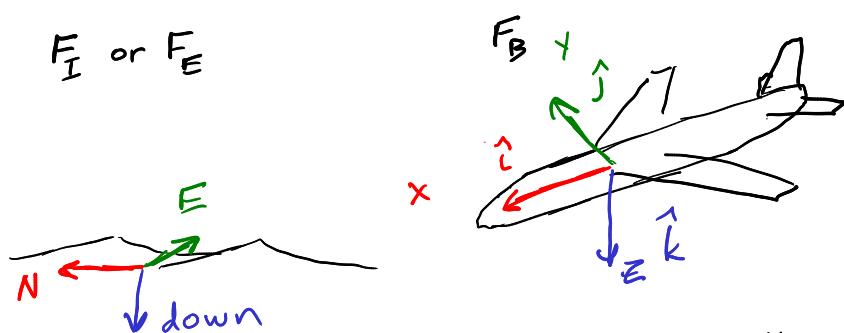


## Notation and Conventions



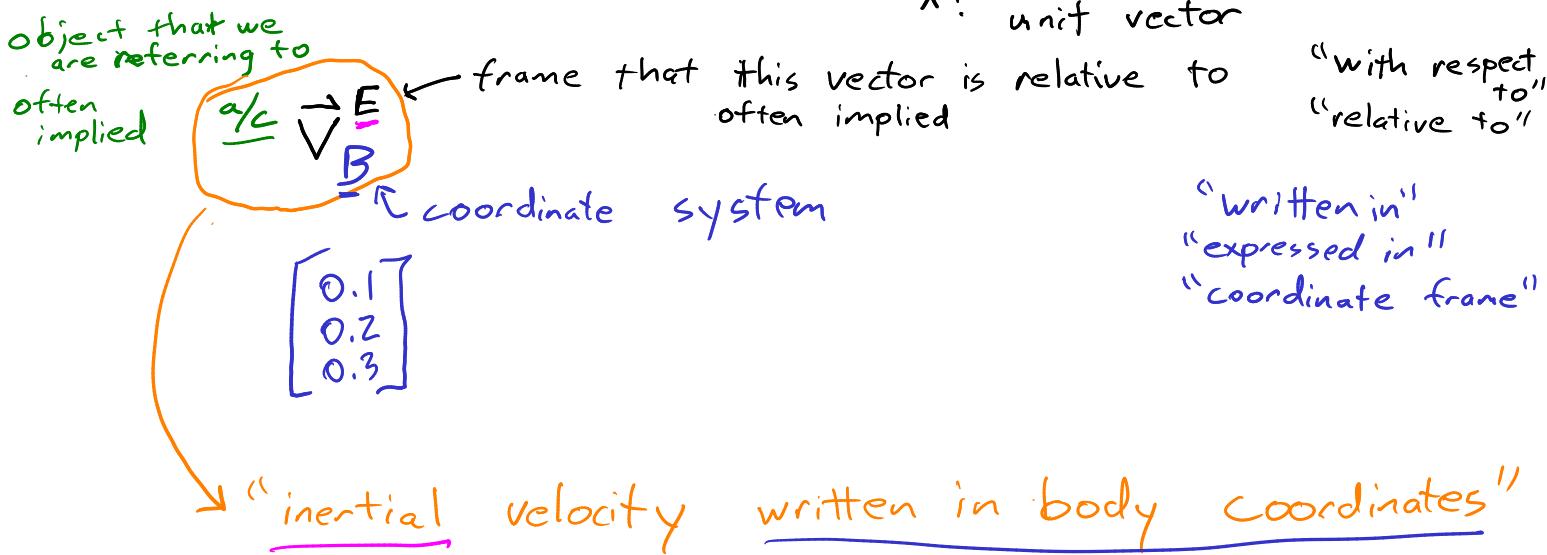
- Frame of reference: Collection of  $\geq 3$  points w/ constant distance between each other
- Inertial frame: A frame that translates with constant (possibly 0) velocity and does not rotate
  - Newton's second law is valid in an inertial frame
- Coordinate System: Three orthogonal unit vectors that allow measurement and vector representation

- Vector Notation

$\rightarrow$  or bold : vector

$^{\wedge}$  : unit vector

"with respect to"  
"relative to"



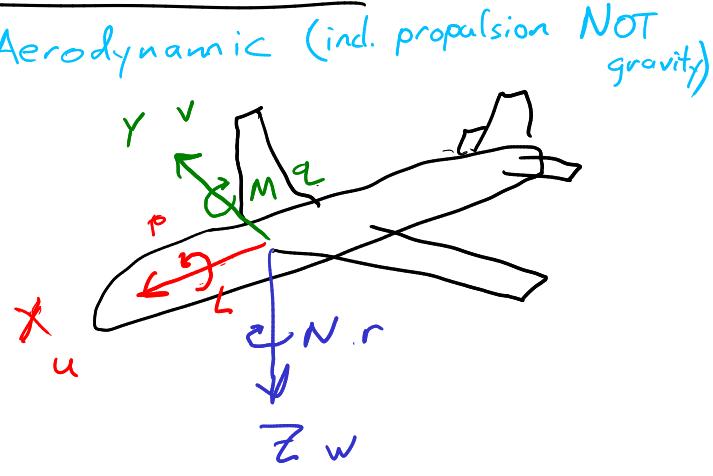
$$\vec{v}_B^W$$

$$\vec{v}_E^E$$

# Forces, Moments, and Velocities

$$\vec{A} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\vec{A}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

$$\vec{V}^E = u^E\hat{i} + v^E\hat{j} + w^E\hat{k}$$

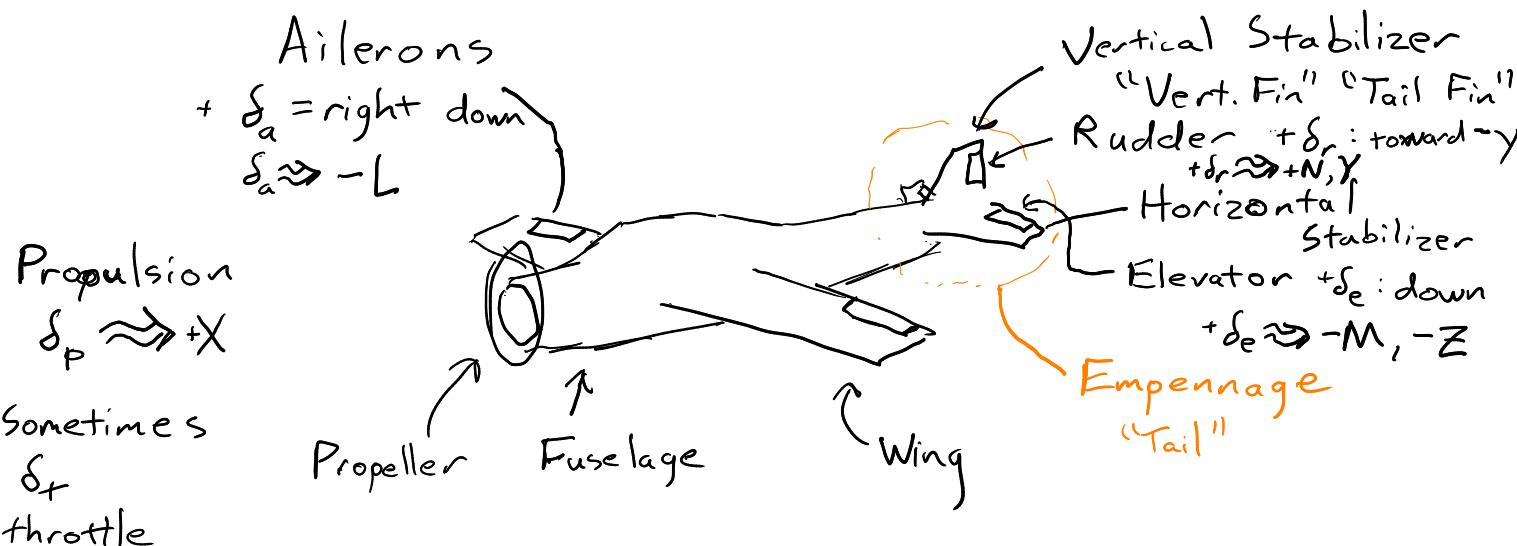
$$\vec{V}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix}$$

$$\vec{\omega}^E = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$V_g = |\vec{V}^E| = \sqrt{u^{E2} + v^{E2} + w^{E2}}$$

## Anatomy of an Airplane aka. "Conventional A/C"



## Wind

Aerodynamic Forces and Moment (i.e. not gravity) are functions of the A/C velocity w.r.t. the Air

$$\vec{V}^W$$

by convention

$$\vec{V} \equiv \vec{V}^W$$

$$\vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$V = |\vec{V}|$$

$$\vec{V}^E = \vec{V}^{(w)} + \vec{W}^{(E)}$$

$$\text{when } \vec{W} = \vec{0}, \vec{V} = \vec{V}^E$$

## Example

Wind blowing east @ 8m/s

A/C pointing north and traveling northward with respect to the wind at 60m/s

What is the A/C velocity vector relative to earth in NED

$$\begin{aligned}\vec{V}_E^E &= \vec{V}_E + \vec{W}_E \\ &= \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 60 \\ 8 \\ 0 \end{bmatrix}\end{aligned}$$

~~$\vec{V}_B + \vec{V}_E$~~   
Illegal!

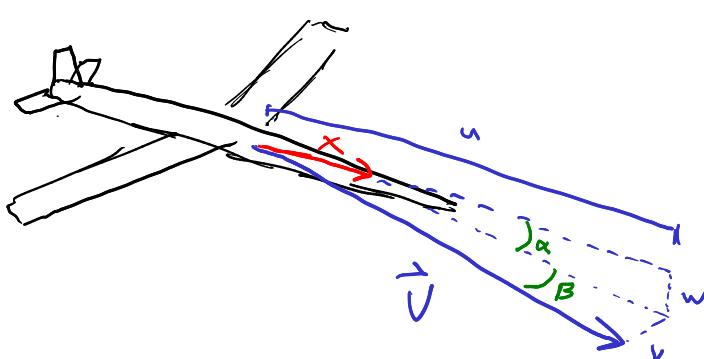
## Wind Angles

Angle of Attack

$$\alpha = \tan\left(\frac{w}{u}\right)$$

Sideslip Angle

$$\beta = \sin^{-1} \frac{v}{V}$$



$$u = V \cos \beta \cos \alpha$$

$$v = V \sin \beta$$

$$w = V \cos \beta \sin \alpha$$

## Orientation

axis

- " 1"  $\Phi$ : roll
- " 2"  $\Theta$ : pitch
- " 3"  $\Psi$ : yaw

E.G.  $45^\circ$

$90^\circ$

$45^\circ$

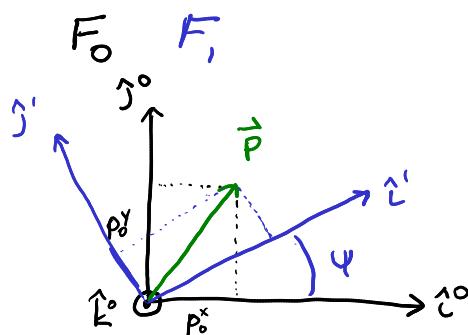
By convention A/C orientation is defined by a 3-2-1 sequence of rotations through  $\Psi-\Theta-\Phi$

Key task: changing the coordinate sys. that a vector is written in

Ex.

know:  $\vec{V}_B^E$

want:  $\vec{V}_E^E$



$$\vec{p}_0 = \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$$\vec{p}_1 = \begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix}$$

$$\vec{p} = p_0^x \hat{i}^0 + p_0^y \hat{j}^0 + p_0^z \hat{k}^0$$

$F_1$  is related to  $F_0$  by a 3-rotation through  $\Psi$

want  $p_1^x$  in terms of  $\vec{p}$

$$\begin{aligned} p_1^x &= \vec{p} \cdot \hat{i}^1 \\ &= p_0^x \hat{i}^0 \cdot \hat{i}^1 + p_0^y \hat{j}^0 \cdot \hat{i}^1 + p_0^z \hat{k}^0 \cdot \hat{i}^1 \\ &= p_0^x \cos \Psi + p_0^y \sin \Psi + p_0^z \cdot 0 \end{aligned}$$

$$p_1^x = [\cos \Psi \quad \sin \Psi \quad 0] \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$R_E^B$   $R_B^E$

$$R_1(\Psi) = \begin{bmatrix} \hat{i}^1 & \hat{j}^1 & \hat{k}^1 \\ \hat{i}^0 \cdot \hat{i}^1 & \hat{j}^0 \cdot \hat{i}^1 & \hat{k}^0 \cdot \hat{i}^1 \\ \hat{i}^0 \cdot \hat{j}^1 & \hat{j}^0 \cdot \hat{j}^1 & \hat{k}^0 \cdot \hat{j}^1 \\ \hat{i}^0 \cdot \hat{k}^1 & \hat{j}^0 \cdot \hat{k}^1 & \hat{k}^0 \cdot \hat{k}^1 \end{bmatrix}$$

$$\begin{bmatrix} p_1^x \\ p_1^y \\ p_1^z \end{bmatrix} = \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0^x \\ p_0^y \\ p_0^z \end{bmatrix}$$

$$\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$$

$R_3(\Psi)$   
c angle  
axis

$$R_1(\Phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}$$

$$R_2(\Theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}$$

# Properties of Direction Cosine Matrices (DCMs)

$$\vec{p}_B = R_A^B \vec{p}_A$$

Book:  $\vec{p}_B = L_{BA} \vec{p}_A$

## 1. Chaining

$$\begin{aligned}\vec{p}_C &= R_B^C \vec{p}_B \\ &= R_B^C R_A^B \vec{p}_A \\ &\quad \boxed{R_A^C = R_B^C R_A^B}\end{aligned}$$

## 2. Inverse

$$\vec{p}_B = R_A^B \vec{p}_A$$

$$(R_A^B)^{-1} \vec{p}_B = (R_A^B)^{-1} R_A^B \vec{p}_A$$

$$R_B^A \vec{p}_B = \vec{p}_A$$

$$R_B^A = (R_A^B)^{-1}$$

Since DCMs are orthonormal (columns are orthogonal unit vectors)

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_A^B (R_A^B)^T = I$$

$$(R_A^B)^{-1} = (R_A^B)^T$$

$$\boxed{R_B^A = R_A^B{}^T}$$

## Earth-to-body DCM

3-2-1 rotation through  $\psi, \theta, \phi$

$$\begin{aligned}\vec{p}_B &= R_1(\phi) R_2(\theta) R_3(\psi) \vec{p}_E \\ \boxed{\vec{p}_B = R_E^B \vec{p}_E}\end{aligned}$$

$$\begin{cases} c_\theta = \cos \theta \\ s_\theta = \sin \theta \end{cases}$$

$$R_E^B = \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

## Example:

Want: pilot  $\vec{p}_E$

know  $\vec{p}_E^E$ , pilot  $\vec{p}_B^B$ ,  $R_E^B$

$$\begin{aligned}\text{pilot } \vec{p}_E &= \vec{p}_E^E + \text{pilot } \vec{p}_B \\ \text{pilot } \vec{p}_E &= \vec{p}_E^E + \text{pilot } \vec{p}_B \times \boxed{(R_E^B)^T}\end{aligned}$$

# Kinematics

Kinematics: "Geometry of motion" (no forces)

Dynamics/Kinetics: Effects of forces and moments on an object

## Vector derivatives

$\dot{\vec{p}}$ : shorthand  $\dot{\vec{p}}^E$

$\frac{d}{dt} \vec{p} \equiv$  time rate of change of  $\vec{p}$  ← "Vector derivative"

$$\vec{v}^E \equiv \frac{d}{dt} \vec{p}$$

$\dot{\vec{p}}_B \equiv$  time rate of change of elements of  $\vec{p}_B$   
"coordinates"

if  $\vec{p}_B = \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix}$  then  $\dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix}$

Question

$$\vec{v}_B^E = \begin{bmatrix} u^E \\ v^E \\ w^E \end{bmatrix} \quad ? \quad \dot{\vec{p}}_B = \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \end{bmatrix} \leftarrow$$

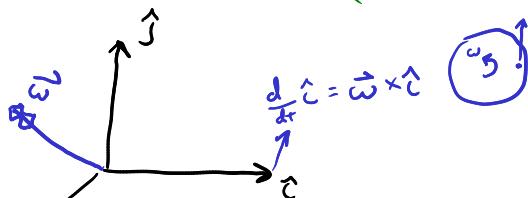
Not always true

$$\vec{p}_B = x_B \hat{i}_B + y_B \hat{j}_B + z_B \hat{k}_B$$

$$\rightarrow \dot{\vec{p}}_B = \dot{x}_B \hat{i}_B + \dot{y}_B \hat{j}_B + \dot{z}_B \hat{k}_B$$

$$\left( \frac{d}{dt} \vec{p} \right)_B = \dot{x}_B \hat{i}_B + \frac{x_B \frac{d}{dt} \hat{i}_B}{\cancel{+}} + \dot{y}_B \hat{j}_B + \frac{y_B \frac{d}{dt} \hat{j}_B}{\cancel{+}} + \dot{z}_B \hat{k}_B + \frac{z_B \frac{d}{dt} \hat{k}_B}{\cancel{+}}$$

what is  $\left( \frac{d}{dt} \hat{i} \right)_B$



$$\vec{\omega}_B = \vec{\omega}_B^E = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\text{Green terms} = x_B (\vec{\omega}_B \times \hat{i}_B) + y_B (\vec{\omega}_B \times \hat{j}_B) + z_B (\vec{\omega}_B \times \hat{k}_B) \\ = \vec{\omega}_B \times \vec{p}_B$$

$$\boxed{\left( \frac{d}{dt} \vec{p} \right)_B = \dot{\vec{p}}_B + \vec{\omega}_B \times \vec{p}_B} = \dot{\vec{p}}_B + \tilde{\omega}_B \vec{p}_B$$

$$\tilde{\omega}_B = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

"Kinematic Transport Theorem"

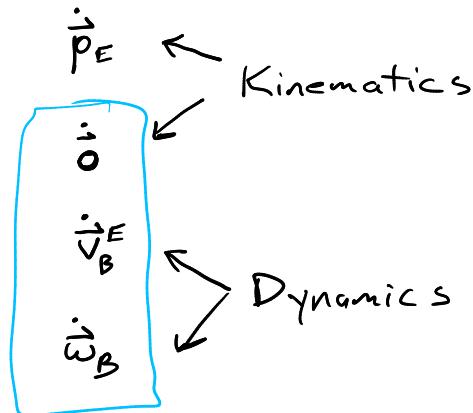
$$\left( \frac{d}{dt} \vec{p} \right)_E = \dot{\vec{p}}_E + \vec{\omega}_E \times \vec{p}_E = \dot{\vec{p}}_E$$

# Aircraft Equations of Motion (EOM)

$$\dot{\vec{x}} = f(+, \vec{x}, \vec{u})$$

$$\begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{bmatrix} \quad \left\{ \begin{array}{l} \vec{p}_E = \vec{p}_E^E \\ \vec{o} \text{ "pseudo-vector" array of numbers} \\ \vec{v}_B^E \\ \vec{\omega}_B^E \end{array} \right.$$

Need



## Translational Kinematics

$$\dot{\vec{p}}_E^E = \frac{d}{dt} \vec{p}_E - \vec{\omega}_E^E \times \vec{p}_E^E = \frac{d}{dt} \vec{p}_E = \vec{v}_E^E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{p}}_E = (R_E^B)^T \vec{v}_B$$

## Rotational Kinematics

want:  $\dot{\vec{o}} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$  have:  $\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

Diagram illustrating the aircraft coordinate frames and rotation angles:

- Body frame  $B$ :  $\hat{i}^B, \hat{j}^B, \hat{k}^B$
- Earth frame  $E$ :  $\hat{i}^E, \hat{j}^E, \hat{k}^E$
- Roll ( $\phi$ ):  $\hat{i}^B \rightarrow \hat{i}^E$
- Pitch ( $\theta$ ):  $\hat{j}^B \rightarrow \hat{j}^E$
- Yaw ( $\psi$ ):  $\hat{k}^B \rightarrow \hat{k}^E$
- Transformation:  $R_E^B = R_1(\phi)R_2(\theta)R_3(\psi)$
- Angular velocity:  $\vec{\omega}_B = \dot{\psi} \hat{k}^B + \dot{\theta} \hat{j}^B + \dot{\phi} \hat{i}^B$
- Angular velocity components:  $\vec{\omega}_B = \dot{\psi} R_E^B \hat{k}_E^E + \dot{\theta} R_E^B \hat{j}_E^E + \dot{\phi} R_E^B \hat{i}_E^E$
- Angular velocity matrix:  $\vec{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$
- Invert:  $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix}}_{\text{invert}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & s\phi + \theta & c\phi + \theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$\boxed{\dot{\vec{\omega}} = T \vec{\omega}_B}$

Eqn. 4.4, 7  
in book

$T$  = "attitude influence matrix"

# Dynamics

$$\vec{f} \quad \vec{G}$$

## Translational Dynamics

Newton's 2nd Law

$$\vec{f} = m \vec{a}$$

$$\vec{f} = m \frac{d}{dt} \vec{v}^E$$

want  $\dot{\vec{v}}_B^E$

$$\left( \frac{d}{dt} \vec{v}_B^E \right) = \dot{\vec{v}}_B^E + \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{v}}_B^E = \frac{d}{dt} \vec{v}_B^E - \vec{\omega}_B \times \vec{v}_B^E$$

$$\boxed{\dot{\vec{v}}_B^E = \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E}$$

$\underbrace{\vec{\omega}_B \vec{v}_B^E}$

## Rotational Dynamics

"Euler's 2nd Law"

$$\frac{d}{dt} \vec{h} = \vec{G}$$

↑                   ↑  
angular              moment  
momentum         

$$\vec{h} = I \vec{\omega}$$

$$I = I_B \text{ book}$$

$$I = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

Want  $\dot{\vec{\omega}}_B$

$$\frac{d}{dt} \vec{h}_B = \dot{\vec{h}}_B + \vec{\omega}_B \times \vec{h}_B = \vec{G}_B$$

$$I \dot{\vec{\omega}}_B + \vec{\omega}_B \times I \vec{\omega}_B = \vec{G}_B$$

$$\boxed{\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \dot{\vec{\omega}}_B = I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B)}$$

# A/C EOM

$$\dot{\vec{X}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B \\ \dot{\vec{\omega}}_B \end{bmatrix} = \begin{bmatrix} (R_E^B)^T \vec{v}_B^E \\ T \vec{\omega}_B \\ \vec{f}_B - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

## Quadrotors

$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u_E \\ v_E \\ w_E \\ p \\ q \\ r \end{bmatrix} \quad \left\{ \begin{array}{l} \vec{p}_E \\ \vec{o} \\ \vec{v}_B^E \\ \vec{\omega}_B \end{array} \right\}$$

A/C EOM

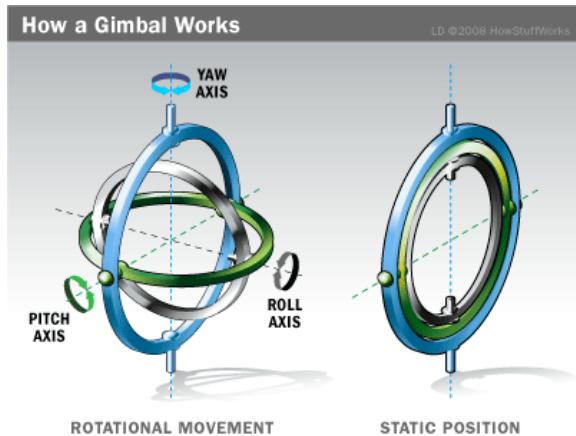
$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} (R_B^E)^T \vec{v}_B \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1}(\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & s\phi + \theta & c\phi + \theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}$$

From Quiz MC2

$$T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{\omega}_B = \begin{bmatrix} 0 \\ 10^\circ/s \\ 0 \end{bmatrix}$$

$$\vec{o} = \begin{bmatrix} 0 \\ 0 \\ 10^\circ/s \end{bmatrix}$$



## Homework PI Monospinner

$\vec{F}$ : sum of all forces acting on A/C

$$\vec{F} = \vec{A} + g\vec{f}$$

aerodynamic forces      gravity

$\vec{G}$ : sum of all moments acting about the G.G. of A/C

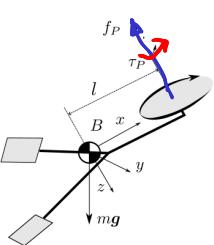
### Multirotor Case

$$\vec{f} = \vec{d} + \vec{c} + \vec{g}$$

drag      A      control

$$\vec{G} = \vec{d} + \vec{c}$$

drag      control



### Monospinner Assignment

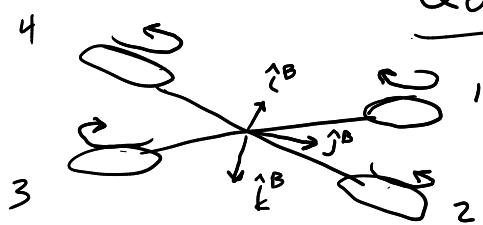
$$\vec{d} = 0 \quad (\text{by assumption})$$

$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ -f_P \end{bmatrix}$$

$$\vec{d} = 0 \quad (\text{by assumption})$$

$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ -\tau_P \end{bmatrix} + \vec{P}_B \times \vec{f}_B$$

### Quadrotor Case



$$\vec{c}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix}$$

$$\vec{c}_G_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix}$$

Since the quadrotor is symmetric about the  $\hat{x}_B - \hat{z}_B$  and  $\hat{y}_B - \hat{z}_B$  planes

$$I_{xy} = I_{yz} = I_{xz} = 0$$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

$$I^{-1} = \begin{bmatrix} 1/I_x & 0 & 0 \\ 0 & 1/I_y & 0 \\ 0 & 0 & 1/I_z \end{bmatrix}$$

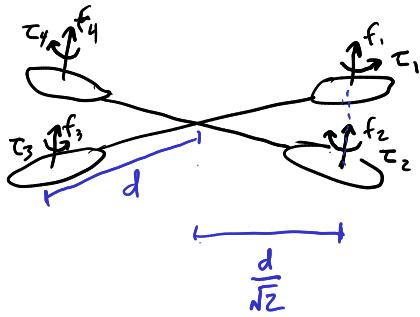
$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \underbrace{\begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix}}_{\vec{d}_B} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

# Control Forces and Moments



$${}^C \vec{f}_B = \begin{bmatrix} 0 \\ 0 \\ Z_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -f_1 - f_2 - f_3 - f_4 \end{bmatrix}$$

$${}^C \vec{G}_B = \begin{bmatrix} L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} \frac{d}{N\sqrt{2}} (-f_1 - f_2 + f_3 + f_4) \\ \frac{d}{N\sqrt{2}} (f_1 - f_2 - f_3 + f_4) \\ -\tau_1 + \tau_2 - \tau_3 + \tau_4 \end{bmatrix}$$

$w_i$

$$f_i = k_f C_L(w_i)^2$$

$$\tau_i = k_\tau C_D(w_i)^2$$

$$\boxed{\tau_i = k_m f_i}$$

$$k_m = \frac{k_\tau C_D}{k_f C_L}$$

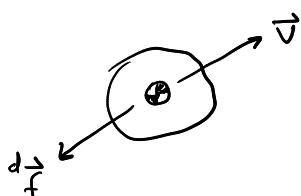
control forces + Moments  $\iff$  individual rotor forces

$$\begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -\frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} \\ \frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & -\frac{d}{N\sqrt{2}} & \frac{d}{N\sqrt{2}} \\ -k_m & k_m & -k_m & k_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

$\downarrow$  invert

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} Z_c \\ L_c \\ M_c \\ N_c \end{bmatrix}$$

