Anatomy of a Random Variable

Outline

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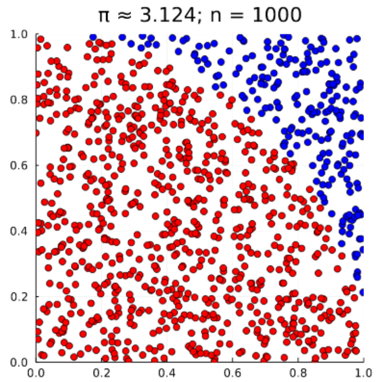

- A Motivating Example: Monte Carlo Integration

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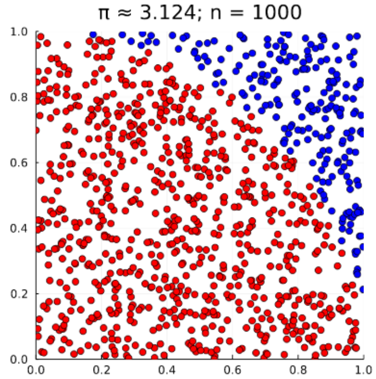
- Rigorous Definitions of a Random Variable



$$X : \Omega \to E$$

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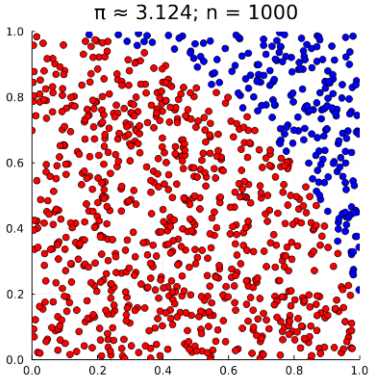
- A Motivating Example: Monte Carlo Integration
- Rigorous Definitions of a Random Variable
- Law of large numbers and the Central Limit Theorem



$$X : \Omega \to E$$

$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right)$$

Monte Carlo Integration

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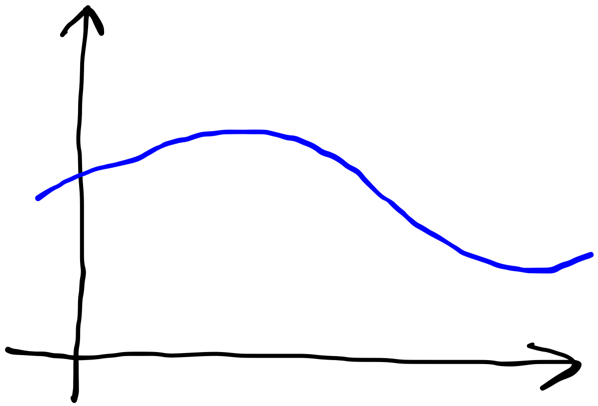

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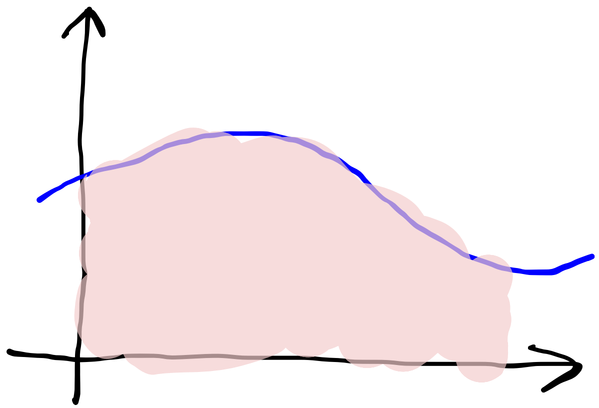

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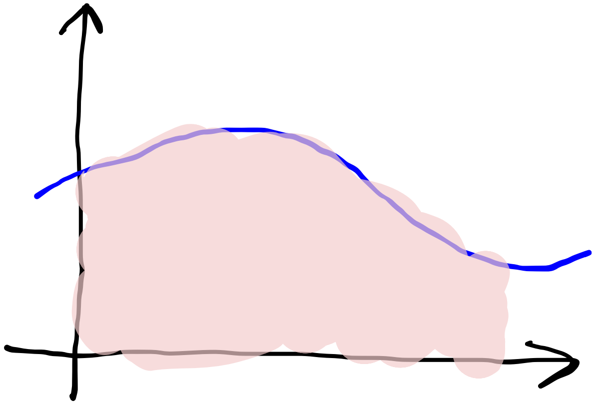

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$$I = \int_{\Omega} f(x)\mu(dx)$$

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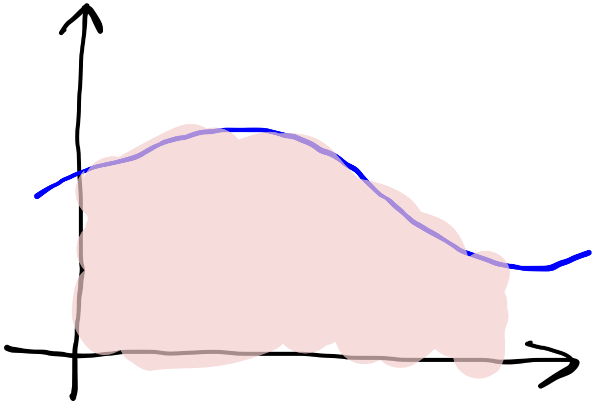

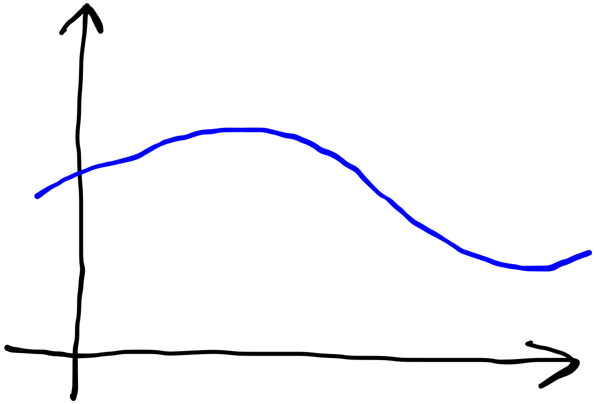

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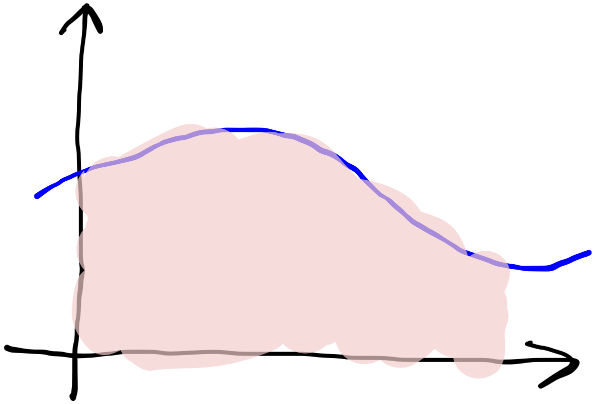

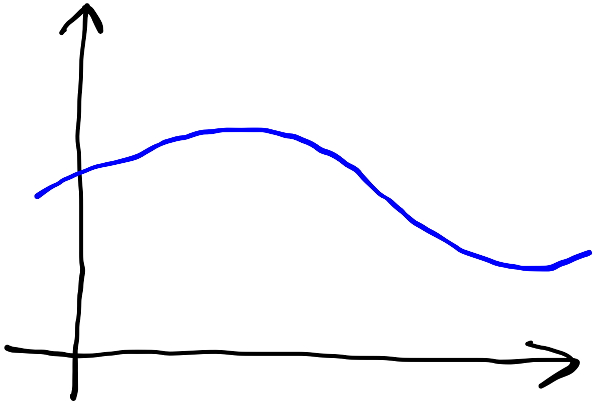

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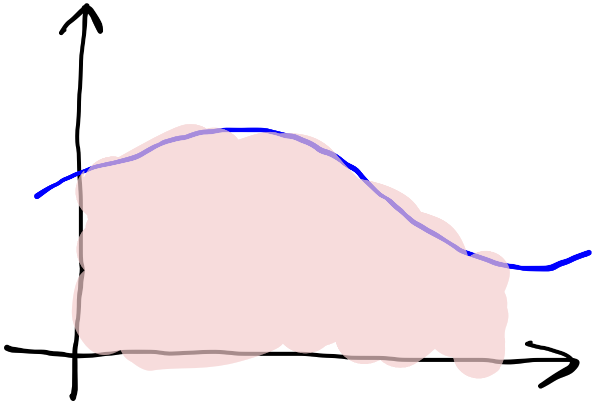

