THE AXIOM OF CHOICE AND MAXIMAL IDEALS IN RING THEORY

ZEN TAMURA

ABSTRACT. In this expository paper, we show that the Axiom of Choice is equivalent to the statement "every commutative ring with unity has a maximal ideal".

1. INTRODUCTION

Since Zermelo first formulated the Axiom of Choice in 1904, the Axiom of Choice has had a large impact on modern mathematics. The Axiom of Choice is used to justify results in various fields of mathematics such as analysis (e.g. the existence of a subset of $\mathbb{R}$ which is not Lebesgue-measurable; see [3, Proposition 9.32]), algebra (the main theorem in this paper), topology (Tychonoff's theorem; see [7]), and geometry (Banach-Tarski paradox; see [7]). While some results that follow from the Axiom of Choice are deceptively simple (e.g. a countable union of countable sets is countable; see [5, Theorem 6Q]), others seem to defy intuition. For example, the Axiom of Choice implies that every set can be well-ordered [5]. It is hard (if not impossible) to think of an explicit ordering of $\mathbb{R}$, for example, that results in a well-ordering of $\mathbb{R}$. The Banach-Tarski paradox in geometry is another paradoxical result that follows from the Axiom of Choice.

An important result by Cohen in 1963 [4] established the independence of the Axiom of Choice from the other standard Zermelo-Fraenkel axioms of set theory. For a more complete list of results derived from the Axiom of Choice and a detailed review of the history of the Axiom of Choice, see [7].

In this exposition, we focus on a statement in ring theory that is equivalent to the Axiom of Choice:

Main Theorem. *In Zermelo-Fraenkel set theory, the statement "every commutative ring with unity has a maximal ideal" is equivalent to the Axiom of Choice*

The structure of the paper is as follows. In Section 2 we provide necessary definitions. In Section 3 we provide a proof of the Main Theorem. In Section 4 we provide some examples.

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2. BACKGROUND

We assume basic familiarity with Zermelo-Fraenkel set theory and abstract algebra at the level of MATH 1530. For further set theory background described below, we generally follow [5]. For algebra background beyond the scope of MATH 1530, we generally follow either [1] or [9].

We first provide some additional set theory background.

Remark 2.1 (Set theory notation)**.** Let A be a set.

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(1) $\mathcal{P}(A)$ denotes the powerset of A.
(2) Let $X \subseteq A$. X^C or $A \setminus X$ denotes the complement of X in A.
(3) Let f be a function. $\mathrm{dom} f$ denotes the domain of f.

Definition 2.2. Let R be a relation and A a set. We define the following:
(1) R is *reflexive on A* if for all $x \in A$, $(x, x) \in R$.
(2) R is *transitive* if for all x, y, z, $((x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R)$.
(3) R is *irreflexive* if for all $x, (x, x) \notin R$.

Definition 2.3 (Partial ordering)**.** A relation R is a *partial ordering relation* if:
(1) R is a transitive relation, and
(2) R is irreflexive.

Definition 2.4 (Linear ordering)**.** A relation R is a *linear ordering* on a set A if:
(1) R is a transitive relation, and
(2) R *satisfies trichotomy on A*: for all $x, y \in A$, exactly one of the following is true: $(x, y) \in R, x = y, (y, x) \in R$.

Definition 2.5 (Chain)**.** A set B is a *chain* if for all $C, D \in B$, $C \subseteq B$ or $D \subseteq C$.

Definition 2.6 (Closed under chain union)**.** A set A is *closed under chain union* if whenever $B \subseteq A$ and B is a chain, then $\bigcup B \in A$.

Definition 2.7 (Maximal element relative to inclusion)**.** Let A be a set and $M \in A$. M is *a maximal element of A relative to inclusion* if it does not have a proper superset in A.

Definition 2.8 (Structure)**.** A *structure* is an ordered pair (A, R) consisting of a set A and a binary relation R on A. If R is a partial (linear) ordering on A, then (A, R) is a *partially* (or *linearly*) *ordered structure*.

Definition 2.9. (Tree, branch) A *tree* is a partially ordered structure $(T, <)$ such that for every $t_i \in T$, the set $\hat{t_i} = \{t_j \in T \mid t_j < t_i\}$ is linearly ordered.

A *branch* in a tree $(T, <)$, is a maximal linearly ordered subset.

Definition 2.10 (Comparability)**.** Let (A, R) be an ordered structure. Two elements $r, t \in A$ are *comparable* if either $(r, t) \in R$ or $(t, r) \in R$.

