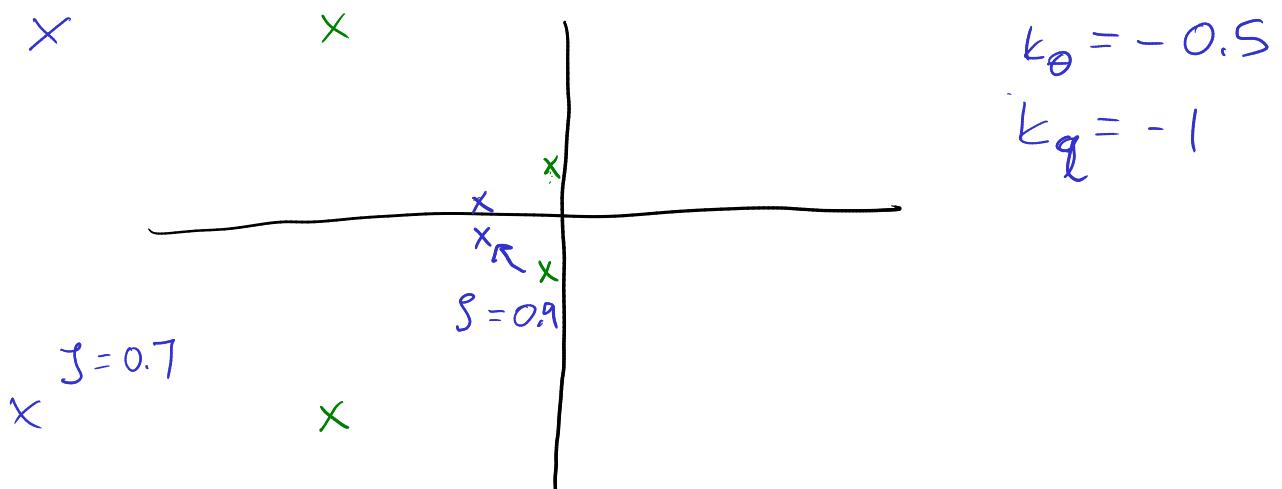
2. Low steady-state error $|\theta - \theta_r| \rightarrow 0$



