Signal Processing | Homework 1

Anton Zaitsev | 0230981826

October 11, 2023

Exercise 1

Show properties of Fourier Transform 1 and 3.

Property 1

Fourier Transform of a translated signal is Fourier Transform of original signal multiplied by e^{-ivh}:

$$F(\tau_h f)(v) = F(f)(v)e^{-ivh}, \tau_h(f) = f(t-h)$$

Fourier Transform of a signal $f(t)$ is:

$$F(f)(v) = \int_{\mathbb{R}} f(t)e^{-ivt}dt$$

Fourier Transform of translated signal $f(t-h)$ is:

$$F(f(t-h))(v) = F(\tau_h f)(v) = \int_{\mathbb{R}} f(t-h)e^{-ivt}dt$$

Let $a = t - h$. Then Fourier Transform for a translated signal $f(t-h) = f(a)$ is:

$$F(\tau_h f)(v) = \int_{\mathbb{R}} f(a)e^{-iv(a+h)}da$$

Since e^{-ivh} is a constant, we can take it out of the integral with respect to a:

$$F(\tau_h f)(v) = e^{-ivh} \int_{\mathbb{R}} f(a)e^{-iva}da$$

Since $\int_{\mathbb{R}} f(a)e^{-iva}da = \int_{\mathbb{R}} f(t)e^{-ivt}dt$ we have:

$$F(\tau_h f)(v) = e^{-ivh} \int_{\mathbb{R}} f(t)e^{-ivt}dt = F(f)(v)e^{-ivh}$$

□

Property 3

Fourier Transform of derivative of a signal is derivative of Fourier Transform of original signal, or Fourier Transform of original signal multiplied by iv:

$$F(f')(v) = (F(f))'(v) = ivF(f)(v)$$

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(f)e^{ivt}dv \tag{1.1}$$

$$\frac{d}{dt}f(t) = \frac{d}{dt}\frac{1}{2\pi} \int_{\mathbb{R}} F(f)e^{ivt}dv$$

By using the Leibniz Integral Rule, we can move the derivative inside the integral, since the derivative is over t and we are integrating over v:

$$\frac{d}{dt}f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(f)\frac{d}{dt}e^{ivt}dv = \frac{1}{2\pi} \int_{\mathbb{R}} ivF(f)e^{ivt}dv$$

Let $G(f) = ivF(f)$. Then:

$$\frac{d}{dt}f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} G(f)e^{ivt}dv \tag{1.2}$$

From equation 1.1 we can notice that the Fourier Transform of a signal $f(t)$ is $F(f)$ and from the equation 1.2 the Fourier Transform of a derivative of a signal $f'(t)$ is $G(f) = ivF(f)$. Thus, Fourier Transform of a derivative of a signal is Fourier Transform of original signal multiplied by iv.

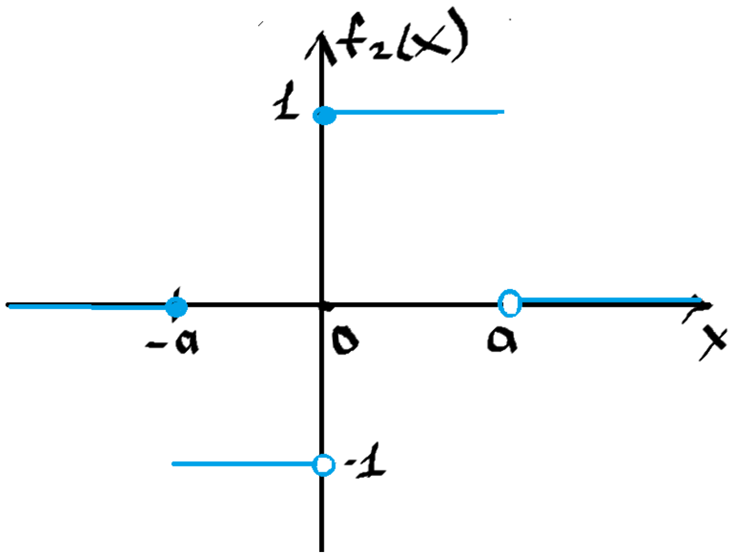
□

Exercise 2

Compute the support, verify if the signal is stable, of finite energy and compute the Fourier Transform, if defined.

$$f_2(t) = \begin{cases} -1 & -a \leq t < 0 \\ 1 & 0 \leq t < a \quad , a > 0 \\ 0 & otherwise \end{cases}$$

Figure 1: Graph of $f_2(t)$



Note: Solid blue line represents the graph of the function $f_2(t)$.

