Element Mapping

Pertinent reading: *Chapter 10* in textbook

Department of Civil and Environmental Engineering

Presented by:

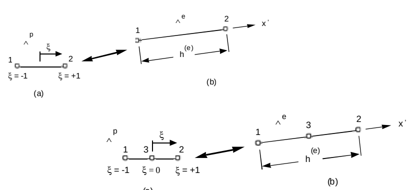
V.N. Kaliakin

University of Delaware

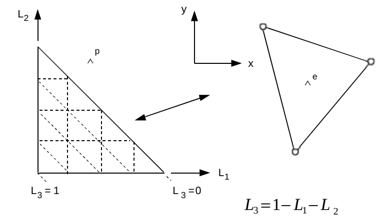
General Idea Behind Element Mapping

- Establish a ***unique***, one-to-one mapping between the "parent" domain $\wedge^p$ and the actual (distorted) element domain $\wedge^e$.

One-Dimensional Elements

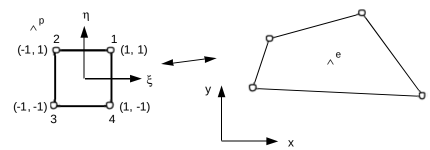


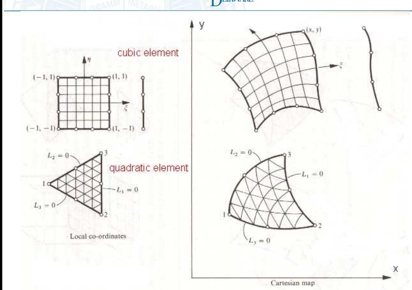
Two-Dimensional Elements



$$L_3 = 1 - L_1 - L_2$$

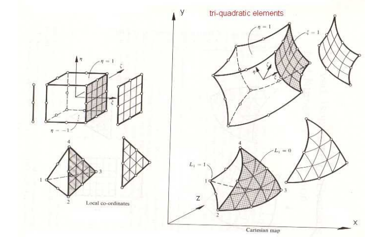
Two-Dimensional Elements





Two-Dimensional Mapping of Some Elements (after Zienkiewicz, 1977)

Three-Dimensional Elements



Three-Dimensional Mapping of Some Elements (after Zienkiewicz, 1977)

General Aspects of Element Mapping

- The mapping from *natural* (ξ, η, ζ) coordinates (in the "parent" domain) to *global* Cartesian coordinates (x, y, z) is expressed by

$$x = f_1(\xi, \eta, \zeta)$$
$$y = f_2(\xi, \eta, \zeta)$$
$$z = f_3(\xi, \eta, \zeta)$$

General Aspects of Element Mapping

- The mapping from *global* Cartesian coordinates (x, y, z) to *natural* coordinates (ξ, η, ζ) is expressed by

$$\xi = f_4(x, y, z)$$
$$\eta = f_5(x, y, z)$$
$$\zeta = f_6(x, y, z)$$

General Aspects of Element Mapping

Recall the *Jacobian Matrix*

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}$$

General Aspects of Element Mapping

The functions f_1 to f_6 will be *single-valued* provided:

1. The first partial derivatives in f_1 to f_6 are *continuous*.
2. The *Jacobian* determinant does not pass through zero (i.e., does *not change sign*) within Ω^e.

Treatment of Derivatives

- Recall that derivatives with respect to the *global* (x, y, z) coordinates are present in the equations associated with the element "properties" (stiffness) matrix $\mathbf{K}^{(e)}$.
- These derivatives must now be related to ones with respect to the *natural* coordinates (ξ, η, ζ).

Treatment of Derivatives

- The following approach is most expedient:

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \xi}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \zeta}$$

Treatment of Derivatives

- The relation between derivatives in the global and natural coordinates thus requires *inversion* of the Jacobian Matrix.

