Some Basic Statistical Theorems

Review

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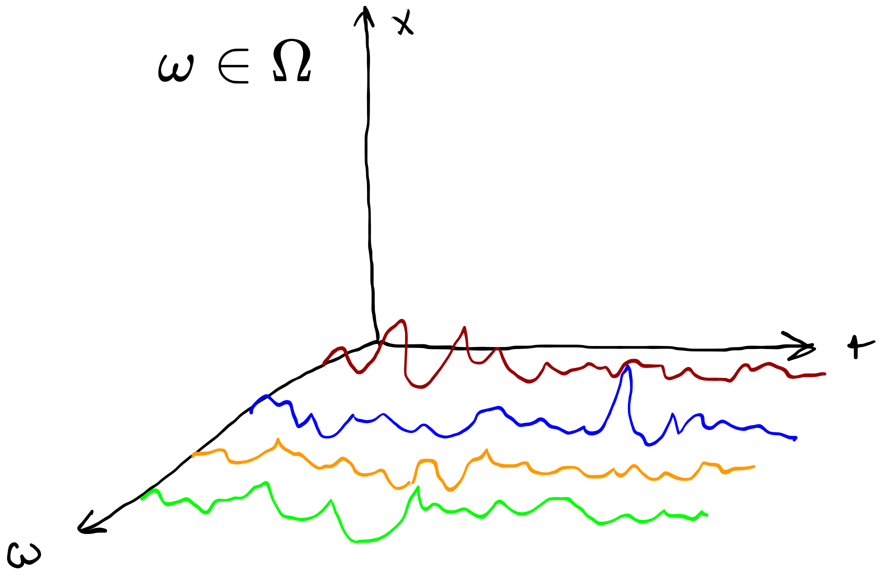

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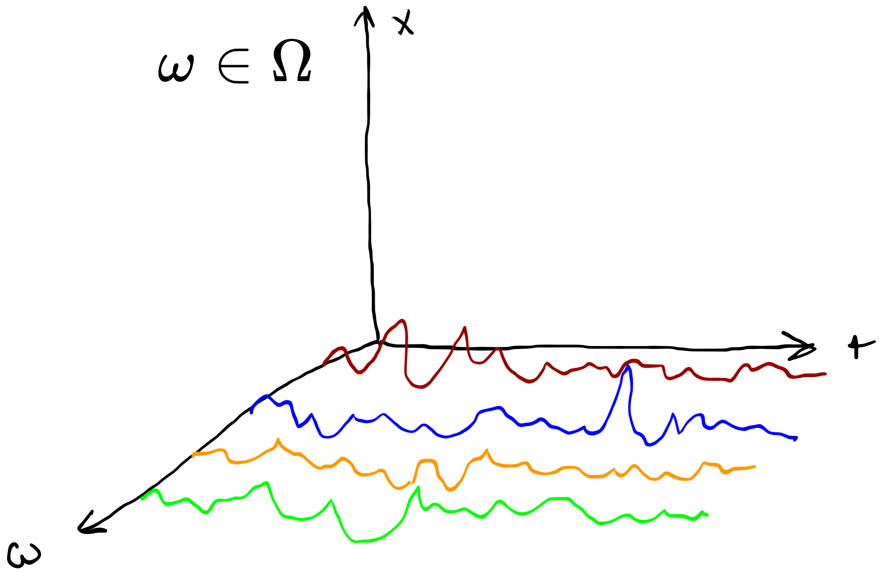

