Example: Indeterminate Beam Solved Using Bernoulli-Euler Elements

Consider the beam shown in Figure 1. Besides the applied distributed and concentrated loads, the beam experiences a support settlement. The beam is characterized by an elastic modulus of $E = 10 \text{ x } 10^3$ kips/in^2[1]. The moment of inertia for the cross-section is $I = 320$ in^4.

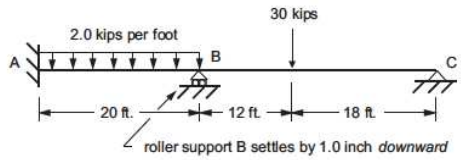

Figure 1: Beam subjected to applied loads and a support settlement.

The coarsest mesh contains *three* element and *four* nodes as shown in Figure 2. It follows that this model contains *eight* (8) global degrees of freedom; viz.,

$$\hat{\boldsymbol{\phi}}_{\boldsymbol{n}} = \begin{Bmatrix} \hat{w}_1 & \hat{\theta}_1 & \hat{w}_2 & \hat{\theta}_2 & \hat{w}_3 & \hat{\theta}_3 & \hat{w}_4 & \hat{\theta}_4 \end{Bmatrix}^T$$

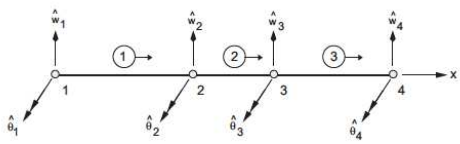

Figure 2: Finite element model of beam subjected to applied loads and a support settlement.

Next, consider the contributions of the respective elements. Units of *kips* and *inches* are used throughout the problem.

[1] 1 kip = 1000 pounds.

- **Element 1**:

$$h^{(1)} = 240 \text{ inches}$$

Thus,

$$\frac{EI}{\left(h^{(1)}\right)^3} = \frac{(10 \text{ x } 10^3 \text{ ksi})(320 \text{ in}^4)}{(240 \text{ inches})^3} = 2.315 \text{ x } 10^{-1} \text{ k/in}$$

The element stiffness matrix is thus

$$\mathbf{K^{(1)}} = \begin{bmatrix} 2.778 \text{ x } 10^0 & 3.333 \text{ x } 10^2 & -2.778 \text{ x } 10^0 & 3.333 \text{ x } 10^2 \\ 3.333 \text{ x } 10^2 & 5.333 \text{ x } 10^4 & -3.333 \text{ x } 10^2 & 2.667 \text{ x } 10^4 \\ -2.778 \text{ x } 10^0 & -3.333 \text{ x } 10^2 & 2.778 \text{ x } 10^0 & -3.333 \text{ x } 10^2 \\ 3.333 \text{ x } 10^2 & 2.667 \text{ x } 10^4 & -3.333 \text{ x } 10^2 & 5.333 \text{ x } 10^4 \end{bmatrix}$$

Since $\bar{q}^{(1)} = -2.0$ k/in, the associated element force vector is

$$\mathbf{q^{(1)}} = \frac{(-2.0 \text{ k/in})(1 \text{ ft/12 in})(240 \text{ in})}{2} \begin{Bmatrix} 1.0 \\ (240 \text{ in})/6 \\ 1.0 \\ -(240 \text{ in})/6 \end{Bmatrix} = 20 \begin{Bmatrix} 1.0 \\ -40.0 \\ -1.0 \\ 40.0 \end{Bmatrix}$$

- **Element 2**:

$$h^{(2)} = 144 \text{ inches}$$

Thus,

$$\frac{EI}{\left(h^{(1)}\right)^3} = \frac{(10 \text{ x } 10^3 \text{ ksi})(320 \text{ in}^4)}{(144 \text{ inches})^3} = 1.072 \text{ x } 10^0 \text{ k/in}$$

The element stiffness matrix is thus

$$\mathbf{K^{(2)}} = \begin{bmatrix} 1.286 \text{ x } 10^1 & 9.259 \text{ x } 10^2 & -1.286 \text{ x } 10^1 & 9.259 \text{ x } 10^2 \\ 9.259 \text{ x } 10^2 & 8.889 \text{ x } 10^4 & -9.259 \text{ x } 10^2 & 4.444 \text{ x } 10^4 \\ -1.286 \text{ x } 10^1 & -9.259 \text{ x } 10^2 & 1.286 \text{ x } 10^1 & -9.259 \text{ x } 10^2 \\ 9.259 \text{ x } 10^2 & 4.444 \text{ x } 10^4 & -9.259 \text{ x } 10^2 & 8.889 \text{ x } 10^4 \end{bmatrix}$$

