High Dimensional Statistics | Prof. Dr. Podolskij Mark | Homework 3

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Exercise 3

Let data $X \sim \mathbb{P}_\theta, \theta \in \Theta$. Let θ_0 true parameter, $\hat{\theta}_{\mathrm{ML}}$ the maximum likelihood estimator of θ_0. Let $f : \Theta \to \tilde{\Theta}$ a bijective function. We denote $\mathcal{L}_{\mathbb{X}}(\theta_0)$ as the likelihood function of the parameter θ_0 and data $\mathbb{X} = (X_1, \ldots, X_n)$.

We denote $\tilde{\theta}_0 = f(\theta_0)$. Since f is bijective, the inverse of f exists, i.e. $\theta_0 = f^{-1}(f(\theta_0))$ and thus: $\theta_0 = f^{-1}(\tilde{\theta}_0)$. By the definition of the likelihood function, we have the following:

$$\mathcal{L}_{\mathbb{X}}(\theta_0) = \prod_{i=1}^{n} p_{\theta_0}(X_i) = \prod_{i=1}^{n} p_{f^{-1}(\tilde{\theta}_0)}(X_i) = \mathcal{L}_{\mathbb{X}}(f^{-1}(\tilde{\theta}_0)), \tag{1}$$

with $p_{\theta_0}(X_i)$ - probability density function of X_i.

Since $\hat{\theta}_{\mathrm{ML}}$ is the maximum likelihood estimator of θ_0, it verifies:

$$\mathcal{L}_{\mathbb{X}}(\hat{\theta}_{\mathrm{ML}}) \geq \mathcal{L}_{\mathbb{X}}(\theta), \forall \theta \in \Theta \tag{2}$$

In particular, $\mathcal{L}_{\mathbb{X}}(\hat{\theta}_{\mathrm{ML}}) \geq \mathcal{L}_{\mathbb{X}}(\theta_0)$. Thus, (1) is **maximized** when:

$$\theta_0 = \hat{\theta}_{\mathrm{ML}} \iff f^{-1}(\tilde{\theta}_0) = \hat{\theta}_{\mathrm{ML}} \iff \tilde{\theta}_0 = f(\hat{\theta}_{\mathrm{ML}})$$

Therefore, the maximum likelihood estimator for $\tilde{\theta}_0 = f(\theta_0)$ is $f(\hat{\theta}_{\mathrm{ML}})$.