THE AXIOM OF CHOICE AND MAXIMAL IDEALS IN RING THEORY

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Abstract. In this expository paper, we show that the Axiom of Choice is equivalent to the statement "every commutative ring with unity has a maximal ideal".

1. Introduction

Since Zermelo first formulated the Axiom of Choice in 1904, the Axiom of Choice has had a large impact on modern mathematics. The Axiom of Choice is used to justify results in various fields of mathematics such as analysis (e.g. the existence of a subset of $\mathbb{R}$ which is not Lebesgue-measurable; see [3, Proposition 9.32]), algebra (the main theorem in this paper), topology (Tychonoff's theorem; see [7]), and geometry (Banach-Tarski paradox; see [7]). While some results that follow from the Axiom of Choice are deceptively simple (e.g. a countable union of countable sets is countable; see [5, Theorem 6Q]), others seem to defy intuition. For example, the Axiom of Choice implies that every set can be well-ordered [5]. It is hard (if not impossible) to think of an explicit ordering of $\mathbb{R}$, for example, that results in a well-ordering of $\mathbb{R}$. The Banach-Tarski paradox in geometry is another paradoxical result that follows from the Axiom of Choice.

An important result by Cohen in 1963 [4] established the independence of the Axiom of Choice from the other standard Zermelo-Fraenkel axioms of set theory. For a more complete list of results derived from the Axiom of Choice and a detailed review of the history of the Axiom of Choice, see [7].

In this exposition, we focus on a statement in ring theory that is equivalent to the Axiom of Choice:

Main Theorem. *In Zermelo-Fraenkel set theory, the statement "every commutative ring with unity has a maximal ideal" is equivalent to the Axiom of Choice*

The structure of the paper is as follows. In Section 2 we provide necessary definitions. In Section 3 we provide a proof of the Main Theorem. In Section 4 we provide some examples.

Acknowledgements. We thank Professor Madeline Brandt for helpful suggestions.

2. Background

We assume basic familiarity with Zermelo-Fraenkel set theory and abstract algebra at the level of MATH 1530. For further set theory background described below, we generally follow [5]. For algebra background beyond the scope of MATH 1530, we generally follow either [1] or [9].

We first provide some additional set theory background.

Remark 2.1 (Set theory notation)**.** Let A be a set.

(1) $\mathcal{P}(A)$ denotes the powerset of A.

(2) Let $X \subseteq A$. X^C or $A \setminus X$ denotes the complement of X in A.
(3) Let f be a function. $\mathrm{dom} f$ denotes the domain of f.

Definition 2.2. Let R be a relation and A a set. We define the following:
(1) R is *reflexive on* A if for all $x \in A$, $(x, x) \in R$.
(2) R is *transitive* if for all x, y, z, $((x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R)$.
(3) R is *irreflexive* if for all $x, (x, x) \notin R$.

Definition 2.3 (Partial ordering)**.** A relation R is a *partial ordering relation* if:
(1) R is a transitive relation, and
(2) R is irreflexive.

Definition 2.4 (Linear ordering)**.** A relation R is a *linear ordering* on a set A if:
(1) R is a transitive relation, and
(2) R *satisfies trichotomy on* A: for all $x, y \in A$, exactly one of the following is true: $(x, y) \in R, x = y, (y, x) \in R$.

Definition 2.5 (Chain)**.** A set B is a *chain* if for all $C, D \in B$, $C \subseteq B$ or $D \subseteq C$.

Definition 2.6 (Closed under chain union)**.** A set A is *closed under chain union* if whenever $B \subseteq A$ and B is a chain, then $\bigcup B \in A$.

Definition 2.7 (Maximal element relative to inclusion)**.** Let A be a set and $M \in A$. M is *a maximal element of A relative to inclusion* if it does not have a proper superset in A.

Definition 2.8 (Structure)**.** A *structure* is an ordered pair (A, R) consisting of a set A and a binary relation R on A. If R is a partial (linear) ordering on A, then (A, R) is a *partially* (or *linearly*) *ordered structure*.

Definition 2.9. (Tree, branch) A *tree* is a partially ordered structure $(T, <)$ such that for every $t_i \in T$, the set $\hat{t_i} = \{t_j \in T \mid t_j < t_i\}$ is linearly ordered.

