Linear Programming

A *linear program* is an optimization problem where $f(\vec{x})$, $\vec{g}(\vec{x})$, and $\vec{h}(\vec{x})$ are affine functions of $\vec{x}$

Example

Three goods: x_1, x_2, x_3 $\quad$ $x_1 \le \bar{x}_1$

Volume per unit: v_1, v_2, v_3 $\quad$ max volume $\bar{V}$

Price per unit: p_1, p_2, p_3

Weight per unit: w_1, w_2, w_3 $\quad$ max weight $\vec{w}$

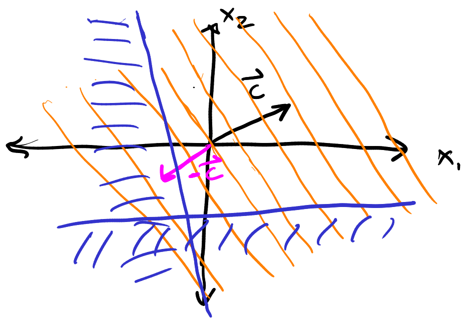
$$\begin{aligned} \underset{\vec{x}}{\text{maximize}} \quad & \sum_i p_i x_i \\ \text{subject to} \quad & \sum_i v_i x_i \le \bar{V} \\ & \sum_i w_i x_i \le \vec{w} \end{aligned}$$

$$\underset{\vec{x}}{\text{minimize}} \ \|A\vec{x} - \vec{b}\|_1 \quad \Rightarrow \quad \begin{aligned} \underset{\vec{x}, \vec{s}}{\text{minimize}} \quad & \sum_i s_i \\ \text{subject to} \quad & A\vec{x} - \vec{b} \le \vec{s} \\ & A\vec{x} - \vec{b} \ge -\vec{s} \end{aligned}$$

$$\underset{\vec{x}}{\text{minimize}} \ \|A\vec{x} - \vec{b}\|_\infty \quad \Rightarrow \quad \begin{aligned} \underset{\vec{x}, t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & A\vec{x} - b \le t\vec{1} \\ & A\vec{x} - b \ge -t\vec{1} \end{aligned}$$

Geometry

objective: $\vec{c}^T\vec{x}$



Constraints: $h_i(\vec{x}) = \vec{a}_i^T\vec{x} + b_i$

$g_i(\vec{x}) = \vec{a}_i^T\vec{x} + b_i$

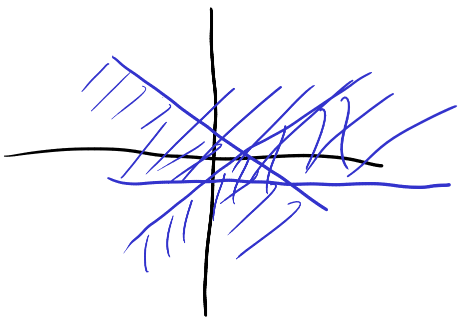
hyperplane

half space

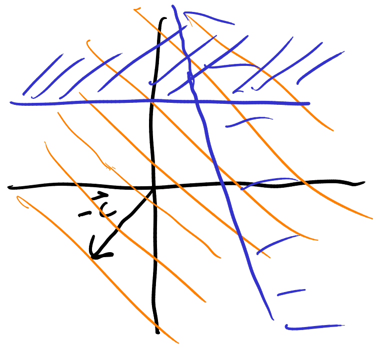
Break: If $\vec{c} \neq 0$, will there ever be optimal solutions that are interior points?