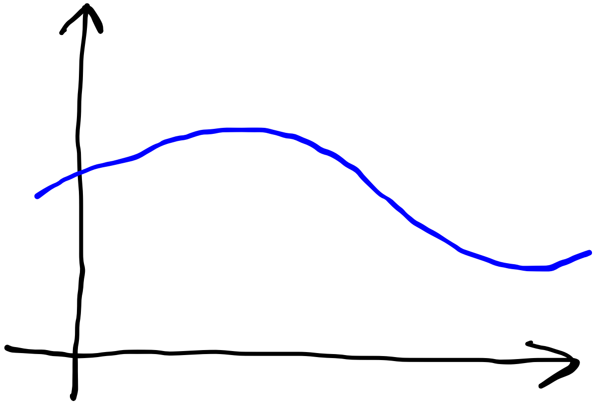

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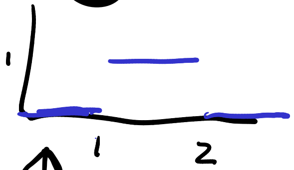
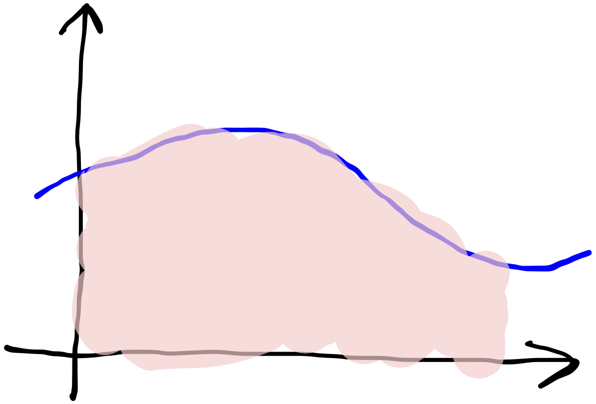
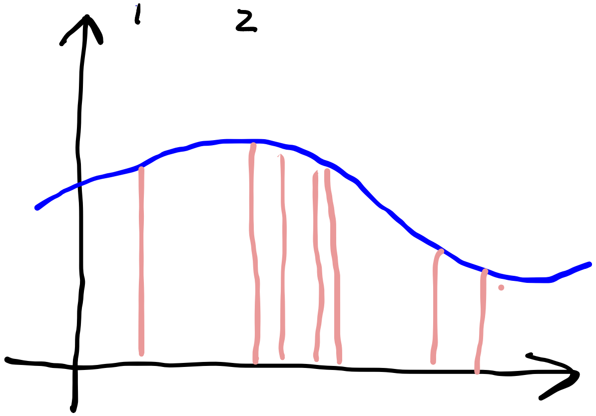
Monte Carlo Integration

$1_{1 \le x \le 2}(x)$



$1_{\{prop\}}(x) = \begin{cases} 1 & \text{if } prop(x) \\ 0 & \text{ow} \end{cases}$





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How accurate is this?

Random Variables

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Given a **probability space** $(\Omega, \mathcal{F}, P)$, and a **measurable space** $(E, \mathcal{E})$, an E-valued **random variable is a measurable function** $X : \Omega \to E$.

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What is this function?

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Example: Coin World

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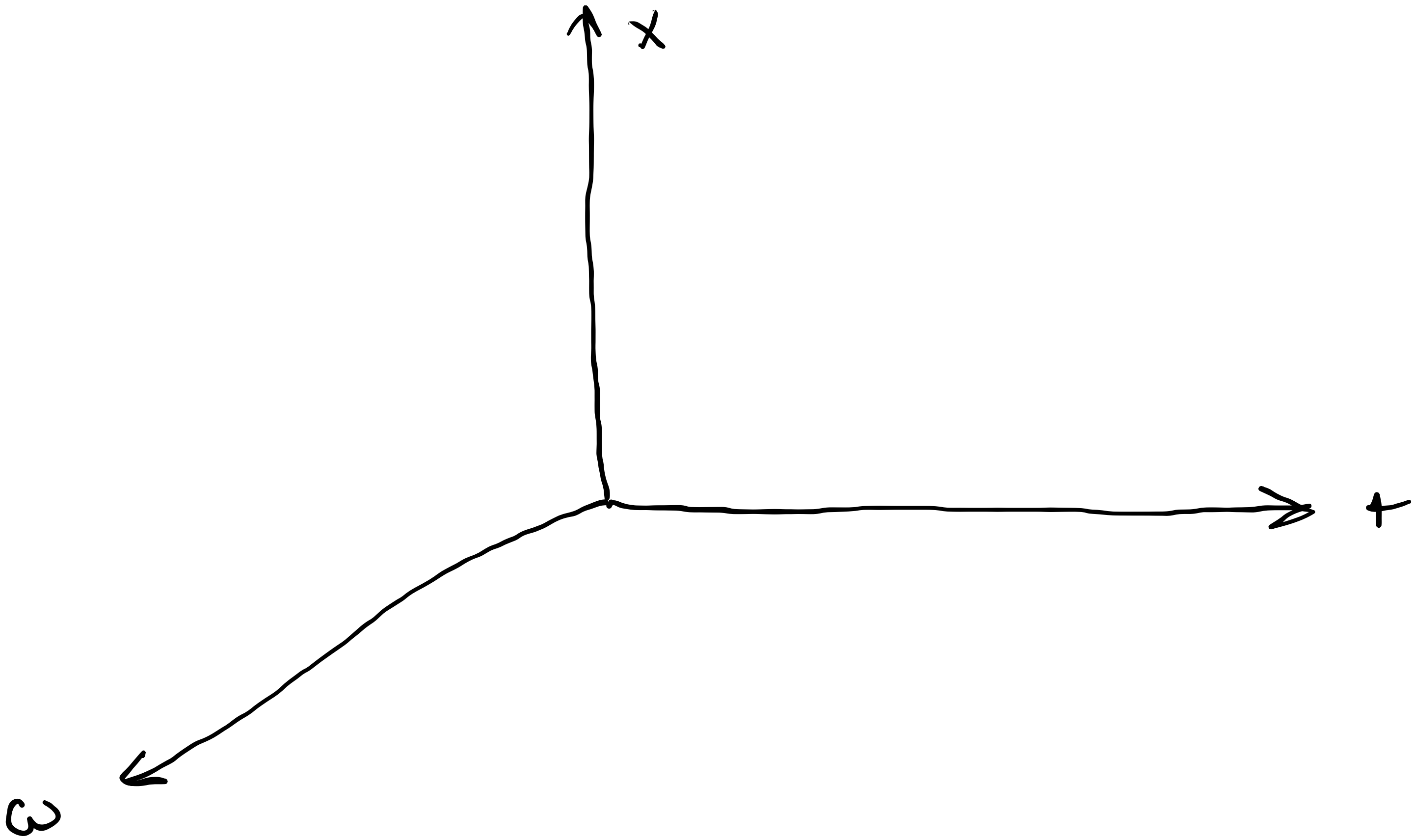
stochastic process $\{x_t\}$