Definition 2.11 (Initial segment)**.** Let $<$ be some ordering (partial or linear) on a set A and $r \in A$. The *initial segment up to r* is the set $\{x \in A \mid x < r\}$.

Axiom 2.12 (Axiom of Choice)**.** There are many equivalent formulations of the Axiom of Choice. We list several popular ones.

The following are equivalent (see [5, Theorem 6M], [8, Lemma 1] for proofs of equivalence):
(1) *Choice function axiom*. For any set A, there is a function F (a "choice function for A") such that $\mathrm{dom} F = \mathcal{P}(A) - \varnothing$ such that for every $\varnothing \neq B \subseteq A, F(B) \in B$.
(2) *Choice set axiom*. Let $\mathcal{A}$ be a set of sets such that (a) each member of $\mathcal{A}$ is nonempty and (b) if $a, b \in \mathcal{A}$ then $a \cap b = \varnothing$. Then there is a set C containing exactly one element from each $B \in \mathcal{A}$. (In other words, $B \in A \implies$ there exists $x \in B$ such that $C \cap B = \{x\}$).
(3) *Zorn's Lemma*. If A is a set closed under chain union, then there is some $M \in A$ that is maximal with respect to inclusion.

(4) *Subfunction axiom*. For any relation R, there is a function $F \subseteq R$ with $\mathrm{dom}F = \mathrm{dom}R$.
(5) *Tree*. Every tree has a branch.

In the proof of the main theorem, we will use Zorn's Lemma and *Tree*.

We also provide additional background in ring theory.

Remark 2.13 (Ring theory notation)**.** Let R, S be rings.

(1) If R and S are isomorphic, we denote $R \cong S$.
(2) Let $X = \{x_1, \ldots, x_n\} \in R$. We denote the ideal generated by X in R as $\langle X \rangle = \{a_1x_1 + \cdots + a_nx_n \mid a_i \in R\}$. Note that some authors (including key papers we follow throughout this exposition) denote $\langle X \rangle$ as XR.

Definition 2.14 (Polynomial rings over several variables)**.** Let R be a ring and $X = \{x_1, \ldots, x_n\}$ be a set. A *monomial* m_i over X is a product of $x_1, \ldots, x_n$ of the form: $m_i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ where the exponents $i_\alpha (1 \leq \alpha \leq n)$ are non-negative integers. Using vector notation, denote $i = (i_1, \ldots, i_n)$ and $x = (x_1, \ldots, x_n)$. Then we can denote the monomial as $m_i = x^i$ symbolically.

A *polynomial* in the variables $x_1, \ldots, x_n$ with coefficients in R (denoted $R[X]$) is a linear combination of finitely many monomials. Using vector notation, each $f(x) \in R[X]$ can be written as $f(x) = \sum a_i x^i$, where the sum runs over a finite number of $n-$tuples $(i_1, \ldots, i_n)$ and $a_i = a_{(i_1,\ldots,i_n)} \in R$.

Definition 2.15 (Localization of rings)**.** Let R be a ring with unity 1.

(1) A subset $S \subseteq R$ is *multiplicatively closed* if $1 \in S$ and for all $a, b \in S$, $ab \in S$.
(2) Let $S \subseteq R$ be a multiplicatively closed set. For $a, s, a', s' \in R$, define a relation T on $R \times S$ such that $((a, s), (a', s')) \in T \iff$ there is an element $u \in S$ such that $u(as' - a's) = 0$. T is an equivalence relation and we denote the equivalence class of a pair $(a, s) \in R \times S$ as $\frac{a}{s}$. ($\frac{a}{s} = [(a, s)]_T$ using notation from class).
(3) The set of all equivalence classes:

$$S^{-1}R := \left\{ \frac{a}{s} \mid a \in R, s \in S \right\}$$

is the *localization of* R *at* S. $S^{-1}R$ is a ring with addition $(+)$ and multiplication $(\cdot)$ defined as follows:

$$\frac{a}{s} + \frac{a'}{s'} := \frac{as' + a's}{ss'},$$
$$\frac{a}{s} \cdot \frac{a'}{s'} := \frac{aa'}{ss'}.$$

Proposition 2.16. *Let R be a commutative ring with unity and M a maximal ideal in R. If $c \in M$, then c is not a unit.*

3. Main Theorem

Main Theorem. *The statement "every commutative ring with unity has a maximal ideal" is equivalent to the Axiom of Choice.*

The proof is divided into two propositions: 3.1 and 3.2, which each correspond to one direction of the equivalence statement. Each proposition is further divided into several lemmas and claims.

