CIEG 401/601

Introduction to the Finite Element Method

Homework #4
Due on Thursday, April 19, 2018.

• Problem 8

Consider the two-node (linear) cable element resting on elastic foundation that was discussed in class. It is desired to modify the element equations so as to allow for a *linear* variation in the foundation stiffness; viz.,

$$\beta^{(e)} = \frac{1}{2}(1-\xi)\beta_1^{(e)} + \frac{1}{2}(1+\xi)\beta_2^{(e)}$$

where ξ is the natural coordinate, and $\beta_1^{(e)}$ and $\beta_2^{(e)}$ represent the values of the foundation stiffness at nodes 1 and 2, respectively.

Following a procedure similar to that presented in lecture, derive the modified element "foundation" matrix $\mathbf{M^{(e)}}$. Begin your solution with the general expression for this matrix; viz.,

$$\mathbf{M^{(e)}} = \int_0^{h^{(e)}} \mathbf{N}^T \beta^{(e)} \mathbf{N}\, dx$$

where $\mathbf{N} = \begin{Bmatrix} N_1 & N_2 \end{Bmatrix}$, with N_1 and N_2 as defined in lecture. Perform your calculations in terms of the *natural* coordinate ξ. What check do you have on your resulting $\mathbf{M^{(e)}}$?

• Problem 9

Consider once again the two-node (linear) cable element (resting on an elastic foundation) discussed in class. You are asked to change the approximation for transverse displacement to a *quadratic* one. The resulting element, which is shown in Figure **??**, has three nodes and three transverse displacement degrees of freedom.

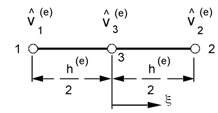

Figure 1: Typical quadratic cable element.

The associated element interpolation functions, written in terms of the *natural* ξ coordinate, are[1]

$$N_1 = \frac{1}{2}\xi(\xi-1) \quad ; \quad N_2 = \frac{1}{2}\xi(\xi+1) \quad ; \quad N_3 = (1-\xi^2)$$

Derive the equations associated with the quadratic cable element. Follow the 5 step procedure presented in Chapter 7 of the textbook and discussed in class lectures. Please note the following:

1. Steps 1 to 3 are *unchanged* from the development presented in class for the linear cable element. Consequently, there is *no need* to repeat these steps.
2. Step 4 is changed because of the use of quadratic interpolation functions in lieu of linear ones. Since the convergence criteria have been checked and verified for the linear element, and since a higher-order element is now under consideration, there is *no need* to repeat this step.

Present your equations for the quadratic cable element in vector-matrix form.

[1]Details pertaining to the derivation of these interpolation functions are given in Chapter 9 of the textbook.

• Problem 10

Consider once again the prismatic bar analyzed in Problem 6. The bar is prismatic and has essential boundary conditions at both of its ends (Figure **??**).

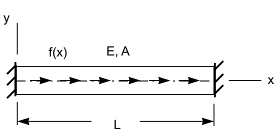

Figure 2: Prismatic bar with homogeneous essential boundary conditions at both ends.

Assume the same values as in Problem 6; i.e., an elastic modulus $E = 2.07\,\mathrm{x}\,10^8\,kPa$, a cross-sectional area $A = 3.20\,\mathrm{x}\,10^{-3}\,m^2$ and a body force in the *positive* x direction $b_x = 75.0\,kN/m^3$ (consequently, $f(x) = b_x A$). The length of the bar is again $L = 3.05$ meters.

Please do the following:

1. Consider a mesh of two *linear* elements of equal length. Form the respective element arrays and assemble them to form the global arrays. Impose the two homogeneous essential nodal constraints. Finally, solve for the unknown displacement at the middle (unconstrained) node. Written in vector-matrix form, the element equations for the linear element are as follows:

$$\frac{E^{(e)}A^{(e)}}{h^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \hat{u}_1^{(e)} \\ \hat{u}_2^{(e)} \end{Bmatrix} = \frac{b_x^{(e)}A^{(e)}h^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

2. Consider next a mesh consisting of a single *quadratic* bar element with its middle node located equidistant from its ends (see Figure **??**).

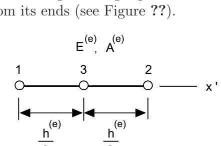

Figure 3: Typical quadratic bar element.

Form the element arrays (which, in this case, are identical to the global arrays). Impose the two homogeneous essential nodal constraints. Finally, solve for the unknown

displacement at the middle (unconstrained) node. Written in vector-matrix form, the element equations for the quadratic element are as follows:

$$\frac{E^{(e)}A^{(e)}}{3h^{(e)}}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{Bmatrix} \hat{u}_1^{(e)} \\ \hat{u}_2^{(e)} \\ \hat{u}_3^{(e)} \end{Bmatrix} = \frac{b_x^{(e)}A^{(e)}h^{(e)}}{6}\begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}$$

3. Compare your approximate solutions with each other and with the exact one.[2]
4. Discuss your results.

[2]The exact solution was given as part of the complete solution to Problem 4.