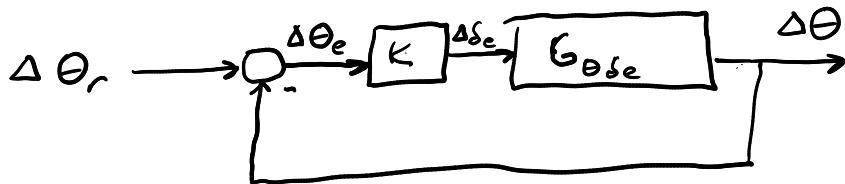
Previously

$$\Delta\delta_e = -k_\theta \Delta\theta - k_q \Delta q$$

k_θ proportional k_q derivative gain



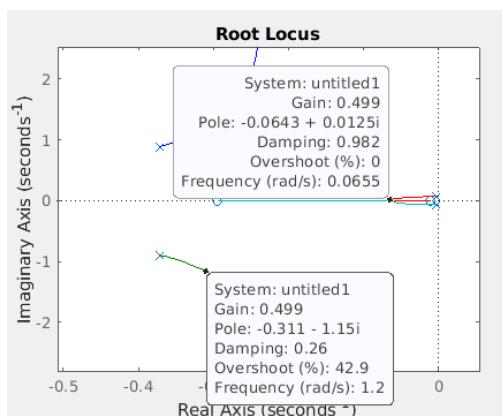
Controller 1



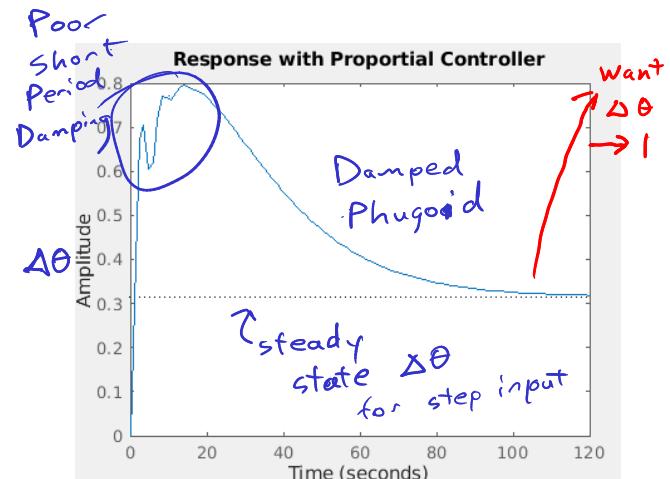
Proportional controller

$$J = K$$

$$\Delta \theta_e = K(\Delta \theta_r - \Delta \theta)$$

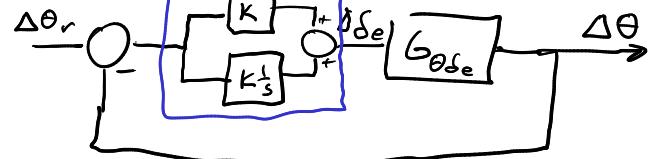


Choose $K = -0.5$



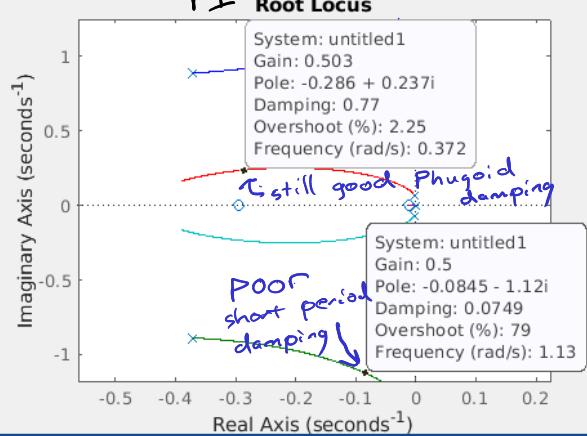
Controller 2: PI

$$J = K \left(1 + \frac{1}{s} \right)$$

