Enumeration (Combinatorics)

Combinatorics is an art of counting sets.

Definition 1.1: Cardinal

The cardinal of a set A is a number of elements contained in A.

Types of Sets

- Countable set: can list all elements in a sequence indexed by integers.
  - Finite set: the number of elements in the set is finite. Example: $A = \{1, 2, 3, 6, 7\}, card(A) = 5$.
  - Infinite set: the number of elements in the set is infinite. Example: set of $\mathbb{N} = \{1, 2, 3, ...\}$
- Uncountable set: set that cannot be indexed by the sequence of integers. Example: $\mathbb{R}, \{0, 1\}^{\mathbb{N}} = 0010101..., 0101101111..., \mathcal{P}(\mathbb{N})$.

Remark: $\mathbb{N}$ is countable because you can establish a one-to-one correspondence (bijection) between its elements and the positive integers. You can list them in a sequence without missing any, and you can count them. This correspondence can be represented as $f : \mathbb{N} \to \mathbb{N}$, where $f(n) = n$. Each natural number corresponds to itself, making it easy to count and list them. $\mathbb{R}$ is uncountable because it has a higher cardinality than $\mathbb{N}$ or any other countable set. Cantor's diagonal argument is one way to demonstrate that $\mathbb{R}$ is uncountable. It shows that you cannot list all real numbers in a countable sequence. If you try to list them as a decimal or binary expansion, you can always construct a real number that is not in your list, simply by changing at least one digit from each number in your list. This process is infinitely recursive and ensures that you can never list all real numbers.

Basic Rules

Bijection

In set theory, a bijection is a function between two sets that establishes a one-to-one correspondence between their elements. More formally, a function $f : A \to B$ is considered a bijection if the following conditions are met:

1. Injective (One-to-One): For every pair of distinct elements x and y in set A, $f(x)$ and $f(y)$ are also distinct in set B. In other words, no two elements in set A map to the same element in set B.

2. Surjective (Onto): For every element y in set B, there exists an element x in set A such that $f(x) = y$. In other words, the function f covers all elements in set B.

3. Bijective: If a function f is both injective and surjective, it is called a bijection. This means that there is a one-to-one correspondence between

the elements of sets A and B, and every element in set B has a unique pre-image in set A.

In practical terms, bijections are often used to prove that two sets have the same number of elements or to establish a correspondence between elements in different contexts. They are also used in the study of functions and transformations in mathematics.

If A and B are in bijection, then

$$card(A) = card(B)$$

Product

Let A, B be two finite sets. We define the product $A \cdot B$ as $A \cdot B = \{(a, b) | a \in A, b \in B\}$ and

$$card(A \cdot B) = card(A) \cdot card(B)$$

Disjoint

Let A, B be two subsets of a set Ω. If A, B are disjoint, then

$$card(A \cup B) = card(A) + card(B)$$

Joint

If A, B are joint, then:

$$card(A \cup B) = card(A) + card(B) - card(A \cap B)$$

Since $card(A)+card(B)$ counts $card(A\cap B)$ two times, we subtract one $card(A\cap B)$.

If A, B, C are joint, then:

$$card(A \cup B \cup C) = card(A) + card(B) + card(C) -$$
$$-card(A \cap B) - card(A \cap C) - card(B \cap C) + card(A \cap B \cap C)$$

Example:

Throw 2 dice. We want to find the probability that we get atleast one 6.

$1/6 + 1/6 - 1/36 = 11/36$, where $1/36 = A \cap B$

Definition 1.2: Indicator Function

Let Ω be a set and $A \subset \Omega$. We denote by 1_A the indicator function of the set A s.t.

$$1_A :$$
$$\Omega \to \{0, 1\}$$
$$\omega \to \begin{cases} 1, \omega \in A \\ 0, \omega \notin A \end{cases}$$

Example:

$\Omega = \mathbb{R}, a < b \in \mathbb{R}$

$1_{\overline{A}} = 1 - 1_A, \overline{A} = \Omega \setminus A$

$$card(A) = \sum_{\omega \in \Omega} 1_A(\omega) = \sum_{\omega \in A} 1$$

Definition 1.3: Power Set

Let Ω be a set. We denote by $\mathcal{P}(\Omega)$ the power set of Ω. It is defined by $\mathcal{P}(\Omega) = \{A | A \subset \Omega\}$ (nested sets).

Example:
$\Omega = \{1, 2, 3\}$
$\mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \emptyset\}$

Theorem 1.1: Cardinal of Power Set

$$card(\mathcal{P}(\Omega)) = 2^{card(\Omega)}$$

Proof:
The set $\mathcal{P}(\Omega)$ can be put in bijection with the set $\{0, 1\} \times \{0, 1\} \times ... \times \{0, 1\} = \{0, 1\}^{\Omega} = \{0, 1\}^{card(\Omega)}$

$\mathcal{P}(\Omega) \to \{\{0, 1\} \times \{0, 1\} \times ... \times \{0, 1\}\}$

We map each element of Ω to either 0 or 1. Thus:
$card(\mathcal{P}(\Omega)) = card(\{0, 1\}^{card(\Omega)}) = card(\{0, 1\})^{card(\Omega)} = 2^{card(\Omega)}$

Diagonal Counter Argument

Set $\{0, 1\}^{\mathbb{N}}$ is uncountable. $\{0, 1\}^{\mathbb{N}} = \{$ infinite sequence of 0s or 1s $\}$

Proof:
Assume $\{0, 1\}^{\mathbb{N}}$ is countable.

$s_1 = (0110001...)$
$s_2 = (0011001...)$
$s_3 = (0110101...)$
$s_4 = (0000001...)$
... $s_n = (0000001...1)$
Make matrix S, where each row is $s_1, s_2, ..., s_n$
Let $s = (0010...1)$ - where each digit corresponds to diagonal element of S.
Let $s_I = (1101...0)$ be an inverse of s.
We claim $s_I \neq s_n, \forall n \in \mathbb{N}$ (s_I is different from $s_n, \forall n \in \mathbb{N}$).
This is true, because n_{th} digit of s_I is different from the n_{th} digit of s_n. Thus, $s_I \notin S$, making $\{0, 1\}^{\mathbb{N}}$ uncountable.

Definition 1.4: Permutation

A set Ω is not ordered, there is no repetition:
$\{a, b, c\} = \{b, c, a\} = \{a, c, b\} = \{a, b, c, b, c, a\} = ...$

A permutation is a way to order the elements of Ω. Alternatively, it is a bijetion from Ω to $\{1, 2, 3, ..., n\}$ where $n = card(\Omega)$

Theorem 1.2: Permutation

If Ω has n elements, then there are $n!$ number of permutations of Ω.

$(w_1, w_2, w_3, ..., w_n)$
$n, n-1, n-2, ..., 1$
n possible ways to choose w_1, $n-1$ possible ways to choose w_2, $\ldots$, 1 possible way to choose $w_n = n!$

Question: how many anagrams of the word PROBA ?
Answer: 5!

Question: how many anagrams of the word ECOLE ?
Answer: 5!/2!

Question: how many anagrams of the word MISSISSIPPI ?
Answer: $\frac{11!}{4!4!2!}$

Definition 1.5: Arrangement

An arrangement of k elements from n elements of a set Ω is an ordered sequence of k distinct elements of Ω. In arrangement, the order of selection matters.

Example:

$\Omega = \{a, b, c, d, e, f\}$
$(a, b, c), (a, f, e), (a, e, f)$ are 3 distinct arrangments of 6 elements of Ω.