$x(t)$



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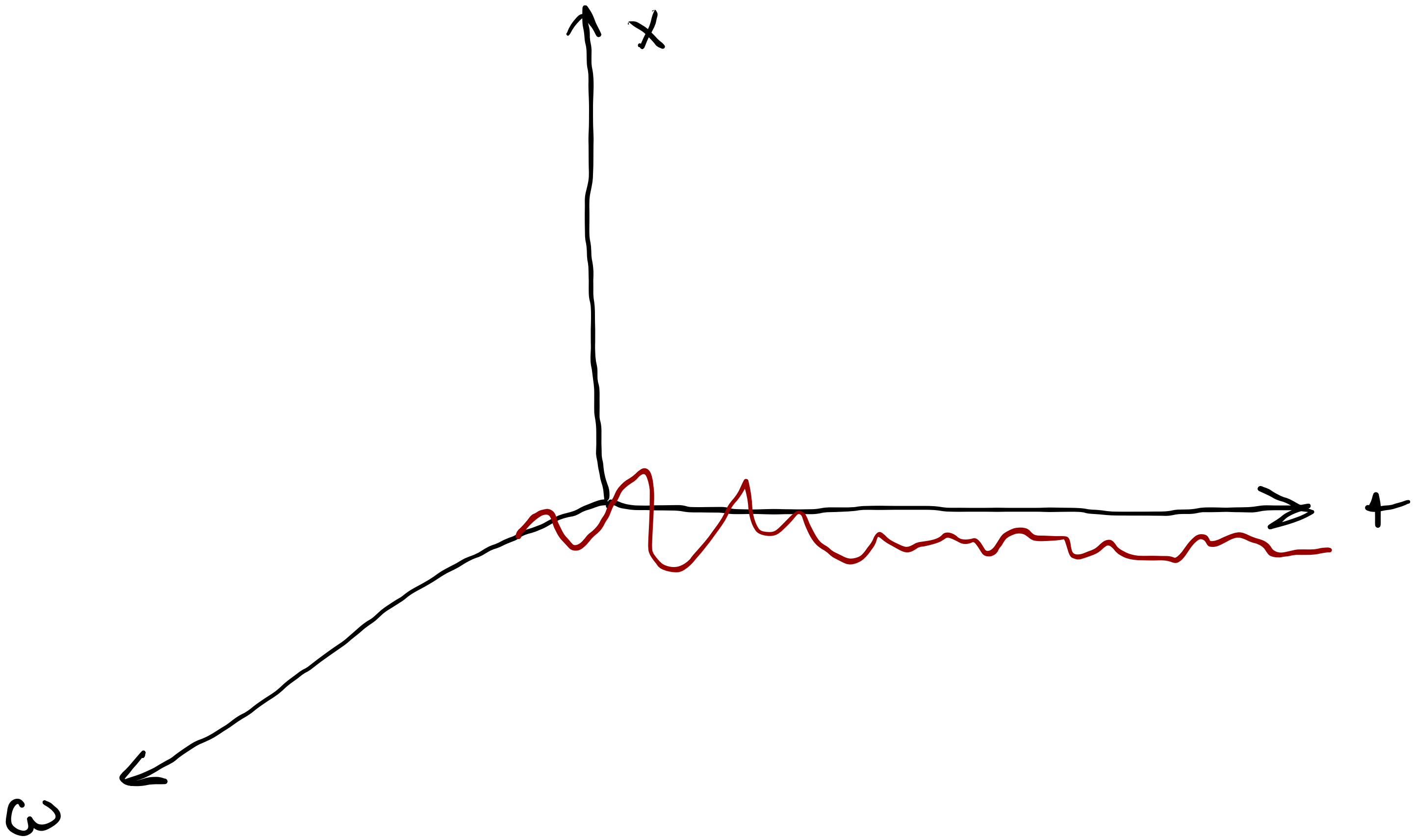

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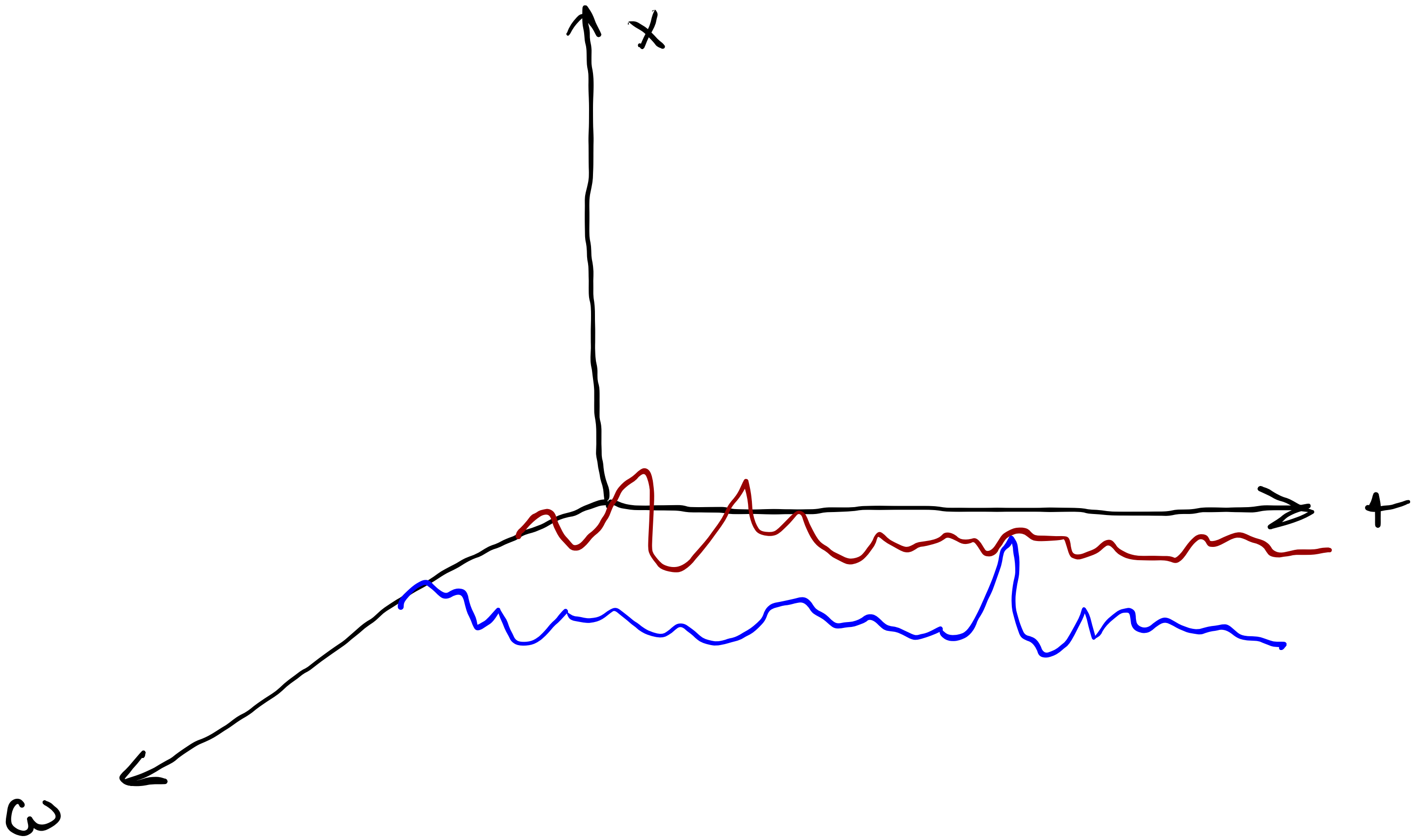

