Important Math Review Points

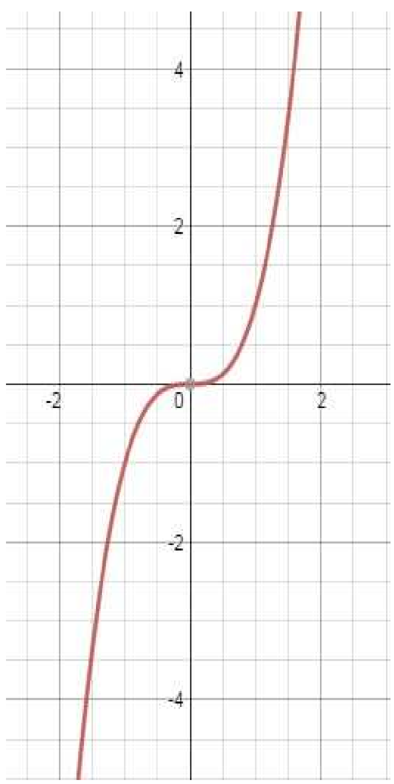
February 22, 2015

1 Increasing/Nondecreasing functions

A function is increasing if its graph climbs steadily upward. More precisely:

Defintion. A function f on the real line is *increasing* (*nondecresing*) if, whenever $x_1 < x_2$ ($x_1 \leq x_2$), $f(x_1) < f(x_2)$ ($f(x_1) \leq f(x_2)$).

Example: $f(x) = x^3$ is an increasing function.

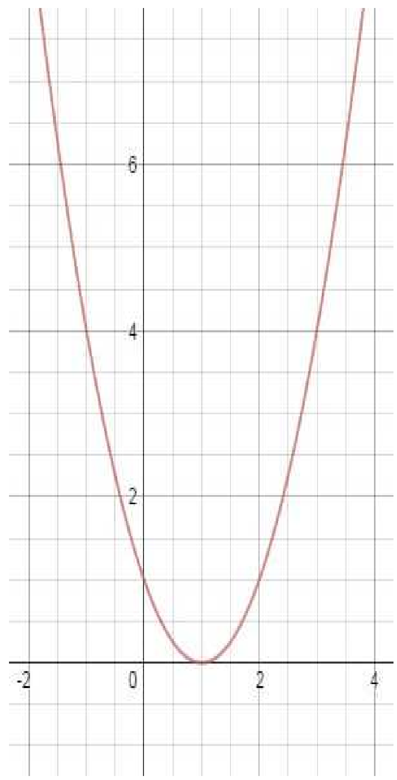


Question Is $f(x) = x^2$ increasing?

2 Eventually Nondecreasing Functions

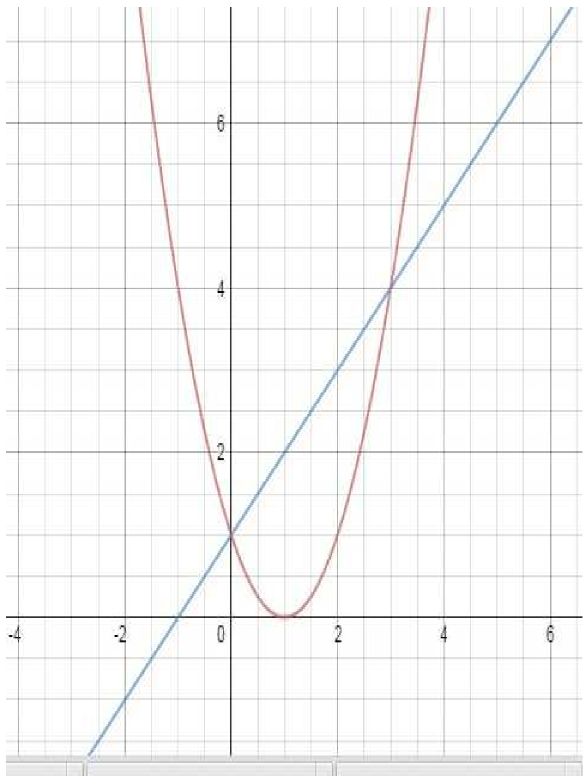
A function is eventually nondecreasing if for all values beyond a certain point on the x-axis, the graph steadily climbs. More precisely,
Definition. A function f is *eventually nondecreasing* if for some real number x_0, f is increasing on $[x_0, \infty)$. In other words, for some x_0 we have that whenever $x_0 \leq x_1 \leq x_2$, then $f(x_0) \leq f(x_1) \leq f(x_2)$.

