THE BOIJ–SÖDERBERG CONJECTURES

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1. Introduction

The Boij–Söderberg conjectures, which are now theorems, concern a description of the cone of Betti tables of modules. In this section we recall the notion of a Betti table, then provide an outline of the description of the cone.

We fix a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$.

1.1. **Free resolutions.** Let M be an R-module. A **free resolution** of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

each F_i a free R-module.

Now R is graded in the usual way by degree. If M is a graded module, we may require the resolution to respect this structure. A **graded free resolution** is a free resolution as above, in which each F_i is a graded free R-module and each map $F_{i+1} \to F_i$ has degree zero.

Example 1. Let $R = \mathbb{K}[x, y]$ and $I = (x^2, y^3)$. Let $M = R/I$. A graded free resolution of M is given by

$$0 \to R(-5) \xrightarrow{\begin{pmatrix} y^3 & -x^2 \end{pmatrix}} R(-2) \oplus R(-3) \xrightarrow{\begin{pmatrix} x^2 & y^3 \end{pmatrix}} R \to R/I \to 0.$$

There is not a unique graded free resolutions of M, but there is a unique (up to non-canonical isomorphism) **minimal** graded free resolution of M, that is a resolution which does not admit any map to a graded free resolution which is not an isomorphism of complexes.

1.2. **Betti tables.** The **graded Betti numbers** $\beta_{i,j} = \beta_{i,j}(M)$ are a device to keep track of which shifts appear in the minimal graded free resolution of M. To be precise, $\beta_{i,j}$ is the multiplicity of the summand $R(-j)$ in the ith step F_i of the minimal graded free resolution of M.

For the example above, $M = \mathbb{K}[x, y]/(x^2, y^3)$, we have $\beta_{0,0} = \beta_{1,2} = \beta_{1,3} = \beta_{2,5} = 1$ and all other $\beta_{i,j} = 0$.

The graded Betti numbers do not capture all the information of the resolution; indeed, non-isomorphic modules can have the same graded Betti numbers. Still, they are important invariants which do capture some important information about the module.

Date: June 23, 2008.

Now, it is tempting to arrange the $\beta_{i,j}$ in a table by simply placing $\beta_{i,j}$ in the position at the ith row and jth column. It is conventional, however, to re-index the table's entries as follows: First, the columns correspond to the steps in the resolution, indexed by i. In the position at row x and column y, where $x, y \geq 0$, we place the Betti number $\beta_{y,x+y}$. Equivalently, $\beta_{i,j}$ is placed into the entry of the table at row $j - i$ and column i.

For the example above, the Betti table is:

	0	1	2
0	1	0	0
1	0	1	0
2	0	1	0
3	0	0	1

We write $\beta(M)$ for this table.

1.3. **Fundamental question.** Consider the collection of graded R-modules M such that all the nonzero entries of $\beta(M)$ lie in some fixed region, say the rectangle consisting of the first a rows and b columns. (This is a simplifying assumption to avoid dealing with tables with infinitely many rows and columns.)[1] Then we may regard $\beta(M)$ as a vector in the vector space $\mathbb{Q}^{a \times b}$ of $a \times b$ matrices with rational entries. Two things are apparent: Every entry of $\beta(M)$ is nonnegative, and every entry is an integer.

It is natural to ask if the converse is true: Given a vector T in $\mathbb{Q}^{a \times b}$, regarded as a table, with nonnegative integer entries, is there necessarily an R-module M with $\beta(M) = T$? The answer is no. One then asks, which such T are realized as Betti tables? This is a much harder question.

It is a fact that $\beta(M \oplus N) = \beta(M) + \beta(N)$, the sum on the right being the usual entry-by-entry sum of matrices. So the set of T which are realized as Betti tables is closed under sums. Then it is a commutative semigroup. It is torsion-free: no positive multiple of $T \neq 0$ is zero. One difficulty is that this semigroup is far from saturated. That is, there may be a table T which is not realized as a Betti table, but some positive multiple mT, $m \in \mathbb{Z}_{\geq 0}$, is equal to some $\beta(M)$.

