Interpolation Functions for Lagrangian Elements

Pertinent reading: *Section 9.2* in textbook

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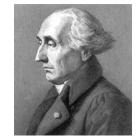
Lagrangian Elements

- Elements comprising the so-called Lagrangian family are systematically derived with the aid of *one-dimensional polynomials* of the *Lagrange form*.

Lagrangian Elements

- Although they carry they name of *Joseph Louis Lagrange*, such polynomials were actually discovered by *Edward Waring* in 1779 and later rediscovered by *Leonard Euler* in 1783.



Lagrangian Elements

- One-dimensional polynomial in the Lagrange form:

$$\Lambda_i^m(\xi)=\frac{\prod\limits_{q=1,q\neq i}^{m+1}\left(\xi-\xi_q\right)}{\prod\limits_{q=1,q\neq i}^{m+1}\left(\xi_i-\xi_q\right)}$$

m = denotes the order of the polynomial

i = represents the local (i.e., element) node number

ξ = natural coordinate

$$=\frac{(\xi-\xi_1)(\xi-\xi_2)\,!\,(\xi-\xi_{i-1})(\xi-\xi_{i+1})\,!\,(\xi-\xi_{m+1})}{(\xi_i-\xi_1)(\xi_i-\xi_2)!\;(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})!\;(\xi_i-\xi_{m+1})}$$

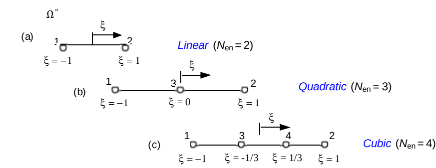
Lagrangian Elements

- From the previous equation it is evident that polynomials of the Lagrange form satisfy the basic "Kronecker delta" requirement; that is,

$$\Lambda_i^m(\xi_j)=\delta_{ij}$$

One-Dimensional Lagrangian Elements

- Family of one-dimensional Lagrangian elements ("parent" domains shown).



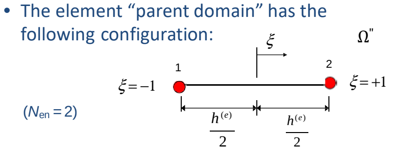
One-Dimensional Lagrangian Elements

- For a general one-dimensional Lagrangian element containing N_{en} nodes, the interpolation function associated with node i will be the Lagrange polynomial of degree (N_{en} - 1) that takes on the value of one at node i and the value of zero at the remaining nodes.
- This is written as

$$N_i=\Lambda_i^{(N_{en}-1)}(\xi)\qquad ;\quad i=1,2,!,N_{en}$$

Linear One-Dimensional Lagrangian Element

- The interpolation functions for the *linear* (two-node) *Lagrangian* element are next derived.
- The element "parent domain" has the following configuration:



Linear One-Dimensional Lagrangian Element

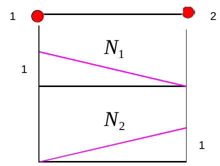
- For node 1:

$$N_1=\Lambda_1^{(1)}(\xi)=\frac{\xi-\xi_2}{\xi_1-\xi_2}=\frac{\xi-1}{-1-1}=\frac{1}{2}(1-\xi)$$

- For node 2:

$$N_2=\Lambda_2^{(1)}(\xi)=\frac{\xi-\xi_1}{\xi_2-\xi_1}=\frac{\xi-(-1)}{1-(-1)}=\frac{1}{2}(1+\xi)$$

Linear One-Dimensional Lagrangian Element

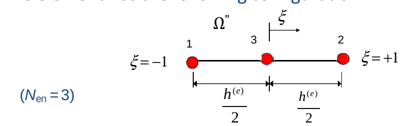


Linear One-Dimensional Lagrangian Element

- The interpolation functions for the linear element are rather easily derived using the "matrix inversion" approach (recall that this was done in conjunction with the discussion of Chapter 7).
- The real benefit of the Lagragian approach lies in the ease with which interpolation functions for *higher-order* elements are derived.

Quadratic One-Dimensional Lagrangian Element

- The interpolation functions for the *quadratic* (three-node) Lagrangian element are thus next derived.
- The element has the following configuration:



Quadratic One-Dimensional Lagrangian Element

- For node 1:

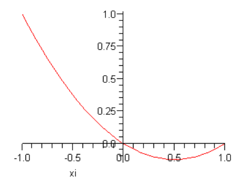
$$N_1 = \Lambda_1^{(2)}(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{(\xi-1)(\xi-0)}{(-1-1)(-1-0)} = \frac{1}{2}\xi(\xi-1)$$

Check the "Kronecker delta" property:

$$N_1(-1) = \frac{1}{2}(-1)(-1-1) = 1 \quad ; \quad N_1(1) = \frac{1}{2}(1)(1-1) = 0$$

$$N_1(0) = \frac{1}{2}(0)(0-1) = 0$$

Quadratic One-Dimensional Lagrangian Element



Quadratic One-Dimensional Lagrangian Element

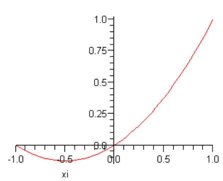
- For node 2:

$$N_2 = \Lambda_2^{(2)}(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} = \frac{1}{2}\xi(\xi+1)$$