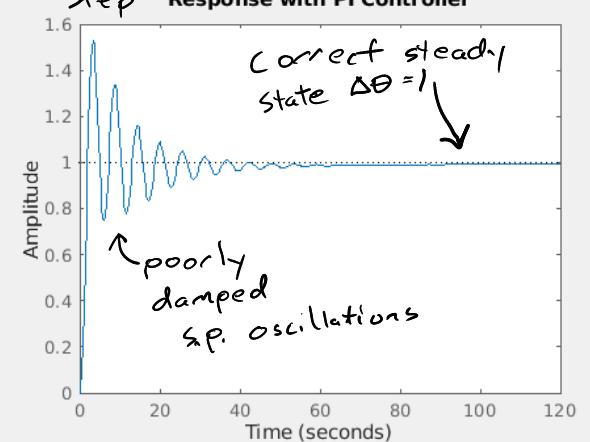


J

PI Root Locus

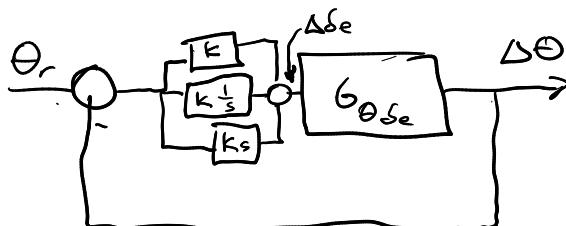


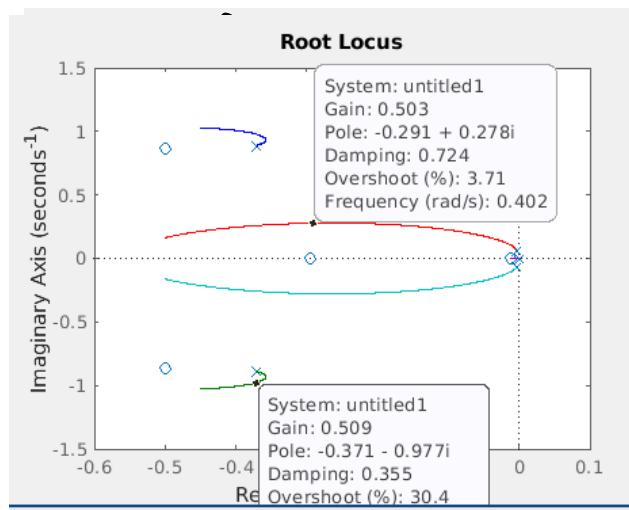
Step Response with PI Controller



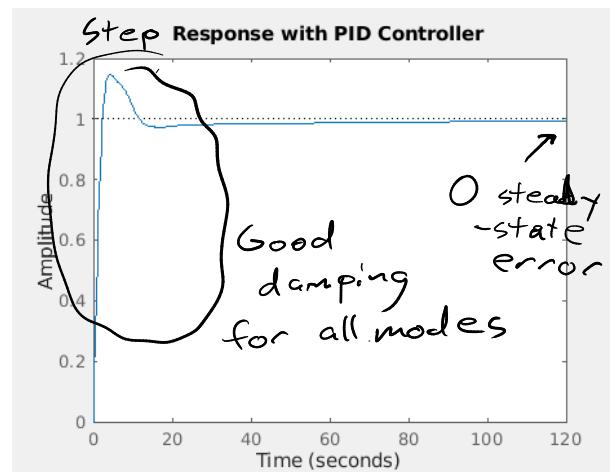
Controller 3: PID

$$J = K \left(1 + \frac{1}{s} + \frac{1}{s^2} \right)$$



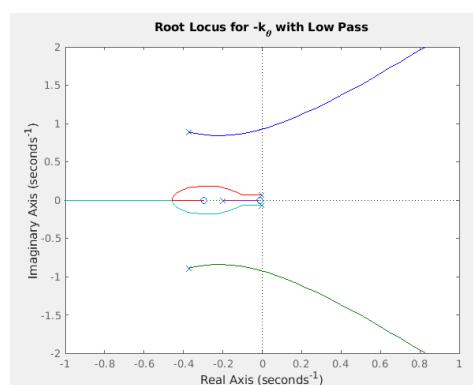
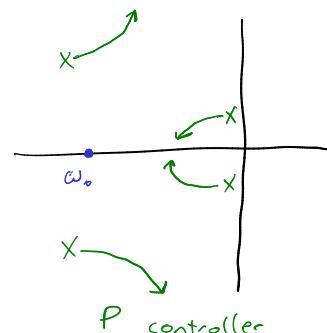
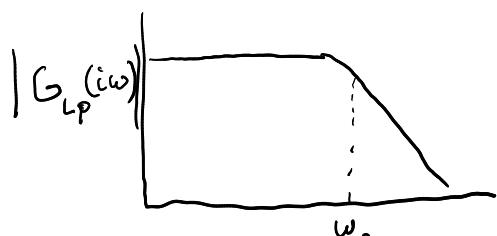
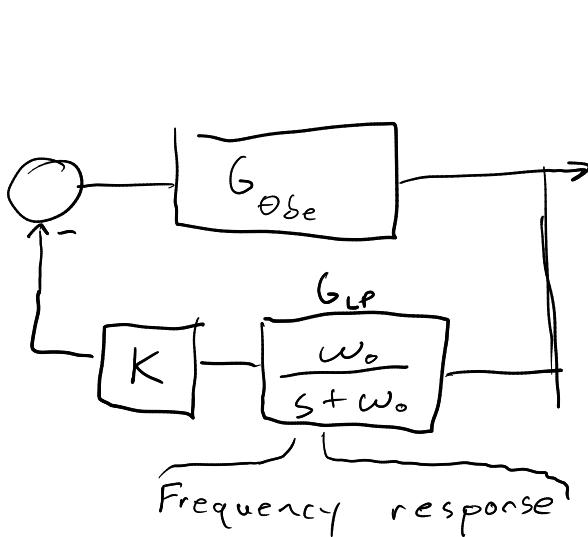
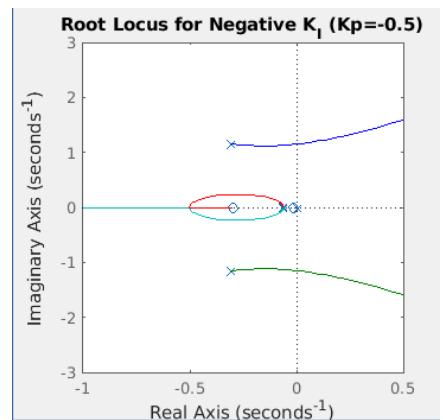
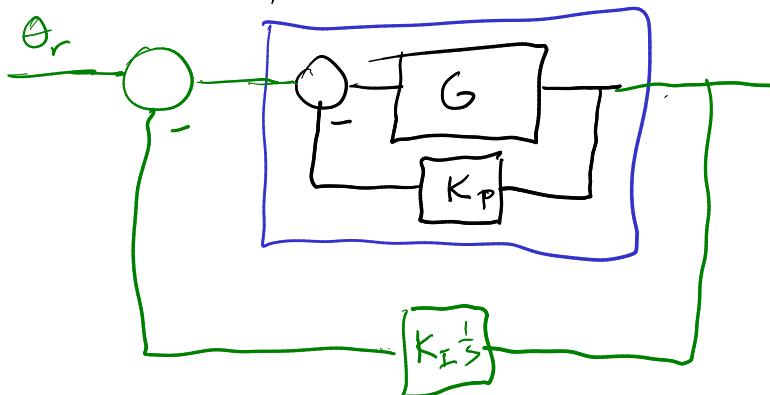


