Aerospace Dynamics and Control Systems

ASEN 3728: Part 2 Notes

Contents

1 Aircraft Dynamics: Longitudinal and Lateral/Directional

Aircraft dynamics are traditionally divided into two main categories based on the motion characteristics:

1.1 Longitudinal Dynamics

Aircraft longitudinal dynamics primarily involve motion in the vertical plane and include:

- Altitude changes
- Forward speed variations
- Pitch attitude and rotation

Longitudinal dynamics involve symmetric motion about the aircraft's plane of symmetry.

1.2 Lateral/Directional Dynamics

Lateral and directional dynamics involve asymmetric motion and include:

- Roll rotation and bank angle
- Yaw rotation and heading changes
- Sideslip motion perpendicular to the flight path

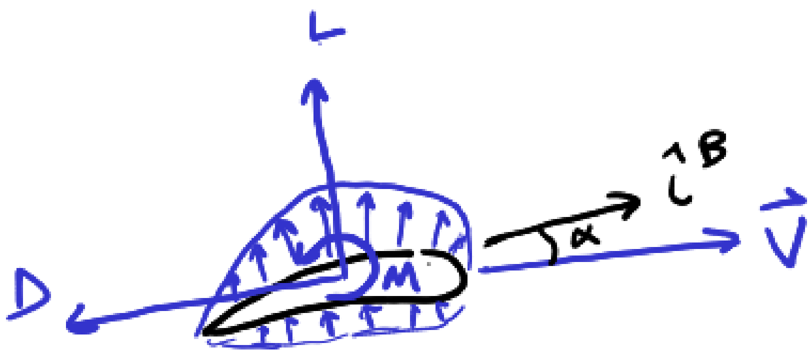

Figure 1: Basic aerodynamic forces and moments on an airfoil section. L = lift, D = drag, V = velocity, alpha = angle of attack, B = body reference frame.

2 Longitudinal Forces and Moments

The primary aerodynamic forces and moments affecting longitudinal motion:

2.1 Lift

The lift force acts perpendicular to the relative wind and is defined as:

$$L = \frac{1}{2}\rho V_a^2 S C_L \tag{1}$$

where:

- ρ = air density (kg/m^3 or slugs/ft^3)
- V_a = airspeed (m/s or ft/s)
- S = wing reference area (m^2 or ft^2)
- C_L = lift coefficient (dimensionless)

2.2 Drag

The drag force acts parallel to the relative wind in the opposite direction of flight:

$$D = \frac{1}{2}\rho V_a^2 S C_D \tag{2}$$

where C_D is the drag coefficient (dimensionless).

2.3 Pitch Moment

The pitching moment acts about the y-axis (wing axis) and is defined as:

$$M = \frac{1}{2}\rho V_a^2 S \bar{c} C_m \tag{3}$$

where:

- $\bar{c}$ = mean aerodynamic chord (m or ft)
- C_m = pitching moment coefficient (dimensionless)

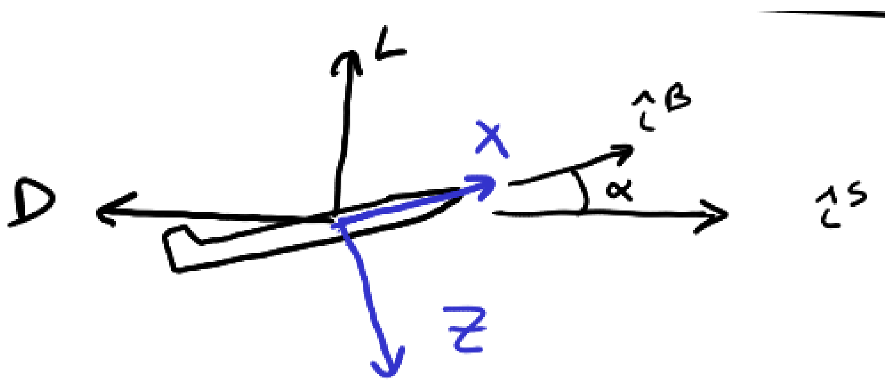

Figure 2: Coordinate transformation between wind and body axes. L = lift, D = drag, X and Z represent forces in the body axis, alpha = angle of attack.

3 Non-dimensionalization

In aircraft dynamics, variables are often non-dimensionalized to make them independent of aircraft size and flight conditions. This enables comparison between different aircraft.

Table 1: Non-dimensionalization of Aircraft Variables

Variable	Divisor	Non-dim Variable
X, Y, Z	$\frac{1}{2}\rho V^2 S$	C_x, C_y, C_z
W	$\frac{1}{2}\rho V^2 S$	C_W
M	$\frac{1}{2}\rho V^2 S\bar{c}$	C_m
L, N	$\frac{1}{2}\rho V^2 Sb$	C_l, C_n
u, v, w	V	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$\frac{2V}{\bar{c}}$	$\hat{\dot{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$\frac{2V}{b}$	$\hat{\dot{\beta}}, \hat{p}, \hat{r}$
m	$\rho S\bar{c}/2$	μ
I_y	$\rho S(\bar{c})^3$	$\hat{I}_y$
I_x, I_z, I_{xz}	$\rho S(b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

These non-dimensional parameters help in analyzing stability derivatives and comparing different aircraft designs.

4 Lift Modeling

4.1 Lift Coefficient Dependencies

The lift coefficient C_L depends on several parameters:

- Angle of attack (α)
- Pitch rate (q)
- Control surface deflection (δ_e for elevator)

4.2 First-order Taylor Series Expansion

The lift coefficient can be expanded using a first-order Taylor series:

$$\boxed{L = \frac{1}{2}\rho V_a^2 S \left(C_{L_{zero}} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q}\hat{q} + \frac{\partial C_L}{\partial \delta_e}\delta_e \right) \qquad (4)}$$

Where $\hat{q} = \frac{q\bar{c}}{2V_a}$ is the non-dimensionalized pitch rate and $C_{L_{zero}}$ is the lift coefficient at zero angle of attack.

