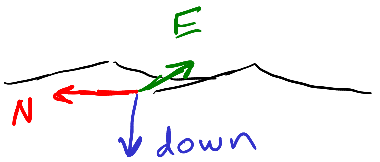
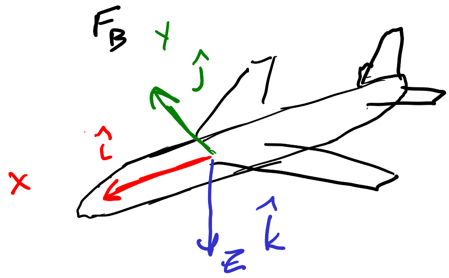
Notation and Conventions

F_I or F_E



F_B



- Frame of reference : Collection of ≥ 3 points w/ constant distance between each other

- Inertial frame : A frame that translates with constant (possibly 0) velocity and does not rotate
  - Newton's second law is valid in an inertial frame

- Coordinate System : Three orthogonal unit vectors that allow measurement and vector representation

- Vector Notation

$\rightarrow$ or **bold** : vector

$\wedge$: unit vector

object that we are referring to

often implied

$_{a/c}\vec{V}^{E}_{B}$

frame that this vector is relative to

often implied

"with respect to"
"relative to"

coordinate system

"written in"
"expressed in"
"coordinate frame"

$$\begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

"inertial velocity written in body coordinates"

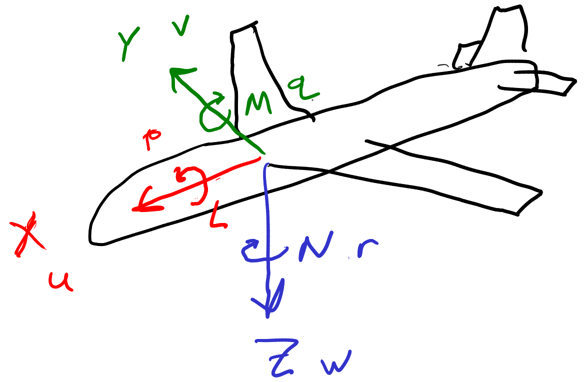
$\vec{V}^{W}_{B}$ $\qquad$ $\vec{V}^{E}_{E}$

Forces, Moments, and Velocities

$$\vec{A} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

Aerodynamic (incl. propulsion NOT gravity)

$$\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

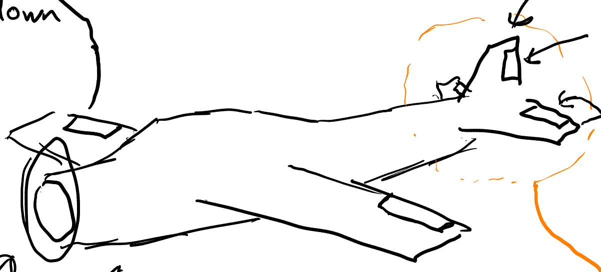
$$\vec{V}^E_B = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{\omega}^E_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$V_g = |\vec{V}^E| = \sqrt{u^{E2} + v^{E2} + w^{E2}}$$

Anatomy of an Airplane aka "Conventional A/C"



Propulsion
$\delta_p \rightsquigarrow +X$

Sometimes
δ_t
throttle

Wind

Aerodynamic Forces and Moment (i.e. not gravity) are functions of the A/C velocity w.r.t. the Air

$\vec{V}^{W}$

by convention $\vec{V} \equiv \vec{V}^{W}$

$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

$V = |\vec{V}|$

$$\boxed{\vec{V}^{E} = \vec{V}^{(W)} + \vec{W}^{(E)}}$$

When $\vec{W} = \vec{0}$, $\vec{V} = \vec{V}^{E}$

Example

Wind blowing east @ 8 m/s

A/c pointing north and traveling northward with respect to the wind at 60 m/s

What is the A/c velocity vector relative to earth in NED

$$\vec{V}_E^{E} = \vec{V}_E + \vec{W}_E = \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 8 \\ 0 \end{bmatrix}$$

$\cancel{\vec{V}_B + \vec{W}_E}$

Illegal!

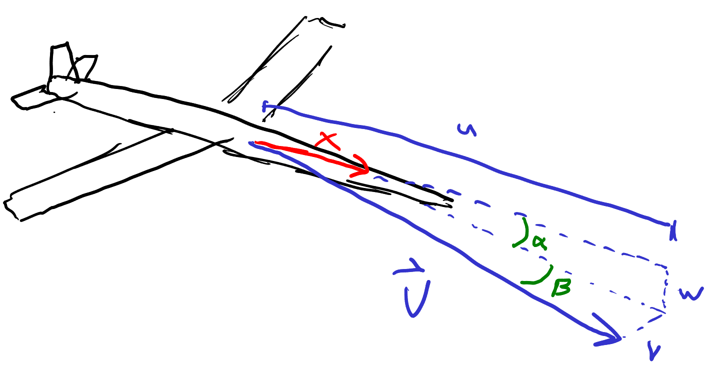
Wind Angles

Angle of Attack

$\alpha \equiv \operatorname{atan}\left(\frac{w}{u}\right)$

Sideslip Angle

$\beta \equiv \sin^{-1} \frac{v}{V}$



$u = V \cos\beta \cos\alpha$

$v = V \sin\beta$

$w = V \cos\beta \sin\alpha$

Orientation

axis

"1" ϕ : roll

"2" θ : pitch

"3" ψ : yaw

E.G. 45°

90°

45°

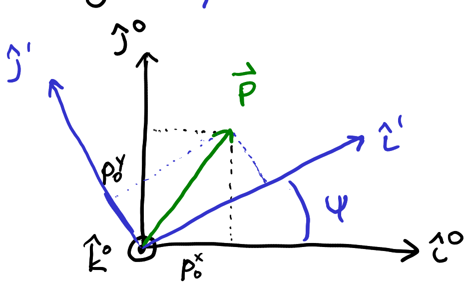
By convention A/C orientation is defined by a 3-2-1 sequence of rotations throug $\psi - \theta - \phi$

Key task: changing the coordinate sys. that a vector is written in

Ex.

know: $\vec{V}_B^E$ want: $\vec{V}_E^E$

F_0 F_1



$$\vec{p}_0 = \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix} \quad \text{know} \qquad \vec{p}_1 = \begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix}$$

$$\vec{p} = p_0^x \hat{i}^0 + p_0^y \hat{j}^0 + p_0^z \hat{k}^0$$

F_1 is related to F_0 by a 3-rotation through ψ

want p_1^x in terms of $\vec{p}$

$$p_1^x = \vec{p} \cdot \hat{i}'$$

$$= p_0^x \hat{i}^0 \cdot \hat{i}' + p_0^y \hat{j}^0 \cdot \hat{i}' + p_0^z \hat{k}^0 \cdot \hat{i}'$$

$$= p_0^x \cos\psi + p_0^y \sin\psi + p_0^z \cdot 0$$

$$p_1^x = \begin{bmatrix} \cos\psi & \sin\psi & 0 \end{bmatrix} \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

General

$$\boxed{R_0^1 = \begin{array}{c} \begin{array}{ccc} \hat{i}^0 & \hat{j}^0 & \hat{k}^0 \end{array} \\ \begin{bmatrix} \hat{i}^0\cdot\hat{i}' & \hat{j}^0\cdot\hat{i}' & \hat{k}^0\cdot\hat{i}' \\ \hat{i}^0\cdot\hat{j}' & \hat{j}^0\cdot\hat{j}' & \hat{k}^0\cdot\hat{j}' \\ \hat{i}^0\cdot\hat{k}' & \hat{j}^0\cdot\hat{k}' & \hat{k}^0\cdot\hat{k}' \end{bmatrix} \end{array}}$$

$$\begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix} = \underbrace{\begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\psi)} \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

axis, angle

R_E^B R_B^E

$$\begin{matrix} c\theta & s\theta \\ -s\theta & c\theta \end{matrix}$$

$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

Properties of Direction Cosine Matrices (DCMs)

$\vec{p}_B = R_A^B \vec{p}_A$

Book: $\vec{p}_B = L_{BA}\vec{p}_A$

1. Chaining

$\vec{p}_C = R_B^C \vec{p}_B \qquad \vec{p}_B = R_A^B \vec{p}_A$

$= \underbrace{R_B^C R_A^B}_{R_A^C} \vec{p}_A$

$$\boxed{R_A^C = R_B^C R_A^B}$$

2. Inverse

$\vec{p}_B = R_A^B \vec{p}_A$

$(R_A^B)^{-1}\vec{p}_B = \cancel{(R_A^B)^{-1} R_A^B} \vec{p}_A$

$R_B^A \vec{p}_B = \vec{p}_A \qquad R_B^A = (R_A^B)^{-1}$

Since DCMs are orthonormal (columns are orthogonal unit vectors)

$\begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix}\begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_A^B (R_A^B)^T = I$

$(R_A^B)^{-1} = (R_A^B)^T$

$$\boxed{R_B^A = R_A^{B\,T}}$$

Earth-to-body DCM

3-2-1 rotation through ψ, θ, ϕ

$\vec{p}_B = \underbrace{R_1(\phi)R_2(\theta)R_3(\psi)}\vec{p}_E$

$$\boxed{\vec{p}_B = R_E^B \vec{p}_E}$$

$c_\theta = \cos\theta$
$s_\theta = \sin\theta$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

Example:

Want: ${}^{pilot}\vec{p}_E^{\,E}$ know ${}^{B}\vec{p}_E^{\,E}$, ${}^{pilot}\vec{p}_B^{\,B}$ ×, R_E^B

${}^{pilot}\vec{p}^{\,E} = {}^{B}\vec{p}^{\,E} + {}^{pilot}\vec{p}^{\,B}$

${}^{pilot}\vec{p}_E^{\,E} = {}^{B}\vec{p}_E^{\,E} + {}^{pilot}\vec{p}_E^{\,B}$ × $= {}^{B}\vec{p}_E^{\,E} + R_B^E\, {}^{pilot}\vec{p}_B^{\,B}$

$(R_E^B)^T$

Kinematics

Kinematics: "Geometry of motion" (no forces)

Dynamics/kinetics: Effects of forces and moments on an object

Vector derivatives

$\vec{p}$: shorthand $\vec{p}^E$

$\frac{d}{dt}\vec{p} \equiv$ time rate of change of $\vec{p}$ ← "Vector derivative"

$\vec{V}^E \equiv \frac{d}{dt}\vec{p}$

$\dot{\vec{p}}_B \equiv$ time rate of change of elements "coordinates" of $\vec{p}_B$

if $\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$ then $\dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

Question

$$\vec{V}^E_B = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \stackrel{?}{=} \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$$

Not always true

$\hat{i}_B$: shorthand for $\hat{i}^B_B$

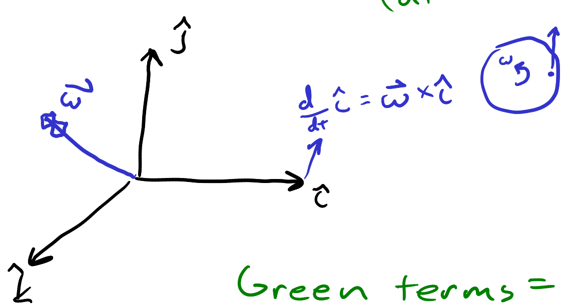
Recall: $\frac{d}{dt}(uv) = u\frac{d}{dt}v + v\frac{d}{dt}u$

$$\vec{p}_B = x_B\hat{i}_B + y_B\hat{j}_B + z_B\hat{k}_B$$

$$\dot{\vec{p}}_B = \dot{x}_B\hat{i}_B + \dot{y}_B\hat{j}_B + \dot{z}_B\hat{k}_B$$

$$\left(\frac{d}{dt}\vec{p}\right)_B = \boxed{\dot{x}_B\hat{i}_B} + x_B\frac{d}{dt}\hat{i}_B + \boxed{\dot{y}_B\hat{j}_B} + y_B\frac{d}{dt}\hat{j}_B + \boxed{\dot{z}_B\hat{k}_B} + z_B\frac{d}{dt}\hat{k}_B$$

what is $\left(\frac{d}{dt}\hat{i}\right)_B$



$$\vec{\omega}_B = \vec{\omega}^E_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Green terms $= x_B(\vec{\omega}_B\times\hat{i}_B) + y_B(\vec{\omega}_B\times\hat{j}_B) + z_B(\vec{\omega}_B\times\hat{k}_B)$

$= \vec{\omega}_B\times\vec{p}_B$

$$\boxed{\left(\frac{d}{dt}\vec{p}\right)_B = \dot{\vec{p}}_B + \vec{\omega}_B\times\vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B\vec{p}_B \qquad \tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

"Kinematic Transport Theorem"

$$\left(\frac{d}{dt}\vec{p}\right)_E = \dot{\vec{p}}_E + \cancelto{0}{{}^E\vec{\omega}^E_E}\times\vec{p}_E = \dot{\vec{p}}_E$$

Aircraft Equations of Motion (EOM)

$\dot{\vec{x}} = f(t, \vec{x}, \vec{u}) \longrightarrow$ Need

$$\begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{p}_E = \vec{p}_E^{\,E} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\Theta} \text{ ``pseudo-vector'' array of numbers} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{V}_B^E \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\omega}_B^E \end{matrix}$$

Need:

$\dot{\vec{p}}_E$ — Kinematics

$\dot{\vec{\Theta}}$ — Kinematics

$\dot{\vec{V}}_B^E$ — Dynamics

$\dot{\vec{\omega}}_B$ — Dynamics

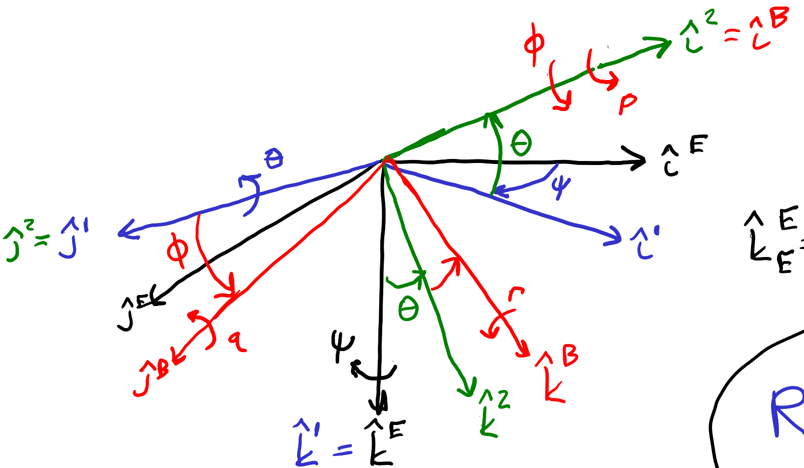
Translational Kinematics

$$\dot{\vec{p}}_E^{\,E} = \frac{d}{dt}\vec{p}_E - \cancelto{0}{{}^{E}\vec{\omega}_E^{E}} \times \vec{p}_E^{\,E} = \frac{d}{dt}\vec{p}_E = \vec{V}_E^E = R_B^E \vec{V}_B^E$$

$$\boxed{\dot{\vec{p}}_E = \left(R_E^B\right)^T \vec{V}_B^E}$$

Rotational Kinematics

want: $\dot{\vec{\Theta}} = \begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}$ have: $\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$



$$\vec{\omega} = p\,\hat{i}^B + q\,\hat{j}^B + r\,\hat{k}$$

$$\vec{\omega} = \dot\psi\,\hat{k}^E + \dot\theta\,\hat{j}^1 + \dot\phi\,\hat{i}^B$$

$$\vec{\omega}_B = \dot\psi\,\hat{k}^E_B + \dot\theta\,\hat{j}^1_B + \dot\phi\,\hat{i}^B_B$$

$$\hat{k}^E_E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{\omega}_B = \dot\psi\,\underbrace{R_E^B}\,\hat{k}^E_E + \dot\theta\,\underbrace{R_1^B}\,\hat{j}^1_1 + \dot\phi\,\hat{i}^B_B$$

$$R_E^B = \underbrace{R_1(\phi) R_2(\theta) R_3(\psi)}_{B \quad 2 \quad 1 \quad E} \qquad R_1^B = R_1(\phi) R_2(\theta)$$

$$\vec{\omega}_B = \dot\psi\, R_1(\phi) R_2(\theta) R_3(\psi) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot\theta\, R_1(\phi) R_2(\theta) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot\phi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & s\phi c\theta \\ 0 & -s\phi & c\phi c\theta \end{bmatrix}}_{\text{invert}} \underbrace{\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix}}_{\dot{\vec{\Theta}}}$$

$t\theta = \tan\theta$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}}_{} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Eqn. 4.4, 7 in book

$$\boxed{\dot{\vec{o}} = T\,\vec{\omega}_B}$$

T = "attitude influence matrix"

Dynamics $\vec{f}$ $\vec{G}$

Translational Dynamics

Newton's 2nd Law

$$\vec{f} = m\vec{a}$$

$$\vec{f} = m\,\frac{d}{dt}\vec{v}^E \qquad \text{want } \dot{\vec{v}}^E_B$$

$$\left(\frac{d}{dt}\vec{v}^E\right)_B = \dot{\vec{v}}^E_B + {}^B\vec{\omega}^E_B \times \vec{v}^E_B$$

$$\dot{\vec{v}}^E_B = \frac{d}{dt}\vec{v}^E_B - \vec{\omega}_B \times \vec{v}^E_B$$

$$\boxed{\dot{\vec{v}}^E_B = \frac{\vec{f}_B}{m} - \underbrace{\vec{\omega}_B \times \vec{v}^E_B}_{\tilde{\omega}_B \vec{v}^E_B}}$$

Rotational Dynamics

"Euler's 2nd Law"

$$\frac{d}{dt}\vec{h} = \vec{G}$$

angular momentum moment

$I = I_B$ book

$$\vec{h} = I\vec{\omega}$$

$$\mathbf{I} = \begin{pmatrix} \int (y^2+z^2)\,\mathrm{dm} & -\int xy\,\mathrm{dm} & -\int xz\,\mathrm{dm} \\ -\int xy\,\mathrm{dm} & \int (x^2+z^2)\,\mathrm{dm} & -\int yz\,\mathrm{dm} \\ -\int xz\,\mathrm{dm} & -\int yz\,\mathrm{dm} & \int (x^2+y^2)\,\mathrm{dm} \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

Want $\dot{\vec{\omega}}_B$

$$\frac{d}{dt}\vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \times \vec{h}_B = \vec{G}_B$$

$$I\dot{\vec{\omega}}_B + \vec{\omega}_B \times I\vec{\omega}_B = \vec{G}_B$$

$$\boxed{\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \dot{\vec{\omega}}_B = I^{-1}\left(\vec{G}_B - \vec{\omega}_B \times I\vec{\omega}_B\right)}$$

A/C EOM

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E^E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} \left(R_E^B\right)^T \vec{v}_B^E \\ T\vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}\left(\vec{G}_B - \vec{\omega}_B \times I\vec{\omega}_B\right) \end{bmatrix}$$

Quadrotors

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{p}_E \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\Theta} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{v}^E_B \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \vec{\omega}_B \end{matrix}$$

A/C EOM

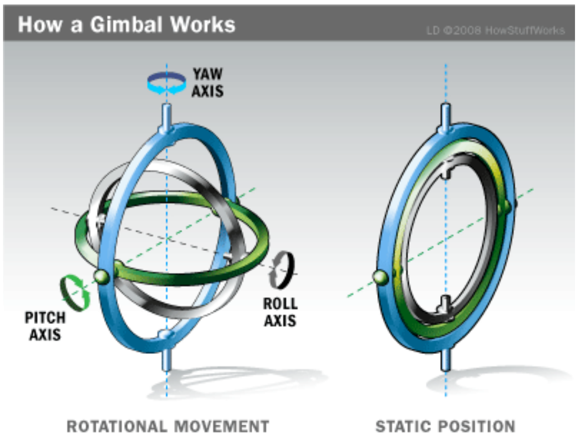
$$\boxed{\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}^E_E \\ \dot{\vec{\Theta}} \\ \dot{\vec{v}}^E_B \\ \dot{\vec{\omega}}^E_B \end{bmatrix} = \begin{bmatrix} (R^B_E)^T \vec{v}^E_B \\ T\vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}^E_B \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I\vec{\omega}_B) \end{bmatrix}}$$

$$T = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}$$

From Quiz MC2

$$T = \begin{bmatrix} & 0 & \\ & 0 & \\ & 1 & \end{bmatrix} \quad \vec{\omega}_B = \begin{bmatrix} 0 \\ 10°/s \\ 0 \end{bmatrix}$$

$$\vec{\Theta} = \begin{bmatrix} 0 \\ 0 \\ 10°/s \end{bmatrix}$$



Homework P1 Monospinner

$\vec{F}$: sum of all forces acting on A/C

$\vec{F} = \vec{A} + {}^g\vec{F}$

aerodynamic forces ($\vec{A}$), gravity (${}^g\vec{F}$)

$\vec{G}$: sum of all moments acting about the C.G. of A/C

Multirotor Case

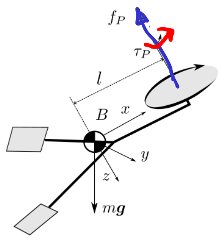
$$\vec{f} = \underbrace{{}^{d}\vec{f} + {}^{c}\vec{f}}_{A} + {}^{g}\vec{f}$$

drag ... control

$$\vec{G} = {}^{d}\vec{G} + {}^{c}\vec{G}$$

drag ... control

Monospinner Assignment



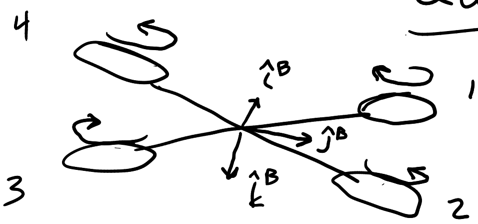
$${}^{d}\vec{f} = 0 \quad \text{(by assumption)}$$

$${}^{c}\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ -f_P \end{bmatrix}$$

$${}^{d}\vec{G} = 0 \quad \text{(by assumption)}$$

$${}^{c}\vec{G}_B = \begin{bmatrix} 0 \\ 0 \\ -\tau_P \end{bmatrix} + {}^{P}\vec{P}_B^{B} \times \vec{f}_B$$

Quadrotor Case



$${}^{c}\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} \qquad {}^{c}\vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since the quadrotor is symmetric about the $\hat{i}^B$-$\hat{k}^B$ and $\hat{j}^B$-$\hat{k}^B$ planes

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \qquad I^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

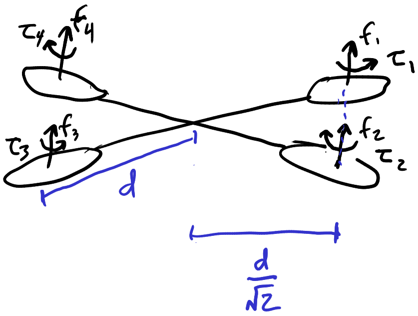
$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + \underbrace{g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix}}_{} + \frac{1}{m} \overbrace{\begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix}}^{{}^{d}\vec{f}_B} + \frac{1}{m} \overbrace{\begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}}^{{}^{c}\vec{f}_B}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Control Forces and Moments



$${}^c\vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix}$$

$${}^c\vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} \frac{d}{\sqrt{2}}(-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{\sqrt{2}}(f_1 - f_2 - f_3 + f_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{bmatrix}$$

ω_i

$f_i = k_f C_L (\omega_i)^2$

$\tau_i = k_\tau C_D (\omega_i)^2$

$$\boxed{\tau_i = k_m f_i}$$

$k_m = \frac{k_\tau C_D}{k_f C_L}$

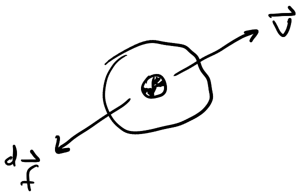
control forces + Moments $\Longleftrightarrow$ individual rotor forces

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & -\frac{d}{\sqrt{2}} & \frac{d}{\sqrt{2}} \\ -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$\Downarrow$ invert

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

Drag Forces

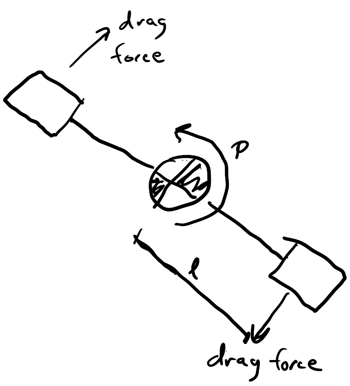


$$d\vec{f} = -D \frac{\vec{v}}{V_a} \qquad V_a = |\vec{v}|$$

$$D = \frac{1}{2}\rho V_a^2 C_D A = \nu V_a^2$$

$$d\vec{f}_B = \begin{bmatrix} X_d \\ Y_d \\ Z_d \end{bmatrix} = -\nu V_a^2 \frac{\vec{V}_B}{V_a} = \boxed{-\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}}$$

Drag Moments



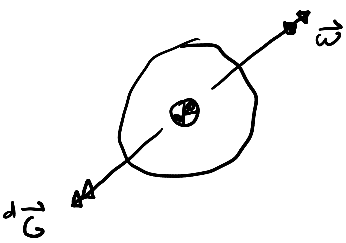
$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$L_{drag} = -2 l f_{drag}$$

$$= \underbrace{-2 l \frac{1}{2} \rho C_D A}_{\mu} (l p)^2 \, \text{sign}(p)$$

$$= -\mu p |p|$$

General case



$$\boxed{d\vec{G}_B = \begin{bmatrix} L_d \\ M_d \\ N_d \end{bmatrix} = -\mu |\vec{\omega}| \begin{bmatrix} p \\ q \\ r \end{bmatrix}}$$

Quadrotor Linear Model

$\dot{\vec{x}} = f(t, \vec{x}, \vec{u})$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}^E_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}^E_B \\ \dot{\vec{\omega}}^E_B \end{bmatrix} = \begin{bmatrix} (R^B_E)^T \vec{v}^E_B \\ T\vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}^E_B \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I\vec{\omega}_B) \end{bmatrix}$$