A *branch* in a tree $(T, <)$, is a maximal linearly ordered subset.

Definition 2.10 (Comparability)**.** Let (A, R) be an ordered structure. Two elements $r, t \in A$ are *comparable* if either $(r, t) \in R$ or $(t, r) \in R$.

Definition 2.11 (Initial segment)**.** Let $<$ be some ordering (partial or linear) on a set A and $r \in A$. The *initial segment up to* r is the set $\{x \in A \mid x < r\}$.

Axiom 2.12 (Axiom of Choice)**.** There are many equivalent formulations of the Axiom of Choice. We list several popular ones.

The following are equivalent (see [5, Theorem 6M], [8, Lemma 1] for proofs of equivalence):
(1) *Choice function axiom.* For any set A, there is a function F (a "choice function for A") such that $\mathrm{dom} F = \mathcal{P}(A) - \varnothing$ such that for every $\varnothing \neq B \subseteq A, F(B) \in B$.
(2) *Choice set axiom.* Let $\mathcal{A}$ be a set of sets such that (a) each member of $\mathcal{A}$ is nonempty and (b) if $a, b \in \mathcal{A}$ then $a \cap b = \varnothing$. Then there is a set C containing exactly one element from each $B \in \mathcal{A}$. (In other words, $B \in A \implies$ there exists $x \in B$ such that $C \cap B = \{x\}$).
(3) *Zorn's Lemma.* If A is a set closed under chain union, then there is some $M \in A$ that is maximal with respect to inclusion.
(4) *Subfunction axiom.* For any relation R, there is a function $F \subseteq R$ with $\mathrm{dom} F = \mathrm{dom} R$.

(5) *Tree*. Every tree has a branch.

In the proof of the main theorem, we will use Zorn's Lemma and *Tree*.

We also provide additional background in ring theory.

Remark 2.13 (Ring theory notation)**.** Let R, S be rings.

(1) If R and S are isomorphic, we denote $R \cong S$.
(2) Let $X = \{x_1, \ldots, x_n\} \in R$. We denote the ideal generated by X in R as $\langle X \rangle = \{a_1x_1 + \cdots + a_nx_n \mid a_i \in R\}$. Note that some authors (including key papers we follow throughout this exposition) denote $\langle X \rangle$ as XR.

Definition 2.14 (Polynomial rings over several variables)**.** Let R be a ring and $X = \{x_1, \ldots, x_n\}$ be a set. A *monomial* m_i over X is a product of $x_1, \ldots, x_n$ of the form: $m_i = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ where the exponents $i_\alpha (1 \leq \alpha \leq n)$ are non-negative integers. Using vector notation, denote $i = (i_1, \ldots, i_n)$ and $x = (x_1, \ldots, x_n)$. Then we can denote the monomial as $m_i = x^i$ symbolically.

A *polynomial* in the variables $x_1, \ldots, x_n$ with coefficients in R (denoted $R[X]$) is a linear combination of finitely many monomials. Using vector notation, each $f(x) \in R[X]$ can be written as $f(x) = \sum a_i x^i$, where the sum runs over a finite number of $n-$tuples $(i_1, \ldots, i_n)$ and $a_i = a_{(i_1,\ldots,i_n)} \in R$.

Definition 2.15 (Localization of rings)**.** Let R be a ring with unity 1.

(1) A subset $S \subseteq R$ is *multiplicatively closed* if $1 \in S$ and for all $a, b \in S$, $ab \in S$.
(2) Let $S \subseteq R$ be a multiplicatively closed set. For $a, s, a', s' \in R$, define a relation T on $R \times S$ such that $((a, s), (a', s')) \in T \iff$ there is an element $u \in S$ such that $u(as' - a's) = 0$. T is an equivalence relation and we denote the equivalence class of a pair $(a, s) \in R \times S$ as $\frac{a}{s}$. ($\frac{a}{s} = [(a, s)]_T$ using notation from class).
(3) The set of all equivalence classes:

$$S^{-1}R := \left\{ \frac{a}{s} \mid a \in R, s \in S \right\}$$

is the *localization of* R *at* S. $S^{-1}R$ is a ring with addition $(+)$ and multiplication $(\cdot)$ defined as follows:

$$\frac{a}{s} + \frac{a'}{s'} := \frac{as' + a's}{ss'},$$
$$\frac{a}{s} \cdot \frac{a'}{s'} := \frac{aa'}{ss'}.$$