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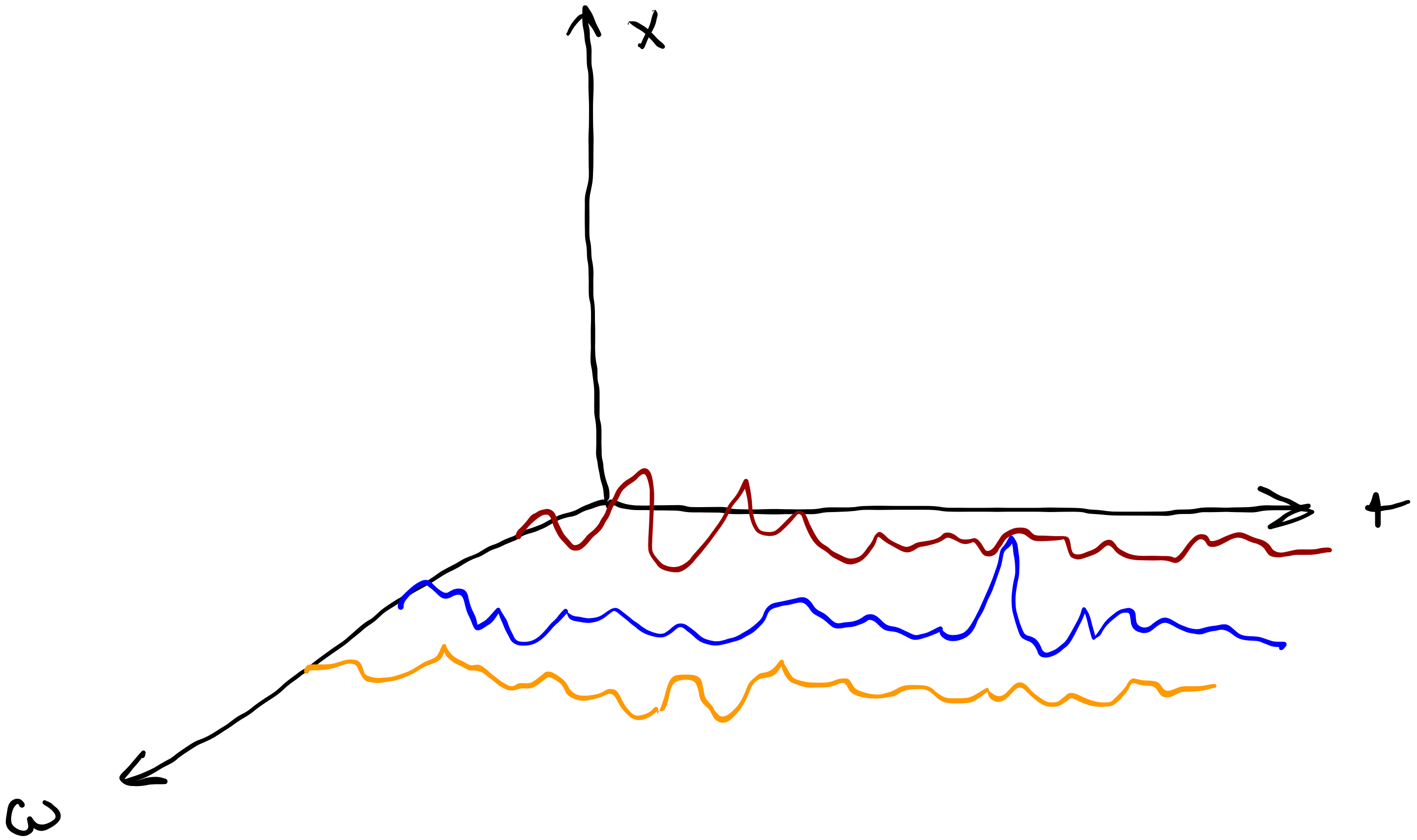

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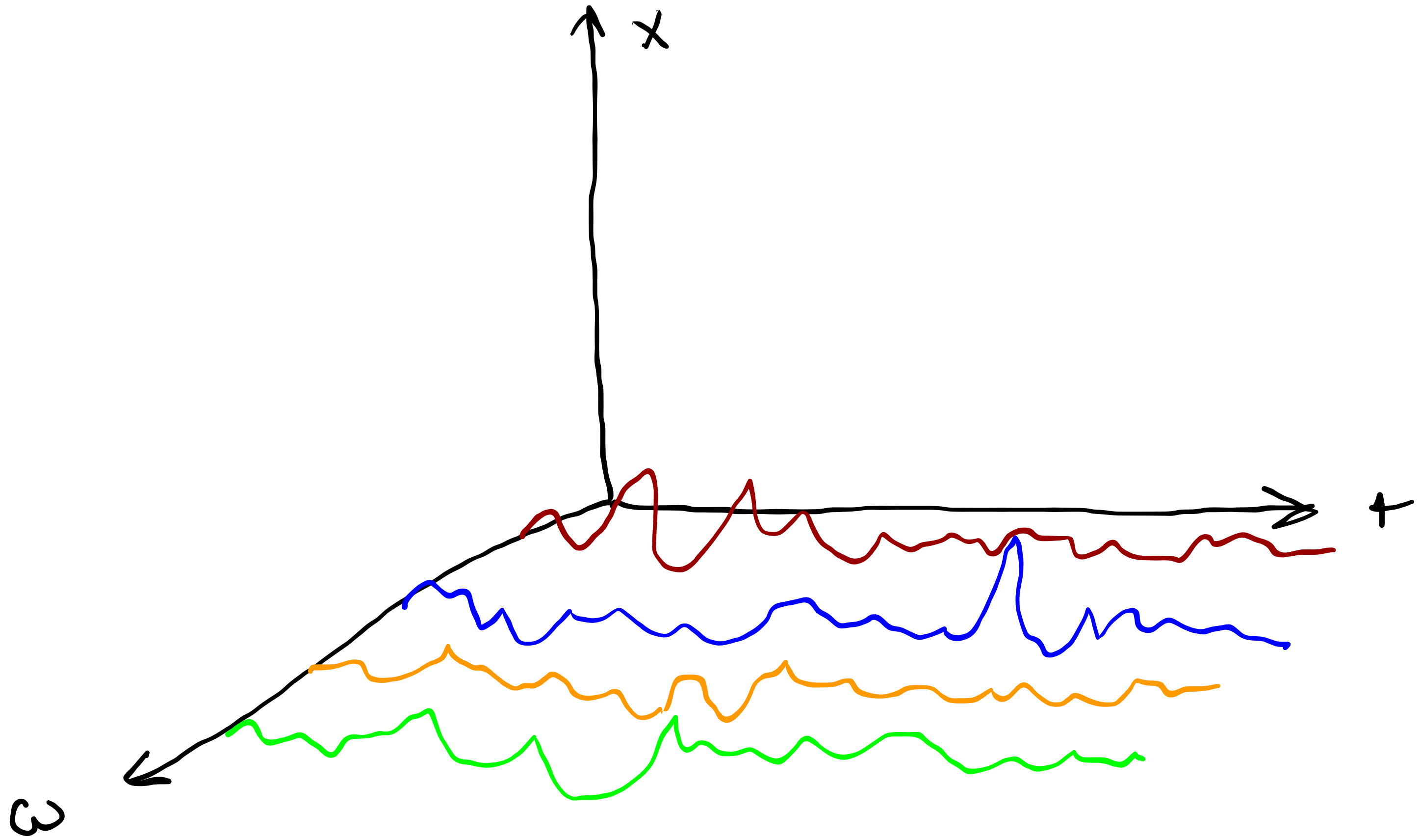