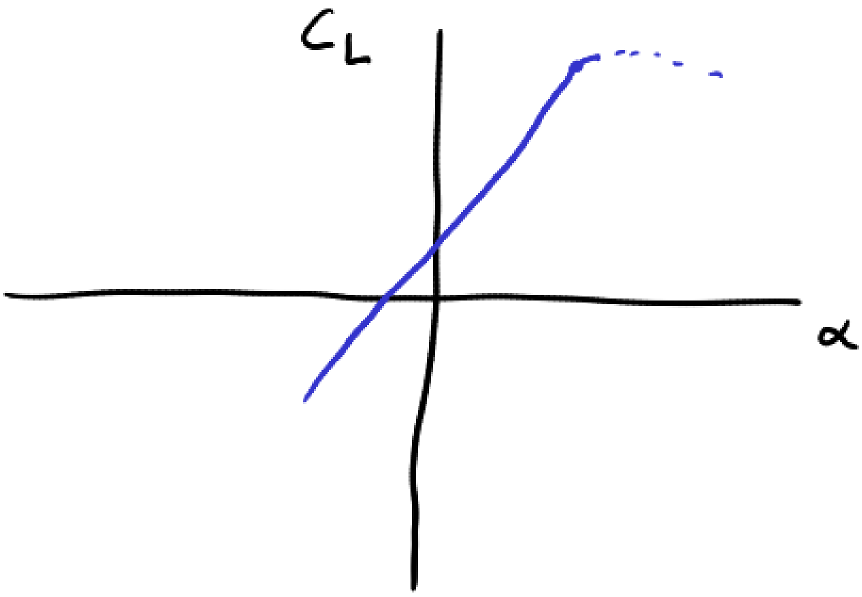

Figure 3: Lift coefficient (C_L) versus angle of attack (α) showing the linear relationship and lift-curve slope in the normal operating range.

4.3 Linear Approximation

For small angles of attack (linear region), the relationship between C_L and α is approximately linear:

$$C_{L_\alpha} = a = \frac{\partial C_L}{\partial \alpha} \tag{5}$$

Where a represents the lift-curve slope.

4.4 Estimating Lift-Curve Slope

For typical subsonic wings, the lift-curve slope can be estimated using:

$$C_{L_\alpha} \approx \frac{\pi AR}{1 + \sqrt{1 + (\frac{AR}{2})^2}} \tag{6}$$

where AR is the aspect ratio of the wing (wingspan^2/wing area).

5 Drag Modeling

5.1 Drag Components

Aircraft drag consists of two primary components:

- Parasitic drag (form drag, skin friction, interference drag)
- Induced drag (resulting from lift production)

5.2 Drag Polar

The relationship between drag and lift coefficients is often modeled using the parabolic drag polar:

$$C_D = C_{D_{min}} + K(C_L - C_{L_{min}})^2 \tag{7}$$

where:

- $C_{D_{min}}$ = minimum drag coefficient
- $C_{L_{min}}$ = lift coefficient at minimum drag
- K = induced drag factor

The induced drag factor relates to the wing efficiency:

$$K = \frac{1}{\pi e AR} \tag{8}$$

where e is Oswald's efficiency factor (typically 0.7-0.85 for conventional aircraft).

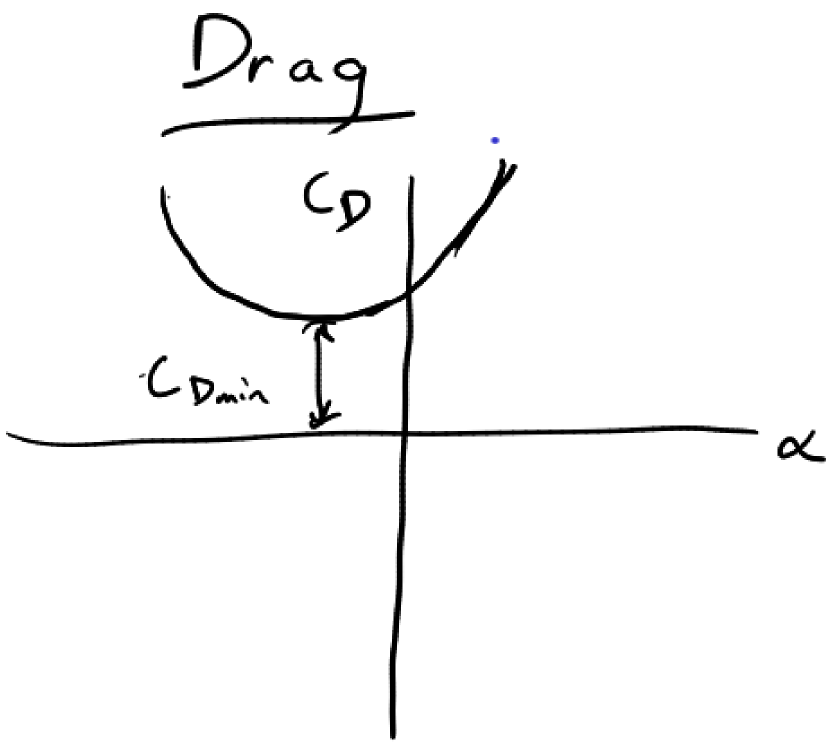

Figure 4: Drag polar showing the relationship between drag coefficient (C_D) and lift coefficient (C_L). The parabolic curve illustrates minimum drag at $C_{L_{min}}$ and the increase in drag due to lift production.

