Solving Matrix Games

Finding Nash Equilibria

Book: Algorithmic Game Theory

L.H. algorithm

Review

Player 2

Player 1		stag	hare
	stag	10,10	0,6
	hare	6,0	5,5

Nash Equilibrium

Every player plays a best response.

$M \equiv$ set of actions for P1

$N \equiv$ set of actions for P2

$$A = \begin{bmatrix} 10 & 0 \\ 6 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 10 & 6 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

x = strategy for P1

y = strat for P2

payoffs: P1: $x^T A y$, P2: $x^T B y$

Calculating Mixed NE

Every action in the support of a NE must be a best response.

Mathematically (in a NE):

$$x_i > 0 \implies (Ay)_i = u = \max\{(Ay)_k \mid k \in M\}$$

To get another player to play a mixed NE strategy, you must make them <u>indifferent</u> between their actions

$$\begin{bmatrix} 10 & 0 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad [x_1 \ \ x_2] \begin{bmatrix} 10 & 6 \\ 0 & 5 \end{bmatrix}$$

P1 indifferent:

$$10y_1 + 0y_2 = u$$
$$6y_1 + 5y_2 = u$$
$$y_1 + y_2 = 1$$

$$6y_1 + 5y_2 = 10y_1$$
$$5y_2 = 4y_1$$
$$5 - 5y_1 = 4y_1$$
$$5 = 9y_1$$

$$\boxed{y_1 = \frac{5}{9} \qquad y_2 = \frac{4}{9}}$$

P2 indifferent:

$$10x_1 + 0x_2 = v$$
$$6x_1 + 5x_2 = v$$
$$x_1 + x_2 = 1$$

$$\boxed{x_1 = \frac{5}{9} \qquad x_2 = \frac{4}{9}}$$

$M = \{1,2,3\}$ $\qquad$ $N = \{4,5\}$

$$A = \begin{array}{c} \begin{bmatrix} \boxed{3} & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \qquad B = \begin{array}{cc} 4 & 5 \\ \multicolumn{2}{c}{\begin{bmatrix} \boxed{3} & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix}} \end{array}$$

mixed NE: $x = [4/5, 1/5, 0]$ $\quad y = [2/3, 1/3]$

$Ay = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ $\qquad x^T B = [14/5, 14/5]$

mixed NE: $x = [0, 1/3, 2/3]$ $\quad y = [1/3, 2/3]$

$Ay = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$ $\qquad x^T B = [8/3, 8/3]$

$x = [1,0,0]$, $y = [1,0]$ is a pure NE

P1 indifferent

$$3y_4 + 3y_5 = u$$
$$2y_4 + 5y_5 = u$$
$$0y_4 + 6y_5 = u$$
$$y_4 + y_5 = 1$$

one of P1's actions not in the NE's support

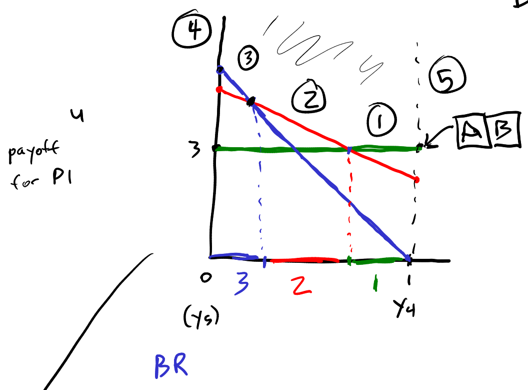
<u>Def</u>: A two-player game is called <u>nondegenerate</u> if no mixed strategy of support size k has more than k pure best responses.

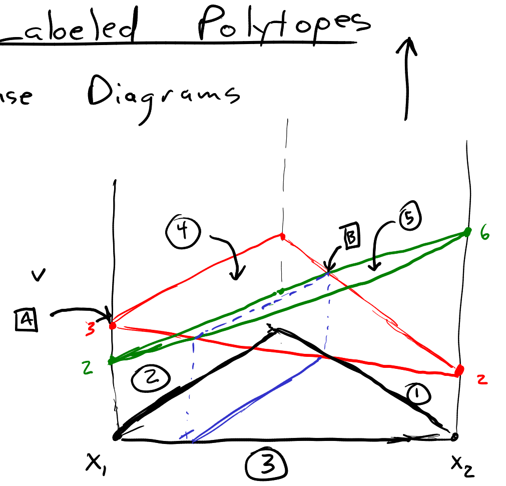
<u>Prop</u>: In any N.E. (x,y) of a nondegenerate bimatrix game, x and y have supports of equal size.