- differential
- first order
- ordinary (not partial)
- coupled
- nonlinear

simulate

Linearization

$\vec{x} \approx \vec{x}_0 + \Delta\vec{x}$ $\qquad \dot{\vec{x}} \approx \Delta\dot{\vec{x}} \qquad \vec{u} \approx \vec{u}_0 + \Delta\vec{u}$

trim condition

For a quadrotor, "Hover" trim condition

dot means any value

$$\vec{x}_0 = \begin{bmatrix} x_{E,0} \\ y_{E,0} \\ z_{E,0} \\ \phi_0 \\ \theta_0 \\ \psi_0 \\ u^E_0 \\ v^E_0 \\ w^E_0 \\ p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix} Z_{c,0} \\ L_{c,0} \\ M_{c,0} \\ N_{c,0} \end{bmatrix} = \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Want Linear EOM

$\Delta\dot{x} = A\Delta x + B\Delta u$

Approach: Use first-order Taylor Series approx

$y = f(x,u)$

$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$

$$\underbrace{y_0}_{\text{cancel}} + \Delta y \approx \underbrace{f(x_0,u_0)}_{\text{cancel}} + \frac{\partial f}{\partial x}\Big|_0 \Delta x + \frac{\partial f}{\partial u}\Big|_0 \Delta u + \cancel{H.O.T.}$$

ignore

2 Approaches for finding Taylor series

1. Calculate partial derivatives (always work)
2. Substitute $x = x_0 + \Delta x$ use small number approximations (sometimes faster)

$\sin(\Delta x) \approx \Delta x$

$\cos(\Delta x) \approx 1$

$\Delta x \Delta u \approx 0$

$$\begin{pmatrix}\dot x_E\\ \dot y_E\\ \dot z_E\end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta\end{pmatrix}\begin{pmatrix}u^E\\ v^E\\ w^E\end{pmatrix}$$

$$\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix} = \begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\begin{pmatrix}\dot u^E\\ \dot v^E\\ \dot w^E\end{pmatrix} = \begin{pmatrix}rv^E - qw^E\\ pw^E - ru^E\\ qu^E - pv^E\end{pmatrix} + g\begin{pmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{pmatrix} + \frac{1}{m}\begin{pmatrix}X_d\\ Y_d\\ Z_d\end{pmatrix} + \frac{1}{m}\begin{pmatrix}0\\ 0\\ Z_c\end{pmatrix}$$

$$\begin{pmatrix}\dot p\\ \dot q\\ \dot r\end{pmatrix} = \begin{pmatrix}\frac{I_y - I_z}{I_x}qr\\ \frac{I_z - I_x}{I_y}pr\\ \frac{I_x - I_y}{I_z}pq\end{pmatrix} + \begin{pmatrix}\frac{1}{I_x}L_d\\ \frac{1}{I_y}M_d\\ \frac{1}{I_z}N_d\end{pmatrix} + \begin{pmatrix}\frac{1}{I_x}L_c\\ \frac{1}{I_y}M_c\\ \frac{1}{I_z}N_c\end{pmatrix}$$

Example

$$\dot\Theta = \overbrace{\cos(\phi)q - \sin(\phi)r}^{f(\phi,q,r)}$$

Approach 1: $\cancel{\dot\theta_0}^{0} + \Delta\dot\theta = \cancel{\cos(\phi_0)q_0}^{0} - \cancel{\sin(\phi_0)r_0}^{0} + \frac{\partial f}{\partial\phi}\Big|_0\Delta\phi + \frac{\partial f}{\partial q}\Big|_0\Delta q + \frac{\partial f}{\partial r}\Big|_0\Delta r$

$$= \cancel{\left(\frac{\partial\cos(\phi)}{\partial\phi}q_0 - \frac{\partial\sin\phi}{\partial\phi}r_0\right)}^{0}\Delta\phi + \cos(\phi_0)\Delta q - \cancel{\sin(\phi_0)}^{0}\Delta r$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

Approach 2: $\dot\theta_0 + \Delta\dot\theta = \cos(\cancel{\phi_0}^{0}+\Delta\phi)(\cancel{q_0}^{0}+\Delta q) - \sin(\cancel{\phi_0}^{0}+\Delta\phi)(\cancel{r_0}^{0}+\Delta r)$

$$= 1\Delta q - \cancel{\Delta\phi\Delta r}^{0}$$

$$\boxed{\Delta\dot\theta = \Delta q}$$

Harder

$$\dot w^E = \underbrace{qu^E - pv^E + g\cos\theta\cos\phi + \frac{1}{m}\overbrace{Z_d}^{Z_d(u,v,w)} + \frac{1}{m}Z_c}_{f(\phi,\theta,u,v,w,p,q,Z_c)}$$

Assume no wind
$u=u^E$, $v=v^E$, $w=w^E$

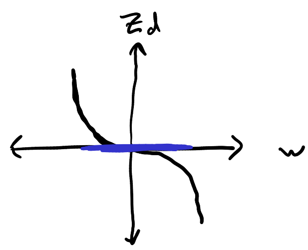
$$\Delta\dot w = \cancel{\frac{\partial f}{\partial\phi}\Big|_0\Delta\phi}^{0} + \cancel{\frac{\partial f}{\partial\theta}\Big|_0\Delta\theta} + \underbrace{\frac{\partial f}{\partial u}\Big|_0\Delta u + \frac{\partial f}{\partial v}\Big|_0\Delta v + \frac{\partial f}{\partial w}\Big|_0\Delta w}_{0} + \cancel{\frac{\partial f}{\partial p}\Big|_0}^{0}\Delta p + \cancel{\frac{\partial f}{\partial q}\Big|_0}^{0}\Delta q + \underbrace{\frac{\partial f}{\partial Z_c}\Big|_0}_{\frac{1}{m}}\Delta Z_c$$

$$Z_d = -\nu w\sqrt{u^2+v^2+w^2}$$

Simple case: assume $u,v=0$

$$Z_d = -\nu w|w| = -\nu w^2\,\mathrm{sign}(w)$$

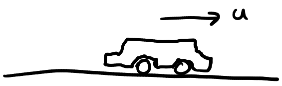
$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\nu\left(2w\,\mathrm{sign}(w) + w^2\frac{\partial}{\partial w}\mathrm{sign}(w)\right)\Big|_0$$



$$\frac{\partial Z_d}{\partial w}\Big|_0 = 0$$

No drag force in Linear model!

Simple "Non hover" example



Trim condition

$u_0 = 30\,m/s$ steady

EOM $\dot{u} = \underbrace{\frac{e_f}{m} - \frac{\nu u|u|}{m}}_{g}$

$\dot{u}_0 = 0 = \frac{e_{f_0}}{m} - \frac{\nu u_0|u_0|}{m}$

$$\Delta\dot{u} = \frac{\partial g}{\partial u}\Big|_0 \Delta u = \frac{\partial(-\nu u^2 \operatorname{sign}(u))}{\partial u}\Big|_0 \Delta u = -\nu\left(2u\operatorname{sign}(u) + \cancel{u^2 \frac{\partial \operatorname{sign}(u)}{\partial u}}^{\,0}\right)\Big|_0 \Delta u$$

$$= -\nu 2 u_0 \Delta u$$

$$\boxed{\Delta\dot{u} = -\nu 2 u_0 \Delta u}$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\nu\left(\sqrt{u^2+v^2+w^2} + w\frac{1}{2}\frac{1}{\sqrt{u^2+v^2+w^2}}2w\right)\Big|_0 = -\nu\,\frac{u^2+v^2+2w^2}{\sqrt{u^2+v^2+w^2}}\Big|_0$$

looks like $\frac{0}{0}$

can solve with spherical coordinates

$u = r\cos\theta\sin\phi$
$v = r\sin\theta\sin\phi$
$w = r\cos\phi$

$$\lim_{u,v,w\to 0} -\nu\frac{u^2+v^2+2w^2}{\sqrt{u^2+v^2+w^2}}$$

$$\lim_{r\to 0}\frac{r^2\cdot\text{stuff}}{r} = 0$$

$$\frac{\partial Z_d}{\partial u}\Big|_0 = 0$$

$$\frac{\partial Z_d}{\partial v}\Big|_0 = 0$$

$$\boxed{\Delta\dot{w}^E = \frac{\Delta Z_c}{m}}$$

$$\begin{pmatrix}\Delta\dot{x}_E\\ \Delta\dot{y}_E\\ \Delta\dot{z}_E\end{pmatrix} = \begin{pmatrix}\Delta u\\ \Delta v\\ \Delta w\end{pmatrix}$$

$$\rightarrow \begin{pmatrix}\Delta\dot{\phi}\\ \Delta\dot{\theta}\\ \Delta\dot{\psi}\end{pmatrix} = \begin{pmatrix}\Delta p\\ \Delta q\\ \Delta r\end{pmatrix}$$

Drag force would show up here

$$\rightarrow \begin{pmatrix}\Delta\dot{u}\\ \Delta\dot{v}\\ \Delta\dot{w}\end{pmatrix} = g\begin{pmatrix}-\Delta\theta\\ \Delta\phi\\ 0\end{pmatrix} + \frac{1}{m}\begin{pmatrix}0\\ 0\\ \Delta Z_c\end{pmatrix}$$

$$\begin{pmatrix}\Delta\dot{p}\\ \Delta\dot{q}\\ \Delta\dot{r}\end{pmatrix} = \begin{pmatrix}\frac{1}{I_x}\Delta L_c\\ \frac{1}{I_y}\Delta M_c\\ \frac{1}{I_z}\Delta N_c\end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{y}_E \\ \Delta\dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{\phi} \\ \Delta\dot{\theta} \\ \Delta\dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{v} \\ \Delta\dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta\theta \\ \Delta\phi \\ 0 \end{pmatrix} + \frac{1}{m}\begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x}\Delta L_c \\ \frac{1}{I_y}\Delta M_c \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

Longitudinal

$$\begin{pmatrix} \Delta\dot{x}_E \\ \Delta\dot{u} \\ \Delta\dot{\theta} \\ \Delta\dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g\Delta\theta \\ \Delta q \\ \frac{1}{I_y}\Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{pmatrix}$$

Vertical

$$\begin{pmatrix} \Delta\dot{z}_E \\ \Delta\dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m}\Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta\dot{\psi} \\ \Delta\dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z}\Delta N_c \end{pmatrix}$$

state space model

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} + D\vec{u}$$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$$

$$u \longrightarrow \boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}} \longrightarrow y$$

$$\begin{bmatrix} \Delta\dot{\phi} \\ \Delta\dot{p} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \frac{1}{I_x}\Delta L_c \end{bmatrix} \qquad \begin{aligned} \Delta p &= \Delta\dot{\phi} \\ \Delta\dot{p} &= \Delta\ddot{\phi} \end{aligned}$$

$$\Delta\ddot{\phi} = \frac{1}{I_x}\Delta L_c$$

Solutions? $\Delta\phi(t)$

Assume ΔL_c is constant

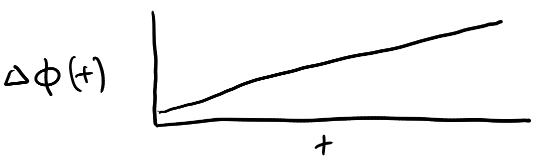
$$\Delta\dot{\phi}(t) = \Delta\dot{\phi}_0 + \int_{\tau=0}^{t}\Delta\ddot{\phi}(t)\,d\tau = \Delta\dot{\phi}_0 + \frac{1}{I_x}\Delta L_c t$$

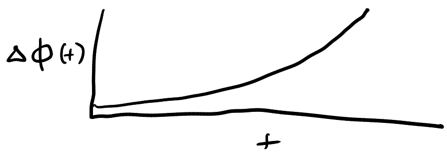
$$\Delta\phi(t) = \Delta\phi_0 + \int_{\tau=0}^{t}\Delta\dot{\phi}(t)\,d\tau = \Delta\phi_0 + \Delta\dot{\phi}_0 t + \frac{1}{2}\frac{1}{I_x}\Delta L_c t^2$$

Exactly at hover $\Delta\phi_0 = \Delta\dot{\phi}_0 = \Delta L_c = 0$

$$\Delta\phi(t) = 0$$

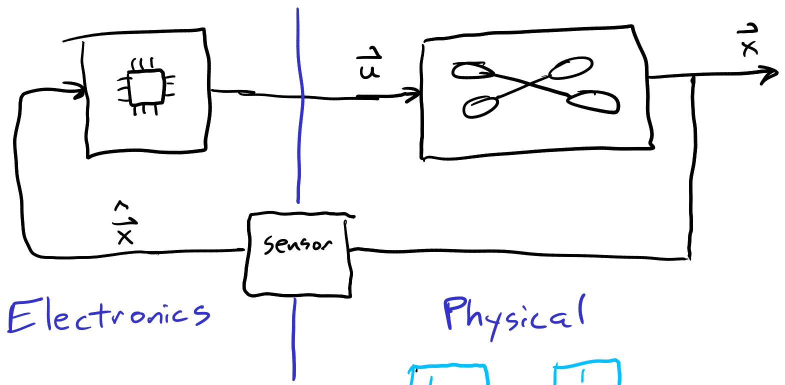
If $\Delta\dot{\phi}_0 > 0$, $\Delta L_c = 0$ If $\Delta L_c > 0$

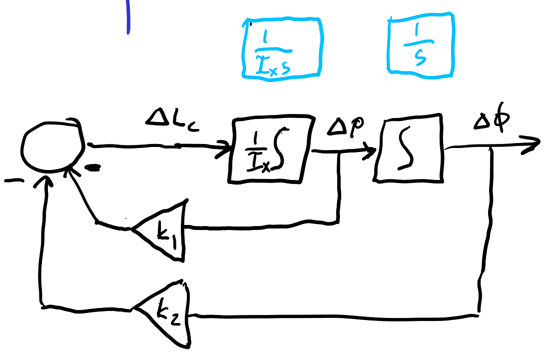




If initial conditions are nonzero (always in real life), vehicle will crash

Solution: Feedback Control





$$\Delta L_c = -k_1 \Delta p - k_2 \Delta\phi$$

Deriv. gain

prop. gain

$$\Delta\ddot{\phi} = \frac{1}{I_x}\Delta L_c$$

$$= \frac{1}{I_x}\left(-k_1 \Delta p - k_2 \Delta\phi\right)$$

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\boxed{\Delta\ddot{\phi} + \frac{k_1}{I_x}\Delta\dot{\phi} + \frac{k_2}{I_x}\Delta\phi = 0} \Longleftrightarrow \boxed{\Delta\ddot{\phi} + 2\zeta\omega_n\Delta\dot{\phi} + \omega_n^2\Delta\phi = 0}$$

$$\zeta = \frac{k_1}{2\sqrt{k_2 I_x}} \qquad \omega_n = \sqrt{\frac{k_2}{I_x}}$$

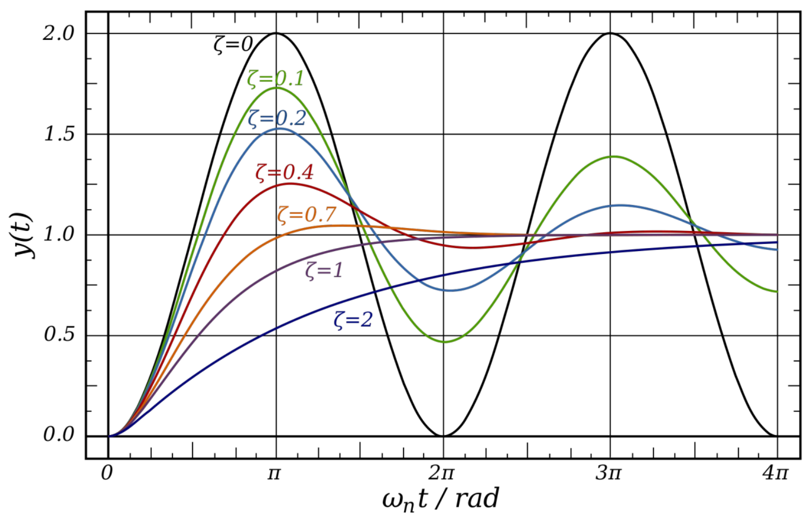
$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

If λ are real and distinct

$$\boxed{\Delta\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}}$$

If λ are complex

$$\boxed{\Delta\phi(t) = e^{-\zeta\omega_n t}\left(C_3\sin(\omega_d t) + C_4\cos(\omega_d t)\right)}$$



$$\begin{bmatrix}\Delta\dot\phi \\ \Delta\dot p\end{bmatrix} = \begin{bmatrix}\Delta p \\ \frac{1}{I_x}\Delta L_c\end{bmatrix} \Longrightarrow \underbrace{\begin{bmatrix}\Delta\dot\phi \\ \Delta\dot p\end{bmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\Delta\phi \\ \Delta p\end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix}0 \\ \frac{1}{I_x}\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}\Delta L_c\end{bmatrix}}_{\vec{u}}$$

$$\underbrace{[\Delta\phi]}_{\vec{Y}} = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}\Delta\phi \\ \Delta p\end{bmatrix}}_{\vec{x}} + \underbrace{[0]}_{D}\underbrace{[\Delta L_c]}_{\vec{u}}$$

$$\vec{u} = [\Delta L_c] = -K\vec{x} = -\begin{bmatrix}k_2 & k_1\end{bmatrix}\begin{bmatrix}\Delta\phi \\ \Delta p\end{bmatrix}$$

$$\dot{\vec{x}} = \underbrace{A\vec{x} - BK\vec{x}}_{} = \underbrace{(A - BK)}_{A^{cl}}\vec{x}$$

$$A^{cl} = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 \\ \frac{1}{I_x}\end{bmatrix}\begin{bmatrix}k_2 & k_1\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x}\end{bmatrix}$$

$\dot{\vec{x}} = A^{cl}\vec{x}$ what are solutions of $\dot{\vec{x}} = A^{cl}\vec{x}$?

scalar case

$\dot{x} = ax \Longrightarrow x(t) = x(0)e^{at}$

analogously

$$\boxed{\dot{\vec{x}} = A\vec{x} \Longrightarrow \vec{x}(t) = e^{At}\vec{x}(0)}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 \ldots \quad \text{(Taylor Series)}$$

Modal Analysis

Eigenvalues and Eigenvectors

$$A\vec{v}_i = \lambda_i\vec{v}_i$$

Suppose that $\vec{x}_0 = \vec{v}_i$

$$\vec{x}(t) = \left(I + tA + \frac{t^2}{2!}A^2 \ldots\right)\vec{v}_i$$

$$= \vec{v}_i + t\lambda_i v_i + \frac{t^2}{2!}\lambda_i^2\vec{v}_i \ldots$$

$$\vec{x}(t) = \vec{v}_i e^{\lambda_i t}$$

$$\boxed{\begin{array}{l}\text{If } \vec{x}_0 = \sum_i q_i\vec{v}_i \\ \text{then } \vec{x}(t) = \sum_i q_i\vec{v}_i e^{\lambda_i t}\end{array}}$$

$\vec{q} = V^{-1}\vec{x}$

matrix of eigenvector columns

Finding Eigenvalues

$A\vec{v}_i = \lambda_i \vec{v}_i$

$(A - \lambda_i I)\vec{v}_i = 0$ only has nontrivial solutions if $|A - \lambda_i I| = 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

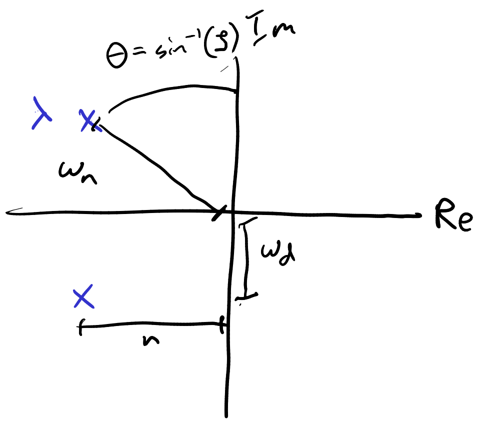
solve with quadratic formula

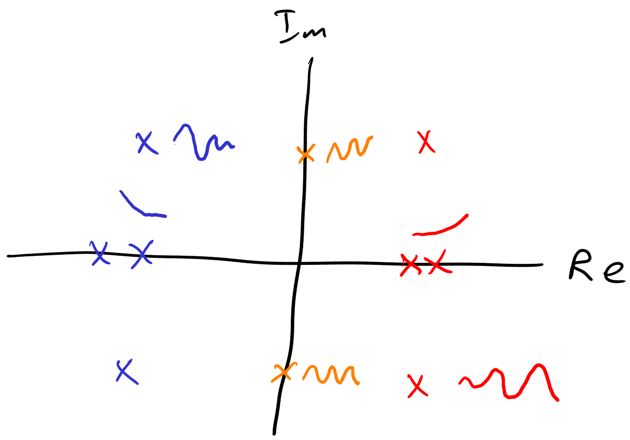
$$|A^{cl} - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} - \lambda \end{vmatrix} = \lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{I_x} \pm \sqrt{\frac{k_1^2}{4I_x} - \frac{k_2}{I_x}}$$

$$= n \pm i\omega_d$$

$$= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$





Solutions to linear ODEs

$$\dot{x} = ax \Rightarrow x(0)e^{at}$$

$$\ddot{x} + \underbrace{2\zeta\omega_n}_{a}\dot{x} + \underbrace{\omega_n^2}_{b}x = 0 \Longrightarrow$$ characteristic eqn $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

quad. form

$$\lambda = -\zeta\omega_n \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$ if λ real and distinct

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \sin(\omega_d t) + C_4 \cos(\omega_d t)\right)$$

$$\dot{\vec{x}} = A\vec{x} \Rightarrow e^{At}\vec{x}(0)$$

or

$$\sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$\vec{v}_i$ are eigenvectors

λ_i are eigenvalues

if $\vec{x}(0) = a\vec{v}_1 + b\vec{v}_2$

$$\vec{x}(t) = a\vec{v}_1 e^{\lambda_1 t} + b\vec{v}_2 e^{\lambda_2 t}$$

$\vec{v}_i$ Eigenvectors: "shape" of the mode
which state variables are actively changing

λ_i Eigenvalues : "speed" of the mode
how fast does it oscillate, decay, or diverge

Linear Control Design Process

1. Derive EOM
2. Linearize and Separate EOM
3. Design Control Architecture
4. Choose Gain Values
5. Testing in Linear Simulation
6. Test in Nonlinear Sim

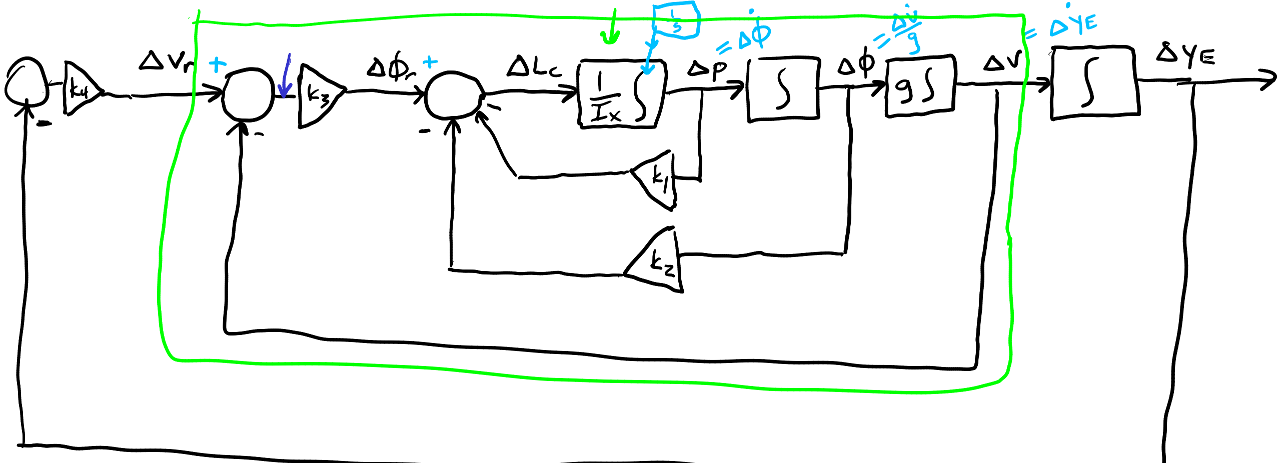
Choose Gain Values:

1. PID tuning
2. Pole Assignment
3. Root Locus
4. Optimal Control (LQR)

$$\underbrace{\begin{bmatrix} \Delta\dot{y}_E \\ \Delta\dot{v} \\ \Delta\dot{\phi} \\ \Delta\dot{p} \end{bmatrix}}_{\dot{\vec{x}}} = \begin{bmatrix} \Delta v \\ g\Delta\phi \\ \Delta p \\ \frac{1}{I_x}\Delta L_c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta\phi \\ \Delta p \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{I_x} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \Delta L_c \end{bmatrix}}_{\vec{u}}$$

$$\vec{y} = \underset{= I}{C}\vec{x} + \underset{= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}{D}\vec{u}$$

$I_x = 7 \times 10^{-5}$ kgm^2



<u>Control Law</u>

$$\longrightarrow \quad \Delta L_c = -k_1\Delta p - k_2\Delta\phi + k_3(-k_4\Delta Y_E - \Delta V)$$

$$= -k_1\Delta p - k_2\Delta\phi - k_3k_4\Delta Y_E - k_3\Delta V$$

$$[\Delta L_c] = \vec{u} = -K\vec{x} = -\underbrace{\left[k_3k_4 \;\vdots\; k_3 \;\vdots\; k_2 \;\vdots\; k_1\right]} \begin{bmatrix} \Delta Y_E \\ \Delta V \\ \Delta\phi \\ \Delta p \end{bmatrix}$$

$$A^{cl} = A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \qquad \dot{\vec{x}} = A^{cl}\vec{x}$$