It has been conjectured that if $mT = \beta(M)$ for some M, then $(m+1)T = \beta(M')$ for some M'. This is supported by some examples and computations of Eisenbud–Schreyer, etc, etc.

We turn our attention to a somewhat more tractable problem: We set aside the problem of saturation (that is, which lattice points along each ray are actually realized), and consider the problem of determining which rays contain a $\beta(M)$.

[1] We will not use any of the following information, but the curious reader may ask what conditions we are imposing on M by requiring $\beta(M)$ to fit into a rows and b columns. That $\beta(M)$ fits into a rows is the condition that M have regularity at most a, and that $\beta(M)$ fits into b columns is the condition that M have projective dimension b. If M is Cohen–Macaulay then the projective dimension of M is equal to its codimension.

1.4. **The cone of Betti tables.** Let us consider the (rational) convex cone generated by all the tables $\beta(M)$, for all graded R-modules M such that $\beta(M)$ fits in the fixed $a \times b$ rectangle.

Equivalently, we may consider the union of the set of rays which contain a $\beta(M)$. Indeed, this union is certainly closed under multiplication by positive rational scalars, and one sees that it is convex (over $\mathbb{Q}$) as follows. Let $\beta(M)$, $\beta(N)$ be Betti tables on two rays and let $t = p/q \in \mathbb{Q} \cap [0,1]$. Then we have

$$t\beta(M) + (1-t)\beta(N) = \frac{1}{q}\Big(p\beta(M) + (q-p)\beta(N)\Big) = \frac{1}{q}\beta(M^{\oplus p} \oplus N^{\oplus q-p}),$$

so the corresponding ray again lies in the union we are considering.

Therefore the convex cone generated by all the tables $\beta(M)$ is exactly the union of all the rays which contain at $\beta(M)$.

Since every entry of each $\beta(M)$ is nonnegative, this cone lies in the positive orthant and so the cone is pointed; that is, it does not contain any positive-dimensional linear subspace.

The Boij–Soderberg conjectures give a detailed combinatorial description of this cone.

1.5. **Pure tables.** The cone of Betti tables may not be finitely generated; that is, there may be infinitely many extremal rays. It seems rather difficult to directly try to describe that cone. In order to manage these complexities, one introduces the so-called **pure Betti tables**.

One begins by considering Betti tables (as always fitting into the $a \times b$ rectangle) which have at most one nonzero entry in each column. These tables are determined by specifying for each column, the row in which that column has a nonzero entry and the value of that entry.

It is a theorem of Herzog–Kuhl that if M is a module whose Betti table has at most one nonzero entry in each column, then the nonzero entries and their rows satisfy certain equations. In fact, given the sequence $(d_0, \ldots, d_s)$ of integers such that the column i has its nonzero entry in row $d_i - i$, then $d_0 < d_1 < \cdots < d_s$ and there is a constant $c = c_M \in \mathbb{Q}_{>0}$ such that

$$\beta_{i,d_i} = c \prod_{j \neq i} \frac{1}{|d_i - d_j|}.$$

That is, the table $\beta(M)$ is determined up to positive rational multiple by the degree sequence $(d_0, \ldots, d_s)$.

We now define a **pure Betti table** as follows. Given a degree sequence $(d_0, \ldots, d_s)$ with $d_0 < \cdots < d_s$, the associated pure table has entries given by the formulas above, with $c = 1$.

The first Boij–Soderberg conjecture is that each pure table is realized as the Betti table of a module. Thus the rays through the pure tables lie in the cone spanned by the Betti tables. It follows that the pure tables span a subcone of the Betti cone.

A priori this subcone may be strictly smaller, and still not finitely generated. In fact, however, the second Boij–Soderberg conjecture states that the cone generated by the Betti tables is equal to the cone generated by the pure tables.

In order to approach this statement, one defines a partial order on the collection of degree sequences and studies the combinatorics of the resulting partially ordered set. This yields a combinatorially described simplicial decomposition of the cone generated by the pure tables.

The second Boij–Soderberg conjecture states more precisely that each Betti table lies in a unique simplicial piece of this decomposition of the pure cone.