Theorem 1.3: Arrangement

There are

$$A_n^k = n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$$

ways to arrange k elements among n elements.

Definition 1.6: Combination

A combination of k elements from n elements of Ω is a subset of Ω with k elements. In combination, the order does not matter ($\{1,2\} = \{2,1\}$).

Example:
$\Omega = \{1,2,3\}$
$k = 2, C(n,k) = \{\{1,2\},\{1,3\},\{2,3\}\}$ - the number of combinations of k elements among a set of n elements ($C(n,k)$ is called "n" choose "k").

Theorem 1.4: Combination

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Proof:
Given a combination of k elements, we can form exactly $k!$ distinct arrangements by permuting the k elements chosen. This gives the equality:

$A_n^k = \frac{n!}{(n-k)!}$

$k!\binom{n}{k} = A_k^n = \frac{n!}{k!(n-k)!}$

Rules / Facts

$$\binom{n}{k} = \binom{n}{n-k}$$

Choosing k elements out of n elements is the same as choosing $n-k$ elements out of n elements.

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$$

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}$$

$$n\binom{n}{k} = n\binom{n-1}{k-1}$$

Vandermonde equality

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-1}$$

Theorem 1.5: Newton Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} C(n,k)x^k y^{n-k}$$

Theorem 1.6: Balls and Bags | How Many Solutions to an Equation, Where Each n is Non-Negative Integer

$$k_1 + ... + k_i = n, k \in \mathbb{N}_0, n \in \mathbb{N}$$
$$\binom{n+i-1}{i-1}$$

Exercises

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$2^n = (1+1)^n$
$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$
$=> 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$

How many anagrams of the words MATHS, SCHOOL, PILITIPILIT are there?

1. MATHS: $\frac{5!}{1!1!1!1!1!}$
2. SCHOOL: $\frac{6!}{2!}$
3. PILITIPILIT: $\frac{11!}{2!5!2!2!}$

We split the 52 card game among 4 players (A,B,C,D), so that each player has 13 cards. Compute the number of distinct game configurations.

$\binom{52}{13}\binom{52-13}{13}\binom{52-13-13}{13}\binom{52-13-13-13}{13} = \binom{52}{13}\binom{39}{13}\binom{26}{13} = \frac{52!}{13!13!13!13!}$

This is the same problem as computing the number of anagrams for the word $(A_1, ..., A_{13})(B_1, ..., B_{13})(C_1, ..., C_{13})(D_1, ..., D_{13})$

We can say that player A is the same as player B, then:

$\frac{\frac{52!}{13!13!13!13!}}{4!}$

Marie wants to decorate her garden. She wants to plant a row of 8 flowers. She bought 3 green flowers and 5 blue ones. 1. In how many distinct ways she can plant her flowers? 2. In how many ways she can plant her flowers if 2 green flowers cannot be planted consecutively?

1. $\frac{8!}{3!5!}$

8 positions for flowers, remove repeats for 3 green and 5 blue.

2. We have 6 positions for 3 green flowers so that they are not planted consecutively.

Thus: $\binom{6}{3}$

We have 9 digit access code. 1. How many possible codes there are? 2. How many codes contain digit 4? 3. How many codes contain digit 4 exactly 1 time? 4. Same questions if the digits are all different.

1. 9 possible digits for each out of 4 positions: 9^4
2. First compute how many codes do not contain digit 4: 8^4. Then subtract from all codes number of codes that do not contain digit 4: $9^4 - 8^4$
3. We have 4 possibilities: $4xyz, x4yz, xy4z, xyz4$, where x, y, z can have 8 digits. Thus: $8^4 \cdot 4$
4.
   1. $9 * 8 * 7 * 6$
   2. $9 * 8 * 7 * 6 - 8 * 7 * 6 * 5$
   3. $8 * 7 * 6 * 4$ (same as 4.2.)

We have 9 digit and 2 letters among 3 letters (A,B,C) access code. 1. How many possible codes there are? 2. How many codes if all characters are distinct?

1. $9^4 * 3^2 * \frac{6!}{2!4!}$
   $\frac{6!}{2!4!}$ is how many positions there are for 2 letters in 6 digit code.
2. $9 * 8 * 7 * 6 * 3 * 2 * \frac{6!}{2!4!}$

Compute the number of solutions to the equation: $n_1 + n_2 + n_3 + n_4 = 6$**, where** n_i **are non-negative integers**

Balls and bags problem. General solution for $k_1 + ... + k_i = n$ is: $\binom{n+i-1}{i-1}$
Hence: $\binom{6+4-1}{4-1}$

Why? We have 6 balls and 3 sticks, in total 9 items. We need to arrange 3 sticks among 9 positions in any position we want, thus we have combination $\binom{9}{3}$.

If we were not to count 0 as a solution, then we would have to arrange 3 sticks among 5 positions (between balls), so that there are no 0 balls between sticks.

Vandermonde formula. Show by a 1. combinatorial proof and by a 2. algebraic proof that: $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-1}$

1. Imagine we have n girls and m boys. We need to form a team of k pupils. Thus: $0 \leq k \leq m+n$. We can do this with $\binom{m+n}{k}$.

But, we can also form k pupils with i boys and $k-i$ girls: $\binom{m}{i}\binom{n}{k-i}$. Since we fixed i, but i can be 0, 1, ..., m, or in other words we can have 0 boys, k girls; 1 boy, $k-1$ girls; 2 boys, $k-2$ girls, we have a sum: $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$

Thus: $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-1}$

2. Remember Netwon binomial formula:

$\sum_{k=0}^{m+n} \binom{m+n}{k} x^k = \sum_{i=0}^{m} \binom{m}{i} x^i \sum_{j=0}^{n} \binom{n}{j} x^j$

If we expand right side we get:

$((\binom{m}{0}x^0 + \binom{m}{1}x^1 + \binom{m}{2}x^2 + ... + \binom{m}{i}x^i) \cdot ((\binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + ... + \binom{n}{j}x^j) = ((\binom{m}{0}\binom{n}{0})x^0 + ((\binom{m}{0}\binom{n}{1} + \binom{m}{1}\binom{n}{0})x + ((\binom{m}{0}\binom{n}{2} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{0})x^2 + ...$

Coefficients for x^k are:

$\binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + ... + \binom{m}{k}\binom{n}{0} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$

Therefore, by comparing the coefficients of x^k, we get:

$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$

$\binom{m+n}{k}$ is the coefficient for any x^k in $(1+x)^{m+n} \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$, but for $(1+x)^m(1+x)^n$ it is the summation $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$

Probability Space

Probability theory aims to quantify the notion of uncertainty. The notion of uncertainty reflects a lack of information that prevents us from predicting the outcome of an experiment with absolute certainty.

Let's take the example of a coin toss. Intuitively, we'll come up tails about half of the time. But what's the reality ? The coin toss, as a physical system, can be considered deterministic. If I could describe with absolute precision the action exerted by my hand on the coin, I could deduce its exact trajectory and predict on which face the coin would fall. Let's assume that the coin's trajectory depends on a parameter $\lambda \in [0, 1]$. Since we have no information on the parameter λ, we'll assume that it is chosen randomly according to some law (for example, a uniform law). In this case, we'd observe that for about half the values of λ the coin will land on heads, and for the other half it will land on tails.

Probability theory enables us to quantify a lack of information about the parameters of a given experiment. This lack of information translates into uncertainty about the outcome of the experiment. **To give ourselves a framework for**

making probabilities is to give ourselves a model of the uncertainty we face.