$\mathcal{F}$

$\vdots$

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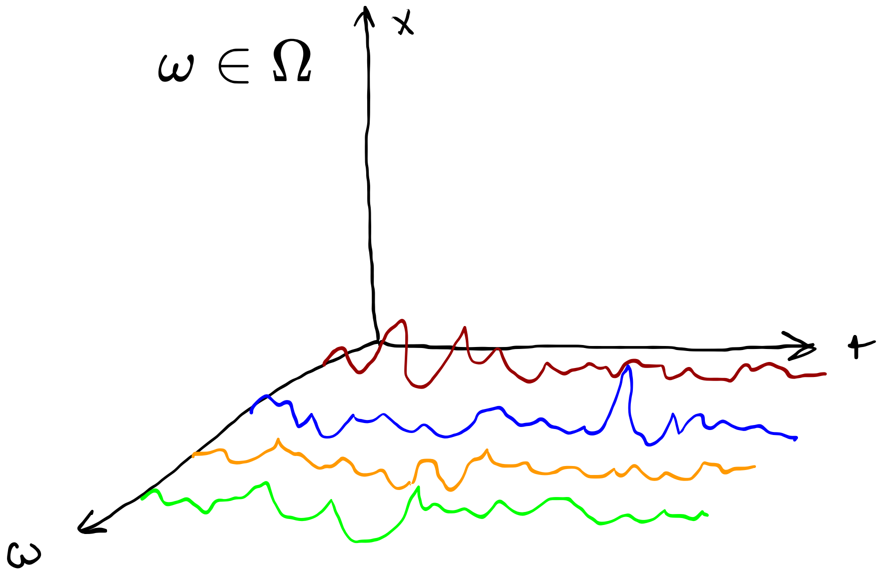

$$\mathcal{F} = \sigma(\{\ \}) = \{\Omega, \ , \ , \emptyset\}$$

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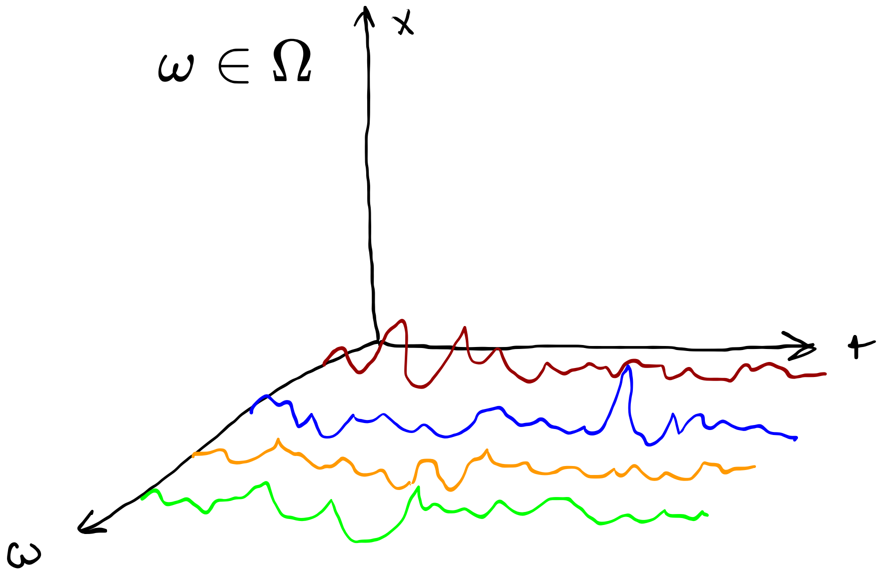

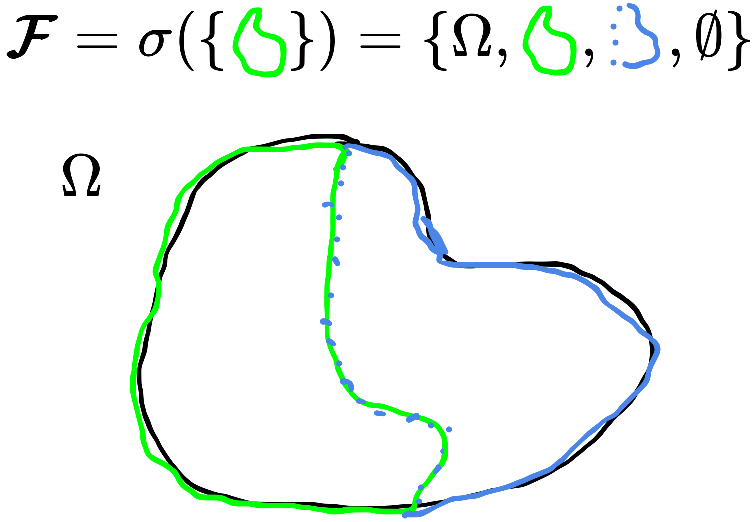

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Convergence:

- Sure ("pointwise")
- Almost Sure
- In Probability
- Weak ("in distribution"/"in law")

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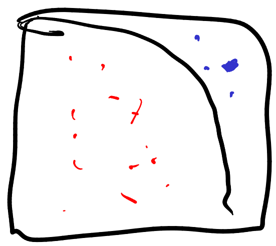

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$F_{X_n}(\alpha) \to F_X(\alpha)$ for each continuity point.