2. Construction of modules with pure resolutions

3. Decomposition of pure cone

Throughout the next two sections, we work entirely by example, with 3×3 Betti tables. The proofs will extend naturally to the general case. The Betti tables under consideration will have the form

$$\begin{matrix} * & * & * \\ * & * & * \\ * & * & * \end{matrix}\,.$$

These appear to form a nine-dimensional vector space, but we are interested only in Betti tables of modules with codimension 3-1=2; this imposes two linearly independent conditions on the possible tables, so we are actually considering only a seven-dimensional vector space.

The pure degree sequences all lie between (0,1,2) and (2,3,4) in the componentwise partial order (i.e., if $d = (d_1, d_2, d_3)$ and $d' = (d'_1, d'_2, d'_3)$, we say that $d \leq d'$ if $d_i \leq d'_i$ for all i). The poset of such degree sequences is complicated, but at least we can say that every maximal chain begins with (0,1,2), ends with (2,3,4), and has length (2-0)+(3-1)+(4-2)=6. Since the number of terms in such a chain is equal to the dimension of the vector space, we might hope that the pure diagrams in any chain form a basis for the space of codimension-two tables. This will turn out to be the case.

Consider the chain π of degree sequences and pure diagrams given in Table 1.

Proposition 2. *The tables $\{\pi_0, \ldots, \pi_6\}$ are linearly independent.*

Proof. Each successive table introduces a nonzero Betti number in a new position, e.g., $b_{1,2}(\pi_3) \neq 0$ while $b_{1,2}(\pi_0) = b_{1,2}(\pi_1) = b_{1,2}(\pi_2) = 0$. □

Thus, every Betti table of a CM module is a linear combination of the pure Betti tables coming from a maximal chain. It is not the case, though, that everything may be expressed as a *positive* linear combination of these tables. Every chain generates a cone; we will show that these cones fit together into a simplicial fan.

Proposition 3. *Let C and C' be the cones generated by chains ρ and ρ'. Then C and C' intersect in a common face.*

(0,1,2)	$\pi_0 = \begin{matrix}1&2&1\\ .&.&.\\ .&.&.\end{matrix}$
(0,1,3)	$\pi_1 = \begin{matrix}2&3&.\\ .&.&1\\ .&.&.\end{matrix}$
(0,1,4)	$\pi_2 = \begin{matrix}3&4&.\\ .&.&.\\ .&.&1\end{matrix}$
(0,2,4)	$\pi_3 = \begin{matrix}1&.&.\\ .&2&.\\ .&.&1\end{matrix}$
(1,2,4)	$\pi_4 = \begin{matrix}.&.&.\\ 2&3&.\\ .&.&1\end{matrix}$
(1,3,4)	$\pi_5 = \begin{matrix}.&.&.\\ 1&.&.\\ .&3&2\end{matrix}$
(2,3,4)	$\pi_6 = \begin{matrix}.&.&.\\ .&.&.\\ 1&2&1\end{matrix}$

TABLE 1. A chain of Betti tables

(0,1,2)	(0,1,3)	(0,1,4)	(0,2,4)	(1,2,4)	(1,3,4)	(2,3,4)
$\pi_0 = \begin{matrix}1&2&1\\ .&.&.\\ .&.&.\end{matrix}$	$\pi_1 = \begin{matrix}2&3&.\\ .&.&1\\ .&.&.\end{matrix}$	$\pi_2 = \begin{matrix}3&4&.\\ .&.&.\\ .&.&1\end{matrix}$	$\pi_3 = \begin{matrix}1&.&.\\ .&2&.\\ .&.&1\end{matrix}$	$\pi_4 = \begin{matrix}.&.&.\\ 2&3&.\\ .&.&1\end{matrix}$	$\pi_5 = \begin{matrix}.&.&.\\ 1&.&.\\ .&3&2\end{matrix}$	$\pi_6 = \begin{matrix}.&.&.\\ .&.&.\\ 1&2&1\end{matrix}$

Proof. Consider the maximal chains π (as above) and ρ given below:

$\pi_0 = \begin{matrix}1&2&1\\ .&.&.\\ .&.&.\end{matrix}$	$\rho_0 = \begin{matrix}1&2&1\\ .&.&.\\ .&.&.\end{matrix}$
$\pi_1 = \begin{matrix}2&3&.\\ .&.&1\\ .&.&.\end{matrix}$	$\rho_1 = \begin{matrix}2&3&.\\ .&.&1\\ .&.&.\end{matrix}$
$\pi_2 = \begin{matrix}3&4&.\\ .&.&.\\ .&.&1\end{matrix}$	$\rho_2 = \begin{matrix}1&.&.\\ .&3&2\\ .&.&.\end{matrix}$
$\pi_3 = \begin{matrix}1&.&.\\ .&2&.\\ .&.&1\end{matrix}$	$\rho_3 = \begin{matrix}1&.&.\\ .&2&.\\ .&.&1\end{matrix}$
$\pi_4 = \begin{matrix}.&.&.\\ 2&3&.\\ .&.&1\end{matrix}$	$\rho_4 = \begin{matrix}1&.&.\\ .&.&.\\ .&4&3\end{matrix}$
$\pi_5 = \begin{matrix}.&.&.\\ 1&.&.\\ .&3&2\end{matrix}$	$\rho_5 = \begin{matrix}.&.&.\\ 1&.&.\\ .&3&2\end{matrix}$
$\begin{matrix}.&.&.\end{matrix}$	$\begin{matrix}.&.&.\end{matrix}$

intersect in a common face (namely, the cone on $\{\pi_0, \pi_1, \pi_3, \pi_5, \pi_6\}$, the diagrams common to the two chains). The proof will generalize to any pair of chains.

Let β be a Betti table in both $C(\pi)$ and $C(\rho)$, so that we can write $\beta = \sum a_i \pi_i = \sum b_i \rho_i$, with all a_i and all b_i nonnegative. Choose β so that the total number nonzero coefficients in these expressions is minimal, and let i be the first index where either a_i or b_i is nonzero. By symmetry, we may assume that a_i is nonzero.

If $i = 0$, we have $\pi_0 = \rho_0$; thus, by the minimality assumption, $b_0 = 0$. (If not, and supposing $a_0 \geq b_0$, we could replace β with $\beta - b_0\rho_0$.) Thus, $\beta = \sum_{i>0} b_i \rho_i$, so β must have the form $\begin{matrix} * & * & - \\ & * & * \\ & * & * \end{matrix}$. On the other hand, $\beta_{2,2} \geq a_0$, contradicting the assumption that $a_0 \neq 0$.

If $i = 2$, we have $\pi_2 \neq \rho_2$, so we have no information about b_2. However, we still have $\beta = \sum_{i \geq 2} b_i \rho_i$, so β has the form $\begin{matrix} * & - & - \\ & * & * \\ & * & * \end{matrix}$. On the other hand, $\beta_{1,1} \geq 4a_2$, contradicting the assumption that $a_2 = 0$.

If i is anything else, we apply one of the above arguments, depending on whether or not $\pi_i = \rho_i$. □

Thus the union of the cones generated by the maximal chains is a simplicial fan. Because it is convex, this fan is the cone on all the pure diagrams.

Question 4. Why is it convex?

4. Facet equations for pure cone

We can describe the cone on the pure diagrams in terms of its facets. Since this is a simplicial fan, the facets all occur as "exterior" facets of the cones $C(\rho)$ arising from the maximal chains in the previous sections. Thus, it suffices to examine these facets. Again, we restrict our attention to the chain π.

Since $C(\pi)$ is a simplicial cone on seven points, its facets are the cones generated by any six of these points, i.e., by removing one term from the chain. The facet is *interior* if it is a facet of more than one such cone, i.e., if there are two or more ways to extend the resulting chain to a maximal one, and *exterior* if there is only one such way (i.e., if π is the only chain above it).

We need some more notation.

Definition 5. We say that stepping from π_k to π_{k+1} *drops the* i, j^{th} *Betti number* if $b_{i,j}(\pi_k) \neq 0$ but $b_{i,j}(\pi_{k+1}) = 0$.