Check the "Kronecker delta" property:

$$N_\$(-1) = \frac{(}{\$}(-1)(-1+1) = 0 \, ; N_\$(1) = \frac{(}{\$}(1)(1+1) = 1; N_\$(0) = \frac{(}{\$}(0)(1+0) = 0$$

Quadratic One-Dimensional Lagrangian Element



Quadratic One-Dimensional Lagrangian Element

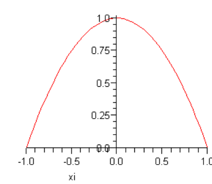
- For node 3:

$$N_+ = \Lambda_+^{(\$)}(\xi) = \frac{(\xi-\xi_()(\xi-\xi_\$)}{(\xi_+-\xi_()(\xi_+-\xi_\$)} = \frac{(\xi+1)(\xi-1)}{(0+\xi_()(0-1)} = (1-\xi^\$)$$

Check the "Kronecker delta" property:

$$N_+(-1) = (1-(-1)^\$) = 0 \, ; N_+(1) = (1-(1)^\$) = 0 \, ; N_+(0) = (1-(0)^\$) = 1$$

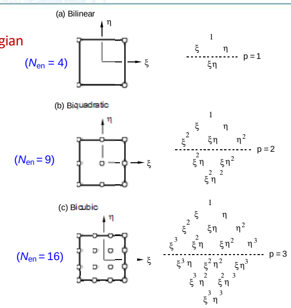
Quadratic One-Dimensional Lagrangian Element



Two-Dimensional Lagrangian Elements

- In general, two-dimensional Lagrangian elements are *quadrilateral* in shape.
- The first three members of the two-dimensional *Lagrangian element family*, along with their respective *Pascal triangles*, are shown in the following figure.

Family of Two-Dimensional Lagrangian Elements



Two-Dimensional Lagrangian Elements

- The present development is limited to the two-dimensional *parent domain* (a bi-unit square) $\wedge^p$: $\xi, \eta \in [-1, 1]$.
- As shown in Chapter 10, this domain is easily mapped to actual *quadrilateral* element geometries, possibly with curved edges.
- As such, the interpolation functions derived herein apply to *any* two-dimensional element possessing the requisite number of nodes.

Two-Dimensional Lagrangian Elements

- The interpolation functions for such elements are derived by forming *products of two one-dimensional polynomials of the Lagrange form* in the following manner:

$$N_i(\xi,\eta) = \Lambda_j^{(N_{en\xi}-1)}(\xi) * \Lambda_k^{(N_{en\eta}-1)}(\eta)$$

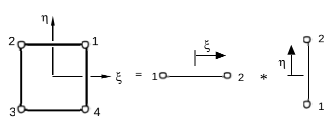
$$N_i(\xi,\eta) = \Lambda_j^{(N_{en\xi}-1)}(\xi) * \Lambda_k^{(N_{en\eta}-1)}(\eta)$$

- i denotes the node number in the parent element domain $\wedge^p$, (I = 1, 2, ……, N_e).
- j denotes the node number for a *one-dimensional* interpolation function parallel to the ξ-axis,
- k is the node number for a *one-dimensional* interpolation function parallel to the η-axis.
- $N_{en\xi}$ and $N_{en\eta}$ are the number of element nodes in the ξ- and η-directions, respectively.
- $N_{en\xi}$ and $N_{en\eta}$ are *not* necessarily equal.

Two-Dimensional Lagrangian Elements

- The relation between i, j and k is *element specific*.
- We now present some details pertaining to the development of interpolation functions for two-dimensional Lagrangian elements.

Bi-Linear Lagrangian Element



- Since $N_{en\xi} = N_{en\eta} = 2$, the element interpolation functions are determined from the following expression:

$$N_i(\xi,\eta) = \Lambda_j^{(1)}(\xi) * \Lambda_k^{(1)}(\eta)$$

Bi-Linear Lagrangian Element

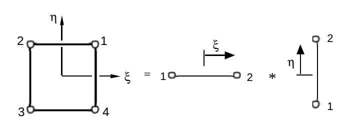
- The relationship between the indices

i (the node number in the *parent* element),

j (the index associated with the ξ-axis in a one-dimensional element), and

k (the index associated with the η-axis in a one-dimensional element) is

Bi-Linear Lagrangian Element



i	j	k
1	2	2
2	1	2
3	1	1
4	2	-1

Bi-Linear Lagrangian Element

Recall: $N_i(\xi,\eta)=\Lambda_j^{(1)}(\xi)*\Lambda_k^{(1)}(\eta)$

- The element interpolation functions are thus

$$N_1=\Lambda_2^{(1)}(\xi)*\Lambda_2^{(1)}(\eta)=\frac{(\xi-\xi_1)}{(\xi_2-\xi_1)}\frac{(\eta-\eta_1)}{(\eta_2-\eta_1)}$$
$$=\frac{(\xi+1)}{(1+1)}\frac{(\eta+1)}{(1+1)}=\frac{1}{4}(1+\xi)(1+\eta)$$