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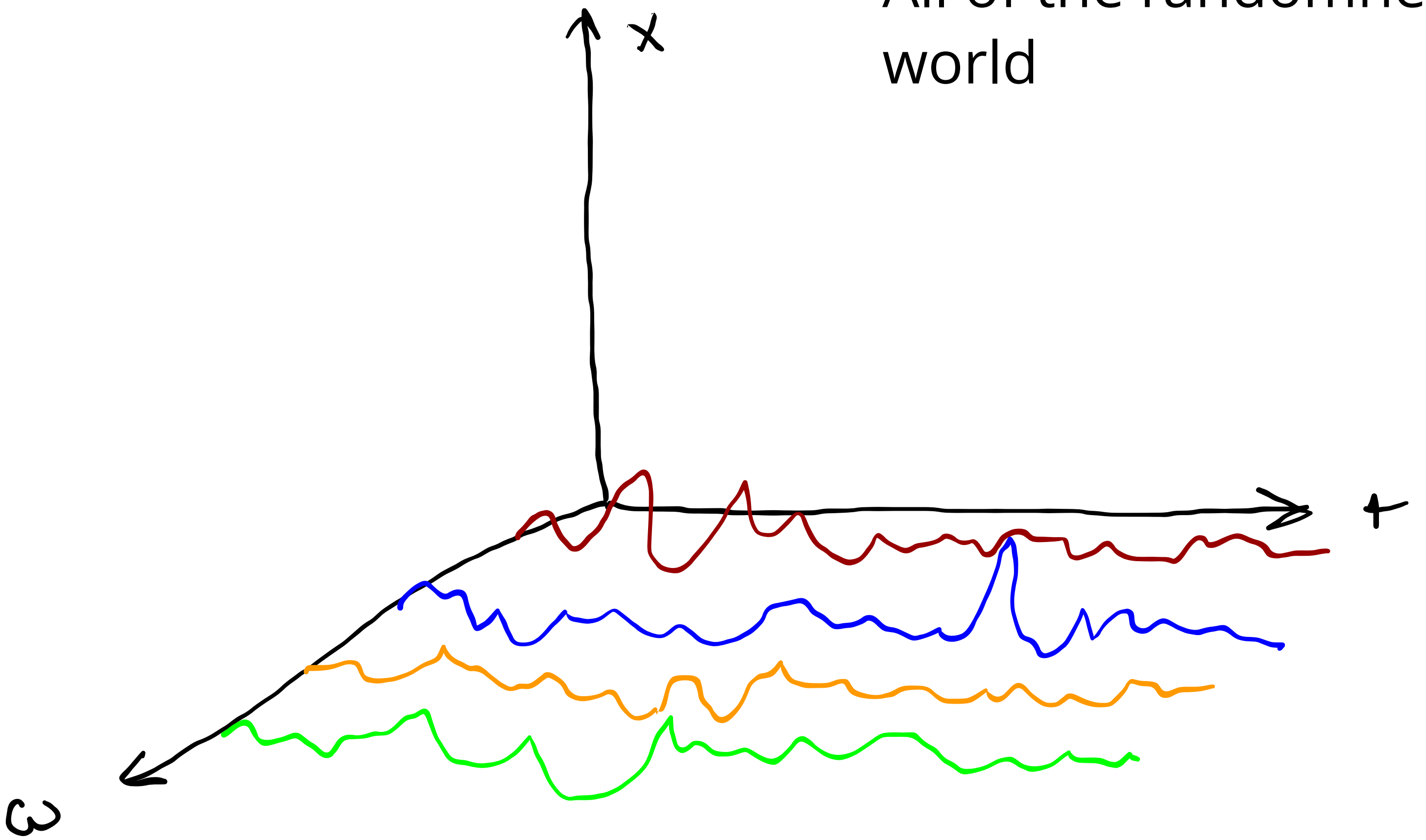

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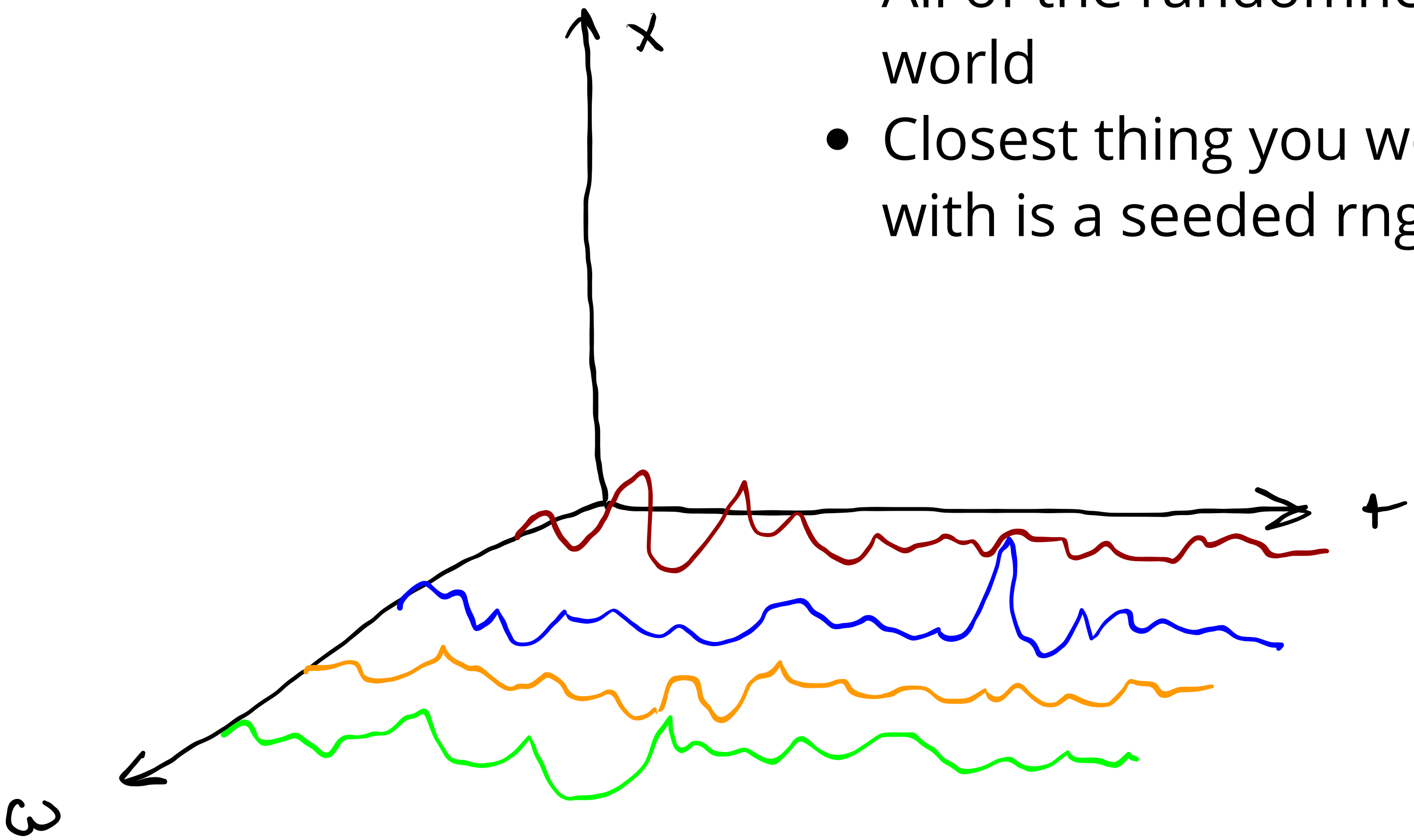