Since $\bar{q}^{(2)} = 0$, the associated element force vector is simply

$$\mathbf{q^{(2)}} = \mathbf{0}$$

- **Element 3**:

$$h^{(3)} = 216 \text{ inches}$$

Thus,

$$\frac{EI}{\left(h^{(1)}\right)^3} = \frac{(10 \text{ x } 10^3 \text{ ksi})(320 \text{ in}^4)}{(216 \text{ inches})^3} = 3.175 \text{ x } 10^{-1} \text{ k/in}$$

The element stiffness matrix is thus

$$\mathbf{K^{(3)}} = \begin{bmatrix} 3.810 \text{ x } 10^0 & 4.115 \text{ x } 10^2 & -3.810 \text{ x } 10^0 & 4.115 \text{ x } 10^2 \\ 4.115 \text{ x } 10^2 & 5.926 \text{ x } 10^4 & -4.115 \text{ x } 10^2 & 2.963 \text{ x } 10^4 \\ -3.810 \text{ x } 10^0 & -4.115 \text{ x } 10^2 & 3.810 \text{ x } 10^0 & -4.115 \text{ x } 10^2 \\ 4.115 \text{ x } 10^2 & 2.963 \text{ x } 10^4 & -4.115 \text{ x } 10^2 & 5.926 \text{ x } 10^4 \end{bmatrix}$$

The associated element force vector is

$$\mathbf{q^{(3)}} = \mathbf{0}$$

The element equations are next assembled. First this is shown symbolically, with the respective element contributions shown in different colors.

$$\mathbf{K} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0 & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} + K_{11}^{(2)} & K_{34}^{(1)} + K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} & 0 & 0 \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} + K_{21}^{(2)} & K_{44}^{(1)} + K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} & 0 & 0 \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} + K_{11}^{(3)} & K_{34}^{(2)} + K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\ 0 & 0 & K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} + K_{21}^{(3)} & K_{44}^{(2)} + K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\ 0 & 0 & 0 & 0 & K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\ 0 & 0 & 0 & 0 & K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix}$$

$$\mathbf{q} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \\ q_3^{(1)} + q_1^{(2)} \\ q_4^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_1^{(3)} \\ q_4^{(2)} + q_2^{(3)} \\ q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix}$$

Substituting explicit values for the entries in the respective element stiffness matrices and right-hand side vectors gives the following global arrays:

$$\mathbf{K}\hat{\boldsymbol{\phi}}_{\boldsymbol{n}} = \mathbf{q}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K_{11}} & \mathbf{K_{12}} \\ \mathbf{K_{21}} & \mathbf{K_{22}} \end{bmatrix}$$

with

$$\mathbf{K_{11}} = \begin{bmatrix} 2.778 \text{ x } 10^0 & 3.333 \text{ x } 10^2 & -2.778 \text{ x } 10^0 & 3.333 \text{ x } 10^2 \\ 3.333 \text{ x } 10^2 & 5.333 \text{ x } 10^4 & -3.333 \text{ x } 10^2 & 2.667 \text{ x } 10^4 \\ -2.778 \text{ x } 10^0 & -3.333 \text{ x } 10^2 & 1.564 \text{ x } 10^1 & 5.926 \text{ x } 10^2 \\ 3.333 \text{ x } 10^2 & 2.667 \text{ x } 10^4 & 5.926 \text{ x } 10^2 & 1.422 \text{ x } 10^5 \end{bmatrix}$$

$$\mathbf{K_{12}} = \begin{bmatrix} 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \\ -1.286 \text{ x } 10^1 & 9.259 \text{ x } 10^2 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \\ -9.259 \text{ x } 10^2 & 4.444 \text{ x } 10^4 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \end{bmatrix}$$

$$\mathbf{K_{21}} = \begin{bmatrix} 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & -1.286 \text{ x } 10^1 & -9.259 \text{ x } 10^2 \\ 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 9.259 \text{ x } 10^2 & 4.444 \text{ x } 10^4 \\ 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 & 0.000 \text{ x } 10^0 \end{bmatrix}$$