Proposition 2.16. *Let R be a commutative ring with unity and M a maximal ideal in R. If $c \in M$, then c is not a unit.*

3. Main Theorem

Main Theorem. *The statement "every commutative ring with unity has a maximal ideal" is equivalent to the Axiom of Choice.*

The proof is divided into two propositions: 3.1 and 3.2, which each correspond to one direction of the equivalence statement. Each proposition is further divided into several lemmas and claims.

Proposition 3.1. *Zorn's Lemma implies that every commutative ring with unity has a maximal ideal.*

We generally follow the proof given in [10, Theorem 7].

Proof. Let R be a commutative ring with unity 1, and $I \subset R$ be a proper ideal. We argue that I is contained in a maximal ideal M. Consider the set

$$P = \{A \in R \mid A \text{ is a proper ideal of } R \wedge I \subseteq A\}.$$

To prove the proposition, it suffices to show that P has a maximal element. We use Zorn's Lemma.
Claim: P is closed under chain union.
Proof: Let L be a totally ordered subset of P. Let $M_L = \bigcup_{J \in L} J$. We argue that $M_L \in P$ by showing: (1) M_L is a proper subset of R (i.e. $M_L \neq P$), (2) M_L is an ideal, and (3) M_L contains I.

(1) We argue $1 \notin M_L$ by contradiction. Suppose $1 \in M_L$. Then there exists a $J_1 \in L$ such that $1 \in J_1$ by the definition of union. This implies that $J_1 = R$ so J_1 is not a proper ideal, which is a contradiction because $J_1 \in P$ and P is the set of proper ideals of R. Therefore, $1 \notin M_L$ so M_L is a proper subset of R.

(2) We first show M_L is a subring of R. Let $a, b \in M_L$. Then there exists $J_a, J_b \in L$ such that $a \in J_a, b \in J_b$. L is a totally ordered susbet of P so $J_a \subseteq J_b$ or $J_b \subseteq J_a$. Without loss of generality, assume $J_b \subseteq J_a$. Then $a, b \in J_a$ and J_a is an ideal so $a - b \in J_a \subseteq M_L$ and $ab \in J_a \subseteq M_L$. Therefore, $a - b, ab \in M_L$ so M_L is a subring of R.

Next, assume $r \in R, a \in M_L$. $a \in M_L$ so there exists $J_a \in L$ such that $a \in J_a$. J_a is an ideal so $ra \in J_a \subseteq M_L$. Therefore, M_L is an ideal.

(3) For any $J \in L$, $I \subseteq J \subseteq M_L$ so $I \subseteq M_L$. From (1)-(3), $M_L \in P$, so P is closed under chain union. ■

P is closed under chain union, so by Zorn's Lemma, there exists a set $M \in P$ that is maximal with respect to inclusion. Such a set M is a maximal ideal of R. □

Proposition 3.2. *If every commutative ring with unity contains a maximal ideal, then this implies the Axiom of Choice.*

We follow the proof given in [8]. For an alternative proof, see [2].

To prove the proposition, we first construct a ring. Let $(T, <)$ be a tree. Construct a ring R as follows. Let $\mathbb{Q}$ be the field of rationals. Let $\mathbb{Q}[T]$ be the polynomial ring with elements $t_i \in T$ as indeterminates. For $G = \{t_1, \ldots, t_k\} \subseteq T$, let $\langle G \rangle = \{a_1 t_1 + \cdots a_k t_k \mid a_i \in \mathbb{Q}[T]\} = \{q_1 m_1 t_1 + \cdots + q_k m_k t_k \mid q_i \in \mathbb{Q}, m_i \text{ are monomials over } T\}$ denote the ideal generated by G in $\mathbb{Q}[T]$.

Claim: For $G \subseteq T$, $\langle G \rangle$ is a prime ideal in $\mathbb{Q}[T]$.
Proof: Let $G \subseteq T$. It suffices to show $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain. Using the First Isomorphism Theorem for rings, we argue $\mathbb{Q}[T]/\langle G \rangle \cong \mathbb{Q}$, which shows that $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain because $\mathbb{Q}$ is an integral domain.

