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June 23, 2022

Exam

Exercise 1: Consider the linear regression model

$$Y_i = b_0 + b_1 t_i + \varepsilon_i, \qquad i = 1, \dots, n,$$

where (ε_i) is a sequence of i.i.d $\mathcal{N}(0, \sigma^2)$-distributed random variables. We assume that there exists a pair (i, j) such that $t_i \neq t_j$. Compute the least squares estimator $\widehat{b}_0$ and determine its variance.

Exercise 2: In this exercise we want to show that determining Lasso is a *convex* optimisation problem. Recall that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a convex function if for any $t \in (0, 1)$ it holds that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad \text{for all } x, y \in \mathbb{R}^k.$$

(i) For $b \in \mathbb{R}^k$ show that the map $b \mapsto \|b\|_1$ is convex.

(ii) For $Y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times k}$ show that the map $b \mapsto \|Y - Xb\|_2^2$ is convex.

Exercise 3: We recall the definition of the Lasso estimator:

$$\widehat{b}(\lambda) = \text{argmin}_{b \in \mathbb{R}^k} \left(n^{-1} \|Y - Xb\|_2^2 + \lambda \|b\|_1 \right)$$

Prove that $\widehat{b}(\lambda) = 0 \in \mathbb{R}^k$ if $\lambda \geq 2n^{-1} \max_{1 \leq j \leq k} |\langle X^{(j)}, Y \rangle|$ where $X^{(j)}$ is the jth column of the matrix X.

Exercise 4: Let $\Sigma \in \mathbb{R}^{k \times k}$ be a covariance matrix with eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_k \geq 0$. Let $v_1 \in \mathbb{R}^k$ be an eigenvector that corresponds to the largest eigenvalue λ_1; in particular $\|v_1\|_2 = 1$. Consider an arbitrary vector $x_0 \in \mathbb{R}^k$ with $\langle x_0, v_1 \rangle \neq 0$, and define the sequence

$$x_i := \Sigma \cdot y_{i-1}, \qquad y_i := x_i / \|x_i\|_2, \qquad i \geq 1$$

with $y_0 = x_0 / \|x_0\|_2$. Prove that

$$\lim_{i \to \infty} y_i = v_1 \qquad \text{or} \qquad \lim_{i \to \infty} y_i = -v_1.$$

Hint: Use the representation $x_0 = \sum_{j=1}^{k} a_j v_j$ where v_j is an eigenvector corresponding to eigenvalue λ_j and $a_j = \langle x_0, v_j \rangle$.