$$\mathbf{K_{22}} = \begin{bmatrix} 1.667 \text{ x } 10^1 & -5.144 \text{ x } 10^2 & -3.810 \text{ x } 10^0 & 4.115 \text{ x } 10^2 \\ -5.144 \text{ x } 10^2 & 1.481 \text{ x } 10^5 & -4.115 \text{ x } 10^2 & 2.963 \text{ x } 10^4 \\ -3.810 \text{ x } 10^0 & -4.115 \text{ x } 10^2 & 3.810 \text{ x } 10^0 & -4.115 \text{ x } 10^2 \\ 4.115 \text{ x } 10^2 & 2.963 \text{ x } 10^4 & -4.115 \text{ x } 10^2 & 5.926 \text{ x } 10^4 \end{bmatrix}$$

and

$$\mathbf{q} = \begin{Bmatrix} -2.000 \text{ x } 10^1 \\ -8.000 \text{ x } 10^2 \\ -2.000 \text{ x } 10^1 \\ 8.000 \text{ x } 10^2 \\ 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 \end{Bmatrix}$$

The half-bandwidth of $\mathbf{K}$ is next computed. From equation (8.13) in the textbook,

$$n_{bw} = (\text{dif}_{max} + 1)(\text{node}_{dof})$$

where dif_{max} = the maximum difference between any two node numbers associated with a particular element (computed over *all* elements), and node_{dof} = the number of unknowns per node. Thus, for the present mesh, $\text{dif}_{max} = 1$ and $\text{node}_{dof} = 2$, giving

$$n_{bw} = (1 + 1)(2) = 4$$

In terms of global degrees of freedom, the nodal specifications are $\hat{w}_1 = 0.0$, $\hat{\theta}_1 = 0.0$, $\hat{w}_2 = -1.0$ in, $\hat{M}_2 = 0.0$, $\hat{V}_3 = -30$ kips, $\hat{M}_3 = 0.0$, $\hat{w}_4 = 0.0$, and $\hat{M}_4 = 0.0$. These specifications are next applied using the *elimination* approach at the *global* level. The resulting global stiffness matrix is thus

$$\mathbf{K} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.422 \text{ x } 10^5 & -9.259 \text{ x } 10^2 & 4.444 \text{ x } 10^4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -9.259 \text{ x } 10^2 & 1.667 \text{ x } 10^1 & -5.144 \text{ x } 10^2 & 0.0 & 4.115 \text{ x } 10^2 \\ 0.0 & 0.0 & 0.0 & 4.444 \text{ x } 10^4 & -5.144 \text{ x } 10^2 & 1.481 \text{ x } 10^5 & 0.0 & 2.963 \text{ x } 10^4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 4.115 \text{ x } 10^2 & 2.963 \text{ x } 10^4 & 0.0 & 5.926 \text{ x } 10^4 \end{bmatrix}$$

$$\mathbf{q} = \begin{Bmatrix} 0.0 \\ 0.0 \\ -1.0 \\ 8.000 \text{ x } 10^2 - (-1.0)(5.926 \text{ x } 10^2) \\ 0.000 \text{ x } 10^0 - (-1.0)(-1.286 \text{ x } 10^1) - 30.0 \\ 0.0 - (-1.0)(9.259 \text{ x } 10^2) \\ 0.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 \\ -1.000 \text{ x } 10^0 \\ 1.393 \text{ x } 10^3 \\ -4.286 \text{ x } 10^1 \\ 9.259 \text{ x } 10^2 \\ 0.000 \text{ x } 10^0 \\ 0.000 \text{ x } 10^0 \end{Bmatrix}$$

Solving this set of simultaneous equations gives

$$\hat{\boldsymbol{\phi}}_{\boldsymbol{n}} = \begin{Bmatrix} \hat{w}_1 \\ \hat{\theta}_1 \\ \hat{w}_2 \\ \hat{\theta}_2 \\ \hat{w}_3 \\ \hat{\theta}_3 \\ \hat{w}_4 \\ \hat{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} 0.000E+00 \\ 0.000E+00 \\ -1.000E+00 \\ -1.916E-02 \\ -5.140E+00 \\ -1.443E-02 \\ 0.000E+00 \\ 4.291E-02 \end{Bmatrix}$$