## Drag Forces

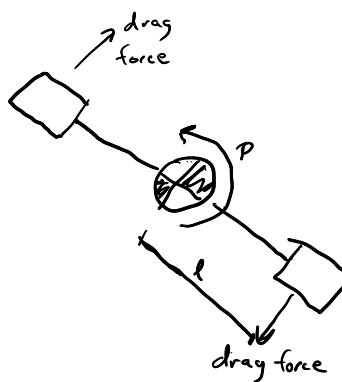


$$d_f = -D \frac{\vec{v}}{V_a} \quad V_a = |\vec{v}|$$

$$D = \frac{1}{2} \rho V_a^2 C_D A = \nu V_a^2$$

$${}^D \vec{f}_B = \begin{bmatrix} X_d \\ Y_d \\ Z_d \end{bmatrix} = -\nu V_a^2 \frac{\vec{v}_B}{V_a} = -\nu V_a \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

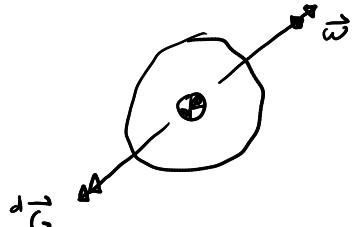
## Drag Moments



$$\begin{aligned} L_{drag} &= -2l f_{drag} \\ &= -2l \frac{1}{2} \rho C_D A (l_p)^2 \underbrace{\mu}_{\mu} \text{ sign}(p) \\ &= -\mu p l p l \end{aligned}$$

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

General case



$${}^D \vec{G}_B = \begin{bmatrix} L_d \\ M_d \\ N_d \end{bmatrix} = -\mu |\vec{\omega}| \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

# Quadrotor Linear Model

$$\dot{\vec{x}} = f(\vec{x}, \vec{u})$$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{\vec{p}}_E \\ \dot{\vec{o}} \\ \dot{\vec{v}}_B^E \\ \dot{\vec{\omega}}_B^E \end{bmatrix} = \begin{bmatrix} (R_E^B)^T \vec{v}_B^E \\ T \vec{\omega}_B \\ \frac{\vec{f}_B}{m} - \vec{\omega}_B \times \vec{v}_B^E \\ I^{-1} (\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B) \end{bmatrix}$$

- differential
  - first order
  - ordinary (not partial)
  - coupled
  - nonlinear
- simulate

## Linearization

$$\vec{x} \approx \vec{x}_0 + \Delta \vec{x}$$

$$\dot{\vec{x}} \approx \Delta \dot{\vec{x}}$$

$$\vec{u} \approx \vec{u}_0 + \Delta \vec{u}$$

$\overset{\text{trim}}{\text{condition}}$

For a quadrotor, "Hover" trim condition  
dot means any value

$$\vec{x}_0 = \begin{bmatrix} x_{E,0} \\ y_{E,0} \\ z_{E,0} \\ \phi_0 \\ \theta_0 \\ \psi_0 \\ u_E^0 \\ v_E^0 \\ w_E^0 \\ p_0 \\ q_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} z_{c,0} \\ l_{c,0} \\ m_{c,0} \\ n_{c,0} \end{bmatrix} = \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Want Linear EOM

$$\Delta \dot{\vec{x}} = A \Delta \vec{x} + B \Delta \vec{u}$$

Approach: Use first-order Taylor Series approx

$$y = f(x, u)$$

$$y_0 + \Delta y = f(x_0 + \Delta x, u_0 + \Delta u)$$

$$\cancel{y_0 + \Delta y \approx f(x_0, u_0) + \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u + \text{H.O.T.}}$$

ignore

## 2 Approaches for finding Taylor series

1. Calculate partial derivatives (always work)

2. Substitute  $x = x_0 + \Delta x$  use small number approximations (sometimes faster)

$$\begin{aligned} \sin(\Delta x) &\approx \Delta x \\ \cos(\Delta x) &\approx 1 \\ \Delta x \Delta u &\approx 0 \end{aligned}$$

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\psi & \sin\theta \cos\psi - \cos\phi \sin\psi & \cos\phi \sin\theta \cos\psi + \sin\phi \sin\psi \\ \cos\theta \sin\psi & \sin\theta \cos\psi + \cos\phi \sin\psi & \cos\phi \sin\theta \sin\psi - \sin\phi \cos\psi \\ -\sin\theta & \sin\phi \cos\theta & \cos\phi \cos\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X_d \\ Y_d \\ Z_d \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ Z_c \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{I_y - I_z}{I_x} qr \\ \frac{I_z - I_x}{I_y} pr \\ \frac{I_x - I_y}{I_z} pq \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_d \\ \frac{1}{I_y} M_d \\ \frac{1}{I_z} N_d \end{pmatrix} + \begin{pmatrix} \frac{1}{I_x} L_c \\ \frac{1}{I_y} M_c \\ \frac{1}{I_z} N_c \end{pmatrix}$$

Example  $f(\phi, q, r)$

$$\dot{\theta} = \cos(\phi)q - \sin(\phi)r$$

Approach 1:  $\dot{\theta}_0 + \Delta\dot{\theta} = \cos(\phi_0)q_0 - \sin(\phi_0)r_0 + \frac{\partial f}{\partial \phi}\Big|_0 \Delta\phi + \frac{\partial f}{\partial q}\Big|_0 \Delta q + \frac{\partial f}{\partial r}\Big|_0 \Delta r$

$$= \left( \frac{\partial \cos(\phi)}{\partial \phi} q_0 - \frac{\partial \sin(\phi)}{\partial \phi} r_0 \right) \Delta\phi + \cos(\phi_0) \Delta q - \sin(\phi_0) \Delta r$$

$\Delta\dot{\theta} = \Delta q$

Approach 2:  $\dot{\theta}_0 + \Delta\dot{\theta} = \cos(\phi_0 + \Delta\phi)(q_0 + \Delta q) - \sin(\phi_0 + \Delta\phi)(r_0 + \Delta r)$

$$= 1 \Delta q - \Delta\phi \Delta r$$

$\Delta\dot{\theta} = \Delta q$

Harder

$$\dot{w}^E = \underbrace{qu^E - pv^E + g \cos\theta \cos\phi}_{f(\phi, \theta, u, v, w, p, q, Z_c)} + \frac{1}{m} Z_d + \frac{1}{m} Z_c$$

Assume no wind  
 $u=u^E, v=v^E, w=w^E$

$$\Delta\dot{w} = \frac{\partial f}{\partial \phi}\Big|_0 \Delta\phi + \underbrace{\frac{\partial f}{\partial \theta}\Big|_0 \frac{\partial f}{\partial u}\Big|_0 \Delta u + \frac{\partial f}{\partial v}\Big|_0 \Delta v + \frac{\partial f}{\partial w}\Big|_0 \Delta w}_{0} + \frac{\partial f}{\partial p}\Big|_0 \Delta p + \frac{\partial f}{\partial q}\Big|_0 \Delta q + \frac{\partial f}{\partial Z_c}\Big|_0 \Delta Z_c$$

$$Z_d = -\sqrt{w \sqrt{u^2 + v^2 + w^2}}$$

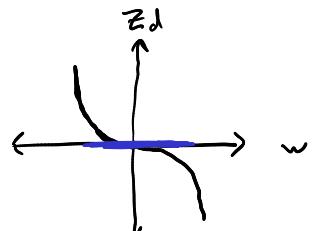
Simple case: assume  $u, v = 0$

$$Z_d = -\sqrt{w} |w| = -\sqrt{w^2} \text{sign}(w)$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = -\sqrt{w} \left( 2w \text{sign}(w) + w^2 \frac{\partial}{\partial w} \text{sign}(w) \right)\Big|_0$$

$$\frac{\partial Z_d}{\partial w}\Big|_0 = 0$$

No drag force in Linear model!



# Simple "Non hover" example

EOM  $\dot{u} = \frac{e_f}{m} - \frac{\gamma u |u|}{m}$

Trim condition

$$u_0 = 30 \text{ m/s steady}$$

$$\dot{u}_0 = 0 = \frac{e_f}{m} - \frac{\gamma u_0 |u_0|}{m}$$

$$\Delta \dot{u} = \frac{\partial g}{\partial u} \Big|_0 \Delta u = \frac{\partial (-\gamma u^2 \text{sign}(u))}{\partial u} \Big|_0 \Delta u = -\gamma (2u \text{sign}(u) + u^2 \frac{\partial \text{sign}(u)}{\partial u}) \Big|_0 \Delta u = -\gamma 2u_0 \Delta u$$

$$\boxed{\Delta \dot{u} = -\gamma 2u_0 \Delta u}$$

$$\frac{\partial Z_d}{\partial w} \Big|_0 = -\gamma \left( \sqrt{u^2 + v^2 + w^2} + w \frac{1}{2} \frac{1}{\sqrt{u^2 + v^2 + w^2}} 2w \right) \Big|_0 = -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}} \Big|_0$$

looks like  $\frac{\partial}{\partial}$

$$\lim_{u,v,w \rightarrow 0} -\gamma \frac{u^2 + v^2 + 2w^2}{\sqrt{u^2 + v^2 + w^2}}$$

$$\lim_{r \rightarrow 0} \frac{r^2 \cdot \text{stuff}}{r} = 0$$

can solve with spherical coordinates

$$u = r \cos \theta \sin \phi$$

$$v = r \sin \theta \sin \phi$$

$$w = r \cos \phi$$

$$\frac{\partial Z_d}{\partial u} \Big|_0 = 0$$

$$\frac{\partial Z_d}{\partial v} \Big|_0 = 0$$

$$\boxed{\Delta \dot{w}^E = \frac{\Delta Z_d}{m}}$$

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{y}_E \\ \Delta \dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \Delta \dot{\phi} \\ \Delta \dot{\theta} \\ \Delta \dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

Drag force would show up here

$$\rightarrow \begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta \theta \\ \Delta \phi \\ 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x} \Delta L_c \\ \frac{1}{I_y} \Delta M_c \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{y}_E \\ \Delta \dot{z}_E \end{pmatrix} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{\phi} \\ \Delta \dot{\theta} \\ \Delta \dot{\psi} \end{pmatrix} = \begin{pmatrix} \Delta p \\ \Delta q \\ \Delta r \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{u} \\ \Delta \dot{v} \\ \Delta \dot{w} \end{pmatrix} = g \begin{pmatrix} -\Delta \theta \\ \Delta \phi \\ 0 \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ \Delta Z_c \end{pmatrix}$$

$$\begin{pmatrix} \Delta \dot{p} \\ \Delta \dot{q} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{I_x} \Delta L_c \\ -\frac{1}{I_y} \Delta M_c \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

Longitudinal

$$\begin{pmatrix} \Delta \dot{x}_E \\ \Delta \dot{u} \\ \Delta \dot{\theta} \\ \Delta \dot{q} \end{pmatrix} = \begin{pmatrix} \Delta u \\ -g \Delta \theta \\ \Delta q \\ \frac{1}{I_u} \Delta M_c \end{pmatrix}$$

Lateral

$$\begin{pmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \end{pmatrix} = \begin{pmatrix} \Delta v \\ q \Delta \phi \\ \Delta p \\ \frac{1}{I_r} \Delta L_c \end{pmatrix}$$

Vertical

$$\begin{pmatrix} \Delta \dot{z}_E \\ \Delta \dot{w} \end{pmatrix} = \begin{pmatrix} \Delta w \\ \frac{1}{m} \Delta Z_c \end{pmatrix}$$

Spin

$$\begin{pmatrix} \Delta \dot{\psi} \\ \Delta \dot{r} \end{pmatrix} = \begin{pmatrix} \Delta r \\ \frac{1}{I_z} \Delta N_c \end{pmatrix}$$

State space model

$$\begin{aligned} \dot{\vec{x}} &= A \vec{x} + B \vec{u} \\ \vec{y} &= C \vec{x} + D \vec{u} \end{aligned}$$

$$\ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = \omega_n^2 u$$

$$\begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \frac{1}{I_x} \Delta L_c \end{bmatrix}$$

$$\begin{aligned} \Delta p &= \Delta \dot{\phi} \\ \Delta \dot{p} &= \Delta \ddot{\phi} \end{aligned}$$

$$\Delta \ddot{\phi} = \frac{1}{I_x} \Delta L_c$$

Solutions?  $\Delta \phi(+)$

Assume  $\Delta L_c$  is constant

$$\Delta \dot{\phi}(+) = \Delta \dot{\phi}_o + \int_{\tau=0}^{+} \Delta \dot{\phi}(t) dt = \Delta \dot{\phi}_o + \frac{1}{I_x} \Delta L_c +$$

$$\Delta \phi(+) = \Delta \phi_o + \int_{\tau=0}^{+} \Delta \dot{\phi}(+) dt = \Delta \phi_o + \Delta \dot{\phi}_o t + \frac{1}{2} \frac{1}{I_x} \Delta L_c t^2$$

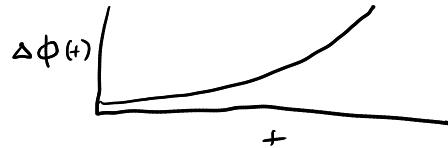
Exactly at hover  $\Delta \phi_o = \Delta \dot{\phi}_o = \Delta L_c = 0$

$$\Delta \phi(+) = 0$$

If  $\Delta \dot{\phi}_o > 0$ ,  $\Delta L_c = 0$

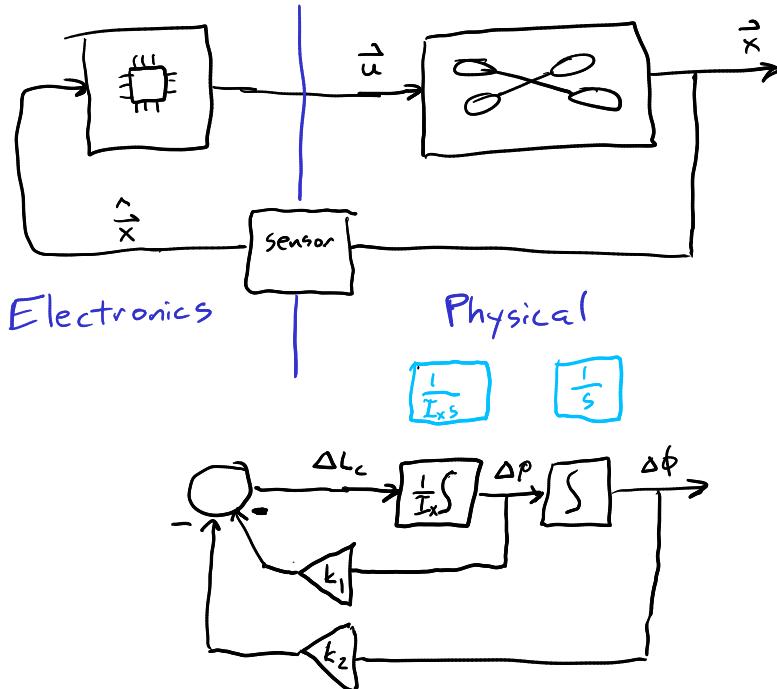


If  $\Delta L_c > 0$



If initial conditions are nonzero (always in real life), vehicle will crash

# Solution: Feedback Control



$$\Delta L_c = -k_1 \Delta p - k_2 \Delta \phi$$

$k_1$  Deriv. gain       $k_2$  prop. gain

$$\dot{\Delta \phi} = \frac{1}{I_x} \Delta L_c$$

$$= \frac{1}{I_x} (-k_1 \Delta p - k_2 \Delta \phi)$$

$$\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0$$

$$\ddot{\Delta \phi} + \frac{k_1}{I_x} \dot{\Delta \phi} + \frac{k_2}{I_x} \Delta \phi = 0$$

$$\ddot{\Delta \phi} + 2\zeta\omega_n \dot{\Delta \phi} + \omega_n^2 \Delta \phi = 0$$

If  $\lambda$  are real and distinct

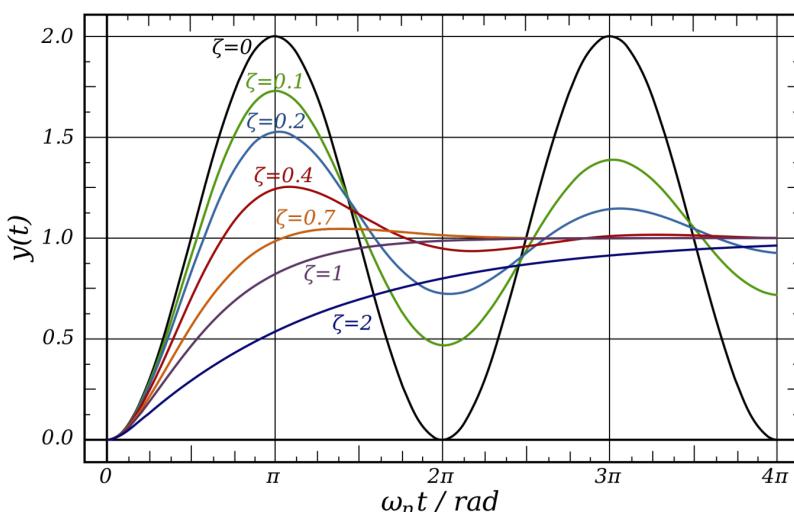
$$\Delta \phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

If  $\lambda$  are complex

$$\zeta = \frac{k_1}{2\sqrt{k_2 I_x}} \quad \omega_n = \sqrt{\frac{k_2}{I_x}}$$

$$\lambda = -\zeta\omega_n \pm i\omega_n \sqrt{1-\zeta^2}$$

$$\Delta \phi(t) = e^{-\zeta\omega_n t} (C_3 \sin(\omega_d t) + C_4 \cos(\omega_d t))$$



$$\begin{bmatrix} \dot{\Delta\phi} \\ \dot{\Delta p} \end{bmatrix} = \begin{bmatrix} \Delta p \\ \frac{1}{I_x} \Delta L_c \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\vec{x}} \\ \vec{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta L_c \end{bmatrix}$$

$$\begin{array}{c} \dot{\vec{y}} \\ \vec{y} \end{array} = \begin{array}{c} C \\ \vec{x} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Delta L_c \end{bmatrix} \quad D \quad \vec{u}$$

$$\vec{u} = \begin{bmatrix} \Delta L_c \end{bmatrix} = -K \vec{x} = -[k_2 \ k_1] \begin{bmatrix} \Delta\phi \\ \Delta p \end{bmatrix}$$

$$\dot{\vec{x}} = \underbrace{A \vec{x} - BK \vec{x}}_{A^{cl}} = \underbrace{(A - BK)}_{A^{cl}} \vec{x}$$

$$A^{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{I_x} \end{bmatrix} [k_2 \ k_1] = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} \end{bmatrix}$$

$$\dot{\vec{x}} = A^{cl} \vec{x} \quad \text{what are solutions of } \vec{x} = A^{cl} \vec{x}?$$

scalar case

$$\dot{x} = ax \Rightarrow x(t) = x(0)e^{at}$$

analogously

$$\dot{\vec{x}} = A \vec{x} \Rightarrow \vec{x}(+) = e^{A+} \vec{x}(0)$$

$$e^{At} = I^+ + A + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 \dots \quad (\text{Taylor Series})$$

## Modal Analysis

Eigenvalues and Eigen vectors

$$A \vec{v}_i = \lambda_i \vec{v}_i$$

Suppose that  $\vec{x}_0 = \vec{v}_i$

$$\begin{aligned} \vec{x}(+) &= (I^+ + A + \frac{t^2}{2!} A^2 + \dots) \vec{v}_i \\ &= \vec{v}_i + \lambda_i \vec{v}_i + \frac{t^2}{2!} \lambda_i^2 \vec{v}_i \dots \end{aligned}$$

$$\vec{x}(+) = \vec{v}_i e^{\lambda_i t}$$

$$\text{If } \vec{x}_0 = \sum_i q_i \vec{v}_i$$

$$\text{then } \vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$$\vec{q} = V^{-1} \vec{x}$$

matrix of eigenvector columns

# Finding Eigenvalues

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$(A - \lambda_i I) \vec{v}_i = 0$$

only has nontrivial solutions if  
 $|A - \lambda_i I| = 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$$

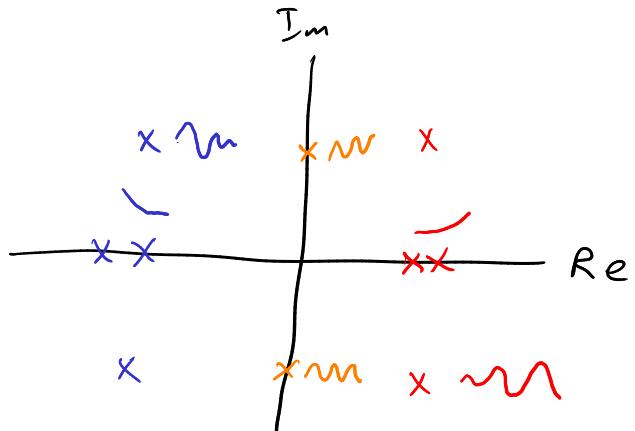
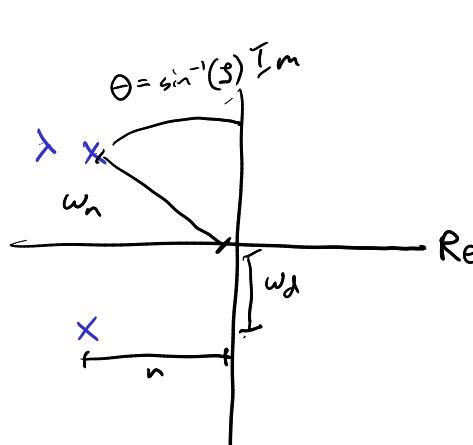
solve with quadratic formula

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -\frac{k_2}{I_x} & \frac{k_1}{I_x} - \lambda \end{vmatrix} = \lambda^2 + \frac{k_1}{I_x}\lambda + \frac{k_2}{I_x} = 0$$

$$\lambda = -\frac{k_1}{I_x} \pm \sqrt{\frac{k_1^2}{4I_x} - \frac{k_2}{I_x}}$$

$$= n \pm i\omega_d$$

$$= -\zeta\omega_n \pm i\omega_n \sqrt{1 - \zeta^2}$$



# Solutions to linear ODEs

$$\dot{x} = ax \Rightarrow x(0)e^{at}$$

$$\ddot{x} + \frac{2\zeta\omega_n}{a}\dot{x} + \frac{\omega_n^2}{b}x = 0 \Rightarrow \text{characteristic eqn} \quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

quad. form

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad \text{if } \lambda \text{ real and distinct}$$

$$x(t) = e^{-\zeta\omega_n t} (C_3 \sin(\omega_d t) + C_4 \cos(\omega_d t))$$

$$\dot{\vec{x}} = A\vec{x} \Rightarrow e^{At}\vec{x}(0)$$

$$\sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$\vec{v}_i$  are eigenvectors

$\lambda_i$  are eigenvalues

$$\text{if } \vec{x}(0) = a\vec{v}_1 + b\vec{v}_2$$

$$\vec{x}(t) = a\vec{v}_1 e^{\lambda_1 t} + b\vec{v}_2 e^{\lambda_2 t}$$

$\vec{v}_i$  Eigenvectors: "shape" of the mode

which state variables are actively changing

$\lambda_i$  Eigenvalues : "speed" of the mode

how fast does it oscillate, decay, or diverge

## Linear Control Design Process

1. Derive EOM

2. Linearize and Separate EOM

3. Design Control Architecture

4. Choose Gain Values

5. Testing in Linear Simulation

6. Test in Nonlinear Sim

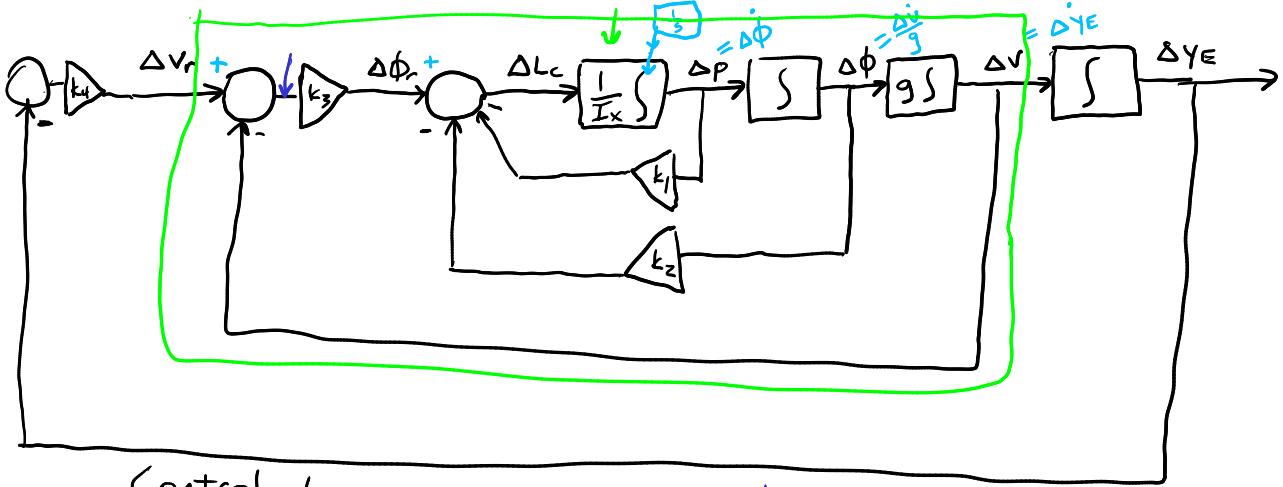
1. PID tuning  
2. Pole Assignment  
3. Root Locus

4. Optimal Control (LQR)

$$\begin{array}{c} \rightarrow \\ \dot{\vec{x}} \end{array} = \begin{bmatrix} \Delta \dot{y}_E \\ \Delta \dot{v} \\ \Delta \dot{\phi} \\ \Delta \dot{p} \\ \hline \end{bmatrix} = \begin{bmatrix} \Delta v \\ g \Delta \phi \\ \Delta p \\ \hline \frac{1}{I_x} \Delta L_C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta y_E \\ \Delta v \\ \Delta \phi \\ \Delta p \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline \end{bmatrix} \begin{bmatrix} \Delta L_C \end{bmatrix}$$

$$\vec{y} = \begin{matrix} \text{"} \\ \text{I} \\ \text{"} \\ \text{D} \\ \text{"} \end{matrix} \vec{x} + \begin{matrix} \text{"} \\ \text{I} \\ \text{"} \\ \text{D} \\ \text{"} \end{matrix} \vec{u}$$

$$I_x = 7 \times 10^{-5} \text{ kg m}^2$$



Control Law

$$\rightarrow \Delta L_c = -k_1 \Delta P - k_2 \Delta \phi + k_3 (-k_4 \Delta y_E - \Delta V)$$

$$= -k_1 \Delta P - k_2 \Delta \phi - k_3 k_4 \Delta y_E - k_3 \Delta V$$

$$[\Delta L_c] = \vec{u} = -K \vec{x} = -\underbrace{[k_3 k_4 | k_3 | k_2 | k_1]}_{\text{Matrix}} \begin{bmatrix} \Delta y_E \\ \Delta V \\ \Delta \phi \\ \Delta P \end{bmatrix}$$

$$A^{cl} = A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_3 k_4}{I_x} & \frac{-k_3}{I_x} & \frac{-k_2}{I_x} & \frac{-k_1}{I_x} \end{bmatrix} \quad \dot{\vec{x}} = A^{cl} \vec{x}$$

Choosing Gains

1. Choose  $k_1$  and  $k_2$  with the "pole placement" strategy

Pole placement

- Decide where we want eigenvalues (poles) to be
- Solve for  $k_1$  and  $k_2$

For 2nd order system

$$\lambda = -j\omega_n \pm i\omega_n \sqrt{1 - \zeta^2}$$

$$|A^{cl} - \lambda I| = 0$$

$$\lambda^2 + \frac{k_1}{I_x} \lambda + \frac{k_2}{I_x} = 0$$

$$\begin{bmatrix} \Delta \dot{\phi} \\ \Delta \dot{P} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{I_x} & -\frac{k_1}{I_x} \end{bmatrix} \begin{bmatrix} \Delta \phi \\ \Delta P \end{bmatrix}$$

