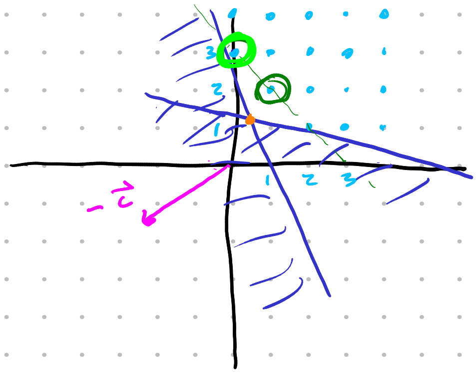
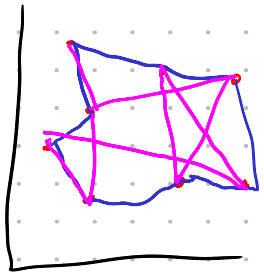
Discrete Optimization

Integer Linear program (ILP)

$$\underset{\vec{x}}{\text{minimize}} \; \vec{c}^T\vec{x}$$
$$\text{subject to} \quad A\vec{x} \le b$$
$$\vec{x} \ge 0$$
$$\vec{x} \in \mathbb{Z}^n$$



Travelling Salesman Problem (TSP)



Find shortest "tour" (path passing through all points)

Formulate as ILP

$$\underset{\vec{x}\,\vec{u}}{\text{minimize}} \; \sum_{ij} d_{ij} x_{ij}$$

$x_{ij} \in \{0,1\}$
$= \begin{cases} 1 & \text{if path contains } i \to j \\ 0 & \text{o.w.} \end{cases}$

subject to $\quad x_{ij} \in \{0,1\}$

$\sum_j x_{ij} = 1 \quad \forall i$ "exit" $\qquad \sum_{ji} x_{ij} = n$

$\sum_i x_{ij} = 1 \quad \forall j$ "entry" $\qquad \forall i$

Miller-Tucker-Zemlin

$u_i \in \{2, \ldots, n\}$ $\qquad \forall i \quad x_{ii} = 0$

want if $x_{ij} = 1$
$\underline{u_j - u_i = 1}$

$u_i - u_j + 1 \le (n-1)(1 - x_{ij}) \qquad \forall i,j$

if $x_{ij} = 1$
$u_i - u_j + 1 \le 0$
$u_i - u_j \le -1$
$u_j - u_i \ge 1$

when applied to all i,j forces
$u_j - u_i = 1$

if $x_{ij} = 0$

$u_i - u_j + 1 \le n - 1$

worst case
$u_i = n, \; u_j = 2$

does not constrain u

MILP "Mixed Integer LP"

$$\underset{\vec{x}}{\text{minimize}} \quad \vec{c}^T \vec{x}$$

$$A\vec{x} \le \vec{b} \quad \leftarrow \quad A\vec{x} = \vec{b}$$

$$\vec{x} \ge 0$$

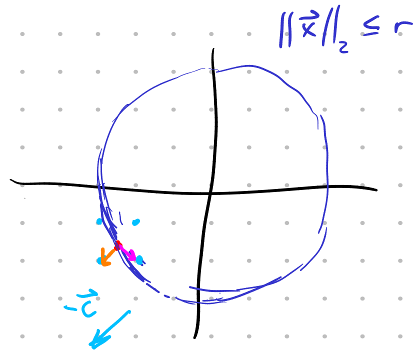
$$\vec{x}_D \in \mathbb{Z}^n$$

Rounding

relax $\vec{x}_D \in \mathbb{Z}^n$ constraint to $\vec{x}_D \in \mathbb{R}^n$

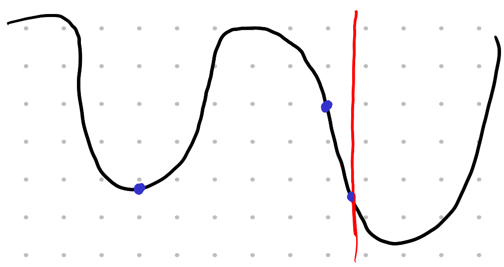
solve

round to nearest integer $\vec{x}_D$



Problems

1) result might be infeasible
2) nearest feasible integral solution might be much worse than optimal