Bi-Linear Lagrangian Element

$$N_2=\Lambda_1^{(1)}(\xi)*\Lambda_2^{(1)}(\eta)=\frac{(\xi-\xi_2)}{(\xi_1-\xi_2)}\frac{(\eta-\eta_1)}{(\eta_2-\eta_1)}$$
$$=\frac{(\xi-1)}{(-1-1)}\frac{(\eta+1)}{(1+1)}=\frac{1}{4}(1-\xi)(1+\eta)$$

Bi-Linear Lagrangian Element

$$N_3=\Lambda_1^{(1)}(\xi)*\Lambda_1^{(1)}(\eta)=\frac{(\xi-\xi_2)}{(\xi_1-\xi_2)}\frac{(\eta-\eta_2)}{(\eta_1-\eta_2)}$$
$$=\frac{(\xi-1)}{(-1-1)}\frac{(\eta-1)}{(-1-1)}=\frac{1}{4}(1-\xi)(1-\eta)$$

Bi-Linear Lagrangian Element

$$N_4=\Lambda_2^{(1)}(\xi)*\Lambda_1^{(1)}(\eta)=\frac{(\xi-\xi_1)}{(\xi_2-\xi_1)}\frac{(\eta-\eta_2)}{(\eta_1-\eta_2)}$$
$$=\frac{(\xi+1)}{(1+1)}\frac{(\eta-1)}{(-1-1)}=\frac{1}{4}(1+\xi)(1-\eta)$$

Bi-Linear Lagrangian Element

Check the "Kronecker delta" property

(NOTE: must use natural coordinates in the "parent" domain $\wedge^{p}$).

$$N_1(1,1)=\frac{1}{4}(1+1)(1+1)=\frac{1}{4}(2)(2)=1$$
$$N_1(-1,1)=\frac{1}{4}(1-1)(1+1)=0$$
$$N_1(-1,-1)=\frac{1}{4}(1-1)(1-1)=0$$
$$N_1(1,-1)=\frac{1}{4}(1+1)(1-1)=0$$

etc.

Bi-Linear Lagrangian Element

- The second check is associated with the completeness criterion; viz.,

$$\sum_{i=1}^{4}N_i=\frac{1}{4}(1+\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1-\eta)$$
$$=\frac{1}{2}(1+\eta)+\frac{1}{2}(1-\eta)=1$$

Bi-Linear Lagrangian Element

- In closing, the interpolation functions associated with the bi-linear Lagrangian element can be written in the following compact form:

$$N_m=\frac{1}{4}(1+\xi_m\xi)(1+\eta_m\eta)$$

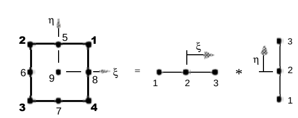
Bi-Linear Lagrangian Element

- The derivatives of the interpolation functions with respect to the *natural coordinates* follow directly; viz.,

$$\frac{\partial N_m}{\partial\xi}=\frac{1}{4}\xi_m(1+\eta_m\eta)$$
$$\frac{\partial N_m}{\partial\eta}=\frac{1}{4}\eta_m(1+\xi_m\xi)$$

Bi-Quadratic Lagrangian Element

- The schematic relationship between the bi-quadratic, two-dimensional interpolation functions and two one-dimensional quadratic interpolation functions is shown below.



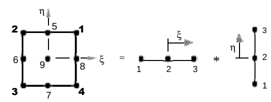
Bi-Quadratic Lagrangian Element

- The interpolation functions are determined from the expression

$$N_i(\xi,\eta) = \Lambda_j^2(\xi) \bullet \Lambda_k^2(\eta)$$

where the relationship between the indices *i*, *j*, and *k* is given in the following table.

Bi-Quadratic Lagrangian Element



i	*j*	*k*
1	3	3
2	1	3
3	1	1
4	3	1
5	2	3
6	1	2
7	2	1
8	3	2
9	2	2

Bi-Quadratic Lagrangian Element

- For node 1:

$$N_1 = \Lambda_3^2(\xi) * \Lambda_3^2(\eta) = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} \frac{(\eta-\eta_1)(\eta-\eta_2)}{(\eta_3-\eta_1)(\eta_3-\eta_2)}$$

$$= \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} \frac{(\eta+1_1)(\eta-0)}{(1+1)(1-0)}$$

$$= \frac{1}{4}\xi\eta(1+\xi)(1+\eta)$$

For nodes 2 to 9, see pp. 316 to 317 in textbook.