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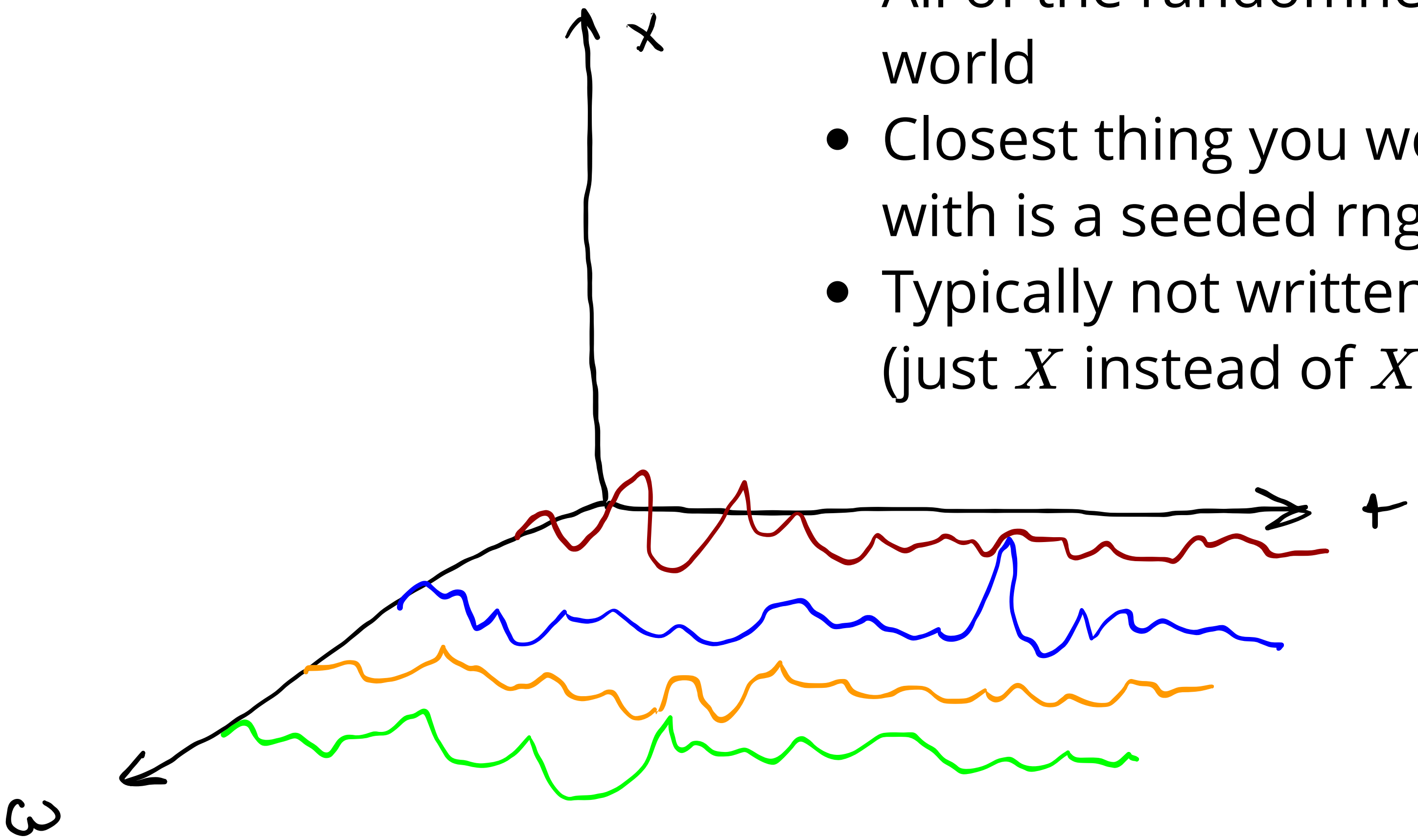

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- Typically not written expressly (just X instead of $X(\omega)$)

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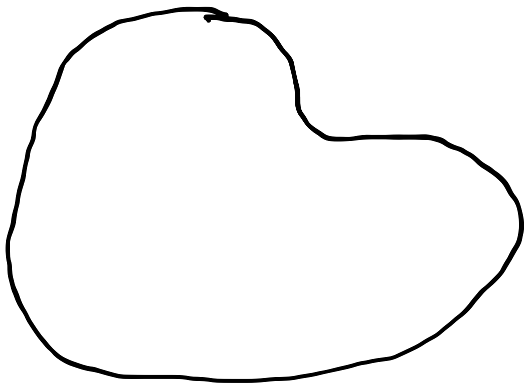
Ω



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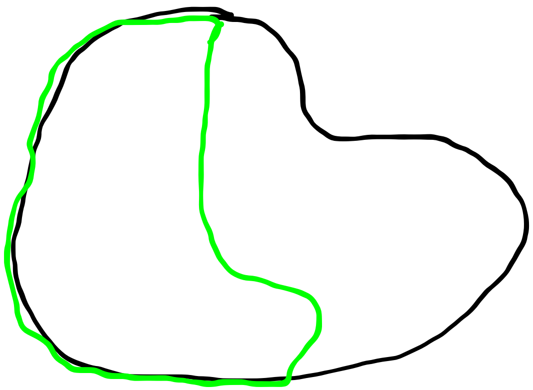

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$$\mathcal{F} = \sigma(\{\ \}) = \{\Omega,\ ,\ ,\emptyset\}$$

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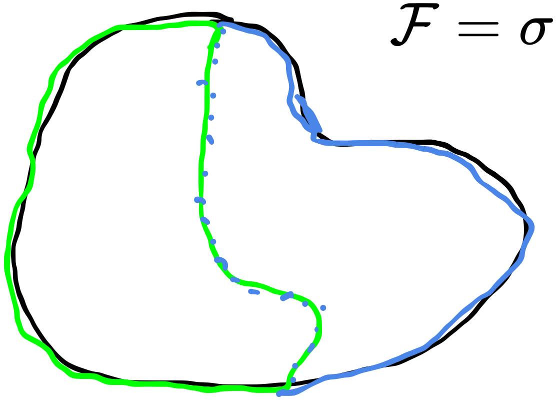

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$(-\infty, 1)$ $(2, \infty)$

- Is $[1, 2]$ in $\mathcal{B}$? Yes

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- Is π in $\mathcal{B}$?

$(-\infty, \pi)$ (π, ∞)

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2. $P(\Omega) = 1$.
3. (Countable additivity) $P(A) = \sum_{n=1}^{\infty} P(A_n)$ whenever $A = \cup_{n=1}^{\infty} A_n$ is a countable union of disjoint sets $A_n \in \mathcal{F}$

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$$f^{-1}(A) \in \mathcal{F}$$

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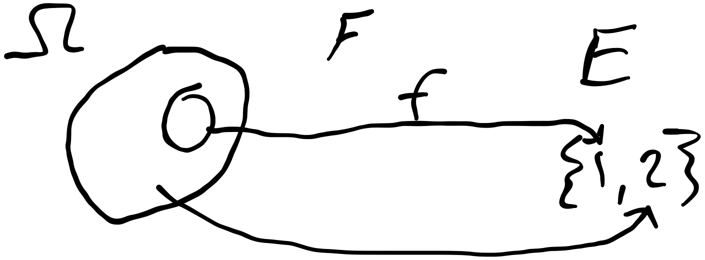

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Random Variables

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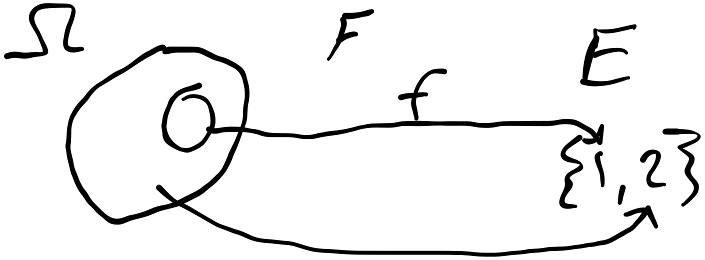

$$f^{-1}(A) \in \mathcal{F}$$

Are there functions that are not Borel-measurable?