Define $\psi : \mathbb{Q}[T] \to \mathbb{Q}$ such that $\psi(f(t)) = f(0)$ for $f(t) \in \mathbb{Q}[T]$. We first show ψ is well-defined. Assume $f(t) = g(t)$ for $f(t), g(t) \in \mathbb{Q}[T]$. Using vector notation, denote $f(t) = a_m t^m + \cdots + a_1 t + a_0$, $g(t) = b_m t^m + \cdots + b_1 t + b_0$. $f(t) = g(t)$ by assumption so $a_i = b_i$ for all i. $\psi(f(t)) = f(0) = a_0$ and $\psi(g(t)) = g(0) = b_0$ but $a_0 = b_0$ so $\psi(f(t)) = \psi(g(t))$. Therefore, ψ is well-defined.

We next argue that ψ is a homomorphism. Let $f(t), g(t) \in \mathbb{Q}[T]$ with the same notation as before. $\psi(f(t)+g(t)) = (f+g)(0) = a_0 + b_0 = \psi(f(t)) + \psi(g(t))$. Furthermore, if we let $r \in \mathbb{Q}$, then $\psi(r \cdot f(t)) = r \cdot f(0) = ra_0 = r\psi(f(t))$, so ψ is a homomorphism.

We next argue that ψ is surjective. Let $q \in \mathbb{Q}$. Then for $f(t) = a_m t^m + \cdots + a_1 t + a_0 \in \mathbb{Q}[T]$, by choosing $a_0 = q$, there exists $f(t) \in \mathbb{Q}[T]$ such that $\psi(f(t)) = q$. Therefore, ψ is surjective.

ψ is a homomorphism, so by the First Isomorphism Theorem for rings, $\phi : \mathbb{Q}[T]/\ker\psi \to \psi(\mathbb{Q}[T])$ is an isomorphism. We have shown ψ is surjective so $\psi(\mathbb{Q}[T]) = \mathbb{Q}$, and therefore, $\mathbb{Q}[T]/\ker\psi \cong \mathbb{Q}$.

We next show that $\ker\psi = \langle G \rangle$. Let $f(t) \in \ker\psi$. Then $f(0) = 0$. This implies that $f(t) \in \langle G \rangle$.

Let $f(t) \in \langle G \rangle$. $f(t) = a_1 t_1 + \cdots + a_k t_k = q_1 m_1 t_1 + \cdots + q_k m_k t_k$ for $a_i \in \mathbb{Q}[T]$ and $q_i \in \mathbb{Q}$, m_i monomials over T. If we let $l_i = m_i t_i$, then each monomial $l_i \neq 1$ so $f(0) = 0$. Therefore, $f(t) \in \ker\psi$.

Therefore, $\mathbb{Q}[T]/\langle G \rangle \cong \mathbb{Q}$. This shows that $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain, and thus $\langle G \rangle$ is a prime ideal.

■

Let $L \subseteq \mathcal{P}(T)$ be the set of linearly ordered subsets of T. Define

$$S = \mathbb{Q}[T] \setminus \bigcup_{G \in L} \langle G \rangle = \left(\bigcup_{G \in L} \langle G \rangle \right)^C.$$

Let $H = \bigcup_{G \in L} \langle G \rangle$.

Claim: $H = \bigcup_{G \in L} \langle G \rangle$ is a prime ideal.

Proof: Let $a, b \in \mathbb{Q}[T]$ and $ab \in H$. Then there exists $G_i \in L$ such that $ab \in \langle G_i \rangle$. $\langle G_i \rangle$ is a prime ideal so $a \in \langle G_i \rangle$ or $b \in \langle G_i \rangle$. This implies that $a \in H$ or $b \in H$ so $H = \bigcup_{G \in L} \langle G \rangle$ is a prime ideal.

■

Claim: S is multiplicatively closed.

Proof: We first show that $1 \in S$ by contradiction. Suppose $1 \notin S$. Then $1 \in H$ by definition of S. This implies that $H = \mathbb{Q}[T]$, which is a contradiction because prime ideals are *proper* ideals by definition.