In order to define a framework for doing probability we need three ingredients.

1. **Universe** of possible configurations for the experiment or game. For a dice roll - trajectory of the hand, for card game - set of all possible decks.
2. **Notion of information** - set of events we consider. Say we have a survey: people between 0 and 20 years old, between 20 and 50 years old, over 50 years old. Here, the **universe is all the people** surveyed. Our survey allows us to quantify the probability of occurrence of events such as "The person is under 50". On the other hand, we have no access to quantities such as "The person is between 40 and 60", let alone "The person is a man". This notion of information seems a little superfluous, as it always seems possible to "enlarge" the experiment in order to access information we don't have. It is almost impossible and bad to consider all information because:
    1. This distorts/complicates the model (e.g. when rolling a dice we do not care how many flips a dice does, the essential information is only the final value of the die).
    2. When modeling a quantity that evoles with time.
    3. There are mathematical obstructions to considering "all information" when the space Ω is uncountable (see Banach-Tarski paradox: have 1 sphere with radius $r = 1$, cut the sphere into uncountable number of pieces, glue pieces back together, get 2 spheres with radius $r = 1$).
3. **Measure of probability**. It quantifies the probability of a given event to occur. In the case of a well-mixed pack of cards, for example, we can choose a uniform probability over all the decks of cards, meaning that each card shuffle has a probability $1/(52!)$, but if it's badly mixed, the probability may be chosen differently. Modeling an experiment therefore requires choosing the probability with which each event occurs. This measure is supposed to respect the physical reality of the experiment as closely as possible, which can prove complicated. A statistical test can be used to check the agreement between a theoretical probabilistic model and a real experiment repeated a large number of times

Definition 2.1: Probability Space

A probability space is a triplet (Ω, F, P) where: - Ω is the universe, set of possible configurations of the experiment, simply a set; - F is a σ-algebra on Ω. F represents the information we can acquire during the experiment. An element in F is called an **event**. An event is the subset of Ω universe. Therefore, F is a set of events. It is possible to apply certain operations between events : union, intersection, difference, complementary, etc. In other words, the set F is stable by a number of operations; - P is a probability measure. It is used to quantify the probability of a given event occurring. For a given event A, it associates a number between 0 and 1, denoted $P(A)$, which reflects the probability of the event A occurring.

Definition 2.2: σ-algebra

A σ-algebra F on a space Ω is a subset of $P(\Omega)$ (the power set of Ω) such that:

1. $\Omega \in F$ (the universe is an event)
2. If $A \in F$, then $\overline{A} \in F$ (stability by passing to the complementary)
3. If $(A_n)_{n\geq 0} \in F$, then $\bigcup_{n\geq 0} A_n \in F$ (stability by countable union)

σ-algebra - everything you can construct with union, intersection, complementary, ... of events

Definition 2.3: Measurable Space

The pair (Ω, F) is a measurable space (events in F are measurable).

Definition 2.4: Trivial / Discrete σ-algebras

The σ-algebra $\{\emptyset, \Omega\}$ is called the trivial σ-algebra. It is the smallest σ-algebra on Ω that we can consider. The σ-algebra $P(\Omega)$ is called the discrete σ-algebra. It is the largest σ-algebra on Ω that can be considered.

Definition 2.5: Smallest $\sigma-$algebra

Let C be a subset of $P(\Omega)$. Let $\sigma(C)$ be the smallest σ-algebra containing C. It is the intersection of all σ-algebras containing C. For example, if Ω is countable and C is the set of singletons, it's easy to see that $\sigma(C) = P(\Omega)$.

Let $\Omega = [0, 1]$. If I is an interval with ends a and b (not necessarily open or closed), then $\mu(I) = b - a$. This definition is "consistent" with the notion of a probability space: $\mu(\Omega) = 1$.

I_1 and I_2 are disjoint intervals such that $I_1 \cup I_2$ is an interval (e.g., $I_1 = [a, b]$ and $I_2 = [b, c]$), then: $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$

Definition 2.6: Borelian σ-algebra

Let $B([0, 1])$ be the σ-algebra generated by the intervals included in $[0, 1]$. It is called the Borelian σ-algebra on $[0, 1]$. Similarly, $B(\mathbb{R})$ is the σ-algebra generated by the intervals in $\mathbb{R}$.

$B([0, 1]) \neq \mathcal{P}([0, 1]), \mathcal{P}([0, 1])$ is much bigger.

$C = \{[a, b] | a, b \in [0, 1]\}$
$B([0, 1]) = \sigma(C)$

Definition 2.7: Probability Measure P

A probability measure P on a measurable space $(\Omega,\ F)$ is an application:

$P : F \to [0,1]\ A \to P(A)$

Such that:

1. $P(\Omega) = 1$ (the universe is an event with probability 1)
2. If $(A_n)_{n\geq 0}$ is a countable family of pairwise disjoint events, then: $P(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} P(A_n)$

Definition 2.8: Almost-Surely / Negligible

If A is an event in a probability space $(\Omega,\ F,\ P)$ such that $P(A) = 1$, we say that A is realized **almost-surely** (sometimes denoted as a.s.). Conversely, if A is an event in a probability space $(\Omega,\ F,\ P)$ such that $P(A) = 0$, we'll say that A is a **negligible** event.

Theorem 2.1: Caratheodory Extension

There exists a unique probability measure (still denoted μ) on $(\Omega, B([0,1]))$ such that: - $([0,1], B([0,1]), \mu)$ is a probability space. - $\mu(I) = b - a$ for any interval I of extremity a and b.

In other words, if we decide that the measure of an interval is exactly its length, then we can extend this measure to the entire σ-algebra generated by intervals:countable unions of intervals, singletons,... and much more.

Conditional Probability and Independence

When we want to model two distinct quantities by a probabilistic model, it often happens that these two quantities are **correlated**.

Definition 2.9: Conditional Probability

Let (Ω, F, P) be a probability space, and B an event of non-zero measure. Then for any event A, we call the quantity:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

It indicates the probability of an event A occurring knowing that event B has occurred.

$$(\Omega, F, P) \to (B, \mathcal{F}_B, P(\cdot|B))$$

$$\mathcal{F}_B = \{A \cap B \mid A \in F\}$$

Definition 2.10: Complete System of Events

A family $\{B_1, \ldots, B_n\}$ is said to be a complete system of events if:

- $\forall i \in \{1, \ldots, n\}, P(B_i) \neq 0$ $(B_i \neq\ )$
- $\forall i, j \in \{1, \ldots, n\}, P(B_i \cap B_j) = 0$ $(B_i \cap B_j =)$
- $P\left(\bigcup_{i=1}^{n} B_i\right) = 1$ or equivalently $\sum_{i=1}^{n} P(B_i) = 1$ $(=> \cup B_i = \Omega)$

In other words, the family $\{B_1, \ldots, B_n\}$ forms a probabilistic partition of the universe Ω. It is possible to adapt the definition for a countable family of events.

For example, if $0 < P(A) < 1$, then the family $\{A, A\}$ forms a complete system of events. Similarly, provided the following events have non-zero probabilities, the following family: $\{A \cap B, \overline{A} \cap B, A \cap \overline{B}, \overline{A} \cap \overline{B}\}$ is a complete system of events.