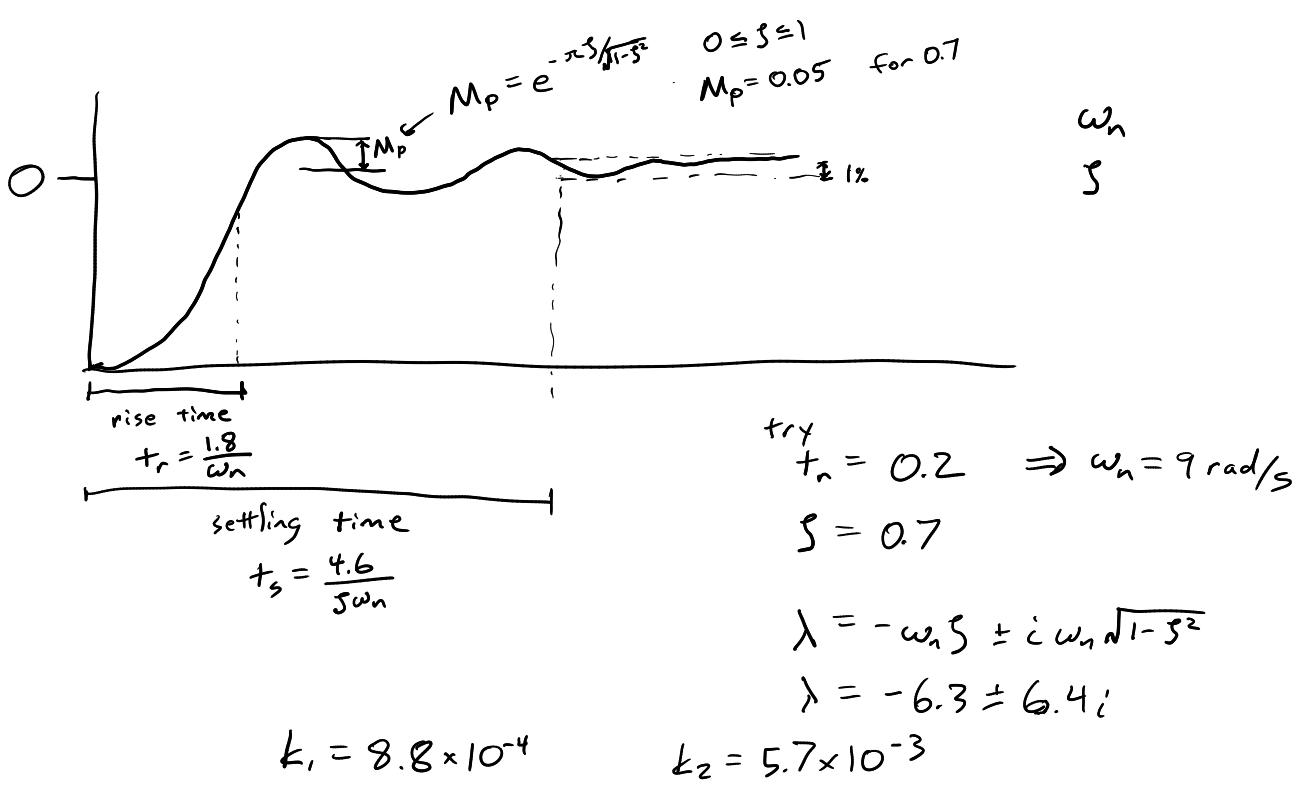
$$A^{cl}$$

analogous to

$$\lambda^2 + 2j\omega_n \lambda + \omega_n^2 = 0$$

$$k_1 = 2j\omega_n I_x$$

$$k_2 = \omega_n^2 I_x$$



$$\dot{\vec{x}} = A^{cl} \vec{x} \leftarrow \begin{bmatrix} \Delta \phi \\ \Delta p \end{bmatrix}$$

solutions look like

$$\ddot{\Delta \phi} + \frac{\Delta \phi}{2\zeta\omega_n} + \frac{1}{\omega_n^2} = 0$$

solutions look like

$$\phi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

( $C, \lambda$  might be complex  $\Rightarrow \sin/\cos$ )

$\lambda_i$  are solutions to

$$\lambda^2 + [2\zeta\omega_n]\lambda + [\omega_n^2] = 0$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i$$

$$\vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t} \quad \Longleftrightarrow$$

$\lambda_i$  are solutions to

$$|A^{cl} - \lambda I| = 0$$

if  $A^{cl}$  is  $2 \times 2$

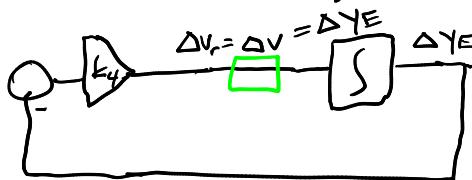
$$\lambda^2 + \frac{k_1}{I_x} \lambda + \frac{k_2}{I_x} = 0$$

$$\frac{k_1}{I_x}$$

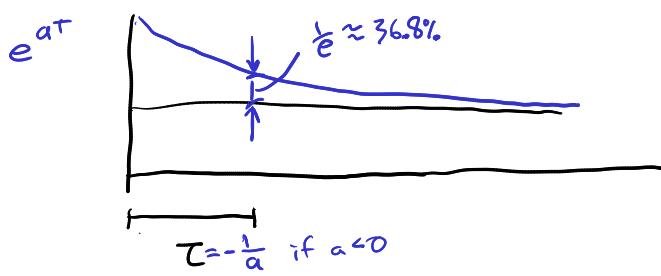
$$\frac{k_2}{I_x}$$

$$k_1 = 2\zeta\omega_n I_x \quad k_2 = \omega_n^2 I_x$$

Choose  $k_4$  with pole placement



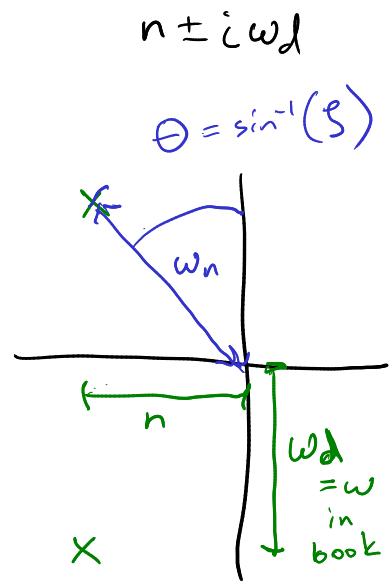
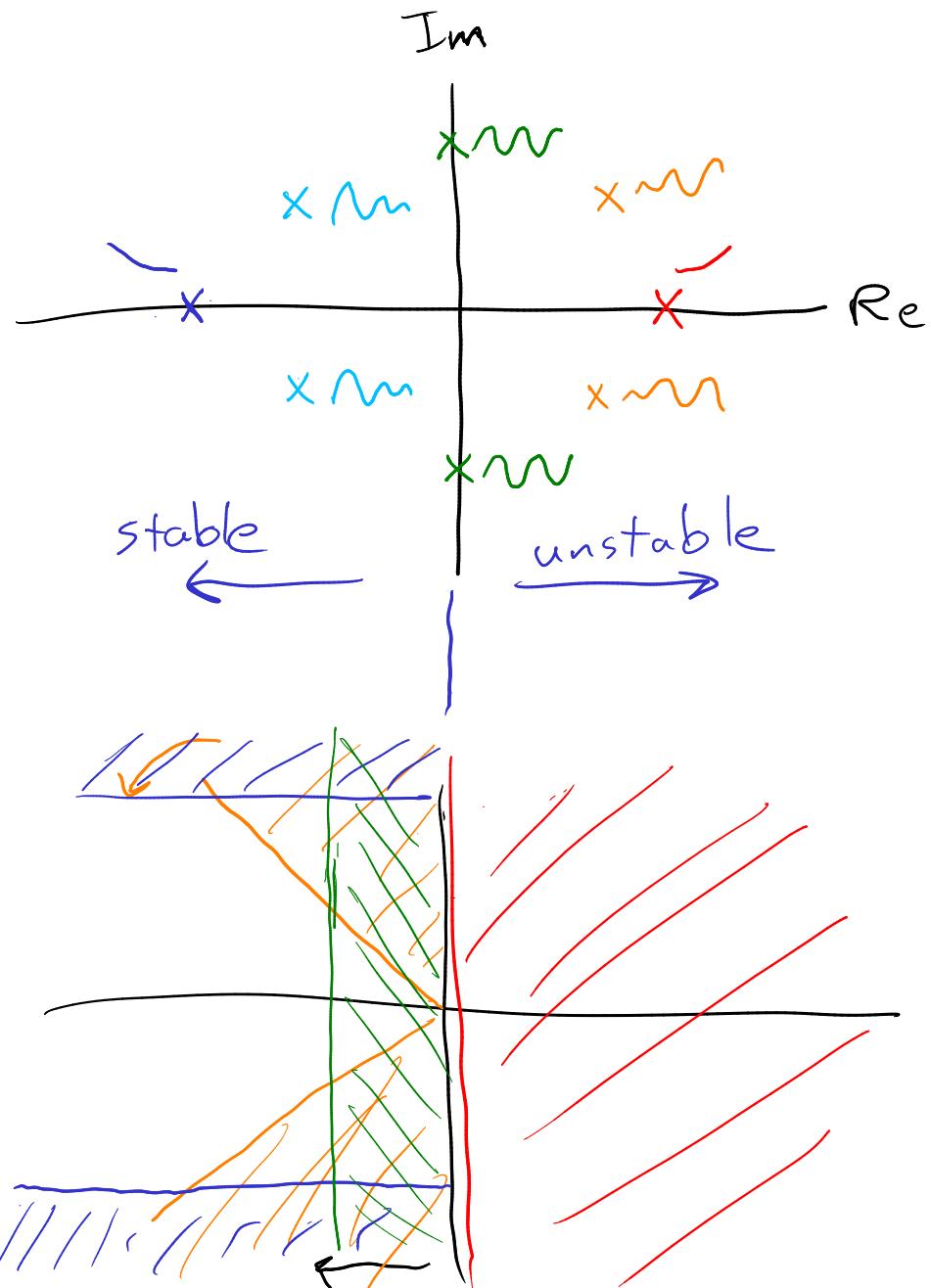
$$\Delta \dot{y}_E = -k_4 \Delta y_E \quad \Longrightarrow \quad \Delta y_E(+) = \Delta y_E(0) e^{-k_4 t}$$



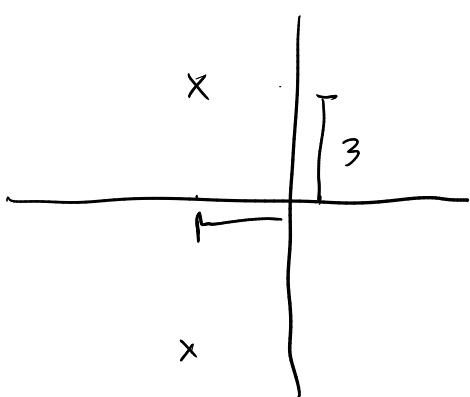
$$T_y = \frac{1}{k_4}$$

inner loop  $(\phi, p)$  has settling time of  $\frac{4.6}{\zeta\omega_n} \approx 0.7$   
choose  $T_y$  10x larger  $\approx 7$  sec.  $\approx 5$

$$T_y = 5 \Rightarrow k_4 = 0.2$$



$$\omega_d = \omega_n \sqrt{1 - \beta^2}$$



# Conventional A/C Dynamics

## Longitudinal

Altitude  
Speed  
Pitch

## Lateral/Directional

Roll  
Yaw  
Sideslip

### Long. Forces and Moments

Lift

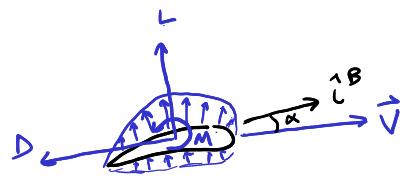
$$L = \frac{1}{2} \rho V_a^2 S \overset{\text{density}}{\cancel{C_L}} \overset{\text{Wing Area}}{\cancel{C_L}}$$

Drag

$$D = \frac{1}{2} \rho V_a^2 S \overset{\text{airspeed}}{\cancel{C_D}}$$

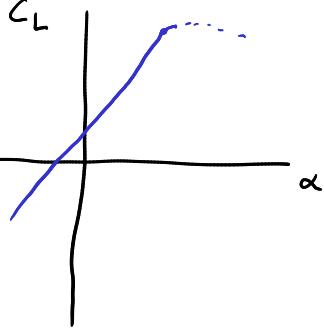
Pitch Moment

$$M = \frac{1}{2} \rho V_a^2 S \bar{z} C_m$$



Variable	Divisor	Non-dim Variable
$X, Y, Z$	$\frac{1}{2} \rho V^2 S$	$C_x, C_y, C_z$
$W$	$\frac{1}{2} \rho V^2 S$	$C_W$
$M$	$\frac{1}{2} \rho V^2 S \bar{c}$	$C_m$
$L, N$	$\frac{1}{2} \rho V^2 S \bar{b}$	$C_l, C_n$
$u, v, w$	$V$	$\hat{u}, \hat{v}, \hat{w}$
$\dot{\alpha}, q$	$2V/\bar{c}$	$\dot{\hat{\alpha}}, \hat{q}$
$\dot{\beta}, p, r$	$2V/b$	$\dot{\hat{\beta}}, \hat{p}, \hat{r}$
$m$	$\rho S \bar{c}/2$	$\mu$
$I_y$	$\rho S (\bar{c}/2)^3$	$\hat{I}_y$
$I_x, I_z, I_{xz}$	$\rho S (b/2)^3$	$\hat{I}_x, \hat{I}_z, \hat{I}_{xz}$

### Lift



$$C_L(\alpha, q, \delta_e)$$

1st order Taylor series

$$L = \frac{1}{2} \rho V_a^2 S \left( C_{L_{\text{zero}}} + \frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e} \delta_e \right)$$

### Stability Derivatives

$$C_{a,b} = \left. \frac{\partial \text{nondimensionalized } a}{\partial \text{nondimensionalized } b} \right|_{\substack{\text{condition} \\ (\text{usually trim})}}$$

- based on linear assumptions
- main tool for connecting aerodynamics + dynamics
- determined by A/C geometry
- only accurate in a linear region (e.g. small  $\alpha$ )

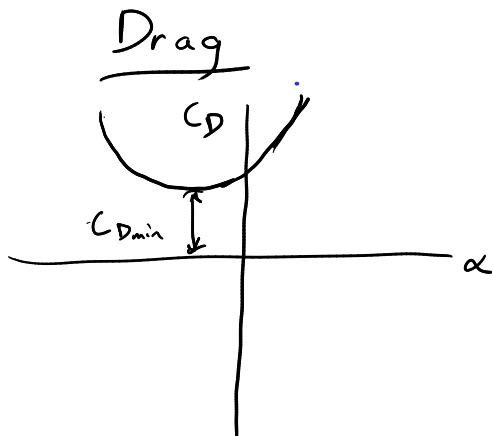


### Estimated

- Geometric Data  $\rightarrow C_{L\alpha} \approx \frac{\pi AR}{1 + \sqrt{1 + (\frac{AR}{2})^2}}$
- Wind Tunnel
- Flight Test
- CFD
- Other Aircraft

$$L = \frac{1}{2} \rho V_a^2 S (C_{L_{zero}} + C_{L\alpha} \alpha + C_{Lq} \hat{q} + C_{L\delta_e} \delta_e)$$

$\hat{q} = q \frac{C}{2V_a}$



parasitic + induced  
 $\propto C_L^2$

$$C_D = C_{D_{min}} + K (C_L(\alpha, q, \delta_e) - C_{L_{min}})^2$$

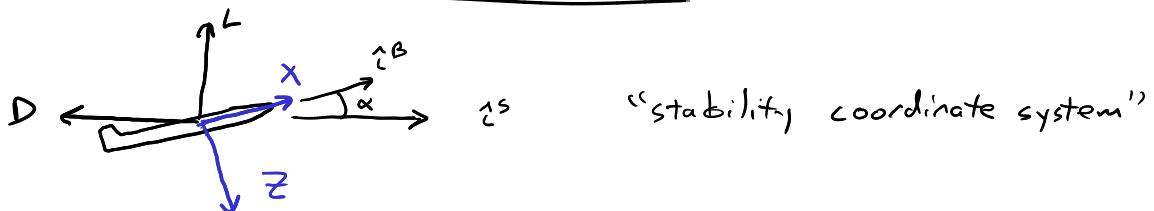
$$K = \frac{1}{\pi e AR}$$

Oswald's  
Efficiency

### Pitch Moment

$$M \approx \frac{1}{2} \rho V_a^2 S C (C_{m_{zero}} + C_{m\alpha} \alpha + C_{mq} \hat{q} + C_{m\delta_e} \delta_e)$$

### Lift + Drag $\rightarrow$ Body coordinates



$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} -D \\ -L \end{bmatrix}$$

# Longitudinal Stability

So far :  $C_L$ ,  $C_D$ ,  $C_m$

Now :  $C_{L\alpha}$ ,  $C_{m\alpha}$ ,  $C_{L_{se}}$ ,  $C_{m_{se}}$

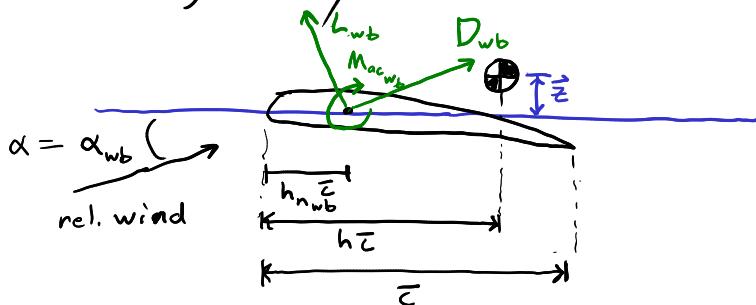
static margin :  $b_n - h$

Steady state forces  
trim, static stability

Three contributors 1. Wing / body

2. Propulsion (often small)  
3. Tail

Wing / body



Aerodynamic Center

$$\frac{\partial \text{Moment}}{\partial \alpha} = 0$$

Location stays constant  
Moment usually  $< 0$

Center of Pressure

$$\text{Moment} = 0$$

Changes w/  $\alpha$

Neutral Point

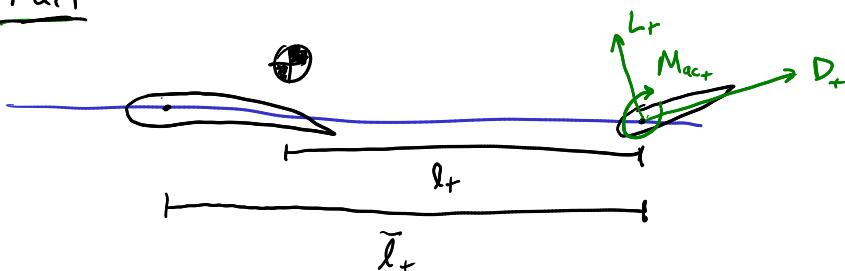
C.G. location that yields  $C_{m\alpha} = 0$   
A.C. of entire A/C

$$M_{wb} = M_{ac_{wb}} + (\underline{L \cos \alpha} + \underline{D \sin \alpha})(h - h_{nwb}) \bar{c} + (\underline{L \sin \alpha} - \underline{D \cos \alpha}) \bar{z}$$

only keep most important terms + nondimensionalize

$$C_{m_{wb}} = C_{m_{ac_{wb}}} + C_{L_{wb}} (h - h_{nwb})$$

Tail



$$L = L_{wb} + L_+$$

$$= C_{L_{wb}} \left( \frac{1}{2} \rho V^2 S \right) + C_{L_+} \left( \frac{1}{2} \rho V^2 S_+ \right) \Rightarrow C_L = C_{L_{wb}} + \underbrace{\frac{S_+}{S} C_{L_+}}$$

$$M_+ = -l_+ L_+ = -l_+ C_{L_+} \left( \frac{1}{2} \rho V^2 S_+ \right) \Rightarrow C_{m_+} = \underbrace{-l_+ \frac{S_+}{S} C_{L_+}}$$

$$= -V_H C_{L_+}$$

("volume ratio")

more convenient  
b/c C.G. can change

$$\bar{V}_H = \frac{l_+}{\bar{c}} \frac{S_+}{S} \Rightarrow V_H = \bar{V}_H - \frac{S_+}{S} (h - h_{nwb})$$

$$C_{m+} = -\bar{V}_H C_{L+} + C_{L+} \frac{S_+}{S} (h - h_{nwb})$$

Wing/body

$$\rightarrow C_m = \underbrace{C_{m_{acwb}}}_{\text{Tail}} + \underbrace{C_L (h - h_{nwb})}_{\text{Tail}} - \bar{V}_H C_{L+} + C_{mp}$$

propulsion

$$C_{m\alpha} = \frac{\partial C_{m_{acwb}}}{\partial \alpha} + C_{L\alpha} (h - h_{nwb}) - \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} + \frac{\partial C_{mp}}{\partial \alpha}$$

Want  $h_n \equiv cg$  location where  $C_{m\alpha} = 0$

$$0 = C_{L\alpha} (h_n - h_{nwb}) - \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} + \frac{\partial C_{mp}}{\partial \alpha}$$

$$h_n = h_{nwb} + \frac{1}{C_{L\alpha}} \left( \bar{V}_H \frac{\partial C_{L+}}{\partial \alpha} - \frac{\partial C_{mp}}{\partial \alpha} \right)$$

tail correction

$$C_{m\alpha} = C_{L\alpha} (h - h_n)$$

$C_{m\alpha} < 0$  for stability

Static margin

$K_n \equiv h_n - h$  must be  $> 0$  for stability

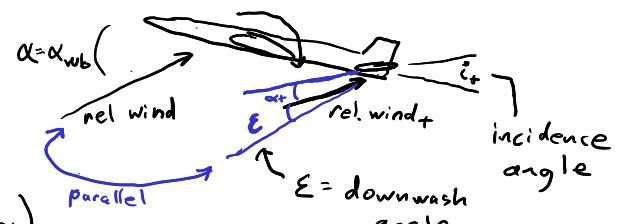
back to  $h_n$  eq.

Lift curve slope  $a \equiv C_{L\alpha}$

$$C_{L_{wb}} = a_{wb} \alpha_{wb} = a_{wb} \alpha$$

$$C_{L+} = a_+ \alpha_+$$

$$\alpha_+ = \alpha - i_+ - (\varepsilon_{zero} + \frac{\partial \varepsilon}{\partial \alpha} \alpha)$$



$$\frac{\partial C_{L+}}{\partial \alpha} = a_+ \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right)$$

$$h_n = h_{nwb} + \frac{a_+}{a} \bar{V}_H \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) - \frac{1}{a} \frac{\partial C_{mp}}{\partial \alpha}$$

$$C_{L\alpha} = a = a_{wb} \left[ 1 + \frac{a_+ S_+}{a_{wb} S} \left( 1 - \frac{\partial \varepsilon}{\partial \alpha} \right) \right]$$

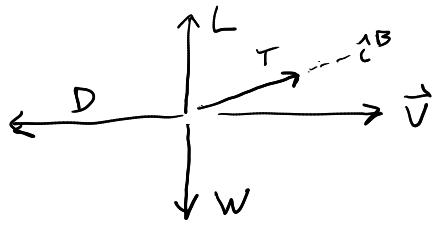
### Longitudinal Control

$\delta_e$  changes  $C_{L+}$

$$C_{L_{\delta e}} = \frac{\partial C_{L+}}{\partial \delta_e} \frac{S_+}{S} = a_e \frac{S_+}{S}$$

$$C_{m_{\delta e}} = -a_e \bar{V}_H + C_{L_{\delta e}} (h - h_{nwb})$$

## Linear Trim Estimation



For Linear trim,  $T=D$ ,  $L=W$

$$C_{L_{\text{trim}}} = \frac{W}{\frac{1}{2} \rho V^2 S} = C_{L_{\text{zero}}} + C_{L_{\alpha}} \alpha_{\text{trim}} + C_{L_{\delta e}} \delta_{\text{e,trim}}$$

$$C_{m_{\text{trim}}} = C_{m_{\text{zero}}} + C_{m_{\alpha}} \alpha_{\text{trim}} + C_{m_{\delta e}} \delta_{\text{e,trim}} = 0$$

$$\begin{bmatrix} C_{L\alpha} & C_{L\delta e} \\ C_{m\alpha} & C_{m\delta e} \end{bmatrix} \begin{bmatrix} \alpha_{\text{trim}} \\ \delta_{\text{e,trim}} \end{bmatrix} = \begin{bmatrix} C_{L_{\text{trim}}} \\ -C_{m_{\text{zero}}} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{\text{trim}} \\ \delta_{\text{e,trim}} \end{bmatrix} = \begin{bmatrix} C_{L\alpha} & C_{L\delta e} \\ C_{m\alpha} & C_{m\delta e} \end{bmatrix}^{-1} \begin{bmatrix} C_{L_{\text{trim}}} \\ -C_{m_{\text{zero}}} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Cramer's rule)

$$\alpha_{\text{trim}} = \frac{C_{m_{\text{zero}}} C_{L\delta e} + C_{m_{\delta e}} C_{L_{\text{trim}}}}{\Delta}$$

$$\Delta = C_{L\alpha} C_{m\delta e} - C_{L\delta e} C_{m\alpha}$$

$$\delta_{\text{e,trim}} = -\frac{C_{m_{\text{zero}}} C_{L\alpha} + C_{m_{\alpha}} C_{L_{\text{trim}}}}{\Delta}$$

# Longitudinal Linear Model

$$\dot{\vec{p}}_E = R_B^E \vec{v}_B^E$$

$$\dot{\vec{o}} = T \vec{\omega}_B$$

$$\vec{v}_B^E = \frac{\vec{f}_e}{m} - \vec{\omega}_B \times \vec{v}_B^E$$

$$\dot{\vec{\omega}} = I^{-1} [\vec{G}_B - \vec{\omega}_B \times I \vec{\omega}_B]$$

2 differences

1. Aerodynamic Forces

2. I more complex

Symmetry about x-z axis  $\Rightarrow I_{xy} = I_{yz} = 0$

$$I_B^{-1} = \begin{bmatrix} I_z & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{bmatrix} \quad I_{xz} \neq 0$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma} \quad \Gamma_4 = \frac{I_{xz}}{\Gamma} \quad \Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma} \quad \Gamma_5 = \frac{I_z - I_x}{I_y} \quad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \quad \Gamma_6 = \frac{I_{xz}}{I_y} \quad \Gamma = I_x I_z - I_{xz}^2$$



$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi c_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi \sec \theta & \cos \phi \sec \theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin \theta \\ \cos \theta \sin \phi \\ \cos \theta \cos \phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\ \Gamma_7 pq - \Gamma_8 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Linearize about trim state

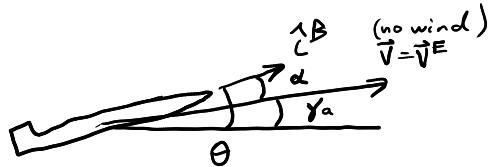
Inputs:  $U_0, h_0, \gamma_{a_0}$

airspeed  $= V = V_a$

altitude

air-relative flight-path angle

$\alpha_0, \delta_{e0}, \delta_{r0}$



$$\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u_E \\ v_E \\ w_E \\ p \\ q \\ r \end{bmatrix} = \vec{x}_0 + \vec{\Delta x}$$

$$\vec{x}_0 = \begin{bmatrix} \cdot \\ -h_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ U_0^E \\ V_0^E \\ W_0^E \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For linearization:  $\vec{V} = \vec{V}^E$  (no wind)  
assume  $w_0^E = 0$

$$\vec{u} = \begin{bmatrix} \delta_e \\ \delta_\alpha \\ \delta_r \\ \delta_\tau \end{bmatrix}$$

$$\vec{u}_0 = \begin{bmatrix} \delta_{e0} \\ 0 \\ 0 \\ \delta_{r0} \end{bmatrix}$$