Treatment of Integrals

- Recall that integrals with respect to the global (x, y, z) coordinates are present in the equations associated with the element "properties" (stiffness) matrix $\mathbf{K}^{(e)}$ and the right hand side "forcing" vector $\mathbf{q}^{(e)}$.

$$\mathbf{K}^{(e)} = \int_{\Omega} \mathbf{B}^T \mathbf{D}\mathbf{B}\, d\Omega \; ; \; \mathbf{q}^{(e)} = \int_{\Omega} \mathbf{N}^T f^{(e)}\, d\Omega$$

Treatment of Integrals

- When elements have curved boundaries, the integrals associated with the element matrices are most easily evaluated using suitable parent domains Ω^p in natural coordinate space.
- There is thus no need to deal with equations for the curved boundaries.

Treatment of Integrals

- This evaluation of integrals is, however, not without its limitations.
- As the order of the element increases, so does the complexity of terms involved in the integration.
- The mapping used to distort the shape of Ω^p involves *inversion* of the Jacobian matrix.

Treatment of Integrals

- The complexity of the associated integrals makes their exact evaluation *impractical*.
- In such cases the integration must be approximated using *numerical integration* or *quadrature*.
- Additional details pertaining to numerical integration are given in *Appendix F* of the textbook.

Treatment of Integrals

In general

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i) w_i$$

Known quadrature location

Known quadrature weight

n = number of quadrature points

Treatment of Integrals

- For FE applications the most efficient approach to numerical integration of *quadrilaterals* & *hexahedra* typically involves the use of Gaussian quadrature.

Carl Friedrich Gauss (1777-1855)

Treatment of Integrals

- The idea behind Gaussian quadrature is to choose the quadrature points x_i and the weighting coefficients w_i in an *optimal* manner.
- Gaussian quadrature formulae use polynomials *orthogonal* to the polynomial being integrated.
- For certain special intervals $[a, b]$ the orthogonal polynomials have names such as *Chebyshev*, *Hermite*, *Jacobi*, *Laguerre* and *Legendre*.

Treatment of Integrals

- Gaussian formulae with n points in a given coordinate direction achieve a *degree of precision* of $(2n - 1)$.
- That is, a complete polynomial of degree $(2n - 1)$ will be integrated *exactly*.

Treatment of Integrals

- For *Newton-Cotes* formulae (e.g., Simpson's rule, etc.), for n odd, the highest degree of precision is $(n + 1)$.

Treatment of Integrals

- For FE applications, the *Legendre* polynomials represent the most commonly used set of orthogonal polynomials.

Adrien-Marie Legendre (1752-1833).

Treatment of Integrals

- Some values of the quadrature points x_i and the weighting coefficients w_i associated with *Gauss-Legendre* quadrature are listed in Table F.1 in *Appendix* F of the textbook.

Treatment of Integrals

- In the case of *distorted triangles* and *tetrahedra*, numerical integration must be used.
- The derivation of an integration formula of a given degree of precision again involves the determination of weighting coefficients w_i, and the number and location of the corresponding quadrature points x_i in the triangular or tetrahedral "parent" domain.

Treatment of Integrals

- In theory, Gauss-Legendre quadrature could be extended to triangles by treating them as quadrilaterals with one degenerate edge of zero length.
- However, such formulae lack symmetry with respect to the directions of the edges.
- Consequently, it is advantageous to use *specially developed*, *symmetric* formulae (see Tables F.4 and F.5 in *Appendix* F of the textbook).