Outline for Today

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1. Concentration Inequalities

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2. Proof of the weak law of large numbers and $\frac{1}{\sqrt{N}}$ convergence rates

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2. Proof of the weak law of large numbers and $\frac{1}{\sqrt{N}}$ convergence rates
3. Central limit theorem
4. Importance sampling

Concentration Inequalities

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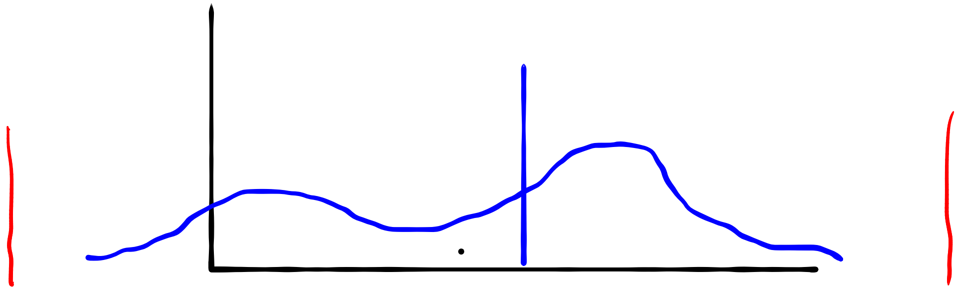

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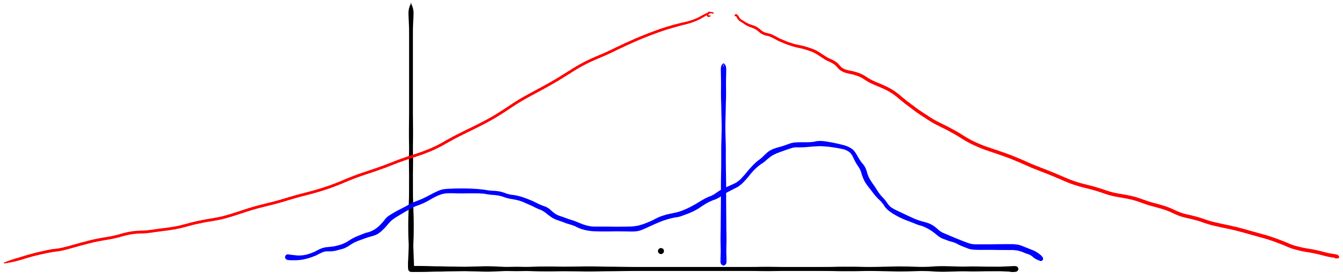

Concentration inequalities take the form

$$P(X \geq t) \leq \phi(t)$$

where ϕ goes to zero (quickly) as $t \to \infty$

Concentration Inequalities

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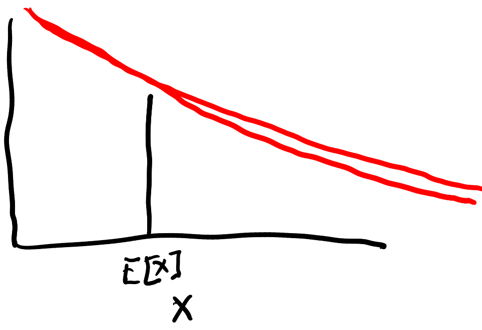
Markov's Inequality:

If $X \geq 0$, then

$$P(X \geq t) \leq \frac{E[X]}{t} \quad \forall t \geq 0$$

X=1



Proof:

$$E[X] = \int_{-\infty}^{\infty} x\,p(x)\,dx$$

$$= \int_0^{\infty} x\,p(x)\,dx$$

$$= \underbrace{\int_0^{a} x\,p(x)\,dx}_{\geq 0} + \int_a^{\infty} x\,p(x)\,dx$$

$$\geq \int_a^{\infty} x\,p(x)\,dx \geq \int_a^{\infty} a\,p(x)\,dx = a\int_a^{\infty} p(x)\,dx$$

$$= a\,P(X \geq a)$$

$$\therefore \frac{E[X]}{a} \geq P(X \geq a)$$

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Let X be *any* real-valued random variable with $\mathrm{Var}(X) < \infty$. Then

$$P(|X - E[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

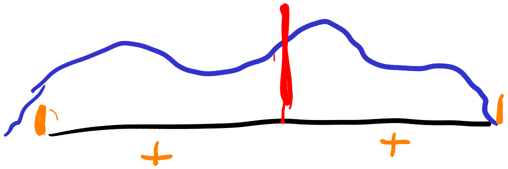
Proof: Recall $\mathrm{Var}(X) = E[(X - E[X])^2]$

Use Markov's inequality on $(X-E[X])^2$

$$P\big(\overset{Y}{(X-E[X])^2} \geq \overset{\gamma}{t^2}\big) \leq \frac{E[(X-E[X])^2]}{t^2} \qquad \frac{E[Y]}{\gamma}$$

$$P(|X-E[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$$

$$0.05 = \frac{\mathrm{Var}(X)}{t^2}$$



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Chebyshev's Inequality:

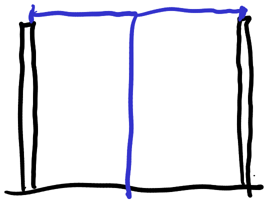
Let X be *any* real-valued random variable with $\mathrm{Var}(X) < \infty$. Then

$$P(|X - E[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

Very general, but still loose

Concentration Inequalities

k	Min. % within *k* standard deviations of mean	Max. % beyond *k* standard deviations from mean
1	0%	100%
$\sqrt{2}$	50%	50%
1.5	55.56%	44.44%
2	75%	25%
$2\sqrt{2}$	87.5%	12.5%
3	88.8889%	11.1111%
4	93.75%	6.25%
5	96%	4%
6	97.2222%	2.7778%
7	97.9592%	2.0408%
8	98.4375%	1.5625%
9	98.7654%	1.2346%
10	99%	1%



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68% For Normal Distribution

95%

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Chernoff Bound: If the moment-generating function M_X exists, then

$$P(X \geq a) \leq \frac{E[e^{tX}]}{e^{ta}} \quad \forall\, t > 0$$

$\forall a$

infinite number of bounds, one for each t

and

$$P(X \leq a) \leq \frac{E[e^{tX}]}{e^{ta}} \quad \forall\, t < 0$$

Proof: Apply Markov's inequality to e^{tX}

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}}$$

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Tighter than Markov and Chebyshev

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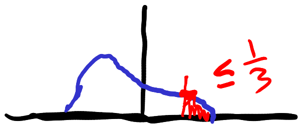
Let Y be a r.v. that takes values in $[-1, 1]$ with mean -0.5. Give an upper bound on the probability that $Y \geq 0.5$.

Let $Z = Y+1$ $\quad P(Y \geq 0.5) = P(Z \geq 1.5) \leq \frac{0.5}{1.5} = \frac{1}{3}$



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Let X_i be independent identically distributed r.v.s with mean μ and variance σ^2. If $Q_N \equiv \frac{1}{N} \sum_{i=1}^{N} X_i$, then $Q_N \to_p \mu$.