Examples to Linearize

$$\dot{\theta} = \cos \phi q - \sin \phi r$$

$$\dot{\theta}_0 + \Delta \dot{\theta} = \cos \phi_0 q_0 - \sin \phi_0 r_0 + \left. \frac{\partial}{\partial \phi} (\cos \phi q - \sin \phi r) \right|_0 \Delta \phi$$

$$+ \left. \frac{\partial}{\partial q} (\cos \phi q - \sin \phi r) \right|_0 \Delta q$$

$$+ \left. \frac{\partial}{\partial r} (\cos \phi q - \sin \phi r) \right|_0 \Delta r$$

$$= -\sin \phi_0 q_0 - \cos \phi_0 r_0 + \cos \phi_0 \Delta q - \sin \phi_0 \Delta r$$

$\Delta \dot{\theta} = \Delta q$

$$\dot{u} = r v - q w - g \sin \theta + \frac{X}{m}$$

$$\dot{u}_0 + \Delta \dot{u} = \cancel{r_0 v_0} - \cancel{q_0 w_0} - g \sin \theta_0 + \cancel{\frac{X_0}{m}} + \left. \frac{\partial}{\partial r} rv \right|_0 \Delta r + \left. \frac{\partial}{\partial v} rv \right|_0 \Delta v + \left. \frac{\partial}{\partial q} (-qw) \right|_0 \Delta q$$

$$+ \left. \frac{\partial}{\partial w} (-qw) \right|_0 \Delta w + \cancel{\left. \frac{\partial}{\partial \theta} (g \sin \theta) \right|_0 \Delta \theta} + \left. \frac{\partial}{\partial X} \left( \frac{X}{m} \right) \right|_0 \Delta X$$

$$= \cancel{v_0 \Delta r} + \cancel{p_0 \Delta v} - \cancel{w_0 \Delta q} - \cancel{q_0 \Delta w} - g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta X$$

$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{1}{m} \Delta X$

## Lateral

$$\rightarrow \Delta\dot{\phi} = \Delta p + \Delta r \tan \theta_0$$

$$\rightarrow \Delta\dot{\theta} = \Delta q$$

## Long.

$$\rightarrow \Delta\dot{u} = -g \cos \theta_0 \Delta\theta + \frac{\Delta X}{m}$$

$$\rightarrow \Delta\dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta\phi + \frac{\Delta Y}{m}$$

$$\rightarrow \Delta\dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta\theta + \frac{\Delta Z}{m}$$

$$\rightarrow \Delta\dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N$$

$$\rightarrow \Delta\dot{q} = \frac{\Delta M}{I_y}$$

$$\rightarrow \Delta\dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N$$

Dimensional stab derivs.

$$\Delta X = X_u \Delta u + X_w \Delta w + \Delta X_c$$

$$\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + \Delta Y_c$$

$$\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}} \Delta \dot{w} + Z_q \Delta q + \Delta Z_c$$

$$\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + \Delta L_c$$

$$\Delta M = M_u \Delta u + M_w \Delta w + M_{\dot{w}} \Delta \dot{w} + M_q \Delta q + \Delta M_c$$

$$\Delta N = N_v \Delta v + N_p \Delta p + N_r \Delta r + \Delta N_c$$

$$X_u \equiv \left. \frac{\partial X}{\partial u} \right|_0$$

$$C_{X_u} \equiv \left. \frac{\partial C_x}{\partial u} \right|_0$$

$$\hat{\omega} = \frac{w}{u_0} \approx \alpha$$

$$\alpha = \tan(\frac{\Delta w}{u_0 + \Delta u}) \approx \frac{w}{u_0}$$

Table 4.4

Longitudinal Dimensional Derivatives

	X	$X_u$	Z	M
u	$\rho u_0 S C_{w_0} \sin \theta_0 + \frac{1}{2} \rho u_0 S C_{x_u}$		$-\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$	$\frac{1}{2} \rho u_0 \bar{c} S C_{m_u}$
w		$\frac{1}{2} \rho u_0 S C_{x_\alpha}$	$\frac{1}{2} \rho u_0 S C_{z_\alpha}$	$\frac{1}{2} \rho u_0 \bar{c} S C_{m_\alpha}$
q	$\frac{1}{4} \rho u_0 \bar{c} S C_{x_q}$		$\frac{1}{4} \rho u_0 \bar{c} S C_{z_q}$	$\frac{1}{4} \rho u_0 \bar{c}^2 S C_{m_q}$
w	$\frac{1}{4} \rho c S C_{x_{\dot{w}}}$		$\frac{1}{4} \rho c S C_{z_{\dot{w}}}$	$\frac{1}{4} \rho c^2 S C_{m_{\dot{w}}}$

$$Z_u \equiv \left. \frac{\partial Z}{\partial u} \right|_0$$

$$Z = \frac{1}{2} \rho V^2 S C_Z$$

$$\begin{aligned} \left. \frac{\partial Z}{\partial u} \right|_0 &= \frac{1}{2} \rho S \left( \left. \frac{\partial V^2}{\partial u} \right|_0 C_Z + V^2 \left. \frac{\partial C_Z}{\partial u} \right|_0 \right) \\ &= \frac{1}{2} \rho S \left( Z_{u_0} C_{Z_0} + u_0^2 \left. \frac{\partial C_Z}{\partial u} \right|_0 \right) \end{aligned}$$

$$Z_u = -\rho u_0 S C_{w_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{z_u}$$

$$\hat{u} = \frac{u}{V} = \frac{u}{u_0} \quad u = \hat{u} u_0$$

$$\left. \frac{\partial C_Z}{\partial u} \right|_0 = \left. \frac{\partial C_Z}{u_0 \hat{u}} \right|_0 = \frac{1}{u_0} C_{Z_u}$$

$$C_{Z_0} = -C_{w_0} \cos \theta_0$$

Table 5.1

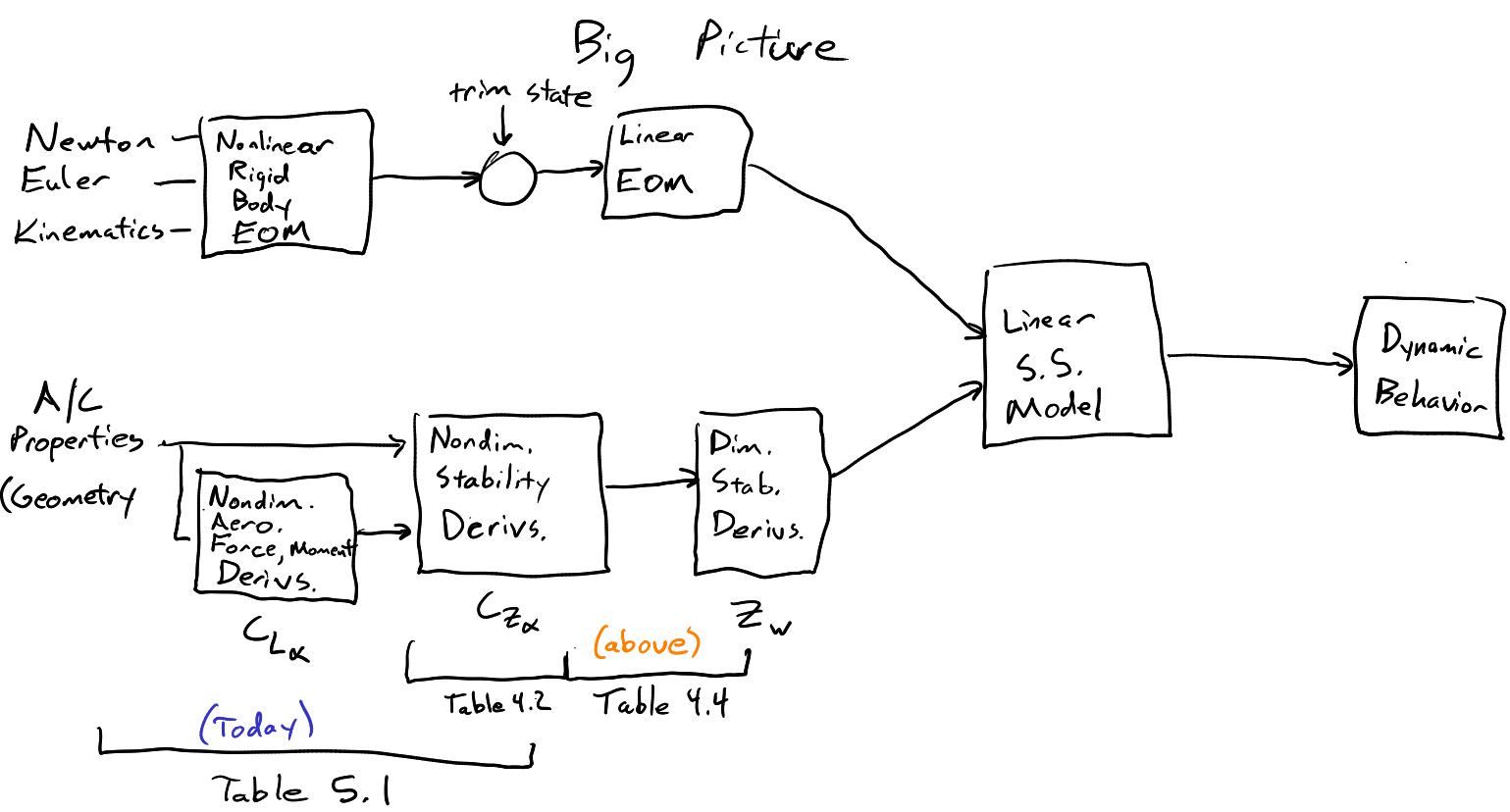
Summary—Longitudinal Derivatives

	$C_x$	$C_z$	$C_m$
$\hat{u}^\dagger$	$\mathbf{M}_0 \left( \frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$*-2a_i V_H \frac{\partial \epsilon}{\partial \alpha}$	$*-2a_i V_H \frac{l_i}{c} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$*-2a_i V_H$	$*-2a_i V_H \frac{l_i}{c}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\dagger C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$



# Nondimensional Stability Derivatives

Table 5.1  
Summary—Longitudinal Derivatives

	$C_x$	$C_z$	$C_m$
$\hat{u}^+$	$\mathbf{M}_0 \left( \frac{\partial C_T}{\partial \mathbf{M}} - \frac{\partial C_D}{\partial \mathbf{M}} \right) - \rho u_0^2 \frac{\partial C_D}{\partial p_d} + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right)$	$-\mathbf{M}_0 \frac{\partial C_L}{\partial \mathbf{M}} - \rho u_0^2 \frac{\partial C_L}{\partial p_d} - C_{T_u} \frac{\partial C_L}{\partial C_T}$	$\mathbf{M}_0 \frac{\partial C_m}{\partial \mathbf{M}} + \rho u_0^2 \frac{\partial C_m}{\partial p_d} + C_{T_u} \frac{\partial C_m}{\partial C_T}$
$\alpha$	$C_{l_0} - C_{D_\alpha}$	$-(C_{L_\alpha} + C_{D_0})$	$-a(h_n - h)$
$\dot{\alpha}$	Neg.	$* -2a_t V_H \frac{\partial \epsilon}{\partial \alpha}$	$* -2a_t V_H \frac{l_t}{c} \frac{\partial \epsilon}{\partial \alpha}$
$\hat{q}$	Neg.	$* -2a_t V_H$	$* -2a_t V_H \frac{l_t}{c}$

Neg. means usually negligible.

\*means contribution of the tail only, formula for wing-body not available.

$$\dagger C_{T_u} = \frac{(\partial T / \partial u)_0}{\frac{1}{2} \rho u_0 S} - 2C_{T_0}; C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0$$

$\alpha$ -derivatives

$$\boxed{C_{m_\alpha} = C_{L_\alpha} (h - h_n)}$$

$$C_{z_\alpha}$$

$$Z = -L \cos \alpha - D \sin \alpha$$

$$C_z = -(C_L \cos \alpha + C_D \sin \alpha)$$

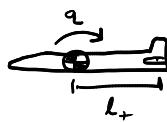
$$\approx -(C_L + C_D \alpha)$$

$$C_{z_\alpha} = \frac{\partial C_z}{\partial \alpha} \Big|_0 = -(C_{L_\alpha} + C_{D_0} + \cancel{C_{D_\alpha}})$$

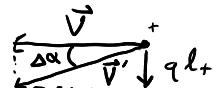
$$\boxed{C_{z_\alpha} = -(C_{L_\alpha} + C_{D_0})}$$

$q$ -derivatives

Wing-body | Tail



velocity observed by tail



$$\Delta C_{L_t} = \alpha_t \Delta \alpha \xrightarrow{\text{Lift curve slope of tail}} \alpha_t + \tan^{-1} \left( \frac{q l_t}{u_0} \right) \approx \alpha_t + \frac{q l_t}{u_0}$$

$$\Delta C_L = \frac{S_t}{S} \Delta C_{L_t}$$

$$= \frac{S_t}{S} \alpha_t + \frac{q l_t}{u_0}$$

$$(C_{z_q})_{tail}$$

$$C_{z_q} = \frac{\partial C_z}{\partial q} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_L}{\partial q} \Big|_0 = -\frac{2u_0}{c} \frac{\partial C_L}{\partial q} \Big|_0$$

$$(C_{z_q})_{tail} = -\frac{2u_0}{c} \alpha_t \frac{S_t}{S} \frac{l_t}{u_0} = \boxed{-2\alpha_t V_H}$$

$$V_H = \frac{S_t l_t}{S c}$$

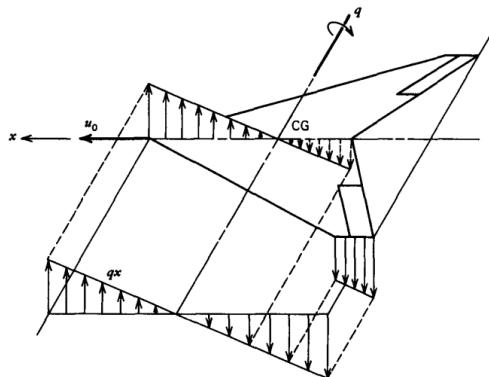
$(C_{m\dot{\alpha}})_{tail}$ 

$$\Delta C_m = -V_H \Delta C_{L+} = \alpha_+ V_H \frac{q l_+}{u_0}$$

$$C_{m\dot{\alpha}} \equiv \frac{\partial C_m}{\partial \dot{\alpha}} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_m}{\partial q} \Big|_0$$

$$(C_{m\dot{\alpha}})_{tail} = -2\alpha_+ V_H \frac{l_+}{c}$$

## Wing - Body



measure in  
wind tunnel  
or CFD

Figure 5.4 Wing velocity distribution due to pitching.

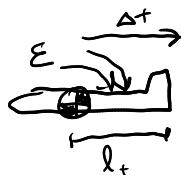
## $\dot{\alpha}$ derivatives

### Unsteady effects

Wing-body → determined by initial response  
or oscillation of wing in wind tunnel or flight test

## Tail

### Downwash lag



$$\begin{aligned} -\Delta\alpha_+ &= \Delta\varepsilon = -\frac{\partial\varepsilon}{\partial\alpha} \dot{\alpha} \Delta t \\ &= -\frac{\partial\varepsilon}{\partial\alpha} \dot{\alpha} \frac{l_+}{u_0} \end{aligned}$$

$$\Delta C_{L+} = \alpha_+ \Delta\alpha_+ = \alpha_+ \dot{\alpha} \frac{l_+}{u_0} \frac{\partial\varepsilon}{\partial\alpha}$$

$$\Delta C_L = \alpha_+ \dot{\alpha} \frac{l_+ S_+}{u_0 S} \frac{\partial\varepsilon}{\partial\alpha}$$

$$(C_{Z\dot{\alpha}})_{tail} = \frac{\partial C_Z}{\partial \dot{\alpha} c} \Big|_0 = \frac{2u_0}{c} \frac{\partial C_Z}{\partial \dot{\alpha}} \Big|_0 = \boxed{-2\alpha_+ \frac{l_+ S_+}{c S} \frac{\partial\varepsilon}{\partial\alpha}}$$

$$(C_{m\dot{\alpha}})_{tail} = -2\alpha_+ V_H \frac{l_+}{c} \frac{\partial\varepsilon}{\partial\alpha}$$

## u derivatives

3 important factors:

- Compressibility: Mach Number

- Dynamic Pressure:  $p_d = \frac{1}{2} \rho V^2$

- Thrust

- Different from the dynamic pressure in nondimensionalization.

Changes in  $C_L$ ,  $C_D$  etc. due to changes in dynamic pressure.

$$C_{X_u} = \left. \frac{\partial C_x}{\partial u} \right|_0$$

$$C_{*u} = \underbrace{\left. \frac{\partial C_*}{\partial M} \right|_0 \left. \frac{\partial M}{\partial u} \right|_0}_{+} + \underbrace{\left. \frac{\partial C_*}{\partial p_d} \right|_0 \left. \frac{\partial p_d}{\partial u} \right|_0}_{+} + \underbrace{\left. \frac{\partial C_*}{\partial C_T} \right|_0 \left. \frac{\partial C_T}{\partial u} \right|_0}_{+}$$

$$M = \frac{V}{a}$$

$$\left. \frac{\partial M}{\partial u} \right|_0 = u_0 \left. \frac{\partial M}{\partial u} \right|_0 = \frac{u_0}{a} \left. \frac{\partial V}{\partial u} \right|_0 = M_0 \quad \text{Mach number at trim}$$

$$\left. \frac{\partial p_d}{\partial u} \right|_0 = u_0 \left. \frac{\partial p_d}{\partial u} \right|_0 = u_0 \frac{1}{2} \rho \left. \frac{\partial V^2}{\partial u} \right|_0 = u_0 \rho u_0 = \rho u_0^2$$

$$C_T = \frac{T}{\frac{1}{2} \rho V^2 S}$$

$$\begin{aligned} \left. \frac{\partial C_T}{\partial u} \right|_0 &= u_0 \left. \frac{\partial C_T}{\partial u} \right|_0 = u_0 \left( \frac{\partial T / \partial u}{\frac{1}{2} \rho V^2 S} - \frac{Z T}{\frac{1}{2} \rho V^3 S} \right) \Big|_0 \\ &= \frac{\partial T / \partial u}{\frac{1}{2} \rho u_0 S} \Big|_0 - Z C_{T_0} \end{aligned}$$

3 Cases

$$C_{T_0} = C_{D_0} + C_{W_0} \sin \theta_0$$

Gliding:  $C_{T_u} = 0$

Constant Thrust (Jet):

Constant Power (Prop):

$$T V = \text{constant}$$

$$\left. \frac{\partial T}{\partial u} \right|_0 = - \frac{T_0}{u_0}$$

$$C_{T_u} = -3 C_{T_0}$$

$C_{X_u}$ 

$$C_x \approx C_T - C_D$$

$$\frac{\partial C_x}{\partial M} \Big|_o = \frac{\partial C_T}{\partial M} \Big|_o - \frac{\partial C_D}{\partial M} \Big|_o$$

$$\frac{\partial C_x}{\partial p_d} \Big|_o = \frac{\partial C_T}{\partial p_d} \Big|_o - \frac{\partial C_D}{\partial p_d} \Big|_o$$

$$\frac{\partial C_x}{\partial C_T} \Big|_o = 1 - \frac{\partial C_D}{\partial C_T} \Big|_o$$

$$C_{X_u} = M_o \left( \frac{\partial C_T}{\partial M} - \frac{\partial C_D}{\partial M} \right) \Big|_o - \rho u_o^2 \frac{\partial C_D}{\partial p_d} \Big|_o + C_{T_u} \left( 1 - \frac{\partial C_D}{\partial C_T} \right) \Big|_o$$

 $C_{Z_u}$ 

Assume  $C_z = -C_L$

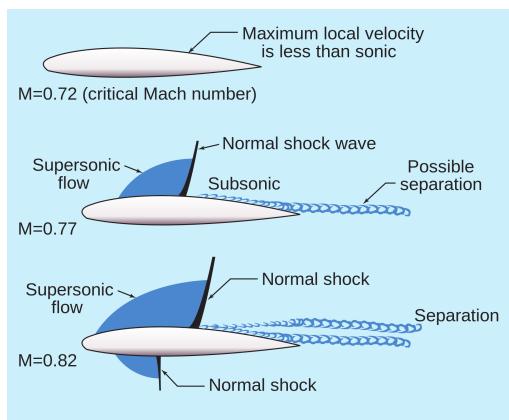
$$C_{Z_u} = -M_o \frac{\partial C_L}{\partial M} \Big|_o - \rho u_o^2 \frac{\partial C_L}{\partial p_d} \Big|_o - C_{T_u} \frac{\partial C_L}{\partial C_T} \Big|_o$$

small except  
for transonic

 $C_{m_u}$ 

$$C_{m_u} = M_o \frac{\partial C_m}{\partial M} \Big|_o + \rho u_o^2 \frac{\partial C_m}{\partial p_d} \Big|_o + C_{T_u} \frac{\partial C_m}{\partial C_T} \Big|_o$$

Mach Tuck



## Longitudinal Modes

$$\dot{\vec{x}} = A\vec{x}$$

$n \times n$

(assume that  $A$  has  $n$  distinct non zero eigenvalues)

$$\vec{x}(t) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

Full State Space

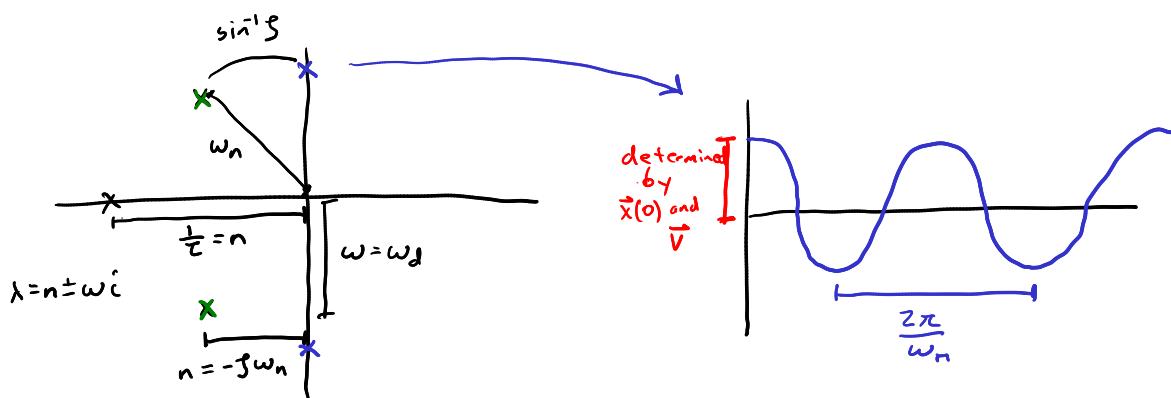
$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

What does this mean?

A single real-valued  $(\lambda, \vec{v})$  pair, or a pair of complex-valued  $((\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2))$  describes a mode.

$\lambda$ : "speed" of mode  
 $\vec{v}$ : "shape" of mode



### Eigenvectors

$$A\vec{v}_i = \vec{v}_i \lambda_i$$

$$(A - \lambda_i I)\vec{v}_i = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$|A - \lambda_i I| = 0 = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0 \quad \lambda_1 = -1, \quad \lambda_2 = -2$$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \propto$$

$$\text{let } (\vec{v}_1)_1 = 1$$

$$(\vec{v}_1)_2 = -1$$

$$(\vec{v}_1)_1 + (\vec{v}_1)_2 = 0$$

$$(A - \lambda_2 I)\vec{v}_2 = 0$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \vec{v}_2 = 0 \quad \therefore \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \propto$$

$$\vec{x}(0) = \sum_i q_i \vec{v}_i e^{\lambda_i t} = \sum_i q_i \vec{v}_i = q_1 \vec{v}_1 + q_2 \vec{v}_2 + \dots + q_n \vec{v}_n = \sqrt{\vec{q}}$$

$$\vec{x}(0) = \sqrt{\vec{q}}$$

$$\vec{q} = V^{-1} \vec{x}(0)$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

## Conventional A/C Long. Modes

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$h_0 = 40k \text{ ft}$$

$$u_0 = 774 \text{ ft/s}$$

$$\gamma_0 = \theta_0 = \alpha_0 = 0$$

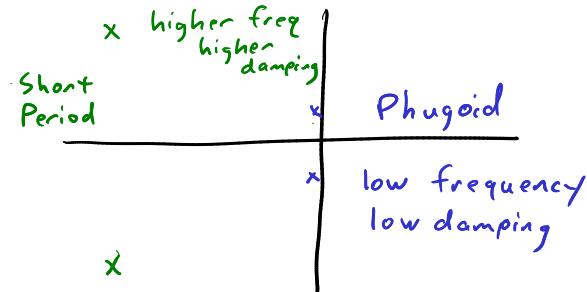
$$\mathbf{A}_{lon} = \begin{pmatrix} -0.006868 & 0.01395 & 0 & -32.2 \\ -0.09055 & -0.3151 & 773.98 & 0 \\ 0.0001187 & -0.001026 & -0.4285 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = -0.37 \pm 0.89i$$

$$\omega_n = 0.96 \quad \zeta = 0.38$$

$$\lambda_{3,4} = -0.0033 \pm 0.067i$$

$$\omega_n = 0.067 \quad \zeta = 0.049$$



$$\vec{v}_{1,2} = \begin{bmatrix} 0.02 \pm 0.016i \\ 0.9996 \\ -0.0001 \pm 0.0011i \\ 0.0011 \mp 0.0004i \end{bmatrix}$$

$$\begin{aligned} \Delta U & \\ \Delta W & \\ \Delta q & \\ \Delta \theta & \end{aligned}$$

$$\vec{v}_{3,4} = \begin{bmatrix} -0.9983 \\ -0.057 \pm 0.0097i \\ -0.0001 \mp 0i \\ 0.0001 \pm 0.0021i \end{bmatrix}$$

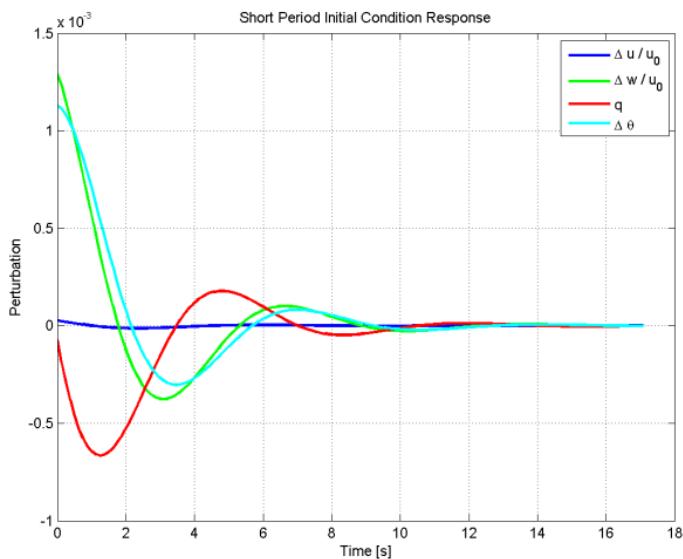
$$\vec{x}(0) = 0.5 \vec{v}_1 + 0.5 \vec{v}_2 = \operatorname{Re}(\vec{v}_{1,2})$$

$$\lambda_{1/2} = -0.372 + 0.888i$$

$$\zeta = 0.387$$

$$\omega_n = 0.962$$

$$\mathbf{x}(0) = Re(\mathbf{v}_1) = \begin{pmatrix} 0.0211 \\ 0.9996 \\ -0.0001 \\ 0.0011 \end{pmatrix}$$



