Solution for Exercise 9.5

Since $N_{en\xi} = 3$ and $N_{en\eta} = 2$, the element interpolation functions are determined from the following expression:

$$N_i(\xi,\eta) = \Lambda_j^{(2)}(\xi) * \Lambda_k^{(1)}(\eta) \tag{1}$$

where the relationship between the indices i, j, and k is given in Table 1.

Thus

$$N_1 = \Lambda_3^2 * \Lambda_2^1 = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} * \frac{(\eta-\eta_1)}{(\eta_2-\eta_1)} = \frac{1}{4}\xi(1+\xi)(1+\eta) \tag{2}$$

$$N_2 = \Lambda_1^2 * \Lambda_2^1 = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} * \frac{(\eta-\eta_1)}{(\eta_2-\eta_1)} = \frac{1}{4}\xi(\xi-1)(1+\eta) \tag{3}$$

$$N_3 = \Lambda_1^2 * \Lambda_1^1 = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} * \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} = \frac{1}{4}\xi(1-\xi)(\eta-1) \tag{4}$$

$$N_4 = \Lambda_3^2 * \Lambda_1^1 = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} * \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} = \frac{1}{4}\xi(1+\xi)(1-\eta) \tag{5}$$

$$N_5 = \Lambda_2^2 * \Lambda_2^1 = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} * \frac{(\eta-\eta_1)}{(\eta_2-\eta_1)} = \frac{1}{2}(1-\xi^2)(1+\eta) \tag{6}$$

$$N_6 = \Lambda_2^2 * \Lambda_1^1 = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} * \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} = \frac{1}{2}(1-\xi^2)(1-\eta) \tag{7}$$

Careful inspection of N_1 to N_6 reveals that they satisfy Equation (9.1) in the textbook; viz.,

$$N_i\left(\xi_j,\eta_j\right) = \delta_{ij} \tag{8}$$

To begin the check of completeness, begin with the interpolation functions themselves; viz.,

$$\begin{aligned}
\sum_{i=1}^{6} N_i &= \frac{(1+\eta)}{4}\left[\xi(1+\xi)+\xi(\xi-1)+2(1-\xi^2)\right] + \frac{(1-\eta)}{4}\left[\xi(1-\xi)+\xi(1+\xi)+2(1-\xi^2)\right] \\
&= \frac{1}{2}(1+\eta) + \frac{1}{2}(1-\eta) \\
&= \frac{1}{2}(1+\eta+1-\eta) = 1
\end{aligned} \tag{9}$$

Thus the "rigid body mode" corresponding to a constant state of $\hat{\phi}$ shall be exactly represented by the element.

We next consider constant states of the first derivative of $\hat{\phi}(\xi,\eta)$. For the special case of constant state of $\hat{\phi}$, we must show that

$$\sum_{i=1}^{6}\frac{\partial N_i}{\partial \xi} = 0 \quad \text{and} \quad \sum_{i=1}^{6}\frac{\partial N_i}{\partial \eta} = 0 \tag{10}$$

The derivatives of the interpolation functions with respect to the local coordinates are easily shown to be

Table 1: Relation Between Indices for Six-Node Lagrangian Transition Element

i	j	k
1	3	2
2	1	2
3	1	1
4	3	1
5	2	2
6	2	1

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(1+2\xi)(1+\eta) \quad ; \quad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}\xi(1+\xi) \tag{11}$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(2\xi-1)(1+\eta) \quad ; \quad \frac{\partial N_2}{\partial \eta} = \frac{1}{4}\xi(\xi-1) \tag{12}$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1-2\xi)(\eta-1) \quad ; \quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}\xi(1-\xi) \tag{13}$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(1+2\xi)(1-\eta) \quad ; \quad \frac{\partial N_4}{\partial \eta} = -\frac{1}{4}\xi(1+\xi) \tag{14}$$

$$\frac{\partial N_5}{\partial \xi} = -\xi(1+\eta) \quad ; \quad \frac{\partial N_5}{\partial \eta} = \frac{1}{2}(1-\xi^2) \tag{15}$$

$$\frac{\partial N_6}{\partial \xi} = -\xi(1-\eta) \quad ; \quad \frac{\partial N_6}{\partial \eta} = -\frac{1}{2}(1-\xi^2) \tag{16}$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{6}\frac{\partial N_i}{\partial \xi} &= \frac{1}{4}(1+2\xi)*2 + \frac{1}{4}(2\xi-1)*2 + (-2)\xi \\
&= \frac{1}{2} + \xi + \xi - \frac{1}{2} - 2\xi = 0
\end{aligned} \tag{17}$$

with $\sum_{i=1}^{6}\frac{\partial N_i}{\partial \eta} = 0$ evident by inspection.

We next must show that for $\hat{\phi}_i = e\xi_i$, where $i = 1, \cdots 6$ and e is a constant, that $\hat{\phi} = e\xi$. Similarly, for $\hat{\phi}_i = g\eta_i$, we must show that $\hat{\phi} = g\eta$, where g is a constant. From Figure 9.29 it follows that the nodal values ξ_i and η_i are as given in Table 2.

Thus,

Table 2: Nodal Values of ξ and η in the Parent Element Domain

i	ξ_i	η_i
1	1	1
2	-1	1
3	-1	-1
4	1	-1
5	0	1
6	0	-1

$$\begin{aligned}
\hat{\phi} &= \sum_{i=1}^{6} N_i\hat{\phi}_i^{(e)} \\
&= e\sum_{i=1}^{6} N_i\hat{\xi}_i \\
&= e\left[\frac{1}{4}\xi(1+\xi)(1+\eta)*(1) + \frac{1}{4}\xi(\xi-1)(1+\eta)*(-1) + \frac{1}{4}\xi(1-\xi)(\eta-1)*(-1)\right] \\
&+ e\left[\frac{1}{4}\xi(1+\xi)(1-\eta)*(1) + 0 + 0\right] \\
&= e\left[\frac{1}{2}\xi(1+\xi) + \frac{1}{2}\xi(1-\xi)\right] = e\xi
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\hat{\phi} &= \sum_{i=1}^{6} N_i\hat{\phi}_i^{(e)} \\
&= g\sum_{i=1}^{6} N_i\hat{\eta}_i \\
&= g\left[\frac{1}{4}\xi(1+\xi)(1+\eta)*(1) + \frac{1}{4}\xi(\xi-1)(1+\eta)*(1) + \frac{1}{4}\xi(1-\xi)(\eta-1)*(-1)\right] \\
&+ g\left[\frac{1}{4}\xi(1+\xi)(1-\eta)*(-1) + \frac{1}{4}(1-\xi^2)(1+\eta)*(1) + \frac{1}{4}(1-\xi^2)(1-\eta)*(-1)\right] \\
&= g\left[\xi^2\eta + \eta(1-\xi^2)\right] = g\eta
\end{aligned} \tag{19}$$

The six-node element under consideration is thus *complete* but is *conditionally* compatible (i.e., the sides with quadratic approximation are not compatible with those having a linear one). As such, the use of this element is only justifiable for as a transition element.