Parametric Mapping

- It remains to **explicitly** define the mapping described analytically by

$$x = f_1(\xi,\eta,\zeta)$$
$$y = f_2(\xi,\eta,\zeta)$$
$$z = f_3(\xi,\eta,\zeta)$$

Parametric Mapping

- A very common and convenient form of element mapping is of the *parametric kind*.
- In parametric mapping the relationship between the natural coordinates (ξ, η, ζ) and the global coordinates (x, y, z) is written employing the *same* type of element interpolation functions as used to approximate the primary dependent variables over a given element; that is,

$$\hat{\phi}^{(e)} = \sum_{m=1}^{N_{dof}} \hat{\phi}_m^{(e)} N_m(\xi,\eta,\zeta)$$

Parametric Mapping

- If $G_m(\xi, \eta, \zeta)$ represents a suitable interpolation function for the element geometry, then the mapping relationship for each element is written as

$$x = \sum_{m=1}^{N_{pt}} x_m^{(e)} G_m(\xi,\eta,\zeta)$$
$$y = \sum_{m=1}^{N_{pt}} y_m^{(e)} G_m(\xi,\eta,\zeta)$$
$$z = \sum_{m=1}^{N_{pt}} z_m^{(e)} G_m(\xi,\eta,\zeta)$$

Parametric Mapping

$$x = \sum_{m=1}^{N_{pt}} x_m^{(e)} G_m(\xi,\eta,\zeta), \quad y = \sum_{m=1}^{N_{pt}} y_m^{(e)} G_m(\xi,\eta,\zeta), \quad z = \sum_{m=1}^{N_{pt}} z_m^{(e)} G_m(\xi,\eta,\zeta)$$

where

$\left(x_m^{(e)}, y_m^{(e)}, z_m^{(e)}\right)$ are the global (x, y, z) coordinates of the point into which point m in (ξ, η, ζ) natural coordinates shall be mapped, and

N_{pt} = the number of points in the element used to define its geometry.

Parametric Mapping

- In general, the local (element) node numbers 1 to N_{pt} will be mapped to some different global numbers that are assigned to the entire mesh.

Parametric Mapping

- For a *parametric* mapping, the general expressions for the derivatives of x with respect to the natural coordinates are

$$\frac{\partial x}{\partial \xi} = \sum_{m=1}^{N_{pt}} x_m^{(e)} \frac{\partial G_m}{\partial \xi}$$
$$\frac{\partial x}{\partial \eta} = \sum_{m=1}^{N_{pt}} x_m^{(e)} \frac{\partial G_m}{\partial \eta}$$
$$\frac{\partial x}{\partial \zeta} = \sum_{m=1}^{N_{pt}} x_m^{(e)} \frac{\partial G_m}{\partial \zeta}$$

Parametric Mapping

- The function $G_m(\xi, \eta, \zeta)$ must also be chosen so that the following condition is satisfied:

$$x(\xi_i,\eta_i,\zeta_i) = x_i^{(e)}$$
$$y(\xi_i,\eta_i,\zeta_i) = y_i^{(e)}$$
$$z(\xi_i,\eta_i,\zeta_i) = z_i^{(e)}$$

Parametric Mapping

- This imposes the following restriction of the geometric interpolation functions:

$$G_m(\xi_i,\eta_i,\zeta_i) = \delta_{mi}$$

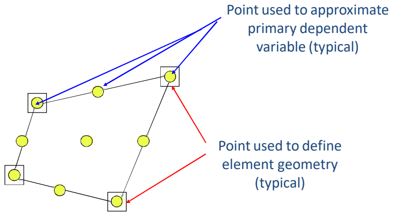
Parametric Mapping

- N_{pt} does *not* necessarily equal N_{en}, which is the number of nodes in an element that are used in the approximation of primary dependent variables (and associated with the interpolation functions N_m).
- In light of the observations regarding the functions G_m and N_m, the following ***three*** types of parametric mappings are possible:

Parametric Mapping

- If *fewer* nodes are used to define the element geometry than are used to approximate the primary dependent variables, then $N_{pt} < N_{en}$.
- The interpolation functions G_m are *different* from the functions N_m.
- Such an element is said to be *subparametric*.

Parametric Mapping



Parametric Mapping

- If the number of nodes used to define the element geometry is *greater* than the number of nodes used to approximate the primary dependent variables, then $N_{pt} > N_{en}$.
- The interpolation functions G_m are again *different* from the functions N_m.
- Such an element is said to be *superparametric*.