Phasor Plot: plot of eigenvector in complex plane

Aside: Polar Coordinates of a complex number

$$z = a + bi$$

$$z = r e^{i\phi}$$

$$z = r \angle \phi$$

$$r = \sqrt{a^2 + b^2} \quad \phi = \text{atan} z(b, a)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

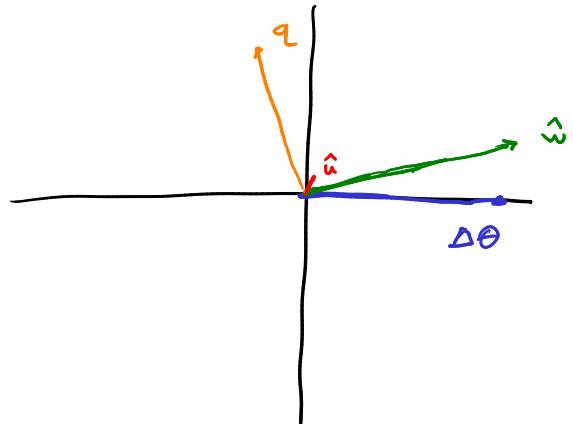
For consistent phasor plots,

- normalize so that  $\Delta\theta = 1$
- nondimensionalize  $u$  and  $w$

Short Period

$$\vec{V}'_{1,2} = \vec{V}_{1,2} / (\vec{V}_{1,2})_4 = \begin{bmatrix} & & \\ & \vdots & \\ & & 1.0 \end{bmatrix}$$

$$\hat{V}_{1,2} = \begin{bmatrix} 0.016 \pm 0.024i \\ 1.02 \pm 0.36i \\ -0.37 \pm 0.99i \\ 1.0 \end{bmatrix} \quad \begin{aligned} \hat{u} &= \frac{\Delta u}{u_0} \\ \hat{w} &= \frac{\Delta w}{w_0} \approx \alpha \\ q & \\ \Delta\theta & \end{aligned} \Rightarrow$$

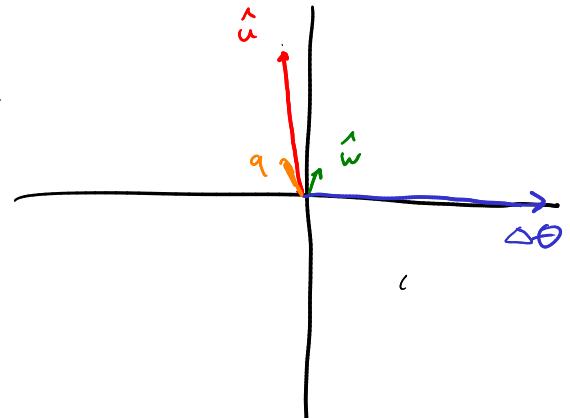


# Phugoid Mode

phugoid = "flight"



$$\vec{v}_{3,9} = \begin{bmatrix} 0.62 < 92^\circ \\ 0.036 < 83^\circ \\ 0.067 < 93^\circ \\ 1.0 < 0 \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{\omega} = \alpha \\ q \\ \Delta\theta \end{matrix}$$



$\lambda_{3,4} \Rightarrow \begin{cases} \text{low frequency} \\ \text{low damping} \end{cases}$

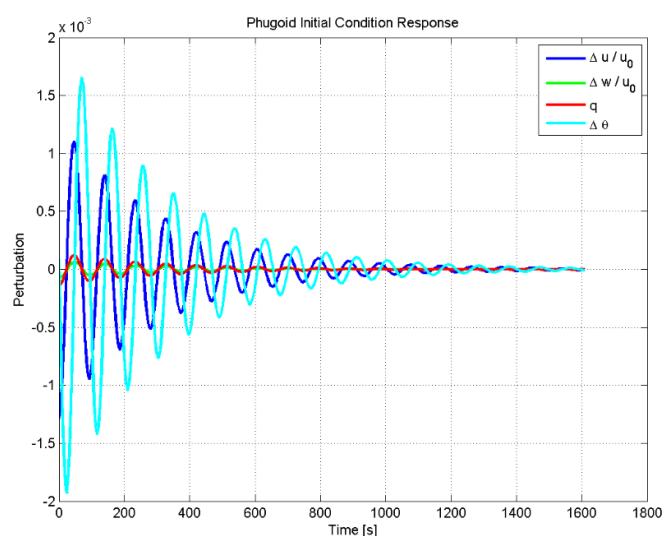
$\vec{v}_{3,4} \Rightarrow \begin{cases} \text{Large } \hat{u} \text{ and } \theta \text{ oscillations out of phase (90° offset)} \\ \text{small } \alpha \end{cases}$

$$\lambda_{3/4} = -3.29e-03 + 6.72e-02i$$

$$\zeta = 0.0489 \leftarrow \text{poorly damped}$$

$$\omega_n = 0.0673 \leftarrow \text{slow response}$$

$$\mathbf{x}(0) = Re(\mathbf{v}_3) = \begin{pmatrix} -0.9983 \\ -0.0573 \\ -0.0001 \\ 0.0001 \end{pmatrix}$$



# Longitudinal Mode Approximations

$$\vec{v} = \begin{bmatrix} a+bi \\ b \end{bmatrix}$$

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + B_{lon} \vec{u}_{lon}$$

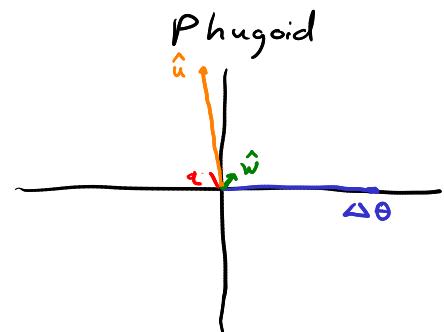
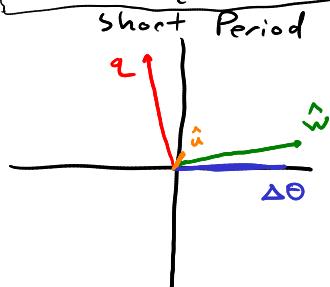
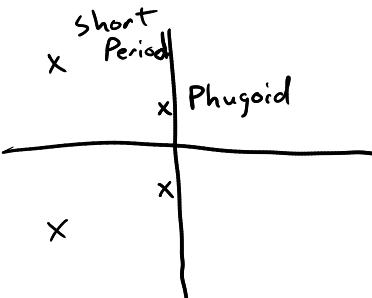
$$\dot{\vec{y}} = C \vec{x}_{lon} + D \vec{u}_{lon}$$

$\downarrow I \quad \downarrow O$

$$\vec{x}_{lon} = \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$\vec{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

$$\hat{w} = \frac{\Delta w}{u_0}$$



## Short period Approx

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\vec{x}}_{lon} = A_{lon} \vec{x}_{lon} + \vec{c}_{lon}$$

Assume:  $\Delta u = 0$

$$Z_w \ll m$$

$$Z_q \ll m u_0$$

$$\theta_0 = 0$$

No vertical motion

$$\Delta \theta = \Delta \alpha = \frac{\Delta w}{u_0}$$

$$\vec{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \vec{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$A_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_w} & \frac{Z_w}{m - Z_w} & \frac{Z_q + mu_0}{m - Z_w} & \frac{-mg \sin \theta_0}{m - Z_w} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_w} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_w} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_w)} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$\begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{Z_w}{m} \\ \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_w} \right] \end{bmatrix}}_{A_{sp}} \underbrace{\begin{bmatrix} u_0 \\ \frac{1}{I_y} \left[ M_q + M_{\dot{w}} u_0 \right] \end{bmatrix}}_{\lambda} \begin{bmatrix} \Delta w \\ \Delta q \end{bmatrix}$$

$$|A_{sp} - \lambda I| = \lambda^2 - \underbrace{\left[ \frac{Z_w}{m} + \frac{1}{I_y} (M_q + M_{\dot{w}} u_0) \right]}_{-2 \zeta \omega_n} \lambda - \underbrace{\frac{1}{I_y} (u_0 M_w - \frac{M_q Z_w}{m})}_{-\omega_n^2} = 0$$

How does this relate to size and shape?

### Dimensional Stab. Deriv.

$$Z_w = \frac{\partial Z}{\partial w} \Big|_0 = \frac{1}{2} \rho u_0 S C_{Z\alpha}$$

Table 4.4

$$M_w$$

$$M_w$$

$$M_q$$

### Nondim. Stab. Deriv.

$$C_{Z\alpha}$$

$$C_m\alpha$$

$$C_m\alpha$$

$$C_mq$$

### A/C Properties

$$= -C_{L\alpha} - C_{D\alpha}$$

Table 5.1

$$= C_{L\alpha} (h - h_n)$$

....

$$(C_m)_{tail} = -2 \alpha_r V_H \frac{l + c}{c}$$

How accurate is this approximation?

For 747  
@ cruise

### Full A<sub>lon</sub>

$$\lambda_{1,2} = -0.372 \pm 0.888i$$

$$g = 0.387$$

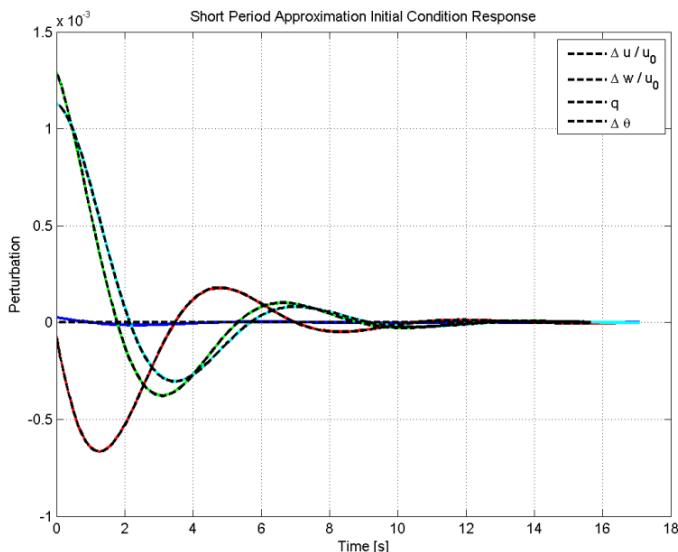
$$\omega_n = 0.962$$

### S.P. Approx

$$\lambda_{sp} = -0.371 \pm 0.889i$$

$$g = 0.385$$

$$\omega_n = 0.963$$



### Phugoid Mode

### Lanchester (1908)

Assume conservation of energy

$$E = \frac{1}{2} m V^2 - mg \Delta z_E = \frac{1}{2} m u_0^2$$

$$V^2 = 2g \Delta z_E + u_0^2$$

$$C_L = C_{L0} = C_{W0}$$

$$L = \frac{1}{2} \rho V^2 S C_L = \frac{1}{2} \rho u_0^2 S C_{W0} + \rho g S C_{W0} \Delta z_E = W + \rho g S C_{W0} \Delta z_E$$

Newton's 2nd Law in z

$$W - L = m \Delta \ddot{z}_E$$

$$W - (W + \rho g S C_{W0} \Delta z_E) = m \Delta \ddot{z}_E$$

$$\Delta \ddot{z}_E + \underbrace{\frac{\rho g S C_{W0}}{m} \Delta z_E}_{\omega_n^2} = 0$$

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{\rho g S C_{W_0}}} = 2\pi \sqrt{\frac{\frac{1}{2} u_0^2 m g}{g^2 \frac{1}{2} \rho u_0^2 S C_{W_0}}} = \boxed{\pi \sqrt{2} \frac{u_0}{g}}$$

$$\boxed{T = 0.138 u_0 \text{ if } u_0 \text{ in f/s} \\ = 0.453 u_0 \text{ if } u_0 \text{ in m/s}}$$

for 747

$$\underline{\text{Full Aer}}$$
  
 $T = 93s$

$$\underline{\text{Lanchester}}$$
  
 $T = 107s$

### "2x2" Phugoid Approximation

Dynamics of Flight, Eq. (4.9,18)  $\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_w} \\ \frac{\Delta M_c}{I_y} + \frac{M_w}{I_y} \frac{\Delta Z_c}{(m - Z_w)} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & 0 & 0 & 0 \\ \frac{Z_u}{m - Z_w} & \frac{1}{I_y} \left[ M_u + \frac{M_w Z_u}{m - Z_w} \right] & 0 & 0 \\ 0 & 0 & \frac{1}{I_y} \left[ M_q + \frac{M_w (Z_q + mu_0)}{m - Z_w} \right] & \frac{-g \cos \theta_0}{I_y (m - Z_w)} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \Delta \alpha &= 0 \\ \Delta q &\text{ small} \\ Z_w &\ll m \\ Z_q &\ll mu_0 \\ \theta_0 &= 0 \end{aligned}$$

$$\rightarrow \begin{bmatrix} \Delta u \\ \Delta w = \Delta \alpha u_0 \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & 0 & -g \\ \frac{Z_u}{m} & u_0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

$$0 = \frac{Z_u}{m} \Delta u + u_0 \Delta q$$

$$\Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\Delta \dot{\theta} = \Delta q = -\frac{Z_u}{mu_0} \Delta u$$

$$\boxed{\begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{mu_0} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}}$$

$A_{ph}$

$$|A_{ph} - \lambda I| = \lambda^2 - \underbrace{\frac{X_u}{m} \lambda}_{-2\beta \omega_n} - \underbrace{\frac{Z_u g}{m u_0}}_{-\omega_n^2} = 0$$

$$Z_u = -\rho u_0 S C_{W_0} \cos \theta_0 + \frac{1}{2} \rho u_0 S C_{Z_u}$$

$$C_{Z_u} = -M_o \frac{\partial C_L}{\partial M} \Big|_0 - \rho u_0^2 \frac{\partial C_L}{\partial p_d} \Big|_0 - C_{T_u} \frac{\partial C_L}{\partial C_T} \Big|_0$$

assume  $\frac{\text{small}}{\text{small}} \quad \frac{\text{small}}{\text{small}} \quad \frac{\text{small}}{\text{small}}$

$$C_{Z_u} = 0 \quad \therefore \boxed{Z_u = -\rho u_0 S C_{W_0}}$$

$$X_u = \rho u_0 S C_{w_0} \sin \theta_0^0 + \frac{1}{2} \rho u_0 S C_{x_0}$$

$$C_{x_0} = -2 C_{T_0}$$

$$C_{T_0} = C_{D_0} + C_{w_0} \sin \theta_0^0$$

$$\underline{X_u = -\rho u_0 S C_{D_0}}$$

$$\boxed{\omega_u = \sqrt{\frac{-Z_u g}{m u_0}} = \sqrt{\frac{\rho S C_{w_0} g}{m}}}$$

$$\zeta = \frac{-X_u}{2} \sqrt{\frac{-u_0}{m Z_u g}} = \frac{\rho u_0 S C_{D_0}}{2} \sqrt{\frac{u_0}{2mg \rho u_0 S C_{L_0}}}$$

$$\text{substitute } mg = \frac{1}{2} \rho u_0^2 S C_{L_0}$$

$$\boxed{\zeta = \frac{1}{\sqrt{2}} \frac{C_{D_0}}{C_{L_0}}}$$

Same as Lanchester

Note: in class I included this  $\frac{1}{2}$  incorrectly. The final expression for  $\zeta$  is still correct.

and cancel

High  $\frac{L}{D} \Rightarrow$  less damping

(b/c less energy loss)

747

Full  $A_{6x1}$

$$\lambda_{3,4} = -3.29 \times 10^{-3} \pm 6.72 \times 10^{-2} i$$

$$\zeta = 0.0489$$

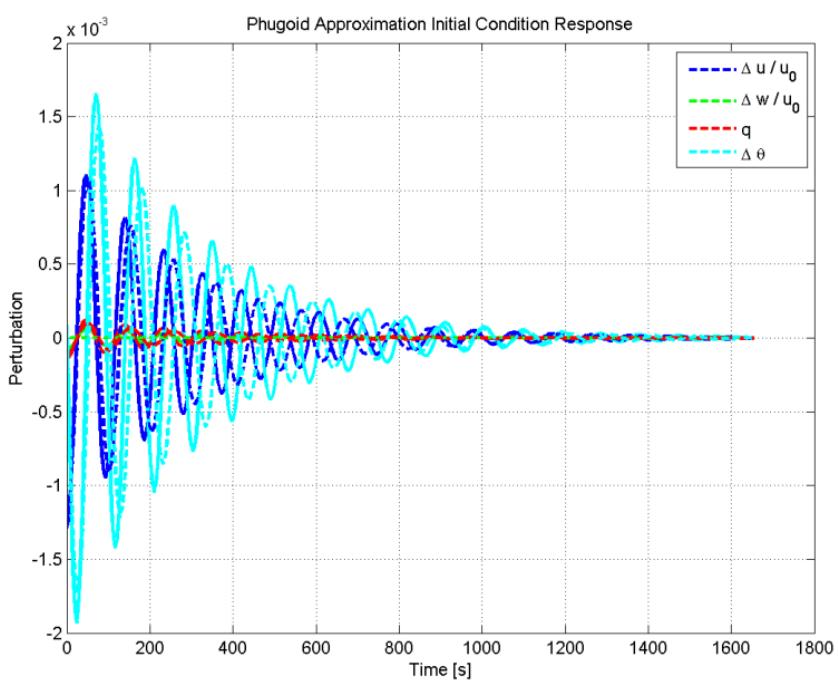
$$\omega_n = 0.0673$$

$2 \times 2$  Ph. Approx

$$\lambda_{ph} = -3.43 \times 10^{-3} \pm 6.11 \times 10^{-2} i$$

$$\zeta = 0.0561$$

$$\omega_n = 0.0612$$



# Longitudinal Control

Dynamics of Flight, Eq. (4.9,18)

$$\dot{\mathbf{x}}_{lon} = \mathbf{A}_{lon} \mathbf{x}_{lon} + \mathbf{c}_{lon}$$

$$\mathbf{x}_{lon} = \begin{pmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{pmatrix} \quad \mathbf{c}_{lon} = \begin{pmatrix} \frac{\Delta X_c}{m} \\ \frac{\Delta Z_c}{m - Z_{\dot{w}}} \\ \frac{\Delta M_c}{I_y} + \frac{M_{\dot{w}}}{I_y} \frac{\Delta Z_c}{(m - Z_{\dot{w}})} \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lon} = \begin{pmatrix} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \cos \theta_0 \\ \frac{Z_u}{m - Z_{\dot{w}}} & \frac{Z_w}{m - Z_{\dot{w}}} & \frac{Z_q + mu_0}{m - Z_{\dot{w}}} & \frac{-mg \sin \theta_0}{m - Z_{\dot{w}}} \\ \frac{1}{I_y} \left[ M_u + \frac{M_{\dot{w}} Z_u}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_w + \frac{M_{\dot{w}} Z_w}{m - Z_{\dot{w}}} \right] & \frac{1}{I_y} \left[ M_q + \frac{M_{\dot{w}} (Z_q + mu_0)}{m - Z_{\dot{w}}} \right] & \frac{-M_{\dot{w}} mg \sin \theta_0}{I_y (m - Z_{\dot{w}})} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\vec{c}_{lon} = B_{lon} \vec{u}_{lon}$$

$$\vec{u}_{lon} = \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_t \end{bmatrix}$$

← elevator  
 + = down  
 ← throttle  
 + = more thrust

dimensional control derivatives

$$\Delta X_c = X_{\delta e} \Delta \delta_e + X_{\delta t} \Delta \delta_t$$

$$\Delta Z_c = Z_{\delta e} \Delta \delta_e + Z_{\delta t} \Delta \delta_t \quad \text{often } 0$$

$$\Delta M_c = M_{\delta e} \Delta \delta_e + M_{\delta t} \Delta \delta_t$$

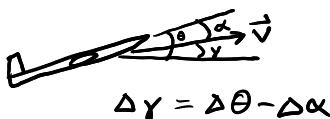
$$B_{lon} = \begin{bmatrix} \frac{X_{\delta e}}{m} & \frac{X_{\delta t}}{m} \\ \frac{Z_{\delta e}}{m - Z_{\dot{w}}} & \frac{Z_{\delta t}}{m - Z_{\dot{w}}} \\ \frac{M_{\delta e} + M_{\dot{w}} Z_{\delta e}}{I_y} & \frac{M_{\delta t} + M_{\dot{w}} Z_{\delta t}}{I_y} \\ 0 & 0 \end{bmatrix}$$

	$\delta_e$	$\delta_t$
$\Delta u$	-0.000187	9.66
$\Delta w$	-17.85	0
$\Delta q$	-1.158	0
$\Delta \theta$	0	0

$$\dot{\vec{x}}_{lon} = \mathbf{A}_{lon} \vec{x}_{lon} + \mathbf{B}_{lon} \vec{u}_{lon}$$

C and D depend on output

E.g. flight path angle  $\gamma$



$$\Delta \gamma = \Delta \theta - \Delta \alpha$$

$$\Delta \gamma = \underbrace{\begin{bmatrix} 0 & -\frac{1}{u_0} & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}}_D$$

$$\vec{\gamma} = [\Delta \gamma] = C_{\gamma} \vec{x}_{lon} + \underbrace{[0]}_D \vec{u}_{lon}$$

Open-loop step response to control inputs

$$B_{\delta e} = B_{lon}(:, 1)$$

$$B_{\delta t} = B_{lon}(:, 2)$$

$$\delta_e: 1^\circ$$

$$\delta_t: \frac{1}{6} = 0.05 \text{ rad}$$

$\delta_e \Rightarrow \gamma$  little change

$\delta_t \Rightarrow \gamma$  significant increase

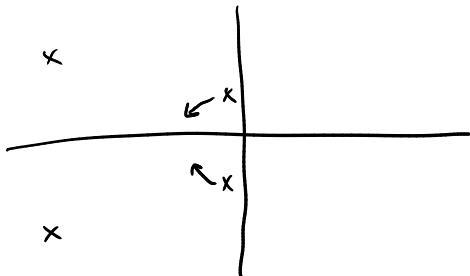
## Longitudinal stability augmentation

Realistic main goal:

- increase phugoid damping

On homework: change s.p. damping ratio

- not usually a real-life goal
- easy to do by hand



Increase Phugoid damping with

$$\Delta \delta_e = -k_\theta \Delta \theta$$

$$\dot{\vec{x}}_{ph} = \begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta \theta} \end{bmatrix} = \underbrace{A_{ph} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}}_{\text{previous}} + B_{ph, \delta e} \Delta \delta_e$$

$$B_{\text{lon}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}$$

for 747

Derivation of  $A_{ph}$  and  $B_{ph, \delta e}$

Assume  $\dot{q}$  only depends on  $\alpha$ ,  $q$ , and  $\delta e$   
solve for  $\dot{q}=0$  (after short period has damped out)

$$\dot{\vec{q}} = \frac{M_w}{I_Y} \Delta w + \frac{M_{\delta e}}{I_Y} \Delta \delta e + \frac{M_q}{I_Y} \Delta q$$

$$\Rightarrow \Delta w = -\frac{M_{\delta e}}{M_w} \Delta \delta e$$

solve for  $\dot{\Delta w} = 0$  (after s.p. oscillations damp out)

$$\dot{\Delta w} = 0 = \frac{Z_u}{m} \Delta u + \frac{Z_w}{m} \Delta w + \frac{m u_o}{m} \Delta \dot{q} + \frac{Z_{\delta e}}{m} \Delta \delta e$$

solve for  $\dot{\Delta \theta}$ , substitute  $\Delta w$  above

$$\frac{m u_o}{m} \dot{\Delta \theta} = -\frac{Z_u}{m} \Delta u - \frac{Z_w}{m} \left( -\frac{M_{\delta e}}{M_w} \Delta \delta e \right) - \frac{Z_{\delta e}}{m} \Delta \delta e$$

$$\dot{\Delta \theta} = -\frac{Z_u}{m u_o} \Delta u + \left( \frac{Z_w}{m u_o} \frac{M_{\delta e}}{M_w} - \frac{Z_{\delta e}}{m u_o} \right) \Delta \delta e$$

$$\dot{\Delta u} = \frac{X_u}{m} \Delta u + \frac{X_{\delta e}}{m} \Delta \delta e$$

$$\boxed{\begin{bmatrix} \dot{\Delta u} \\ \dot{\Delta \theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} & -g \\ -\frac{Z_u}{m u_o} & 0 \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix} + \begin{bmatrix} \frac{X_{\delta e}}{m} \\ \frac{Z_w M_{\delta e}}{m u_o} - \frac{Z_{\delta e}}{m u_o} \end{bmatrix} \Delta \delta e}$$

$$A_{ph} = \begin{bmatrix} -0.0069 & -32.2 \\ 0.0001 & 0 \end{bmatrix}$$

$$B_{ph, \delta e} = \begin{bmatrix} 0 \\ -0.0002 \\ -0.44 \end{bmatrix}$$

$$\Delta \delta_e = -k_\theta \Delta \theta$$

$$\vec{u} = -K_{ph, \theta} \vec{x}_{ph}$$

$$= -[0 \ k_\theta] \begin{bmatrix} \Delta u \\ \Delta \theta \end{bmatrix}$$

$$\begin{aligned}
 A^{cl} &= A_{ph} - B_{ph\theta_e} K_{ph\theta} \\
 &= A_{ph} - \begin{bmatrix} 0 \\ -0.44 \end{bmatrix} [0 \quad k_\theta] \\
 &= A_{ph} - \begin{bmatrix} 0 & 0 \\ 0 & -0.44 k_\theta \end{bmatrix} \\
 &= \begin{bmatrix} -0.0069 & -32.2 \\ -0.0001 & 0.44 k_\theta \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 |A^{cl} - \lambda I| &= (-0.0069 - \lambda)(0.44 k_\theta - \lambda) - (-0.0001)(-32.2) = 0 \\
 &= \lambda^2 + \underbrace{(-0.44 k_\theta + 0.0069)\lambda}_{2\zeta\omega_n} - \underbrace{0.003 k_\theta - 0.0032}_{-\omega_n^2} = 0
 \end{aligned}$$

$$\zeta = 0.7 \Rightarrow k_\theta = -0.2$$

Plug back into  $4 \times 4$

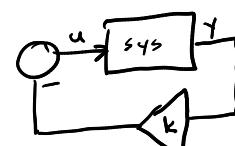
$$\begin{aligned}
 \Delta \delta_e &= -K_\theta \vec{x}_{ion} \\
 &= -[0 \ 0 \ 0 \ k_\theta] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}
 \end{aligned}$$

$$A^{cl} = A_{ion} - \underline{B_{\delta e}} K_\theta$$


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nlocus assumes  $\underline{u} = -k y$

$$\Delta \delta_e = -k_\theta \Delta \theta$$



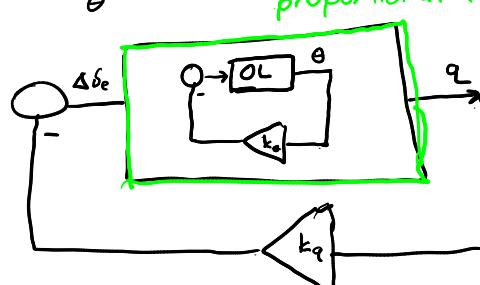
$$\begin{aligned}
 y = \Delta \theta &= [0 \ 0 \ 0 \ 1] \begin{bmatrix} \vec{x}_{ion} \\ C_\theta \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix} \\
 D &= 0
 \end{aligned}$$


---

derivative gain

$$\Delta \delta_e = \underbrace{-k_\theta \Delta \theta}_{\text{proportional}} - \underbrace{k_q \Delta q}_{\text{derivative}}$$