Sometimes there are gaurantees on how close the solution is

If $A \in \mathbb{Z}^{n \times m}$

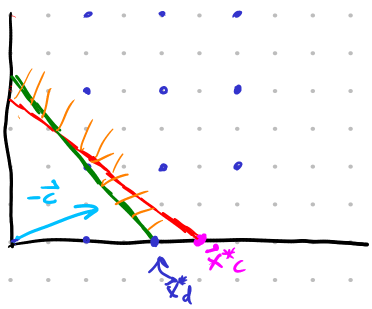
$\| x_d^* - x_c^* \| \le n \times$ max absolute value of determinants of submatrices of A

relaxed solution

optimal discrete solution

S.O.T.A. "branch + cut"

Cutting Plane



Introduce new constraint that

1) excludes x_c^*
2) includes all other discrete solutions

Partition $\vec{x}_c^*$ into (B, V) $\quad \vec{x}_B^* \quad \vec{x}_V^* = 0$

all of the non-integral components

For each $b \in B$ where x_b^* is non-integral introduce a new constraint

$$\underbrace{x_b^* - \lfloor x_b^* \rfloor}_{\geq 0} - \underbrace{\sum_{v \in V} \left(\bar{A}_{bv} - \lfloor \bar{A}_{bv} \rfloor \right) x_v}_{0} \leq 0 \qquad \bar{A}_{bv} = A_B^{-1} A_v$$

$$x_k + \sum_{v \in V} \left(\lfloor \bar{A}_{bv} \rfloor - \bar{A}_{bv} \right) x_v = \lfloor x_b^* \rfloor - x_b^*$$

$$x_k \in \mathbb{N}$$

Consider the integer program:

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & 2x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & \begin{bmatrix} 0.5 & -0.5 & 1.0 \\ 2.0 & 0.5 & -1.5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2.5 \\ -1.5 \end{bmatrix} \\ & \mathbf{x} \geq \mathbf{0} \qquad \mathbf{x} \in \mathbb{Z}^3 \end{aligned}$$

x_c^* $V = \{2\}$

The relaxed solution is $\mathbf{x}^* \approx [0.818, 0, 2.091]$, yielding: $B = \{1,3\}$

$$\mathbf{A}_{\mathcal{B}} = \begin{bmatrix} 0.5 & 1 \\ 2 & -1.5 \end{bmatrix} \qquad \mathbf{A}_{\mathcal{V}} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} \qquad \bar{\mathbf{A}} = \begin{bmatrix} -0.091 \\ -0.455 \end{bmatrix}$$

From equation (19.7), the constraint for x_1 with slack variable x_4 is:

$$\begin{aligned} x_4 + (\lfloor -0.091 \rfloor - (-0.091))x_2 &= \lfloor 0.818 \rfloor - 0.818 \\ x_4 - 0.909x_2 &= -0.818 \end{aligned}$$

The constraint for x_3 with slack variable x_5 is:

$$\begin{aligned} x_5 + (\lfloor -0.455 \rfloor - (-0.455))x_2 &= \lfloor 2.091 \rfloor - 2.091 \\ x_5 - 0.545x_2 &= -0.091 \end{aligned}$$

The modified integer program has:

$$\mathbf{A} = \begin{bmatrix} 0.5 & -0.5 & 1 & 0 & 0 \\ 2 & 0.5 & -1.5 & 0 & 0 \\ 0 & -0.909 & 0 & 1 & 0 \\ 0 & -0.545 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2.5 \\ -1.5 \\ -0.818 \\ -0.091 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Solving the modified LP, we get $\mathbf{x}_c^* \approx [0.9, 0.9, 2.5, 0.0, 0.4]$. Since this point is not integral, we repeat the procedure with constraints:

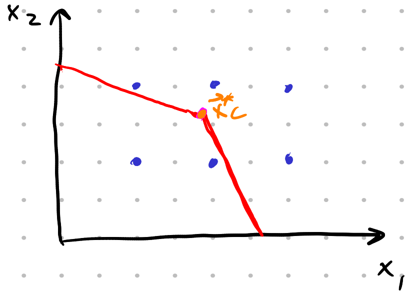
$$\begin{aligned} x_6 - 0.9x_4 &= -0.9 & \qquad x_7 - 0.9x_4 &= -0.9 \\ x_8 - 0.5x_4 &= -0.5 & \qquad x_9 - 0.4x_4 &= -0.4 \end{aligned}$$

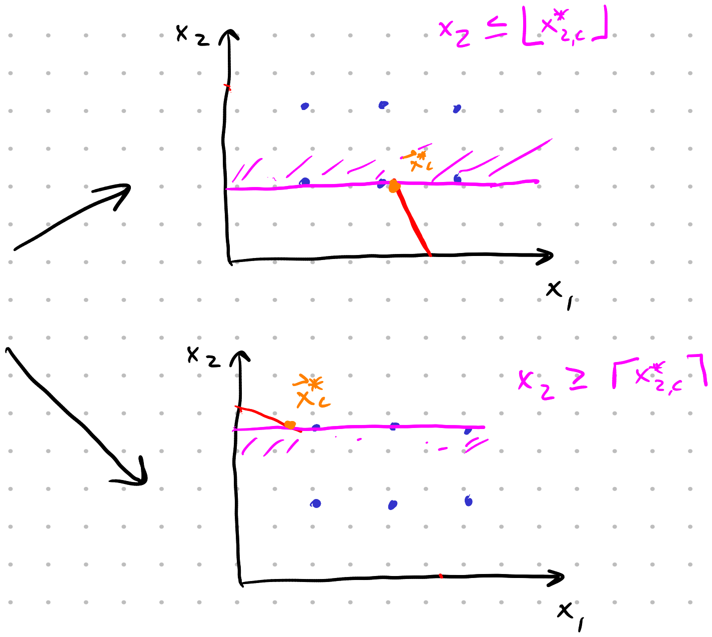
and solve a third LP to obtain: $\mathbf{x}^* = [1, 2, 3, 1, 1, 0, 0, 0, 0]$ with a final solution of $\mathbf{x}_i^* = [1, 2, 3]$.