<u>Choosing Gains</u>

1. Choose k_1 and k_2 with the "pole placement" strategy

Pole placement

- Decide where we want eigenvalues (poles) to be
- Solve for k_1 and k_2

For 2nd order system

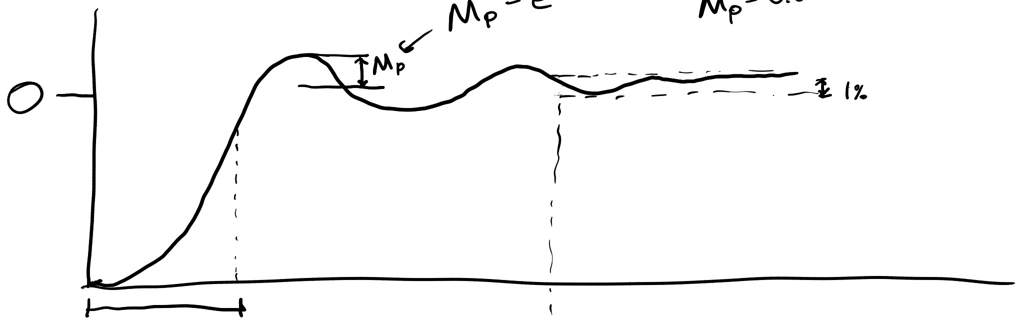
$$\begin{bmatrix} \Delta\dot\phi \\ \Delta\dot p \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} \end{bmatrix}}_{A^{cl}} \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix}$$

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$|A^{cl} - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0 \qquad \text{analogous to} \qquad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$k_1 = 2\zeta\omega_n I_x \qquad k_2 = \omega_n^2 I_x$$



$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ $\quad 0 \le \zeta \le 1$

$M_p = 0.05$ for 0.7

ω_n

ζ

rise time

$t_r = \frac{1.8}{\omega_n}$

settling time

$t_s = \frac{4.6}{\zeta\omega_n}$

try

$t_r = 0.2 \Rightarrow \omega_n = 9\ \text{rad/s}$

$\zeta = 0.7$

$\lambda = -\omega_n\zeta \pm i\,\omega_n\sqrt{1-\zeta^2}$

$\lambda = -6.3 \pm 6.4i$

$k_1 = 8.8\times10^{-4}$ $\qquad k_2 = 5.7\times10^{-3}$

$\dot{\vec{x}} = \underset{2\times2}{A^{cl}}\vec{x}$ $\quad \begin{bmatrix}\Delta\phi \\ \Delta p\end{bmatrix}$

solutions look like

$\vec{x}(0) = \sum_i q_i \vec{v}_i$

$\vec{x}(t) = \sum_i q_i \vec{v}_i \underline{e^{\lambda_i t}}$

λ_i are solutions to

$|A^{cl} - \lambda I| = 0$

if A^{cl} is 2×2

$\lambda^2 + \underbrace{\quad}_{\boxed{\frac{k_1}{I_x}}}\lambda + \underbrace{\quad}_{\boxed{\frac{k_2}{I_x}}} = 0$

$\Delta\ddot{\phi} + \underbrace{\quad}_{2\zeta\omega_n}\Delta\dot{\phi} + \underbrace{\quad}_{\omega_n^2} = 0$

solutions look like

$\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

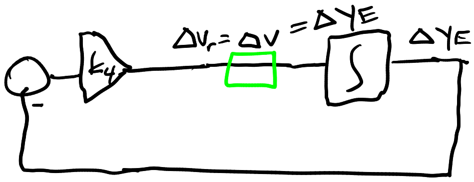
(C, λ might be complex $\Rightarrow$ sin/cos)

λ_i are solutions to

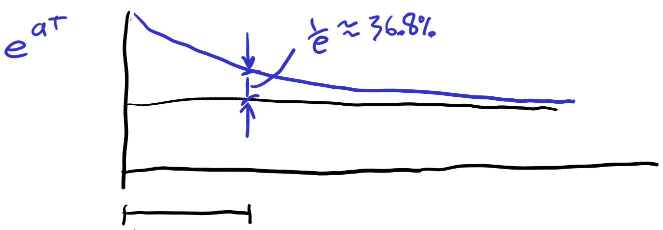
$\lambda^2 + \boxed{2\zeta\omega_n}\lambda + \boxed{\omega_n^2} = 0$

$k_1 = 2\zeta\omega_n I_x$ $\qquad k_2 = \omega_n^2 I_x$

Choose k_4 with pole placement



$\Delta\dot{Y}_E = -k_4 \Delta Y_E \Longrightarrow \Delta Y_E(t) = \Delta Y_E(0)\,e^{-k_4 t}$

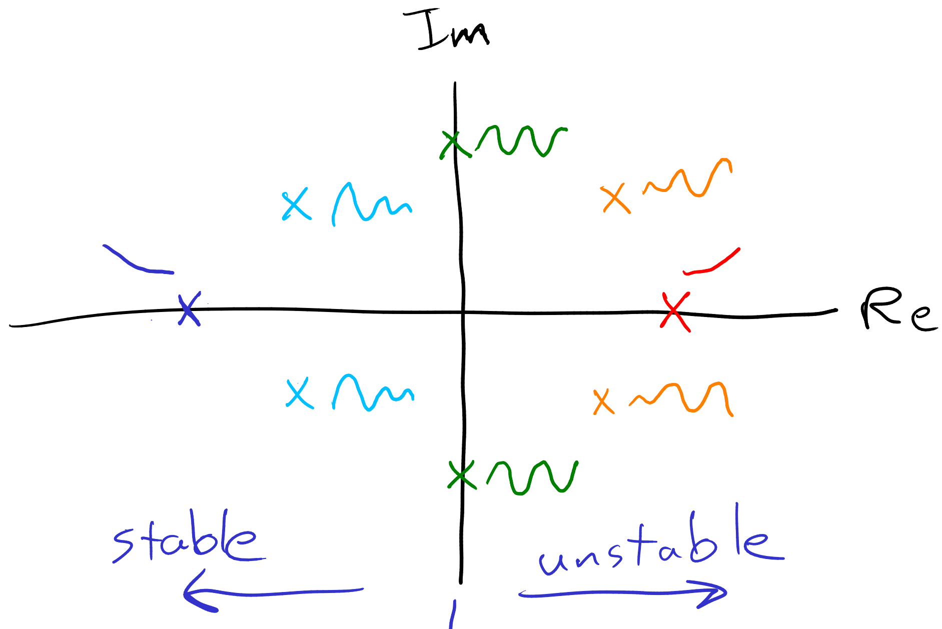


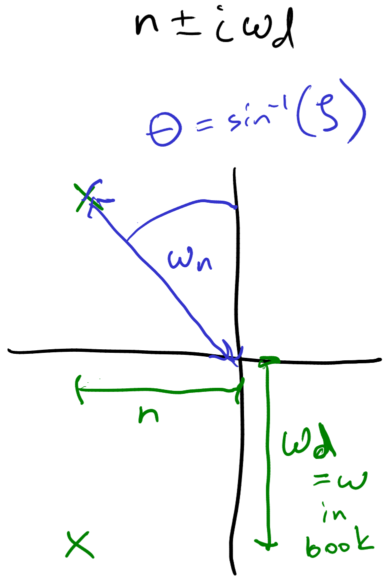
$\tau_y = \frac{1}{k_4}$

inner loop (ϕ, p) has settling time of $\frac{4.6}{\zeta\omega_n} \approx 0.7$

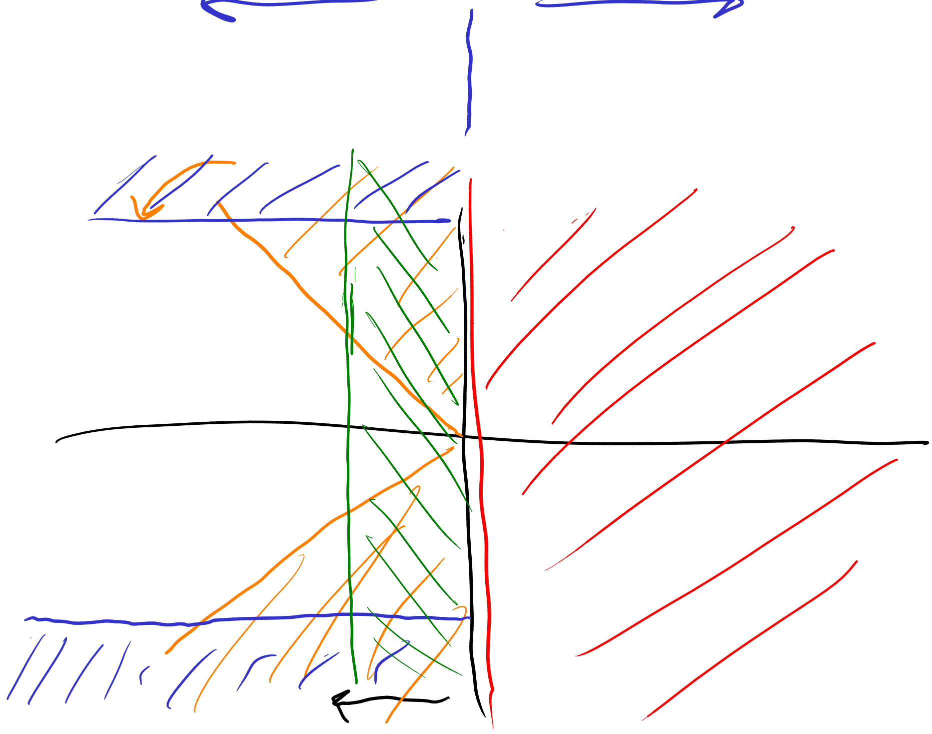
choose τ_y 10x larger ~7sec, ≈5

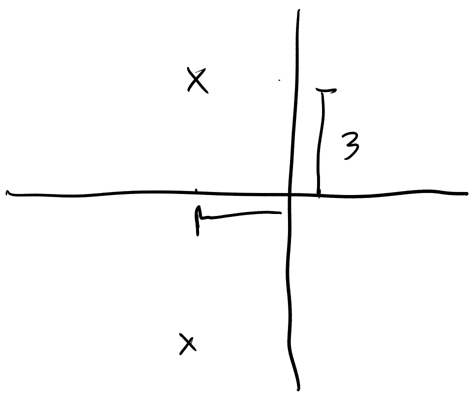
$\tau_y = 5 \Rightarrow k_4 = 0.2$





$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$





Conventonal A/C Dynamics

Longitudinal	**Lateral/Directional**
Altitude	Roll
Speed	Yaw
Pitch	Sideslip

Long. Forces and Moments

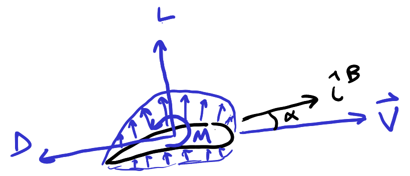
Lift $\quad L = \frac{1}{2}\rho V_a^2 S C_L$

density, Wing Area, airspeed

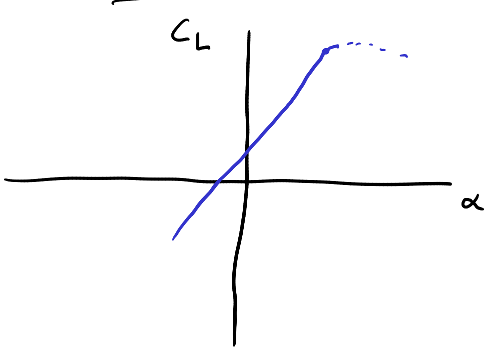
Drag $\quad D = \frac{1}{2}\rho V_a^2 S C_D$

Pitch Moment $\quad M = \frac{1}{2}\rho V_a^2 S \bar{c} C_m$

Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 S\bar{b}$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c}/2)^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$



Lift



$C_L(\alpha, q, \delta_e)$

1st order Taylor series

$$L = \frac{1}{2}\rho V_a^2 S \left(C_{L_{zero}} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e}\delta_e \right)$$

Stability Derivatives

$$C_{a_b} \equiv \left.\frac{\partial \text{ nondimensionalized } a}{\partial \text{ nondimensionalized } b}\right|_{\substack{\text{condition} \\ \text{(usually trim)}}}$$

- based on linear assumptions
- main tool for connecting aerodynamics + dynamics
- determined by A/C geometry
- only accurate in a linear region (e.g. small α)

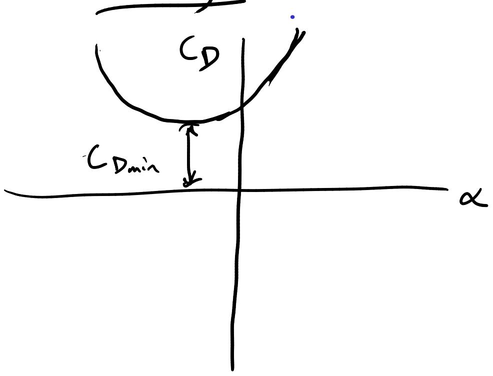
Estimated

- Geometric Data $\longrightarrow$ $C_{L_\alpha} \approx \dfrac{\pi AR}{1+\sqrt{1+\left(\frac{AR}{2}\right)^2}}$
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$\boxed{L = \tfrac{1}{2}\rho V_a^2 S\left(C_{L_{zero}} + C_{L_\alpha}\alpha + C_{L_q}\hat{q} + C_{L_{\delta e}}\delta e\right)}$$

$$\hat{q} = q\frac{\bar{c}}{2V_a}$$

Drag



parasitic + induced
$\propto C_L^2$

$$\boxed{C_D = C_{D_{min}} + K\left(C_L(\alpha,q,\delta e) - C_{L_{min}}\right)^2}$$

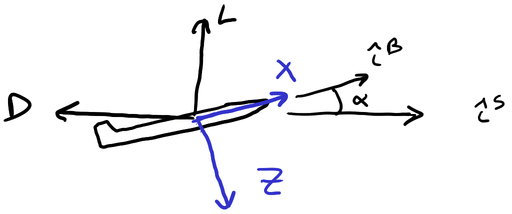
$$K = \frac{1}{\pi e AR}$$

e: Oswald's Efficiency

Pitch Moment

$$M \approx \tfrac{1}{2}\rho V_a^2 S\bar{c}\left(C_{m_{zero}} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta e}}\delta e\right)$$

Lift + Drag $\longrightarrow$ Body coordinates



"stability coordinate system"

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} -D \\ -L \end{bmatrix}$$

Longitudinal Stability

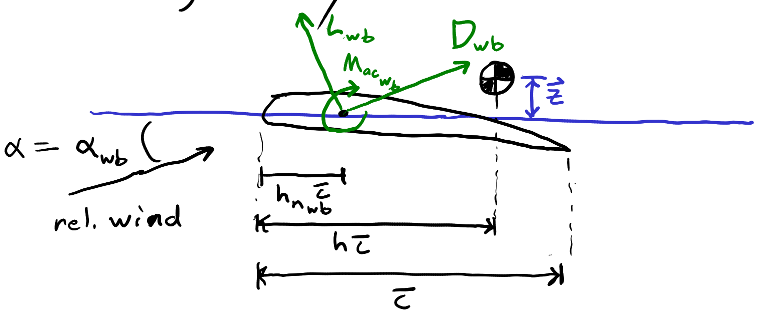
So far : C_L, C_D, C_m — Steady state forces

Now : C_{L_α}, C_{m_α}, $C_{L_{\delta_e}}$, $C_{m_{\delta_e}}$ — trim, static stability

static margin : $h_n - h$

Three contributors
1. Wing/body
2. Propulsion (often small)
3. Tail

Wing/body



Aerodynamic Center
$\frac{\partial \text{Moment}}{\partial \alpha} = 0$
Location stays constant
Moment usually < 0

Center of Pressure
Moment $= 0$
Changes w/ α

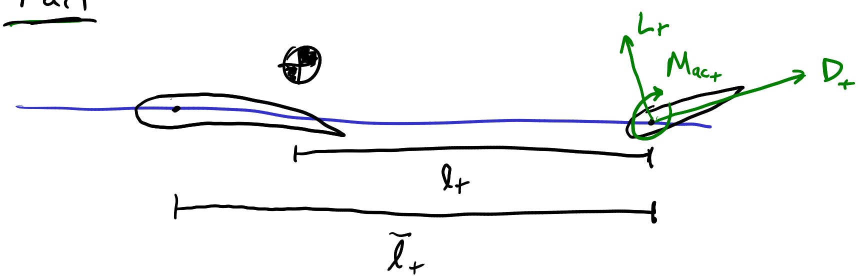
Neutral Point
C.G. location that yields $C_{m_\alpha} = 0$
A.C. of entire A/c

$$M_{wb} = M_{ac_{wb}} + (L\cos\alpha + D\sin\alpha)(h - h_{n_{wb}})\bar{c} + (L\sin\alpha - D\cos\alpha)\bar{z}$$

only keep most important terms + nondimensionalize

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_{wb}})$$

Tail



$$L = L_{wb} + L_t$$
$$= C_{L_{wb}}\left(\tfrac{1}{2}\rho V^2 S\right) + C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right) \implies \boxed{C_L = C_{L_{wb}} + \frac{S_t}{S} C_{L_t}}$$
$$= C_L\left(\tfrac{1}{2}\rho V^2 S\right)$$

$$M_t = -l_t L_t = -l_t C_{L_t}\left(\tfrac{1}{2}\rho V^2 S_t\right) \implies C_{m_t} = \underbrace{\frac{-l_t}{\bar{c}}\frac{S_t}{S}}_{} C_{L_t}$$

$$= -V_H C_{L_t}$$

"volume ratio"

more convenient b/c C.G. can change

$$\bar{V}_H = \frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S} \implies V_H = \bar{V}_H - \frac{S_t}{S}(h - h_{n_{wb}})$$

$$C_{m_t} = -\bar{V}_H C_{L_t} + C_{L_t}\frac{S_t}{S}(h - h_{n_{wb}})$$

$$\rightarrow C_m = \overbrace{C_{m_{ac_{wb}}} + C_L(h - h_{n_{wb}})}^{\text{Wing/body}} \underbrace{- \bar{V}_H C_{L_t}}_{\text{Tail}} + \underbrace{C_{m_p}}_{\text{propulsion}}$$

$$C_{m_\alpha} = \underbrace{\cancel{\frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha}}}_{0} + C_{L_\alpha}(h - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

Want $h_n \equiv$ cg location where $C_{m_\alpha} = 0$

$$0 = C_{L_\alpha}(h_n - h_{n_{wb}}) - \bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}$$

$$h_n = \underline{h_{n_{wb}}} + \frac{1}{C_{L_\alpha}}\underbrace{\left(\bar{V}_H \frac{\partial C_{L_t}}{\partial \alpha} - \frac{\partial C_{m_p}}{\partial \alpha}\right)}_{\text{tail correction}}$$

$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h - h_n)}$$

$C_{m_\alpha} < 0$ for stability

static margin

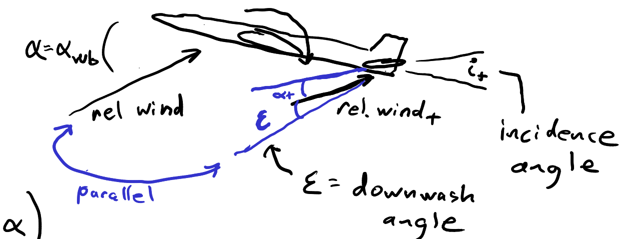
$K_n \equiv h_n - h$ must be > 0 for stability

back to h_n eq.

Lift curve slope $a \equiv C_{L_\alpha}$

$$C_{L_{wb}} = a_{wb}\alpha_{wb} = a_{wb}\alpha$$

$$C_{L_t} = a_t \alpha_t$$



$$\alpha_t = \alpha - i_t - \left(\varepsilon_{zero} + \frac{\partial \varepsilon}{\partial \alpha}\alpha\right)$$

$$\frac{\partial C_{L_t}}{\partial \alpha} = a_t\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)$$

$$\boxed{h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{m_p}}{\partial \alpha}}$$

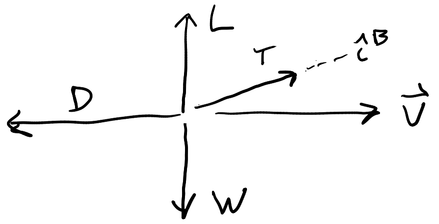
$$\boxed{C_{L_\alpha} = a = a_{wb}\left[1 + \frac{a_t S_t}{a_{wb} S}\left(1 - \frac{\partial \varepsilon}{\partial \alpha}\right)\right]}$$

Longitudinal Control

δ_e changes C_{L_t}

$$\boxed{\begin{aligned} C_{L_{\delta_e}} &= \frac{\partial C_{L_t}}{\partial \delta_e}\frac{S_t}{S} = a_e \frac{S_t}{S} \\ C_{m_{\delta_e}} &= -a_e \bar{V}_H + C_{L_{\delta_e}}(h - h_{n_{wb}}) \end{aligned}}$$

Linear Trim Estimation



For Linear trim, $T = D$, $L = W$

0 for Linear Trim Estimation

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V^2 S} = \cancelto{0}{C_{L_{zero}}} + C_{L_\alpha}\alpha_{trim} + C_{L_{\delta e}}\delta_{e_{trim}}$$

$$C_{m_{trim}} = C_{m_{zero}} + C_{m_\alpha}\alpha_{trim} + C_{m_{\delta e}}\delta_{e_{trim}} = 0$$

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_\alpha} & C_{L_{\delta e}} \\ C_{m_\alpha} & C_{m_{\delta e}} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{trim}} \\ -C_{m_{zero}} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Cramer's rule)

$$\alpha_{trim} = \frac{C_{m_{zero}} C_{L_{\delta e}} + C_{m_{\delta e}} C_{L_{trim}}}{\Delta}$$

$$\Delta = C_{L_\alpha} C_{m_{\delta e}} - C_{L_{\delta e}} C_{m_\alpha}$$

$$\delta_{e_{trim}} = -\frac{C_{m_{zero}} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta}$$

Longitudinal Linear Model

$$\dot{\vec{p}}_E = R_B^E \vec{V}_B^E$$

$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\dot{\vec{V}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{V}_B^E$$

$$\dot{\vec{\omega}} = I^{-1}\left[\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B\right]$$

2 differences

1. Aerodynamic Forces
2. I more complex

symmetry about x-z axis $\Rightarrow I_{xy} = I_{yz} = 0$

$I_{xz} \neq 0$

$$I_B^{-1} = \begin{bmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}\left(I_x - I_y + I_z\right)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x\left(I_x - I_y\right) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z\left(I_z - I_y\right) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$I_{yz} \neq 0$

$I_{xy} \neq 0$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

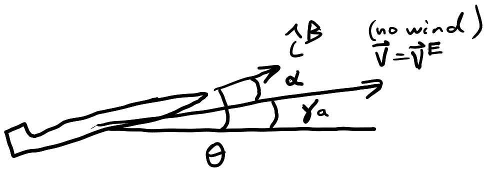
$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Linearize about trim state

Inputs: u_0, h_0, γ_{a_0} $\longrightarrow$ $\alpha_0, \delta_{e_0}, \delta_{t_0}$

airspeed $= V = V_a$; altitude ; air-relative flight-path angle



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \Delta \vec{x} \qquad \vec{x}_0 = \begin{bmatrix} \cdot \\ \cdot \\ -h_0 \\ 0 \\ \theta_0 \\ \cdot \\ u_0^E \\ 0 \\ w_0^E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For linearization: $\vec{V} = \vec{V}^E$ (no wind)

assume $w_0^E = 0$

$$\vec{u} = \begin{bmatrix} \delta_e \\ \delta_a \\ \delta_r \\ \delta_t \end{bmatrix} \qquad \vec{u}_0 = \begin{bmatrix} \delta_{e_0} \\ 0 \\ 0 \\ \delta_{t_0} \end{bmatrix}$$

Examples to Linearize

$$\dot{\theta} = \cos\phi \, q - \sin\phi \, r$$

$$\cancel{\dot{\theta}_0}^{\,0} + \Delta\dot{\theta} = \cancel{\cos\phi_0 \, q_0}^{\,0} - \cancel{\sin\phi_0 \, r_0}^{\,0} + \frac{\partial}{\partial\phi}(\cos\phi \, q - \sin\phi \, r)\Big|_0 \Delta\phi$$
$$+ \frac{\partial}{\partial q}(\cos\phi \, q - \sin\phi \, r)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial r}(\cos\phi \, q - \sin\phi \, r)\Big|_0 \Delta r$$
$$= -\cancel{\sin\phi_0}^{\,0} q_0 - \cancel{\cos\phi_0 r_0}^{\,0} + \cancel{\cos}^{\,1}\phi_0 \Delta q - \cancel{\sin}^{\,0}\phi_0 \Delta r$$

$$\boxed{\Delta\dot{\theta} = \Delta q}$$

$$\dot{u} = rv - qw - g\sin\theta + \frac{X}{m}$$

$$\cancel{\dot{u}_0}^{\,0} + \Delta\dot{u} = \cancel{r_0 v_0}^{\,0} - \cancel{q_0 w_0}^{\,0} \underbrace{- g\sin\theta_0 + \frac{X_0}{m}}_{\text{cancel}}{}^{0} + \frac{\partial}{\partial r} rv\Big|_0 \Delta r + \frac{\partial}{\partial v} rv\Big|_0 \Delta v + \frac{\partial}{\partial q}(-qw)\Big|_0 \Delta q$$
$$+ \frac{\partial}{\partial w}(-qw)\Big|_0 \Delta w + \frac{\partial}{\partial \theta}(g\sin\theta)\Big|_0 \Delta\theta + \frac{\partial}{\partial X}\left(\frac{X}{m}\right)\Big|_0 \Delta X$$
$$= \cancel{v_0 \Delta r}^{\,0} + \cancel{r_0 \Delta v}^{\,0} - \cancel{w_0 \Delta q}^{\,0} - \cancel{q_0 \Delta w}^{\,0} - g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X$$

$$\boxed{\Delta\dot{u} = -g\cos\theta_0 \Delta\theta + \frac{1}{m}\Delta X}$$

Lateral

$$\Delta\dot\phi = \Delta p + \Delta r \tan\theta_0$$

$$\Delta\dot\theta = \Delta q$$

Long.

$$\Delta\dot u = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$

$$\Delta\dot v = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$

$$\Delta\dot w = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$

$$\Delta\dot p = \Gamma_3\Delta L + \Gamma_4\Delta N$$

$$\Delta\dot q = \frac{\Delta M}{I_y}$$

$$\Delta\dot r = \Gamma_4\Delta L + \Gamma_8\Delta N$$

Dimensional stab derivs.

