Derive the equations associated with the quadratic (three-node) cable on elastic foundation element shown in Figure 1. The element has three transverse displacement degrees of freedom.

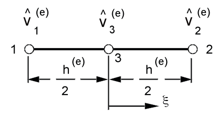

Figure 1. Typical quadratic cable element resting in an elastic foundation.

From Section 9.2.1, the element interpolation functions and their derivatives with respect to natural coordinates are

$$N_1 = \frac{1}{2}\xi(\xi - 1) \quad ; \quad N_2 = \frac{1}{2}\xi(\xi + 1) \quad ; \quad N_3 = (1 - \xi^2) \tag{1}$$

$$\frac{dN_1}{d\xi} = \frac{1}{2}(2\xi - 1) \quad ; \quad \frac{dN_2}{d\xi} = \frac{1}{2}(2\xi + 1) \quad ; \quad \frac{dN_3}{d\xi} = -2\xi \tag{2}$$

In matrix form,

$$\mathbf{N} = \left\{ \frac{1}{2}\xi(\xi - 1) \quad \frac{1}{2}\xi(\xi + 1) \quad (1 - \xi^2) \right\} \tag{3}$$

and

$$\mathbf{B} = \frac{2}{h^{(e)}} \left\{ \frac{1}{2}(2\xi - 1) \quad \frac{1}{2}(2\xi + 1) \quad -2\xi \right\} \tag{4}$$

The general expressions for the element stiffness and foundation matrices and the element forcing vector are

$$\mathbf{K^{(e)}} = \int_{-1}^{1} \mathbf{B}^T T^{(e)} \mathbf{B} \left(\frac{h^{(e)}}{2} \, d\xi \right) \tag{5}$$

$$\mathbf{M^{(e)}} = \int_{-1}^{1} \mathbf{N}^T k^{(e)} \mathbf{N} \left(\frac{h^{(e)}}{2} \, d\xi \right) \tag{6}$$

$$\mathbf{q^{(e)}} = \int_{-1}^{1} \mathbf{N}^T \bar{f}^{(e)} \left(\frac{h^{(e)}}{2} \, d\xi \right) \tag{7}$$

Assuming $T^{(e)}$, $k^{(e)}$, and $\bar{f}^{(e)}$ constant over the element, equations (5) to (7) become

$$\mathbf{K^{(e)}} = \frac{T^{(e)} h^{(e)}}{2} \int_{-1}^{1} \begin{bmatrix} B_1B_1 & B_1B_2 & B_1B_3 \\ B_2B_1 & B_2B_2 & B_2B_3 \\ B_3B_1 & B_3B_2 & B_3B_3 \end{bmatrix} d\xi \tag{8}$$

$$\mathbf{M^{(e)}} = \frac{k^{(e)} h^{(e)}}{2} \int_{-1}^{1} \begin{bmatrix} N_1N_1 & N_1N_2 & N_1N_3 \\ N_2N_1 & N_2N_2 & N_2N_3 \\ N_3N_1 & N_3N_2 & N_3N_3 \end{bmatrix} d\xi \tag{9}$$

$$\mathbf{q^{(e)}} = \frac{\bar{f}^{(e)} h^{(e)}}{2} \int_{-1}^{1} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} d\xi \tag{10}$$

Substituting for B_1, B_2, B_3, N_1, N_2, and N_3 from equations (3) and (4), integrating the resulting expressions and evaluating the limits of integration gives the following element matrices:

$$\boxed{\mathbf{K^{(e)}} = \frac{7T^{(e)}}{3h^{(e)}} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}}$$

$$\boxed{\mathbf{M^{(e)}} = \frac{k^{(e)} h^{(e)}}{30} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix}}$$

$$\boxed{\mathbf{q^{(e)}} = \frac{\bar{f}^{(e)} h^{(e)}}{6} \begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}}$$