Proposition 3.1. *Zorn's Lemma implies that every commutative ring with unity has a maximal ideal.*

We generally follow the proof given in [10, Theorem 7].

Proof. Let R be a commutative ring with unity 1, and $I \subset R$ be a proper ideal. We argue that I is contained in a maximal ideal M. Consider the set

$$P = \{A \in R \mid A \text{ is a proper ideal of } R \wedge I \subseteq A\}.$$

To prove the proposition, it suffices to show that P has a maximal element. We use Zorn's Lemma.

Claim: P is closed under chain union.

Proof: Let L be a totally ordered subset of P. Let $M_L = \bigcup_{J \in L} J$. We argue that $M_L \in P$ by showing: (1) M_L is a proper subset of R (i.e. $M_L \neq P$), (2) M_L is an ideal, and (3) M_L contains I.

(1) We argue $1 \notin M_L$ by contradiction. Suppose $1 \in M_L$. Then there exists a $J_1 \in L$ such that $1 \in J_1$ by the definition of union. This implies that $J_1 = R$ so J_1 is not a proper ideal, which is a contradiction because $J_1 \in P$ and P is the set of proper ideals of R. Therefore, $1 \notin M_L$ so M_L is a proper subset of R.

(2) We first show M_L is a subring of R. Let $a, b \in M_L$. Then there exists $J_a, J_b \in L$ such that $a \in J_a, b \in J_b$. L is a totally ordered susbet of P so $J_a \subseteq J_b$ or $J_b \subseteq J_a$. Without loss of generality, assume $J_b \subseteq J_a$. Then $a, b \in J_a$ and J_a is an ideal so $a - b \in J_a \subseteq M_L$ and $ab \in J_a \subseteq M_L$. Therefore, $a - b, ab \in M_L$ so M_L is a subring of R.

Next, assume $r \in R, a \in M_L$. $a \in M_L$ so there exists $J_a \in L$ such that $a \in J_a$. J_a is an ideal so $ra \in J_a \subseteq M_L$. Therefore, M_L is an ideal.

(3) For any $J \in L$, $I \subseteq J \subseteq M_L$ so $I \subseteq M_L$. From (1)-(3), $M_L \in P$, so P is closed under chain union. ■

P is closed under chain union, so by Zorn's Lemma, there exists a set $M \in P$ that is maximal with respect to inclusion. Such a set M is a maximal ideal of R. □

Proposition 3.2. *If every commutative ring with unity contains a maximal ideal, then this implies the Axiom of Choice.*

We follow the proof given in [8]. For an alternative proof, see [2].

To prove the proposition, we first construct a ring. Let $(T, <)$ be a tree. Construct a ring R as follows. Let $\mathbb{Q}$ be the field of rationals. Let $\mathbb{Q}[T]$ be the polynomial ring with elements $t_i \in T$ as indeterminates. For $G = \{t_1, \dots, t_k\} \subseteq T$, let $\langle G \rangle = \{a_1 t_1 + \cdots a_k t_k \mid a_i \in \mathbb{Q}[T]\} = \{q_1 m_1 t_1 + \cdots + q_k m_k t_k \mid q_i \in \mathbb{Q}, m_i \text{ are monomials over } T\}$ denote the ideal generated by G in $\mathbb{Q}[T]$.

Claim: For $G \subseteq T, \langle G \rangle$ is a prime ideal in $\mathbb{Q}[T]$.

Proof: Let $G \subseteq T$. It suffices to show $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain. Using the First Isomorphism Theorem for rings, we argue $\mathbb{Q}[T]/\langle G \rangle \cong \mathbb{Q}$, which shows that $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain because $\mathbb{Q}$ is an integral domain.

Define $\psi : \mathbb{Q}[T] \to \mathbb{Q}$ such that $\psi(f(t)) = f(0)$ for $f(t) \in \mathbb{Q}[T]$. We first show ψ is well-defined. Assume $f(t) = g(t)$ for $f(t), g(t) \in \mathbb{Q}[T]$. Using vector notation, denote $f(t) = a_m t^m + \cdots + a_1 t + a_0$, $g(t) = b_m t^m + \cdots + b_1 t + b_0$. $f(t) = g(t)$ by assumption so $a_i = b_i$ for all i. $\psi(f(t)) = f(0) = a_0$ and $\psi(g(t)) = g(0) = b_0$ but $a_0 = b_0$ so $\psi(f(t)) = \psi(g(t))$. Therefore, ψ is well-defined.

