Assembly of Element Equations

Pertinent reading: *Section 8.3* in textbook

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General Steps in Performing Finite Element Analyses

- Discretization of the solution domain.
- Assembly of *element* equations to form the *global* equations.
- Application of *nodal specifications*.
- Solution of the *global* equations.
- Calculation of the *secondary dependent variables* (e.g., "gradients").
- Post-processing of the results.
- Interpretation of the results.

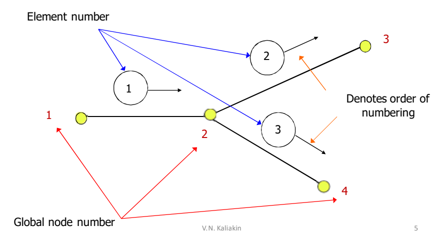
Assembly of Element Equations

- Once the equations are determined for an element, they must next be suitably incorporated into the "global" equations applicable to the entire domain.
- This so-called *assembly process* is performed for *all* the elements in the mathematical model.
- It essentially consists of *mapping* element (*local*) degrees of freedom to *global* degrees of freedom.

Assembly of Element Equations

- The assembly process is based on the premise of *compatibility* or *continuity*.
- That is, in accordance with the compatibility criterion, the values of primary dependent variables, and possibly their derivatives, associated with two adjacent elements must be the same at node points along the common interface.

Example Involving Line Elements

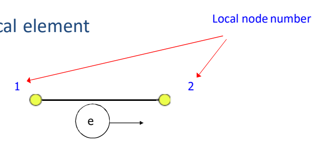


Example Involving Line Elements

- Assume *scalar* nodal unknowns (degrees of freedom).
- Thus only *one* unknown exists at each node.

Example Involving Line Elements

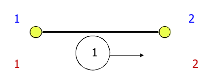
- For a typical element



$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & & \\ 2 & & \end{array}\quad \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1^{(e)} \\ \hat{\phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Example Involving Line Elements

- <u>For element 1</u>:



Local 1 → Global 1

Local 2 → Global 2

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & & \\ 2 & & \end{array}\quad \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1^{(e)} \\ \hat{\phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Example Involving Line Elements

- <u>Contribution to Global Coefficient Matrix</u>

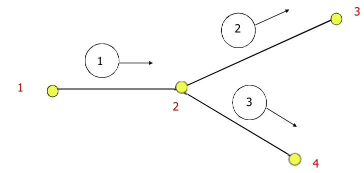
$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & & \end{array} \\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example Involving Line Elements

- Contribution to Global "Forcing" Vector

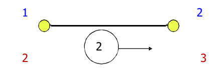
$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \\ \end{array} \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

Example Involving Line Elements



Example Involving Line Elements

- For element 2:



Local 1 → Global 2

Local 2 → Global 3

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & k_{11}^{(e)} & k_{12}^{(e)} \\ 2 & k_{21}^{(e)} & k_{22}^{(e)} \end{array} \quad \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1^{(e)} \\ \hat{\phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Example Involving Line Elements

- Contribution to Global Coefficient Matrix

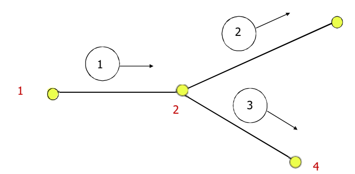
$$\begin{array}{c} \\ \\ 2 \\ 3 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \phantom{k_{11}^{(1)}} & 2 & 3 & \end{array} \\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example Involving Line Elements

- Contribution to Global "Forcing" Vector

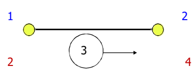
$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \\ 0 \end{Bmatrix} \begin{array}{c} \\ 2 \\ 3 \\ \\ \end{array}$$

Example Involving Line Elements



Example Involving Line Elements

- For element 3:



Local 1 → Global 2

Local 2 → Global 4

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & k_{11}^{(e)} & k_{12}^{(e)} \\ 2 & k_{21}^{(e)} & k_{22}^{(e)} \end{array} \quad \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} \hat{\phi}_1^{(e)} \\ \hat{\phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Example Involving Line Elements

- Contribution to Global Coefficient Matrix

$$\begin{array}{c} \\ \\ 2 \\ \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \phantom{k_{11}^{(1)}} & 2 & & 4 \end{array} \\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)} & k_{12}^{(2)} & k_{12}^{(3)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} & 0 \\ 0 & k_{21}^{(3)} & 0 & k_{22}^{(3)} \end{bmatrix} \end{array}$$

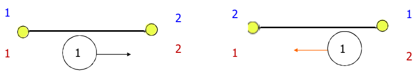
Example Involving Line Elements

- Contribution to Global "Forcing" Vector

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} + f_1^{(3)} \\ f_2^{(2)} \\ f_2^{(3)} \end{Bmatrix} \begin{array}{c} \\ 2 \\ \\ 4 \end{array}$$

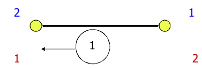
Example Involving Line Elements

- What is the effect of changing the order of numbering for a given element?
- That is,



Example Involving Line Elements

- For element 1:



Local 1 → Global 2

Local 2 → Global 1

$$\begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \end{array} \begin{Bmatrix} \hat{\phi}_1^{(e)} \\ \hat{\phi}_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix}$$

Example Involving Line Elements

- Contribution to Global Coefficient Matrix

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & & \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ \end{array} & \begin{bmatrix} k_{22}^{(1)} & k_{21}^{(1)} & 0 & 0 \\ k_{12}^{(1)} & k_{11}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example Involving Line Elements

- Contribution to Global "Forcing" Vector

$$\begin{Bmatrix} f_2^{(1)} \\ f_1^{(1)} \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ \\ \\ \end{array}$$

Example Involving Line Elements

- There is thus no change as to *where* in the global arrays the non-zero element contributions are summed.
- The only change is the *order* in which values that are summed in these locations.