6 Pitch Moment

6.1 Pitch Moment Coefficient

Similar to lift, the pitch moment coefficient can be expressed as a function of α, q, and δ_e:

$$M \approx \frac{1}{2}\rho V_a^2 S\bar{c}\left(C_{m_{zero}} + C_{m_\alpha}\alpha + C_{m_q}\hat{q} + C_{m_{\delta_e}}\delta_e\right) \tag{9}$$

where:

- $C_{m_{zero}}$ = moment coefficient at zero angle of attack
- $C_{m_\alpha} = \frac{\partial C_m}{\partial \alpha}$ = pitch stability derivative
- $C_{m_q} = \frac{\partial C_m}{\partial \hat{q}}$ = pitch damping derivative
- $C_{m_{\delta_e}} = \frac{\partial C_m}{\partial \delta_e}$ = elevator effectiveness

6.2 Lift and Drag to Body Coordinates

To analyze aircraft motion, aerodynamic forces must be transformed from wind axes to body axes. The transformation equations are:

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -D \\ -L \end{bmatrix} \tag{10}$$

7 Longitudinal Stability

7.1 Static and Dynamic Stability

Aircraft stability has two components:

- Static stability: The initial tendency to return to equilibrium after a disturbance
- Dynamic stability: The time history of motion after a disturbance

7.2 Static Stability Parameters

The key parameters affecting longitudinal static stability include:

- C_L - Lift coefficient
- C_D - Drag coefficient
- C_m - Pitching moment coefficient

7.3 Static Margin

The static margin is a measure of static stability:

$$K_n \equiv h_n - h \tag{11}$$

where:

- h_n = location of neutral point (in chord fractions aft of the leading edge)
- h = location of center of gravity

For static longitudinal stability:

$$C_{m_\alpha} < 0 \text{ and } K_n > 0 \tag{12}$$

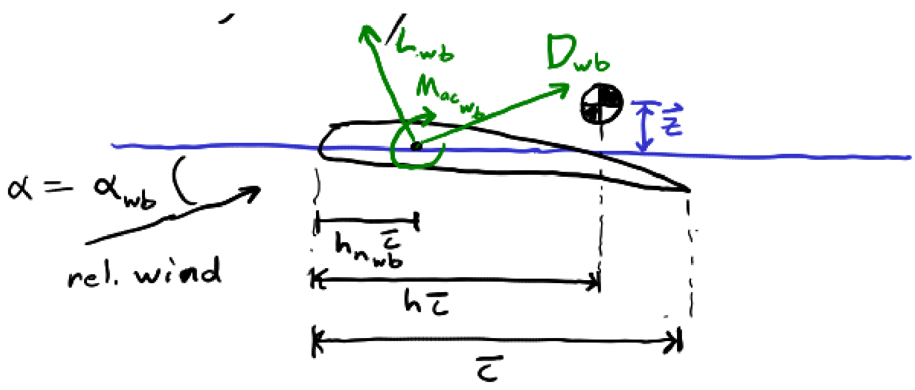

Figure 5: Wing-body configuration showing the relationship between center of gravity location, neutral point, and key stability parameters. The distances are shown in terms of the mean aerodynamic chord ($\bar{c}$).

7.4 Contributors to Longitudinal Stability

Three primary contributors to longitudinal stability:

1. Wing/body combination
2. Propulsion system (often small contribution)
3. Tail (horizontal stabilizer)

7.4.1 Wing/Body

The pitching moment contribution from the wing-body combination:

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_{wb}}) \tag{13}$$

where:

- $C_{m_{ac_{wb}}}$ = pitching moment at the aerodynamic center of the wing-body
- $h_{n_{wb}}$ = neutral point of the wing-body (typically near 25% chord)
- h = center of gravity location (fraction of chord)

7.4.2 Tail

The tail generates lift and contributes to the pitching moment:

$$L_t = C_{L_t}\left(\frac{1}{2}\rho V^2 S_t\right) \tag{14}$$

The tail moment arm creates a pitch moment:

$$M_t = -\ell_t L_t = -\ell_t C_{L_t}\left(\frac{1}{2}\rho V^2 S_t\right) \tag{15}$$

This gives the tail contribution to the pitching moment coefficient:

$$\boxed{C_{m_t} = -V_H C_{L_t} \tag{16}}$$

where $V_H = \frac{\ell_t S_t}{\bar{c}S}$ is the "horizontal tail volume ratio."

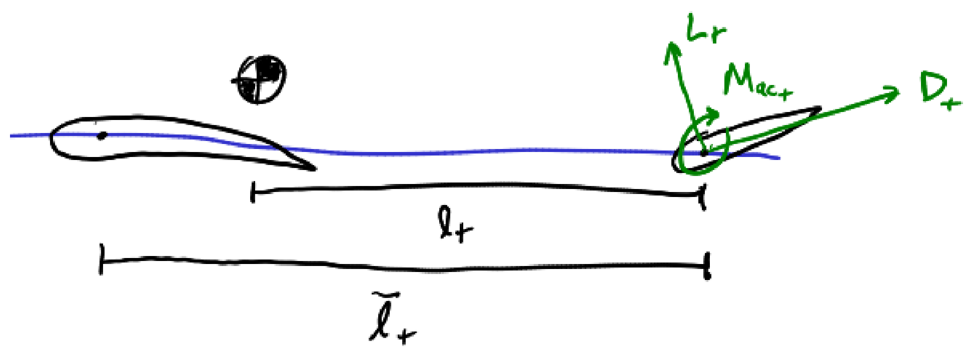

Figure 6: Complete wing-body-tail configuration illustrating the contributions of each component to longitudinal stability. The diagram shows force and moment vectors and the geometric relationships between components.

7.4.3 Total Pitching Moment

The total pitching moment is the sum of all contributions:

$$\boxed{C_m = C_{m_{ac_{wb}}} + C_L(h - h_{n_{wb}}) - V_H C_{L_t} + C_{m_p} \tag{17}}$$

where C_{m_p} is the propulsion contribution to pitching moment.