We next argue that ψ is a homomorphism. Let $f(t), g(t) \in \mathbb{Q}[T]$ with the same notation as before. $\psi(f(t)+g(t)) = (f+g)(0) = a_0 + b_0 = \psi(f(t)) + \psi(g(t))$. Furthermore, if we let $r \in \mathbb{Q}$, then $\psi(r \cdot f(t)) = r \cdot f(0) = ra_0 = r\psi(f(t))$, so ψ is a homomorphism.

We next argue that ψ is surjective. Let $q \in \mathbb{Q}$. Then for $f(t) = a_m t^m + \cdots + a_1 t + a_0 \in \mathbb{Q}[T]$, by choosing $a_0 = q$, there exists $f(t) \in \mathbb{Q}[T]$ such that $\psi(f(t)) = q$. Therefore, ψ is surjective.

ψ is a homomorphism, so by the First Isomorphism Theorem for rings, $\phi : \mathbb{Q}[T]/\ker\psi \to \psi(\mathbb{Q}[T])$ is an isomorphism. We have shown ψ is surjective so $\psi(\mathbb{Q}[T]) = \mathbb{Q}$, and therefore, $\mathbb{Q}[T]/\ker\psi \cong \mathbb{Q}$.

We next show that $\ker\psi = \langle G \rangle$. Let $f(t) \in \ker\psi$. Then $f(0) = 0$. This implies that $f(t) \in \langle G \rangle$.

Let $f(t) \in \langle G \rangle$. $f(t) = a_1 t_1 + \cdots + a_k t_k = q_1 m_1 t_1 + \cdots + q_k m_k t_k$ for $a_i \in \mathbb{Q}[T]$ and $q_i \in \mathbb{Q}$, m_i monomials over T. If we let $l_i = m_i t_i$, then each monomial $l_i \neq 1$ so $f(0) = 0$. Therefore, $f(t) \in \ker\psi$.

Therefore, $\mathbb{Q}[T]/\langle G \rangle \cong \mathbb{Q}$. This shows that $\mathbb{Q}[T]/\langle G \rangle$ is an integral domain, and thus $\langle G \rangle$ is a prime ideal.

■

Let $L \subseteq \mathcal{P}(T)$ be the set of linearly ordered subsets of T. Define

$$S = \mathbb{Q}[T] \setminus \bigcup_{G \in L} \langle G \rangle = \left(\bigcup_{G \in L} \langle G \rangle \right)^C.$$

Let $H = \bigcup_{G \in L} \langle G \rangle$.

Claim: $H = \bigcup_{G \in L} \langle G \rangle$ is a prime ideal.

Proof: Let $a, b \in \mathbb{Q}[T]$ and $ab \in H$. Then there exists $G_i \in L$ such that $ab \in \langle G_i \rangle$. $\langle G_i \rangle$ is a prime ideal so $a \in \langle G_i \rangle$ or $b \in \langle G_i \rangle$. This implies that $a \in H$ or $b \in H$ so $H = \bigcup_{G \in L} \langle G \rangle$ is a prime ideal.

■

Claim: S is multiplicatively closed.

Proof: We first show that $1 \in S$ by contradiction. Suppose $1 \notin S$. Then $1 \in H$ by definition of S. This implies that $H = \mathbb{Q}[T]$, which is a contradiction because prime ideals are *proper* ideals by definition.

Next, let $a, b \in S$. We argue $ab \in S$ by contradiction. Suppose $ab \notin S$. This implies $ab \in H$, which implies $a \in H$ or $b \in H$ because H is a prime ideal. However, $a \in H$ implies $a \notin S$ which is a contradiction. Therefore, $ab \in S$ and S is multiplicatively closed.

■

S is multiplicatively closed so we can define the localization of $\mathbb{Q}[T]$ at S as follows:

$$R = S^{-1}\mathbb{Q}[T] = \left\{ \frac{x}{s} \mid x \in \mathbb{Q}[T], s \in S \right\}.$$

Claim: An element $c = \frac{x}{s} \in R$ is a unit if and only if for all $t_i \in T$, $x \notin \langle \hat{t_i} \rangle$.