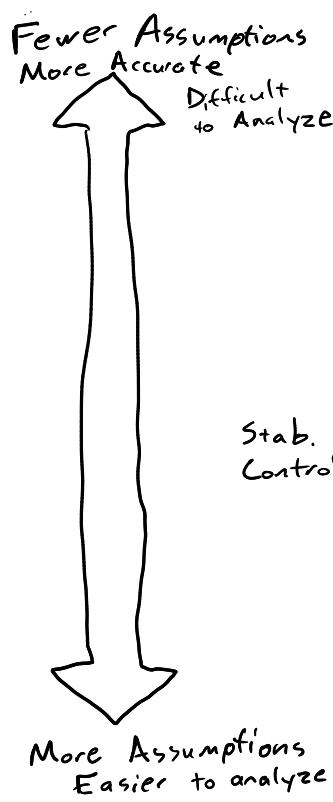
$k_\theta = -0.5$  proportional feedback



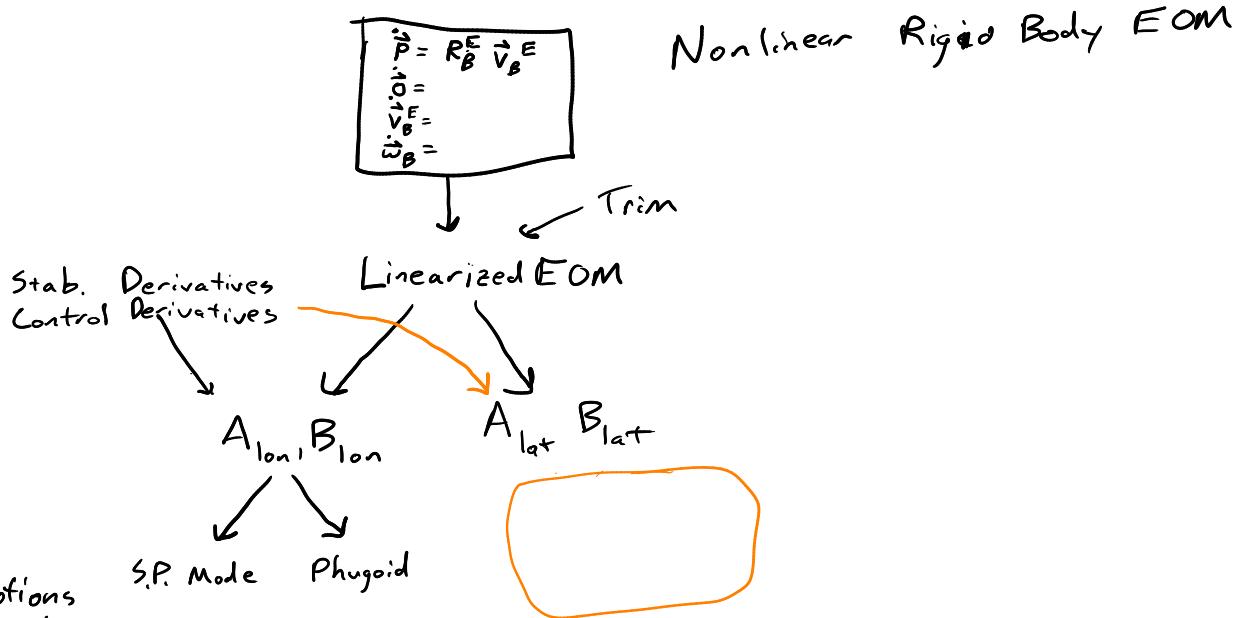
$$\Delta q = [0 \ 0 \ 1 \ 0] \begin{bmatrix} \Delta u \\ \Delta w \\ \Delta q \\ \Delta \theta \end{bmatrix}$$

from root locus choose  $k_q = -1$

# Lateral / Directional Dynamics



## Big Picture



$$\Delta \dot{\phi} = \Delta p + \Delta r \tan \theta_0 \quad \leftarrow \text{Lat}$$

$$\Delta \dot{\theta} = \Delta q \quad \leftarrow \text{long}$$

$$\Delta \dot{u} = -g \cos \theta_0 \Delta \theta + \frac{\Delta X}{m} \quad \leftarrow$$

$$\Delta \dot{v} = -u_0 \Delta r + g \cos \theta_0 \Delta \phi + \frac{\Delta Y}{m} \quad \leftarrow$$

$$\Delta \dot{w} = u_0 \Delta q - g \sin \theta_0 \Delta \theta + \frac{\Delta Z}{m} \quad \leftarrow$$

$$\Delta \dot{p} = \Gamma_3 \Delta L + \Gamma_4 \Delta N \quad \leftarrow$$

$$\Delta \dot{q} = \frac{\Delta M}{I_y} \quad \leftarrow$$

$$\Delta \dot{r} = \Gamma_4 \Delta L + \Gamma_8 \Delta N \quad \leftarrow$$

$$\Gamma_1 = \frac{I_{xz}(I_x - I_y + I_z)}{\Gamma}$$

$$\Gamma_2 = \frac{I_z(I_z - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma}$$

$$\Gamma_4 = \frac{I_{xz}}{\Gamma}$$

$$\Gamma_5 = \frac{I_z - I_x}{I_y}$$

$$\Gamma_6 = \frac{I_{xz}}{I_y}$$

$$\Gamma_7 = \frac{I_x(I_x - I_y) + I_{xz}^2}{\Gamma}$$

$$\Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma = I_x I_z - I_{xz}^2$$

$$Y = \frac{1}{2} \rho V^2 S C_Y(\beta, \rho, r, \delta_a, \delta_r)$$

$$L = \frac{1}{2} \rho V^2 S_b C_L(\beta, \rho, r, \delta_a, \delta_r)$$

$$N = \frac{1}{2} \rho V^2 S_b C_n(\beta, \rho, r, \delta_a, \delta_r)$$

$$Y \approx \frac{1}{2} \rho V_a^2 S \left[ C_{Y_0} + C_{Y_\beta} \beta + C_{Y_p} \left( \frac{b}{2V_a} p \right) + C_{Y_r} \frac{b}{2V_a} r + C_{Y_{\delta_a}} \delta_a + C_{Y_{\delta_r}} \delta_r \right]$$

$$L \approx \frac{1}{2} \rho V_a^2 S b \left[ C_{l_0} + C_{l_\beta} \beta + C_{l_p} \frac{b}{2V_a} p + C_{l_r} \frac{b}{2V_a} r + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r \right]$$

$$N \approx \frac{1}{2} \rho V_a^2 S b \left[ C_{n_0} + C_{n_\beta} \beta + C_{n_p} \frac{b}{2V_a} p + C_{n_r} \frac{b}{2V_a} r + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r \right]$$

For symmetric aircraft,  $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat} \mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ . & 1 & \tan \theta_0 & 0 \end{pmatrix}$$



+  $\beta$ : wind coming from right

$$\beta = \sin^{-1} \left( \frac{\Delta v}{V} \right)$$

$$\beta \approx \frac{\Delta v}{u_0} = \hat{v}$$

**Table 4.5**  
**Lateral Dimensional Derivatives**

	$Y$	$L$	$N$
$v$	$\frac{1}{2} \rho u_0 S C_{y_\beta}$	$\frac{1}{2} \rho u_0 b S C_{l_\beta}$	$\frac{1}{2} \rho u_0 b S C_{n_\beta}$
$p$	$\frac{1}{4} \rho u_0 b S C_{y_p}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_p}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_p}$
$r$	$\frac{1}{4} \rho u_0 b S C_{y_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{l_r}$	$\frac{1}{4} \rho u_0 b^2 S C_{n_r}$

$$L = \frac{1}{2} \rho V^2 S b C_L \checkmark \text{ nondimensional use lower-case for moments}$$

$$\rightarrow L_v \equiv \frac{\partial L}{\partial v} \Big|_0 = \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0^2 S b \frac{\partial C_L}{\partial \beta} \Big|_0 \frac{\partial \beta}{\partial v} \Big|_0$$

$$= \frac{1}{2} \rho u_0 S b C_{l_\beta}$$

$$C_{l_\beta} \equiv \frac{\partial C_L}{\partial \beta} \Big|_0 = \frac{\partial C_L}{\partial \hat{v}} \Big|_0$$

$$\beta = \hat{v} = \frac{\Delta v}{u_0} \Rightarrow \frac{\partial \beta}{\partial v} = \frac{1}{u_0}$$

**Table 5.2**  
**Summary—Lateral Derivatives**

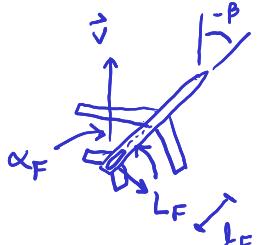
	$C_y$	$C_l$	$C_n$
$\beta$	$* -a_F \frac{S_F}{S} \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$	N.A.	$* a_F V_V \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$
$\hat{p}$	$* -a_F \frac{S_F}{S} \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$	N.A.	$* a_F V_V \left( 2 \frac{z_F}{b} - \frac{\partial \sigma}{\partial \hat{p}} \right)$
$\hat{r}$	$* a_F \frac{S_F}{S} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* a_F \frac{S_F}{S} \frac{z_F}{b} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$	$* -a_F V_V \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$

\*means contribution of the *tail only*, formula for wing-body not available;  $V_F/V = 1$ .

N.A. means no formula available.

### $\beta$ derivatives

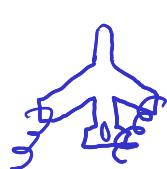
$C_{n_\beta}$ : weathervane derivative / yaw stiffness Sign? +



$$\alpha_F = -\beta + \sigma \quad \text{side wash}$$

$$C_{L_F} = a_F (-\beta + \sigma)$$

$$N_F = - C_{L_F} \frac{1}{2} \rho V_F^2 S_F l_F$$



$$C_{n_F} = - C_{L_F} \frac{S_F l_F}{S b} \left( \frac{V_F}{V} \right)^2$$

$$V_V = \frac{S_F l_F}{S b}$$

$$= - V_V C_{L_F} \left( \frac{V_F}{V} \right)^2$$

$$(C_{n_\beta})_{\text{Tail}} = \frac{\partial C_{n_F}}{\partial \beta} \Big|_0 = - V_V \left( \frac{V_F}{V} \right)^2 \frac{\partial C_{L_F}}{\partial \beta} \Big|_0$$

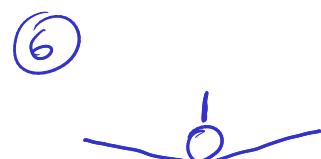
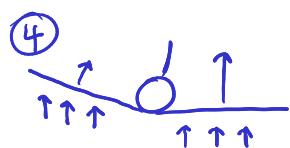
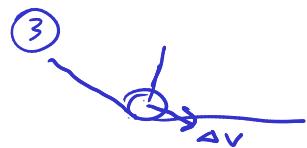
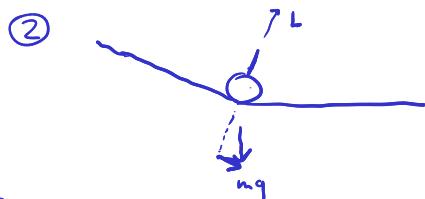
$$a_F \left( -1 + \frac{\partial \sigma}{\partial \beta} \right)$$

$$(C_{n_\beta})_{\text{Tail}} = V_V a_F \left( \frac{V_F}{V} \right)^2 \left( 1 - \frac{\partial \sigma}{\partial \beta} \right)$$

$C_{y_\beta}$ : usually small, similar derivation to  $C_{n_\beta}$  Sign? -

$C_{l_\beta}$ : Dihedral Effect Sign? -

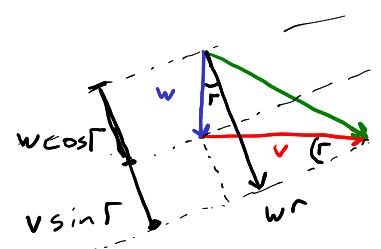
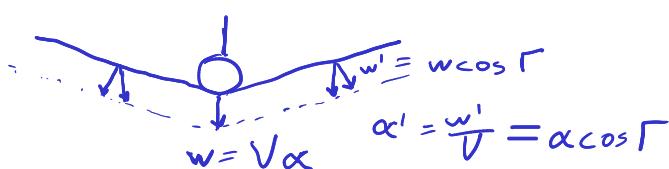
(looking from behind)



#### 4 Significant Factors

1. Dihedral Angle
2. Wing Height
3. Wing Sweep
4. Vertical Tail

#### 1 Dihedral Angle



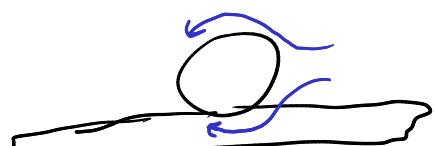
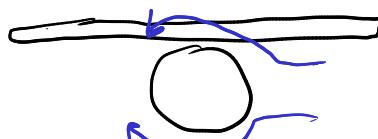
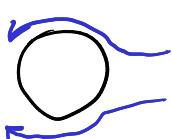
$$w^r = w \cos \Gamma + v \sin \Gamma \approx w + v \Gamma$$

$$\alpha^l \approx \frac{w^l}{u_0} = \alpha - \beta \Gamma$$

$$\alpha^r \approx \frac{w^r}{u_0} = \alpha + \beta \Gamma$$

$$\boxed{\begin{aligned} L \alpha (\alpha^l - \alpha^r) \\ C_{L\beta} \alpha - \Gamma \end{aligned}}$$

#### 2. Wing Height



High Wing: -ve  $C_{L\beta}$  contribution

Low Wing: +ve  $C_{L\beta}$  contribution

### 3. Wing Sweep



$$C_{\ell_B}^A \propto 2 C_L V^2 \sin(z\alpha) \\ + \alpha \Rightarrow + C_{\ell_B}$$

### 4. Tail



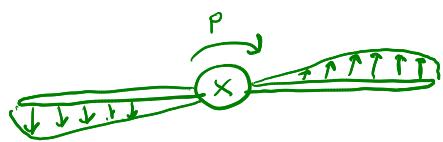
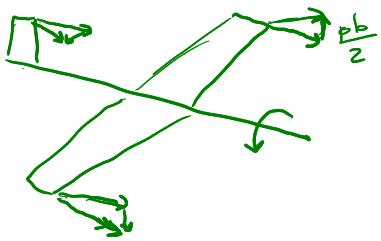
$$\Delta C_L^F = C_{L_F} \frac{S_F z_F}{S_b} = \alpha_F \left( -\beta + \sigma \right) \frac{S_F z_F}{S_b} \\ C_{\ell_B}^F = -\alpha_F \left( 1 - \frac{\partial \sigma}{\partial \beta} \right) \frac{S_F z_F}{S_b} \left( \frac{V_F}{V} \right)^2$$



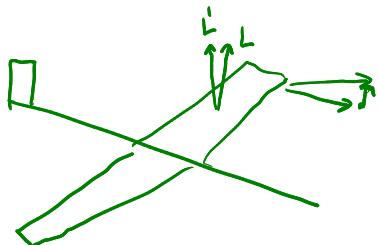
## p-derivatives

$$\hat{p} = \frac{\rho b}{2V}$$

$C_{\ell p}$ : roll damping Sign? -



$C_{n_p}$  Wing Effect



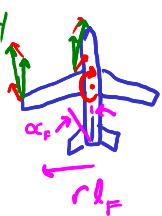
$$(C_{n_p})_{tail} = \alpha_F V_v \left( 2 \frac{z_F}{b} + \frac{\partial \sigma}{\partial \hat{p}} \right)$$

$C_{y_p}$  (usually small)

(similar to derivation for  $(C_{n_B})_{tail}$ )

## r-derivatives

Wing-Body



$$\Delta \alpha_F = \frac{rl_F}{u_0} + r \frac{\partial \sigma}{\partial r}$$

$$= \hat{r} \left( \frac{2l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\frac{\partial \alpha_F}{\partial \hat{r}} = \left( \frac{2l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\vec{\omega} \times \vec{v}_P$$

$$C_y = \frac{Y}{\frac{1}{2} \rho u_0^2 S}$$

$$(C_y)_{tail} = \frac{\frac{1}{2} \rho u_0^2 S_F \alpha_F \hat{r}}{\frac{1}{2} \rho u_0^2 S}$$

$$(C_y)_{tail} = \frac{\partial (C_y)_{tail}}{\partial \hat{r}} \Big|_0 = \alpha_F \frac{S_F}{S} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{\ell r})_{tail} = \alpha_F \frac{S_F}{S} \frac{z_F}{b} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$(C_{nr})_{tail} = -\alpha_F \frac{V_v}{u_0} \left( 2 \frac{l_F}{b} + \frac{\partial \sigma}{\partial \hat{r}} \right)$$

$$\frac{S_F}{S} \frac{l_F}{b}$$

yaw damping

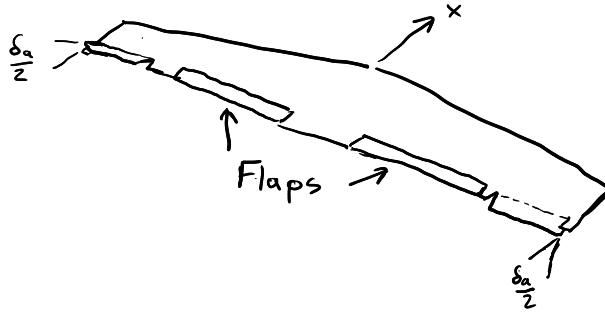


# Lateral Control and Coordinated Turn

## Rudder



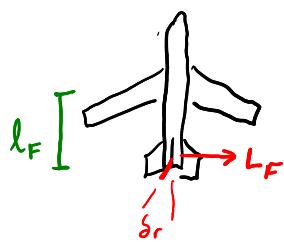
## Aileron



## Rudder Power

$$C_{n_{\delta_r}}$$

$$N_F = -l_F L_F = -l_F \frac{1}{2} \rho V_F^2 S_F C_{L_F}(\alpha_F, \delta_r)$$



$$C_{n_F} = \frac{N_F}{\frac{1}{2} \rho V^2 S_b} = - \underbrace{\frac{l_F S_F}{S_b}}_{V_v} \left( \frac{V_F^2}{V^2} \right) C_{L_F} = -V_v \left( \frac{V_F^2}{V^2} \right) C_{L_F}$$

$$C_{n_{\delta_r}} \equiv \left. \frac{\partial C_{n_F}}{\partial \delta_r} \right|_0 = -V_v \left( \frac{V_F^2}{V^2} \right) \left. \frac{\partial C_{L_F}}{\partial r} \right|_0 = \boxed{-a_r V_v \left( \frac{V_F^2}{V^2} \right)}$$

## Other nondim. rudder control derivatives

$$C_{\gamma_{\delta_r}} \text{ sign? } +$$

$$C_{q_{\delta_r}} \text{ sign? } +$$

## Aileron nondim. control derivatives

$$C_{l_{\delta_a}} \text{ sign? } -$$

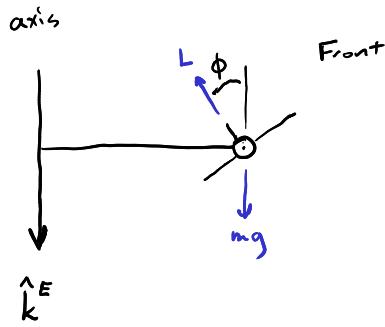
aileron reversal can occur due to wing twist

$$C_{n_{\delta_a}} \text{ sign? can be either}$$

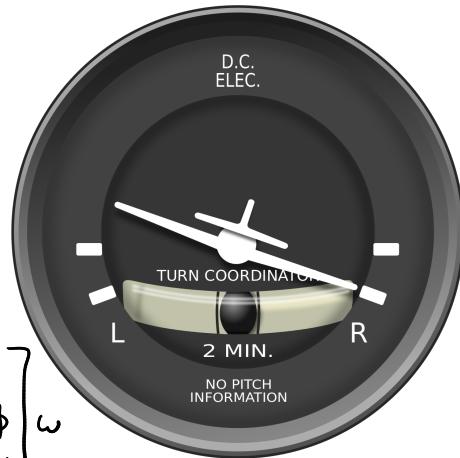
$C_{n_{\delta_a}} > 0$ , called adverse yaw

$$C_{\gamma_{\delta_a}} \text{ usually small}$$

## Coordinated Turn



- angular velocity vector is constant and aligned with inertial  $\hat{z}$
- No aerodynamic forces in a/c  $\gamma$  direction



$$\omega = \frac{u_0}{R}$$

$$a_n = \omega^2 R = \frac{u_0^2}{R}$$

$$\vec{\omega}_E = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

$$\vec{\omega}_B = R_E^B \vec{\omega}_E = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \sin\phi \cos\theta \\ \cos\phi \cos\theta \end{bmatrix} \omega = \begin{bmatrix} -\theta \\ \sin\phi \\ \cos\phi \end{bmatrix} \omega$$

assume  $\theta$  small  
but  $\phi$  is not

$$L \cos\phi = mg$$

$$L \sin\phi = m a_n = m \frac{u_0^2}{R} = m \omega u_0$$

$$\boxed{\tan\phi} = \frac{L \sin\phi}{L \cos\phi} = \frac{m \omega u_0}{mg} = \boxed{\frac{\omega u_0}{g}}$$

From EOM

$$\boxed{Z = -mg \cos\phi - mqu}$$

Assumed, no wind,  $V \ll u$ ,  $V = u = u_0$

$$\text{load factor } n = -\frac{Z}{mg} = \cos\phi + \frac{qu_0}{g}$$

$$= \cos\phi + \frac{\omega u_0 \sin\phi}{g}$$

$$= \cos\phi + \tan\phi \sin\phi$$

$$\boxed{n = \sec\phi}$$

$$n = \frac{L}{w}$$

$$\Delta C_L = \frac{L - mg}{\frac{1}{2} \rho V^2 S} = (n - 1) C_w$$

### Coordinated Turn

$$C_x = 0$$

$$= C_{y_B} \beta + C_{y_p} \hat{p} + C_{y_r} \hat{r} + C_{y_s} \delta_r + C_{y_a} \delta_a$$

$$C_R = 0$$

$$= \vdots$$

$$C_n = 0$$

$$= \vdots$$

$$C_m = 0$$

$$= \vdots$$

$$C_L = (n - 1) C_w =$$

$$\boxed{\beta, p, r, \delta_r, \delta_a \quad \alpha, q, \delta_e}$$

determined by  
 $\omega, \theta, \phi$

$$\begin{array}{l}
 Y \\
 l \\
 n \\
 \hline
 \end{array} \quad
 \begin{bmatrix}
 C_{Y\beta} & C_{Y\delta_r} & 0 \\
 C_{l\beta} & C_{l\delta_r} & C_{l\delta_a} \\
 C_{n\beta} & C_{n\delta_r} & C_{n\delta_a}
 \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \delta_a \end{bmatrix} = \begin{bmatrix} C_{Y\beta} & C_{Yr} \\ C_{l\beta} & C_{lr} \\ C_{n\beta} & C_{nr} \end{bmatrix} \begin{bmatrix} \theta \\ -\cos\phi \end{bmatrix} \frac{\omega b}{2u_0}$$
  

$$\begin{bmatrix}
 C_{m\alpha} & C_{m\delta_e} \\
 C_{L\alpha} & C_{L\delta_e}
 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta\delta_e \end{bmatrix} = - \begin{bmatrix} C_{m\alpha} \\ C_{L\alpha} \end{bmatrix} \frac{\omega \bar{c} \sin\phi}{2u_0} + \begin{bmatrix} 0 \\ (n-1)C_w \end{bmatrix}$$

Steady Sideslip

$\text{from back}$

$$\begin{aligned}
 Y + mg \sin\phi &= 0 \\
 Y + mg\phi &= 0 \\
 L &= 0 \\
 N &= 0
 \end{aligned}$$

$$-mg\phi = Y = Y_v v + Y_p \overset{\circ}{\beta} + Y_r \overset{\circ}{\alpha} + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r$$

$$L = \dots$$

$$N =$$

$$mg\phi + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r = -Y_v v$$

$$\begin{bmatrix}
 Y_{\delta_r} & 0 & mg \\
 L_{\delta_r} & L_{\delta_a} & 0 \\
 N_{\delta_r} & N_{\delta_a} & 0
 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = - \begin{bmatrix} Y_v \\ L_v \\ N_v \end{bmatrix} v \quad \beta = \frac{\phi}{u_0}$$

## Steady Sideslip

For Piper Cherokee



$$\begin{bmatrix}
 280.7 & 0 & 2400 \\
 755.7 & -3821.9 & 0 \\
 -3663.5 & 359 & 0
 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \\ \phi \end{bmatrix} = \begin{bmatrix} 2.991 \\ 102.93 \\ -19.394 \end{bmatrix} v \quad (7.8, 4)$$

It is convenient to express the sideslip as an angle instead of a velocity. To do so we recall that  $\beta = v/u_0$ , with  $u_0$  given above as 112.3 fps. The solution of (7.8,4) is found to be

$$\begin{aligned}
 \delta_r/\beta &= .303 \\
 \delta_a/\beta &= -2.96 \\
 \phi/\beta &= .104
 \end{aligned}$$

We see that a positive sideslip (to the right) of say  $10^\circ$  would entail left rudder of  $3^\circ$  and right aileron of  $29.6^\circ$ . Clearly the main control action is the aileron displacement, without which the airplane would, as a result of the sideslip to the right, roll to the left. The bank angle is seen to be only  $1^\circ$  to the right so the sideslip is almost flat.

