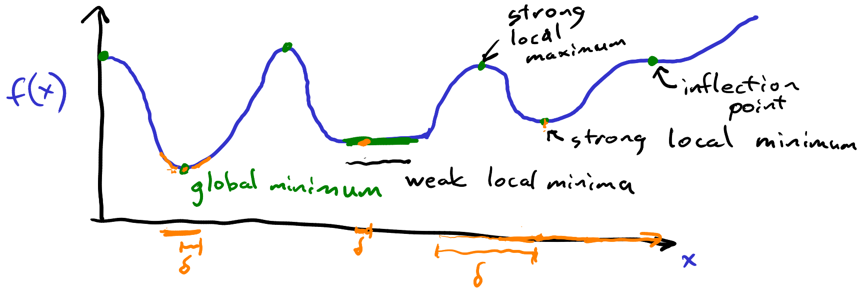
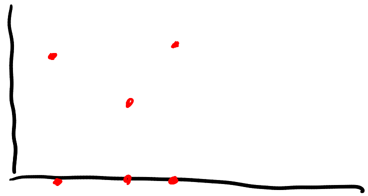
1-Dimensional Optimization



critical points

$f'(x) = 0$



hard to determine

global minimizer

x^* s.t. $f(x^*) \leq f(x) \quad \forall_x \in X$

local minimizer

x^* s.t. $\exists\, \delta > 0$ s.t. $f(x^*) \leq f(x) \quad \forall_x$ s.t. $|x - x^*| < \delta$

strong local minimizer

x^* s.t. $\exists\, \delta > 0$ s.t. $f(x^*) < f(x) \quad \forall_x$ s.t. $|x - x^*| < \delta$

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continuous

$f'(x) = 0$

$f''(x) > 0$ $\Rightarrow$ strong local minimum

$f'(x) = 0$ FONC

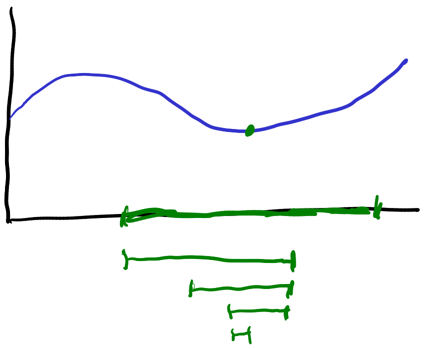
$f''(x) \geq 0$ SONC

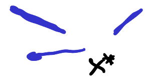
for a local minimum

not sufficient for local minimum (inflection point)

Bracketing

1. Identify interval containing local minimum
2. Shrink interval

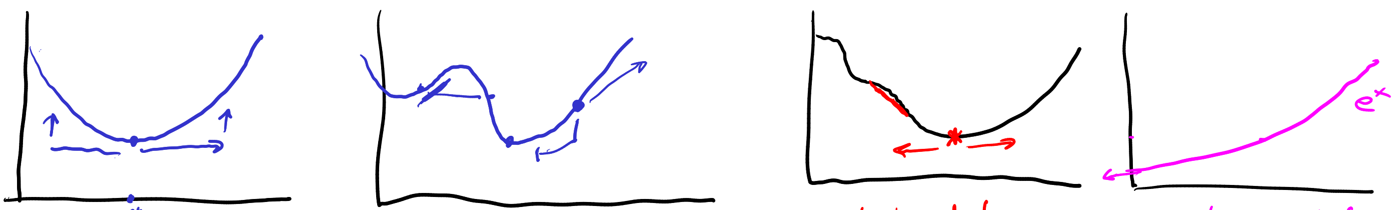




Property: Unimodality

fix 1

A function f is unimodal if $\exists$ unique optimizer x^* such that f is strictly monotonically decreasing for $x \le x^*$ and strictly monotonically increasing for $x \ge x^*$