Theorem 2.2: Law of Total Probability

Let $\{B_1, \ldots, B_n\}$ be a complete system of events, and A be any event. Then:

$$P(A) = \sum_{i=1}^{n} P(B_i) P(A|B_i)$$

Theorem 2.3: Bayes' Formula

Let A and B be two events of non-zero probability. Then we have the identity:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

If the family $\{A_1, \ldots, A_n\}$ is a complete system of events, then for $1 \leq i \leq n$ we have:

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Bayes' formula makes it possible, by knowing the effects, to trace back to a probability on the causes. It is the basis of Bayesian inference.

Definition 2.9: Independence

Event A is independent of event B if the probability of its occurrence does not depend on whether or not B has occurred. In other words: $P(A|B) = P(A)$

For example, if two dice are rolled, knowledge of the value of the first die has no influence on the value of the second die. The events "the first die is a 6" and "the second is even" are independent.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\sigma-$algebras in $\mathcal{F}$, then we say $\mathcal{F}_1 \perp \mathcal{F}_2$ (independent) if

$$\forall A \in \mathcal{F}_1, \forall B \in \mathcal{F}_2 : P(A \cap B) = P(A)P(B)$$

Definition 2.10: Mutual Independence in a Family of Events: Infinite Subset I.

Events $A_1, \dots, A_n$ are said to be mutually independent if for any subset I of $\{1, \dots, n\}$ we have:

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

Similarly, the σ-algebras $F_1, \dots, F_n$ are mutually independent if for any events $A_1, \dots, A_n$ such that: $A_1 \in F_1, \dots, A_n \in F_n$, the events $A_1, \dots, A_n$ are independent.

Independence vs Disjoint

- Independence: Two events A and B are said to be independent if: $P(A \cap B) = P(A) \cdot P(B)$. In simple terms, if two events A and B are independent, the occurrence of one event does not provide any information about the occurrence or non-occurrence of the other event. They are unrelated in terms of probability.
- Disjoint (Mutually Exclusive): Two events A and B are said to be disjoint or mutually exclusive if they cannot occur at the same time. In other words, if one event happens, the other cannot happen simultaneously. Mathematically, A and B are disjoint if $A \cap B = \emptyset$, where $\emptyset$ represents the empty set. Disjoint events are not independent because if one event occurs, it implies that the other event cannot occur. Therefore, there is a strong relationship between them, but it's a relationship of exclusion.

Theorem 2.4: Mutually Independent Events

Let $F_1, \dots, F_n, G_1, \dots, G_m$ be mutually independent σ-algebras. Then:

$$\sigma\left(\bigcap_{i=1}^{n} F_i\right) \perp \sigma\left(\bigcap_{j=1}^{m} G_j\right)$$

For example, let A_1, A_2, A_3, A_4, A_5 be mutually independent events. Then:

$$(A_1 \cup A_3) \perp (A_2 \cap (A_4 \cup A_5))$$

Generally speaking, if we separate a family of mutually independent events into two groups, then an event constructed from the first group and the operations $\cap$, $\cup$, and complement will be independent of an event constructed from the second group and the same operations. Mutual independence must be understood in terms of the independence of information. The information provided by knowledge of A_1 and A_3 is independent of the information generated by knowledge of A_2, A_4, and A_5.

Definition 2.11: Mutual Independence in a Family of Events: Finite Subset I

A family $(A_n)_{n\in\mathbb{N}}$ of events is said to be mutually independent if for any finite subset I of $\mathbb{N}$ we have:

$$P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i)$$

Similarly, the σ-algebras $(F_n)_{n\in\mathbb{N}}$ are mutually independent if for any family of events $(A_n)_{n\in\mathbb{N}}$ such that:

$$\forall n \in \mathbb{N}, A_n \in F_n$$

the events $(A_n)_{n\in\mathbb{N}}$ are mutually independent.

Limit Theorems

Definition 2.12: Limit Superior | Limit Inferior

Let $(A_n)_{n\geq 0}$ be a sequence of sets. The limit superior of the sequence is defined as:

$$\limsup_{n\geq 0} A_n = \bigcap_{n\geq 0} \bigcup_{k\geq n} A_k,$$

and the limit inferior of the sequence is defined as:

$$\liminf_{n\geq 0} A_n = \bigcup_{n\geq 0} \bigcap_{k\geq n} A_k.$$

The interpretations of these two quantities are as follows:

- An element $\omega \in \Omega$ belongs to the set $\limsup_{n\geq 0} A_n$ if and only if ω belongs to an infinite number of events in the sequence $(A_n)_{n\geq 0}$.
- An element $\omega \in \Omega$ belongs to the set $\liminf_{n\geq 0} A_n$ if and only if ω belongs to all events of the sequence $(A_n)_{n\geq 0}$ starting from a certain rank.

Theorem 2.5: Borel-Cantelli

Let $(A_n)_{n\geq 0}$ be a sequence of events in a probability space (Ω, F, P). If

$$\sum_{n\geq 0} P(A_n) < +\infty$$

then

$$P\left(\limsup_{n\geq 0} A_n\right) = 0$$

If the events $(A_n)_{n\geq 0}$ are mutually independent and

$$\sum_{n\geq 0} P(A_n) = +\infty$$

then

$$P\left(\limsup_{n\geq 0} A_n\right) = 1$$

In other words, if the series of terms $(P(A_n))_{n\geq 0}$ converges, then the events in the sequence $(A_n)_{n\geq 0}$ will almost surely only occur a finite number of times.

Conversely, if we add the assumption of independence of the sequence $(A_n)_{n\geq 0}$ and the series of terms $(P(A_n))_{n\geq 0}$ diverges, then the events of the sequence $(A_n)_{n\geq 0}$ occur infinitely often almost surely.

Examples: 1. Sequence: $P(A_1) = 1/2, P(A_2) = 1/4, P(A_3) = 1/8, ..., P(A_n) = 1/2^n$
$\sum P(A_n) = \sum \frac{1}{2^n} < \infty$
=> $P(\limsup_{n\geq 0} A_n) = 0$

2. Sequence: $P(A_1) = 1, P(A_2) = 1/2, P(A_3) = 1/4, ..., P(A_n) = 1/n$
   $\sum P(A_n) = \sum \frac{1}{n} = +\infty$
   => $P(\limsup_{n\geq 0} A_n) = 1$

This remains unchanged if we don't change the tail (asymptotic events depend on A_n).

$P(x_n$ converges to $E[x]) = 0$ or 1, for 1 - theorem of large numbers.

In the following, $(\Omega, \mathcal{F}, P)$ is a probability space, and $(F_n)_{n\geq 0}$ is a sequence of mutually independent σ-algebras included in $\mathcal{F}$.

Definition 2.13: Asymptotic σ-algebra

We define the asymptotic σ-algebra:

$$F_\infty = \bigcup_{n\geq 0} \sigma\left(\bigcup_{k\geq n} F_k\right)$$

The asymptotic σ-algebra groups events that depend only on the "tail" of the σ-algebra sequence $(F_n)_{n\geq 0}$, i.e., that do not depend on the first elements of the σ-algebra sequence. An event in F_∞ is called an asymptotic event.

Theorem 2.6: Kolmogorov's 0-1 Law

Let A be an event of the asymptotic σ-algebra F_∞. Then

$$P(A) \in \{0, 1\}$$

In other words, an event in the asymptotic σ-algebra either almost surely occurs, or almost surely never occurs. There is no in-between possibility. On the other hand, it can be difficult to know in which case we are. Typical examples of asymptotic events are: "Is a sequence of independent random variables bounded? Does it converge?"