Proof: ($\implies$) By contradiction. Let $c = \frac{x}{s} \in R$ be a unit. Suppose there exists $t_i \in T$ such that $x \in \langle \hat{t_i} \rangle$. By assumption, $c = \frac{x}{s}$ is a unit so in particular, $x \in \mathbb{Q}[T]$ has a multiplicative inverse, $x^{-1} \in Q[T]$. $\langle \hat{t_i} \rangle$ is an ideal, so $1 = x^{-1}x \in \langle \hat{t_i} \rangle$, which implies that $\langle \hat{t_i} \rangle = R$, which is a contradiction because $\hat{t_i} = \{t_1, \ldots, t_{i-1}\}$ and $\langle \hat{t_i} \rangle$ is a proper ideal.

($\impliedby$) Let $c = \frac{x}{s} \in R$. Assume for all $t_i \in T$, $x \notin \langle \hat{t_i} \rangle$. Then there exits some $y \in R$ such that $yx \notin \langle \hat{t_i} \rangle$. In particular, there is a $y \in R$ such that $yx = 1$. Therefore, c is a unit.

■

Suppose R has a maximal ideal M. Let $c = \frac{x}{s} \in M$. $x \in \mathbb{Q}[T]$ so x can be written as $x = q_1m_1 + \ldots + q_nm_n$ for $q_i \in \mathbb{Q}$ and $m_1, \ldots, m_n$ distinct monomials over T. By Proposition 2.16, c is not a unit so by the previous claim, there exists $t_i \in T$ such that $x \in \langle \hat{t_i} \rangle$. This implies that there exists at least one finite linearly ordered set $\hat{t_i} \subseteq T$ by the definition of a tree. Denote $A = \hat{t_i}$. A is not necessarily unique, but there are at most finitely many choices for it, say $A_1, \ldots, A_k$. Let $E(c) = \max_{1 \leq i \leq k} A_i$. We observe that if $t_i \in E(c)$ then $c \in \langle \hat{t_i} \rangle$. If $c = 0$, then A is empty and hence $E(c)$ is also empty. If $c \neq 0$ and d is any other element in M which involves the monomials $m_1, \ldots, m_n$ (same monomials invovled in x), then for every $t_j \in E(d)$, there is a $t_i \in E(c)$ such that $t_i < t_j$. Define D as the set of $t \in T$ such that for every non-zero $c \in M$, there is a $r \in E(c)$ which is comparable with t.

Lemma 3.3. $D \subseteq M$.

Proof. By contradiction. Let $t \in D$. Suppose $t \notin M$. M is maximal in R, so by a similar argument given in class, there exists $a \in R$, $c \in M$ such that $at + c = 1$. The singleton $\{t\}$ is linearly ordered so $\{t\} \in L$, which implies that t is not a unit. This implies $c \neq 0$. By definition of D, there is a $r \in E(c)$ which is comparable with t, so $c \in \langle \hat{r} \rangle$. If $r < t$, then $\langle \hat{r} \rangle \subseteq \langle \hat{t} \rangle$ so $c, t \in \langle \hat{t} \rangle$. Because $\langle \hat{t} \rangle$ is an ideal, $1 = at + c \in \langle \hat{t} \rangle$, which is a contradiction. If $t < r$, then similarly, $c, t \in \langle \hat{r} \rangle$, so $1 \in \langle \hat{r} \rangle$, which is also a contradiction. Therefore, $D \subseteq M$. $\square$

As an immediate corollary, we have the following.

Corollary 3.4. $\langle D \rangle \subseteq M$.

Lemma 3.5. *The set D is a linearly ordered initial segment of T.*

We omit the proof of this lemma and defer to [8].

Lemma 3.6. $M \subseteq \langle D \rangle$.

Proof. By contradiction. Suppose $\langle D \rangle \subset M$, where $\subset$ is a proper inclusion. Then there exists some $c = \frac{x}{s} \in (\langle D \rangle)^C = M \setminus \langle D \rangle$. Without loss of generality, we can assume c is a non-zero element of $\mathbb{Q}[T]$.

We first claim that for all $t \in D$, $t \notin E(c)$. We argue by contradiction. Suppose there exists $t \in D$ such that $t \in E(c)$. This implies $c \in \langle \hat{t} \rangle$. By Lemma 3.5, $\langle \hat{t} \rangle \subseteq \langle D \rangle$, so $c \in \langle D \rangle$, which is a contradiction because of the initial choice of c as $c \notin \langle D \rangle$.