unimodal and convex

not unimodal and not convex

unimodal not convex

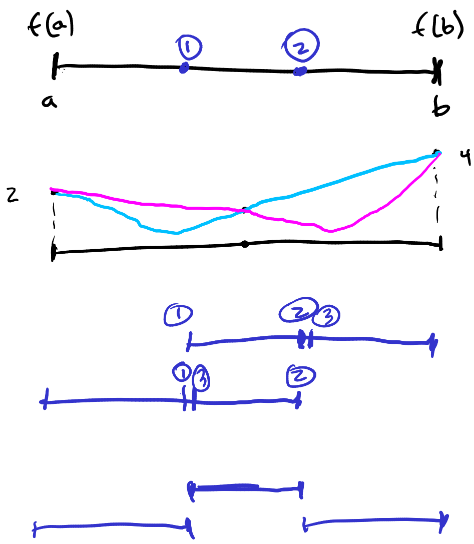
not unimodal convex

fix 2

→ ~~f unimodal → bracketing finds a global minimum~~

→ f unimodal ⟹ if a bracketing algorithm finds a local minimum that is a global

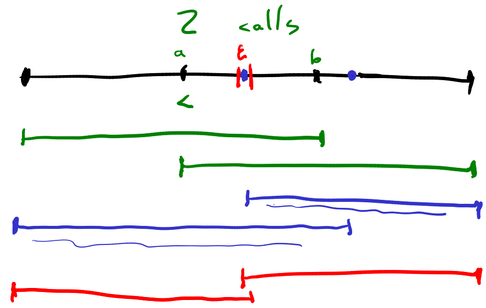
Break:

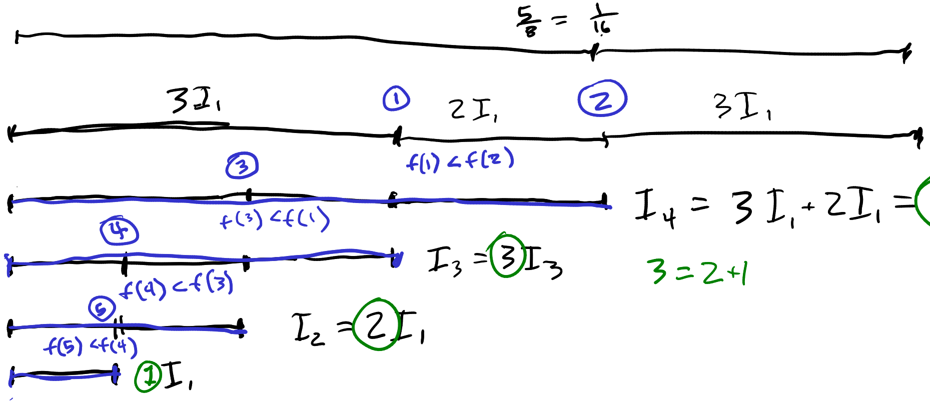


unimodal function

have 3 function calls

how do you use the function calls to shrink maximally





$I_5 = 8 I_1$ $\quad 8 = 5 + 3$

$I_4 = 3 I_1 + 2 I_1 = 5 I_1$ $\quad 5 = 3 + 2$

$I_3 = 3 I_3$ $\quad 3 = 2 + 1$

$I_2 = 2 I_1$

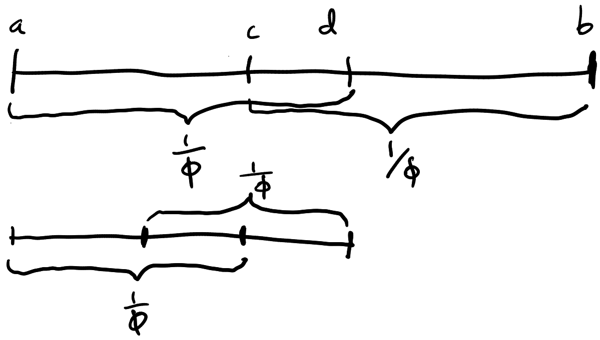
$$F_n = \begin{cases} 1 & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{o.w.} \end{cases}$$

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} \qquad \phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \qquad \text{"Golden Ratio"}$$

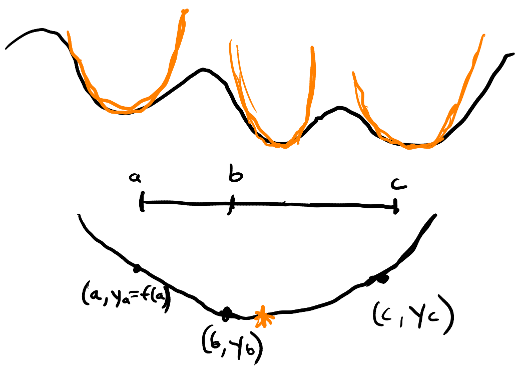
$$\frac{F_n}{F_{n-1}} = \frac{\frac{\phi^n - (1-\phi)^n}{\cancel{\sqrt{5}}}}{\frac{\phi^{n-1} - (1-\phi)^{n-1}}{\cancel{\sqrt{5}}}} = \frac{\phi^n - (\overset{<1}{1-\phi})^n}{\phi^{n-1} - (1-\phi)^{n-1}}$$