Choose $K = -0.5$

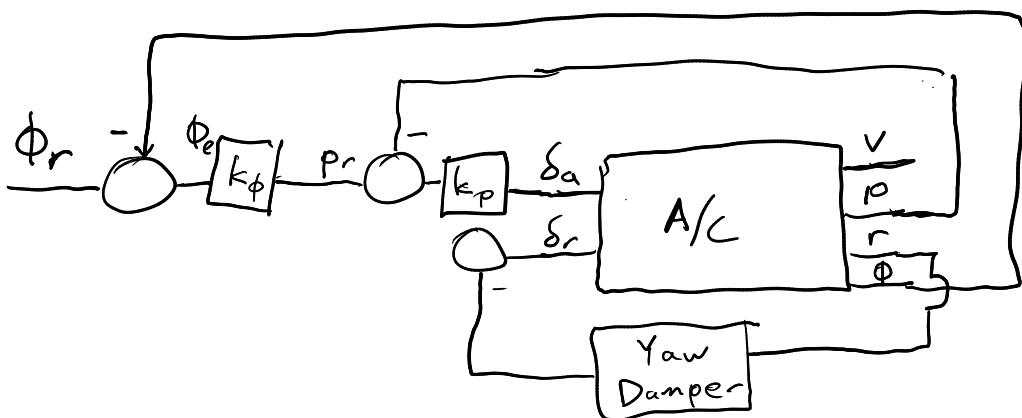


$$j = -0.5 \left(1 + \frac{1}{s} + s \right)$$

If you wanted to tune individually

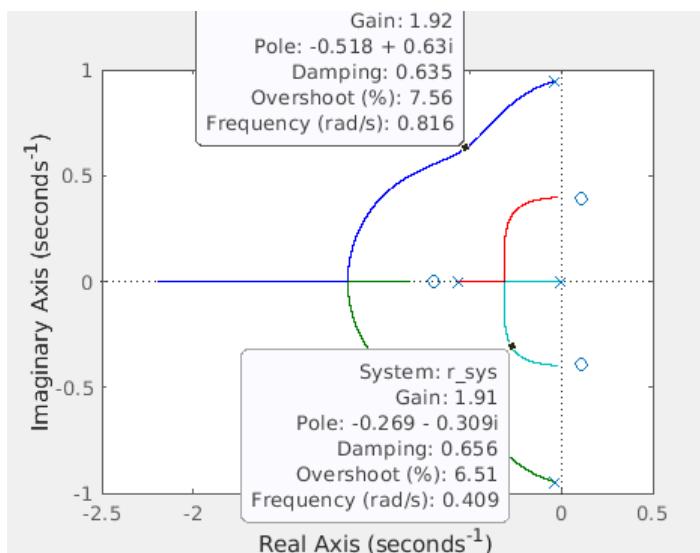


Designing a Roll Controller



Part 1: Yaw Damper

$$\delta_r = -k_r r$$



choose.
 $k_r = -1.9$

Problem : In steady right turn, $r > 0$
 with $k_r = -1.9$, $\delta_r > 0$

Want: $\delta_r \rightarrow 0$ at low frequency
 $\delta_r = -k_r r$ at dutch roll frequency