$$\Delta X = X_u\Delta u + X_w\Delta w + \Delta X_c$$

$$\Delta Y = Y_v\Delta v + Y_p\Delta p + Y_r\Delta r + \Delta Y_c$$

$$\Delta Z = Z_u\Delta u + Z_w\Delta w + Z_{\dot w}\Delta\dot w + Z_q\Delta q + \Delta Z_c$$

$$\Delta L = L_v\Delta v + L_p\Delta p + L_r\Delta r + \Delta L_c$$

$$\Delta M = M_u\Delta u + M_w\Delta w + M_{\dot w}\Delta\dot w + M_q\Delta q + \Delta M_c$$

$$\Delta N = N_v\Delta v + N_p\Delta p + N_r\Delta r + \Delta N_c$$

Dimensional stability Derivative

$$X_u \equiv \left.\frac{\partial X}{\partial u}\right|_0$$

Nondim Stab Deriv.

$$C_{x_u} \equiv \left.\frac{\partial C_x}{\partial \hat u}\right|_0$$

$$\hat w = \frac{w}{u_0} \approx \alpha$$

$$\alpha = \operatorname{atan}\left(\frac{\Delta w}{u_0+\Delta u}\right) \approx \frac{w}{u_0}$$

Table 4.4
Longitudinal Dimensional Derivatives

	X X_u	Z	M
u	$\rho u_0 S C_{w_0}\sin\theta_0 + \frac{1}{2}\rho u_0 S C_{x_u}$	$-\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}$	$\frac{1}{2}\rho u_0 \bar c S C_{m_u}$
w	$\frac{1}{2}\rho u_0 S C_{x_\alpha}$ X_w	$\frac{1}{2}\rho u_0 S C_{z_\alpha}$	$\frac{1}{2}\rho u_0 \bar c S C_{m_\alpha}$
q	$\frac{1}{4}\rho u_0 \bar c S C_{x_q}$	$\frac{1}{4}\rho u_0 \bar c S C_{z_q}$	$\frac{1}{4}\rho u_0 \bar c^2 S C_{m_q}$
$\dot w$	$\frac{1}{4}\rho \bar c S C_{x_{\dot\alpha}}$	$\frac{1}{4}\rho \bar c S C_{z_{\dot\alpha}}$	$\frac{1}{4}\rho \bar c^2 S C_{m_{\dot\alpha}}$

$$Z_u \equiv \left.\frac{\partial Z}{\partial u}\right|_0 \qquad Z = \frac{1}{2}\rho V^2 S C_z$$

$$\left.\frac{\partial Z}{\partial u}\right|_0 = \frac{1}{2}\rho S\left(\left.\frac{\partial V^2}{\partial u}\right|_0 C_z + V^2\left.\frac{\partial C_z}{\partial u}\right|_0\right)$$

$$= \frac{1}{2}\rho S\left(2u_0 C_{z_0} + u_0^2\left.\frac{\partial C_z}{\partial u}\right|_0\right)$$

$$\boxed{Z_u = -\rho u_0 S C_{w_0}\cos\theta_0 + \frac{1}{2}\rho u_0 S C_{z_u}}$$

$$\hat u = \frac{u}{V} = \frac{u}{u_0} \qquad u = \hat u u_0$$

$$\left.\frac{\partial C_z}{\partial u}\right|_0 = \left.\frac{\partial C_z}{u_0\partial\hat u}\right|_0 = \frac{1}{u_0}C_{z_u}$$

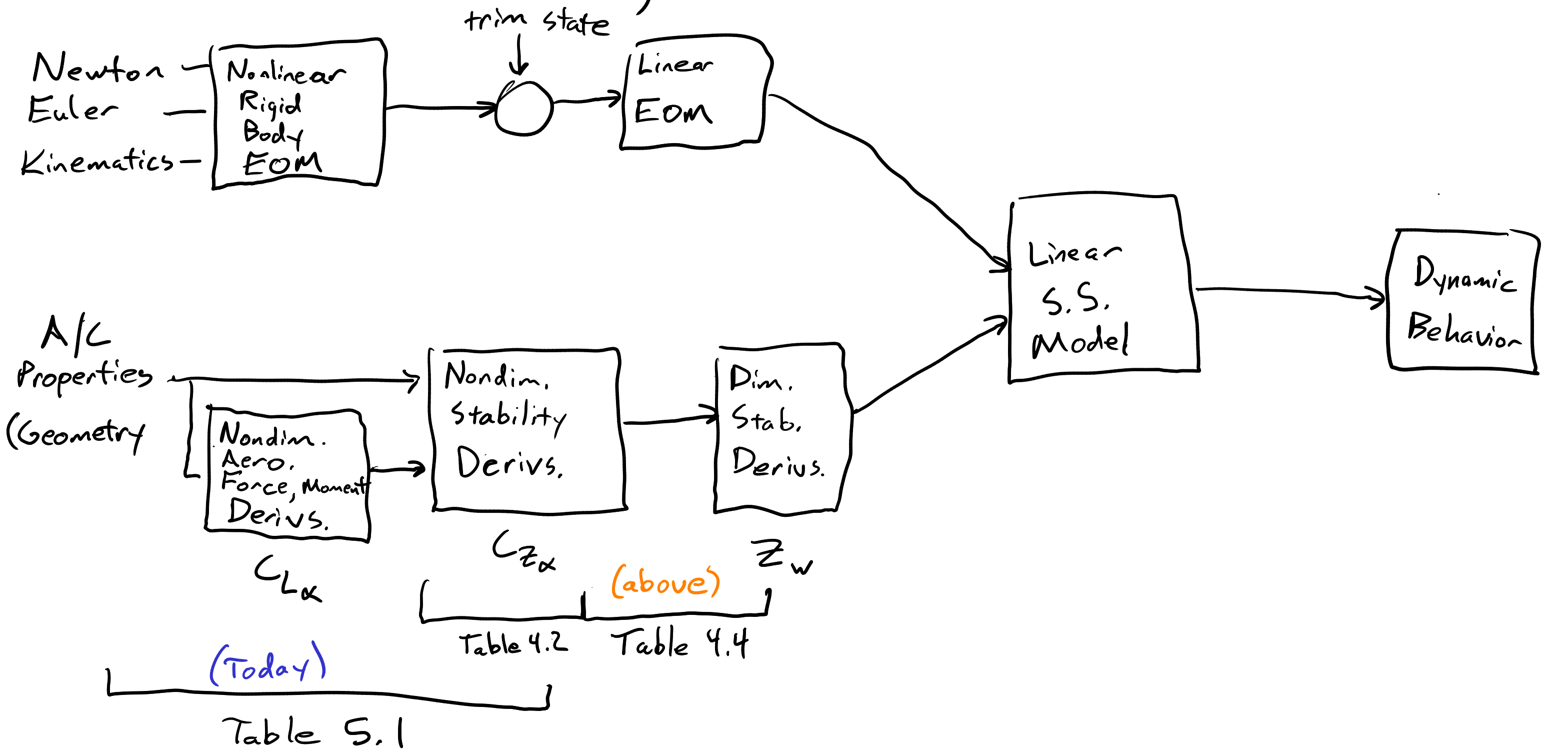
$$C_{z_0} = -C_{w_0}\cos\theta_0$$

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat u$†	$M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right) - \rho u_0^2\frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)$	$-M_0\frac{\partial C_L}{\partial M} - \rho u_0^2\frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$M_0\frac{\partial C_m}{\partial M} + \rho u_0^2\frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{L_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot\alpha$	Neg.	$*-2a_t V_H\frac{\partial\epsilon}{\partial\alpha}$	$*-2a_t V_H\frac{l_t}{\bar c}\frac{\partial\epsilon}{\partial\alpha}$
$\hat q$	Neg.	$*-2a_t V_H$	$*-2a_t V_H\frac{l_t}{\bar c}$

Neg. means usually negligible.
*means contribution of the *tail only*, formula for wing-body not available.
$\dagger C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0}$; $C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

Big Picture



Newton
Euler
Kinematics
Nonlinear Rigid Body EOM
trim state
Linear EOM
Linear S.S. Model
Dynamic Behavior
A/C Properties (Geometry
Nondim. Aero. Force, Moment Derivs.
Nondim. Stability Derivs.
Dim. Stab. Derivs.
C_{L_α}
C_{Z_α}
Z_w
(above)
Table 4.2
Table 4.4
(Today)
Table 5.1

Nondimensional Stability Derivatives

Table 5.1
Summary—Longitudinal Derivatives

	C_x	C_z	C_m
$\hat{u}$†	$\mathbf{M}_0\left(\frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}}\right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u}\left(1-\frac{\partial C_D}{\partial C_T}\right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2\frac{\partial C_L}{\partial p_d} - C_{T_u}\frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0\frac{\partial C_m}{\partial\mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u}\frac{\partial C_m}{\partial C_T}$
α	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha}+C_{D0})$	$-a(h_n-h)$
$\dot{\alpha}$	Neg.	$*-2a_tV_H\frac{\partial\epsilon}{\partial\alpha}$	$*-2a_tV_H\frac{l_t}{\bar c}\frac{\partial \epsilon}{\partial\alpha}$
$\hat{q}$	Neg.	$*-2a_tV_H$	$*-2a_tV_H\frac{l_t}{\bar c}$

Neg. means usually negligible.
*means contribution of the *tail only,* formula for wing-body not available.
†$C_{T_u} = \frac{(\partial T/\partial u)_0}{\frac{1}{2}\rho u_0 S} - 2C_{T_0}$; $C_{T_0} = C_{D_0} + C_{w_0}\sin\theta_0$

α-derivatives

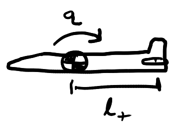
$$\boxed{C_{m_\alpha} = C_{L_\alpha}(h-h_n)}$$

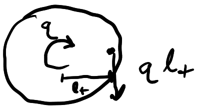
C_{z_α}

$$Z = -L\cos\alpha - D\sin\alpha$$
$$C_z = -(C_L\cos\alpha + C_D\sin\alpha)$$
$$\approx -(C_L + \underline{C_D\alpha})$$
$$C_{z_\alpha} \equiv \left.\frac{\partial C_z}{\partial\alpha}\right|_0 = -\left(C_{L\alpha} + C_{D_0} + \cancel{\alpha_0}^{\,0} C_{D\alpha}\right)$$
$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

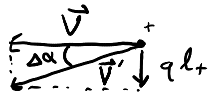
q-derivates

Wing-body | $\boxed{\text{Tail}}$





velocity observed by tail



Lift curve slope of tail

$$\Delta C_{L_+} = a_+\Delta\alpha = a_+\tan^{-1}\left(\frac{ql_+}{u_0}\right) \approx a_+\frac{ql_+}{u_0}$$
$$\Delta C_L = \frac{S_+}{S}\Delta C_{L_+}$$
$$= \frac{S_+}{S}a_+\frac{ql_+}{u_0}$$

$\underline{(C_{z_q})_{tail}}$

$$C_{z_q} = \left.\frac{\partial C_z}{\partial\hat{q}}\right|_0 = \left.\frac{2u_0}{\bar c}\frac{\partial C_z}{\partial q}\right|_0 = \left.-\frac{2u_0}{\bar c}\frac{\partial C_L}{\partial q}\right|_0$$

$$(C_{z_q})_{tail} = -\frac{2u_0}{\bar c}a_+\frac{S_+}{S}\frac{l_+}{u_0} = \boxed{-2a_+V_H} \qquad V_H = \frac{S_+ l_+}{S\bar c}$$

$(C_{m_q})_{tail}$ $\qquad \Delta C_m = -V_H \Delta C_{L_t} = a_t V_H \frac{q l_t}{u_0}$

$$C_{m_q} \equiv \left.\frac{\partial C_m}{\partial \hat{q}}\right|_0 = \frac{2u_0}{\bar{c}} \left.\frac{\partial C_m}{\partial q}\right|_0$$

$$(C_{m_q})_{tail} = \boxed{-2 a_t V_H \frac{l_t}{\bar{c}}}$$

Wing-Body

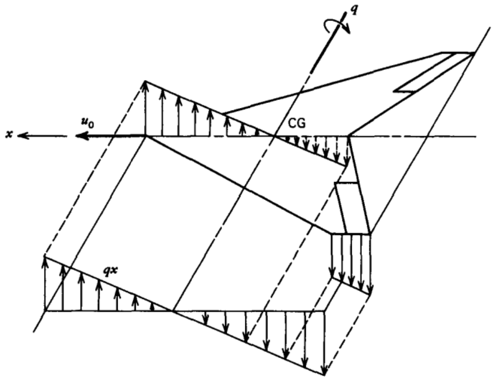


Figure 5.4 Wing velocity distribution due to pitching.

measure in wind tunnel or CFD

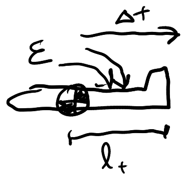
$\dot{\alpha}$ derivatives

Unsteady effects

Wing-body $\longrightarrow$ determined by initial response or oscillation of wing in wind tunnel or flight test

Tail

Downwash lag



$$-\Delta\alpha_t = \Delta\varepsilon = -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \Delta t$$

$$= -\frac{\partial \varepsilon}{\partial \alpha} \dot{\alpha} \frac{l_t}{u_0}$$

$$\Delta C_{L_t} = a_t \Delta\alpha_t = a_t \dot{\alpha} \frac{l_t}{u_0} \frac{\partial \varepsilon}{\partial \alpha}$$

$$\Delta C_L = a_t \dot{\alpha} \frac{l_t S_t}{u_0 S} \frac{\partial \varepsilon}{\partial \alpha}$$

$$(C_{z_{\dot{\alpha}}})_{tail} \equiv \left.\frac{\partial C_z}{\partial \frac{\dot{\alpha}\bar{c}}{2u_0}}\right|_0 = \frac{2u_0}{\bar{c}} \left.\frac{\partial C_z}{\partial \dot{\alpha}}\right|_0 = \boxed{-2 a_t \frac{l_t S_t}{\bar{c} S} \frac{\partial \varepsilon}{\partial \alpha}}$$

$$\boxed{(C_{m_{\dot{\alpha}}})_{tail} = -2 a_t V_H \frac{l_t}{\bar{c}} \frac{\partial \varepsilon}{\partial \alpha}}$$

u derivatives

3 important factors:

- Compressibility: Mach Number
- Dynamic Pressure: $p_d = \frac{1}{2}\rho V^2$
- Thrust

– Different from the dynamic pressure in nondimensionalization. Changes in C_L, C_D etc. due to changes in dynamic pressure.

$$C_{X_u} \equiv \frac{\partial C_X}{\partial \hat{u}}\Big|_0$$

$* \in \{X, Z, M\}$

$$C_{*_u} = \underbrace{\frac{\partial C_*}{\partial M}\Big|_0 \frac{\partial M}{\partial \hat{u}}\Big|_0} + \underbrace{\frac{\partial C_*}{\partial p_d}\Big|_0 \frac{\partial p_d}{\partial \hat{u}}\Big|_0} + \underbrace{\frac{\partial C_*}{\partial C_T}\Big|_0 \frac{\partial C_T}{\partial \hat{u}}\Big|_0}$$

$M = \frac{V}{a}$

$$\frac{\partial M}{\partial \hat{u}}\Big|_0 = u_0 \frac{\partial M}{\partial u}\Big|_0 = \frac{u_0}{a}\frac{\partial V}{\partial u}\Big|_0 = M_0$$

Mach number at trim

$$\frac{\partial p_d}{\partial \hat{u}}\Big|_0 = u_0 \frac{\partial p_d}{\partial u}\Big|_0 = u_0 \frac{1}{2}\rho \frac{\partial V^2}{\partial u}\Big|_0 = u_0 \rho u_0 = \rho u_0^2$$

$C_T = \frac{T}{\frac{1}{2}\rho V^2 S}$

$$\frac{\partial C_T}{\partial \hat{u}}\Big|_0 = u_0 \frac{\partial C_T}{\partial u}\Big|_0 = u_0 \left(\frac{\partial T/\partial u}{\frac{1}{2}\rho V^2 S} - \frac{2T}{\frac{1}{2}\rho V^3 S}\right)\Big|_0$$

$$= \frac{\partial T/\partial u}{\frac{1}{2}\rho u_0 S}\Big|_0 - 2C_{T_0}$$

$C_{T_0} = C_{D_0} + C_{W_0}\sin\theta_0$

3 Cases

Gliding: $C_{T_u} = 0$

Constant Thrust (Jet): $\boxed{C_{T_u} = -2C_{T_0}}$

Constant Power (Prop):

$TV = \text{constant}$

$$\frac{\partial T}{\partial u}\Big|_0 = -\frac{T_0}{u_0}$$

$$\boxed{C_{T_u} = -3C_{T_0}}$$

$\underline{C_{x_u}}$ $\qquad C_x \approx C_T - C_D$

$$\left.\frac{\partial C_x}{\partial M}\right|_0 = \left.\frac{\partial C_T}{\partial M}\right|_0 - \left.\frac{\partial C_D}{\partial M}\right|_0$$

$$\left.\frac{\partial C_x}{\partial p_d}\right|_0 = \left.\frac{\partial C_T}{\partial p_d}\right|_0 - \left.\frac{\partial C_D}{\partial p_d}\right|_0$$

$$\left.\frac{\partial C_x}{\partial C_T}\right|_0 = 1 - \left.\frac{\partial C_D}{\partial C_T}\right|_0$$

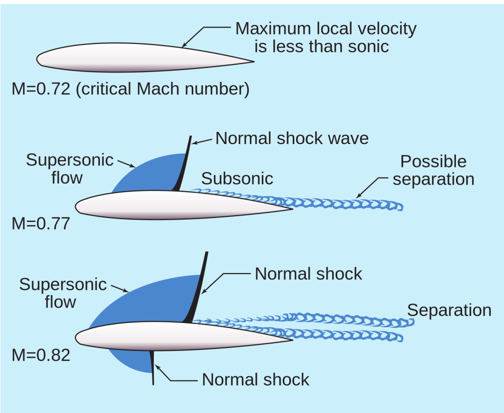
$$\boxed{C_{x_u} = M_0\left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M}\right)\Big|_0 - \rho u_0^2 \left.\frac{\partial C_D}{\partial p_d}\right|_0 + C_{T_u}\left(1 - \frac{\partial C_D}{\partial C_T}\right)\Big|_0}$$

$\underline{C_{z_u}}$ $\qquad$ Assume $\quad C_z = -C_L$

$$\boxed{C_{z_u} = \underbrace{-M_0 \left.\frac{\partial C_L}{\partial M}\right|_0}_{\text{small except for transonic}} - \rho u_0^2 \left.\frac{\partial C_L}{\partial p_d}\right|_0 - C_{T_u} \left.\frac{\partial C_L}{\partial C_T}\right|_0}$$

$\underline{C_{m_u}}$

$$\boxed{C_{m_u} = \underbrace{M_0 \left.\frac{\partial C_m}{\partial M}\right|_0}_{\text{Mach Tuck}} + \rho u_0^2 \left.\frac{\partial C_m}{\partial p_d}\right|_0 + C_{T_u} \left.\frac{\partial C_m}{\partial C_T}\right|_0}$$



Longitudinal Modes

Full State Space

$$\dot{\vec{x}} = \underset{n \times n}{A} \vec{x}$$

$$\boxed{\dot{\vec{x}} = A\vec{x}} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

(assume that A has n distinct non zero eigenvalues)

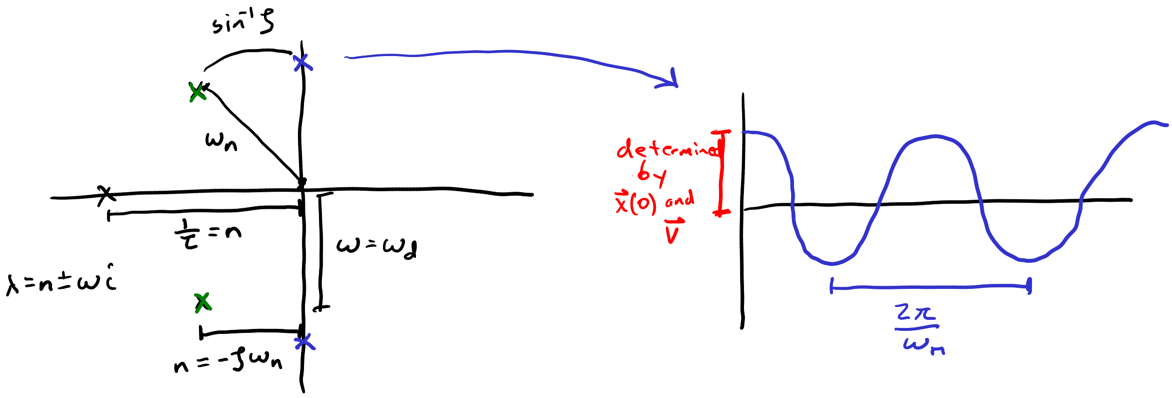
$$\boxed{\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}}$$

What does this mean?

A single real-valued $(\lambda, \vec{v})$ pair, or a pair of complex-valued $((\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2))$ describes a <u>mode</u>.

λ: "speed" of mode

$\vec{v}$: "shape" of mode



Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$(A - \lambda_i I)\vec{v}_i = 0$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \qquad \lambda_1 = -1, \quad \lambda_2 = -2$$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0 \qquad (\vec{v}_1)_1 + (\vec{v}_1)_2 = 0$$

let $(\vec{v}_1)_1 = 1$

$(\vec{v}_1)_2 = -1$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \alpha$$

$$(A - \lambda_2 I)\vec{v}_2 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \vec{v}_2 = 0 \quad \therefore \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \alpha$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i \cancel{e^{\lambda 0}}^{\,1} = \sum_i q_i \vec{v}_i = q_1 \vec{v}_1 + q_2 \vec{v}_2 + \dots q_n \vec{v}_n = V \vec{q}$$

$$V = [\vec{v}_1, \vec{v}_2 \dots \vec{v}_n] \qquad \vec{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$\boxed{\begin{aligned} \vec{x}(0) &= V\vec{q} \\ \vec{q} &= V^{-1}\vec{x}(0) \end{aligned}}$$

Conventional A/C Long. Modes

Dynamics of Flight, Eq. (4.9,18) $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mu_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}}Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}}Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + mu_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}}mg\sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$h_0 = 40\text{k ft}$

$u_0 = 774 \text{ ft/s}$

$\gamma_0 = \theta_0 = \alpha_0 = 0$

$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

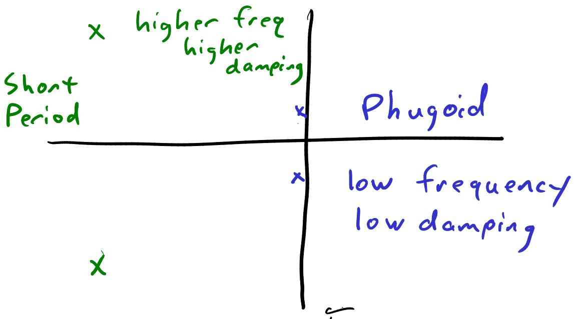
$\lambda_{1,2} = -0.37 \pm 0.89i$

$\omega_n = 0.96 \quad \zeta = 0.38$

$\lambda_{3,4} = -0.0033 \pm 0.067i$

$\omega_n = 0.067$

$\zeta = 0.049$

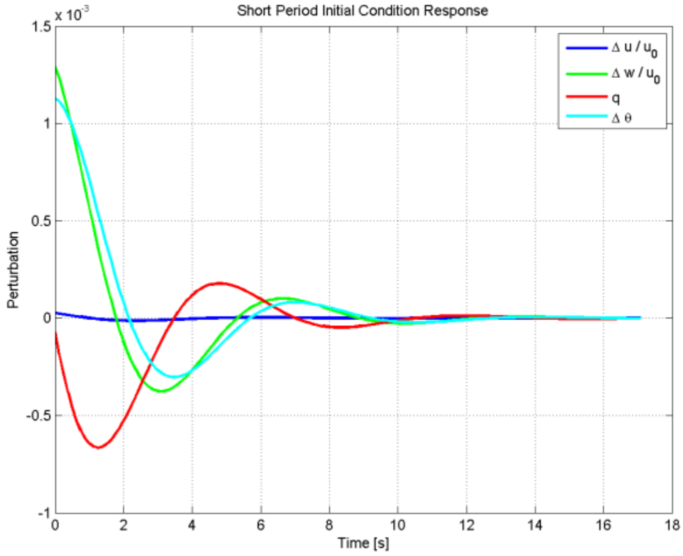


$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \mp 0.0004i \end{bmatrix} \begin{matrix} \Delta U \\ \Delta W \\ \Delta q \\ \Delta \theta \end{matrix} \qquad \vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097 \\ -0.0001 \mp 0i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

$$\vec{x}(0) = 0.5\vec{v}_1 + 0.5\vec{v}_2 = Re(\vec{v}_{1,2})$$

$$\lambda_{1/2} = -.372 + .888i$$
$$\zeta = 0.387$$
$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



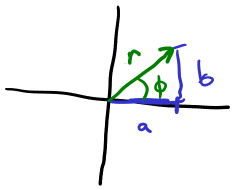
Phasor Plot: plot of eigenvector in complex plane

Aside: Polar Coordinates of a complex number

$z = a + bi$

$z = re^{i\phi}$

$z = r\angle\phi$



$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

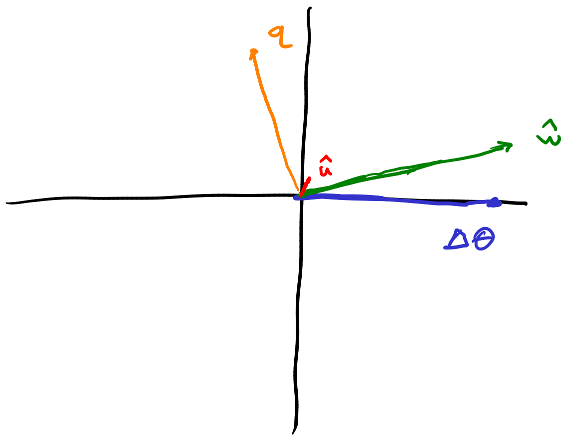
$r = \sqrt{a^2+b^2}$ $\quad \phi = \text{atan2}(b, a)$

For consistent phasor plots,
- normalize so that $\Delta\theta = 1$
- nondimensionalize u and w

Short Period

$$\vec{v}'_{1,2} = \vec{v}_{1,2} / (\vec{v}_{1,2})_4 = \begin{bmatrix} \vdots \\ \vdots \\ 1.0 \end{bmatrix}$$