Next, let $a, b \in S$. We argue $ab \in S$ by contradiction. Suppose $ab \notin S$. This implies $ab \in H$, which implies $a \in H$ or $b \in H$ because H is a prime ideal. However, $a \in H$ implies $a \notin S$ which is a contradiction. Therefore, $ab \in S$ and S is multiplicatively closed.

■

S is multiplicatively closed so we can define the localization of $\mathbb{Q}[T]$ at S as follows:

$$R = S^{-1}\mathbb{Q}[T] = \left\{ \frac{x}{s} \mid x \in \mathbb{Q}[T], s \in S \right\}.$$

Claim: An element $c = \frac{x}{s} \in R$ is a unit if and only if for all $t_i \in T$, $x \notin \langle \hat{t_i} \rangle$.

Proof: ($\Longrightarrow$) By contradiction. Let $c = \frac{x}{s} \in R$ be a unit. Suppose there exists $t_i \in T$ such that $x \in \langle \hat{t_i} \rangle$. By assumption, $c = \frac{x}{s}$ is a unit so in particular, $x \in \mathbb{Q}[T]$ has a multiplicative inverse, $x^{-1} \in Q[T]$. $\langle \hat{t_i} \rangle$ is an ideal, so $1 = x^{-1}x \in \langle \hat{t_i} \rangle$, which implies that $\langle \hat{t_i} \rangle = R$, which is a contradiction because $\hat{t_i} = \{t_1, \ldots, t_{i-1}\}$ and $\langle \hat{t_i} \rangle$ is a proper ideal.

($\Longleftarrow$) Let $c = \frac{x}{s} \in R$. Assume for all $t_i \in T$, $x \notin \langle \hat{t_i} \rangle$. Then there exits some $y \in R$ such that $yx \notin \langle \hat{t_i} \rangle$. In particular, there is a $y \in R$ such that $yx = 1$. Therefore, c is a unit.

■

Suppose R has a maximal ideal M. Let $c = \frac{x}{s} \in M$. $x \in \mathbb{Q}[T]$ so x can be written as $x = q_1 m_1 + \ldots + q_n m_n$ for $q_i \in \mathbb{Q}$ and $m_1, \ldots, m_n$ distinct monomials over T. By Proposition 2.16, c is not a unit so by the previous claim, there exists $t_i \in T$ such that $x \in \langle \hat{t_i} \rangle$. This implies that there exists at least one finite linearly ordered set $\hat{t_i} \subseteq T$ by the definition of a tree. Denote $A = \hat{t_i}$. A is not necessarily unique, but there are at most finitely many choices for it, say $A_1, \ldots, A_k$. Let $E(c) = \max_{1 \le i \le k} A_i$. We observe that if $t_i \in E(c)$ then $c \in \langle \hat{t_i} \rangle$. If $c = 0$, then A is empty and hence $E(c)$ is also empty. If $c \neq 0$ and d is any other element in M which involves the monomials $m_1, \ldots, m_n$ (same monomials invovled in x), then for every $t_j \in E(d)$, there is a $t_i \in E(c)$ such that $t_i < t_j$. Define D as the set of $t \in T$ such that for every non-zero $c \in M$, there is a $r \in E(c)$ which is comparable with t.

Lemma 3.3. *$D \subseteq M$.*

Proof. By contradiction. Let $t \in D$. Suppose $t \notin M$. M is maximal in R, so by a similar argument given in class, there exists $a \in R$, $c \in M$ such that $at + c = 1$. The singleton $\{t\}$ is linearly ordered so $\{t\} \in L$, which implies that t is not a unit. This implies $c \neq 0$. By definition of D, there is a $r \in E(c)$ which is comparable with t, so $c \in \langle \hat{r} \rangle$. If $r < t$, then $\langle \hat{r} \rangle \subseteq \langle \hat{t} \rangle$ so $c, t \in \langle \hat{t} \rangle$. Because $\langle \hat{t} \rangle$ is an ideal, $1 = at + c \in \langle \hat{t} \rangle$, which is a contradiction. If $t < r$, then similarly, $c, t \in \langle \hat{r} \rangle$, so $1 \in \langle \hat{r} \rangle$, which is also a contradiction. Therefore, $D \subseteq M$. □

As an immediate corollary, we have the following.

