"Element Medley"

Pertinent reading: *Chapter 9* in textbook

Department of Civil and Environmental Engineering

Presented by:

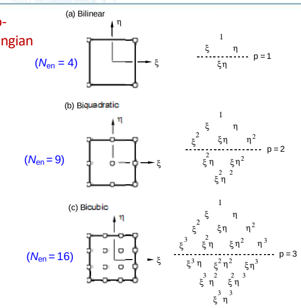
V.N. Kaliakin

UNIVERSITY OF DELAWARE

Lagrangian Elements: A Summary

- Recall that interpolation functions for Lagrangian elements are derived in a *direct*, *systematic* manner.
- However, as the order of the element increases, the element becomes computationally *less efficient*.

Family of Two-Dimensional Lagrangian Elements



Lagrangian Elements: A Summary

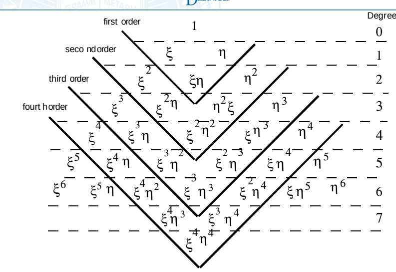
- For example, the interpolation functions for standard 2-d Lagrangian elements are obtained by forming the product of *complete* Lagrange polynomials of degree

$$(N_{en} - 1)$$

in both ξ and η.

Lagrangian Elements: A Summary

- The number of terms in the polynomial resulting from this product is in *excess* of the number required to produce a *complete* polynomial of degree (N_{en} - 1) in ξ and η or in ξ, η and ζ.
- Recall: the rate of convergence is determined by the degree of the *complete* polynomial (see Section 7.5.3).



Pascal Triangle for Lagrange Family of 2-d Elements

Lagrangian Elements: A Summary

- Due to the "symmetry" of terms present in the polynomial, *spatial isotropy* is, however, maintained.
- Spatial isotropy means that the approximating polynomial remains *unchanged* under a linear coordinate transformation (see *Section* 7.5.2).

Lagrangian Elements: A Summary

- The number of terms in the polynomial resulting from this product is in *excess* of the number required to produce a *complete* polynomial of degree (N_{en} - 1) in ξ and η or in ξ, η and ζ.
- In general, a 2-d Lagrangian element complete to degree p contains $(p + 1)^2$ terms.

Lagrangian Elements: A Summary

- Of these, only $(p + 1)(p + 2)/2$ of the terms are needed for completeness.
- The remaining $p(p + 1)/2$ "parasitic terms" typically *do not* improve the rate of convergence.

Lagrangian Elements: A Summary

- The second disadvantage of Lagrangian elements is the presence of *interior nodes* in the quadratic and higher order members of the family.
- Since only edge and vertex nodes are common to adjacent elements, it is possible to eliminate the interior nodes from the element *prior* to assembly by so-called *nodal condensation* (see Section 9.7).

Lagrangian Elements: A Summary

- The shortcomings of Lagrangian elements can be overcome by developing elements with interpolation functions that are capable of producing more closely only the terms present in a *complete* polynomial of appropriate degree.
- This is realized by so-called *serendipity elements*.

Serendipity Elements

Pertinent Reading: *Section 9.3* in the textbook.

Serendipity Elements

- Serendipity is "the effect by which one accidentally discovers something fortunate, especially while looking for something else entirely."
- The word derives from an old Persian fairy tale and was coined in 1754 by Horace Walpole.

Horace Walpole

4th Earl of Orford (1717-1797)

Serendipity Elements

- In serendipity elements the nodes are arranged to lie, as much as possible, only along the element boundaries.
- On the boundaries of serendipity elements the form of the approximation is *identical* to that produced by the Lagrangian family of elements.
- Interelement C^0 continuity of the approximation is thus maintained.

Serendipity Elements

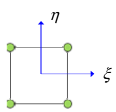
- Originally interpolation functions for serendipity elements were derived by inspection.
- The progression to higher order elements was difficult and required some ingenuity.
- With time, systematic approaches for generating interpolation functions were, however, developed (see p. 322 in text for references).

Serendipity Elements

- Like their Lagrangian counterparts, serendipity elements are *quadrilateral* in shape.
- Although the present development is limited to the two-dimensional *parent domain* ∧p : $\xi, \eta \in [-1, 1]$, this domain is easily mapped to actual *quadrilateral* element geometries (see *Chapter* 10).

Linear Two-Dimensional Serendipity Element

- The linear serendipity element is *identical* to the bi-linear Lagrangian element.
- The interpolating polynomial is complete to degree p = 1.



$$\begin{array}{ccc} & 1 & \\ \xi & & \eta \\ \hline & \xi\eta & \end{array} \quad p = 1$$

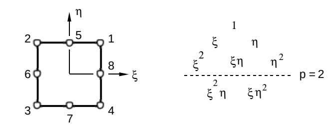
Quadratic Two-Dimensional Serendipity Element

- Beginning with the quadratic element, the interpolation functions for serendipity family *differ* from those associated with Lagrange elements.
- The quadratic serendipity element has *eight* nodes.
- Four terms must thus be added to the interpolating polynomial corresponding to the linear element.