Denote $E(c) = \{t_1, \ldots, t_k\}$. None of these elements $t_1, \ldots, t_k$ are in D so by the definition of D, for each $t_i (1 \leq i \leq k)$ there exists a non-zero element $b_i \in M$ such that for all $r \in E(b_i)$, r is incomparable with t_i (by negating the definition of D). Without further loss of generality, we can assume $b_i \in \mathbb{Q}[T]$. $\mathbb{Q}$ is an infinite field, so we can choose elements $q_1, \ldots, q_k \in \mathbb{Q}$ such that for $x = c + q_1b_1 + \cdots + q_kb_k$, no monomial which occurs with non-zero coefficients in c or some b_i vanishes in x.

Let $w \in E(x)$. Then by the choice of q_i's, there is some t_j such that $t_j < w$. But also, there is some $r \in E(b_j)$ such that $r < w$ so t_j and r are comparable, which contradicts the choice of b_j. Therefore, the initial assumption is false and $M \subseteq \langle D \rangle$. $\square$

We can now prove the main proposition.

Proposition 3.7. *If every commutative ring with unity contains a maximal ideal, then this implies the Axiom of Choice.*

Proof. Assume every commutative ring with unity contains a maximal ideal. We argue that every tree has a branch (see Axiom 2.12). Let $(T, <)$ be a tree and let $R = S^{-1}\mathbb{Q}[T]$. R is a commutative ring with unity so by assumption, it contains a maximal ideal M. By Lemmas 3.3 and 3.6, $M = \langle D \rangle$ for D defined previously.

We argue by contradiction that D is a branch of T. If D is not a branch of T, then by Lemma 3.5, there is a $t_i \in T$ such that $t > r$ for all $r \in D$. Then $M = \langle D \rangle \subsetneq \langle \hat{t_i} \rangle$. M is maximal so this implies that $\langle \hat{t_i} \rangle = R$, which is a contradiction because of the construction of $\langle \hat{t_i} \rangle$. Therefore, $(T, <)$ has a branch D and the proof is complete. $\square$

4. Examples of Rings Without Maximal Ideals

By the main theorem, every commutative ring with unity has a maximal ideal. We provide some examples of rings without maximal ideals. (These examples are not original; see [6]).

Example 4.1. Let F be a field and $F[x]$ denote the ring of polynomials with coefficients in F. Let $F(x)$ denote the field of quotients of $F[x]$. Let

$$S(F) = \left\{ h(x) = \frac{f(x)}{g(x)} \in F(x) \mid f(x), g(x) \in F[x] \text{ and } g(0) \neq 0 \right\}.$$

$S(F)$ has a maximal ideal $R(F) = \langle x \rangle$ in $S(F)$. If F has characteristic zero, then $R(F)$ has no maximal ideals (see [6, Main Theorem & Corollary] for proof).

Example 4.2. Let $(\mathbb{Q}, +)$ be the additive group over $\mathbb{Q}$ with standard addition. If we further define multiplication as $x \cdot y = 0$ for all $x, y \in \mathbb{Q}$, then the ring $R = (\mathbb{Q}, +, \cdot)$ has no maximal ideals because it has no maximal subgroups. Because of the definition of multiplication in R, this ring does not have a unity.

References

[1] M. Artin. *Algebra*. Pearson Education, 2011.

[2] B. Banaschewski. A new proof that "krull implies zorn". *Mathematical Logic Quarterly*, 40(4):478–480, 1994. Available at: https://doi.org/10.1002/malq.19940400405.

[3] A. Browder. *Mathematical Analysis: An Introduction*. Undergraduate Texts in Mathematics. Springer New York, 2012.

[4] P. J. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences*, 50(6):1143–1148, 1963.

[5] H. Enderton. *Elements of Set Theory*. Elsevier Science, 1977.

[6] M. Henriksen. A simple characterization of commutative rings without maximal ideals. *The American Mathematical Monthly*, 82(5):502–505, 1975. Available at: https://doi.org/10.1080/00029890.1975.11993868.

[7] H. Herrlich. *Axiom of Choice*. Axiom of Choice. Springer, 2006.

[8] W. Hodges. Krull implies zorn. *Journal of the London Mathematical Society*, s2-19(2):285–287, 1979. Available at: https://doi.org/10.1112/jlms/s2-19.2.285.

[9] S. Lang. *Algebra*. Springer, New York, NY, 2002.

[10] R. Woodroofe. Zorn's lemma and maximal ideals. Accessed at https://osebje.famnit.upr.si/~russ.woodroofe/wustl-notes/zorn.pdf on May 10, 2023.

Department of Mathematics, Brown University, Providence, RI 02912