Corollary 3.4. *$\langle D \rangle \subseteq M$.*

Lemma 3.5. *The set D is a linearly ordered initial segment of T.*

We omit the proof of this lemma and defer to [8].

Lemma 3.6. *$M \subseteq \langle D \rangle$.*

Proof. By contradiction. Suppose $\langle D \rangle \subset M$, where $\subset$ is a proper inclusion. Then there exists some $c = \frac{x}{s} \in (\langle D \rangle)^C = M \setminus \langle D \rangle$. Without loss of generality, we can assume c is a non-zero element of $\mathbb{Q}[T]$.

We first claim that for all $t \in D$, $t \notin E(c)$. We argue by contradiction. Suppose there exists $t \in D$ such that $t \in E(c)$. This implies $c \in \langle \hat{t} \rangle$. By Lemma 3.5, $\langle \hat{t} \rangle \subseteq \langle D \rangle$, so $c \in \langle D \rangle$, which is a contradiction because of the initial choice of c as $c \notin \langle D \rangle$.

Denote $E(c) = \{t_1, \ldots, t_k\}$. None of these elements $t_1, \ldots, t_k$ are in D so by the definition of D, for each $t_i (1 \le i \le k)$ there exists a non-zero element $b_i \in M$ such that for all $r \in E(b_i)$, r is incomparable with t_i (by negating the definition of D). Without further loss of generality, we can assume $b_i \in \mathbb{Q}[T]$. $\mathbb{Q}$ is an infinite field, so we can choose elements $q_1, \ldots, q_k \in \mathbb{Q}$ such that for $x = c + q_1 b_1 + \cdots + q_k b_k$, no monomial which occurs with non-zero coefficients in c or some b_i vanishes in x.

Let $w \in E(x)$. Then by the choice of q_i's, there is some t_j such that $t_j < w$. But also, there is some $r \in E(b_j)$ such that $r < w$ so t_j and r are comparable, which contradicts the choice of b_j. Therefore, the initial assumption is false and $M \subseteq \langle D \rangle$. □

We can now prove the main proposition.

Proposition 3.7. *If every commutative ring with unity contains a maximal ideal, then this implies the Axiom of Choice.*

Proof. Assume every commutative ring with unity contains a maximal ideal. We argue that every tree has a branch (see Axiom 2.12). Let $(T, <)$ be a tree and let $R = S^{-1}\mathbb{Q}[T]$. R is a commutative ring with unity so by assumption, it contains a maximal ideal M. By Lemmas 3.3 and 3.6, $M = \langle D \rangle$ for D defined previously.

We argue by contradiction that D is a branch of T. If D is not a branch of T, then by Lemma 3.5, there is a $t_i \in T$ such that $t > r$ for all $r \in D$. Then $M = \langle D \rangle \subsetneq \langle \hat{t_i} \rangle$. M is maximal so this implies that $\langle \hat{t_i} \rangle = R$, which is a contradiction because of the construction of $\langle \hat{t_i} \rangle$. Therefore, $(T, <)$ has a branch D and the proof is complete. $\square$

4. Examples of Rings Without Maximal Ideals

By the main theorem, every commutative ring with unity has a maximal ideal. We provide some examples of rings without maximal ideals. (These examples are not original; see [6]).

Example 4.1. Let F be a field and $F[x]$ denote the ring of polynomials with coefficients in F. Let $F(x)$ denote the field of quotients of $F[x]$. Let

$$S(F) = \left\{ h(x) = \frac{f(x)}{g(x)} \in F(x) \mid f(x), g(x) \in F[x] \text{ and } g(0) \neq 0 \right\}.$$

$S(F)$ has a maximal ideal $R(F) = \langle x \rangle$ in $S(F)$. If F has characteristic zero, then $R(F)$ has no maximal ideals (see [6, Main Theorem & Corollary] for proof).

Example 4.2. Let $(\mathbb{Q}, +)$ be the additive group over $\mathbb{Q}$ with standard addition. If we further define multiplication as $x \cdot y = 0$ for all $x, y \in \mathbb{Q}$, then the ring $R = (\mathbb{Q}, +, \cdot)$ has no maximal ideals because it has no maximal subgroups. Because of the definition of multiplication in R, this ring does not have a unity.

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