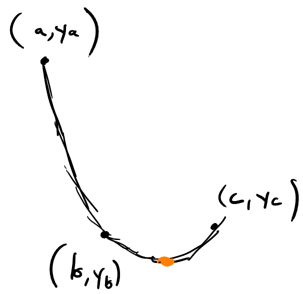
$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \frac{\phi^n}{\phi^{n-1}} = \phi = 1.618$$

Golden Section Search



Quadratic Fit Search

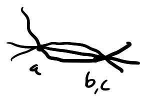




Approximate with $\quad q(x) = p_1 + p_2 x + p_3 x^2$

$$y_a = p_1 + p_2 a + p_3 a^2$$
$$y_b = p_1 + p_2 b + p_3 b^2$$
$$y_c = p_1 + p_2 c + p_3 c^2$$

$$\begin{bmatrix} y_a \\ y_b \\ y_c \end{bmatrix} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

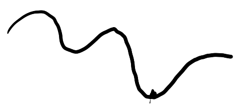


$$\tau_{invert}$$

$$q(x) = y_a \frac{(x-b)(x-c)}{(a-b)(a-c)} + y_b \frac{(x-a)(x-c)}{(b-a)(b-c)} + y_c \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

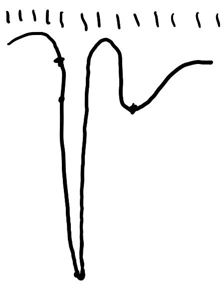
$$q'(x) = 0 \quad \text{at} \quad x^* = \frac{1}{2} \frac{y_a(b^2-c^2) + y_b(c^2-a^2) + y_c(a^2-b^2)}{y_a(b-c) + y_b(c-a) + y_c(a-b)}$$

Break: What property of a function could you use to prove that you have found a global minimum for a function that's not unimodal.



Continuity?

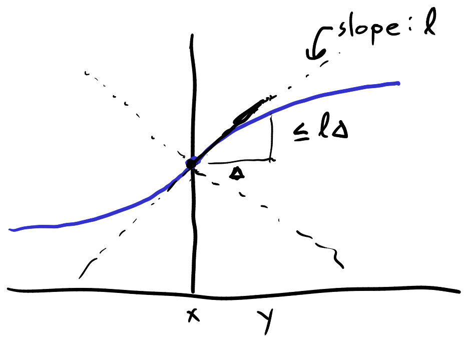
Differentiability?

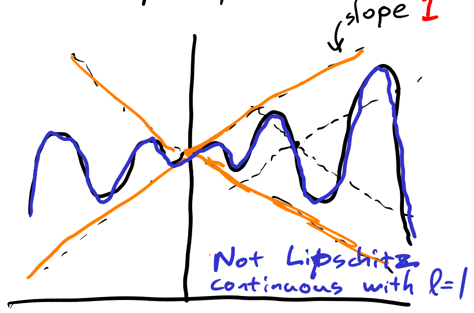


Lipschitz - Continuity

A function f is Lipschitz-continuous on $[a,b]$ if there exists an $\ell > 0$ such that

$$|f(x) - f(y)| \leq \ell \cdot |x-y| \quad \forall x,y \in [a,b]$$





choose orange point as x, and any y and inequality is satisfied

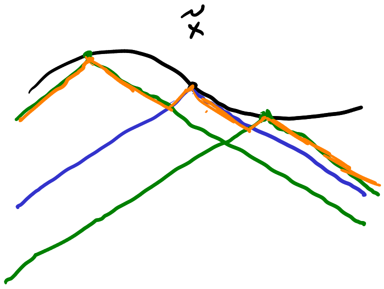
$\frac{1}{x}$ not Lipschitz continuous on $[-1000, 0)$

is L.C. on $[-1000,-1]$

Informally, any $\ell \geq \max_{x \in [a,b]} |f'(x)|$ can be the Lipschitz constant.

If f is L.C. with constant ℓ, and $f(\tilde{x})$ is known

$f(\tilde{x}) - \ell|x - \tilde{x}|$ is a lower bound for f



Schubert - Piyavskii method

- Finds global optimum of a L.C. function