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Proof:

$$\begin{aligned}
\mathrm{Var}(Q_N) &= \mathrm{Var}\left(\frac{1}{N}\sum_i X_i\right) \\
&= \frac{1}{N^2}\mathrm{Var}\left(\sum_i X_i\right) \\
&= \frac{1}{N^2}\sum_i \mathrm{Var}(X_i) \quad \text{(Bienaymé)} \\
&= \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}
\end{aligned}$$

$E[Q_N] = \mu$

$$\begin{aligned}
P(|Q_N - \mu| \geq \varepsilon) &\leq \frac{\mathrm{Var}(Q_N)}{\varepsilon^2} \quad \text{(Chebyshev)} \\
&= \frac{\sigma^2}{N\varepsilon^2}
\end{aligned}$$

$P(|Q_N - \mu| \geq \varepsilon) \to 0$ as $N \to \infty$ $\forall \varepsilon > 0$ $\therefore Q_N \to_p \mu$ $\square$

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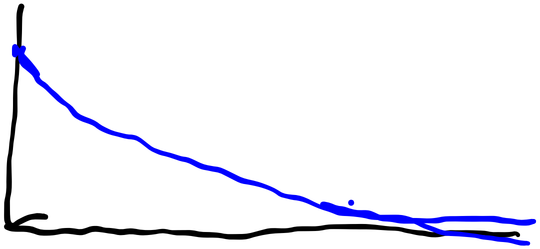
1. Standard deviation decays at $\frac{1}{\sqrt{N}}$ *regardless of dimension*.



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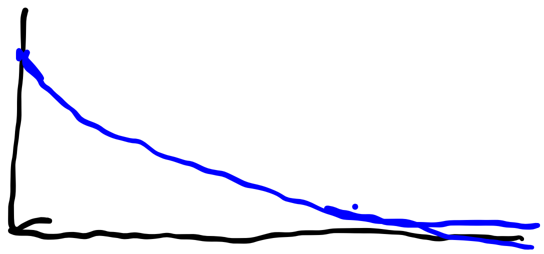

2. You can estimate the "standard error" with

$$SE = \frac{s}{\sqrt{N}}$$

empirical estimate of standard deviaton of X_i

where s is the sample standard deviation.

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We can do much better if we know something about the distribution of Q_N!

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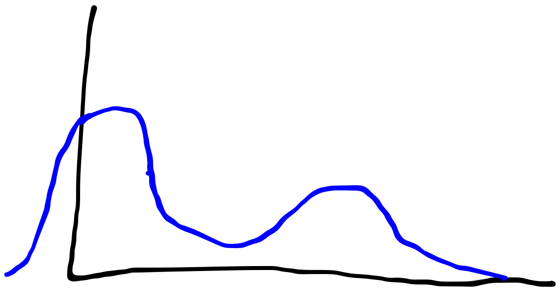
$$Q_1 \stackrel{D}{=} X_i$$



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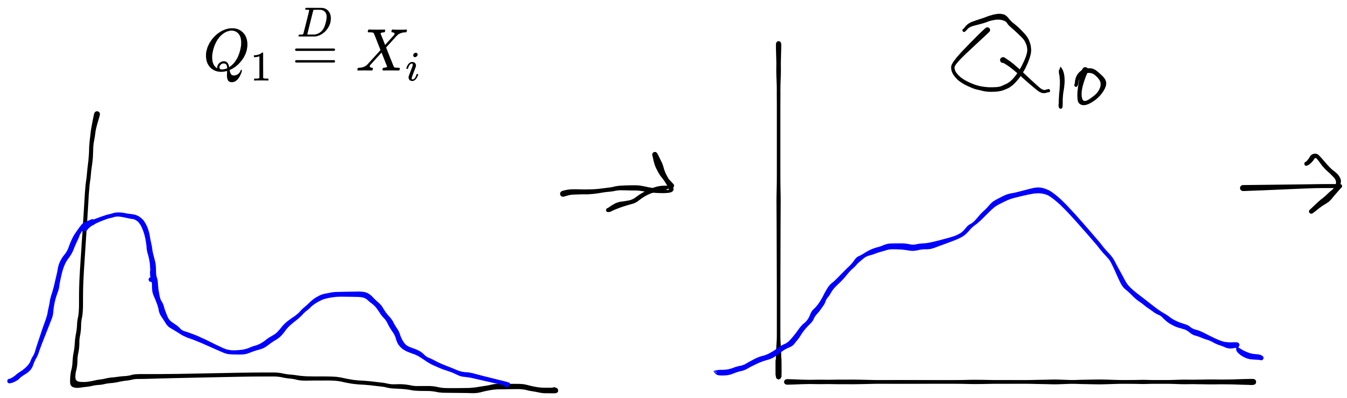