7.5 Neutral Point

The neutral point is the CG location that yields $C_{m_\alpha} = 0$ (neutral stability).

$$h_n = h_{n_{wb}} + \frac{1}{C_{L_\alpha}} \left(V_H \frac{\partial C_{L_t}}{\partial \alpha} - \frac{\partial C_{m_p}}{\partial \alpha} \right) \tag{18}$$

The pitch stability derivative at any CG location is then:

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n) \tag{19}$$

8 Linear Trim Estimation

8.1 Trim Conditions

At trim, all forces and moments are balanced:

- $T = D$ (thrust equals drag)
- $L = W$ (lift equals weight)
- $M = 0$ (zero pitching moment)

8.2 Lift and Moment at Trim

$$C_{L_{trim}} = \frac{W}{\frac{1}{2}\rho V^2 S} = C_{L_{zero}} + C_{L_\alpha}\alpha_{trim} + C_{L_{\delta_e}}\delta_{e_{trim}} \tag{20}$$

$$C_{m_{trim}} = C_{m_{zero}} + C_{m_\alpha}\alpha_{trim} + C_{m_{\delta_e}}\delta_{e_{trim}} = 0 \tag{21}$$

8.3 Solving for Trim Values

These equations can be written in matrix form:

$$\begin{bmatrix} C_{L_\alpha} & C_{L_{\delta_e}} \\ C_{m_\alpha} & C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \alpha_{trim} \\ \delta_{e_{trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{trim}} - C_{L_{zero}} \\ -C_{m_{zero}} \end{bmatrix} \tag{22}$$

Using Cramer's rule:

$$\alpha_{trim} = \frac{C_{m_{zero}} C_{L_{\delta_e}} + C_{m_{\delta_e}} C_{L_{trim}}}{\Delta} \tag{23}$$

$$\delta_{e_{trim}} = -\frac{C_{m_{zero}} C_{L_\alpha} + C_{m_\alpha} C_{L_{trim}}}{\Delta} \tag{24}$$

where $\Delta = C_{L_\alpha} C_{m_{\delta_e}} - C_{L_{\delta_e}} C_{m_\alpha}$

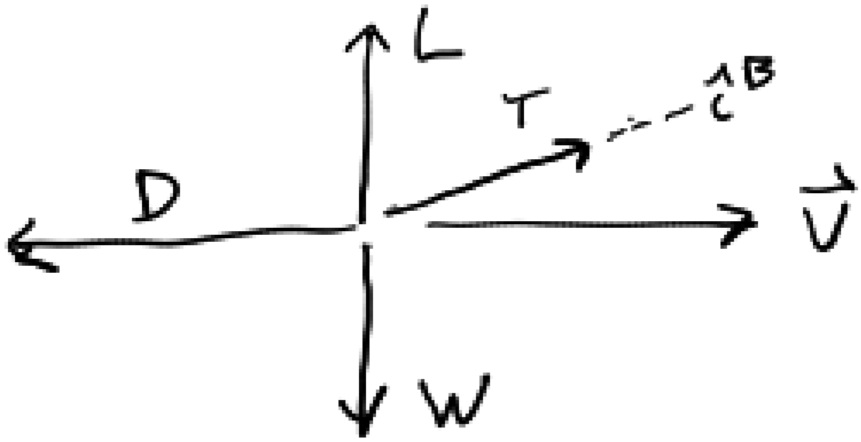

Figure 7: Force balance during trim condition. In linear trim, thrust equals drag (T=D) and lift equals weight (L=W), resulting in no acceleration.

9 Longitudinal Linear Model

9.1 State-Space Representation

The linearized equations of motion can be formulated in state-space form:

$$\boxed{\dot{\vec{x}} = A_{lon}\vec{x}_{lon} + B_{lon}\vec{u}_{lon}} \tag{25}$$

$$\boxed{\vec{y} = C\vec{x}_{lon} + D\vec{u}_{lon}} \tag{26}$$

9.2 State Vector and Input Vector

For longitudinal dynamics, the state and input vectors are:

$$\boxed{\vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta\theta \end{bmatrix}} \tag{27}$$

$$\boxed{\vec{u}_{lon} = \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}} \tag{28}$$

where:

- $\Delta u, \Delta w$ = perturbations in body-axis velocities
- Δq = perturbation in pitch rate
- $\Delta \theta$ = perturbation in pitch angle
- $\Delta \delta_e$ = perturbation in elevator deflection
- $\Delta \delta_t$ = perturbation in throttle setting

9.3 System Matrix

The full longitudinal state matrix has the form:

$$A_{lon} = \begin{bmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mu_0}{m-Z_{\dot{w}}} & -\frac{mg\sin\theta_0}{m-Z_{\dot{w}}} \\ \frac{1}{I_y}[M_u + \frac{M_{\dot{w}}Z_u}{m-Z_{\dot{w}}}] & \frac{1}{I_y}[M_w + \frac{M_{\dot{w}}Z_w}{m-Z_{\dot{w}}}] & \frac{1}{I_y}[M_q + \frac{M_{\dot{w}}(Z_q+mu_0)}{m-Z_{\dot{w}}}] & -\frac{M_{\dot{w}}mg\sin\theta_0}{I_y(m-Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{29}$$

9.4 Control Input Matrix

The input matrix relates control inputs to state derivatives:

$$B_{lon} = \begin{bmatrix} \frac{X_{\delta_e}}{m} & \frac{X_{\delta_t}}{m} \\ \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta_t}}{m-Z_{\dot{w}}} \\ \frac{M_{\delta_e}+M_{\dot{w}}Z_{\delta_e}}{I_y(m-Z_{\dot{w}})} & \frac{M_{\delta_t}+M_{\dot{w}}Z_{\delta_t}}{I_y(m-Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix} \tag{30}$$

10 Longitudinal Modes

10.1 Modal Analysis

The eigensolutions of the system matrix A form the longitudinal modes. For a linear system:

$$\boxed{\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}} \tag{31}$$

where:

- λ_i are the eigenvalues (complex or real numbers)
- $\vec{v}_i$ are the eigenvectors
- q_i are constants determined by initial conditions

Eigenvalues and eigenvectors are found by solving:

$$A\vec{v}_i = \lambda_i \vec{v}_i \tag{32}$$

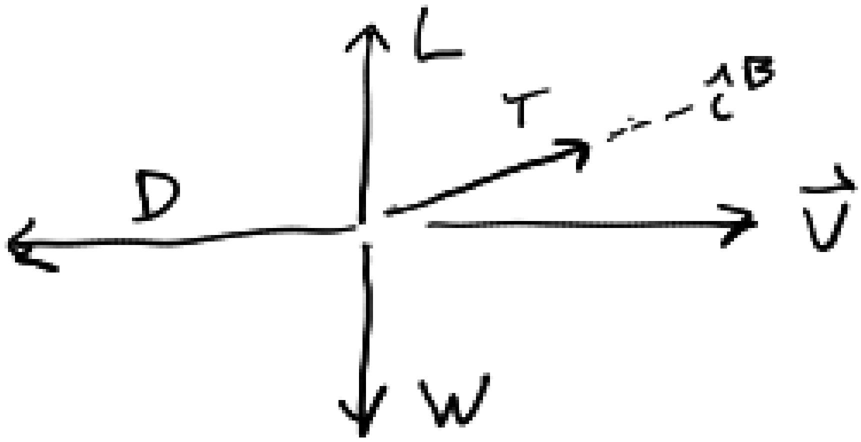

Figure 8: Eigenvalue Diagram

10.2 Short Period Mode

Characteristics:

- Higher frequency (typically 1-10 rad/s)
- Higher damping (typically $\zeta = 0.3$ to 0.7)
- Primarily involves Δw (or α) and Δq
- Results from pitch stiffness and damping
- For a 747 at cruise: $\lambda_{1,2} = -0.37 \pm 0.89i$, $\zeta = 0.38$, $\omega_n = 0.96$ rad/s

Short period eigenvector example:

$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \pm 0.0004i \end{bmatrix} \tag{33}$$

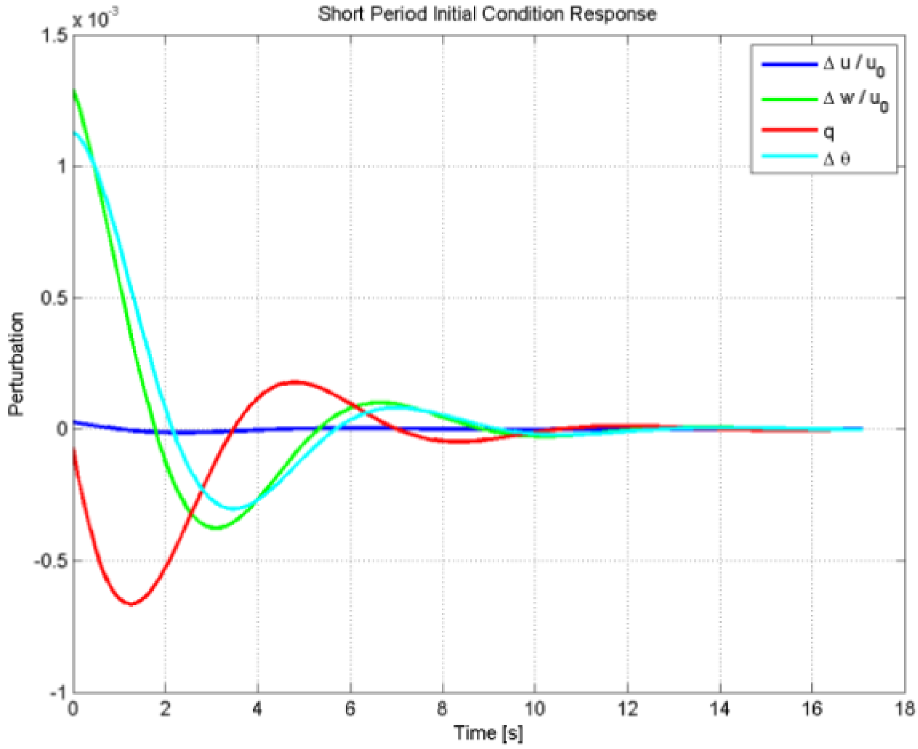

Figure 9: Short Period Initial Condition Response

10.3 Phugoid Mode

Characteristics:

- Lower frequency (typically 0.1-1 rad/s)
- Lower damping (typically $\zeta = 0.02$ to 0.1)
- Primarily involves Δu and $\Delta\theta$ oscillations
- Exchange between potential and kinetic energy
- For a 747 at cruise: $\lambda_{3,4} = -0.0033 \pm 0.067i$, $\zeta = 0.049$, $\omega_n = 0.067$ rad/s

Phugoid eigenvector example:

$$\vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.0573 \pm 0.0097i \\ -0.0001 \pm 0i \\ 0.0001 \pm 0.0024i \end{bmatrix} \tag{34}$$

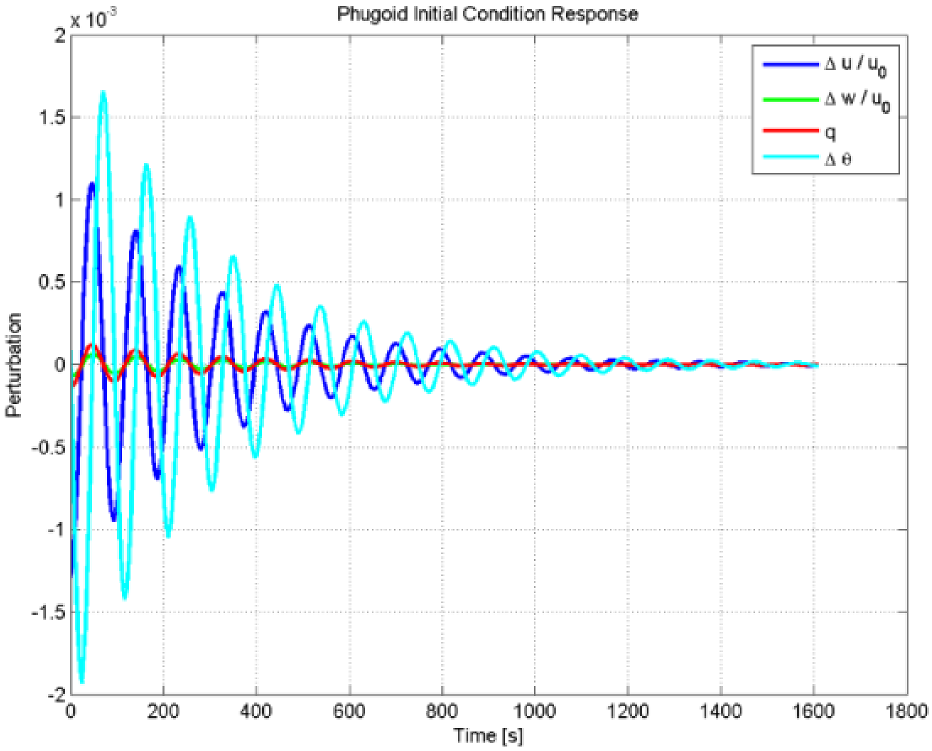

Figure 10: Phugoid Initial Condition Response

11 Longitudinal Mode Approximations

Simplified models help provide insight into the primary dynamics.