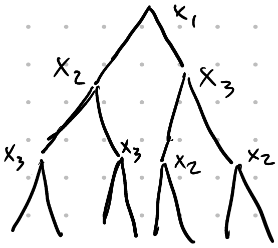
Branch and Bound

add constraints

$x_i \le \lfloor x^*_{i,c} \rfloor \qquad x_i \ge \lceil x^*_{i,c} \rceil$







or

Priority Queue ranked by lower bound

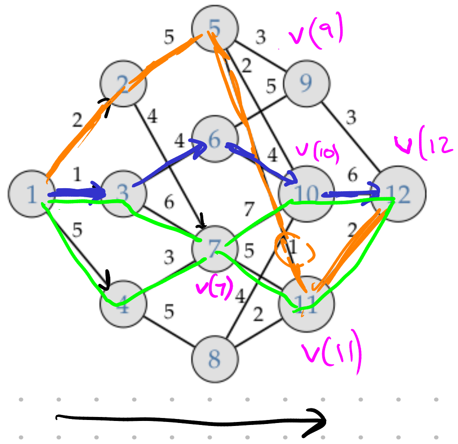
Which i to branch on next

Common heuristic $\vec{x}^*_c = [0.1, 0.4, 0.9]$

```
function minimize_lp_and_y(LP)
    try
        x = minimize_lp(LP)
        return (x, x⋅LP.c)
    catch
        return (fill(NaN, length(LP.c)), Inf)
    end
end
function branch_and_bound(MIP)
    LP = relax(MIP)
    x, y = minimize_lp_and_y(LP)
    n = length(x)
    x_best, y_best, Q = deepcopy(x), Inf, PriorityQueue()
    enqueue!(Q, (LP,x,y), y)
    while !isempty(Q)
        LP, x, y = dequeue!(Q)
        if any(isnan.(x)) || all(isint(x[i]) for i in MIP.D)
            if y < y_best
                x_best, y_best = x[1:n], y
            end
        else
            i = argmax([abs(x[i] - round(x[i])) for i in MIP.D])
            # x_i ≤ floor(x_i)
            A, b, c = LP.A, LP.b, LP.c
            A2 = [A zeros(size(A,1));
                 [j==i for j in 1:size(A,2)]' 1]
            b2, c2 = vcat(b, floor(x[i])), vcat(c, 0)
            LP2 = LinearProgram(A2,b2,c2)
            x2, y2 = minimize_lp_and_y(LP2)
            if y2 ≤ y_best
                enqueue!(Q, (LP2,x2,y2), y2)
            end
            # x_i ≥ ceil(x_i)
            A2 = [A zeros(size(A,1));
                 [j==i for j in 1:size(A,2)]' -1]
            b2, c2 = vcat(b, ceil(x[i])), vcat(c, 0)
            LP2 = LinearProgram(A2,b2,c2)
            x2, y2 = minimize_lp_and_y(LP2)
            if y2 ≤ y_best
                enqueue!(Q, (LP2,x2,y2), y2)
            end
        end
    end
    return x_best
end
```

Shortest Path

travel from node a to b in minimum distance



$$\underset{ij}{\text{minimize}} \ \sum_{ij} c_{ij} x_{ij}$$

subject to $\quad x_{ij} \in \{0, 1\}$

$$\sum_{j \in \text{children}(1)} x_{ij} = 1 \quad \text{"start"}$$

$$\sum_{i \in \text{parents}(n)} x_{in} = 1 \quad \text{"end"}$$

$$\sum_{i \in \text{parents}(k)} x_{ik} - \sum_{j \in \text{children}(k)} x_{kj} = 0$$

$$\underset{x_1, \dots x_n}{\text{minimize}} \left(c(s_1, x_1) + c(s_2, x_2) \dots c(s_n, x_n) \right)$$

$$s_{k+1} = f(s_k, x_k)$$

$$c(s_k, x_k)$$

Greedy Algorithm (not always optimal)

$$\underset{x_i}{\text{minimize}} \ c(s_i, x_i)$$

Bellman's Principle of Optimality

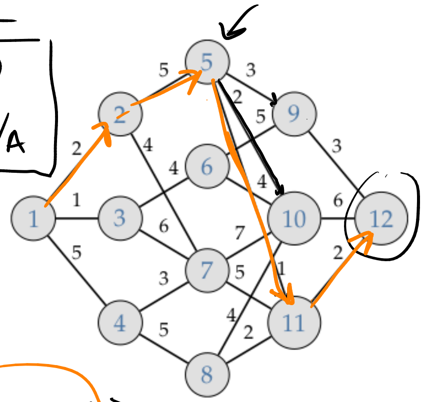
"Every sub-path of an optimal path is optimal"

$$v(s_i) = \underset{x_i \dots x_n}{\text{minimize}} \left(c(s_i, x_i) + \underbrace{c(s_{i+1}, x_{i+1}), \dots c(s_n, x_n)} \right)$$

$$= \underset{x_i}{\text{minimize}} \left(c(s_i, x_i) + v(s_{i+1}) \right)$$

$$= \underset{x_i}{\text{minimize}} \left(c(s_i, x_i) + v(f(s_i, x_i)) \right)$$

	1	2	3	4	5	6	7	8	9	10	11	12
$v(s)$	10	8	12	9	3	8	7	4	3	6	2	0
x^*	2	5	6	8	11	9	11	11	12	12	12	N/A



$t(s,x) = x$

$$v(5) = \underset{x \in \{9,10,11\}}{\text{minimize}} \left(c(5,x) + v(x) \right)$$

$$= \text{minimize} \left\{ 3 + \underset{3}{v(9)},\ 2 + \underset{6}{v(10)},\ 1 + \underset{2}{v(11)} \right\}$$

$$= 3$$

$$v(6) = \underset{x \in \{9,10\}}{\text{minimize}} \left(c(6,x) + v(x) \right)$$

$$= \text{minimize} \left\{ \underbrace{5 + \underset{3}{v(9)}}_{8},\ \underbrace{4 + \underset{6}{v(10)}}_{10} \right\}$$

