• Problem 8

Consider the two-node (linear) cable element resting on elastic foundation that was discussed in class. It is desired to modify the element equations so as to allow for a *linear* variation in the foundation stiffness; viz.,

$$\beta^{(e)} = \frac{1}{2}(1-\xi)\beta_1^{(e)} + \frac{1}{2}(1+\xi)\beta_2^{(e)} \tag{1}$$

where ξ is the natural coordinate, and $\beta_1^{(e)}$ and $\beta_2^{(e)}$ represent the values of the foundation stiffness at nodes 1 and 2, respectively.

Following a procedure similar to that presented in lecture, derive the modified element "foundation" matrix $\mathbf{M^{(e)}}$. Begin your solution with the general expression for this matrix; viz.,

$$\mathbf{M^{(e)}} = \int_0^{h^{(e)}} \mathbf{N}^T \beta^{(e)} \mathbf{N}\, dx \tag{2}$$

where $\mathbf{N} = \left\{ N_1 \quad N_2 \right\}$, with N_1 and N_2 as defined in lecture. Perform your calculations in terms of the *natural* coordinate ξ. What check do you have on your resulting $\mathbf{M^{(e)}}$?

Solution

Since $N_1 = \frac{1}{2}(1-\xi)$ and $N_2 = \frac{1}{2}(1+\xi)$, and substituting for $\beta^{(e)}$ gives

$$\mathbf{M^{(e)}} = \int_{-1}^{1} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \left[\tfrac{1}{2}(1-\xi)\beta_1^{(e)} + \tfrac{1}{2}(1+\xi)\beta_2^{(e)} \right] \left\{ N_1 \quad N_2 \right\} \left(\frac{h^{(e)}}{2} d\xi \right) \tag{3}$$

or

$$\mathbf{M^{(e)}} = \frac{h^{(e)}}{4} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] \begin{bmatrix} N_1 N_1 & N_1 N_2 \\ N_2 N_1 & N_2 N_2 \end{bmatrix} d\xi \tag{4}$$

As an example of the requisite integrations, consider the term

$$\frac{h^{(e)}}{4} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] N_1 N_1\, d\xi$$

Substituting for N_1 gives

$$\begin{aligned} & \frac{h^{(e)}}{4} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] N_1 N_1\, d\xi \\ &= \frac{h^{(e)}}{4} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] \frac{1}{4}(1-\xi)^2\, d\xi \\ &= \frac{h^{(e)}}{16} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] \left(1 - 2\xi + \xi^2 \right)\, d\xi \end{aligned} \tag{5}$$

Integrating all of the terms and then evaluating the limits of integration gives

$$\begin{aligned} & \frac{h^{(e)}}{4} \int_{-1}^{1} \left[(1-\xi)\beta_1^{(e)} + (1+\xi)\beta_2^{(e)} \right] N_1 N_1\, d\xi \\ &= \frac{h^{(e)}}{4} \left[\beta_1^{(e)} + \frac{1}{3}\beta_2^{(e)} \right] \end{aligned} \tag{6}$$

Proceeding in a similar manner gives

$$\boxed{\mathbf{M^{(e)}} = \frac{h^{(e)}}{12} \begin{bmatrix} \left(3\beta_1^{(e)} + \beta_2^{(e)} \right) & \left(\beta_1^{(e)} + \beta_2^{(e)} \right) \\ \left(\beta_1^{(e)} + \beta_2^{(e)} \right) & \left(\beta_1^{(e)} + 3\beta_2^{(e)} \right) \end{bmatrix}} \tag{7}$$

As a check on the results obtained, set $\beta_1^{(e)} = \beta_2^{(e)} = \beta^{(e)}$. The above expression reduces to

$$\mathbf{M^{(e)}} = \frac{h^{(e)}}{12} \begin{bmatrix} 4\beta^{(e)} & 2\beta^{(e)} \\ 2\beta^{(e)} & 4\beta^{(e)} \end{bmatrix} = \frac{\beta^{(e)} h^{(e)}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which is the element "foundation" matrix associated with a *constant* $\beta^{(e)}$.