11.1 Short Period Approximation

For short period motion, we can make the following assumptions:

- $\Delta u = 0$ (speed remains constant)
- $Z_{\dot{w}} \ll m$ (vertical force primarily from gravity)
- $Z_q \ll m u_0$ (pitch rate contribution small)
- $\theta_0 = 0$ (level flight)
- No vertical motion
- $\Delta\theta = \Delta\alpha = \frac{\Delta w}{u_0}$ (small angle approximation)

This results in a simplified 2×2 model:

$$\begin{bmatrix} \Delta\dot{w} \\ \Delta\dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_w}{m} & u_0 \\ \frac{1}{I_y}[M_q + \frac{M_{\dot{w}} Z_w}{m}] & \frac{1}{I_y}[M_q + M_w \frac{u_0}{m}] \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix} \tag{35}$$

11.2 Phugoid Approximation

11.2.1 Lanchester's Model

Based on Lanchester's 1908 model, the phugoid assumes conservation of energy:

$$E = \frac{1}{2}mV^2 - mg\Delta z_E = \frac{1}{2}mu_0^2 \tag{36}$$

This leads to the natural frequency of the phugoid:

$$\boxed{\omega_n = \sqrt{\frac{2g}{u_0}}} \tag{37}$$

And damping ratio:

$$\boxed{\zeta = \frac{1}{\sqrt{2}}\frac{C_{D_0}}{C_{L_0}}} \tag{38}$$

The period depends on airspeed:

$$\boxed{T = \frac{2\pi}{\omega_n} = \pi\sqrt{2}\frac{u_0}{g} = \begin{cases} 0.138u_0 & \text{if } u_0 \text{ in ft/s} \\ 0.453u_0 & \text{if } u_0 \text{ in m/s} \end{cases}} \tag{39}$$

11.2.2 2×2 Phugoid Approximation

A simplified model focusing on the phugoid motion:

$$\boxed{\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta\theta \end{bmatrix}} \tag{40}$$

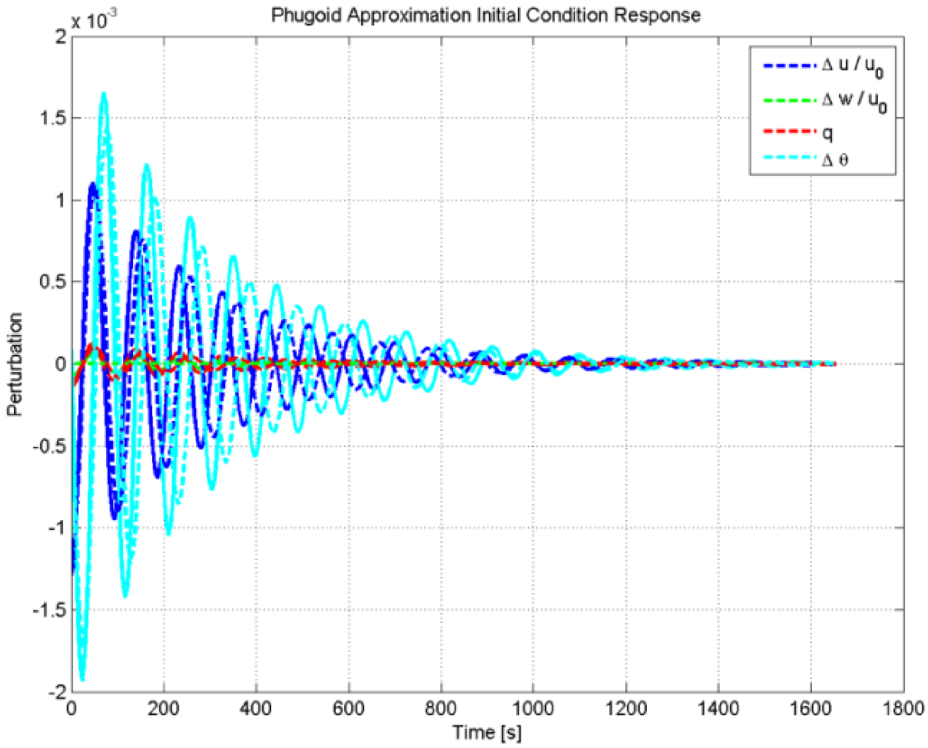

Figure 11: Phugoid Approximation Response

12 Stability Derivatives

12.1 α-Derivatives

12.1.1 Pitch Moment Derivative

$$C_{m_\alpha} = C_{L_\alpha}(h - h_n) \tag{41}$$

12.1.2 Force Derivative

$$C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0}) \tag{42}$$

12.2 q-Derivatives

12.2.1 Tail Contribution

The tail experiences a velocity component due to pitch rate:

$$\Delta\alpha_t = \frac{q\ell_t}{u_0} \tag{43}$$

This creates a lift increment:

$$\Delta C_{L_t} = a_t \Delta\alpha_t = a_t \frac{q\ell_t}{u_0} \tag{44}$$

The q-derivatives for the force and moment coefficients:

$$(C_{Z_q})_{tail} = -2a_t V_H \tag{45}$$

$$(C_{m_q})_{tail} = -2a_t V_H \frac{\ell_t}{\bar{c}} \tag{46}$$

12.3 α-dot Derivatives

12.3.1 Downwash Lag Effect

The tail experiences a delay in downwash changes due to the time it takes for the flow to travel from the wing to the tail:

$$\Delta\alpha_t = \Delta\epsilon = -\frac{\partial\epsilon}{\partial\alpha}\dot{\alpha}\frac{\ell_t}{u_0} \tag{47}$$