Advantages over pdf definition

- Rigorous treatment of deterministic outcomes
- More sophisticated convergence concepts
- Better way of thinking about related random variables (personally, I think)

$$\begin{tikzcd} & \omega \arrow[dl] \arrow[dr] & \\ A \arrow[rr, leftrightarrow] & & B \end{tikzcd}$$

Break

Exercise 1.2.5. *Let $\Omega = \{1, 2, 3\}$. Find a σ-field $\mathcal{F}$ such that $(\Omega, \mathcal{F})$ is a measurable space, and a mapping X from Ω to $\mathbb{R}$, such that X is not a random variable on $(\Omega, \mathcal{F})$.*

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$$f^{-1}(A) \in \mathcal{F}$$

$\sigma(\{1\}) = \{\{1,2,3\}, \phi, \{1\}, \{2,3\}\}$

$X = \mathbb{1}_{\{1,2\}}$

$A = \{0\}$ $X^{-1}(A) = \{3\}$

$X(3) = 0$ $X(1) = 1$ $X(2) = 1$

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14.1

$\Omega = \{1, 2, 3\}$

$\mathcal{F} = \sigma(\{1\}) = \{\{1,2,3\}, \phi, \{1\}, \{2,3\}\}$

$X(\omega) = \mathbb{1}_{\{1,2\}}(\omega) = \begin{cases} 1 & \text{if } \omega \in \{1,2\} \\ 0 & \text{o.w.} \end{cases}$

$X(1) = 1$
$X(2) = 1$
$X(3) = 0$

$A \in \mathcal{E}$ $\qquad$ $E = \mathbb{R}$

$\uparrow$ Borel σ-algebra $= \sigma(\{(a,b) : a, b \in \mathbb{R}\})$

$A = 0$

$X^{-1}(A) = \{3\}$

$X^{-1}(A) \overset{?}{\in} \mathcal{F}$ $\qquad$ No $\qquad$ $\therefore$ X is not measurable on $\mathcal{F}$

$\therefore X$ is not a R.V.

Break

Exercise 1.2.5. *Let* $\Omega = \{1, 2, 3\}$. *Find a* σ*-field* $\mathcal{F}$ *such that* $(\Omega, \mathcal{F})$ *is a measurable space, and a mapping* X *from* Ω *to* $\mathbb{R}$, *such that* X *is not a random variable on* $(\Omega, \mathcal{F})$.

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$$f^{-1}(A) \in \mathcal{F}$$

$$\mathcal{F} = \{\Omega, \emptyset, \{1\}, \{2, 3\}\}$$

$$X = \mathbf{1}_{\{1,2\}}$$

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Convergence

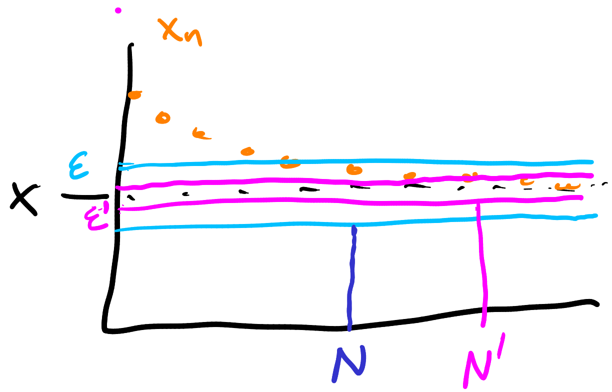
Convergence

Review: For a (deterministic) sequence $\{x_n\}$, we say

$$\lim_{n \to \infty} x_n = x$$

or

$$x_n \to x$$



if, for every $\epsilon > 0$, there exists an N such that $|x_n - x| < \epsilon$ for all $n > N$.

Convergence

In what senses can we talk about random variables converging?

- Sure ("pointwise")
- Almost Sure
- In Probability
- Weak ("in distribution"/"in law")

When are two R.V.'s the same?

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X = Y if $X(\omega) = Y(\omega) \quad \forall \omega \in \Omega$

When are two R.V.'s the same?

P $\quad 0.5, 0.5, 0$

$\Omega = \{1, 2, 3\}$

$X(\omega) = \begin{cases} 1 \text{ if } \omega \in \{1,2\} \\ 0 \text{ ow} \end{cases}$

$Y(\omega) = 1$

$X = Y$ if $X(\omega) = Y(\omega) \quad \forall \omega \in \Omega$

In practice, there are often unimportant ω where this is not true.

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We say that X is *almost surely* the same as Y if $P(\{\omega : X(\omega) \neq Y(\omega)\}) = 0.$

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In practice, there are often unimportant ω where this is not true.

We say that X is *almost surely* the same as Y if $P(\{\omega : X(\omega) \neq Y(\omega)\}) = 0.$

This is denoted $X \overset{a.s.}{=} Y$ and the terms *almost everywhere* (a.e.) and *with probability 1* (w.p.1) mean the same thing.

Sure Convergence

$$X_n(\omega) \to X(\omega) \quad \forall\, \omega \in \Omega$$

Almost Sure Convergence

Almost Sure Convergence

$X_n \overset{a.s.}{\to} X$ if there exists $A \in \mathcal{F}$ with $P(A) = 1$ such that $X_n(\omega) \to X(\omega)$ for each fixed $\omega \in A$.

Almost Sure Convergence

$X_n \stackrel{a.s.}{\to} X$ if there exists $A \in \mathcal{F}$ with $P(A) = 1$ such that $X_n(\omega) \to X(\omega)$ for each fixed $\omega \in A$.

Does sure convergence imply almost sure convergence?

Convergence in Probability

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$X_n \to_p X$ if $P(\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}) \to 0$ for any fixed $\epsilon > 0$.

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Yes.

Convergence in Probability

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Convergence in Probability

Does $X_n \rightarrow_p X$ imply $X_n \overset{a.s}{\rightarrow} X$?

No.

Convergence in Probability

Does $X_n \to_p X$ imply $X_n \stackrel{a.s}{\to} X$?

No.

PROOF. Consider the probability space $\Omega = (0,1)$, with Borel σ-field and the Uniform probability measure U of Example 1.1.11. Suffices to construct an example of $X_n \to_p 0$ such that fixing each $\omega \in (0,1)$, we have that $X_n(\omega) = 1$ for infinitely many values of n. For example, this is the case when $X_n(\omega) = \mathbf{1}_{[t_n, t_n+s_n]}(\omega)$ with $s_n \downarrow 0$ as $n \to \infty$ slowly enough and $t_n \in [0, 1-s_n]$ are such that any $\omega \in [0,1]$ is in infinitely many intervals $[t_n, t_n + s_n]$. The latter property applies if $t_n = (i-1)/k$ and $s_n = 1/k$ when $n = k(k-1)/2 + i$, $i = 1, 2, \ldots, k$ and $k = 1, 2, \ldots$ (plot the intervals $[t_n, t_n + s_n]$ to convince yourself). ∎

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But there exists a subsequence n_k such that $X_{n_k} \overset{a.s.}{\to} X$.

Weak Convergence

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Let $F_X : \mathbb{R} \to [0, 1]$ be the cumulative distribution function of real-valued random variable X.

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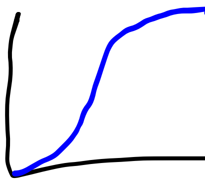
$F_X(x) = P(X \le x)$



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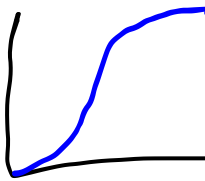

$X_n \xrightarrow{D} X$ if $F_{X_n}(\alpha) \to F_X(\alpha)$ for each fixed α that is a continuity point of F_X.

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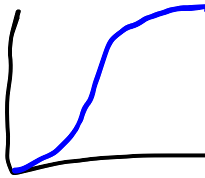

$X_n \xrightarrow{D} X$ if $F_{X_n}(\alpha) \to F_X(\alpha)$ for each fixed α that is a continuity point of F_X.

"Weak convergence", "convergence in distribution", and "convergence in law" all mean the same thing.

Convergence

In what senses can we talk about random variables converging?

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In what senses can we talk about random variables converging?