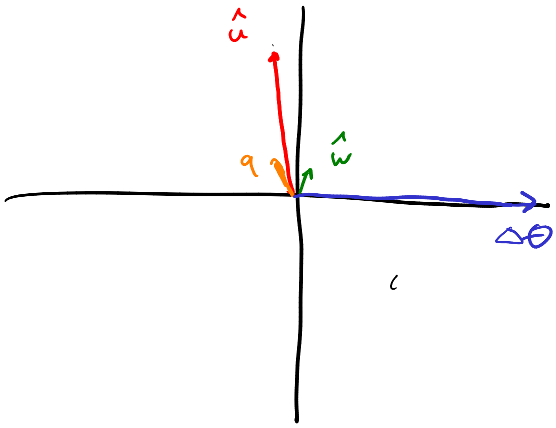
$$\hat{v}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.37 \pm 0.89i \\ 1.0 \end{bmatrix} \quad \begin{matrix} \hat{u} = \frac{\Delta u}{u_0} \\ \hat{w} = \frac{\Delta w}{u_0} \approx \alpha \\ q \\ \Delta\theta \end{matrix} \Rightarrow$$



Phugoid Mode

phugoid = "flight"

$$\hat{v}_{3,4} = \begin{array}{cl} \begin{bmatrix} 0.62 \angle 92^\circ \\ 0.036 \angle 83^\circ \\ 0.067 \angle 93^\circ \\ 1.0 \angle 0 \end{bmatrix} & \begin{array}{l} \hat{u} \\ \hat{w} = \alpha \\ q \\ \Delta\theta \end{array} \end{array}$$



$$\lambda_{3,4} \Rightarrow \begin{cases} \text{low frequency} \\ \text{low damping} \end{cases}$$

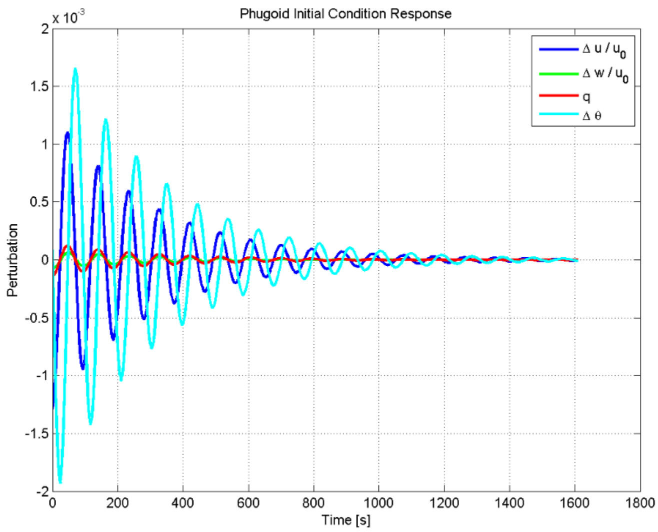
$$\vec{v}_{3,4} \Rightarrow \begin{cases} \text{Large } \hat{u} \text{ and } \theta \text{ oscillations out of phase } (90^\circ \text{ offset}) \\ \text{small } \alpha \end{cases}$$

$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$

$\zeta = 0.0489$ ← poorly damped

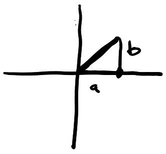
$\omega_n = 0.0673$ ← slow response

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



Longitudinal Mode Approximations

$\vec{v} = \begin{bmatrix} a+bi \\ \\ \\ \end{bmatrix}$



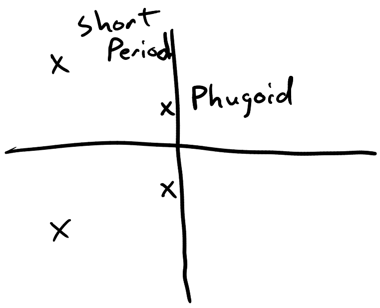
$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + \underbrace{B_{lon}\vec{u}_{lon}}_{\vec{c}_{lon}}$$

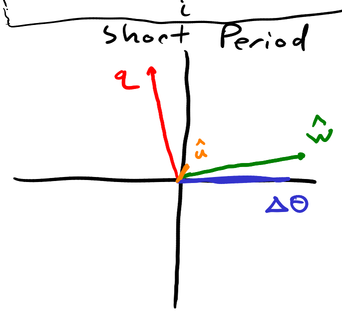
$$\vec{y} = \underset{\downarrow \atop I}{C}\vec{x}_{lon} + \underset{\downarrow \atop 0}{D}\vec{u}_{lon}$$

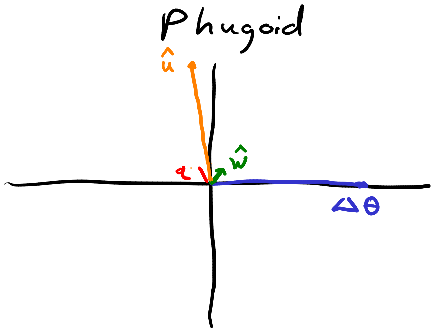
$$\vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$\boxed{\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}}$$

$$\hat{w} \equiv \frac{\Delta w}{u_0}$$







Short Period Approx

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

Assume: $\Delta u = 0$

$Z_{\dot{w}} \ll m$

$Z_q \ll m u_0$

$\theta_0 = 0$

No vertical motion

$\Delta\theta = \Delta\alpha = \frac{\Delta w}{u_0}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - \cancel{Z_{\dot{w}}}} & \frac{\cancel{Z_q} + m u_0}{m - \cancel{Z_{\dot{w}}}} & \frac{-mg \cancel{\sin\theta_0}}{m - \cancel{Z_{\dot{w}}}} \to 0 \\ \frac{1}{I_y}\left[M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right] & \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m - \cancel{Z_{\dot{w}}}}\right] & \frac{1}{I_y}\left[M_q + \frac{M_{\dot{w}}(\cancel{Z_q} + m u_0)}{m - \cancel{Z_{\dot{w}}}}\right] & \frac{-M_{\dot{w}} mg \cancel{\sin\theta_0}}{I_y(m - \cancel{Z_{\dot{w}}})} \to 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} & u_0 \\ \frac{1}{I_y}\left[M_w + \frac{M_{\dot{w}} Z_w}{m}\right] & \frac{1}{I_y}\left[M_q + M_{\dot{w}} u_0\right] \end{bmatrix}}_{A_{sp}} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[\frac{Z_w}{m} + \frac{1}{I_y}(M_q + M_{\dot{w}} u_0)\right]}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{1}{I_y}\left(u_0 M_w - \frac{M_q Z_w}{m}\right)}_{-\omega_n^2} = 0$$

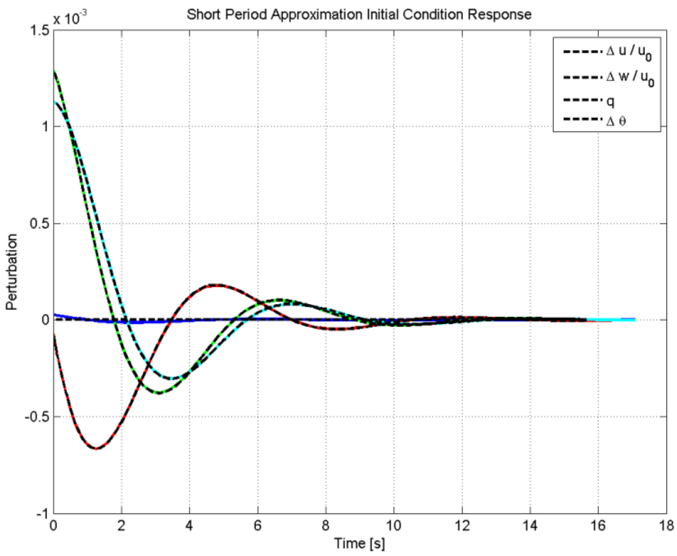
How does this relate to size and shape?

Dimensional Stab. Deriv.	Nondim. Stab. Deriv.	A/C Properties
$Z_w \equiv \left.\frac{\partial Z}{\partial w}\right\|_0 = \underbrace{\frac{1}{2}\rho u_0 S C_{z_\alpha}}_{\text{Table 4.4}}$	C_{z_α}	$= \underbrace{-C_{L_\alpha} - C_{D_0}}_{\text{Table 5.1}}$
M_w	C_{m_α}	$= C_{L_\alpha}(h - h_n)$
$M_{\dot{w}}$	$C_{m_{\dot{\alpha}}}$	....
M_q	C_{m_q}	$(C_{m_q})_{tail} = -2 a_t V_H \frac{l_t}{\bar{c}}$

How accurate is this approximation?

For 747 @ cruise

Full A_{lon}	S.P. Approx
$\lambda_{1,2} = -0.372 \pm 0.888i$	$\lambda_{sp} = -0.371 \pm 0.889i$
$\zeta = 0.387$	$\zeta = 0.385$
$\omega_n = 0.962$	$\omega_n = 0.963$



Phugoid Mode

Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - m g \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2 g \Delta z_E + u_0^2$$

$$C_L = C_{L_0} = C_{W_0}$$

$$L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho u_0^2 S C_{W_0} + \rho g S C_{W_0} \Delta z_E = W + \rho g S C_{W_0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W_0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W_0}}{m}}_{\omega_n^2} \Delta z_E = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{\rho g S C_{W_0}}} = 2\pi\sqrt{\frac{\overbrace{\tfrac{1}{2}u_0^2 m g}^{W}}{g^2 \underbrace{\tfrac{1}{2}\rho u_0^2 S C_{W_0}}_{W}}} = \boxed{\pi\sqrt{2}\,\frac{u_0}{g}}$$

$$\boxed{\begin{aligned} T &= 0.138\,u_0 \quad \text{if } u_0 \text{ in ft/s} \\ &= 0.453\,u_0 \quad \text{if } u_0 \text{ in m/s} \end{aligned}}$$

for 747   <u>Full A_{lon}</u>   <u>Lanchester</u>

$T = 93\,s$ $T = 107\,s$

<u>"2x2" Phugoid Approximation</u>

Dynamics of Flight, Eq. (4.9,18) $\qquad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot w}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot w}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot w})} \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} \boxed{\Delta\dot u} \\ \boxed{\Delta\dot w} \\ \Delta\dot q \\ \boxed{\Delta\dot\theta} \end{bmatrix} \qquad \mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - \cancel{Z_{\dot w}}} & \dfrac{Z_w}{m - Z_{\dot w}} & \dfrac{\cancel{Z_q} + m u_0}{m - \cancel{Z_{\dot w}}} & \dfrac{-mg\sin\theta_0}{m - \cancel{Z_{\dot w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot w}Z_u}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot w}Z_w}{m - Z_{\dot w}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot w}(Z_q + m u_0)}{m - Z_{\dot w}}\right] & \dfrac{-M_{\dot w} m g \sin\theta_0}{I_y(m - Z_{\dot w})} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$-g\cos\theta_0 \to -g$, $\quad -mg\sin\theta_0 \to 0$

$\Delta\alpha = 0$

$\Delta\dot q$ small

$Z_{\dot w} \ll m$

$Z_q \ll m u_0$

$\theta_0 = 0$

$$\rightarrow \begin{bmatrix} \Delta\dot u \\ \Delta\dot w = \Delta\dot\alpha u_0 \to 0 \\ \Delta\dot\theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m}\Delta u + u_0 \Delta q$$

$$\Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\Delta\dot\theta = \Delta q = -\frac{Z_u}{m u_0}\Delta u$$

$$\boxed{\begin{bmatrix} \Delta\dot u \\ \Delta\dot\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_0} & 0 \end{bmatrix}}_{A_{ph}} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}}$$

$$|A_{ph} - \lambda I| = \lambda^2 - \underbrace{\frac{X_u}{m}}_{-2\zeta\omega_n}\lambda - \underbrace{\frac{Z_u g}{m u_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cancel{\cos\theta_0}^{\,1} + \tfrac{1}{2}\rho u_0 S C_{Z_u}$$

$$C_{Z_u} = -\underbrace{M_0 \left.\frac{\partial C_L}{\partial M}\right|_0}_{\text{small}} - \underbrace{\rho u_0^2 \left.\frac{\partial C_L}{\partial p_d}\right|_0}_{\text{small}} - \underbrace{C_{T_u}\left.\frac{\partial C_L}{\partial C_T}\right|_0}_{\text{small}}$$

assume

$$C_{Z_u} = 0 \qquad \therefore\ \underline{Z_u = -\rho u_0 S C_{W_0}}$$

$$X_u = \rho u_0 S C_{w_0} \cancel{\sin\theta_0}^{\,0} + \frac{1}{2}\rho u_0 S C_{x_u}$$

$$C_{x_u} = -2C_{T_0}$$

$$C_{T_0} = C_{D_0} + C_{w_0} \cancel{\sin\theta_0}^{\,0}$$

$$X_u = -\rho u_0 S C_{D_0}$$

$$\boxed{\omega_n = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}}$$

Same as Lanchester

$$\zeta = \frac{-X_u}{2}\sqrt{\frac{-u_0}{m Z_u g}} = \frac{\rho u_0 S C_{D_0}}{2}\sqrt{\frac{u_0}{\cancel{2} m g \rho u_0 S C_{L_0}}}$$

Note: in class I included this $\frac{1}{2}$ incorrectly. The final expression for ζ is still correct.

substitute $mg = \frac{1}{2}\rho u_0^2 S C_{L_0}$ and cancel

$$\boxed{\zeta = \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}}}$$

High $\frac{L}{D} \Rightarrow$ less damping

(b/c less energy loss)

747

Full A_{lon}

$\lambda_{3,4} = -3.29\times10^{-3} \pm 6.72\times10^{-2} i$

$\zeta = 0.0489$

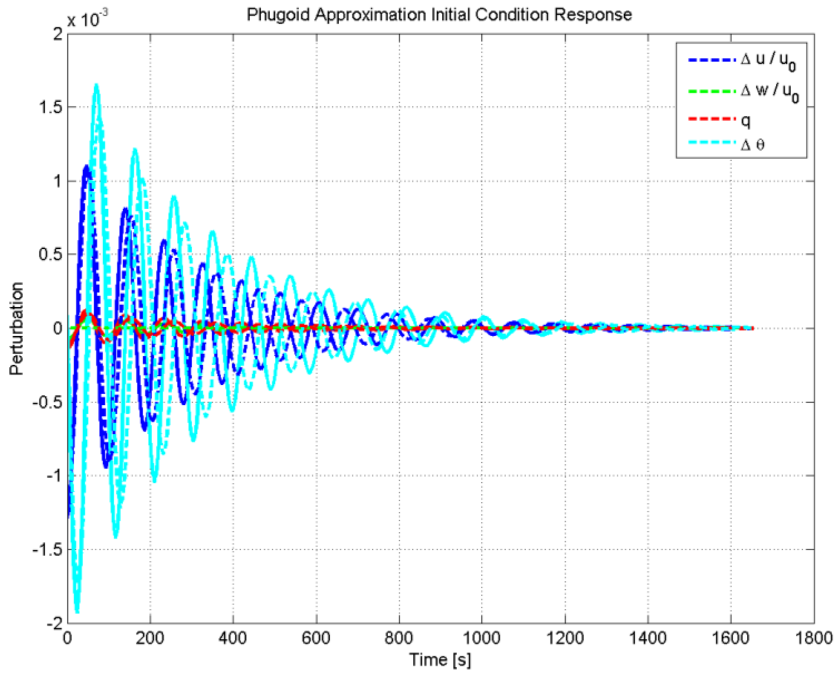
$\omega_n = 0.0673$

2×2 Ph. Approx

$\lambda_{ph} = -3.43\times10^{-3} \pm 6.11\times10^{-2} i$

$\zeta = 0.0561$

$\omega_n = 0.0612$



Longitudinal Control

Dynamics of Flight, Eq. (4.9,18) $\quad \dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon}\mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{pmatrix} \qquad \mathbf{c}_{lon} = \begin{pmatrix} \dfrac{\Delta X_c}{m} \\ \dfrac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \dfrac{\Delta M_c}{I_y} + \dfrac{M_{\dot{w}}}{I_y}\dfrac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\theta_0 \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + m u_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\theta_0}{m - Z_{\dot{w}}} \\ \dfrac{1}{I_y}\left[M_u + \dfrac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_w + \dfrac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}}\right] & \dfrac{1}{I_y}\left[M_q + \dfrac{M_{\dot{w}}(Z_q + m u_0)}{m - Z_{\dot{w}}}\right] & \dfrac{-M_{\dot{w}} m g \sin\theta_0}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\vec{c}_{lon} = B_{lon}\vec{u}_{lon} \qquad \vec{u}_{lon} = \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}$$

elevator + = down

throttle + = more thrust

dimensional control derivatives

$$\Delta X_c = X_{\delta e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t$$

$$\Delta Z_c = Z_{\delta e}\Delta\delta_e + \underline{Z_{\delta_t}}\Delta\delta_t$$

$$\Delta M_c = M_{\delta e}\Delta\delta_e + \underline{M_{\delta_t}}\Delta\delta_t$$

often 0

$$B_{lon} = \begin{bmatrix} \dfrac{X_{\delta e}}{m} & \dfrac{X_{\delta_t}}{m} \\ \dfrac{Z_{\delta e}}{m - Z_{\dot{w}}} & \dfrac{Z_{\delta_t}}{m - Z_{\dot{w}}} \\ \dfrac{M_{\delta e}}{I_y} + \dfrac{M_{\dot{w}} Z_{\delta e}}{I_y(m - Z_{\dot{w}})} & \dfrac{M_{\delta_t}}{I_y} + \dfrac{M_{\dot{w}} Z_{\delta_t}}{I_y(m - Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix}$$

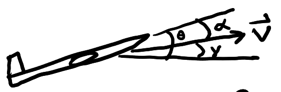
for 747

$$\begin{array}{c|cc} & \delta_e & \delta_t \\ \hline \Delta u & \boxed{-0.000187} & 9.66 \\ \Delta w & -17.85 & 0 \\ \Delta q & -1.158 & 0 \\ \Delta\theta & \boxed{0} & 0 \end{array}$$

$$\dot{\vec{x}}_{lon} = A_{lon}\vec{x}_{lon} + B_{lon}\vec{u}_{lon}$$

C and D depend on output

E.g. flight path angle γ



$$\Delta\gamma = \Delta\theta - \Delta\alpha$$

$$\Delta\gamma = \underbrace{\left[0 \;\; -\tfrac{1}{u_0} \;\; 0 \;\; 1\right]}_{C_\gamma} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

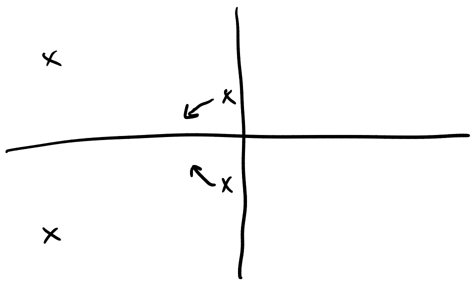
$$\vec{y} = [\Delta\gamma] = C_\gamma \vec{x}_{lon} + \underbrace{[0]}_{D}\vec{u}_{lon}$$

Open-loop step response to control inputs

$$B_{\delta e} = B_{lon}(:,1) \qquad B_{\delta_t} = B_{lon}(:,2)$$

$\delta_e \quad 1^\circ \qquad \delta_t : \frac{1}{6} = 0.05W$

$\delta_e \Rightarrow \gamma$ little change $\qquad \delta_t \Rightarrow \gamma$ significant increase



Longitudinal stability augmentation

Realistic main goal:
- increase phugoid damping

On homework: change s.p. damping ratio
- not usually a real-life goal
- easy to do by hand

Increase Phugoid damping with

$$\Delta\delta_e = -k_\theta \Delta\theta$$

$$\dot{x}_{ph} = \begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \underbrace{A_{ph}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix}}_{\text{previous}} + B_{ph\delta_e}\Delta\delta_e$$

$B_{lon} = \begin{bmatrix} & \\ & \end{bmatrix}$ for 747

Derivation of A_{ph} and $B_{ph,\delta e}$

Assume $\dot{q}$ only depends on α, q, and δe

solve for $\dot{q}=0$ (after short period has damped out)

$$\overset{0}{\cancel{\Delta\dot{q}}} = \frac{M_w}{I_Y}\Delta w + \frac{M_{\delta e}}{I_Y}\Delta\delta e + \frac{M_q}{I_Y}\overset{0}{\cancel{\Delta q}}$$

$$\Rightarrow \Delta w = -\frac{M_{\delta e}}{M_w}\Delta\delta e$$

solve for $\Delta\dot{w} = 0$ (after s.p. oscillations damp out)

$$\Delta\dot{w} = 0 = \frac{Z_u}{m}\Delta u + \frac{Z_w}{m}\Delta w + \frac{m u_o}{m}\overset{\Delta\dot{\theta}}{\cancel{\Delta q}} + \frac{Z_{\delta e}}{m}\Delta\delta e$$

solve for $\Delta\dot{\theta}$, substitute Δw above

$$\frac{m u_o}{m}\Delta\dot{\theta} = -\frac{Z_u}{m}\Delta u - \frac{Z_w}{m}\left(-\frac{M_{\delta e}}{M_w}\Delta\delta e\right) - \frac{Z_{\delta e}}{m}\Delta\delta e$$

$$\boxed{\Delta\dot{\theta} = -\frac{Z_u}{m u_o}\Delta u + \left(\frac{Z_w}{m u_o}\frac{M_{\delta e}}{M_w} - \frac{Z_{\delta e}}{m u_o}\right)\Delta\delta e}$$

$$\boxed{\Delta\dot{u} = \frac{X_u}{m}\Delta u + \frac{X_{\delta e}}{m}\Delta\delta e}$$

$$\boxed{\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{X_u}{m} & -g\\ -\frac{Z_u}{m u_o} & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}\frac{X_{\delta e}}{m}\\ \frac{Z_w\frac{M_{\delta e}}{M_w} - Z_{\delta e}}{m u_o}\end{bmatrix}\Delta\delta_e}$$

$$A_{ph} = \begin{bmatrix}-0.0069 & -32.2\\ 0.0001 & 0\end{bmatrix} \qquad B_{ph\delta e} = \begin{bmatrix}\overset{0}{\cancel{-0.0002}}\\ -0.44\end{bmatrix}$$

$$\Delta\delta e = -k_\theta\Delta\theta$$

$$\vec{u} = -K_{ph\theta}\vec{x}_{ph}$$

$$= -\begin{bmatrix}0 & k_\theta\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix}$$

$$A^{cl} = A_{ph} - B_{ph\delta_e} K_{ph\theta}$$

$$= A_{ph} - \begin{bmatrix} 0 \\ -0.44 \end{bmatrix} \begin{bmatrix} 0 & k_\theta \end{bmatrix}$$

$$= A_{ph} - \begin{bmatrix} 0 & 0 \\ 0 & -0.44k_\theta \end{bmatrix}$$

$$= \begin{bmatrix} -0.0069 & -32.2 \\ -0.0001 & 0.44k_\theta \end{bmatrix}$$

$$|A^{cl} - \lambda I| = (-0.0069 - \lambda)(0.44k_\theta - \lambda) - (-0.0001)(-32.2) = 0$$

$$= \lambda^2 + \underbrace{(-0.44k_\theta + 0.0069)}_{2\zeta\omega_n}\lambda \underbrace{- 0.003k_\theta - 0.0032}_{-\omega_n^2} = 0$$

$$\zeta = 0.7 \implies k_\theta = -0.2$$

Plug back into 4×4

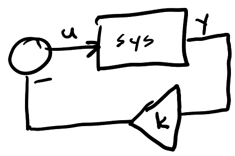
$$\Delta\delta_e = -K_\theta \vec{x}_{lon}$$

$$= -\begin{bmatrix} 0 & 0 & 0 & k_\theta \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

$$A^{cl} = A_{lon} - \underline{B_{\delta e}} K_\theta$$

---

rlocus assumes $\underline{u} = -ky$



$$\Delta\delta_e = -k_\theta \Delta\theta$$

$$y = \Delta\theta = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{C_\theta} \overbrace{\begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}}^{\vec{x}_{lon}}$$