Exercices

Show that a σ-algebra is stable by intersection (1), by ensemblistic difference (2)

1. $\overline{A \cup B} = \overline{A} \cap \overline{B}; \overline{\overline{A} \cup \overline{B}} = A \cap B (A, B \in F, \overline{A}, \overline{B} \in F, A \cup B \in F)$
2. Take arbitrary sets from sigma; ensemblic difference is $A \setminus B = A \bigcap B^C$. We know that sigma is stable by intersection and complementary.

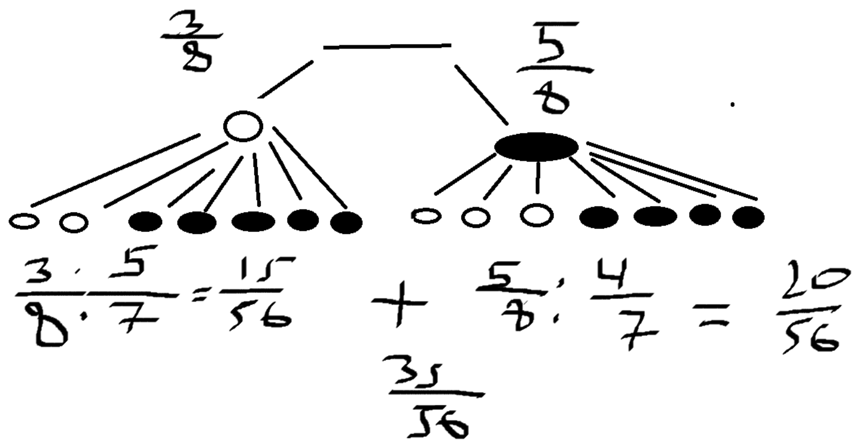
Let Ω be any set. Show that $\{, \Omega\}$ and $\mathcal{P}(\Omega)$ are σ-algebras on Ω. Show that the intersection of a family of σ-algebras is still a σ-algebra. Is the union of two σ-algebras a σ-algebra?

1. By axiom 1: $\Omega \in F$. By axiom 2 and that the set difference between the set and self we have: $\Omega \setminus \Omega = \Omega \bigcap \overline{\Omega} \in F = \in F$
2. Let Ω be a set, $\mathcal{P}(\Omega)$ is its power set. $\forall A, B \in \mathcal{P}(\Omega) : A \cup B \in \mathcal{P}(\Omega)$. $\forall A \in \mathcal{P}(\Omega) : \overline{A} \in \mathcal{P}(\Omega)$. Let $(A_n)_{n \geq 0} \in \mathcal{P}(\Omega)$ be countable infinite sequence of sets. Since power set is closed under countable unions: $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{P}(\Omega)$, we have that $\mathcal{P}(\Omega)$ is sigma-algebra by definition. ### Let $\Omega = \{a, b, c, d, e\}$. Describe the σ-algebra $\sigma(\{\{a, b\}, \{e\}\})$ $\Omega = \{a, b, c, d, e\}$ $F(\{a, b\}, e) = \{\{\Omega\}, \{\}, \{a, b\}, \{e\}, \overline{\{a, b\}} = \{c, d, e\}, \overline{\{e\}} = \{a, b, c, d\}, \{a, b\} \cup \{e\} = \{a, b, e\}, \{c, d\}\}$ ### Let $(\Omega, F, \mathbb{P})$ be a probability space. Show that:
3. $\mathbb{P}() = 0$: A probability measure P on a measurable space (Ω, F) is an application: $P(\Omega) = 1$ (the universe is an event with probability 1). If $(A_n)_{n \geq 0}$ is a countable family of pairwise disjoint events, then: $P(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} P(A_n)$. ALSO: $\forall A \in F : 0 \leq P(A)$ Since $\Omega \cup = \Omega$ we have: $P(\Omega) = P(\Omega \cup) = P(\Omega) + P()$. Since $P(\Omega) = 1$ it follows that $P() = 0$ Or we can prove it like this: $P(\Omega) = P(\Omega \cup) = P(\Omega) + P()$. Since $P(\Omega) = 1$, we have $P() = 0$. Or prove it through complement: $= \overline{\Omega}, P() = 1 - P(\Omega) = 0$
4. If $A \subset B$, show $P(A) \leq P(B)$. $B = A \cup (B \setminus A)$ (make sets disjoint). $=> P(B) = P(A) + P(B \setminus A)$. Since $P(B \setminus A) \geq 0, P(B) \geq P(A)$.
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $P(A \cup B) = P(A) + P(B \setminus A) = P(A) + P(B) - P(A \cap B)$
6. $P(\overline{A}) = 1 - P(A)$. $P(\Omega) = P(A) + P(\overline{A}), 1 = P(A) + P(\overline{A}), P(\overline{A}) = 1 - P(A)$
7. If $(A_n)_{n \geq 0}$ is increasing sequence of events: $P(\bigcup_{n=0}^{\infty} A_n) = \lim_{n \to \infty} P(A_n)$. Since A_n increasing, $n \geq 0 : A_n \subset A_{n+1}$. Set $B_0 = 0, B_n = A_n \setminus A_{n-1}$ (so that they are disjoint). We have $A_n = \bigcup_{k=0}^{n} B_k$. Thus:

$P(\bigcup_{n\geq 0} A_n) = P(\bigcup_{n\geq 0} B_n) = (axiom2) = \sum_{n=0}^{\infty} P(B_n)$. Since $P(B_n) = P(A_n) - P(A_{n-1}) : \sum_{n=0}^{N} P(B_n) = P(B_0) + \sum_{n=0}^{N}(P(A_n) - P(A_{n-1})) = P(B_0) - P(A_0) + P(A_N) = P(A_N).P(\bigcup_{n\geq 0} A_n) = \lim_{N\to\infty} \sum_{n=0}^{N} P(B_n) = \lim_{N\to\infty} P(A_N)$.

Given an urn containing 3 white balls and 5 black balls. Two balls are drawn successively at random. Draw a tree. What is the probability that the second ball drawn is black.

35/56. Since two events are disjoint, we can sum them ($P(\bigcup_{n=0}^{N} A_n) = \sum P(A_n)$)



We have two urns. The first urn contains 2 white balls and 5 black balls. The second urn has 5 white balls and 6 black balls. I choose one of the two urns uniformly at random. Then I draw a ball uniformly at random from the chosen urn. Make a tree. If the ball I've drawn is white, what's the probability that it comes from the first urn?

$P(U_1|W_1) = \frac{P(U_1 \cap W_1)}{P(W_1)}$. $P(W_1) = 1/2*2/7+1/2*5/11 = 57/154$. $P(U_1 \cap W_1) = 1/2*2/7 = 1/7$. $=> P(U_1|W_1) = 1/7*154/57 = 22/57$

Consider a throw of two dice. Consider the events : Event A : "The first die is odd". Event B : "The second die is even". Event C : "The sum of the dice is odd". Show that events A, B and C are pairwise independent, but are not mutually independent.

Events A and B are said to be pairwise independent if the probability of both A and B occurring is equal to the product of their individual probabilities, i.e., $P(A \cap B) = P(A) * P(B)$
Events A, B, and C are said to be mutually independent if the probability of all of them occurring together is equal to the product of their individual probabilities, i.e., $P(A \cap B \cap C) = P(A) * P(B) * P(C)$

We have: $|\Omega| = 36. P(A) = 1/2, P(B) = 1/2, P(C) = 1/2. |A \cap B| = 9, P(A \cap B) = 9/36 = 1/4 => P(A \cap B) = P(A) * P(B). |A \cap C| = 9. P(A \cap C) = 9/36 = 1/4 => P(A \cap C) = P(A) * P(C)$ (same with B). $|A \cap B \cap C| = 9, P(A \cap B \cap C) = 9/36 = 1/4 => P(A \cap B \cap C) \neq P(A) * P(B) * P(C) \neq 1/8$

Random Variable

Random variable X is a mapping to which a configuration, a given outcome (element of Ω) associates an object, a number, a color, etc. **A random variable models the different values that the outcome of a random experiment can take**.