# Lateral Dynamic Modes

$$\dot{\mathbf{x}}_{lat} = \mathbf{A}_{lat}\mathbf{x}_{lat} + \mathbf{c}_{lat}$$

$$\mathbf{x}_{lat} = \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \end{pmatrix} \quad \mathbf{c}_{lat} = \begin{pmatrix} \frac{\Delta Y_c}{m} \\ \Gamma_3 \Delta L_c + \Gamma_4 \Delta N_c \\ \Gamma_4 \Delta L_c + \Gamma_8 \Delta N_c \\ 0 \end{pmatrix}$$

$$\mathbf{A}_{lat} = \begin{pmatrix} \frac{Y_v}{m} & \frac{Y_p}{m} & \left( \frac{Y_r}{m} - u_0 \right) & g \cos \theta_0 \\ \Gamma_3 L_v + \Gamma_4 N_v & \Gamma_3 L_p + \Gamma_4 N_p & \Gamma_3 L_r + \Gamma_4 N_r & 0 \\ \Gamma_4 L_v + \Gamma_8 N_v & \Gamma_4 L_p + \Gamma_8 N_p & \Gamma_4 L_r + \Gamma_8 N_r & 0 \\ 0 & 1 & \tan \theta_0 & 0 \end{pmatrix}$$

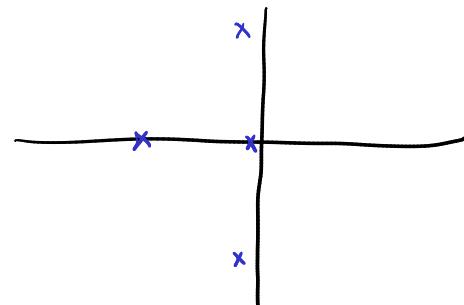


$$\mathbf{A}_{lat} = \begin{pmatrix} -0.0558 & 0 & -774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}_{lat} \mathbf{x}$$

$$\mathbf{x}(+) = \sum_i q_i \vec{v}_i e^{\lambda_i t}$$

	$\lambda_i$	$\zeta$	$\omega_n$
→	$-7.30e - 03$	$1.00e + 00$	$7.30e - 03$
→	$-5.62e - 01$	$1.00e + 00$	$5.62e - 01$
→	$-3.30e - 02 + 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$
→	$-3.30e - 02 - 9.47e - 01i$	$3.49e - 02$	$9.47e - 01$



$$\begin{pmatrix} \mathbf{v}_1 \\ 0.9821 \\ -0.0014 \\ 0.0078 \\ 0.1880 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_2 \\ -0.9972 \\ -0.0367 \\ 0.0021 \\ 0.0652 \end{pmatrix} \quad \begin{pmatrix} \mathbf{v}_{3/4} \\ -1.0000 \\ 0.0019 \mp 0.0032i \\ -0.0001 \pm 0.0011i \\ -0.0035 \mp 0.0019i \end{pmatrix}$$

Augmented State Space Dynamics Matrix

$$\begin{bmatrix} \Delta v \\ \vdots \\ \Delta \psi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{lat} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & \sec \theta_0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & \begin{matrix} \Delta v \\ \Delta p \\ \Delta r \\ \Delta \phi \\ \Delta \psi \\ \Delta \gamma_E \end{matrix} \end{bmatrix}$$

$$\dot{\Delta \psi} = \Delta r \sec \theta_0 \quad \text{if } \theta_0 = 0$$

$$\dot{\Delta \gamma_E} = u_0 \cos \theta_0 \Delta \psi + \Delta v$$

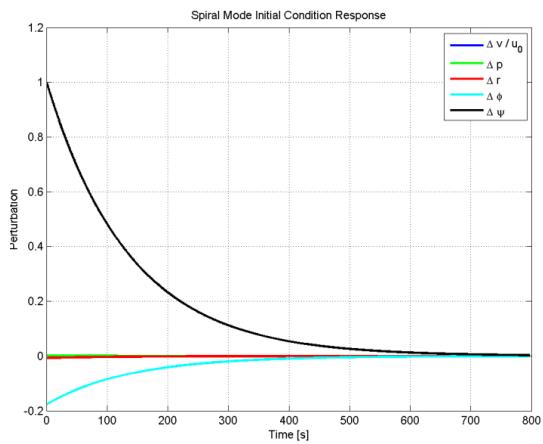
## Spiral Mode

$$\lambda = -0.0073$$

$$\tau = 137 \text{ s}$$

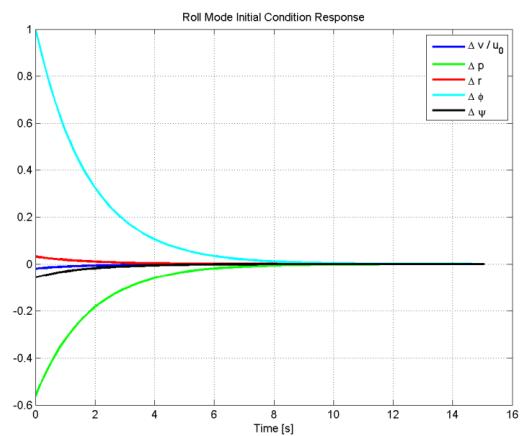
Normalize to  $\Delta\phi = 1$  + nondimensionalize velocity

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} 0.0068 \\ -0.074 \\ 0.04 \\ 1.0 \\ -5.66 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{small} \\ \leftarrow \text{a little} \\ \leftarrow \text{large} \\ \leftarrow \text{large} \end{array}$$



## Roll Mode

$$\hat{\mathbf{v}}_2 = \begin{bmatrix} -0.0198 \\ -0.5625 \\ 0.0316 \\ 1.0 \\ -0.0562 \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array} \quad \begin{array}{l} \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \\ \leftarrow \text{large} \\ \leftarrow \text{small} \end{array}$$



## Dutch Roll

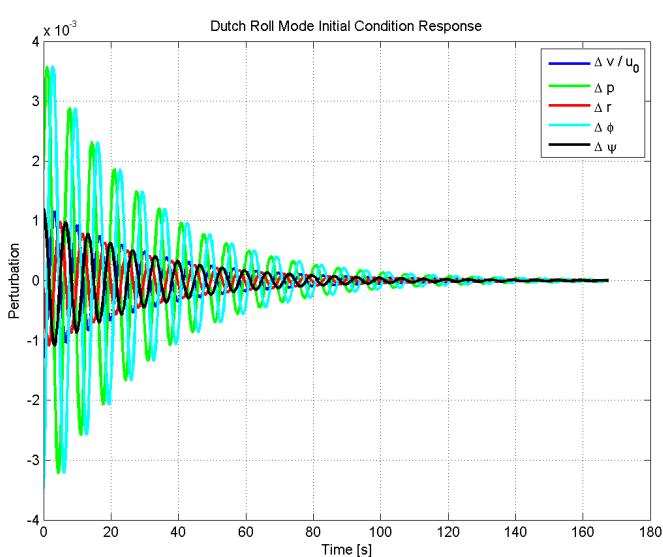
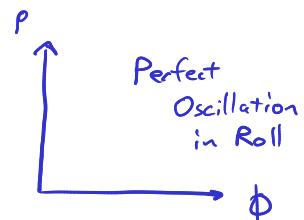
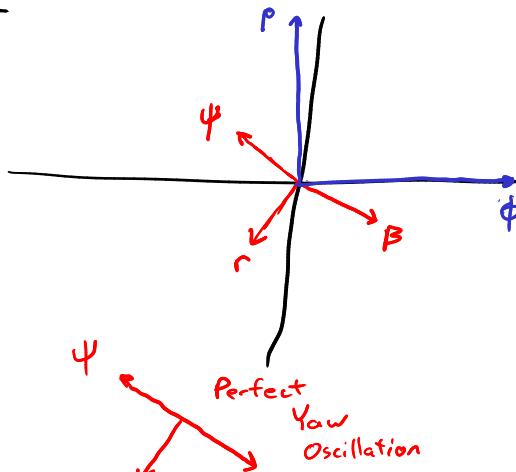
$$\hat{\mathbf{v}}_3 = \begin{bmatrix} 0.321 \angle -28^\circ \\ 0.9471 \angle 92^\circ \\ 0.2915 \angle -112^\circ \\ 1.0 \\ 0.3078 \angle 155^\circ \end{bmatrix} \quad \begin{array}{l} \hat{v} = \beta \\ p \\ r \\ \phi \\ \psi \end{array}$$

$$\lambda_{3,4} = -0.033 \pm 0.947i$$

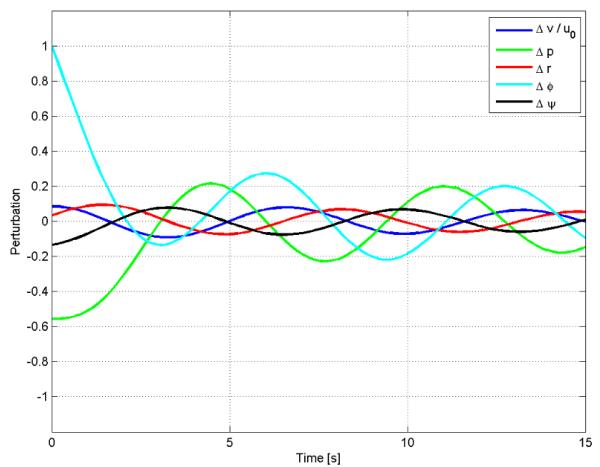
$$f = 0.0349$$

Poor Damping  
relatively fast

$$\omega_n = 0.947$$



$$\mathbf{x}(0) = 0.4 \cdot \operatorname{Re}(\mathbf{v}_r) + 0.4 \cdot \operatorname{Re}(\mathbf{v}_{dr}) + 0.2 \cdot \operatorname{Re}(\mathbf{v}_{spi})$$



## Lateral Mode Approximations

$$A_{lat} = \begin{bmatrix} Y_v & Y_p & Y_r & g \cos \theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{eg. } Y_v = \frac{Y_v}{m}$$

### Roll Approximation

$$r=0 \quad v=0 \quad \dot{\rho} = L_p \rho \quad |A - \lambda I| \quad \text{if } A \text{ is a scalar}$$

$$\lambda_{r, \text{approx}} = L_p \\ = -0.434$$

$$\lambda_r = -0.562 \\ 23\% \text{ difference}$$

### 2x2 Spiral Approximation

$$\rightarrow p=0$$

$$\rightarrow \dot{p}=0$$

Ignore  
side force

$$\begin{bmatrix} v \\ \dot{v} \\ r \\ \dot{r} \\ \phi \end{bmatrix} = \begin{bmatrix} Y_v & Y_p & Y_r & g \cos \theta_0 \\ L_v & L_p & L_r & 0 \\ N_v & N_p & N_r & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \rho \\ r \\ \phi \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ i \\ r \\ \dot{r} \\ \phi \end{bmatrix} = \begin{bmatrix} L_v & L_r \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$0 = L_v v + L_r r \Rightarrow v = -\frac{L_r}{L_v} r$$

$$\dot{r} = -N_v \frac{L_r}{L_v} r + N_r r = \underbrace{\left( \frac{N_r L_v - N_v L_r}{L_v} \right)}_{\lambda_s, \text{approx}} r$$

$$\lambda_s, \text{approx} = -0.0296 \\ \lambda_s = -0.0073$$

not great

### Characteristic-Eqn.-based spiral approx

$$|A_{lat} - \lambda I| = A \lambda^4 + B \lambda^3 + C \lambda^2 + D \lambda + E = 0$$

since  $\lambda_s \ll 1$

$$D \lambda + E = 0$$

$$\lambda_{s, \text{approx}} = -\frac{E}{D}$$

$$E = g [(N_r L_v - N_v L_r) \cos \theta_0 + (N_v L_p - L_v N_p) \sin \theta_0]$$

$$D = -g (L_v \cos \theta_0 + N_v \sin \theta_0) + u_0 (L_v N_p - L_p N_v)$$

$$\text{for 747} \quad \lambda_{s, \text{approx}} = -0.00725 \quad \text{very close!}$$

In Book

Characteristic-based spiral + roll eigenvalue approximation

## Dutch Roll Approx

Assume  $\phi = p = 0$

$$\frac{Y_r}{m} \ll u_0$$

$$\begin{bmatrix} y_v & y_p & y_r^{\approx u_0} & g \cos \theta_0 \\ \dot{z}_v & \dot{z}_p & \dot{z}_r & 0 \\ N_v & N_p & N_r & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \\ r \\ \phi \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} y_v & -u_0 \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r \end{bmatrix}$$

$$\lambda^2 - (y_v + N_r) \lambda + (y_v N_r + u_0 N_v) = 0$$

$$\lambda_{dr, \text{approx}} = -0.1008 \pm 0.9157i$$

$$\lambda_{dr} = -0.033 \pm 0.947i$$

# Laplace Transforms and Transfer Functions

$$\ddot{x}(+) + 2\zeta\omega_n \dot{x}(+) + \omega_n^2 x(+) = \omega_n^2 u(+)$$

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} + Du \end{aligned} \quad \iff \quad G_{yu}(s)$$

## Review: Properties of Laplace transforms

$$\mathcal{L}[x(+)](s) = \int_0^\infty e^{-st} x(+) dt$$

$$x(+) \iff x(s)$$

Appendix A1



$$\dot{x}(+) \iff s x(s) - x(+|_{t=0})$$

$$\int_0^+ x(\tau) d\tau \iff \frac{1}{s} x(s)$$

$$4 \sin(zt) + 2 \cos(zt)$$

$$\alpha x(+) + \beta y(+) \iff \alpha x(s) + \beta y(s)$$

$$\frac{4 \cdot z}{s^2 + z^2} + \frac{2s}{s^2 + 1}$$

## Review: Transfer Function

$$\ddot{x}(+) + 2\zeta\omega_n \dot{x}(+) + \omega_n^2 x(+) = \omega_n^2 u(+)$$

Assume 0 initial conditions

$$s^2 x(s) + 2\zeta\omega_n s x(s) + \omega_n^2 x(s) = \omega_n^2 u(s)$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) x(s) = \omega_n^2 u(s)$$

Transfer Function

$$G_{xu}(s) \equiv \frac{x(s)}{u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

↑  
0 initial conditions

Can you find a TF for any differential equation?

$$\ddot{x}(+) = -\dot{x}(z)^2 + u(+) \quad \text{X no TF for nonlinear diff eq.}$$

↑ nonlinear

## Review: How to use a TF

1. Input  $\rightarrow$  Output

$$u(+) \rightarrow u(s) \rightarrow x(s) = G_{xu}(s) u(s) \rightarrow x(+)$$

$$\frac{s+1}{s^3 + 2s^2 + 3s + 4} \rightarrow \frac{C_1}{s+a_1} + \frac{C_2}{s+a_2} \dots \xrightarrow{\text{table}}$$

↑  
partial frac  
decomposition

Ex.  $u(+) = \text{step function}$



$$u(s) = \frac{1}{s}$$

$$x(s) = G_{xu}(s) u(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{s}$$

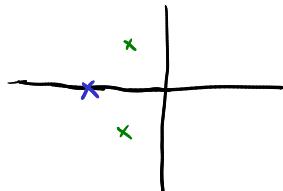
$$x(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

## 2. Stability

Roots of TF denominator are eigenvalues of A matrix  
a.k.a. poles

$\curvearrowleft G(s) \rightarrow \infty$  if  $s$  is at a pole

$$\frac{1}{s+1}, \text{ pole} = -1$$



If all poles are on LHP system is stable

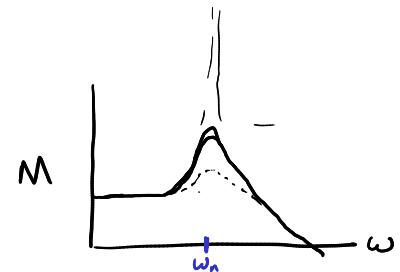
$$G_{xu}(s) = \frac{1}{(s+2)(s^2+2s+2)}$$

$$s=-2 \quad s=-1 \pm i$$

## 3. Steady-State Behavior

### a) Final Value Theorem

If  $sx(s)$  is stable

$$\lim_{s \rightarrow \infty} s x(s) = \lim_{s \rightarrow 0} s x(s)$$


### b) Frequency Response / Harmonic Response

If  $u(t) = A \cos(\omega t)$  the steady-state  $x(t)$  is  $AM \cos(\omega t + \phi)$

$$M = |G_{xu}(i\omega)| \quad \text{and} \quad \phi = \angle G_{xu}(i\omega)$$

$$z = a + bi \quad \text{or} \quad re^{j\phi} \quad \text{or} \quad r \angle \phi$$

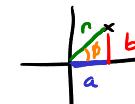
$$G(s) = \frac{1}{s^2 + 2s + 2}$$

$$G(i\omega) = \frac{1}{-\omega^2 + 2i\omega + 2}$$

$$= \frac{1}{\underline{-\omega^2 + 2} + \frac{2i\omega}{b}}$$

$$= \frac{1}{\sqrt{(-\omega^2 + 2)^2 + (2\omega)^2}} e^{i \tan^{-1} \frac{2\omega}{-\omega^2 + 2} \phi}$$

$$= \frac{1}{r} e^{-i\phi}$$



$$\dot{\vec{x}} = A\vec{x} + B_u \quad \rightarrow \quad \vec{y} = C\vec{x} + D_u \quad \iff \quad G_{yu}(s)$$

### State Space to TF

$$ssZ + f$$

$$tf(sys)$$

$$s\vec{x}(s) = A\vec{x}(s) + B u(s)$$

$$(sI - A)\vec{x}(s) = B u(s)$$

$$\vec{x}(s) = (sI - A)^{-1} B u(s)$$

$$y(s) = C\vec{x}(s) + D u(s) \quad \text{assume } D=0 \text{ from here on}$$

$$y(s) = C(sI - A)^{-1} B u(s) + D u(s)$$

$$M^{-1} = \frac{\text{adj}(M)}{|M|} \quad (sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

Adjugate: Transpose of Cofactor Matrix F

$$F_{ij} = (-1)^{i+j} |M_{-i-j}|$$

$\underbrace{-i}_{=i=\text{all except } i}$

If  $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

then  $F = \begin{bmatrix} |ef| & -|df| & \dots \\ |hi| & -|gi| & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

$$\text{adj}(M) = F^T$$

$$G_{yu}(s) = \frac{Y(s)}{U(s)} = \boxed{C(sI - A)^{-1}B} = \boxed{\frac{C \text{adj}(sI - A)B}{|sI - A|}} = \frac{N(s)}{D(s)}$$

Roots of  $D(s)$  are eigenvalues of A

TF to state space

+ 2ss

$$G_{yu}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + \dots + a_{n-1}s + a_n}$$

strictly proper  
 $n > m$

$$\text{Ex: } G_{yu}(s) = \frac{b_0 s + b_1}{s^2 + a_1 s + a_2}$$

multiply by  $\frac{x(s)}{x(s)}$

$$\frac{Y(s)}{U(s)} = G_{yu}(s) = \frac{b_0 s x(s) + b_1 x(s)}{s^2 x(s) + a_1 s x(s) + a_2 x(s)}$$

$$\rightarrow y(t) = b_0 \dot{x}(t) + b_1 x(t) \quad \leftarrow$$

$$u(t) = \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t)$$

$$\rightarrow \ddot{x}(t) = -a_1 \dot{x}(t) - a_2 x(t) + u(t) \quad \leftarrow$$

$$\dot{\vec{x}} = A\vec{x} + Bu \quad \rightarrow$$

$$\vec{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \leftarrow$$

$$y = C\vec{x} + Du \quad \leftarrow$$

$$C = \begin{bmatrix} b_1 & b_0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

$$G_{Y_u}(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + \dots + a_{n-1}s + a_n}$$

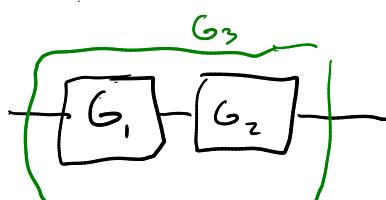
strictly proper  
 $n > m$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & & \\ & & & & 0 & \\ -a_n & \dots & \dots & \dots & -a_1 & \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

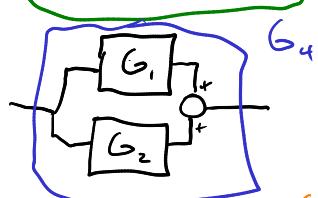
$$C = [b_m \dots b_0 \underbrace{0 \ 0 \ 0}_{\text{if } n > m+1}] \quad D = [0]$$

$\vec{x}$  may not correspond to physical states

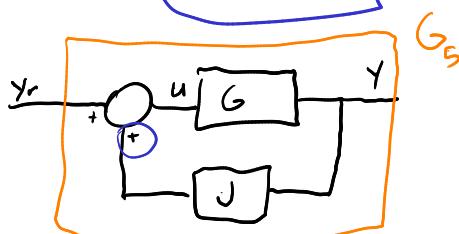
### Review: Block Diagrams



$$G_3 = G_1 G_2$$



$$G_4 = G_1 + G_2$$



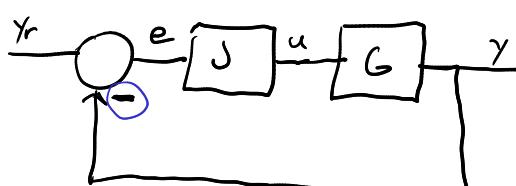
$$G_5 = \frac{Y}{Y_r}$$

$$\begin{aligned} Y &= Gu \\ u &= Y_r + Jy \end{aligned}$$

$$Y = G(Y_r + Jy)$$

$$(1 - JG)Y = GY_r$$

$$\frac{Y}{Y_r} = \boxed{\frac{G}{1 - JG}}$$



$$Y = JGe$$

$$e = Y_r - Y$$

$$Y = JG(Y_r - Y)$$

$$(1 + JG)Y = JGY_r$$

$$\frac{Y}{Y_r} = \boxed{\frac{JG}{1 + JG}}$$

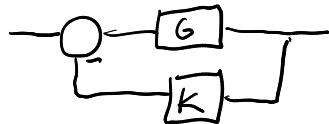
$$\frac{G}{1 + JG}$$

if -ive feedback

$$\frac{JG}{1 - JG}$$

for +ve feedback

## TFs and Root Loci

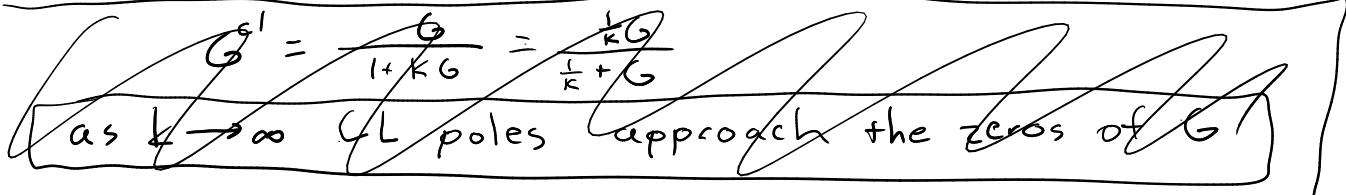


$G \leftarrow$  open loop TF

$$G^{cl} = \frac{G}{1+KG}$$

$$K=0 \Rightarrow G^{cl}=G$$

Root locus starts at poles of the OL systems



$$J=K$$

$$G^{cl} = \frac{G}{1+KG} \quad \text{or} \quad \frac{KG}{1+KG}$$

$$1+KG=0$$

$$\frac{1}{K} + G = 0$$

$$G(s) = \frac{N(s)}{D(s)}$$

$$\text{As } K \text{ gets large } \underline{G(s) = -\frac{1}{K}}$$

When  $K$  is large, two possibilities for CL poles

1.  $N(s)$  is close to zero

CL poles close to O.L. zeros

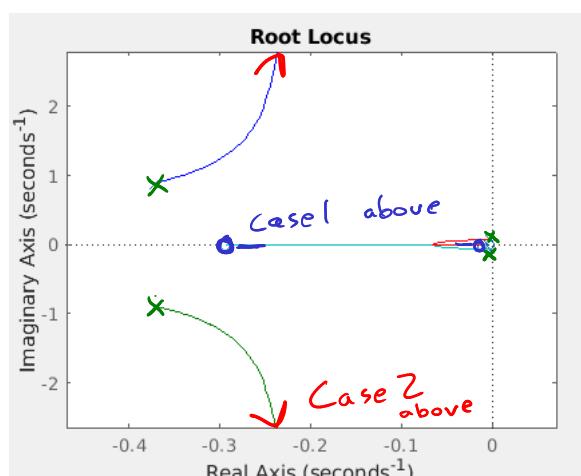
2.  $D(s)$  is very large

Magnitude of CL poles is very large

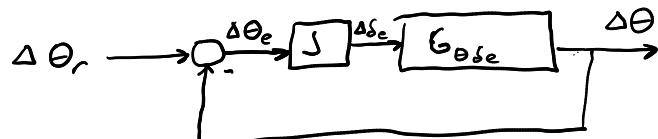
B744 Long dynamics with  $\Delta\delta_e = -K\Delta\theta$

O.L. Poles :  $-0.37 \pm 0.89i$   
 $-0.0033 \pm 0.067i$

O.L. Zeros :  $-0.0113, -0.2948$



## Designing a Pitch - Hold Controller



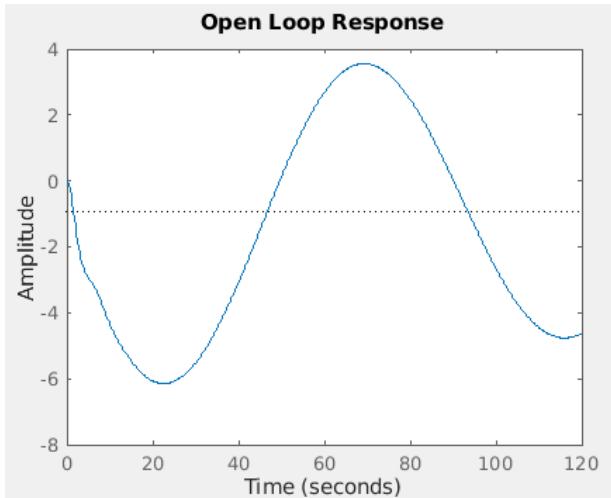
$$\Delta\theta_e = \Delta\theta_r - \Delta\theta$$

Goals

1. Improve Damping

(O.L. Phugoid  
has low damping)

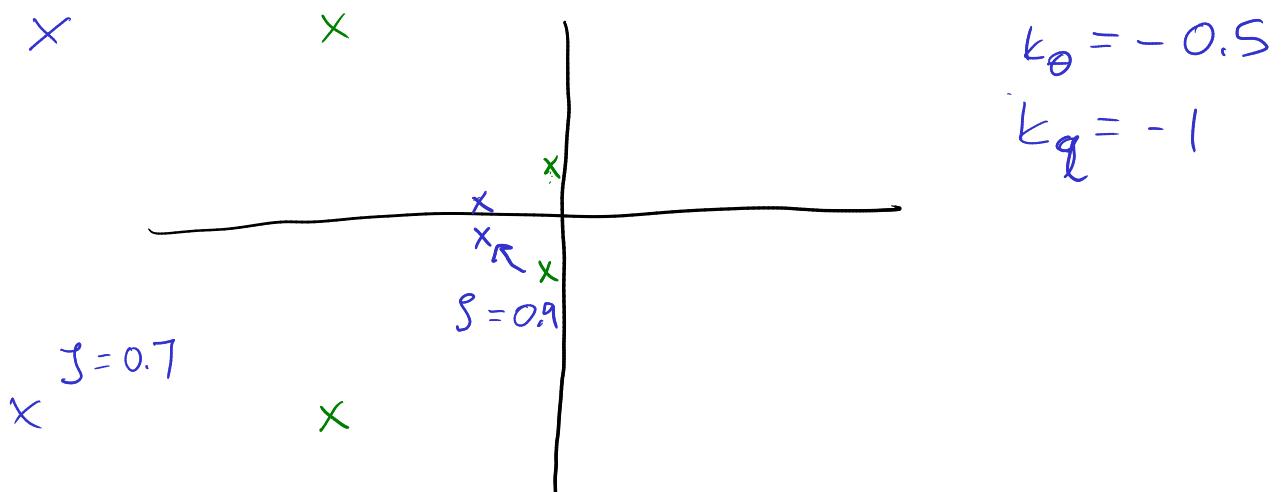
2. Low steady-state error  $|\theta - \theta_r| \rightarrow 0$