$$A = \begin{bmatrix} \boxed{3} & 3 \\ 2 & 5 \\ 0 & 6 \end{bmatrix} \qquad B = \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} \boxed{3} & 2 \\ 2 & 6 \\ 3 & 1 \end{bmatrix} \end{array}$$

Equilibria via Labeled Polytopes

Best Response Diagrams

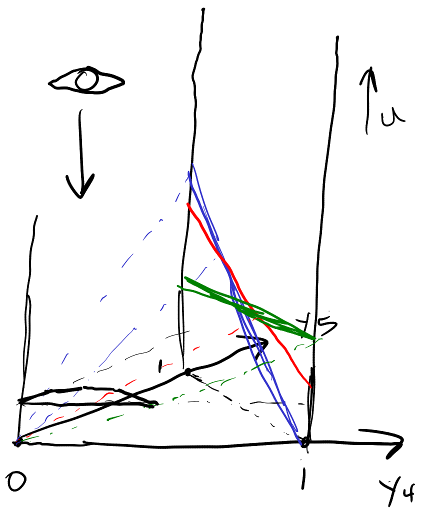


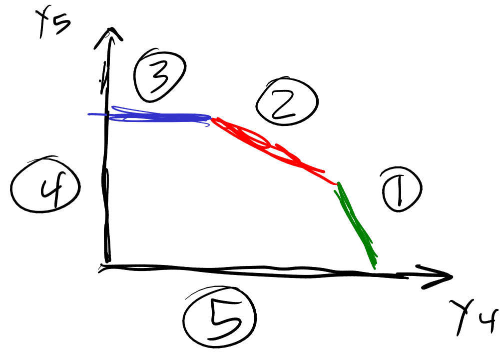


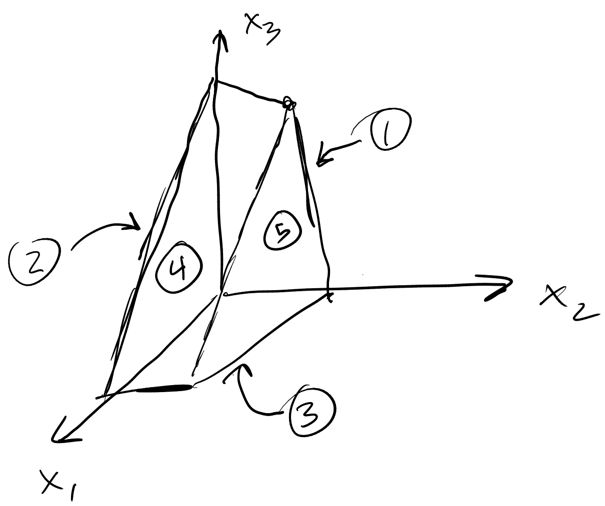
Each vertex gets labels,
A vertex gets label i if – own action i has nonzero prob
or
– if other players action i has zero prob

Pairs of vertices that are jointly fully labeled are NE

A	1	5	2	3	4
B	1	5	5	4	1





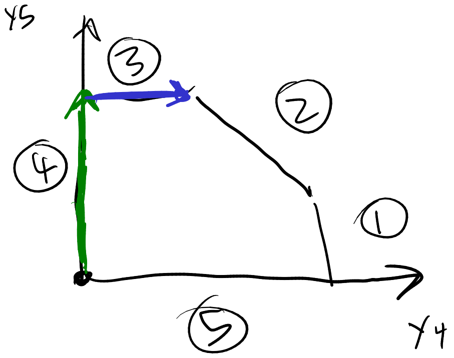


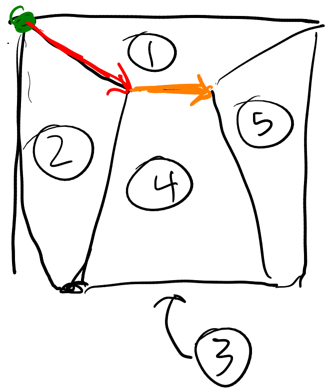
All NE

Examples: lrs Nash
EEE

1 NE

Lemke-Howson (1964)
Porter-Nudelman-Shoham (PNS)





LH

$\boxed{1}$	4	3	1	2	3
$\boxed{2}$	4	3	1	2	4
$\boxed{3}$	2	3	1	2	4
$\boxed{4}$	2	3	1	4	5

found NE!