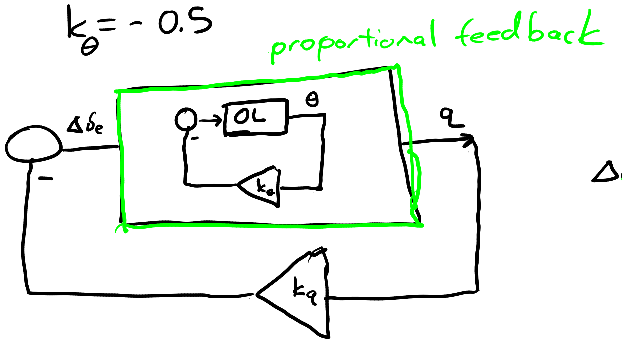
$$D = 0$$

---

derivative gain

$$\Delta\delta_e = -\underbrace{k_\theta \Delta\theta}_{\text{proportional}} - \underbrace{k_q \Delta q}_{\text{derivative}}$$

$$k_\theta = -0.5$$

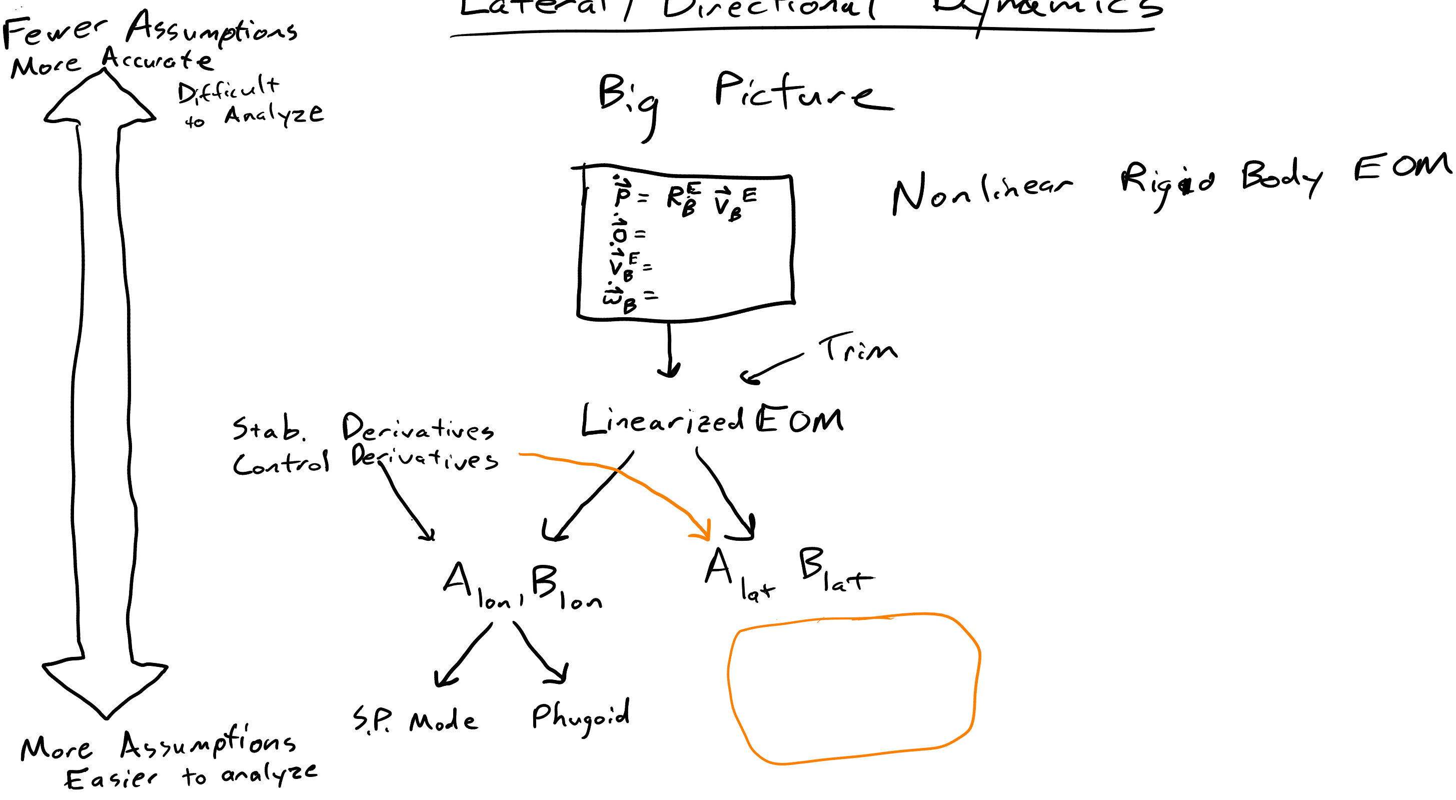


$$\Delta q = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}$$

from root locus choose $k_q = -1$

Lateral / Directional Dynamics

Big Picture



$$\Delta\dot{\phi} = \Delta p + \Delta r \tan\theta_0$$ ← Lat

$$\Delta\dot{\theta} = \Delta q$$ ← long

$$\Delta\dot{u} = -g\cos\theta_0\Delta\theta + \frac{\Delta X}{m}$$ ←

$$\Delta\dot{v} = -u_0\Delta r + g\cos\theta_0\Delta\phi + \frac{\Delta Y}{m}$$ ←

$$\Delta\dot{w} = u_0\Delta q - g\sin\theta_0\Delta\theta + \frac{\Delta Z}{m}$$ ←

$$\Delta\dot{p} = \Gamma_3\Delta L + \Gamma_4\Delta N$$ ←

$$\Delta\dot{q} = \frac{\Delta M}{I_y}$$ ←

$$\Delta\dot{r} = \Gamma_4\Delta L + \Gamma_8\Delta N$$ ←

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

$$Y = \frac{1}{2}\rho V^2 S \, C_y(\beta, p, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2}\rho V^2 S \underbrace{b}_{\text{span}} C_l(\beta, p, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2}\rho V^2 S b \, C_n(\beta, p, r, \delta_a, \delta_r)$$

$\hat{p} = \frac{pb}{2V_a}$

$$Y \approx \frac{1}{2}\rho V_a^2 S \left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$

$$L \approx \frac{1}{2}\rho V_a^2 Sb \left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right]$$

$$N \approx \frac{1}{2}\rho V_a^2 Sb \left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right]$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3\Delta L_c + \Gamma_4\Delta N_c \\ \Gamma_4\Delta L_c + \Gamma_8\Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$+\beta$: wind coming from right

$\beta = \sin^{-1}\left(\frac{\Delta v}{V}\right)$

$\beta \approx \frac{\Delta v}{u_0} = \hat{v}$

Table 4.5
Lateral Dimensional Derivatives

	Y	L	N
v	$\frac{1}{2}\rho u_0 S C_{y_\beta}$	$\frac{1}{2}\rho u_0 b S C_{l_\beta}$	$\frac{1}{2}\rho u_0 b S C_{n_\beta}$
p	$\frac{1}{4}\rho u_0 b S C_{y_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_p}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_p}$
r	$\frac{1}{4}\rho u_0 b S C_{y_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{l_r}$	$\frac{1}{4}\rho u_0 b^2 S C_{n_r}$

$L = \frac{1}{2}\rho V^2 S b C_\ell$ nondimensional, use lower-case for moments

$$L_v \equiv \left.\frac{\partial L}{\partial v}\right|_0 = \frac{1}{2}\rho u_0^2 S b \left.\frac{\partial C_\ell}{\partial v}\right|_0$$

$$= \frac{1}{2}\rho u_0^2 S b \left.\frac{\partial C_\ell}{\partial \beta}\right|_0 \left.\frac{\partial \beta}{\partial v}\right|_0$$

$$\boxed{= \frac{1}{2}\rho u_0 S b C_{\ell_\beta}}$$

$$C_{\ell_\beta} \equiv \left.\frac{\partial C_\ell}{\partial \beta}\right|_0 = \left.\frac{\partial C_\ell}{\partial \hat{v}}\right|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0} \Rightarrow \frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

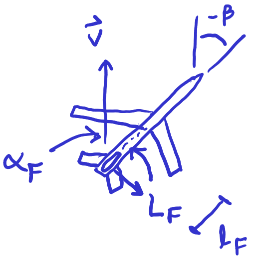
Table 5.2
Summary—Lateral Derivatives

	C_y	C_l	C_n
β	$*-a_F\frac{S_F}{S}\left(1-\frac{\partial\sigma}{\partial\beta}\right)$	N.A.	$*a_F V_V\left(1-\frac{\partial\sigma}{\partial\beta}\right)$
$\hat{p}$	$*-a_F\frac{S_F}{S}\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat{p}}\right)$	N.A.	$*a_F V_V\left(2\frac{z_F}{b}-\frac{\partial\sigma}{\partial\hat{p}}\right)$
$\hat{r}$	$*a_F\frac{S_F}{S}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$	$*a_F\frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$	$*-a_F V_V\left(2\frac{l_F}{b}+\frac{\partial\sigma}{\partial\hat{r}}\right)$

*means contribution of the *tail only*, formula for wing-body not available; $V_F/V = 1$.
N.A. means no formula available.

β derivatives

C_{n_β}: weathervane derivatime / yaw stiffness   Sign? +

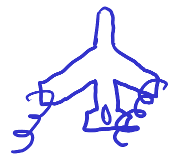


side wash

$$\alpha_F = -\beta + \sigma$$

$$C_{L_F} = a_F(-\beta+\sigma)$$

$$N_F = -\underbrace{C_{L_F}\tfrac{1}{2}\rho V_F^2 S_F}_{L_F}\, l_F$$



$$C_{n_F} = -C_{L_F}\frac{S_F\, l_F}{S\, b}\left(\frac{V_F}{V}\right)^2 \qquad V_V = \frac{S_F l_F}{S b}$$

$$= -V_V C_{L_F}\left(\frac{V_F}{V}\right)^2$$

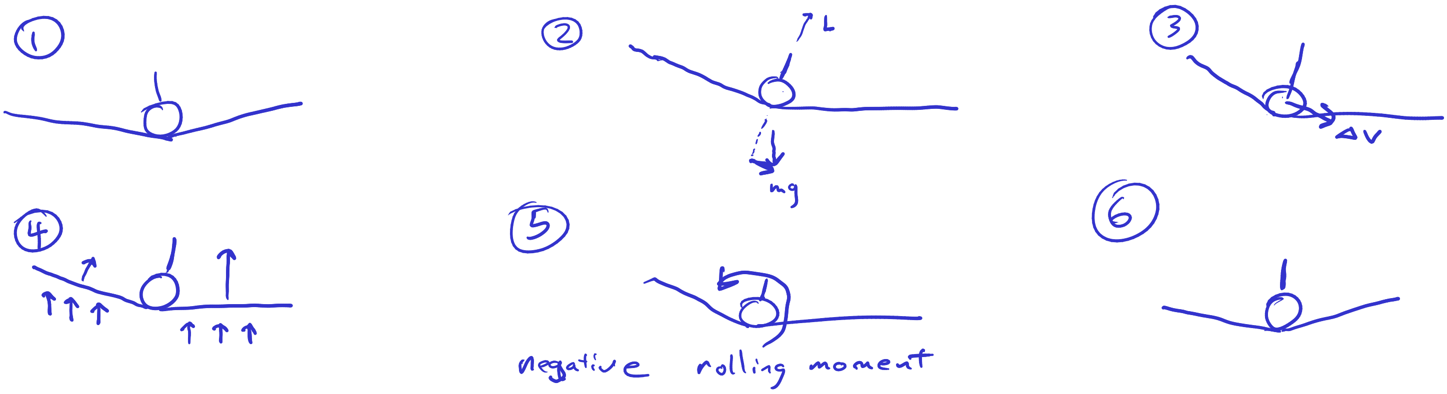
$$\left(C_{n_\beta}\right)_{Tail} = \left.\frac{\partial C_{n_F}}{\partial\beta}\right|_0 = -V_V\left(\frac{V_F}{V}\right)^2 \underbrace{\left.\frac{\partial C_{L_F}}{\partial\beta}\right|_0}_{a_F\left(-1+\frac{\partial\sigma}{\partial\beta}\right)}$$

$$\boxed{\left(C_{n_\beta}\right)_{Tail} = V_V a_F\left(\frac{V_F}{V}\right)^2\left(1-\frac{\partial\sigma}{\partial\beta}\right)}$$

C_{y_β}: usually small, similar derivation to C_{n_β} Sign? −

C_{l_β}: Dihedral Effect   Sign? −

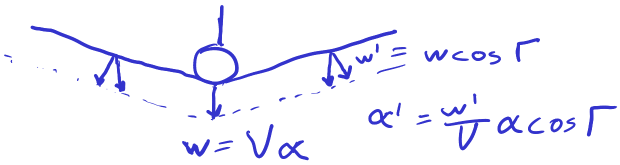
(looking from behind)

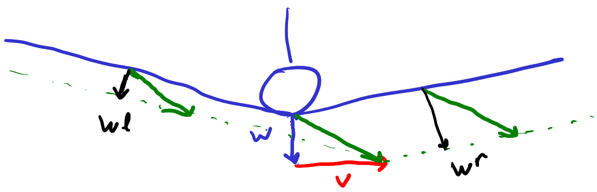


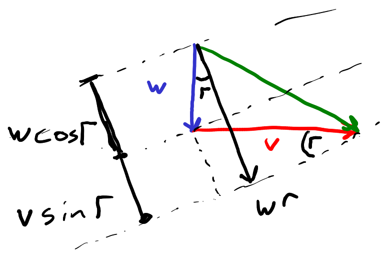
4 Significant Factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

1 Dihedral Angle







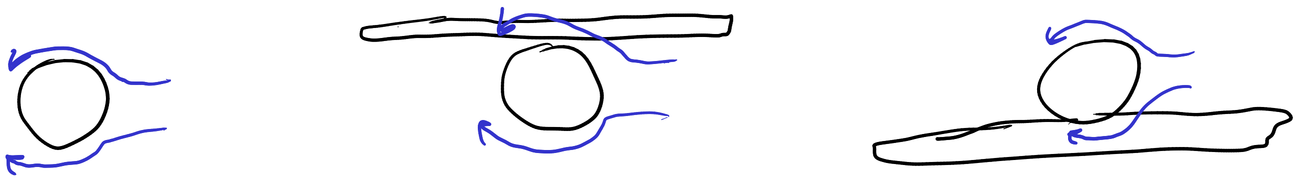
$$w^r = w\cos\Gamma + v\sin\Gamma \approx w + v\Gamma$$

$$\alpha^l \approx \frac{w^l}{u_0} = \alpha - \beta\Gamma \qquad \alpha^r \approx \frac{w^r}{u_0} = \alpha + \beta\Gamma$$

$$L \propto (\alpha^l - \alpha^r)$$

$$\boxed{C_{l_\beta}^{\Gamma} \propto -\Gamma}$$

2. Wing Height



High Wing: -ve C_{l_β} contribution

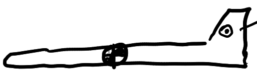
Low Wing: +ve C_{l_β} contribution

3. Wing Sweep

$$C_{l_\beta}^{\Lambda} \propto 2 C_L V^2 \sin(2\Lambda)$$

$$+\Lambda \Rightarrow + C_{l_\beta}$$

4. Tail



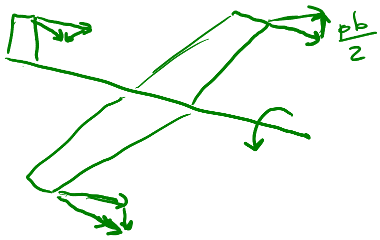
$$\Delta C_l^F = C_{L_F} \frac{S_F z_F}{S b} = a_F\left(-\beta + \sigma\right)\frac{S_F z_F}{S b}$$

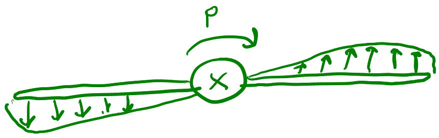
$$C_{l_\beta}^F = -a_F\left(1 - \frac{\partial \sigma}{\partial \beta}\right)\frac{S_F z_F}{S b}\left(\frac{V_F}{V}\right)^2$$

p-derivatives

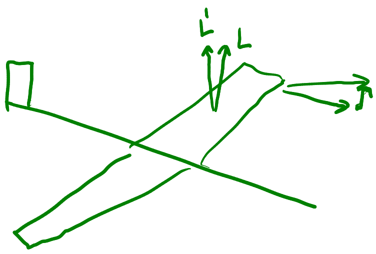
$$\hat{p} = \frac{pb}{2V}$$

C_{l_p}: roll damping Sign? $-$





C_{n_p} Wing Effect



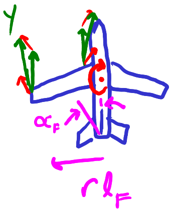
$$\left(C_{n_p}\right)_{tail} = a_F V_V \left(2\frac{z_F}{b} + \frac{\partial\sigma}{\partial\hat{p}}\right)$$

C_{y_p} (usually small)

(similar to derivation for $\left(C_{n_\beta}\right)_{tail}$)

r-derivatives

Wing-Body



$\vec{\omega} \times \vec{r}_P^{F \to B}$

$r l_F$

$$\Delta\alpha_F = \frac{r l_F}{u_0} + r\frac{\partial\sigma}{\partial r}$$

$$= \hat{r}\left(\frac{2 l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$\frac{\partial\alpha_F}{\partial\hat{r}} = \left(\frac{2 l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)$$

$$C_y = \frac{Y}{\frac{1}{2}\rho u_0^2 S} \qquad \left(C_y\right)_{tail} = \frac{\frac{1}{2}\rho u_0^2 S_F a_F \alpha_F}{\frac{1}{2}\rho u_0^2 S}$$

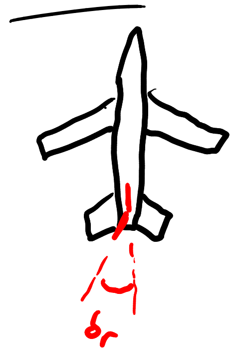
$$\boxed{\left(C_{y_{\hat{r}}}\right)_{tail} = \left.\frac{\partial\left(C_y\right)_{tail}}{\partial\hat{r}}\right|_0 = a_F \frac{S_F}{S}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$

$$\boxed{\left(C_{l_r}\right)_{tail} = a_F \frac{S_F}{S}\frac{z_F}{b}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$

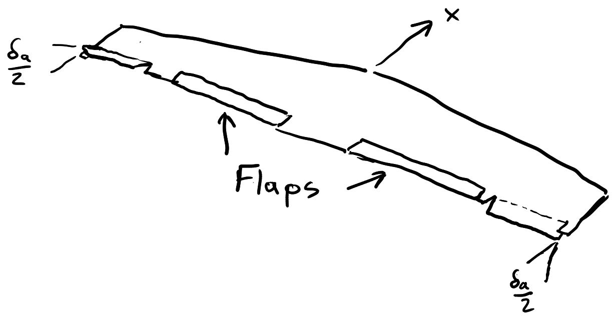
$$\boxed{\left(C_{n_r}\right)_{tail} = -a_F \underbrace{V_V}_{\frac{S_F}{S}\frac{l_F}{b}}\left(2\frac{l_F}{b} + \frac{\partial\sigma}{\partial\hat{r}}\right)}$$ ← yaw damping

Lateral Control and Coordinated Turn

Rudder

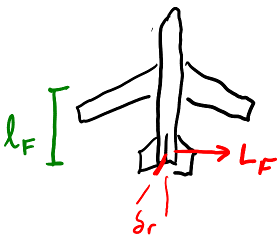


Aileron



Rudder Power

$C_{n_{\delta_r}}$



$$N_F = -l_F L_F = -l_F \tfrac{1}{2}\rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$

$$C_{n_F} = \frac{N_F}{\frac{1}{2}\rho V^2 S b} = -\underbrace{\frac{l_F S_F}{S b}}_{V_V}\left(\frac{V_F^2}{V^2}\right) C_{L_F} = -V_V\left(\frac{V_F^2}{V^2}\right) C_{L_F}$$

$$C_{n_{\delta_r}} = \left.\frac{\partial C_{n_F}}{\partial \delta_r}\right|_0 = -V_V\left(\frac{V_F^2}{V^2}\right) \underbrace{\left.\frac{\partial C_{L_F}}{\partial \delta_r}\right|_0}_{a_r} = \boxed{-a_r V_V\left(\frac{V_F^2}{V^2}\right)}$$

Other nondim. rudder control derivatives

$C_{Y_{\delta_r}}$ sign? $+$

$C_{l_{\delta_r}}$ sign? $+$

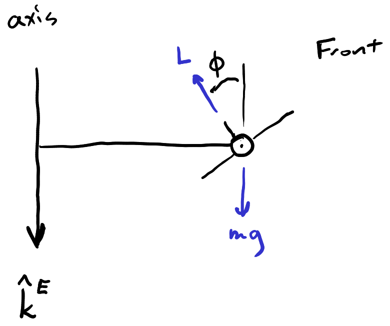
Aileron nondim. control derivatives

$C_{l_{\delta_a}}$ sign? $-$ aileron reversal can occur due to wing twist

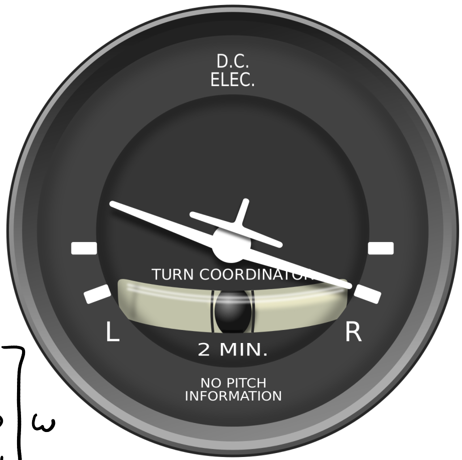
$C_{n_{\delta_a}}$ sign? can be either $C_{n_{\delta_a}} > 0$, called adverse yaw

$C_{Y_{\delta_a}}$ usually small

Coordinated Turn



- angular velocity vector is constant and aligned with inertial z
- No aerodynamic forces in a/c y direction



$$\omega = \frac{u_0}{R} \qquad \vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_B = R^B_E \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} \omega = \begin{bmatrix} -\theta \\ \sin\phi \\ \cos\phi \end{bmatrix} \omega$$

assume θ small but ϕ is not

$$L\cos\phi = mg$$

$$L\sin\phi = m a_n = m\frac{u_0^2}{R} = m\omega u_0$$

$$\boxed{\tan\phi} = \frac{L\sin\phi}{L\cos\phi} = \frac{m\omega u_0}{mg} = \boxed{\frac{\omega u_0}{g}}$$

From EOM

Assumed, no wind, $v \ll u$, $V = u = u_0$

$$Z = -mg\cos\phi - mqu$$

load factor "g's"

$$n = -\frac{Z}{mg} = \cos\phi + \frac{q u_0}{g}$$

$$= \cos\phi + \frac{\omega u_0 \sin\phi}{g}$$

$$= \cos\phi + \tan\phi\sin\phi$$

$$\boxed{n = \sec\phi}$$

$$n = \frac{L}{W}$$

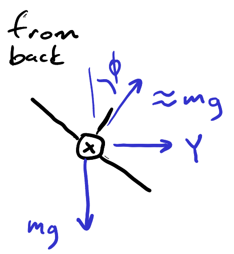
$$\Delta C_L = \frac{L - mg}{\frac{1}{2}\rho V^2 S} = (n-1) C_W$$

determined by ω, θ, ϕ

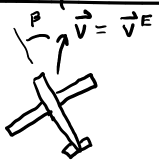
Coordinated Turn		$\beta, p, r, \delta_r, \delta_a$	α, q, δ_e
$C_Y = 0$	$=$	$C_{Y_\beta}\beta + C_{Y_p}\hat{p} + C_{Y_r}\hat{r} + C_{Y_{\delta r}}\delta_r + C_{Y_{\delta a}}\delta_a$	
$C_\ell = 0$	$=$	$\vdots$	
$C_n = 0$	$=$	$\vdots$	
$C_m = 0$	$=$	$\vdots$	
$C_L = (n-1) C_W$	$=$	$\vdots$	

$$
\begin{array}{c} Y \\ l \\ n \end{array}
\begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_\beta} & C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix}
\begin{bmatrix} \beta \\ \delta_r \\ \delta_a \end{bmatrix}
=
\begin{bmatrix} C_{Y_p} & C_{Y_r} \\ C_{l_p} & C_{l_r} \\ C_{n_p} & C_{n_r} \end{bmatrix}
\overbrace{\begin{bmatrix} \theta \\ -\cos\phi \end{bmatrix} \frac{\omega b}{2u_0}}^{\begin{bmatrix} \hat{p} \\ \hat{r} \end{bmatrix}}
$$

$$
\begin{bmatrix} C_{m_\alpha} & C_{m_{\delta e}} \\ C_{L_\alpha} & C_{L_{\delta e}} \end{bmatrix}
\begin{bmatrix} \Delta\alpha \\ \Delta\delta e \end{bmatrix}
= -\begin{bmatrix} C_{m_q} \\ C_{L_q} \end{bmatrix} \overbrace{\frac{\omega \bar{c} \sin\phi}{2u_0}}^{\hat{q}} + \begin{bmatrix} 0 \\ (n-1)C_W \end{bmatrix}
$$



Steady Sideslip



$$Y + mg\sin\phi = 0$$
$$Y + mg\phi = 0$$
$$L = 0$$
$$N = 0$$

$$-mg\phi = Y = Y_v v + Y_p \cancel{p}^{0} + Y_r \cancel{r}^{0} + Y_{\delta a}\delta_a + Y_{\delta_r}\delta_r$$
$$L = \dots$$
$$N =$$

$$mg\phi + Y_{\delta a}\delta_a + Y_{\delta_r}\delta_r = -Y_v v$$

$$
\begin{bmatrix} Y_{\delta r} & 0 & mg \\ L_{\delta r} & L_{\delta a} & 0 \\ N_{\delta r} & N_{\delta a} & 0 \end{bmatrix}
\begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix}
= -\begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} v
$$

βu_0

Steady Sideslip

For Piper Cherokee

$$
\begin{bmatrix} 280.7 & 0 & 2400 \\ 755.7 & -3821.9 & 0 \\ -3663.5 & 359 & 0 \end{bmatrix}
\begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix}
=
\begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix} v \qquad (7.8,4)
$$

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that $\beta \doteq v/u_0$, with u_0 given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\delta_r/\beta = .303$$
$$\delta_a/\beta = -2.96$$
$$\phi/\beta = .104$$

We see that a positive sideslip (to the right) of say 10° would entail left rudder of 3° and right aileron of 29.6°. Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only 1° to the right so the sideslip is almost flat.

Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \qquad \mathbf{c}_{lat} = \begin{pmatrix} \dfrac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

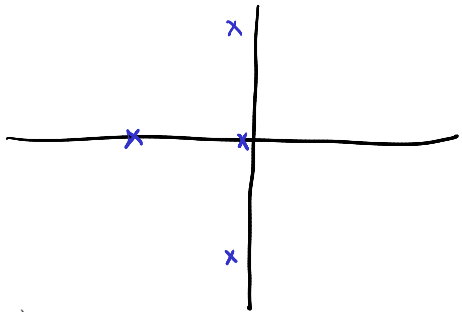
$$\mathbf{A}_{lat} = \begin{pmatrix} \dfrac{Y_v}{m} & \dfrac{Y_p}{m} & \left(\dfrac{Y_r}{m} - u_0\right) & g\cos\theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan\theta_0 & 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\dot{\vec{x}} = A_{lat}\vec{x}$$

$$\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

λ_i	ζ	ω_n
$-7.30e-03$	$1.00e+00$	$7.30e-03$
$-5.62e-01$	$1.00e+00$	$5.62e-01$
$-3.30e-02+9.47e-01i$	$3.49e-02$	$9.47e-01$
$-3.30e-02-9.47e-01i$	$3.49e-02$	$9.47e-01$



$$\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_{3/4} \\ \begin{pmatrix} 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} & \begin{pmatrix} -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} & \begin{pmatrix} -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix} \end{array}$$

Augmented State Space Dynamics Matrix

$$\begin{bmatrix} \Delta\dot{v} \\ \vdots \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & & & & 0 & 0 \\ & A_{lat} & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & \sec\theta_0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & u_0\cos\theta_0 & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta\phi \\ \Delta\psi \\ \Delta y_E \end{bmatrix}$$

1 if $\theta_0 = 0$

$$\Delta\dot{\psi} = \Delta r \sec\theta_0$$

$$\Delta\dot{y}_E = u_0\cos\theta_0\Delta\psi + \Delta v$$

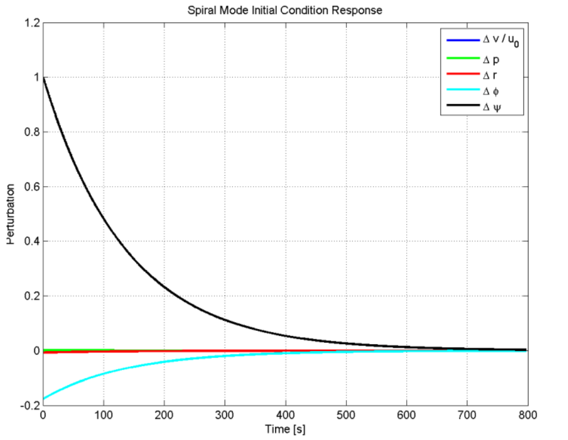
Spiral Mode

$\lambda = -0.0073$

$\tau = 137\,s$

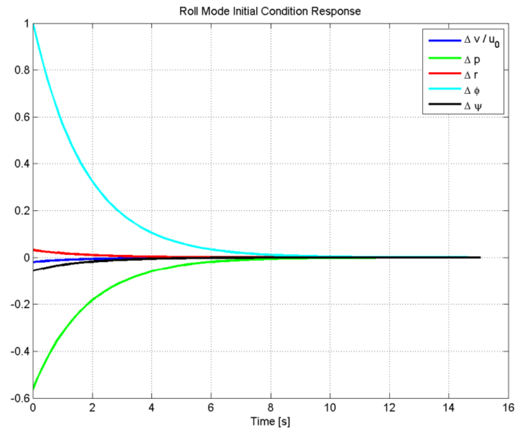
$$\hat{v}_1 = \begin{array}{c} \begin{bmatrix} 0.0068 \\ -0.0074 \\ 0.04 \\ 1.0 \\ -5.66 \end{bmatrix} \end{array} \begin{array}{l} \hat{v} = \beta \leftarrow \text{small} \\ p \leftarrow \text{small} \\ r \leftarrow \text{a little} \\ \phi \leftarrow \text{large} \\ \psi \leftarrow \text{large} \end{array}$$

normalize to $\Delta\phi = 1$ + nondimensionalize velocity



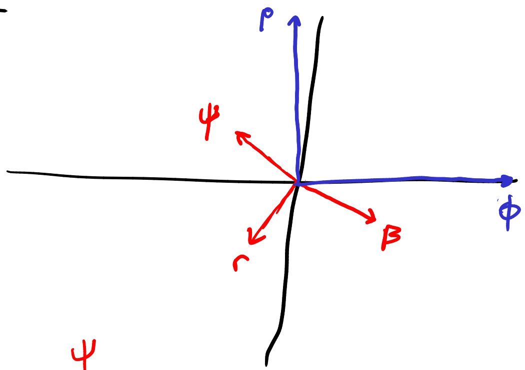
Roll Mode

$$\hat{v}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \begin{array}{l} \hat{v} = \beta \leftarrow \text{small} \\ p \leftarrow \text{large} \\ r \leftarrow \text{small} \\ \phi \leftarrow \text{large} \\ \psi \leftarrow \text{small} \end{array}$$



Dutch Roll

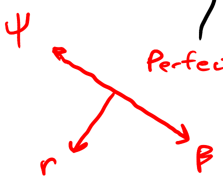
$$\hat{v}_3 = \begin{bmatrix} 0.321\angle -28^\circ \\ 0.9471\angle 92^\circ \\ 0.2915\angle -112^\circ \\ 1.0 \\ 0.3078\angle 155^\circ \end{bmatrix} \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$



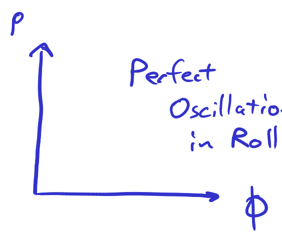
$\lambda_{3,4} = -0.033 \pm 0.947i$

$\zeta = 0.0349$ Poor Damping

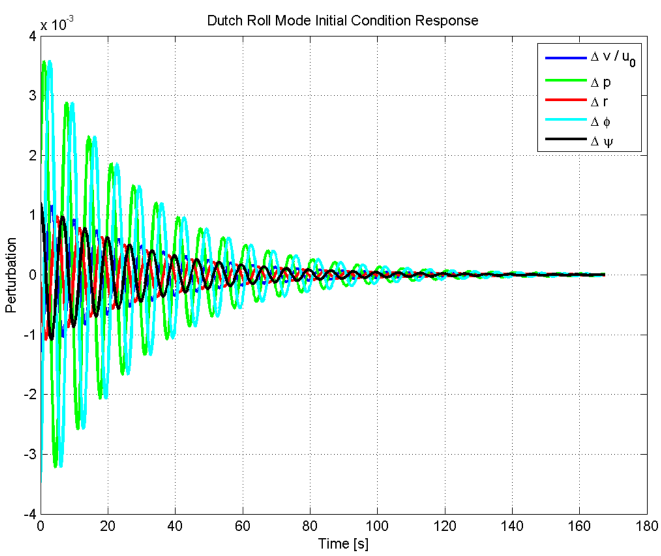
$\omega_n = 0.947$ relatively fast



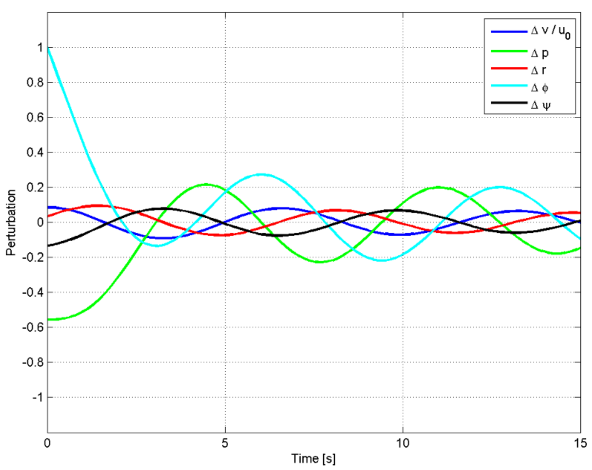
Perfect Yaw Oscillation



Perfect Oscillation in Roll



$$\mathbf{x}(0) = 0.4 \cdot Re(\mathbf{v}_r) + 0.4 \cdot Re(\mathbf{v}_{dr}) + 0.2 \cdot Re(\mathbf{v}_{spi})$$



Lateral Mode Approximations

$$A_{lat} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

e.g. $\mathcal{Y}_v = \frac{Y_v}{m}$

Roll Approximation

$r = 0 \quad v = 0$

$$\dot{p} = \mathcal{L}_p p$$

$|A - \lambda I|$ if A is a scalar $\lambda = A$

$$\lambda_{r,approx} = \mathcal{L}_p = -0.434$$

$\lambda_r = -0.562$

23% difference

2x2 Spiral Approximation

$\rightarrow p = 0$

$\rightarrow \dot{p} = 0$

Ignore side force

$$\begin{bmatrix} \dot{v} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & \mathcal{Y}_p & \mathcal{Y}_r & g\cos\theta_0 \\ \mathcal{L}_v & \mathcal{L}_p & \mathcal{L}_r & 0 \\ \mathcal{N}_v & \mathcal{N}_p & \mathcal{N}_r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_v & \mathcal{L}_r \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = \mathcal{L}_v v + \mathcal{L}_r r \Rightarrow v = -\frac{\mathcal{L}_r}{\mathcal{L}_v} r$$

$$\dot{r} = -\mathcal{N}_v \frac{\mathcal{L}_r}{\mathcal{L}_v} r + \mathcal{N}_r r = \underbrace{\left(\frac{\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r}{\mathcal{L}_v} \right)}_{\lambda_{s,approx}} r$$

$\lambda_{s,approx} = -0.0296$

$\lambda_s = -0.0073$

not great

Characteristic-Eqn.-based spiral approx

$$|A_{lat} - \lambda I| = A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

since $\lambda_s << 1$

$$D\lambda + E = 0$$

$$\lambda_{s,approx} = -\frac{E}{D}$$

$$E = g\left[(\mathcal{N}_r \mathcal{L}_v - \mathcal{N}_v \mathcal{L}_r)\cos\theta_0 + (\mathcal{N}_v \mathcal{L}_p - \mathcal{L}_v \mathcal{N}_p)\sin\theta_0\right]$$

$$D = -g(\mathcal{L}_v \cos\theta_0 + \mathcal{N}_v \sin\theta_0) + u_0(\mathcal{L}_v \mathcal{N}_p - \mathcal{L}_p \mathcal{N}_v)$$

for 747 $\lambda_{s,approx} = -0.00725$ very close!

In Book

Characteristic-based spiral + roll eigen value approximation

Dutch Roll Approx

Assume $\phi = p = 0$ $\qquad$ $\frac{Y_r}{m} \ll u_0$

$$\begin{bmatrix} \mathcal{Y}_v & \cancel{\mathcal{Y}_p} & \overset{u_0}{\cancel{\mathcal{Y}_r}} & \cancel{g\cos\theta_0} \\ \cancel{\mathcal{L}_v} & \cancel{\mathcal{L}_p} & \cancel{\mathcal{L}_r} & \cancel{0} \\ \mathcal{N}_v & \cancel{\mathcal{N}_p} & \mathcal{N}_r & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{0} \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_v & -u_0 \\ \mathcal{N}_v & \mathcal{N}_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$\lambda^2 - (\mathcal{Y}_v + \mathcal{N}_r)\lambda + (\mathcal{Y}_v \mathcal{N}_r + u_0 \mathcal{N}_v) = 0$$

$$\lambda_{dr,\,approx} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

Laplace Transforms and Transfer Functions

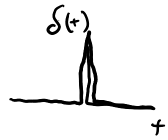
$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 u(t)$$

$$\begin{aligned}\dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} + Du\end{aligned} \quad \Longleftrightarrow \quad G_{yu}(s)$$

Review: Properties of Laplace transforms

$$\mathcal{L}[x(t)](s) = \int_0^\infty e^{-st} x(t)\,dt$$

Appendix A1



$$x(t) \Longleftrightarrow X(s)$$

$$\dot{x}(t) \Longleftrightarrow sX(s) - x(t=0)$$

$$\int_0^t x(\tau)\,d\tau \Longleftrightarrow \frac{1}{s} X(s)$$

$4\sin(2t) + 2\cos(t)$

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha x(s) + \beta y(s)$$

$$\frac{4\cdot 2}{s^2+2^2} + \frac{2s}{s^2+1}$$

Review: Transfer Function

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 u(t)$$

Assume 0 initial conditions

$$s^2 x(s) + 2\zeta\omega_n s x(s) + \omega_n^2 x(s) = \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\, x(s) = \omega_n^2 U(s)$$

Transfer Function

$$\boxed{G_{xu}(s) \equiv \frac{X(s)}{u(s)}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

0 initial conditions

Can you find a TF for any differential equation?

$$\ddot{x}(t) = -\dot{x}(t)^2 + u(t)$$

nonlinear

✗ no TF for nonlinear diff eq.

Review: How to use a TF

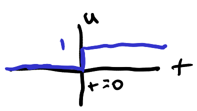
1. Input → Output

$$u(t) \to u(s) \longrightarrow X(s) = G_{xu}(s)\,u(s) \longrightarrow x(t)$$

$$\frac{s+1}{s^3+2s^2+3s+4} \to \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} \cdots \xrightarrow{\text{table}} x(t)$$

partial frac decomposition

Ex. $u(t) =$ step function



$$u(s) = \frac{1}{s}$$

$$X(s) = G_{xu}(s)\,u(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\,\frac{1}{s}$$

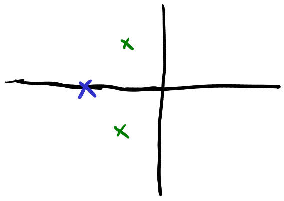
$$X(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

2. Stability

$\underbrace{\text{Roots of TF denominator}}_{\text{a.k.a. poles}}$ are eigenvalues of A matrix

$G(s) \to \infty$ if s is at a pole

$\frac{1}{s+1}$, pole $= -1$



If all poles are on LHP system is stable

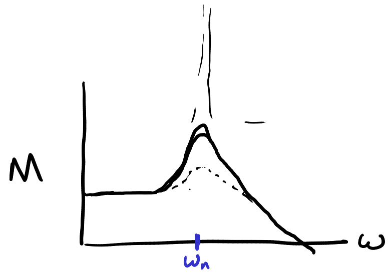
$$G_{xu}(s) = \frac{1}{(s+2)(s^2+2s+2)}$$

$s=-2$ $\qquad$ $s=-1\pm i$

3. Steady-State Behavior

a) Final Value Theorem

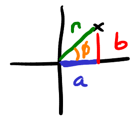
$$\boxed{\text{If } sx(s) \text{ is stable} \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,x(s)}$$



b) Frequency Response / Harmonic Response

$$\boxed{\text{If } u(t) = A\cos(\omega t) \text{ the steady-state } x(t) \text{ is } AM\cos(\omega t+\phi) \quad M = |G_{xu}(i\omega)| \text{ and } \phi = \angle G_{xu}(i\omega)}$$

$z = a + b_i$ or $re^{\phi i}$ or $r\angle\phi$



$$G(s) = \frac{1}{s^2+2s+2}$$

$$G(i\omega) = \frac{1}{-\omega^2+2i\omega+2}$$

$$= \frac{1}{\underbrace{-\omega^2+2}_{a} + \underbrace{2\omega i}_{b}} = \frac{1}{\underbrace{\sqrt{(-\omega^2+2)^2+(2\omega)^2}}_{r}\, e^{i\underbrace{\operatorname{atan}\left(\frac{2\omega}{-\omega^2+2}\right)}_{\phi}}}$$

$$= \frac{1}{r}e^{-i\phi}$$

$$\rightarrow \quad \begin{aligned}\dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} + Du\end{aligned} \quad \Longleftrightarrow \quad G_{yu}(s)$$

State Space to TF $\qquad$ ss2tf $\qquad$ tf(sys)

$$s\vec{x}(s) = A\vec{x}(s) + Bu(s)$$

$$(sI-A)\vec{x}(s) = Bu(s)$$

$$\vec{x}(s) = (sI-A)^{-1}Bu(s)$$

$$y(s) = C\vec{x}(s) + Du(s)$$

assume $D=0$ from here on

$$y(s) = C(sI-A)^{-1}Bu(s) + Du(s)$$

$$M^{-1} = \frac{\mathrm{adj}(M)}{|M|} \qquad (sI-A)^{-1} = \frac{\mathrm{adj}(sI-A)}{|sI-A|}$$

Adjugate: Transpose of Cofactor Matrix F

$$F_{ij} = (-1)^{i+j} |M_{-i-j}|$$

$-i$ = all except i

If $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then $F = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \cdots \\ \vdots & & \end{bmatrix}$

$$\mathrm{adj}(M) = F^T$$

$$G_{yu}(s) = \frac{Y(s)}{U(s)} = \boxed{C(sI-A)^{-1}B} = \boxed{\frac{C\,\mathrm{adj}(sI-A)\,B}{|sI-A|}} = \frac{N(s)}{D(s)}$$

Roots of $D(s)$ are eigenvalues of A

TF to state space

tf2ss

$\leftarrow G_{ode}$

$$G_{yu}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots b_m}{a_0 s^n + \dots\dots a_{n-1}s + a_n}$$

Ex: $G_{yu}(s) = \dfrac{b_0 s + b_1}{s^2 + a_1 s + a_2}$

multiply by $\dfrac{x(s)}{x(s)}$

$$\frac{Y(s)}{u(s)} = G_{yu}(s) = \frac{b_0 s x(s) + b_1 x(s)}{s^2 x(s) + a_1 s x(s) + a_2 x(s)}$$

$\rightarrow$ $y(t) = b_0 \dot{x}(t) + b_1 x(t) \leftarrow$

$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$

$\rightarrow$ $\ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u(t) \leftarrow$

$\dot{\vec{x}} = A\vec{x} + Bu$

$y = C\vec{x} + Du$

$$\rightarrow \underbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}_{\dot{\vec{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \leftarrow$$

$$y = \underbrace{\begin{bmatrix} b_1 & b_0 \end{bmatrix}}_{C} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{[0]}_{D} u \leftarrow$$

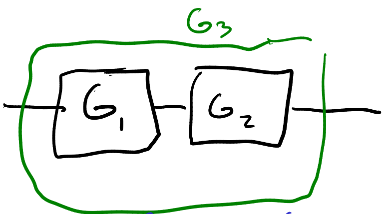
$$G_{y_u}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots b_m}{a_0 s^n + \dots\dots a_{n-1} s + a_n}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ & & & & & 1 \\ -a_n & \cdots & & & \cdots & -a_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

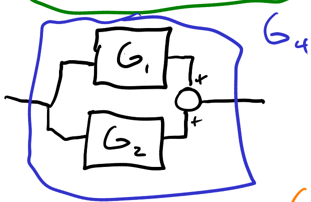
$$C = [\, b_m \cdots \; b_0 \; \underbrace{0\;0\;0}_{\text{if } n > m+1} \,] \qquad D = [0]$$

$\vec{x}$ may not correspond to physical states

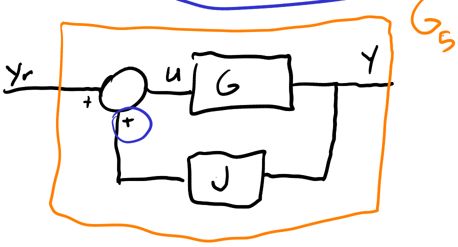
Review: Block Diagrams



$G_3 = G_1 G_2$



$G_4 = G_1 + G_2$



$G_5 = \frac{y}{y_r}$

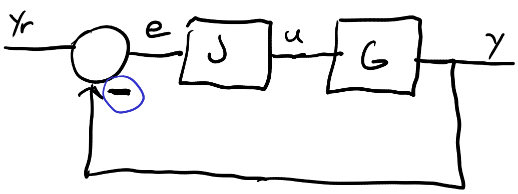
$y = Gu$

$u = y_r + Jy$

$y = G(y_r + Jy)$

$(1 - JG)\, y = G y_r$

$\frac{y}{y_r} = \boxed{\frac{G}{1 - JG}}$



$y = JGe$

$e = y_r - y$

$y = JG(y_r - y)$

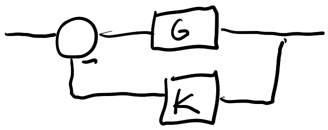
$(1 + JG)\, y = JG y_r$

$\frac{y}{y_r} = \boxed{\frac{JG}{1 + JG}}$

$\frac{G}{1 + JG}$ if -'ve feedback

$\frac{JG}{1 - JG}$ for +'ve feedback

TFs and Root Loci



$G \leftarrow$ open loop TF

$$G^{cl} = \frac{G}{1+KG}$$

$$k = 0 \Rightarrow G^{cl} = G$$

$\boxed{\text{Root locus starts at poles of the OL systems}}$

~~$G^{cl} = \frac{G}{1+KG} = \frac{\frac{1}{K}G}{\frac{1}{K}+G}$~~

~~as $k \to \infty$ CL poles approach the zeros of G~~

$J = K$

$$G^{cl} = \frac{G}{1+KG} \quad \text{or} \quad \frac{KG}{1+KG}$$

$$1 + KG = 0$$

$$\underline{\frac{1}{K} + G = 0}$$

$$G(s) = \frac{N(s)}{D(s)}$$

As K gets large $\underline{G(s) = -\frac{1}{K}}$

When K is large, two possibilities for CL poles

1. N(s) is close to zero

   $\boxed{\text{CL poles close to O.L. zeros}}$

2. D(s) is very large

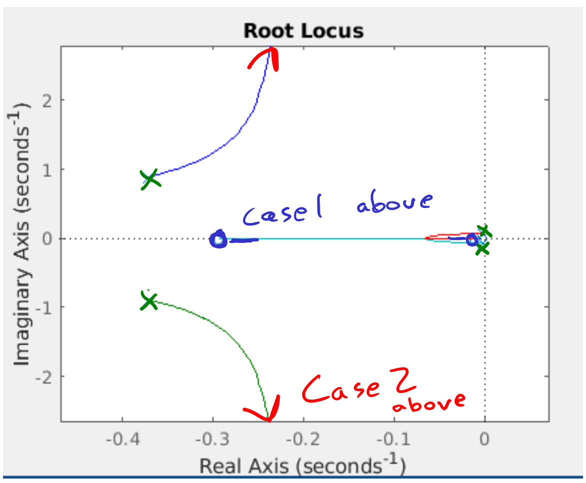
   $\boxed{\text{Magnitude of CL poles is very large}}$

B 744 Long dynamics with $\Delta\delta_e = -k\,\Delta\theta$

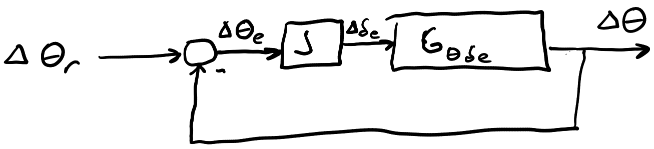
O.L Poles : $-0.37 \pm 0.89i$

$-0.0033 \pm 0.067i$

O.L. Zeros : -0.0113, -0.2948



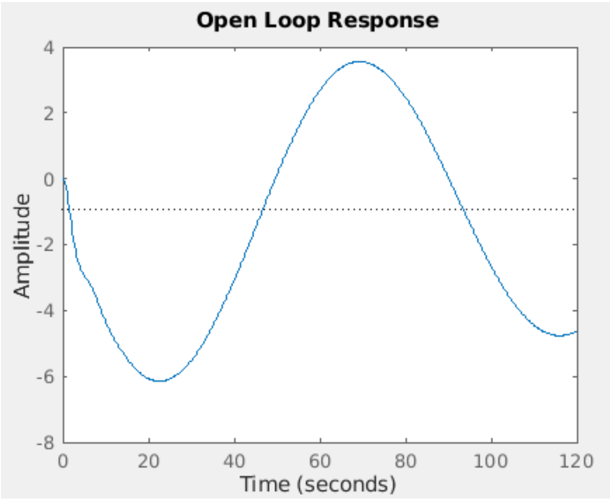
Designing a Pitch-Hold Controller



$\Delta\theta_e = \Delta\theta_r - \Delta\theta$

Goals

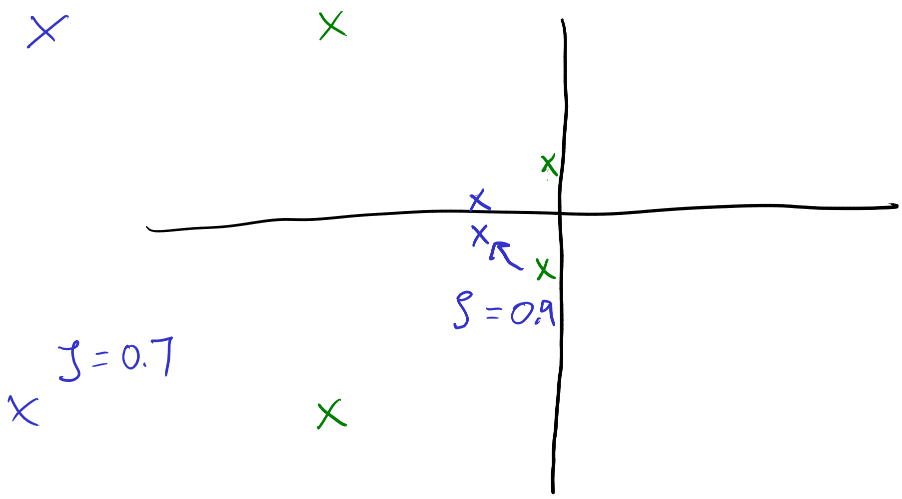
1. Improve Damping (O.L. Phugoid has low damping)
2. Low steady-state error $|\theta - \theta_r| \to 0$