Quadratic Two-Dimensional Serendipity Element

- These include *two missing quadratic terms* (ξ^2 and η^2), and *two cubic terms* ($\xi^2\eta$ and $\xi\eta^2$) that are symmetric in ξ and η in order to preserve *spatial isotropy*.
- The interpolating polynomial is complete to degree p = 2.

Quadratic Two-Dimensional Serendipity Element



Quadratic Two-Dimensional Serendipity Element

- The associated interpolation functions are given in the following table:

Node Type	i	N_i
corner	1, 2, 3, 4	$\frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(\xi_i\xi+\eta_i\eta-1)$
mid-side	5, 7	$\frac{1}{2}(1-\xi^2)(1+\eta_i\eta)$
mid-side	6, 8	$\frac{1}{2}(1+\xi_i\xi)(1-\eta^2)$

Quadratic Two-Dimensional Serendipity Element

where

i	ξ_i	η_i
1	1	1
2	-1	1
3	-1	-1
4	1	-1
5	0	1
6	-1	0
7	0	-1
8	1	0

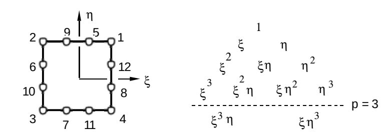
Additional details pertaining to this element are given on pp. 322 to 324 of the textbook.

Cubic Two-Dimensional Serendipity Element

- To progress to the cubic serendipity element from the quadratic one requires the addition of *four* nodes, one per side.
- The corresponding four new terms in the interpolating polynomial include *two missing cubic terms* (ξ^3 and η^3), and *two quartic terms* ($\xi^3\eta$ and $\xi\eta^3$) that are symmetric in ξ and η.

Cubic Two-Dimensional Serendipity Element

- The interpolating polynomial is complete to degree p = 3.



Cubic Two-Dimensional Serendipity Element

- The associated interpolation functions are given in the following table:

Node Type	i	N_i
corner	1, 2, 3, 4	$\frac{1}{32}(1+\xi_i\xi)(1+\eta_i\eta)\left[9(\xi^2+\eta^2)-10\right]$
mid-side	6, 8, 10, 12	$\frac{9}{32}(1+\xi_i\xi)(1-\eta^2)(1+9\eta_i\eta)$
mid-side	5, 7, 9, 11	$\frac{9}{32}(1+\eta_i\eta)(1-\xi^2)(1+9\xi_i\xi)$

Cubic Two-Dimensional Serendipity Element

where

i	ξ_i	η_i
1	1	1
2	-1	1
3	-1	-1
4	1	-1
5	1/3	1
6	-1	1/3
7	-1/3	-1
8	1	-1/3
9	-1/3	1
10	-1	-1/3
11	1/3	-1
12	1	1/3

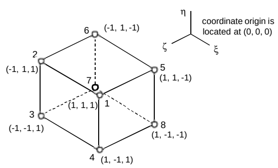
Additional details pertaining to this element are given on pp. 324 to 325 of the textbook.

Three-Dimensional Serendipity Elements

- Like their Lagrangian counterparts, serendipity elements are *hexahedral* in shape.
- Although the present development is limited to the two-dimensional *parent domain* (a tri-unit cube) $\wedge^p$: $\xi, \eta, \zeta \in$ [-1, 1], this domain is easily mapped to actual *hexahedral* element geometries (see *Chapter* 10).

Linear Three-Dimensional Serendipity Element

- The linear serendipity element is *identical* to the tri-linear Lagrangian element (8-node "brick").

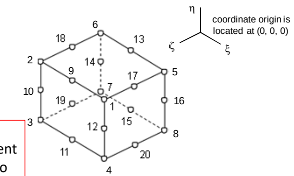


Quadratic Three-Dimensional Serendipity Element

- Beginning with the quadratic element, the interpolation functions for the three-dimensional serendipity family *differ* from those for Lagrangian elements.

Quadratic Three-Dimensional Serendipity Element

- The quadratic serendipity element has *twenty* nodes.



Additional details pertaining to this element are given on pp. 327 to 330 of the textbook.

Triangular Elements

Pertinent Reading: *Section 9.4.2* in the textbook.

Triangular Elements

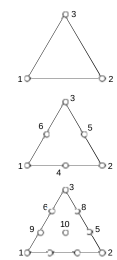
- Triangular elements always make use of *complete* two-dimensional polynomials.
- The nodal locations in such elements are thus arranged in correspondence to the locations of terms in a two-dimensional Pascal triangle.