Example: $f(x) = (x-1)^2$.

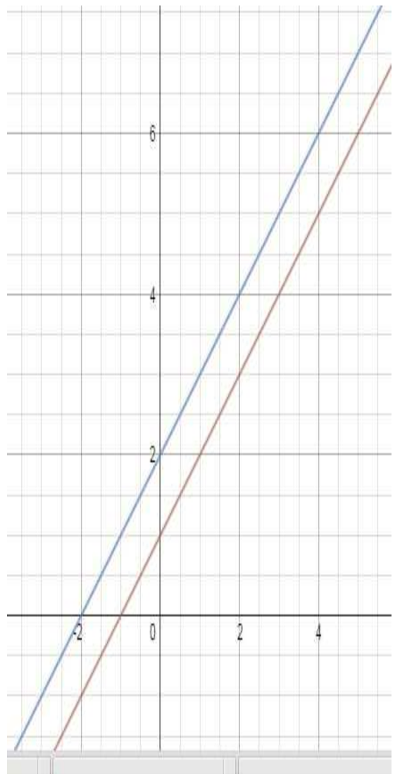


3 Growth Rates of Functions

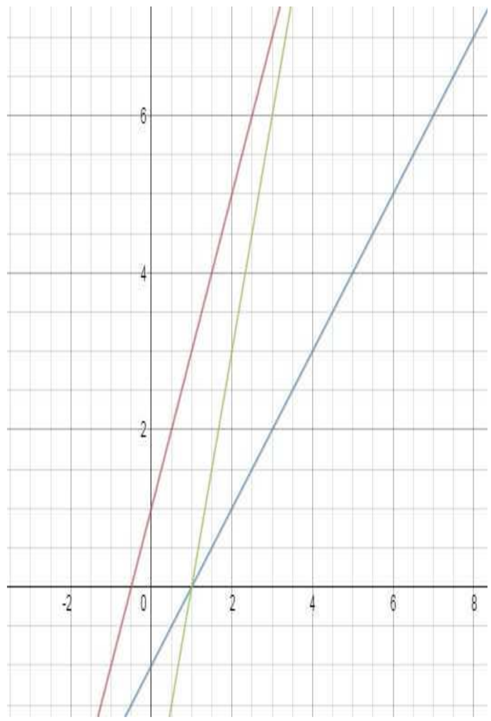
1. Some functions grow faster than others. Example: $f(x) = (x-1)^2$ and $g(x) = x$. Notice when $x = 3$, the quadratic function overtakes the linear function. We say f *is asymptotically greater than* g.



2. Sometimes functions grow at the same rate, but one function lies above the other. Example: $f(x) = x + 1$ and $g(x) = x + 2$.

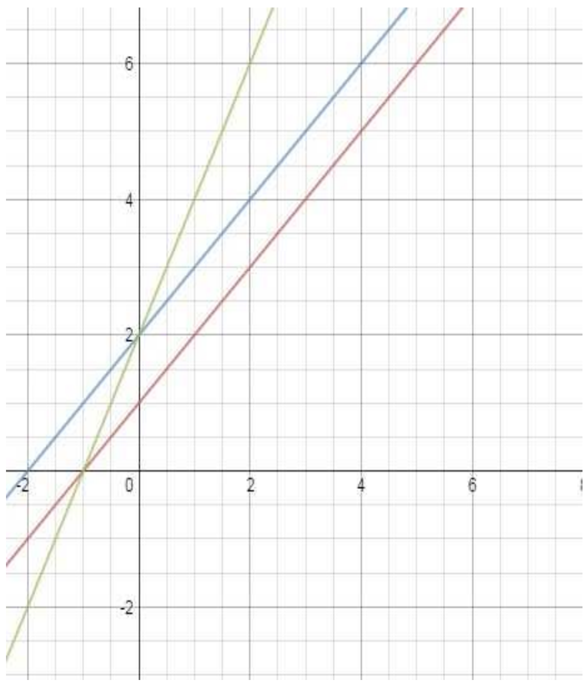


3. Sometimes one function appears to grow faster than another, but if we multiply the second one by an appropriate constant, it eventually overtakes the first. Example: $f(x) = 2x + 1$ and $g(x) = x - 1$ — if we multiply g by 3, we get $3g(x) = 3x - 3$ which eventually overtakes $f(x)$. We say that, *asymptotically, f grows no faster than g* and we can write $f(x)$"$\leq$"$g(x)$.



