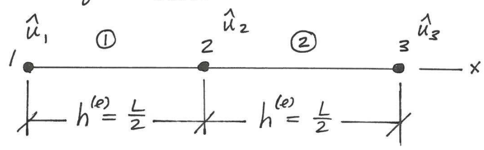
Problem 10

* Mesh consisting of two linear elements



→ For assembly process,

* Element ①:

local	global
1 →	1
2 →	2

* Element ②

local	global
1 →	2
2 →	3

→ Global Arrays:

$$\frac{E^{(e)}A^{(e)}}{h^{(e)}}\begin{bmatrix} 1 & -1 & 0 \\ -1 & (1+1) & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \frac{b_x^{(e)}A^{(e)}h^{(e)}}{2}\begin{Bmatrix} 1 \\ 1+1 \\ 1 \end{Bmatrix}$$

→ Accounting for the nodal specifications using the elimination approach gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2E^{(e)}A^{(e)}}{h^{(e)}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ b_x^{(e)}A^{(e)}h^{(e)} \\ 0 \end{Bmatrix}$$

→ solving for the primary dependent variable gives

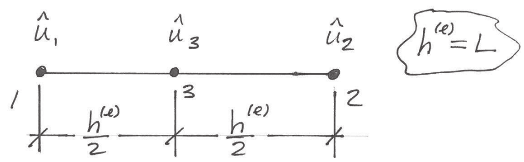
$$\frac{2E^{(e)}A^{(e)}}{h^{(e)}}\hat{u}_2 = b_x^{(e)}A^{(e)}h^{(e)}$$

$$\therefore \quad \hat{u}_2 = \frac{b_x^{(e)}\left(h^{(e)}\right)^2}{2E^{(e)}} = \frac{b_x^{(e)}L^2}{8E^{(e)}}$$

$$= \frac{(75.0\ \mathrm{kN/m^3})(3.05\ \mathrm{m})^2}{8(2.07\times10^8\ \mathrm{kPa})}$$

$$= 4.213\times10^{-7}\ \mathrm{m}$$

* Mesh consisting of one quadratic element



→ Global arrays:

$$\frac{E^{(e)}A^{(e)}}{3h^{(e)}}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \frac{b_x^{(e)}A^{(e)}h^{(e)}}{6}\begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}$$

→ Accounting for nodal specifications using the elimination approach gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{16E^{(e)}A^{(e)}}{3h^{(e)}} \end{bmatrix}\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{2b_x^{(e)}A^{(e)}h^{(e)}}{3} \end{Bmatrix}$$

→ solving for the primary dependent variables gives

$$\frac{16E^{(e)}A^{(e)}}{3h^{(e)}}\hat{u}_3 = \frac{2b_x^{(e)}A^{(e)}h^{(e)}}{3}$$

$$\therefore \quad \hat{u}_3 = \frac{b_x^{(e)}\left(h^{(e)}\right)^2}{8E^{(e)}} = 4.213\times10^{-7}\ \mathrm{m}$$

which is identical to the solution obtained using two linear elements.

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* Exact Solution: (from Exercise 7)

$$u(x) = \frac{b_x}{2E}(L-x)x$$

$$\therefore \quad u\left(\frac{L}{2}\right) = \frac{b_x}{2E}\left(L-\frac{L}{2}\right)\left(\frac{L}{2}\right) = \frac{b_xL^2}{8E}$$

$\Rightarrow$ at the middle of the domain, the two FE solutions are exact!