Stronger

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Implies

Convergence of MC integration

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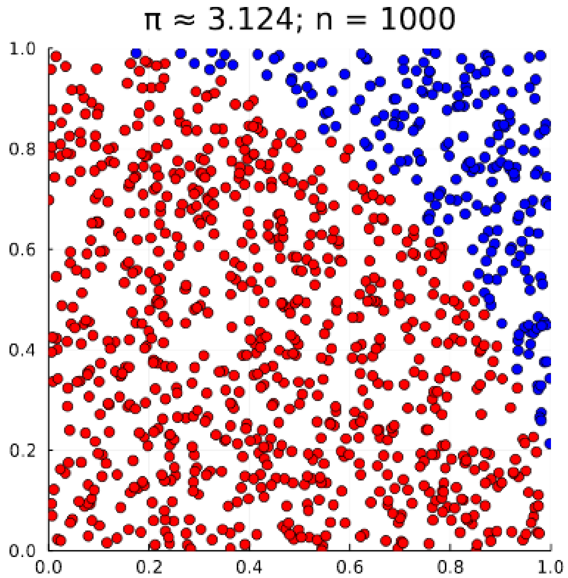
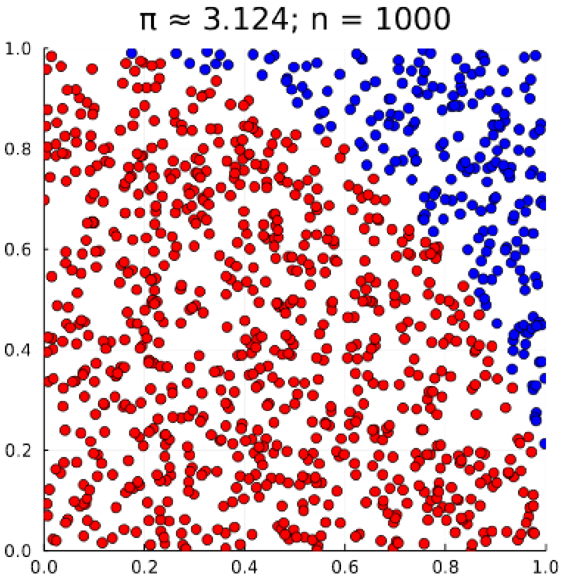
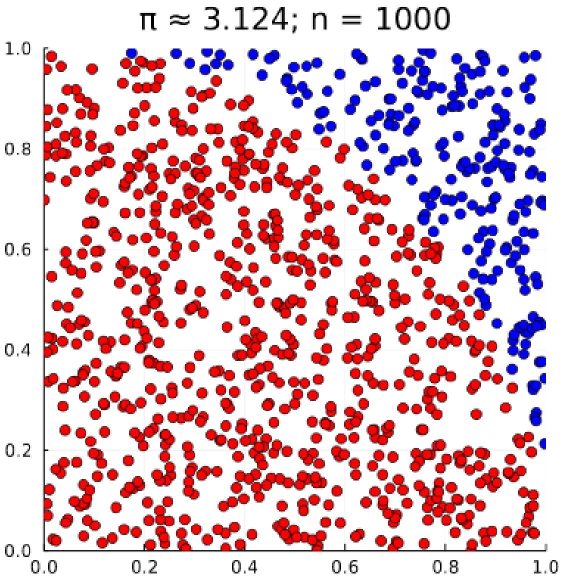


Convergence of MC integration



Let X_i be independent, identically distributed random variables with mean μ, and $Q_N \equiv \frac{1}{N} \sum_{i=1}^{N} X_i$.

Convergence of MC integration



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$$Q_N \stackrel{?}{\to} \mu?$$

Convergence of MC integration

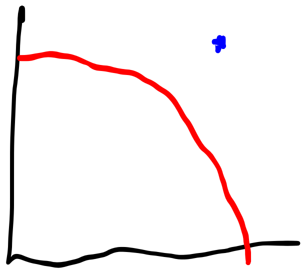
$$Q_N \to \mu \text{ (sure)?}$$

$$Q_N \overset{a.s.}{\to} \mu?$$

$$Q_N \to_p \mu?$$

$$Q_N \overset{D}{\to} \mu?$$

Convergence of MC integration



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Convergence of MC integration

$\exists \omega \in \Omega$ where you always
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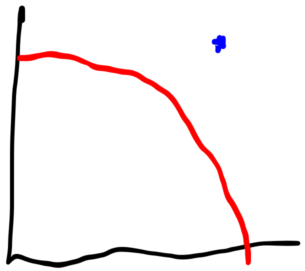

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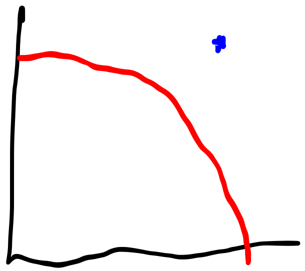

$$Q_N \to \mu \text{ (sure)?}$$

$$Q_N \xrightarrow{a.s.} \mu?$$

Probability that there are enough measurements off in one direction to keep $|Q_N - \mu| > \epsilon$ decays with more samples.

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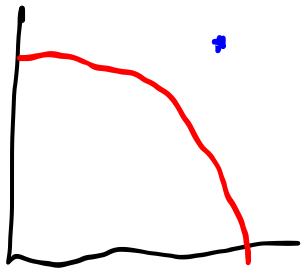

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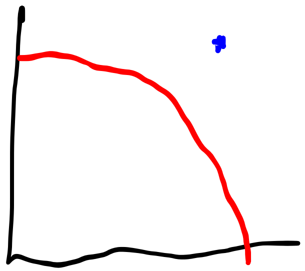

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Weak law of large numbers

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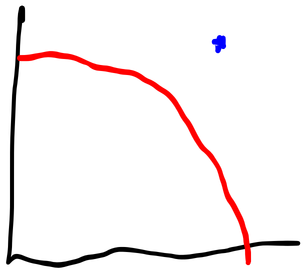

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$Q_N \to_p \mu$? Weak law of large numbers

$Q_N \overset{D}{\to} \mu$?

Convergence *Rate* of M.C. Integration

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How do you quantify $|Q_N - \mu|$?

Convergence *Rate* of M.C. Integration

How do you quantify $|Q_N - \mu|$?

Run M sets of N simulations and plot a histogram of Q_N^j for $j \in \{1, \ldots, M\}$.

Central Limit Theorem

Central Limit Theorem

Lindeberg-Levy CLT: If $\mathrm{Var}[X_i] = \sigma^2 < \infty$, then

$$\sqrt{N}(Q_N - \mu) \xrightarrow{D} \mathcal{N}(0, \sigma)$$

Central Limit Theorem

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After many samples Q_N starts to look distributed like $\mathcal{N}(\mu, \frac{\sigma}{\sqrt{N}})$

Central Limit Theorem

Two somewhat astounding takeaways:

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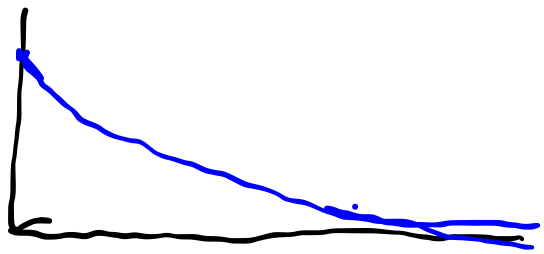
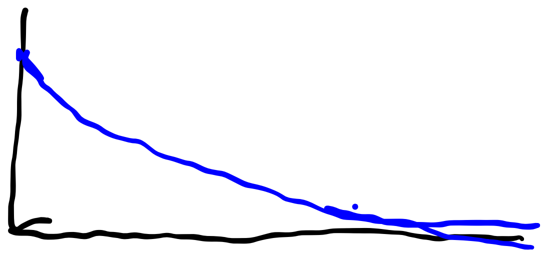
1. Error decays at $\frac{1}{\sqrt{N}}$ *regardless of dimension*.



Central Limit Theorem

Two somewhat astounding takeaways:

1. Error decays at $\frac{1}{\sqrt{N}}$ *regardless of dimension*.



2. You can estimate the "standard error" with
$$SE = \frac{s}{\sqrt{N}}$$
where s is the sample standard deviation.