# 第13周作业及参考答案

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June 5, 2020

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# 习题十二作业

1.求下列函数的傅氏变换:

$$(1)\ f(x)=\sin(\eta x^2).$$

$$(2)\ f(x)=\cos(\eta x^2).$$

其中$\eta>0$.

解 由傅里叶变换的定义以及欧拉公式我们有

$$\begin{aligned} F[\sin(\eta x^2)] &= \int_{-\infty}^{+\infty}\sin(\eta x^2)e^{-\mathrm{i}\omega x}\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}\sin(\eta x^2)\left[\cos(\omega x)+\mathrm{i}\sin(\omega x)\right]\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}\sin(\eta x^2)\cos(\omega x)\mathrm{d}x, \end{aligned}$$

最后一个等号是由于正弦函数$\sin(\omega x)$ 是奇函数. 所以$\sin(\eta x^2)$的傅里叶变换是实函数. 同理可知$\cos(\eta x^2)$的傅里叶变换也是实函数.

再一次利用欧拉公式有

$$e^{\mathrm{i}\eta x^2}=\cos(\eta x^2)+\mathrm{i}\sin(\eta x^2).$$

则

$$
\begin{aligned}
F[\cos(\eta x^2)] + \mathrm{i}F[\sin(\eta x^2)] &= \int_{-\infty}^{+\infty} e^{\mathrm{i}\eta x^2} e^{-\mathrm{i}\omega x}\mathrm{d}x \\
&= e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \int_{-\infty}^{+\infty} e^{\mathrm{i}\eta\left(x-\frac{\omega}{2\eta}\right)^2}\mathrm{d}x \\
&= \frac{1}{\sqrt{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \int_{-\infty}^{+\infty} e^{\mathrm{i}y^2}\mathrm{d}y \\
&= \frac{2}{\sqrt{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \int_{0}^{+\infty} e^{\mathrm{i}y^2}\mathrm{d}y.
\end{aligned}
$$

再再利用欧拉公式, 并由菲涅尔积分(第五章5.2.4节例15)得

$$
\begin{aligned}
\frac{2}{\sqrt{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \int_{0}^{+\infty} e^{\mathrm{i}y^2}\mathrm{d}y &= \frac{2}{\sqrt{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \left[\int_0^{+\infty}\cos(y^2)\mathrm{d}y + \mathrm{i}\int_0^{+\infty}\sin(y^2)\mathrm{d}y\right] \\
&= \frac{2}{\sqrt{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} \left[\frac{1}{2}\sqrt{\frac{\pi}{2}} + \mathrm{i}\frac{1}{2}\sqrt{\frac{\pi}{2}}\right] \\
&= \sqrt{\frac{\pi}{\eta}} e^{-\mathrm{i}\frac{\omega^2}{4\eta}} e^{\mathrm{i}\frac{\pi}{4}} = \sqrt{\frac{\pi}{\eta}} e^{\mathrm{i}\left(\frac{\pi}{4}-\frac{\omega^2}{4\eta}\right)} \\
&= \sqrt{\frac{\pi}{\eta}}\left[\cos\left(\frac{\pi}{4}-\frac{\omega^2}{4\eta}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}-\frac{\omega^2}{4\eta}\right)\right].
\end{aligned}
$$

最后比较实部和虚部我们有

$$F[\cos(\eta x^2)] = \sqrt{\frac{\pi}{\eta}}\cos(\frac{\omega^2}{4\eta} - \frac{\pi}{4}),$$

$$\begin{aligned} F[\sin(\eta x^2)] &= \sqrt{\frac{\pi}{\eta}}\sin(\frac{\pi}{4} - \frac{\omega^2}{4\eta}) = \sqrt{\frac{\pi}{\eta}}\sin(\frac{\pi}{2} - \frac{\omega^2}{4\eta} - \frac{\pi}{4}) \\ &= \sqrt{\frac{\pi}{\eta}}\cos(\frac{\omega^2}{4\eta} + \frac{\pi}{4}). \end{aligned}$$

4. 求解定解问题

$$\begin{cases} u_{tt} + a^2 u_{xxxx} = 0, \\ u(x,0) = f(x) & (-\infty < x < +\infty), \\ u_t(x,0) = 0 & (-\infty < x < +\infty). \end{cases}$$

解 方程和初始条件等号两端关于变量$x$做傅里叶变换得

$$\begin{cases} \dfrac{d^2}{dt^2}\widehat{u} + a^2\omega^4\widehat{u} = 0, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega), \\ \dfrac{d}{dt}\widehat{u}(\omega,0) = 0. \end{cases}$$

解这个关于$t$ 的二阶常微分方程的初值问题得

$$\widehat{u}(\omega,t)=\widehat{f}(\omega)\cos(a\omega^2t).$$

两边做傅里叶逆变换，并利用卷积的性质得

$$u(x,t)=f(x)*F^{-1}[\cos(a\omega^2t)].$$

利用傅里叶变换的对称性质, 和上一题的结果得

$$F^{-1}[\cos(a\omega^2t)]=\frac{1}{2\pi}F[\cos(a\omega^2t)](-x)=\frac{1}{2\sqrt{\pi at}}\cos(\frac{x^2}{4at}-\frac{\pi}{4}).$$

或者所以

$$u(x,t)=\frac{1}{2\sqrt{\pi at}}\int_{-\infty}^{+\infty}f(\xi)\cos\left[\frac{(x-\xi)^2}{4at}-\frac{\pi}{4}\right]\mathrm{d}\xi.$$