This leads to the $\dot{\alpha}$ derivatives:

$$(C_{Z_{\dot{\alpha}}})_{tail} = -2a_t \frac{\ell_t S_t}{\bar{c}S}\frac{\partial\epsilon}{\partial\alpha} \tag{48}$$

$$(C_{m_{\dot{\alpha}}})_{tail} = -2a_t V_H \frac{\ell_t}{\bar{c}}\frac{\partial\epsilon}{\partial\alpha} \tag{49}$$

12.4 u-Derivatives

Three important factors affecting u-derivatives:

- Compressibility (Mach number effects)
- Dynamic pressure changes: $p_d = \frac{1}{2}\rho u^2$
- Thrust characteristics

For thrust effects, there are three cases:

- Gliding: $C_{T_u} = 0$
- Constant thrust (jet):

$$C_{T_u} = -2C_{T_0} \tag{50}$$

- Constant power (propeller):

$$
\boxed{C_{T_u} = -3C_{T_0}} \tag{51}
$$

The force and moment derivatives:

$$
\boxed{C_{X_u} = M_0 \left(\frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) \bigg|_0 - \rho u_0^2 \frac{\partial C_D}{\partial p_d} \bigg|_0 + C_{T_u} \left(1 - \frac{\partial C_D}{\partial C_T} \right) \bigg|_0} \tag{52}
$$

$$
\boxed{C_{Z_u} = -M_0 \frac{\partial C_L}{\partial M} \bigg|_0 - \rho u_0^2 \frac{\partial C_L}{\partial p_d} \bigg|_0 - C_{T_u} \frac{\partial C_L}{\partial C_T} \bigg|_0} \tag{53}
$$

$$
\boxed{C_{m_u} = M_0 \frac{\partial C_m}{\partial M} \bigg|_0 + \rho u_0^2 \frac{\partial C_m}{\partial p_d} \bigg|_0 + C_{T_u} \frac{\partial C_m}{\partial C_T} \bigg|_0} \tag{54}
$$

13 Longitudinal Control

13.1 Control Derivatives

The control derivatives define how control surface deflections affect the aircraft:

13.1.1 Elevator

The elevator primarily affects the tail lift coefficient:

$$
\boxed{C_{L_{\delta_e}} = \frac{\partial C_{L_t}}{\partial \delta_e} \frac{S_t}{S} = a_e \frac{S_t}{S}} \tag{55}
$$

where a_e is the effectiveness of the elevator.
This contributes to the pitching moment:

$$
\boxed{C_{m_{\delta_e}} = -a_e V_H + C_{L_{\delta_e}} (h - h_{n_{wb}})} \tag{56}
$$

13.1.2 Throttle

The throttle affects thrust directly and indirectly affects other forces and moments:

$$
X_{\delta_t} = \frac{\partial T}{\partial \delta_t} \tag{57}
$$

13.2 Control Matrix

For a Boeing 747 at cruise, the control derivatives form the input matrix:

$$B_{lon} = \begin{bmatrix} \frac{X_{\delta_e}}{m} & \frac{X_{\delta_t}}{m} \\ \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta_t}}{m-Z_{\dot{w}}} \\ \frac{M_{\delta_e}}{I_y} + \frac{M_{\dot{w}} Z_{\delta_e}}{I_y(m-Z_{\dot{w}})} & \frac{M_{\delta_t}}{I_y} + \frac{M_{\dot{w}} Z_{\delta_t}}{I_y(m-Z_{\dot{w}})} \\ 0 & 0 \end{bmatrix} \tag{58}$$

Example values for a 747:

$$\begin{bmatrix} \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}_{\delta_e} = \begin{bmatrix} -17.85 \\ -1.158 \\ 0 \end{bmatrix} \tag{59}$$

14 Longitudinal Stability Augmentation

14.1 Realistic Goals

The main goal of longitudinal stability augmentation is typically to:

- Increase phugoid damping (often poorly damped naturally)
- Not significantly alter the short-period dynamics (usually well-damped)
- Maintain good handling qualities

14.2 Feedback Approaches

14.2.1 Pitch Angle Feedback

The simplest form uses pitch angle feedback:

$$\boxed{\Delta \delta_e = -k_\theta \Delta \theta} \tag{60}$$

This creates a closed-loop system:

$$A^{cl} = A_{ph} - B_{ph_{\delta_e}} K_{ph\theta} \tag{61}$$

The characteristic equation becomes:

$$|A^{cl} - \lambda I| = (-0.0069 - \lambda)(0.44k_\theta - \lambda) - (-0.0001)(-32.2) = 0 \tag{62}$$

$$= \lambda^2 + (-0.44k_\theta + 0.0069)\lambda - 0.0032k_\theta - 0.0032 = 0 \tag{63}$$

14.2.2 Pitch Rate Feedback

Adding pitch rate feedback improves damping:

$$\boxed{\Delta\delta_e = -k_\theta \Delta\theta - k_q \Delta q} \quad (64)$$

14.3 Derivation of Phugoid Feedback

For phugoid feedback, we assume the short period has damped out and focus on a simplified model.

Starting with:

$$\Delta\dot{q} = \frac{M_w}{I_y}\Delta w + \frac{M_{\delta_e}}{I_y}\Delta\delta_e + \frac{M_q}{I_y}\Delta q \quad (65)$$

Substituting $\Delta\delta_e = -k_\theta \Delta\theta$:

$$\boxed{\Delta\dot{\theta} = -\frac{Z_u}{mu_0}\Delta u + \left(\frac{Z_w M_{\delta_e}}{mu_0 M_w} - \frac{Z_{\delta_e}}{mu_0}\right)\Delta\delta_e} \quad (66)$$

$$\boxed{\Delta\dot{u} = \frac{X_u}{m}\Delta u + \frac{X_{\delta_e}}{m}\Delta\delta_e} \quad (67)$$

14.4 Closed-Loop System

The closed-loop system matrix is:

$$\boxed{A^{cl} = A_{ph} - B_{ph_{\delta_e}} K_{ph\theta}} \quad (68)$$

For a 747 at cruise with $k_\theta = -0.5$ and $k_q = -0.2$:

$$A^{cl} = \begin{bmatrix} -0.0069 & -32.2 \\ -0.0001 & 0.44k_\theta \end{bmatrix} - \begin{bmatrix} -0.0002 \\ 0.44 \end{bmatrix} [0 \ k_\theta] \quad (69)$$

14.5 Implementation and Gains

For a Boeing 747 at cruise, test data shows the following gains are effective:

- $k_\theta = -0.5$ (proportional gain)
- $k_q = -1.0$ (derivative gain)

These gains increase the phugoid damping ratio from $\zeta = 0.049$ (poor damping) to $\zeta = 0.7$ (good damping) without significantly affecting the short period mode.

