High Dimensional Statistics | Prof. Dr. Podolskij Mark | Homework 2

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Exercise 1

Let $Z \sim \mathcal{N}_n(0, \Sigma)$. We aim to show that $Z^T \Sigma^{-1} Z \sim \mathcal{X}_n^2$.

Proof

First, we can rewrite Z as: $Z = \Sigma^{\frac{1}{2}} X$, where $X \sim \mathcal{N}_n(0, I_n)$ with $\Sigma^{\frac{1}{2}} = PD^{\frac{1}{2}}P^T$, where P an orthogonal matrix and D is a diagonal matrix s.t. $\Sigma = PDP^T$. Since Σ is positive definite, $D = diag(\lambda_1, \ldots, \lambda_n)$ and $D^{\frac{1}{2}} = diag(\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n})$ with $\lambda_1, \ldots, \lambda_n > 0$ - eigenvalues of Σ.

Thus, $Z^T \Sigma^{-1} Z$, which is a quadratic form of a random variable Z, can be rewritten as:

$$
\begin{aligned}
Z^T \Sigma^{-1} Z &= \left(\Sigma^{\frac{1}{2}} X\right)^T \Sigma^{-1} \Sigma^{\frac{1}{2}} X \\
&= X^T \left(\Sigma^{\frac{1}{2}}\right)^T \Sigma^{-1} \Sigma^{\frac{1}{2}} X \\
&= X^T \left(PD^{\frac{1}{2}}P^T\right)^T \left(PDP^T\right)^{-1} PD^{\frac{1}{2}}P^T X \\
&= X^T P \left(D^{\frac{1}{2}}\right)^T P^T (P^T)^{-1} D^{-1} (P)^{-1} PD^{\frac{1}{2}} P^T X
\end{aligned}
$$

Notice that $\left(D^{\frac{1}{2}}\right)^T = D^{\frac{1}{2}}$ since D is diagonal and $P^T(P^T)^{-1} = I_n, P(P)^{-1} = I_n$. Thus:

$$
\begin{aligned}
Z^T \Sigma^{-1} Z &= X^T P \left(D^{\frac{1}{2}}\right)^T P^T (P^T)^{-1} D^{-1} (P)^{-1} PD^{\frac{1}{2}} P^T X \\
&= X^T PD^{\frac{1}{2}} D^{-1} D^{\frac{1}{2}} P^T X
\end{aligned}
$$

Now, the product $D^{\frac{1}{2}} D^{-1} D^{\frac{1}{2}} = I_n$, since D is diagonal (for each diagonal element we would get the multiplication $\sqrt{\lambda_i} \frac{1}{\lambda_i} \sqrt{\lambda_i} = 1$ for $1 \leq i \leq n$). Thus, we get:

$$
\begin{aligned}
Z^T \Sigma^{-1} Z &= X^T PD^{\frac{1}{2}} D^{-1} D^{\frac{1}{2}} P^T X \\
&= X^T PP^T X \\
&= X^T X,
\end{aligned}
$$

with $PP^T = I_n$, since P is an orthogonal matrix.

Since $X^T X = \sum_{i=1}^n X_i^2$, then by the definition of a chi-squared distribution, the sum of n squared random variables following standard normal follows a chi-squared distribution with n degrees of freedom. Thus:

$$
Z^T \Sigma^{-1} Z = \sum_{i=1}^n X_i^2 \sim \mathcal{X}_n^2
$$