Nodal Specifications

Pertinent reading: Section 8.4 in textbook

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General Steps in Performing Finite Element Analyses

- Discretization of the solution domain.
- Assembly of *element* equations to form the *global* equations.
- Application of *nodal specifications*.
- Solution of the *global* equations.
- Calculation of the *secondary dependent variables* (e.g., "gradients").
- Post-processing of the results.
- Interpretation of the results.

Contents of Presentation

- General Remarks.
- Elimination Approach at the Global Level.
- Elimination Approach at the Element Level.
- Penalty Approach at the Global Level.
- Penalty Approach at the Element Level.

General Remarks

- Since it has *not* been modified to account for constraints imposed at boundary and possibly interior nodes, the system property matrix **K** is *singular* at this stage of the analysis.
- In order to specialize the problem, the *nodal constraints* must next be considered.

General Remarks

- In finite element analyses, constraints on the *nodal* values of the primary dependent variables or their derivatives are required in order to account for, or approximate, the actual constraints imposed on the body being analyzed.
- It is only through such constraints that a problem can be *uniquely* posed.

General Remarks

- Since the values of the primary dependent variables, or their derivatives, can be specified at *any node* in the mesh, the previous notion of "boundary conditions" must be generalized to *nodal specifications*.
- Along the total boundary Γ of a body, at nodes along element boundaries, or at any node within an element, *two* types of nodal specifications are typically possible, namely:

General Remarks

- Known values of the primary dependent variables are specified along the portion of Γ (or at an interior node) denoted by Γ_1. These are referred to as *essential* or *Dirichlet* specifications.
- Known values of the gradients of the primary dependent variables are specified along the portion of Γ (or at an interior node) denoted by Γ_2. These are referred to as *natural* or *Neumann* specifications.

General Remarks

- With regard to a *specific* primary dependent variable at a *specific* node in a body, *either* an essential constraint or a natural constraint, *but not both* can be specified.
- Stated mathematically,

$$\Gamma_1 \cup \Gamma_2 = \Gamma, \qquad \Gamma_1 \cap \Gamma_2 = \varnothing$$

Specific Example

- To present the techniques used to specify nodal constraints, consider a hypothetical mesh with *four* degrees of freedom.
- The associated global equations are thus

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$

Specific Example

- These global equations are *symmetric*; in the process of imposing the nodal constraints we do not want to destroy this symmetry.
- We consider two general approaches:

1. *Elimination*.
2. *Penalty*.

- Each of these approaches can be applied at *either* the *global* or *element* level.

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Elimination Approach at the Global Level

- If a known nodal value of the *secondary dependent variable* is specified in degree of freedom *m*,
- Then this "load" is simply *summed* into the *m* th row of the global right hand side vector **q** (recall the $\mathbf{N}^T \hat{q}\,|_{\Gamma}$ term).

Elimination Approach at the Global Level

- If, however, a known nodal value of the *primary dependent variable* is specified in degree of freedom *m*, then the global coefficient matrix **K** must be suitable modified.
- For the present discussion, assume that

$$\hat{\phi}_2 = \bar{\phi}$$

i.e., $m = 2$.

Elimination Approach at the Global Level

- It follows that the global equations should be re-written, placing known quantities on the *right hand side* of the respective equations. That is,

$$\begin{aligned}
k_{11}\hat{\phi}_1 + k_{13}\hat{\phi}_3 + k_{14}\hat{\phi}_4 &= f_1 - k_{12}\bar{\phi}\\
k_{21}\hat{\phi}_1 + k_{23}\hat{\phi}_3 + k_{24}\hat{\phi}_4 &= f_2 - k_{22}\bar{\phi}\\
k_{31}\hat{\phi}_1 + k_{33}\hat{\phi}_3 + k_{34}\hat{\phi}_4 &= f_3 - k_{32}\bar{\phi}\\
k_{41}\hat{\phi}_1 + k_{43}\hat{\phi}_3 + k_{44}\hat{\phi}_4 &= f_4 - k_{42}\bar{\phi}
\end{aligned}$$

Elimination Approach at the Global Level

- In vector-matrix form:

$$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{14}\\ k_{21} & 0 & k_{23} & k_{24}\\ k_{31} & 0 & k_{33} & k_{34}\\ k_{41} & 0 & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1\\ \hat{\phi}_2\\ \hat{\phi}_3\\ \hat{\phi}_4 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}\bar{\phi}\\ f_2 - k_{22}\bar{\phi}\\ f_3 - k_{32}\bar{\phi}\\ f_4 - k_{42}\bar{\phi} \end{Bmatrix}$$

Elimination Approach at the Global Level

- Since the second (*m* th) equation is not necessary for determining

$$\hat{\phi}_1,\ \hat{\phi}_3,\ \hat{\phi}_4$$

we *restore symmetry* by zeroing out the second (*m* th) equation.