Washout / High Pass Filter

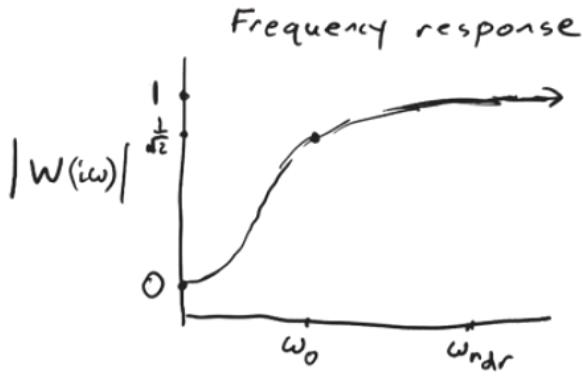
$$W(s) = \frac{s}{s + \omega_0} \quad \text{for } 747 \quad \lambda_{dr} = -0.033 \pm 0.947i$$

$$|W(\omega_i i)| = \left| \frac{\omega_i i}{\omega_i i + \omega_0} \right| = \frac{1}{\sqrt{2}}$$

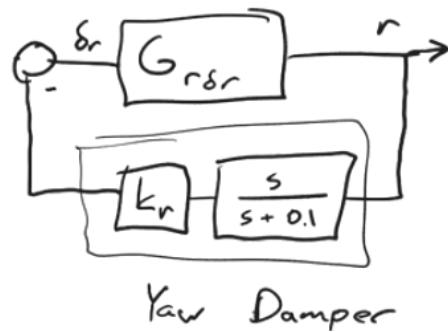
$$\lim_{\omega \rightarrow \infty} |W(\omega i)| = \lim_{\omega \rightarrow \infty} \frac{\omega i}{\omega_i i + \omega_0} = 1$$

$$\omega_{nr} = |\lambda_{dr}| \approx 1$$

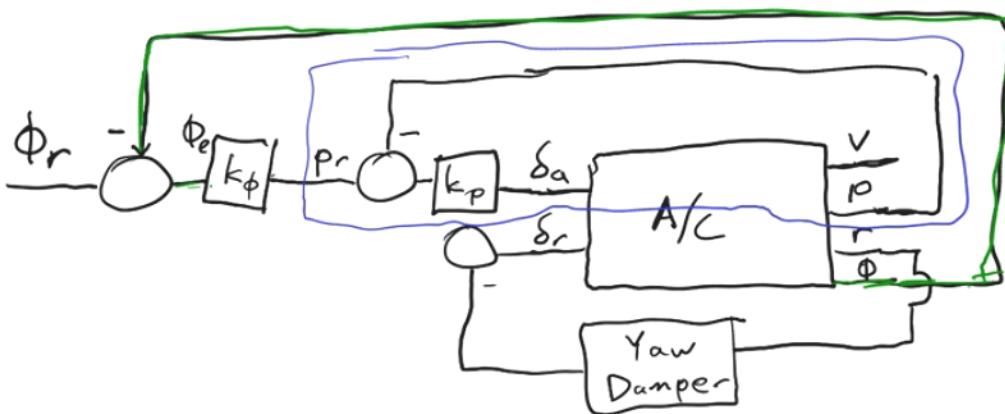
$$|W(0i)| = \frac{0i}{0i + \omega_0} = 0$$



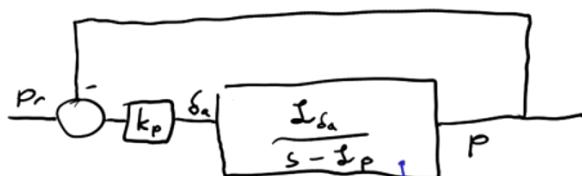
choose $\omega_0 = 0.1$



Part 2: Aileron Controller



Part (a): Inner Loop



$$\delta_a = k_p (p_r - p)$$

Pure Roll Approximation

$$\dot{p} = L_p p + L_{\delta_a} \delta_a$$

$$s p(s) = L_p p(s) + L_{\delta_a} \delta_a(s)$$

$$(s - L_p) p(s) = L_{\delta_a} \delta_a(s)$$

$$G_{p\delta_a}(s) = \frac{p(s)}{\delta_a(s)} = \boxed{\frac{L_{\delta_a}}{s - L_p}}$$

$$A_{lat} = \begin{pmatrix} -0.0558 & 0 & -0.774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta q \\ \Delta g \end{bmatrix}$$

for 747

$$L_p = -0.4342$$

$$L_{\delta_a} = -0.1431$$

$$t \text{ from } B_{lat} = \begin{bmatrix} y_s & y_q \\ L_{\delta_a} & S_{\delta_a} \\ N_{\delta_a} & N_q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix}$$