- The roll of the dice. Consider the random variable $X : \Omega \to \{1, 2, 3, 4, 5, 6\}$ the application to which a given configuration associates the value of the dice.
- The coin toss. Consider the random variable $X : \Omega \to \{"Head", "Tail"\}$ the application to which a given configuration associates the value of the coin.
- The dart game. Consider the random variable $X : \Omega \to \mathbb{R}^+$ the application to which a given configuration associates the distance of the dart from the center of the target.
- The throw of two dice. Consider the random variable $X : \Omega \to \{2, \ldots, 12\}$ the application to which a given configuration associates the sum of the two dice.

For coin game, we have X mapping:

$$X : \Omega \to \{"Head", "Tail"\}, \omega \to x(\omega)$$

We have to ensure that $\{\omega \in \Omega : x(\omega) = "Tail"\}$ (*Notation:* $\{\omega \in \Omega : x(\omega) = "Tail"\} = \{X = "Tail"\}$) is an event, i.e. an element of the σ-algebra $\mathcal{F}$. Indeed, the proposition "the probability that the coin lands on tails is 1/2" is written mathematically:

$$\mathbb{P}(X = "Tails") = 1/2$$

Countable Space

If X takes value in a countable (finite) space we require that for any x in E the set

$$\{X = x\} = \{\omega \in \Omega \,|\, X(\omega) = x\}$$

is an event (i.e., belongs to the σ-algebra $\mathcal{F}$). Since any subset U of E can be written as a countable union of singletons, then:

$$\{X \in U\} = \bigcup_{x \in U} \{X = x\}$$

And thus:

$$\forall U \subset E, \{X \in U\} \subset \mathcal{F}$$

Uncountable Space (X is Real-Valued)

If we ask only that the sets $\{X = x\}$ are events, we can't assert that the set:

$$\{X \in [0,1]\} = \bigcup_{x \in [0,1]} \{X = x\}$$

is an event, since a σ-algebra is only stable by countable union, and the set $[0,1]$ is uncountable. We need to restrict the collection of subsets we're allowed to look at. The minimum we can require is to be able to make sense of propositions like "the probability that my dart arrives within 20cm of the target is 1/2". To achieve this, the set:

$$\{X \leq 20\} = \{\omega \in \Omega \,|\, X(\omega) \leq 20\}$$

is an event. In general, we require that for any $x \in \mathbb{R}$, the set:

$$\{X \leq x\} = \{X \in]-\infty, x]\}$$

is an event. From this, we can that the following subsets of Ω are events (the set of events forms a σ-algebra. So the intersection, union and complement of events are still events):

$$\{X < x\} = \bigcup_{n \geq 0} \left\{X \leq x - \frac{1}{n}\right\}$$

$$\{X > x\} = \{X \leq x\}$$

$$\{X \geq x\} = \{X < x\}$$

$$\{x \leq X \leq y\} = \{X \in [x,y]\} = \{X \geq x\} \cap \{X \leq y\}$$

$$\{X = x\} = \{x \leq X \leq x\}$$

Theorem 3.1: Dynkin System

Let $X : (\Omega, \mathcal{F}, P) \to (E, G)$ be an application. Suppose that the σ-algebra $\mathcal{G}$ is generated by a set C of subsets of E, stable by intersections. This means:

$$\mathcal{G} = \sigma(C)$$

$$\forall U, V \in C, U \cap V \in C$$

Then X is a random variable $\iff$

$$\forall U \in C, \{\omega \in \Omega \,|\, X(\omega) \in U\} \in \mathcal{G}$$

In other words, it's enough to check that $\{X \in U\}$ is an event for all $U \in C$ to deduce that it's an event for all $A \in G$. This allows us to clarify the definition of a random variable in the two cases we're interested in: real random variables and discrete random variables.

Definition 3.1: Discrete Random Variable

Let X be an application of a probability space $(\Omega, \mathcal{F}, P)$ with values in a countable space E, provided with the discrete σ-algebra. This means

$$X : (\Omega, \mathcal{F}, P) \to (E, P(E))$$

We say that X is a discrete random variable with values in E, if for any $e \in E$, the set $\{X = e\} = \{\omega \in \Omega | X(w) = e\}$ is an event of the σ-algebra $\mathcal{F}$.

Definition 3.2: Real Random Variable

Let X be an application of a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}$, provided with the Borelian σ-algebra $\mathcal{B}(\mathbb{R})$ ($B(\mathbb{R})$ is the σ-algebra generated by the intervals in $\mathbb{R}$), i.e.

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

We say that X is a real random variable if for any $x \in \mathbb{R}$, the set $\{X \leq x\} = \{\omega \in \Omega | X(w) \leq x\}$ is an event of the σ-algebra $\mathcal{F}$.

Distribution of Random Variables

Définition 3.2: The Distribution of a Random Variable X

The distribution of a random variable

$$X : (\Omega, \mathcal{F}, P) \to (E, G)$$

is defined for any

$$U : (\Omega, \mathcal{F}, P) \to (E, G)$$

as follows:

$$P(X \in U) = P(\{\omega \in \Omega \,|\, X(\omega) \in U\})$$

Theorem 3.2: Dynkin System: Distribution of a Random Variable

If the σ-algebra G is generated by a set C of subsets of E, stable by intersection, then the distribution of a random variable X is entirely characterized by giving for any $U \in C$, the quantities:

$$P(X \in U)$$

Definition 3.3: Distribution of Discrete Random Variable

Let X be a discrete random variable with values in a countable space E. The distribution of X is entirely characterized by the quantities:

$$P(X = e)$$

for $e \in E$ (X is a collection of quantities $\mathbb{P}(X = e)$)

For example, if the random variable X models the number of people in a queue, then the distribution of X is characterized by giving, for $k \in \mathbb{N}$, the quantities: $P(X = k)$ ($P(X = 0)$ - no one waits in the queue, $P(X = 1)$ - 1 person in the queue, etc.).

Definition 3.4: Distribution of Real Random Variable

Let X be a real random variable. The distribution of X is entirely characterized by the quantities:

$$P(X \leq x)$$

for $x \in \mathbb{R}$.

For example, if the random variable X models the distance of a dart from the center of the dartboard, then its distribution is characterized by giving for $X \in \mathbb{R}^+$ the quantities:

$$P(X \leq x)$$

Cumulative Distribution Function (CDF)

Let X be a real random variable. The function F defined by:

$$F_X : \mathbb{R} \to [0, 1]$$

$$x \mapsto P(X \leq x)$$

is called the cumulative distribution function of the variable X.

The cumulative distribution function characterizes the distribution of X. This function is central to the study of real random variables.

Example:
Let X be a random variable that models the uniform drawing of a real number between 0 and 1. Its CDF is:

We know that $P(X \leq 0) = 0$ since the numbers are $\in [0, 1]$. We also know that $P(X \leq 1) = 1$ since the upper bound is 1. And between $[0, 1]$ is uniform distribution, so the probabilities are the same for each interval $[a, b] \in [0, 1]$.