Triangular Elements

Linear

Quadratic

Cubic



1
x y

1
x y
x^2 xy y^2

1
x y
x^2 xy y^2
x^3 x^2y xy^2 y^3

Triangular Elements

- The number of nodes (N_{en}) contained in a triangular element is equal to

$$N_{en} = \frac{(p+2)!}{2p!}$$

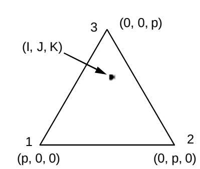
- Thus for the linear element ($p = 1$), $N_{en} = 3!/(2!) = 6/2 = 3$
- For the quadratic element ($p = 2$), $N_{en} = 4!/(2*2!) = 24/4 = 6$
- etc.

Triangular Elements

- The interpolation functions for triangular elements can be derived in a manner quite similar to that used in defining Lagrangian elements.
- For purposes of determining the *order* of the one-dimensional Lagrange interpolating polynomials, the nodes in a triangular element (not necessarily equally spaced) are assigned the indices (I, J, K), where $I + J + K = p$.

Triangular Elements

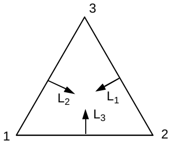
- Each vertex node has one index equal to p, and the other two indices equal to zero.

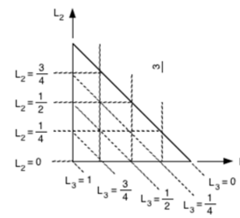


Triangular Elements

- The use of the above notation is restricted only to determining the *order* of the one-dimensional Lagrange polynomials for use in developing the element interpolation functions.
- This notation should not be confused with the (L_1, L_2, L_3) natural ("area") coordinate system introduced in *Appendix* C (Section C.3.2).

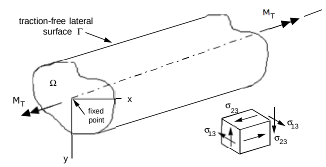
Triangular Elements





NOTE: $L_1 + L_2 + L_3 = 1$

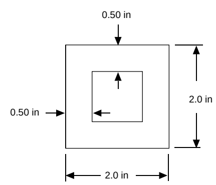
Example: Torsion of Straight, Prismatic Bars

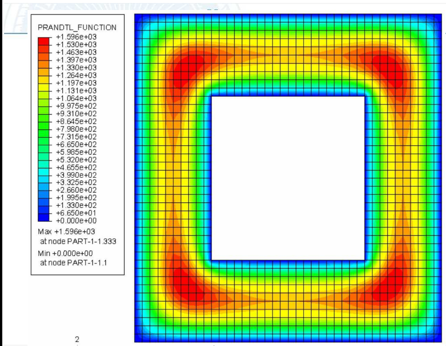


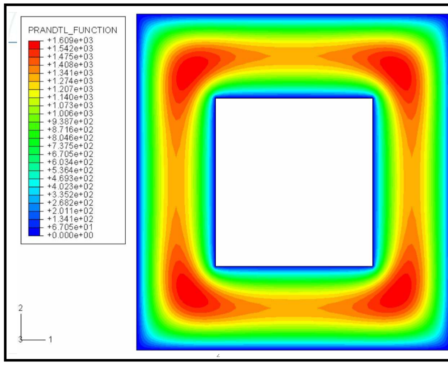
- A *scalar* field problem.
- The primary dependent variable is the *Prandtl stress function* Φ.

Example: Torsion of Straight, Prismatic Bars

Square cross-section







Example: Torsion of Straight, Prismatic Bars

Square cross-section: Conclusions

- Quadrilateral and triangular elements give comparable accuracy for comparable computational effort.
- Linear and quadratic elements give very similar results.

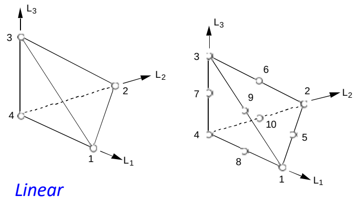
Tetrahedral Elements

Pertinent Reading: *Section 9.4.3* in the textbook.

Tetrahedral Elements

- The three-dimensional counterpart of the triangle is the tetrahedron.
- As with triangles, *complete* interpolating polynomials, in this case in three coordinates, are used for each member of the tetrahedral element family.

Tetrahedral Elements



Linear

Quadratic

Tetrahedral Elements

- Each face of a tetrahedral element contains the *same* number of nodes as found in the corresponding triangular element of the same order.
- As such, in the plane of each tetrahedral face, the order of the 2-d polynomial is the *same* as for the corresponding triangular element.

Tetrahedral Elements

- The process for determining the interpolation functions for triangular and tetrahedral elements is *quite similar*.
- For purposes of determining the *order* of the one-dimensional Lagrange interpolating polynomial associated with an element of order M, the nodes in the element are assigned the indices (I, J, K, L), where

$$I + J + K + L = M.$$

Tetrahedral Elements

- The natural (“volume”) coordinates associated with tetrahedral elements are denoted by L_1, L_2, L_3 and L_4 (see Section C.3.3 of *Appendix* C), where

$$L_1 + L_2 + L_3 + L_4 = 1$$

Tetrahedral Elements

- The interpolation functions for the first two members of the tetrahedral family of elements are derived on pp. 341 and 342 of the textbook.