Previously

$$\Delta\delta_e = \underbrace{-k_\theta}_{\text{proportional}}\Delta\theta - \underbrace{k_q}_{\text{derivative}}\Delta q$$

gain



$k_\theta = -0.5$

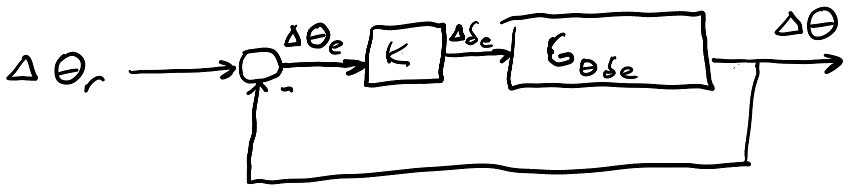
$k_q = -1$

Controller 1

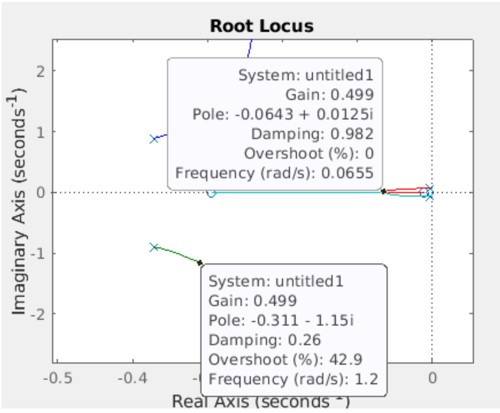
Proportional Controller

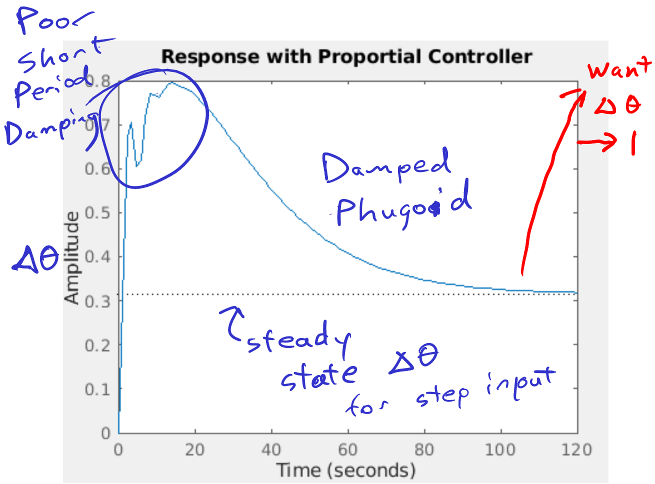
$J = K$

$\Delta\delta_e = K(\Delta\theta_r - \Delta\theta)$



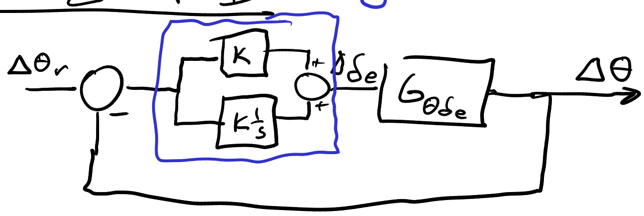
Choose $K = -0.5$

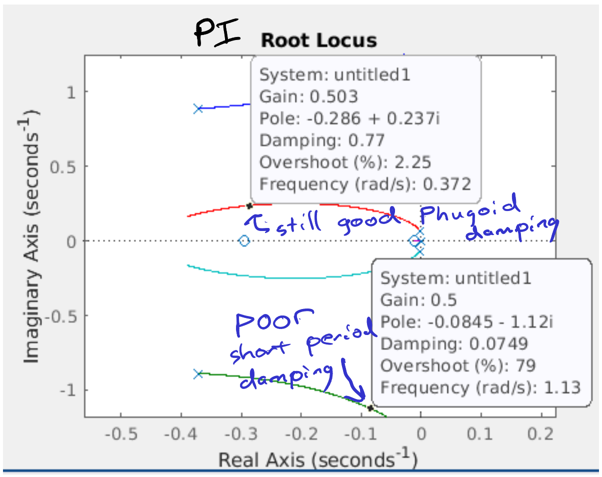


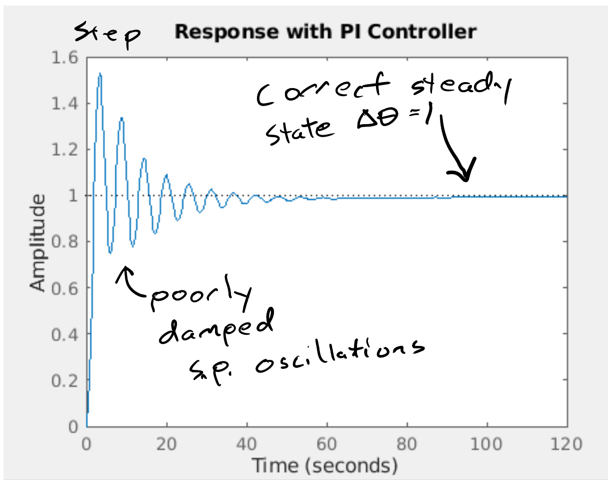


Controller 2: PI

$J = K\left(1 + \frac{1}{s}\right)$

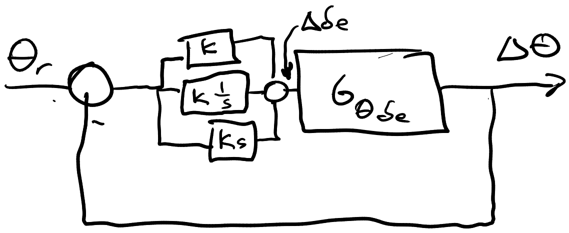


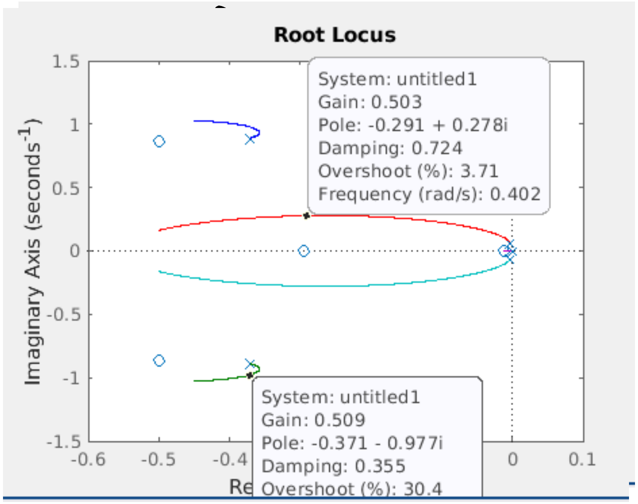




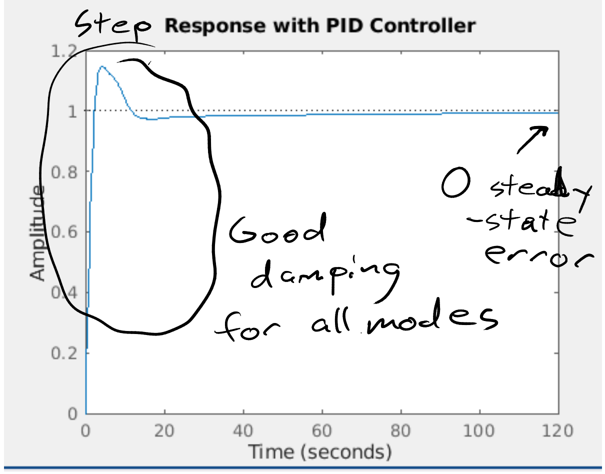
Controller 3: PID

$J = K\left(1 + \frac{1}{s} + s\right)$



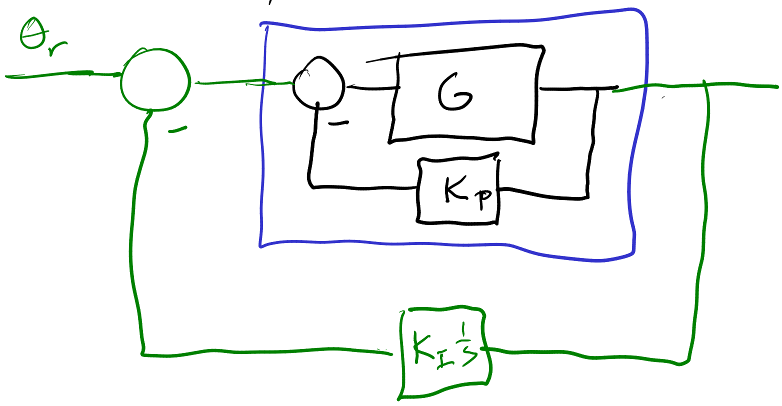


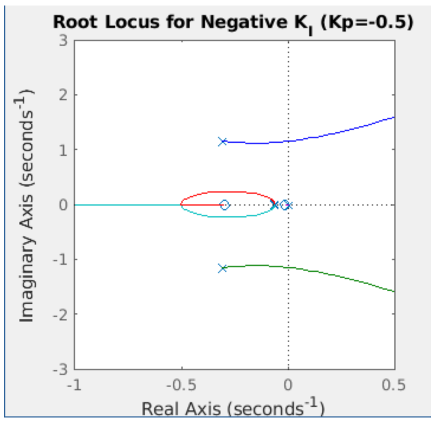
Choose $K = -0.5$

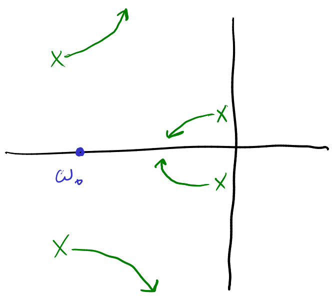


$$J = -0.5\left(1 + \frac{1}{s} + s\right)$$

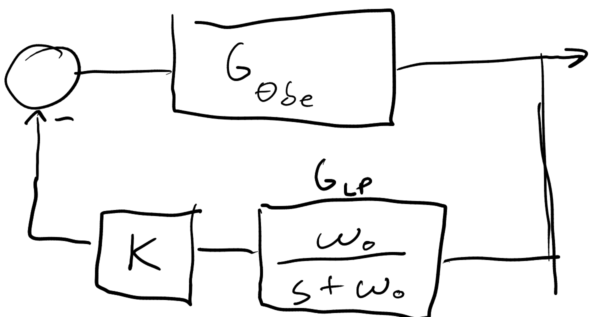
If you wanted to tune individually



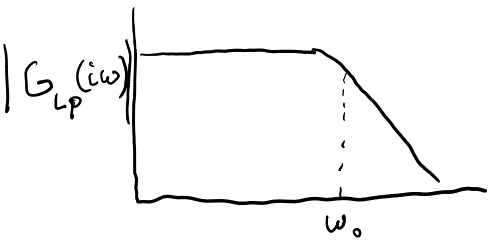


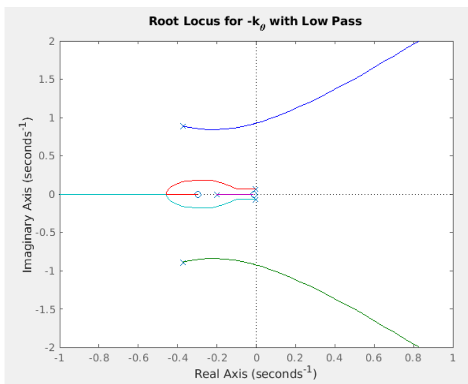


P controller

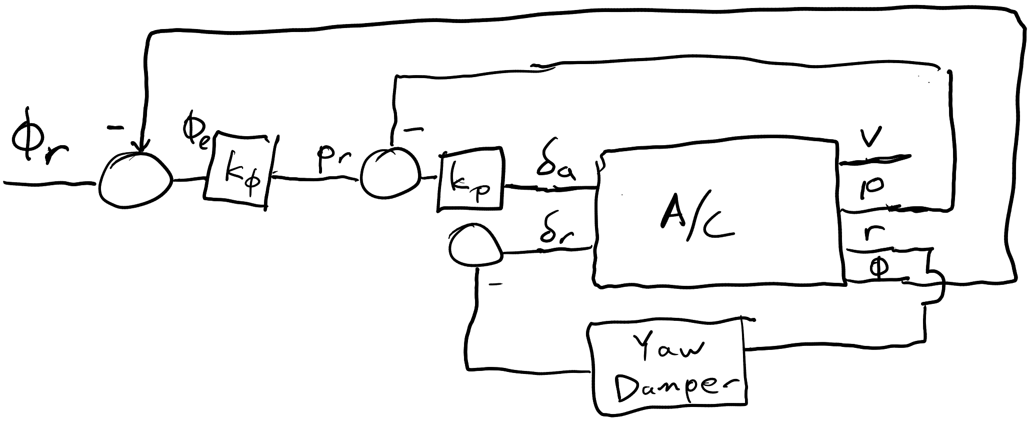


Frequency response



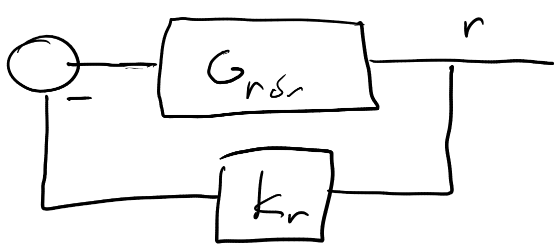


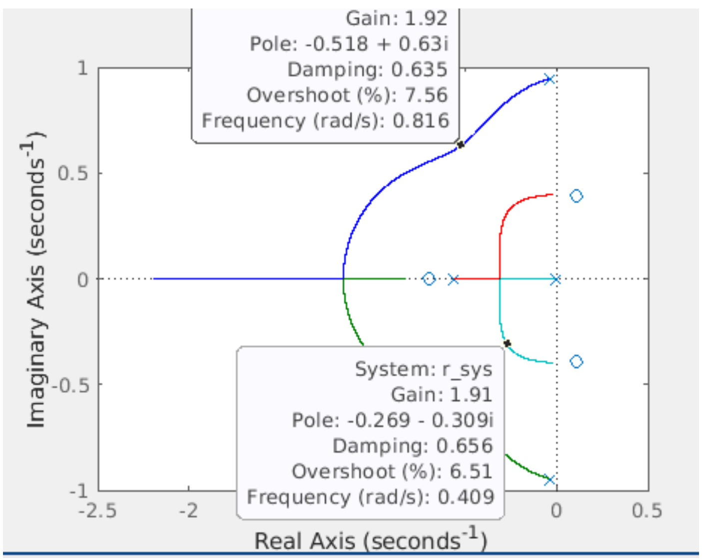
Designing a Roll Controller



Part 1: Yaw Damper

$\delta_r = -k_r r$





choose

$\boxed{k_r = -1.9}$

Problem: In steady right turn, $r > 0$
with $k_r = -1.9$, $\delta_r > 0$

Want: $\delta_r \to 0$ at low frequency
$\delta_r = -k_r r$ at dutch roll frequency

Washout / High Pass Filter

$$W(s) = \frac{s}{s + \omega_0}$$

for 747 $\lambda_{dr} = -0.033 \pm 0.947i$

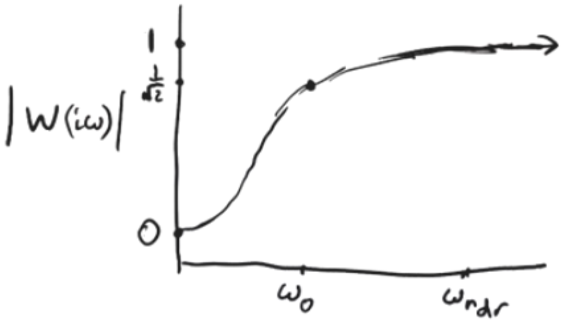
$$\underline{\omega_{n_{dr}} = |\lambda_{dr}| \approx 1}$$

$$|W(\omega_0 i)| = \left|\frac{\omega_0 i}{\omega_0 i + \omega_0}\right| = \frac{1}{\sqrt{2}}$$

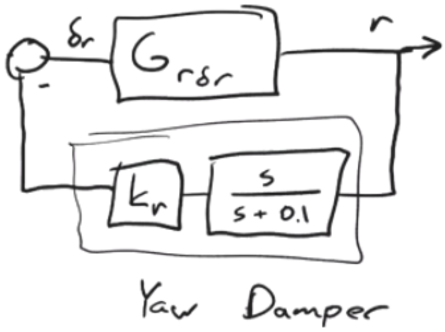
$$\lim_{\omega \to \infty} |W(\omega i)| = \lim_{\omega \to \infty} \frac{\omega i}{\omega i + \omega_0} = 1$$

$$|W(0i)| = \frac{0i}{0i + \omega_0} = 0$$

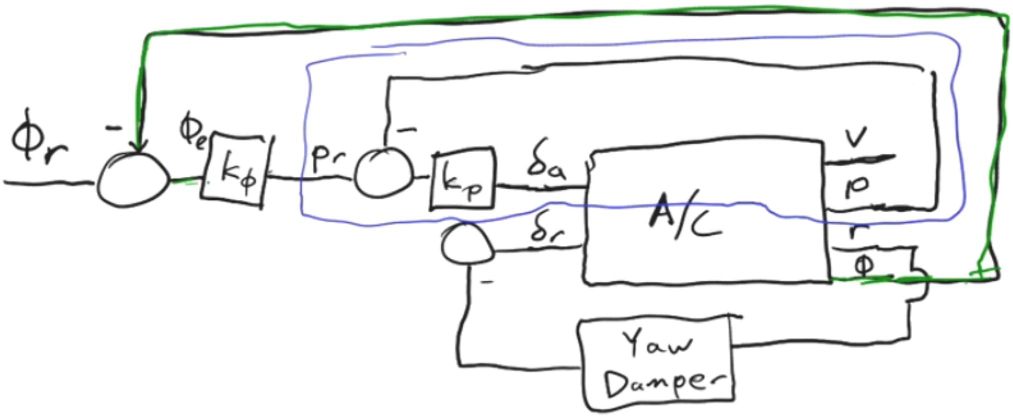
Frequency response



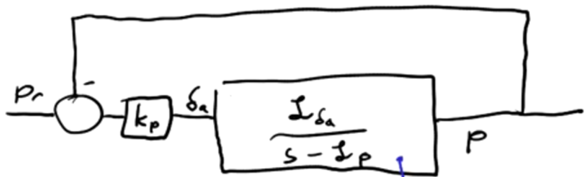
choose $\omega_0 = 0.1$



Part 2: Aileron Controller



Part (a): Inner Loop



$\delta_a = k_p(p_r - p)$

Pure Roll Approximation

$$\dot{p} = \mathcal{L}_p p + \mathcal{L}_{\delta_a} \delta_a$$

$$s\,p(s) = \mathcal{L}_p p(s) + \mathcal{L}_{\delta_a} \delta_a(s)$$

$$(s - \mathcal{L}_p)\,p(s) = \mathcal{L}_{\delta_a} \delta_a(s)$$

$$G_{p\delta_a}(s) = \frac{p(s)}{\delta_a(s)} = \boxed{\frac{\mathcal{L}_{\delta_a}}{s - \mathcal{L}_p}}$$

$$\begin{array}{c} Y \\ \mathcal{L} \\ N \end{array} \quad \mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & \boxed{-0.4342} & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix}$$

for 747

$\mathcal{L}_p = -0.4342$

$\mathcal{L}_{\delta_a} = -0.1431$

from $B_{lat} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ \mathcal{L}_{\delta_a} & \mathcal{L}_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$

Closed Loop

$$G_{pp_r}(s) = \frac{k_p G_{p\delta_a}}{1 + k_p G_{p\delta_a}} = \frac{k_p \frac{\mathcal{L}_{\delta_a}}{s-\mathcal{L}_p}}{1 + k_p \frac{\mathcal{L}_{\delta_a}}{s-\mathcal{L}_p}} = \boxed{\frac{k_p \mathcal{L}_{\delta_a}}{s - \mathcal{L}_p + k_p \mathcal{L}_{\delta_a}} = G_{pp_r}(s)}$$

We care about step response

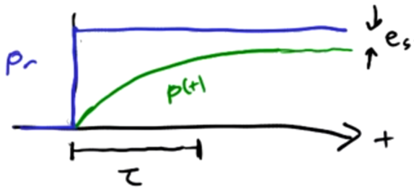
What sign k_p have?

k_p should be negative

$$\delta_a = k_p (p_r - p)$$

$p_r = 1$

$p = 0$



$$e_{ss} = \lim_{t\to\infty} p_r(t) - p(t)$$

steady-state gain

$\lim_{t\to\infty} \frac{p(t)}{p_r(t)}$ if $p_r(t)$ is step

Steady State Gain

step: $p_r(s) = \frac{1}{s}$

$$\lim_{t\to\infty} p(t) = \lim_{s\to 0} s\,p(s) = \lim_{s\to 0} s\,p_r(s)\,G_{pp_r}(s) = \lim_{s\to 0} \cancel{s}\,\frac{1}{\cancel{s}}\,G_{pp_r}(s) = \boxed{G_{pp_r}(0)}$$

only if $s\,p(s)$ is stable

$$G_{pp_r}(0) = -\frac{k_p \mathcal{L}_{\delta a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta a}}$$

as k_p gets more negative $\Rightarrow$ larger steady-state gain

$$G_{pp_r}(s) = \frac{k_p \mathcal{L}_{\delta a}}{s - (\mathcal{L}_p - k_p \mathcal{L}_{\delta a})} \qquad \lambda = \mathcal{L}_p - k_p \mathcal{L}_{\delta a}$$

$$\tau = \left|\frac{1}{\lambda}\right| = \left|\frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta a}}\right|$$

more negative k_p $\Rightarrow$ smaller τ

Why not make $|k_p|$ as large as possible

- saturation
- sensitivity to noise
- wear and tear on actuator

consider $\boxed{k_p = -1}$

$p_e \approx 15$ deg/s $\Rightarrow$ 15° aileron deflection

$$\text{Steady-state gain} = G_{pp_r}(0) = \frac{-k_p \mathcal{L}_{\delta a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta a}} \approx 0.25$$

check if OK Later

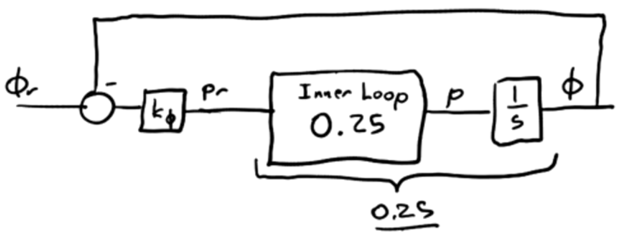
$$\text{Time constant} \quad \tau = \left|\frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta a}}\right| \approx 1.75\ \text{s}$$

check if OK Later

Part (b) Outer Loop

Assume: Inner Loop Much Faster than Outer Loop

$\dot{\phi} = p$



Closed Loop TF

$$G_{\phi\phi_r} = \frac{k_\phi \frac{0.25}{s}}{1 + k_\phi \frac{0.25}{s}} = \frac{0.25 k_\phi}{s + 0.25 k_\phi}$$

k_ϕ should be > 0

2 Goals
- steady-state gain near 1 $\longrightarrow$ $G_{\phi\phi_r}(0) = \frac{0.25 k_\phi}{0.25 k_\phi} = 1$
- low time constant

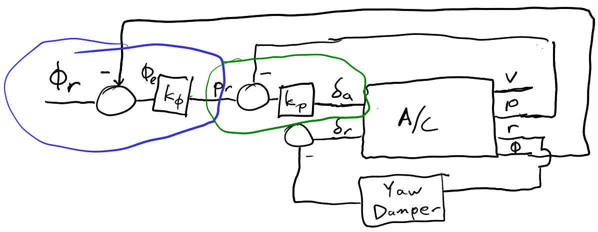
higher $k_\phi \Rightarrow$ lower time constant

$$\boxed{k_\phi = 1.5}$$

Plug back into full linear lateral system (without washout filter)

$$\vec{u} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = -K\vec{x} - K_r[\phi_r]$$

$$= -\begin{bmatrix} 0 & k_p & 0 & k_p k_\phi \\ 0 & 0 & k_r & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} - \begin{bmatrix} -k_p k_\phi \\ 0 \end{bmatrix} [\phi_r]$$



$\rightarrow$ $p_r = k_\phi(\phi_r - \phi)$

$\rightarrow$ $\delta_a = k_p(p_r - p) = k_p(k_\phi(\phi_r - \phi) - p)$

$= k_p k_\phi \phi_r - k_p k_\phi \phi - k_p p$

$\delta_r = -k_r r$

New dynamics equation

$$\dot{\vec{x}} = A'\vec{x} + B'[\phi_r]$$

$$\dot{\vec{x}} = A_{lat}\vec{x} + B_{lat}\vec{u}$$

$$A' = A_{lat} - B_{lat}K$$

$$B' = -B_{lat}K_r$$

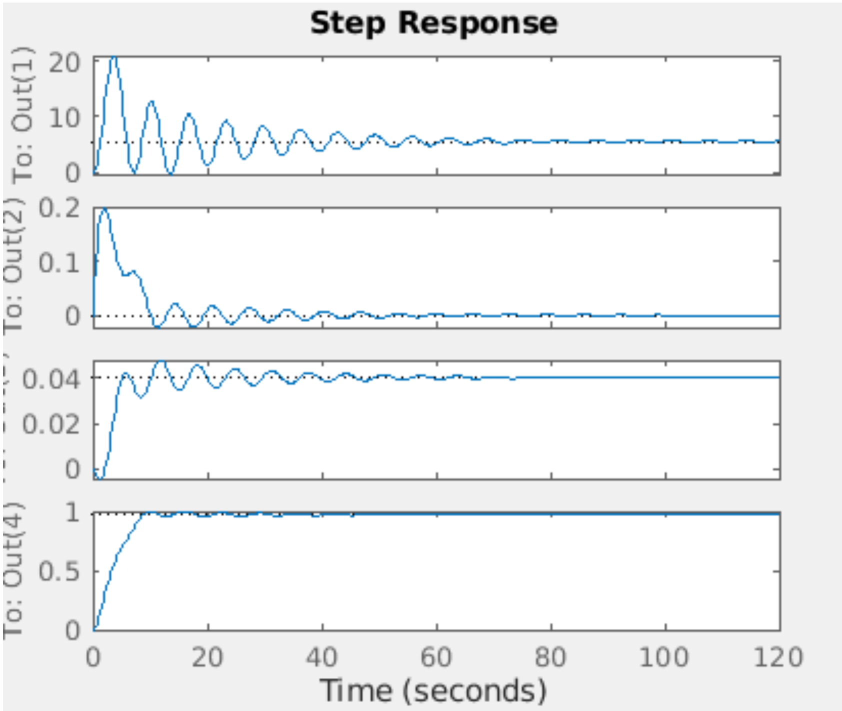
Step response without yaw damper

Δv

Δp

Δr

$\Delta \phi$



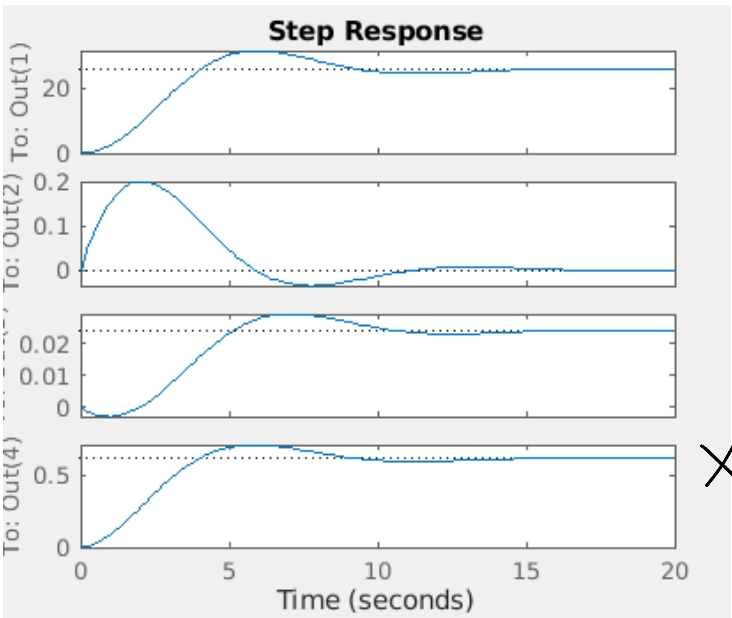
Step response with $\delta_r = -k_r \Delta r$

Δv

Δp

Δr

$\Delta \phi$



$\times\ 0.7$

Step response with Yaw damper with Washout filter