Another example:
Let X be a random variable that models a balanced throw of a die. It is a real random variable since $\{1, 2, 3, 4, 5, 6\} \subset \mathbb{R}$. Its cumulative distribution function is as follows:

In general, when X is a discrete random variable, its cumulative distribution function is piecewise constant, and the points of discontinuity are exactly the values reached by X with non-zero probability.

Theorem 3.1: CDF Properties

The cumulative distribution function F_X of a random variable verifies the following properties:

- F_X is increasing and:
  - $\lim_{x \to -\infty} F_X(x) = 0$
  - $\lim_{x \to +\infty} F_X(x) = 1$
- F_X is continuous on the right: If $(x_n)_{n \geq 0}$ is a decreasing sequence that converges to a real number x, then:
  - $\lim_{n \to +\infty} P(X \leq x_n) = P(X \leq x)$
- F_X admits a left limit at any point: If $(y_n)_{n \geq 0}$ is an increasing sequence that converges to a real number y, then:
  - $\lim_{n \to +\infty} P(X \leq y_n) = P(X < y)$

A function F verifying the last to points is said to be *cadlag* (continious on the right, limit on the left).

Demonstration

Let $x, y \in \mathbb{R}$ be real points s.t. $x \leq y$. Then:

$$\{X \leq x\} \subset \{X \leq y\}$$

$$P(X \leq x) \leq P(X \leq y)$$

Furthermore, let $(x_n)_{n \in \mathbb{N}}$ be a sequence increasing towards infinity. The events $(\{X \leq x_n\})_{n \in \mathbb{N}}$ form an increasing sequence of events whose union is $\mathbb{R}$ any number. Hence:

$$\lim_{n \to +\infty} F_X(x_n) = \lim_{n \to +\infty} P(X \leq x_n)$$

$$= P\left(\bigcup_{n=0}^{\infty} \{X \leq x_n\}\right)$$

$$= P(X \in \mathbb{R})$$
$$= 1$$

The case for $-\infty$ is similar.

Let $(x_n)_{n\in\mathbb{N}}$ be a decreasing sequence towards a real x. Then:

$$\lim_{n\to+\infty} F_X(x_n) = \lim_{n\to+\infty} P(X \leq x_n)$$
$$= P\left(\bigcup_{n=0}^{\infty} \{X \leq x_n\}\right)$$
$$= P(X \leq x)$$
$$= F_X(x)$$

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence increasing towards a real y.

$$\lim_{n\to+\infty} F_X(y_n) = \lim_{n\to+\infty} P(X \leq y_n)$$
$$= P\left(\bigcup_{n=0}^{\infty} \{X \leq y_n\}\right)$$
$$= P(X < y)$$

Quantile Function

Let F be a function verifying the three points of Theorem 3.1. We define the quantile function $F^{-1} : [0,1] \to \mathbb{R}$ by

$$F^{-1}(u) = \inf\{x \in \mathbb{R} \mid F(x) \geq u\}$$

(in general: $F(x) = y \iff x = F^{-1}(y)$)

Let U be a uniform distribution on $[0,1]$. The quantile function F^{-1} verifies the following properties: - If F is continuous and increasing, then F^{-1} is the reciprocal bijection of F. - The random variable $X = F^{-1}(U)$ has the cumulative distribution function F.

Definition 3.6: Atom

Let $x \in \mathbb{R}$. A real random variable X is said to have an atom at the point x if:

$$P(X = x) > 0$$

Take case where X is discrete (throwing dice).

Theorem 3.3: CDF and Atom | Jump of Discontinuity

Let X be a real random variable. The cumulative distribution function of X exhibits a jump of discontinuity at the point x if and only if X has an atom at the point x. In this case, the jump is of size $P(X = x)$.

Example:
U - uniform $\{0, 100\}$ (age)
X - height of Alex at time U
$X \in [50cm, 181cm]$

Another example:
Consider a random variable X representing the outcome of rolling a fair six-sided die. In this case, X can take values from 1 to 6. Each outcome ($1, 2, 3, 4, 5$, or 6) has a probability of 1/6 of occurring, so $P(X = x) = 1/6$ for each x in the range. Since each value has a non-zero probability, X has an atom at each of these points.

Definition 3.7: Random Variable with Density | Probability Density Function of Distribution of X (PDF)

Let X be a real random variable with a cumulative distribution function F_X. The random variable X is said to have a density if there exists an integrable function f_X such that

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$$

The function f_X is called the probability density of the distribution of X.

The function F_X is a primitive of the function f_X. We deduce that F_X is continuous, so X is atom-free. We also deduce the following theorem.

Theorem 3.4: CDF and PDF

Let X be a real random variable. If its cumulative distribution function F_X is continuous and piecewise derivable, then X is a random variable with probability density f and:

$$f_X = (F_X)' = \frac{d}{dx} F_X$$

Conversely, if X is a random variable that has a density, its characteristic function is continuous, so X has no atom. On the other hand, there are atom-free random variables that do not admit a density (Cantor's staircase is a counterexample).

Let a and b be real numbers with $a < b$, and let X be a random variable with density f_X. Then:

$$P(a \leq X \leq b) = \int_{a}^{b} f_X(t)\, dt$$

For example, let X be a uniform random variable on the set $[0, 1]$. For $0 \leq a \leq b \leq 1$, we have:

$$P(a \leq X \leq b) = b - a = \int_a^b 1 \, dt$$

In other words, the density of the uniform distribution is the function $1_{[0,1]}$.

If now $I = [x, x + \varepsilon]$ is a small interval, then we have:

$$P(x \leq X \leq x + \varepsilon) = \int_x^{x+\varepsilon} f_X(t) \, dt \approx \varepsilon f_X(x)$$

Let X be a random variable with density f_X. The probability that X belongs to a small interval around a point $x \in \mathbb{R}$ is proportional to the size of this interval. The proportionality coefficient is exactly $f_X(x)$.

Theorem 3.5: PDF: Positive and Mass

Let X be a random variable with density f_X. Then:

- f_X is positive.
- f_X has mass 1:

$$\int_{-\infty}^{\infty} f_X(t) \, dt = 1$$

Conversely, if f is a function verifying these two properties, then there exists a random variable X of density f.

Demonstration (Proof)

For the first point, we have for ε a small positive real:

$$0 \leq P(x \leq X \leq X + \varepsilon) \approx \varepsilon f(x)$$

And so f is necessarily positive.

For the second point, we have:

$$1 = P(-\infty < X < +\infty) = P(X \in \mathbb{R}) = \int_{-\infty}^{\infty} f(t) \, dt$$

Memoryless Property

Discrete Random Variable

A discrete random variable X is memoryless with respect to a variable a if, (for positive integers a and b) the probability that X is greater than $a+b$ given that X is greater than a is simply the probability of X being greater than b. Symbolically, we write:

$$P(X > a + b|X > a) = P(X > b)$$

To make this specific, let $a = 5$ and $b = 10$. If our probability distribution is memoryless, the probability $X > 15$ if we know $X > 5$ is exactly the same as the probability of $X > 10$.

Note that this is not the same as the probability of X being greater than 15, as it would be if the events $X > 15$ and $X > 5$ were independent.