The chain π drops Betti numbers in the following order:

π_k	Betti number dropped between π_{k-1} and π_k.
$\pi_0 = \begin{matrix} 1 & 2 & 1 \\ . & . & . \\ . & . & . \end{matrix}$	
$\pi_1 = \begin{matrix} 2 & 3 & . \\ . & . & 1 \\ . & . & . \end{matrix}$	$\begin{matrix} . & . & * \\ . & . & . \\ . & . & . \end{matrix}$
$\pi_2 = \begin{matrix} 3 & 4 & . \\ . & . & . \\ . & . & 1 \end{matrix}$	$\begin{matrix} . & . & . \\ . & . & * \\ . & . & . \end{matrix}$
$\pi_3 = \begin{matrix} 1 & . & . \\ . & 2 & . \\ . & . & 1 \end{matrix}$	$\begin{matrix} . & * & . \\ . & . & . \\ . & . & . \end{matrix}$
$\pi_4 = \begin{matrix} . & . & . \\ 2 & 3 & . \\ . & . & 1 \end{matrix}$	$\begin{matrix} * & . & . \\ . & . & . \\ . & . & . \end{matrix}$
$\pi_5 = \begin{matrix} . & . & . \\ 1 & . & . \\ . & 3 & 2 \end{matrix}$	$\begin{matrix} . & . & . \\ . & * & . \\ . & . & . \end{matrix}$
$\pi_6 = \begin{matrix} . & . & . \\ . & . & . \\ 1 & 2 & 1 \end{matrix}$	$\begin{matrix} . & . & . \\ * & . & . \\ . & . & . \end{matrix}$

Notation 6. The facet τ_k is the cone spanned by $\{\pi_0, \dots, \hat{\pi_k}, \dots, \pi_6\}$.

The facet τ_k is exterior if π_k is the only diagram that can be inserted to create a maximal chain.

Thus, τ_0 and τ_6 are exterior, since π_0 and π_6 have the unique degree sequences below π_1 and above π_5, respectively.

Every other facet τ_k corresponds to dropping two consecutive Betti numbers around π_k. For example, τ_1 corresponds to dropping the Betti numbers $\begin{matrix} . & . & * \\ . & . & * \\ . & . & . \end{matrix}$.

τ_k is exterior if the two dropped Betti numbers are in the same row or the same column, and interior otherwise. (For example, τ_2 is interior because π_2 could be replaced with $\begin{matrix} 1 & . & . \\ . & 3 & 2 \\ . & . & . \end{matrix}$) to make a different maximal chain. On the other hand, τ_3 and τ_1 are exterior.

The facet equation for τ_0 is $b_{2,2}$, and the equation for τ_6 is $b_{0,2}$. When the drops are in the same column, the removed diagram is the only one with a nonzero Betti number in that column; the equation for τ_1 is $b_{2,3}$.

When the drops occur in the same row, the equation is more complicated. We express it in terms of a dot product with another diagram by the procedure demonstrated below.

The equation f for τ_3 should evaluate to a positive number on π_3 and to zero on π_k for all other k. To ensure that we get zero on all $\pi_k, k > 3$, we fill in zeroes in the positions where any of these diagrams are nonzero:

0	0	
0	0	0

Dotting f with π_3 yields $f_{0,0} + 0 + 0$, which should be positive; we fill in any positive number for $f_{0,0}$ to ensure this. (We choose 24 here because it is very divisible, which will be handy later.):

24		
0	0	
0	0	0

Dotting f with π_2 yields $72 + 4 * f_{1,1} + 0$, which should be zero; thus $f_{1,1} = -18$:

24	-18	
0	0	
0	0	0

Dotting f with π_1 yields $48 - 54 + f_{2,3}$, which should be zero; thus $f_{2,3} = 6$:

24	-18	
0	0	6
0	0	0

Dotting f with π_0 yields $24 - 36 + f_{2,2}$, which should be zero; thus $f_{2,2} = 12$:

24	-18	
0	0	6
0	0	0

Remark 7. The facet equation f derived above is not unique; since all Betti diagrams of CM modules satisfy the Herzog-Kuhl equations, it is only defined modulo these equations. In particular, the equation we have derived is called the *upper facet equation*. There is also a *lower facet equation* which is defined by filling in zeroes at the top of the diagram and filling in the other positions by working down the chain.

5. Facet equations for Betti cone