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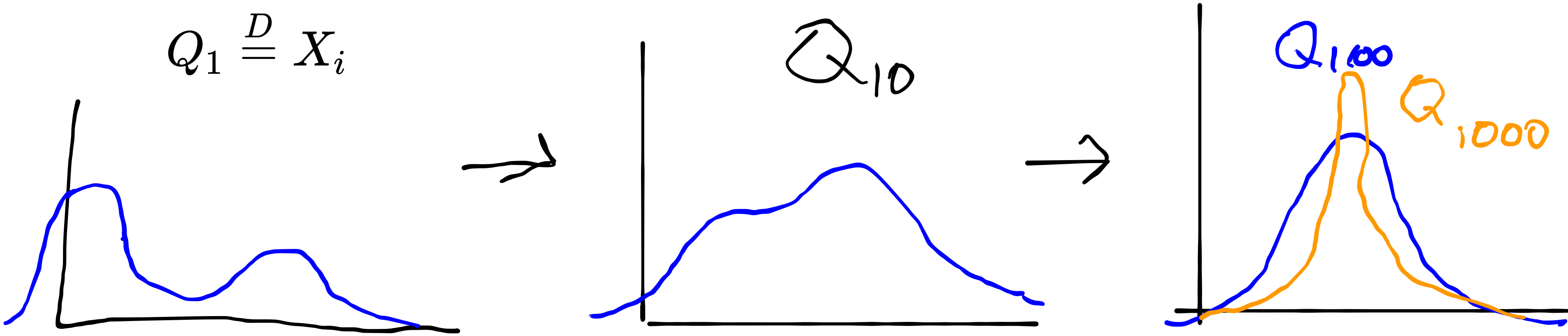

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$$P(|X - \mu| \geq t) = 1 + \mathrm{erf}\left(\frac{t - \mu}{\sqrt{2}\sigma}\right)$$

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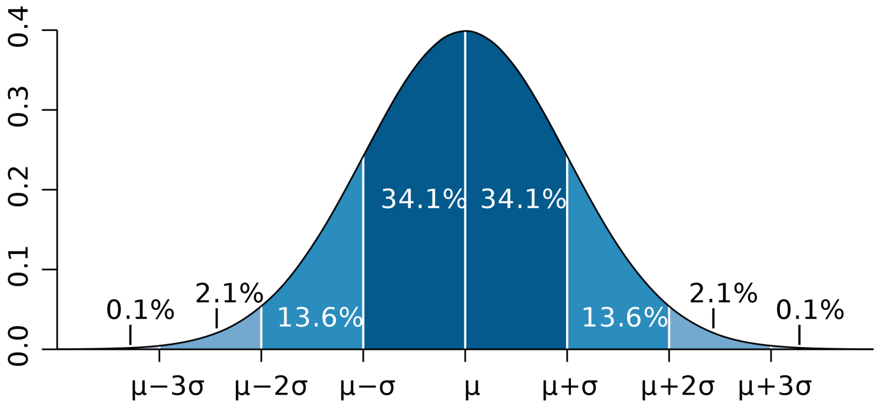

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Idea for approximate confidence interval: estimate $\mathrm{Var}(Q_N)$ with $SE^2 = \frac{s^2}{N}$ and use ~~Chebyshev~~ the central limit theorem.

