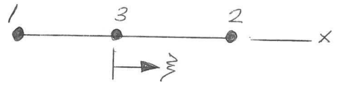
Problem 2



$$N_i = \sin\left[\pi\left(a_i + b_i\xi + c_i\xi^2\right)\right]$$

for $i=1$:

$$N_1(-1) = \sin\left[\pi\left(a_1 - b_1 + c_1\right)\right] = 1$$

$$N_1(0) = \sin\left[\pi(a_1)\right] = 0 \quad \Rightarrow a_1 = 0$$

(or n, where $n=0,1,2,\dots$)

$$N_1(1) = \sin\left[\pi\left(a_1 + b_1 + c_1\right)\right] = 0$$

thus

$$(-b_1 + c_1) = \tfrac{1}{2}$$
$$(b_1 + c_1) = 0 \quad \Rightarrow 2c_1 = \tfrac{1}{2} \Rightarrow \boxed{\begin{array}{l} c_1 = \tfrac{1}{4} \\ b_1 = -\tfrac{1}{4} \end{array}} \begin{array}{l} \text{or } \left(\tfrac{n}{2}+\tfrac{1}{4}\right) \\ -\left(\tfrac{n}{2}+\tfrac{1}{4}\right) \end{array}$$

for $i=2$:

$$N_2(-1) = \sin\left[\pi\left(a_2 - b_2 + c_2\right)\right] = 0$$

$$N_2(0) = \sin\left[\pi(a_2)\right] = 0 \quad \Rightarrow a_2 = 0$$

(or n, where $n=0,1,2,\dots$)

$$N_2(1) = \sin\left[\pi\left(a_2 + b_2 + c_2\right)\right] = 1$$

thus

$$(-b_2 + c_2) = 0$$
$$(b_2 + c_2) = \tfrac{1}{2} \quad \Rightarrow 2c_2 = \tfrac{1}{2} \Rightarrow \boxed{\begin{array}{l} c_2 = \tfrac{1}{4} \\ b_2 = \tfrac{1}{4} \end{array}} \begin{array}{l} \left(\tfrac{n}{2}+\tfrac{1}{4}\right) \\ \left(\tfrac{n}{2}+\tfrac{1}{4}\right) \end{array}$$

for $i=3$:

$$N_3(-1) = \sin\left[\pi\left(a_3 - b_3 + c_3\right)\right] = 0$$

$$N_3(0) = \sin\left[\pi(a_3)\right] = 1 \quad \Rightarrow a_3 = \tfrac{1}{2}$$

(or n; where $n = \tfrac{1}{2}, \tfrac{5}{2}, \dots$) or $\left(2n+\tfrac{1}{2}\right)$

$$N_3(1) = \sin\left[\pi\left(a_3 + b_3 + c_3\right)\right] = 0$$

thus

$$\sin\left[\pi\left(\tfrac{1}{2} - b_3 + c_3\right)\right] = 0$$
$$\sin\left[\pi\left(\tfrac{1}{2} + b_3 + c_3\right)\right] = 0$$

$$\Rightarrow \quad \tfrac{1}{2} - b_3 + c_3 = 0$$
$$\tfrac{1}{2} + b_3 + c_3 = 0$$
$$1 \quad + 2c_3 = 0 \quad \Rightarrow c_3 = -\tfrac{1}{2}$$

or $-\left(n+\tfrac{1}{2}\right)$

$$b_3 = \tfrac{1}{2} + c_3 = \tfrac{1}{2} + \left(-\tfrac{1}{2}\right) = 0$$

Thus $\boxed{a_3 = \tfrac{1}{2},\ b_3 = 0,\ c_3 = -\tfrac{1}{2}}$

Thus,

$$\boxed{\begin{array}{l} N_1 = \sin\left[\tfrac{\pi}{4}\xi(\xi-1)\right] \\[2ex] N_2 = \sin\left[\tfrac{\pi}{4}\xi(\xi+1)\right] \\[2ex] N_3 = \sin\left[\tfrac{\pi}{2}(1-\xi^2)\right] \end{array}}$$

These element interpolation functions do not, in general, satisfy the partition of unity; i.e., $\sum_{i=1}^{3} N_i \neq 1$. (see attached plot)

Remark: for the same 3-node (quadratic) element, the interpolation functions obtained using Lagrangian polynomials are $N_1 = \tfrac{1}{2}\xi(\xi-1)$, $N_2 = \tfrac{1}{2}\xi(\xi+1)$

$$N_3 = 1 - \xi^2.$$