Continious Random Variable

We say that a continuous random variable X (over the range of reals) is memoryless if for every real h, t:

$$P(X > t + h|x > t) = P(X > h)$$

Expectation, Variance, Median

Expectation

Simple Random Variable

Let X be a positive real random variable on a probability space. If X is a simple random variable with:

$$X = \lambda_1 1_{A_1} + \ldots + \lambda_n 1_{A_n} = \sum_{i=1}^{n} \lambda_i 1_{A_i}$$

We define the expected value as:

$$E[X] = \lambda_1 P(A_1) + \ldots + \lambda_n P(A_n) = \sum_{i=1}^{n} \lambda_i P(A_i)$$

In the general case, we define the expected value of positive simple variable as:

$$E[X] = \sup\{E[Y] \,|\, Y \text{ simple and } Y \leq X\}$$

We say that X is integrable (or of finite expectation) if:

$$E[X] < +\infty$$

Quelconque Random Variable | Lebesque Integral

Let X be a real random variable quelconque, not necessarily positive. We decompose X into:

$$X = X1_{X\geq 0} + X1_{X<0}$$

If the positive random variable $X1_{X\geq 0}$ and the positive random variable $(-X1_{X<0})$ have finite expectations, we say that X has a finite expectation and define its mean by:

$$E[X] = E[X1_{X\geq 0}] - E[X1_{X<0}]$$

Expectation Properties

Let X and Y be two integrable random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and λ a real number. Then,

- Linearity : For $\lambda \in \mathbb{R}$, we have

$$E[\lambda X + Y] = \lambda E[X] + E[Y]$$

- Monotonicity : If, for almost all $\omega \in \Omega$, we have $X(\omega) \leq Y(\omega)$, then

$$E[X] \leq E[Y]$$

- If X follows a Bernoulli distribution with parameter p $(P(X = 0) = 1 - p, P(X = 1) = p)$, then

$$E[X] = p$$

For example: $E[1_A] = P[A]$

Expectation of a constant = constant.

Transfer Theorem

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, and g be a function from $\mathbb{R}$ to $\mathbb{R}$. Then, the quantity $E[g(X)]$ depends only on the function g and the distribution of X.

If $F_X = F_Y$, then $E[X] = E[Y]$

Expectation Discrete

Let X be a discrete random variable, and let $(x_n)_{n\in\mathbb{N}}$ be the values it can take. Then, X has finite expectation if and only if:

$$\sum_{n=0}^{\infty} |x_n| P(X = x_n) < +\infty$$

In this case:

$$E[X] = \sum_{n=0}^{\infty} x_n P(X = x_n)$$

Moreover, if g is a function from $\mathbb{R}$ to $\mathbb{R}$ (provided the sum converges absolutely):

$$E[g(X)] = \sum_{n=0}^{\infty} g(x_n) P(X = x_n)$$

Expectation Density

Let X be a real random variable with density f_X. Then, X has finite expectation if and only if:

$$\int_{-\infty}^{\infty} |x| f_X(x) dx < +\infty$$

In this case:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Furthermore, if g is a function from $\mathbb{R}$ to $\mathbb{R}$ (provided the integral converges absolutely):

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Remember that $f_X(x)\varepsilon \approx P(X \in [x, x+\varepsilon])$

Example:
Let $g = 1_{[a,b]}$
$E[g(x)] = E[1_{[a,b]}(x)] = P(a \leq X \leq b) = \int_{-\infty}^{+\infty} 1_{[a,b]}(x) f_X(x) dx = \int_a^b f_X(x) dx = P(a \leq X \leq b)$

Markov's Inequality

Let X be a positive random variable with finite expectation, and α a strictly positive real number. Then,

$$P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$$

More generally, if g is a positive and strictly increasing function, then,

$$P(X \geq \alpha) \leq \frac{E[g(X)]}{g(\alpha)}$$

Median

The median of a random variable is a number m such that:

$$P(X \leq m) \geq \frac{1}{2} \text{ and } P(X \geq m) \geq \frac{1}{2}$$

Variance

Let X be a random variable with finite expectation. We denote $\text{Var}(X)$ as the variance of X, defined as:

$$\text{Var}(X) = E\left[(X - E[X])^2\right]$$

If this quantity is finite, we say X has finite variance. In that case, we denote by $\sigma(X) = \sqrt{\text{Var}(X)}$ its standard deviation.

Let X be a random variable with finite variance, and λ a real number. We have:

- $\text{Var}(\lambda X) = \lambda^2 \text{Var}(X)$
- $\text{Var}(X + \lambda) = \text{Var}(X)$
- X is almost surely (a.s.) constant if and only if $\text{Var}(X) = 0$
- $\text{Var}(X) = E[X^2] - E[X]^2$

If X is a discrete random variable with values in $\{x_1, x_2, \ldots\}$ and finite variance, then:

$$\text{Var}(X) = \left(\int_{\mathbb{R}} t^2 f_X(t)dt\right) - \left(\int_{\mathbb{R}} t f_X(t)dt\right)^2$$

Chebyshev's Inequality

Let X be a random variable with finite variance and β a nonzero real number. By using Markov's inequality, we can show that:

$$P(|X - E[X]| \geq \beta) \leq \frac{\mathrm{Var}(X)}{\beta^2}$$

Probability Distributions

Discrete

Bermoulli

This distribution is associated with an experiment having two possible outcomes: 0 or 1. A random variable X follows a Bernoulli distribution with parameter p if:

$$P(X = 0) = 1 - p \text{ and } P(X = 1) = p$$
$$E[X] = p$$
$$\mathrm{Var}(X) = p(1 - p)$$

Poisson

This distribution is associated with the number of people in a queue. A random variable X follows a Poisson distribution with parameter λ if, for $k \in \mathbb{N}^*$:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$
$$E[X] = \lambda$$
$$\mathrm{Var}(X) = \lambda$$

Binomial

This distribution is associated with the repetition of n independent and identically distributed random variables following a Bernoulli distribution with parameter p. A random variable X follows a binomial distribution with parameters n and p if, for k in the set $\{0, 1, \ldots, n\}$:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

If you toss a biased coin n times (with a probability of landing on heads being p), the random variable X counting the number of heads follows a binomial distribution with parameters n and p.

$$E[X] = np$$
$$\mathrm{Var}(X) = np(1-p)$$

Uniform $\mathbb{U}(a, b)$

This is the uniform distribution over the integers $a, a+1, \ldots, b$. We denote $n = b - a + 1$. A random variable X follows a uniform distribution with parameters a and b if for $k \in \{1, ..., n\}$:

$$P(X = k) = \frac{1}{n}$$
$$E[X] = \frac{n+1}{2}$$
$$\mathrm{Var}(X) = \frac{n^2 - 1}{12}$$

Geometric

This distribution is associated with the distribution of the first occurrence of a repeated experiment. A random variable X follows a geometric distribution with parameter p if for all $k \in N$:

$$P(X = k) = (1-p)^{k-1} p^k$$
$$E[X] = \frac{1}{p}$$
$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

Density

Normal $\mathcal{N}(m, \sigma^2)$

This distribution naturally appears in the central limit theorem. It often models random fluctuations of a parameter around its average value (temperature, white noise, stock prices, etc). Its density is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}}$$
$$m = E[X]$$
$$\sigma^2 = \mathrm{Var}(X)$$

Exponential (Memoryless) $\varepsilon(\lambda)$

This is the only family of memoryless distributions. Atoms die randomly.

$$f_X(t) = \lambda e^{-\lambda t} 1_{[0,+\infty]}(t)$$

$$E[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Uniform on Interval $\mathcal{U}([a,b])$

This is the uniform distribution over a bounded interval $[a,b]$. X follows a uniform distribution over $[a,b]$ if for c,d s.t. $a \leq c \leq d \leq b$:

$$P(c < X < d) = \frac{d-c}{b-a}$$

$$f_X(x) = \frac{1}{b-a} 1_{[a,b]}$$

$$E[X] = \frac{b+a}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$