4. For the functions $f(x) = (x - 1)^2$ and $g(x) = x$, f overtakes g eventually (so, asymptotically, g grows no faster than f) but there is no constant $c > 0$ that would let us scale g so that cg eventually overtakes f. For this reason, it is accurate to say that, asymptotically, f grows faster than g.

5. When functions grow at the same rate, but one function lies above the other, it always turns out that each grows no faster than the other. Example: $f(x) = x + 1$ and $g(x) = x + 2$. Certainly, g is above f everywhere, so $f(x) \leq g(x)$ for all x. But if we multiply f by 2, then $2f(x)$ overtakes $g(x)$ at $x \geq 0$. So g grows no faster than f either. We say in this case that, asymptotically, f and g *grow at the same rate*.



4 Mathematical Induction

The idea: Suppose you wish to prove that some statement $\phi(n)$, which asserts something about each whole number n, is true for every n. For example, to prove that for all $n \geq 0$, $n < 2^n$, we would use "$n < 2^n$" as our statement $\phi(n)$. We wish to show that this statement holds for every n. Suppose now that we can prove two things:

(A) that $\phi(0)$ is true (in our example, this would mean that we can prove $0 < 2^0$);

(B) that, for any n, if $\phi(n)$ happens to be true, then $\phi(n+1)$ must also be true (in our example, this would mean that, if it happens to be true that $n < 2^n$, then it must be true that $n + 1 < 2^{n+1}$).

Mathematical Induction says that, if you can prove both (1) and (2), then you have proven that, for every n, $\phi(n)$ is indeed true.

5 Standard Induction

Suppose $\phi(n)$ is a statement depending on n. If

1. $\phi(0)$ is true, and
2. under the assumption that $n \geq 0$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers n.

Here is a slight generalization:

General Induction Suppose $\phi(n)$ is a statement depending on n and suppose $k \geq 0$ is an integer. If

1. $\phi(k)$ is true, and
2. under the assumption that $n \geq k$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers $n \geq k$.

In General Induction, the step in the proof where $\phi(k)$ is verified is called the *Basis Step*. The second step, where $\phi(n+1)$ is proved assuming $\phi(n)$, is called the *Induction Step*. As we reason during this second step, we will typically need to make use of $\phi(n)$ as an assumption; in this context, $\phi(n)$ is called the *induction hypothesis*.

6 Example of Mathematical Induction

Let $\phi(n)$ be the statement

$$n^2 < 2^n.$$

We prove $\phi(n)$ for all $n \geq 5$. Clearly $\phi(5)$ is true because $5^2 < 2^5$.

For the Induction Step, assume $n^2 < 2^n$; we prove $(n+1)^2 < 2^{n+1}$. As a preliminary, let's observe that

$$(*) \qquad n(n-3) > 0, \ (n > 3)$$

which follows from basic algebra (note that for $0 < n < 3$ the expression is < 0 and for $n > 3$, the expression is > 0). From $(*)$ we may conclude

$$(**) \qquad n^2 > 3n > 2n + 1, \ (n > 3)$$

(note that $3n > 2n + 1$ whenever $n > 1$). Therefore, we can use the induction hypothesis to reach the conclusion as follows:

$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1 \\ &< 2n^2 \qquad \text{(by } (**)\text{)} \\ &< 2 \cdot 2^n \qquad \text{(by Induction Hypothesis)} \\ &= 2^{n+1}. \end{aligned}$$

We have proved the Induction Step. It follows from Mathematical Induction that, for all $n > 4$, $\phi(n)$ holds, that is, $n^2 < 2^n$.

7 The Division Algorithm

(1) Suppose m, n are positive integers. Dividing n by m gives a quotient and remainder.

(2) Example: Divide 17 by 3: Quotient is 5 and remainder is 2. Using integer division and Java mod notation, we can write:

1. quotient $= 17/3$
2. remainder $= 17\,\%\,3$

Using mathematical notation:

1. quotient $= \lfloor 17/3 \rfloor$
2. remainder $= 17 \mod 3$

We can write:

$$17 = \text{quotient} \cdot 3 + \text{remainder} = \lfloor 17/3 \rfloor \cdot 3 + 17 \bmod 3.$$

(3) In general, for any positive integers m, n, there are unique q, r so that

$$n = mq + r \qquad \text{and} \qquad 0 \le r < m.$$

In other words

$$n = m \cdot \lfloor \frac{n}{m} \rfloor + n \bmod m.$$