Closed Loop

$$G_{ppr}(s) = \frac{k_p G_{p\delta_a}}{1 + k_p G_{p\delta_a}} = \frac{k_p \frac{L_{\delta_a}}{s - L_p}}{1 + k_p \frac{L_{\delta_a}}{s - L_p}} = \boxed{\frac{k_p L_{\delta_a}}{s - L_p + k_p L_{\delta_a}} = G_{ppr}(s)}$$

We care about step response

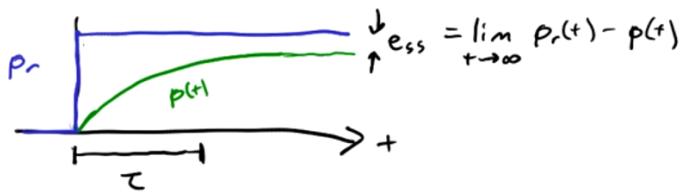
What sign k_p have?

k_p should be negative

$$\delta_a = k_p(p_r - p)$$

$$p_r = 1$$

$$p = 0$$



steady-state gain
 $\lim_{t \rightarrow \infty} \frac{p(t)}{p_r(t)}$ if $p_r(t)$ is step

Steady State Gain

$$\lim_{t \rightarrow \infty} p(t) = \lim_{s \rightarrow 0} s p(s) = \lim_{s \rightarrow 0} s p_r(s) G_{ppr}(s) = \lim_{s \rightarrow 0} \frac{1}{s} G_{ppr}(s) = \boxed{G(0)}$$

only if $s p(s)$ is stable

$$G_{ppr}(0) = -\frac{k_p \mathcal{L}_{\delta_a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}}$$

as k_p gets more negative
 \Rightarrow larger steady-state gain

$$G_{ppr}(s) = \frac{k_p \mathcal{L}_{\delta_a}}{s - (\mathcal{L}_p - k_p \mathcal{L}_{\delta_a})}$$

$$\lambda = \mathcal{L}_p - k_p \mathcal{L}_{\delta_a}$$

$$\tau = \left| \frac{1}{\lambda} \right| = \left| \frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \right|$$

more negative k_p
 \Rightarrow smaller τ

Why not make $|k_p|$ as large as possible

- saturation
- sensitivity to noise
- wear and tear on actuator

consider $k_p = -1$

$P_r \approx 15 \text{ deg/s} \Rightarrow 15^\circ \text{ aileron deflection}$

$$\text{Steady-state gain} = G_{ppr}(0) = \frac{-k_p \mathcal{L}_{\delta_a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \approx 0.25$$

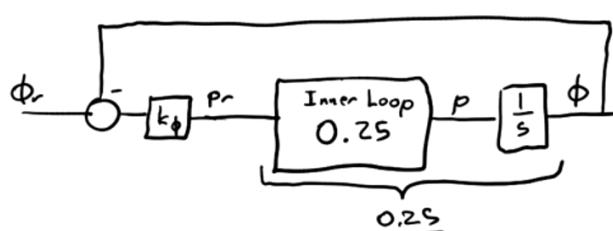
$$\text{Time constant } \tau = \left| \frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \right| \approx 1.75 \text{ s}$$

check if OK Later

check if OK Later

Part (b) Outer Loop

Assume: Inner Loop Much Faster than Outer Loop
 $\dot{\phi} = p$



Closed Loop TF

$$G_{\phi\phi_r} = \frac{k_\phi \frac{0.25}{s}}{1 + k_\phi \frac{0.25}{s}} = \frac{0.25 k_\phi}{s + 0.25 k_\phi}$$

k_ϕ should be > 0

2 Goals

- Steady-state gain near 1 $\rightarrow G_{\phi\phi_r}(0) = \frac{0.25 k_\phi}{0.25 k_\phi} = 1$
- low time constant

higher $k_\phi \Rightarrow$ lower time constant

$$k_\phi = 1.5 \Rightarrow$$