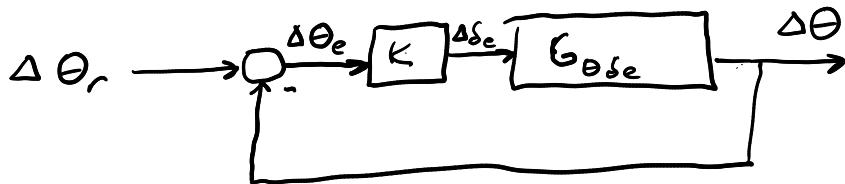
Previously

$$\Delta\delta_e = -k_\theta \Delta\theta - k_q \Delta q$$

$k_\theta$  proportional     $k_q$  derivative gain



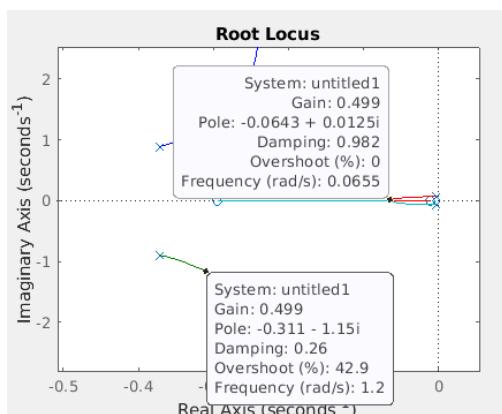
# Controller 1



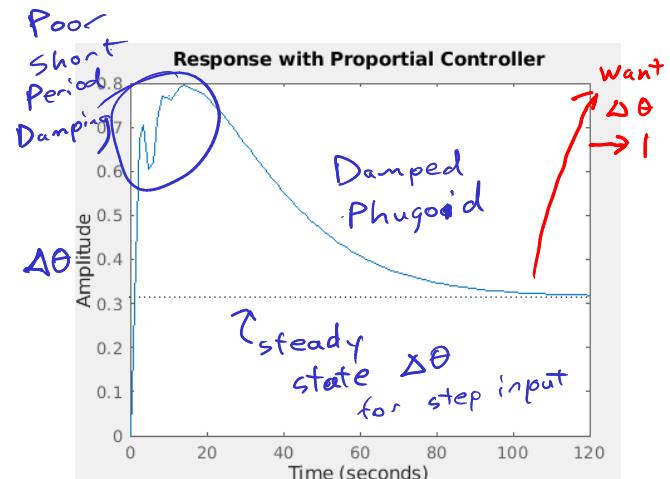
Proportional controller

$$J = K$$

$$\Delta \theta_e = K(\Delta \theta_r - \Delta \theta)$$

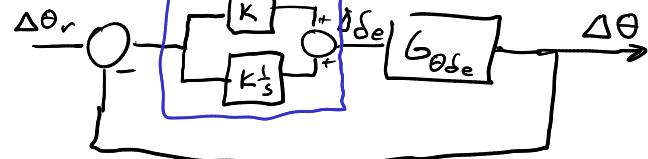


Choose  $K = -0.5$



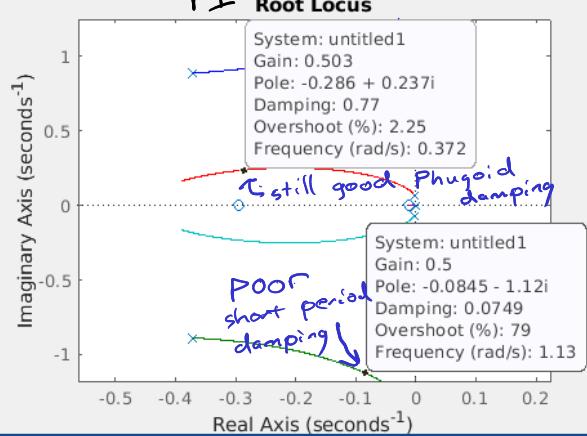
# Controller 2: PI

$$J = K \left( 1 + \frac{1}{s} \right)$$

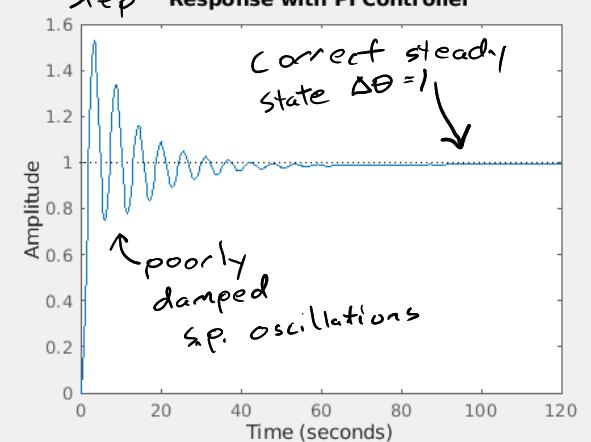


$J$

## PI Root Locus

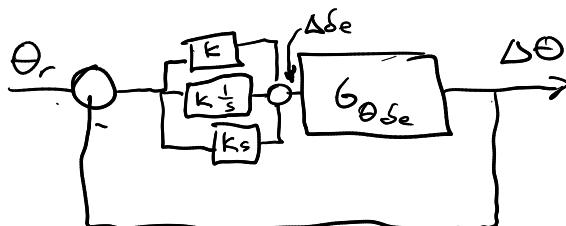


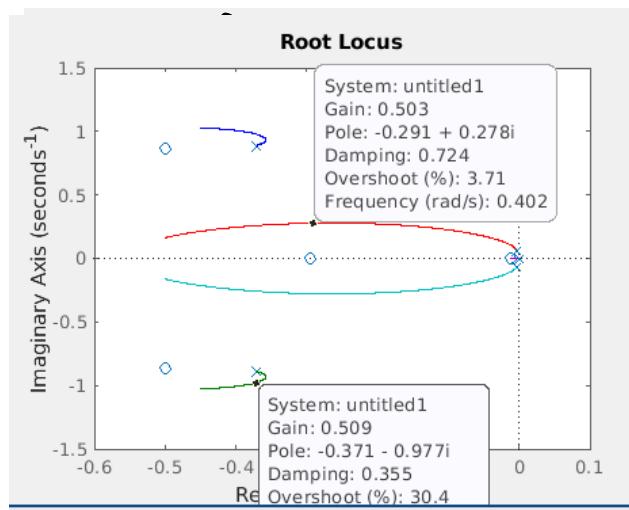
## Step Response with PI Controller



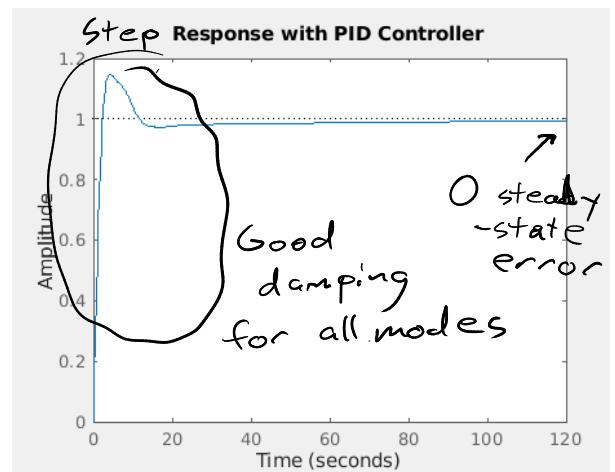
# Controller 3: PID

$$J = K \left( 1 + \frac{1}{s} + \frac{1}{s^2} \right)$$



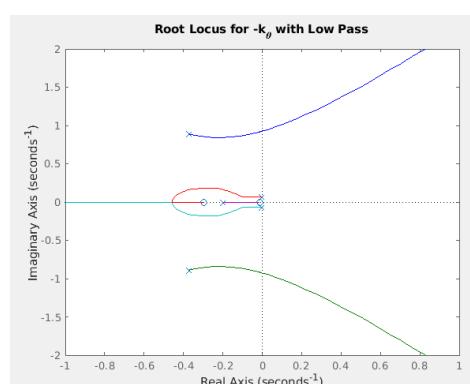
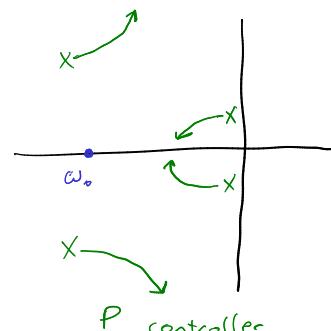
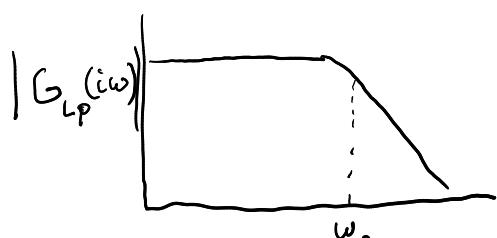
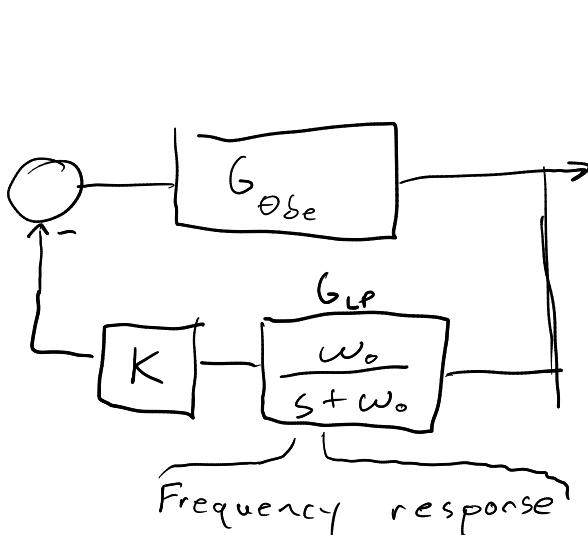
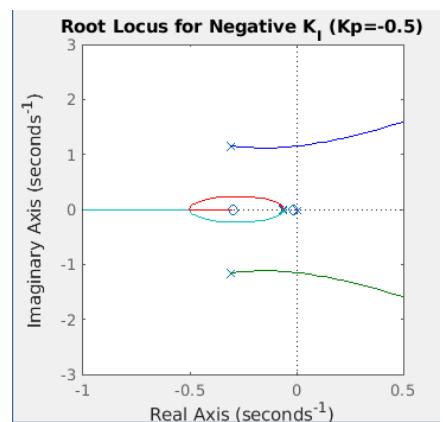
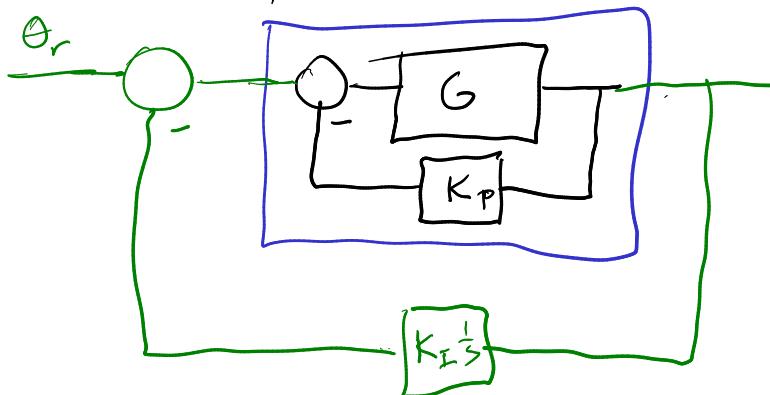


Choose  $K = -0.5$

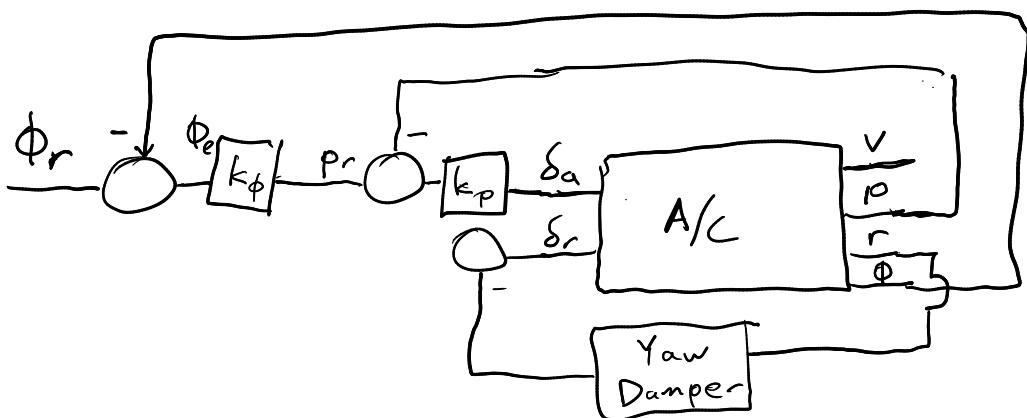


$$j = -0.5 \left( 1 + \frac{1}{s} + s \right)$$

If you wanted to tune individually

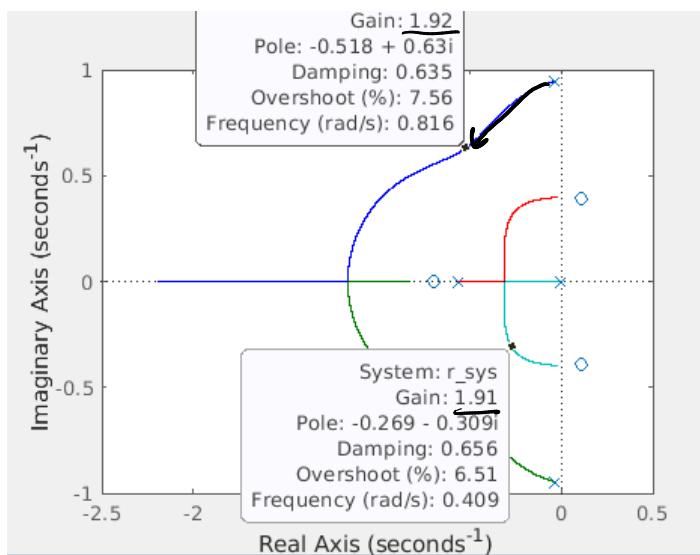


# Designing a Roll Controller



## Part 1: Yaw Damper

$$f_r = -k_r r$$



choose.

$$k_r = -1.9$$

Problem : In steady right turn,  $r > 0$   
 with  $k_r = -1.9$ ,  $\delta_r > 0$

Want:  $S_r \rightarrow 0$  at low frequency  
 $S_r = -k_r r$  at dutch roll frequency

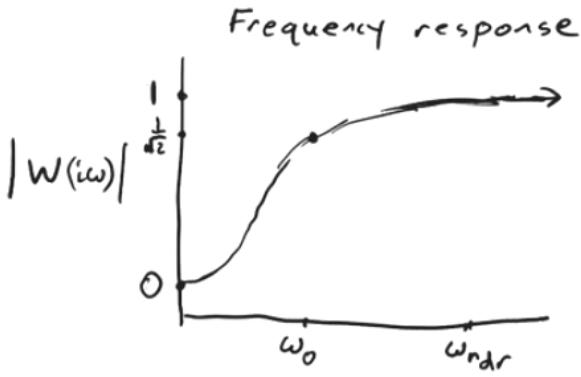
## Washout / High Pass Filter

$$W(s) = \frac{s}{s + w_0} \quad \text{for } 747 \quad \lambda_{dr} = -0.033 \pm 0.947i$$

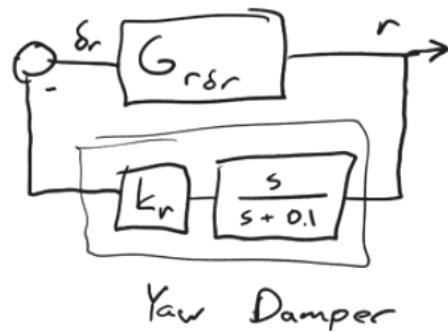
$$|W(\omega_0 i)| = \left( \frac{\omega_0 i}{\omega_0 i + \omega_0} \right) = \frac{1}{\sqrt{2}}$$

$$\lim_{w \rightarrow \infty} |W(w_i)| = \lim_{w \rightarrow \infty} \frac{w_i}{w_i + w_0} = 1$$

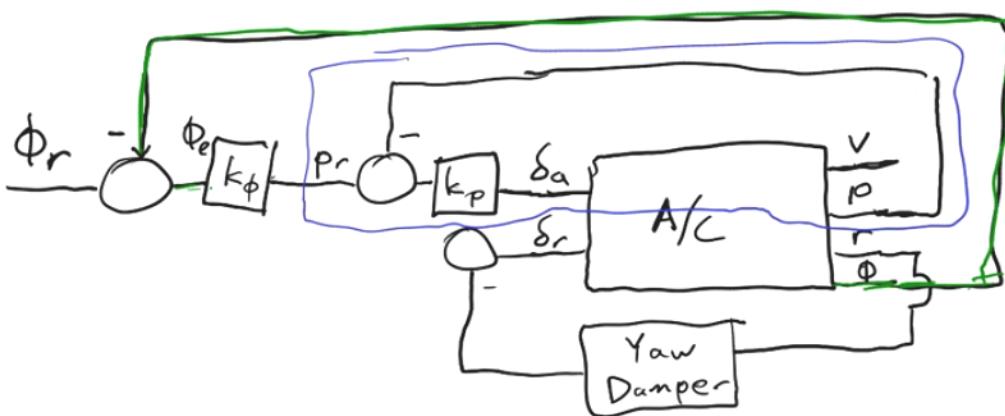
$$|W(O_i)| = \frac{O_i}{O_i + w_b} = 0$$



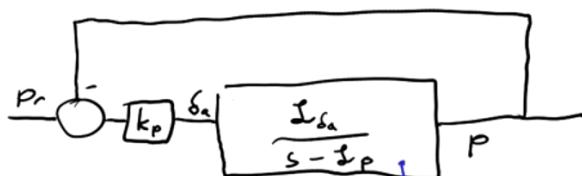
choose  $\omega_0 = 0.1$



## Part 2: Aileron Controller



### Part (a): Inner Loop



$$\delta_a = k_p (p_r - p)$$

### Pure Roll Approximation

$$\dot{p} = L_p p + L_{\delta_a} \delta_a$$

$$s p(s) = L_p p(s) + L_{\delta_a} \delta_a(s)$$

$$(s - L_p) p(s) = L_{\delta_a} \delta_a(s)$$

$$G_{p\delta_a}(s) = \frac{p(s)}{\delta_a(s)} = \frac{L_{\delta_a}}{s - L_p}$$

$$A_{lat} = \begin{pmatrix} -0.0558 & 0 & -0.774 & 32.2 \\ -0.003865 & -0.4342 & 0.4136 & 0 \\ 0.001086 & -0.006112 & -0.1458 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta p \\ \Delta q \\ \Delta g \end{pmatrix}$$

for 747

$$L_p = -0.4342$$

$$L_{\delta_a} = -0.1431$$

$$t \text{ from } B_{lat} = \begin{pmatrix} y_s & y_q \\ L_{\delta_a} & S_{\delta_a} \\ N_{\delta_a} & N_q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_r \\ \delta_a \end{pmatrix}$$

### Closed Loop

$$G_{ppr}(s) = \frac{k_p G_{p\delta_a}}{1 + k_p G_{p\delta_a}} = \frac{k_p \frac{L_{\delta_a}}{s - L_p}}{1 + k_p \frac{L_{\delta_a}}{s - L_p}} = \boxed{\frac{k_p L_{\delta_a}}{s - L_p + k_p L_{\delta_a}} = G_{ppr}(s)}$$

We care about step response

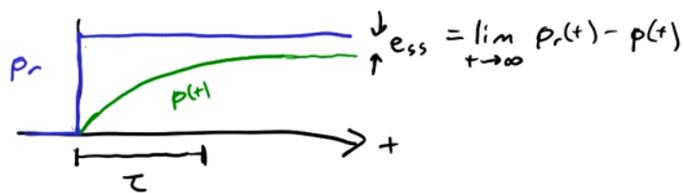
What sign  $k_p$  have?

$k_p$  should be negative

$$\delta_a = k_p(p_r - p)$$

$$p_r = 1$$

$$p = 0$$



steady-state gain  
 $\lim_{t \rightarrow \infty} \frac{p(t)}{p_r(t)}$  if  $p_r(t)$  is step

Steady State Gain

$$\lim_{t \rightarrow \infty} p(t) = \lim_{s \rightarrow 0} s p(s) = \lim_{s \rightarrow 0} s p_r(s) G_{ppr}(s) = \lim_{s \rightarrow 0} \frac{1}{s} G_{ppr}(s) = \boxed{G(0)}$$

only if  $s p(s)$  is stable

$$G_{ppr}(0) = -\frac{k_p \mathcal{L}_{\delta_a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}}$$

as  $k_p$  gets more negative  
 $\Rightarrow$  larger steady-state gain

$$G_{ppr}(s) = \frac{k_p \mathcal{L}_{\delta_a}}{s - (\mathcal{L}_p - k_p \mathcal{L}_{\delta_a})}$$

$$\lambda = \mathcal{L}_p - k_p \mathcal{L}_{\delta_a}$$

$$\tau = \left| \frac{1}{\lambda} \right| = \left| \frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \right|$$

more negative  $k_p$   
 $\Rightarrow$  smaller  $\tau$

Why not make  $|k_p|$  as large as possible

- saturation
- sensitivity to noise
- wear and tear on actuator

consider  $\boxed{k_p = -1}$

$P_e \approx 15 \text{ deg/s} \Rightarrow 15^\circ \text{ aileron deflection}$

$$\text{Steady-state gain} = G_{ppr}(0) = \frac{-k_p \mathcal{L}_{\delta_a}}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \approx 0.25$$

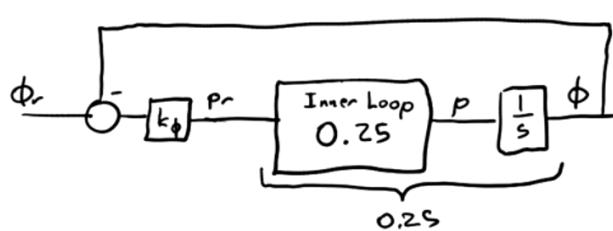
$$\text{Time constant } \tau = \left| \frac{1}{\mathcal{L}_p - k_p \mathcal{L}_{\delta_a}} \right| \approx 1.75 \text{ s}$$

check if OK Later

check if OK Later

## Part (b) Outer Loop

Assume: Inner Loop Much Faster than Outer Loop  
 $\phi = p$



### Closed Loop TF

$$G_{\phi\phi_r} = \frac{k_\phi \frac{0.25}{s}}{1 + k_\phi \frac{0.25}{s}} = \frac{0.25 k_\phi}{s + 0.25 k_\phi}$$

$k_\phi$  should be  $> 0$

2 Goals

- Steady-state gain near 1  $\rightarrow G_{\phi\phi_r}(0) = \frac{0.25 k_\phi}{0.25 k_\phi} = 1$
- low time constant

higher  $k_\phi \Rightarrow$  lower time constant

$$k_\phi = 1.5$$

Plug back into full linear lateral system  
(without washout filter)

$$\vec{u} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = -K \vec{x} - K_r [\phi_r]$$

$$= - \begin{bmatrix} 0 & k_p & 0 & k_p k_\phi \\ 0 & 0 & k_r & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} - \begin{bmatrix} -k_p k_\phi \\ 0 \end{bmatrix} [\phi_r]$$

$$\rightarrow p_r = k_\phi (\phi_r - \phi)$$

$$\rightarrow \delta_a = k_p (p_r - p) = k_p (k_\phi (\phi_r - \phi) - p)$$

$$= \underline{k_p k_\phi \phi_r} - \underline{k_p k_\phi \phi} - \underline{k_p p}$$

$$\delta_r = -k_r r$$

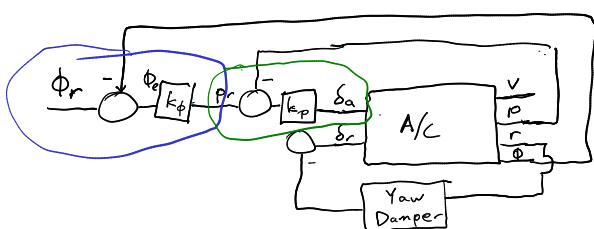
### New dynamics equation

$$\dot{\vec{x}} = A' \vec{x} + B' [\phi_r]$$

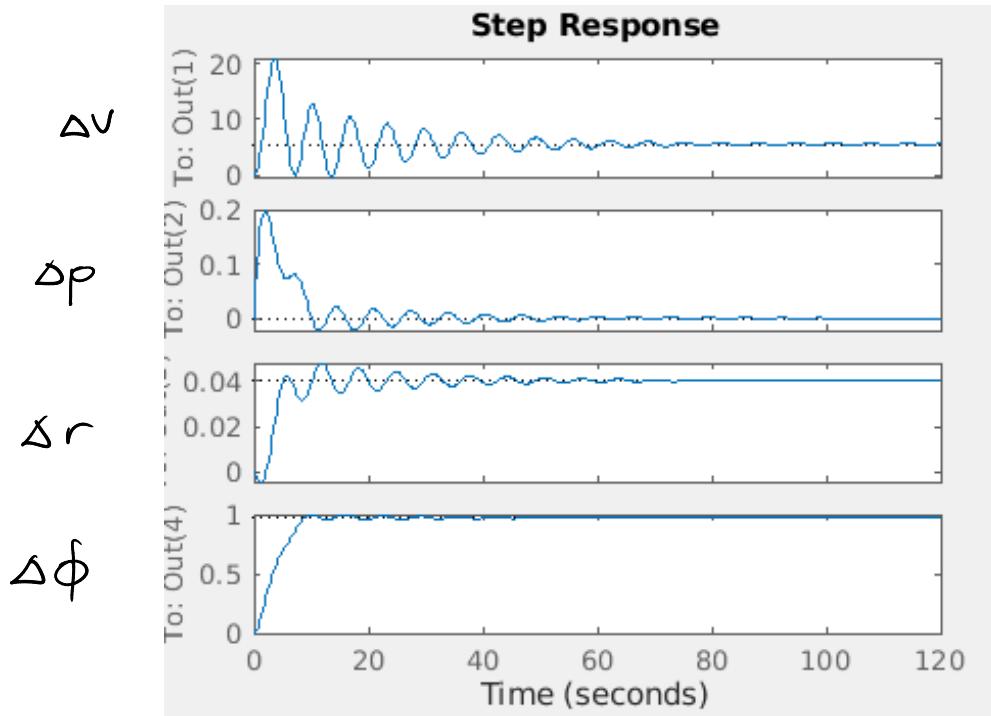
$$A' = A_{lat} - B_{lat} K$$

$$B' = -B_{lat} K_r$$

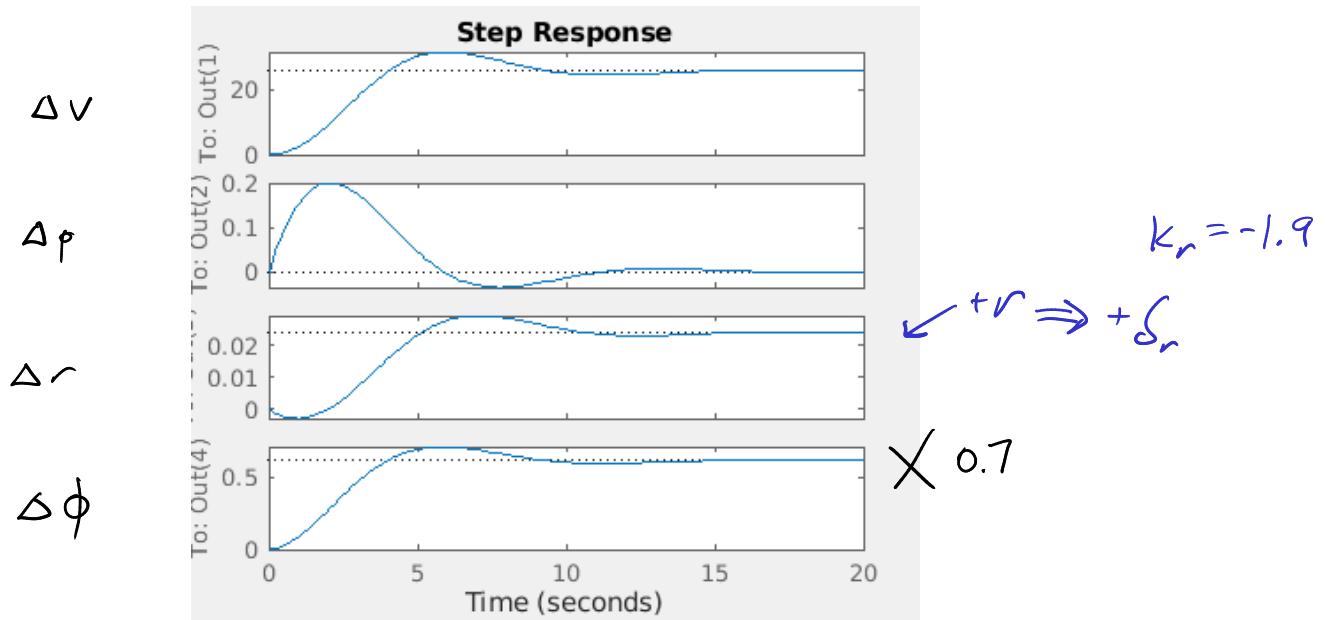
$$\dot{\vec{x}} = A'_{lat} \vec{x} + B'_{lat} \vec{u}$$



Step response without yaw damper



Step response with  $\delta_r = -k_r \Delta r$



Step response with Yaw damper with Washout filter

