HW 3

Ex. 2

$\Rightarrow$:

Assume $\operatorname{corr}(X,Y)=1$.

Let $X' = \frac{X}{\sigma_X}$, $Y' = \frac{Y}{\sigma_Y}$. Then:

$$\begin{aligned}
\operatorname{Var}(X'-Y') &= \operatorname{Var}(X') + \operatorname{Var}(-Y') + 2\operatorname{Cov}(X',-Y') \\
&= \frac{1}{\sigma_X^2}\operatorname{Var}(X) + \frac{1}{\sigma_Y^2}\operatorname{Var}(Y) - \frac{2}{\sigma_X\sigma_Y}\operatorname{Cov}(X,Y) \\
&= 2 - 2\operatorname{corr}(X,Y) \\
&= 0
\end{aligned}$$

We have:

$$\operatorname{Var}(X'-Y') = 0$$

$$\Leftrightarrow \mathbb{E}[\underbrace{(X'-Y'-\mathbb{E}[X']+\mathbb{E}[Y'])^2}_{\geq 0}] = 0$$

$$\Leftrightarrow X'-Y'-\mathbb{E}[X']+\mathbb{E}[Y'] = 0 \quad \mathbb{P}\text{-a.s.}$$

$$\Leftrightarrow X = \underbrace{\frac{\sigma_X}{\sigma_Y}}_{>0} Y + \underbrace{\mathbb{E}[X] - \frac{\sigma_X}{\sigma_Y}\mathbb{E}[Y]}_{\in \mathbb{R}} \quad \mathbb{P}\text{-a.s.}$$

since $\sigma_X, \sigma_Y > 0$
otherwise corr would
not be well-defined

$\Leftarrow$:

Assume $\exists\, a>0, b\in\mathbb{R}$ s.t. $X = aY+b$ $\mathbb{P}$-a.s.

Then we have $\mathbb{P}$-a.s.:

$$\begin{aligned}
\operatorname{corr}(X,Y) &= \frac{\operatorname{Cov}(X,Y)}{\sigma_X\,\sigma_Y} \\
&= \frac{\operatorname{Cov}(aY+b,Y)}{\sigma_{aY+b}\,\sigma_Y} \quad (*)
\end{aligned}$$

We have:

$$\begin{aligned}
\operatorname{Cov}(aY+b,Y) &= a\operatorname{Cov}(Y,Y) \\
&= a\operatorname{Var}(Y)
\end{aligned}$$

$$\begin{aligned}
\sigma_{aY+b} &= \sqrt{\operatorname{Var}(aY+b)} \\
&= |a|\sqrt{\operatorname{Var}(Y)}
\end{aligned}$$

$= a\, \sigma_Y$

Thus:

$$(*) = \frac{a \operatorname{Var}(Y)}{a\, \sigma_Y\, \sigma_Y} = 1$$

Ex. 3

Let θ_0 be the true parameter and $\hat{\theta}_{MLE}$ the MLE of $\theta_0 \in \Theta$. Assume $f: \Theta \to \tilde{\Theta}$ is a bijection. Then $\exists \lambda_0 \in \tilde{\Theta}$ st.: $f(\theta_0) = \lambda_0$

We know by definition of the MLE that:

$$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\operatorname{argmax}} (L(\theta; X))$$

Let us denote $\tilde{L}(\cdot, X)$ the likelihood function wrt. $\lambda = f(\theta) \in \tilde{\Theta}$, for $\theta \in \Theta$.

Since f is a bijection, we know that it is invertible. Thus, we can find the corresponding value $\theta \in \Theta$ associated to $\lambda \in \tilde{\Theta}$ st.:

$$\theta = f^{-1}(\lambda)$$

This allows us to write: for $f(\theta) = \lambda$:

$$\tilde{L}(\lambda; X) = \tilde{L}(f(\theta); X) = L(f^{-1}(\lambda); X) = L(\theta; X)$$

By the definition of $\hat{\theta}_{MLE}$, we have $\forall \theta \in \Theta$:

$$L(\hat{\theta}_{MLE}; X) \geq L(\theta; X)$$

$$\Rightarrow \tilde{L}(f(\hat{\theta}_{MLE}); X) \geq \tilde{L}(\underbrace{f(\theta)}_{\in \tilde{\Theta}}; X)$$

$$\Rightarrow \tilde{L}(f(\hat{\theta}_{MLE}); X) \geq \tilde{L}(\lambda; X) \qquad \forall \lambda \in \tilde{\Theta}$$

$$\Rightarrow f(\hat{\theta}_{MLE}) = \underset{\lambda \in \tilde{\Theta}}{\operatorname{argmax}} (\tilde{L}(\lambda; X))$$

$\Rightarrow f(\hat{\theta}_{MLE})$ is the MLE of $\lambda_0 = f(\theta_0) \in \tilde{\Theta}$