Elimination Approach at the Global Level

$$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{14}\\ 0 & 0 & 0 & 0\\ k_{31} & 0 & k_{33} & k_{34}\\ k_{41} & 0 & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1\\ \hat{\phi}_2\\ \hat{\phi}_3\\ \hat{\phi}_4 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}\bar{\phi}\\ 0\\ f_3 - k_{32}\bar{\phi}\\ f_4 - k_{42}\bar{\phi} \end{Bmatrix}$$

Elimination Approach at the Global Level

- Finally, we *force* the second (*m* th) equation to return the trivial result of $\hat{\phi}_2 = \bar{\phi}$
- This involves setting the *m* th diagonal entry in **K** equal to unity (1.0) and the *m* th right hand side entry equal to $\bar{\phi}$

Elimination Approach at the Global Level

- The resulting equations are thus

$$\begin{bmatrix} k_{11} & 0 & k_{13} & k_{14} \\ 0 & 1.0 & 0 & 0 \\ k_{31} & 0 & k_{33} & k_{34} \\ k_{41} & 0 & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}\bar{\phi} \\ \bar{\phi} \\ f_3 - k_{32}\bar{\phi} \\ f_4 - k_{42}\bar{\phi} \end{Bmatrix}$$

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Elimination Approach at the Element Level

- In this approach the nodal specifications are applied to the element arrays *immediately prior to assembly*.
- Trivial equations associated with degrees of freedom in which primary dependent variables are specified are suitably "marked" - in the solution of the system equations these trivial equations are skipped during the reduction process.
- See Section 8.4.2 for additional details.

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Penalty Approach at the Global Level

- If a known nodal value of the *secondary dependent variable* is specified in degree of freedom *m*,
- Then this "load" is simply *summed* into the *m* th row of the global right hand side vector **q** (recall the $\mathbf{N}''q\$|_{\&}$ term).

Penalty Approach at the Global Level

- If, however, a known nodal value of the *primary dependent variable* is specified in degree of freedom *m*, then the global coefficient matrix **K** must be suitable modified.
- For the present discussion, assume that

$$\hat{\phi}_2 = \bar{\phi}$$

i.e., $m = 2$.

Penalty Approach at the Global Level

- Replace K_{22} by the product GK_{22}, where G is a large positive "*penalty*" number that is known *a priori*.
- In addition, replace q_2 by the product of GK_{22} and the known value of ϕ. Thus,

Penalty Approach at the Global Level

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & Gk_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ Gk_{22}\bar{\phi} \\ f_3 \\ f_4 \end{Bmatrix}$$

Penalty Approach at the Global Level

- Note that all equations except the second, as well as the symmetry of the system equations, are *unaffected*.
- In addition, no computational effort is expended to zero out rows and columns of **K**.

Penalty Approach at the Global Level

- The second (*m* th) equation is now

$$k_{21}\phi_1 + Gk_{22}\phi_2 + k_{23}\phi_3 + k_{24}\phi_4 = Gk_{22}\bar{\phi}$$

Penalty Approach at the Global Level

- Since the magnitude of G has been chosen to be *large*, and since the elements of **K** are typically of the similar order, it follows that the term GK_{22} will *dominate* the left hand side of the previous equation.
- This equation thus effectively reduces to

$$\phi_1 + G\phi_2 + \phi_3 + \phi_4 \approx G\bar{\phi}$$

Penalty Approach at the Global Level

- This equation thus effectively reduces to

$$\phi_1 + G\phi_2 + \phi_3 + \phi_4 \approx G\bar{\phi}$$

- Implying that

$$G\phi_2 \approx G\bar{\phi} \quad \Rightarrow \quad \phi_2 \approx \bar{\phi}$$

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Penalty Approach at the Element Level

- The penalty approach can likewise be implemented at the ***element*** level.
- The requisite logic is actually *simpler* than in the case of the elimination approach.
- See Section 8.4.4 for additional details.