Knapsack

n items
value v_i
weight w_i

$$\underset{\vec{x}}{\text{minimize}}\ -\sum_i \overset{\text{value}}{v_i}\, x_i$$

$$\text{subject to} \quad \sum_i w_i x_i \le w_{max}$$

$$x_i \in \{0,1\}$$

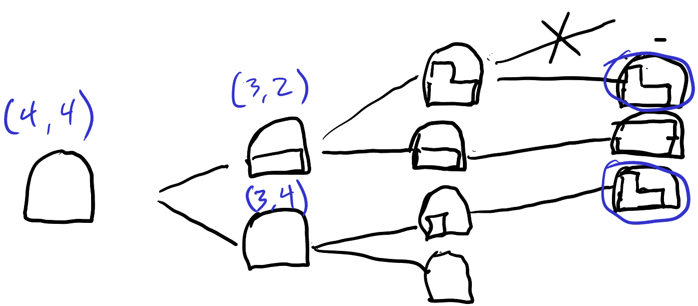
$s = (k, w_{rem})$ — k: number of items left

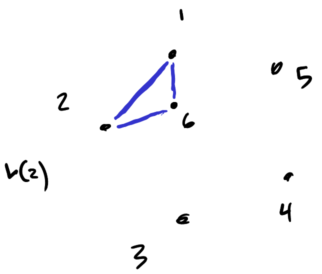
$x = \{0,1\}$ do we include item $n-k$

$$t(s,x) \begin{cases} (k-1, w_{rem} - w_{n-k}) & \text{if } x = 1 \\ (k-1, w_{rem}) & \text{if } x = 0 \end{cases}$$

$$c(s,x) \begin{cases} -v_{k-n} & \text{if } x = 1 \\ 0 & \text{o.w.} \end{cases}$$

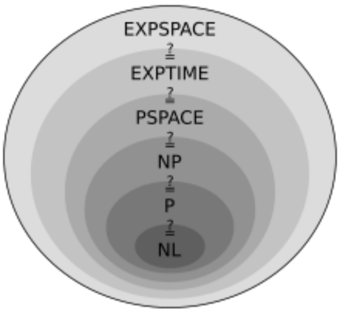
$\boxed{2}$ $\boxed{1}$ $\boxed{2}$ $\boxed{1}$





Problem: some discrete optimization problems appear to be "fundamentally hard" - Zach

Computational Complexity Classes
(for decision problems)
yes/no

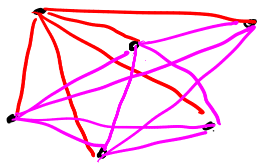


Optimization $\rightarrow$ decision
is there an input with an objective less than y

P
polynomial time

NP
nondeterministic polynomial time

is there a tour of length $\leq \ell$



NP-Complete: A problem is NP-complete if it is in NP and any problem in NP can be translated to it in polynomial time.

There are no known polynomial-time algorithms for any NP-complete problem.

TSP is an NP-complete problem.

Randomization Helps

Simulated Annealing

$x \leftarrow \text{randperm}(1{:}n)$
loop
  $x' \leftarrow \text{change}(x)$ $\leftarrow$ swap 2 points or swap 2 sections
  $\Delta y \leftarrow d(x') - d(x)$
  if $\Delta y \leq 0$ or $\text{rand}() < e^{-\Delta y / t}$
    $x \leftarrow x'$
  reduce t

Genetic Algorithm (use path as chromosome)

Ant Colony Optimization

loop
    for each ant
        run ant   (select next node w.p. $\frac{A(i \to j)}{\sum_{j'} A(i \to j')}$
        if ant reached end   where $A(i \to j) = \tau(i \to j)^{\alpha \leftarrow 1} \eta(i \to j)^{\beta \leftarrow 5}$
            update pheremone along path

τ: pheremone

η: $1/d(i \to j)$