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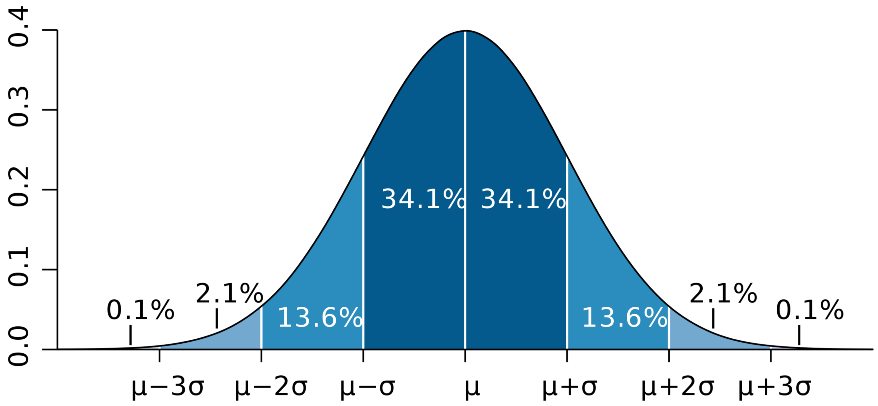

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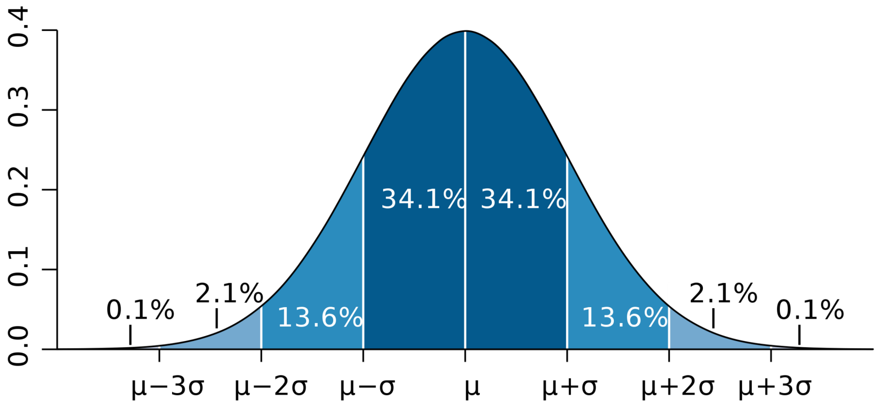

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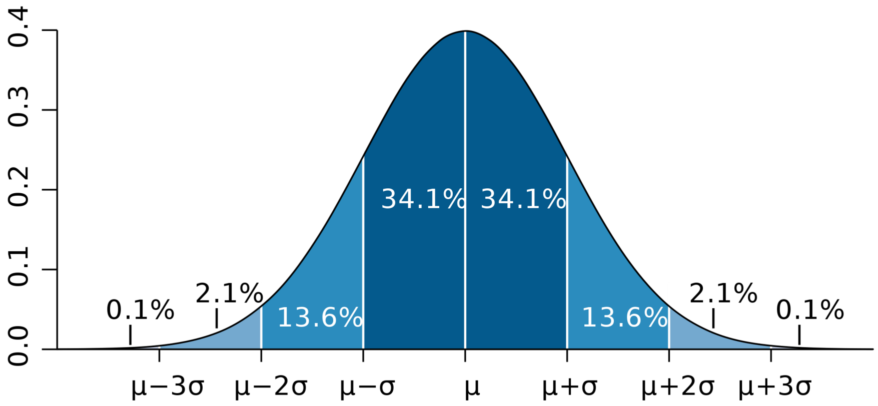

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(Chebyshev gave 4.47)

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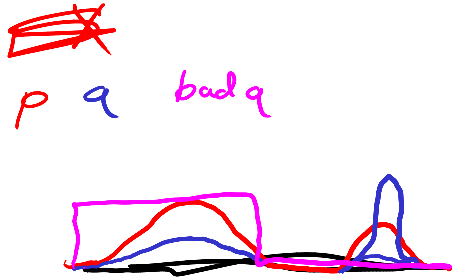

Summary

1. Concentration Inequalities $P(X \geq t) \leq \phi(t)$

2. Law of large numbers $Q_N \rightarrow_p \mu$

3. Central Limit Theorem $Q_N \xrightarrow{D} \mathcal{N}(\mu, \frac{\sigma}{\sqrt{N}})$

4. Importance Sampling $E[X] \approx \dfrac{1}{N} \sum Y_i w_i$ where $w_i = \frac{p(Y_i)}{q(Y_i)}$