and there are no equality constraints

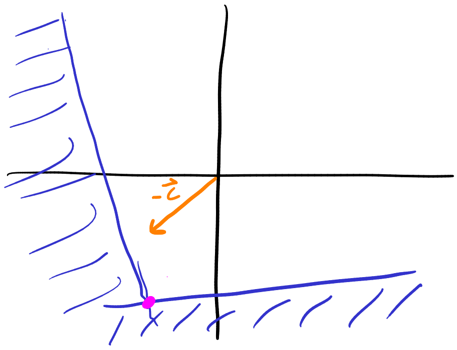
no constraints are active



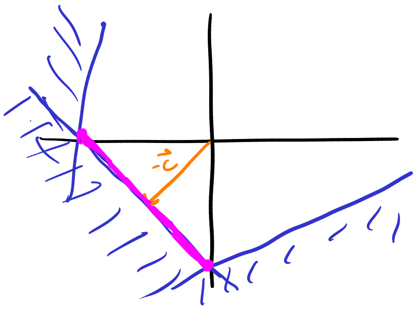
infeasible



unbounded



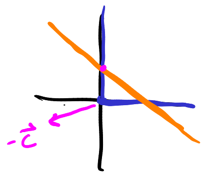
one optimal solution at one vertex

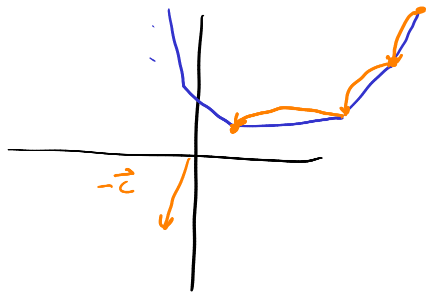


infinite number of optimal solutions on a face of the polytope

Except in degenerate cases, there is at least one vertex that is an optimal solution.

Simplex Method





$$\min_{\vec{x}} \quad \vec{c}^T \vec{x}$$
$$\text{s.t.} \quad A\vec{x} = \vec{b}$$
$$\vec{x} \geq 0$$

Assume that A is an $m \times n$ matrix, $m \leq n$, all rows are linearly independent (preprocessing)

Design variable indices $\{1, \ldots, n\}$ partitioned into two sets:

$i \in V \Rightarrow x_i = 0$ "Active" $n-m$ elements

$i \in B \Rightarrow x_i \geq 0$ "Inactive" m elements

$\vec{x}_B$, $\vec{x}_V = 0$

$$A_B \vec{x}_B = \vec{b} \longrightarrow \vec{x}_B = A_B^{-1} \vec{b}$$

A_B might be singular, $\vec{x}_B$ might not be positive

Two phases
1. Initialization
2. Optimization

KKT conditions $\qquad \mathcal{L}(x,\lambda,\mu) = \vec{c}^T\vec{x} - \vec{\mu}^T\vec{x} - \vec{\lambda}^T(A\vec{x}-\vec{b})$

1. Feasibility: $A\vec{x} = \vec{b}$, $\vec{x} \geq 0$
2. → Dual Feasibility: $\vec{\mu} \geq 0$
3. Complementary Slackness: $\vec{\mu} \odot \vec{x} = 0$
4. Stationarity: $A^T\vec{\lambda} + \vec{\mu} = \vec{c}$

Since problem is convex these are also sufficient conditions

$$A_B^T\vec{\lambda} + \underbrace{\vec{\mu}_B}_{0} = \vec{c}_B \Rightarrow \vec{\lambda} = \overset{(A^T)^{-1}}{A_B^{-T}}\vec{c}_B$$

$$A_V^T\vec{\lambda} + \vec{\mu}_V = \vec{c}_V \qquad \vec{\mu}_V = \vec{c}_V - (A_B^{-1}A_V)^T\vec{c}_B$$

if $\vec{\mu}_V$ contains negative components not optimal!

Optimization Phase

Each step: $\vec{x} \rightarrow \vec{x}'$ "pivot"

select "entering index" $q \in V$

"leaving index" $p \in B$

$\vec{x}'$ must satisfy

$$A\vec{x}' = \vec{b} = A_B\vec{x}_B' + A_q x_q' = A_B\vec{x}_B$$

$$\vec{x}_B' = \vec{x}_B - A_B^{-1}A_q x_q'$$

$p \in B$ becomes active when

$$(\vec{x}_B')_p = 0 = (\vec{x}_B)_p - (A_B^{-1}A_q)_p\, x_q'$$

$$x_q' = \frac{(\vec{x}_B)_p}{(A_B^{-1}A_q)_p}$$

Choose p that minimizes this ratio (minimizes x_q')

choose q based on the effect on objective

$$\vec{c}^T\vec{x}' = \vec{c}_B^T\vec{x}_B' + c_q x_q'$$