Parametric Mapping

- Finally, if the same nodes define the element geometry and the interpolation of the primary dependent variables, then $N_{pt} = N_{en}$.
- The functions G_m and N_m are *identical*.
- In this case, the element is said to be *isoparametric*.

Element Distortions

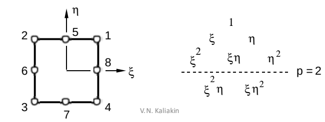
- When "parent" domains are parametrically mapped into distorted element configurations, care must be taken to avoid *non-proper mappings*.
- That is, ones in which a one-to-one relationship between natural and global coordinates ceases to exist.

Element Distortions

- Mathematically, non-proper mappings are characterized by a *change in sign* of the Jacobian determinant.
- Physically, such mappings occur when an element is *excessively distorted* or folds back upon itself.

Element Distortions

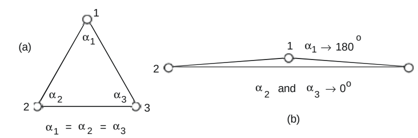
- Desire to quantify the level of distortion present in an element (if possible).
- NOTE: always specify node numbers in the *same order* as defined in the "parent" domain.



Element Distortions

For *linear triangular* elements:

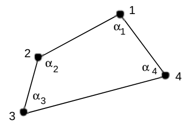
$$\det \mathbf{J} = 2A^{(e)}$$



Element Distortions

- For parametric mappings of *quadrilateral* elements based on *bi-linear* interpolation functions, if the Jacobian determinant is positive at the four corners, then it is positive throughout the element interior.
- Alternately, the necessary condition for a one-to-one mapping is that no interior vertex angle in the distorted element exceed 180 degrees.

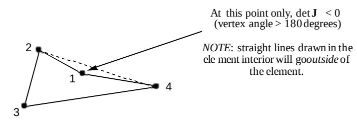
Element Distortions



Remark: in the opinion of Burnett (1988), the interior vertex angles should stay within 20 or 30 degrees of a right angle; otherwise a triangular element would be preferred.

Element Distortions

- If an interior angle exceeds 180 degrees, then the Jacobian determinant *changes sign locally*.

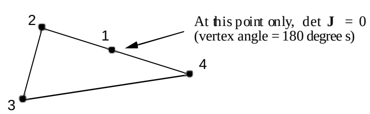


Element Distortions

- For quadratic triangles with either straight or curved sides, the interior angle at each vertex should not equal 0 or 180 degrees.

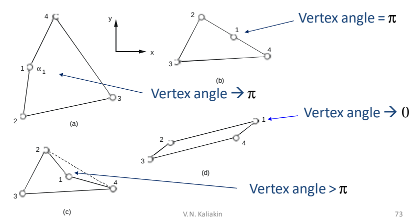
Element Distortions

- If one of these angles equals 180 degrees, then the Jacobian determinant is equal to zero at the associated node.



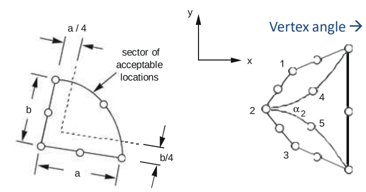
Element Distortions

Thus, for *bi-linear quadrilateral* elements:



Element Distortions

For *quadratic triangular* elements:



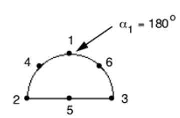
(a) (b)

Element Distortions

- Even if an interior angle is less than, but close to 180 degrees, the element could be *numerically ill-conditioned*.
- That is, it will facilitate the loss of significant digits during computation, and will likely lead to *sporatic* and *unpredictable* results.

Element Distortions

- To avoid a change of sign in the Jacobian determinant for *quadratic triangles* with straight sides, the edge nodes must remain in the *middle quartiles* of the appropriate edge.



Element Distortions