From root locus analysis:

$$\zeta = 0.7 \implies k_\theta \approx -0.2 \quad (70)$$

Adding derivative control:

$$\Delta\delta_e = -k_\theta \Delta\theta - k_q \Delta q - k_{\dot{q}} \Delta\dot{q} \quad (71)$$

15 Combined Longitudinal Dynamics Model

15.1 Big Picture

The complete process from physical aircraft to dynamic behavior involves:

1. Nonlinear rigid body equations of motion
2. Linearization around trim state
3. Linear state-space model
4. Analysis of longitudinal modes
5. Control system design

15.2 Aircraft Properties to Dynamic Behavior

The flow from aircraft physical characteristics to dynamic behavior:

- Aircraft geometry and mass properties
- Aerodynamic forces and moments
- Stability derivatives
- State-space matrices
- Dynamic response and modes

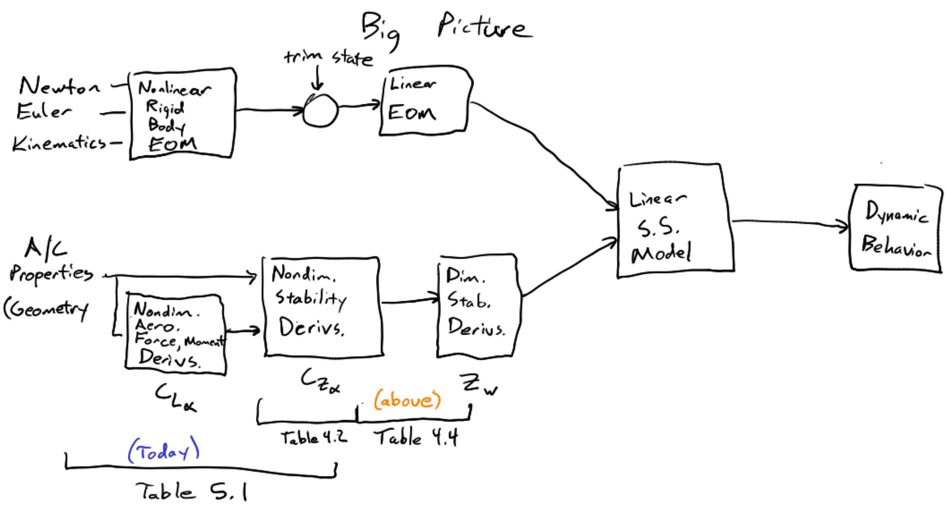

Figure 12: Big Picture Diagram

15.3 Tables of Stability Derivatives

The longitudinal stability derivatives can be organized into tables:

Table 2: Summary of Longitudinal Stability Derivatives

Force/Moment	Speed Derivative	Angle of Attack Derivative
X	X_u	$X_\alpha \approx -C_{D_\alpha}$
Z	Z_u	$Z_\alpha \approx -(C_{L_\alpha} + C_{D_0})$
M	M_u	$M_\alpha = C_{L_\alpha}(h - h_n)$

[Figure: Place the stability derivatives table from page 10 of handwritten notes]

16 Numerical Example: 747 Aircraft

16.1 Aircraft Parameters

Boeing 747 at cruise:

- $h_0 = 40,000$ ft (altitude)
- $u_0 = 774$ ft/s (cruise speed)
- $\gamma_0 = \theta_0 = \alpha_0 = 0$ (level flight)

16.2 System Matrix

$$A_{lon} = \begin{bmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{72}$$

16.3 Eigenvalues and Modes

Short period mode:

- $\lambda_{1,2} = -0.37 \pm 0.89i$
- $\zeta = 0.387$
- $\omega_n = 0.96$ rad/s

Phugoid mode:

- $\lambda_{3,4} = -0.0033 \pm 0.067i$
- $\zeta = 0.0489$
- $\omega_n = 0.0673$ rad/s

16.4 Response Comparison

Comparing the full system to approximations:

- Full A_{lon}: $\lambda_{1,2} = -0.372 \pm 0.888i$, $\zeta = 0.387$, $\omega_n = 0.962$
- Short Period Approximation: $\lambda_{sp} = -0.371 \pm 0.889i$, $\zeta = 0.385$, $\omega_n = 0.963$
- Phugoid Approximation: $\lambda_{ph} = -0.343 \times 10^{-2} \pm 6.11 \times 10^{-2}i$, $\zeta = 0.0561$, $\omega_n = 0.0612$
- Lanchester: $T = 93$s vs. $T = 107$s

17 Summary

This document has covered the key aspects of conventional aircraft longitudinal dynamics:

1. Fundamental forces and moments (lift, drag, pitch moment)
2. Non-dimensionalization of variables
3. Static and dynamic stability
4. Contributors to stability (wing/body, tail, propulsion)
5. Linearization and trim conditions
6. State-space representation
7. Longitudinal modes (short period and phugoid)
8. Stability derivatives
9. Simplified approximations
10. Control system design and stability augmentation

The most important relationships for longitudinal stability:

- Static stability criterion: $C_{m_\alpha} < 0$
- Static margin: $K_n = h_n - h > 0$
- Pitch stiffness: $C_{m_\alpha} = C_{L_\alpha}(h - h_n)$
- Short period frequency: Primarily determined by C_{m_α} and C_{m_q}
- Phugoid frequency: $\omega_n \approx \sqrt{\frac{2g}{u_0}}$
- Phugoid damping: $\zeta \approx \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}}$

These principles form the foundation for understanding aircraft longitudinal dynamics and developing control systems for aircraft stabilization.