Support

The support of the signal $supp(f_2)$ is the closure of $f_2(t)$ where $f_2(t) \neq 0$. Since $f_2(t) \neq 0$ for $t \in [-a, a)$, the support is equal to:

$$supp(f_2) = \overline{[-a, a)} = [-a, a]$$

Stable

To check whether the signal is stable we have to compute $L^1(f(t))$ and make sure it does not diverge.

$$L^1(f_2(t)) = \int_{\mathbb{R}} |f_2(t)|dt = \int_{-a}^{0} |-1|dt + \int_{0}^{a} |1|dt = \int_{-a}^{a} 1dt =$$

$$\lim_{t \to a} t|_{t=-a}^{t=a} = a - (-a) = 2a$$

Since $L^1(f_2(t)) < \infty$, we proved that it is stable.

Finite Energy

To check whether the signal of finite energy, we have to compute L^2 of the signal.

$$L^2(f_2) = \int_{-a}^{0} |(-1)^2|dt + \int_{0}^{a} |(1)^2|dt = \int_{-a}^{a} dt = \lim_{t \to a}[t]|_{t=-a}^{t=a} = a - (-a) = 2a$$

Since $L^2(f_2(t)) < \infty$, we proved that it is of finite energy.

Fourier Transform

We can evaluate Fourier Transform of a signal $f_2(t)$ as follows:

$$F(f_2)(v) = \int_{-a}^{a} f_2(t)e^{-ivt}dt =$$

$$= \int_{-a}^{0} -e^{-ivt}dt + \int_{0}^{a} e^{-ivt}dt = \frac{1}{iv}e^{-ivt}|_{t=-a}^{t=0} - \frac{1}{iv}e^{-ivt}|_{t=0}^{t=a}$$

$$= \frac{1}{iv}(e^{-ivt}|_{t=-a}^{t=0} - e^{-ivt}|_{t=0}^{t=a}) = \frac{1}{iv}(1 - e^{iva} - e^{-iva} + 1) = \frac{2 - e^{iva} - e^{-iva}}{iv}$$

Exercise 3

Compute the support, verify if the signal is stable, of finite energy and compute the Fourier Transform, if defined.

Signal $f_4(t)$

$$f_4(t) = \cos(at)e^{-|t|}$$

To draw a sketch of the signal, we would need to draw in three dimensions. To make this a little easier for us, we can fix variable a and see how functions behaves for various values of a. Let us fix $a_1 = 0, a_2 = 2, a_3 = 4$ and draw functions for these values of a.

Figure 2: Sketch of $f_4(t), a = 0$

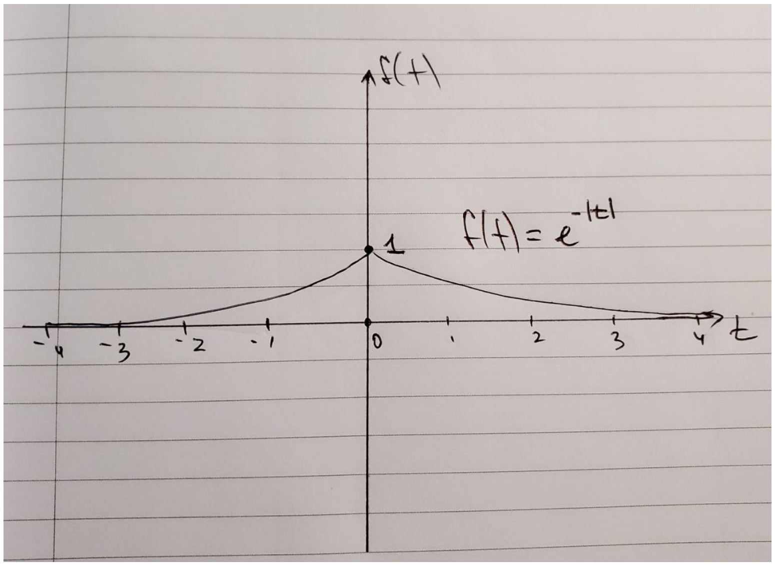


Figure 3: Sketch of $f_4(t), a = 0$

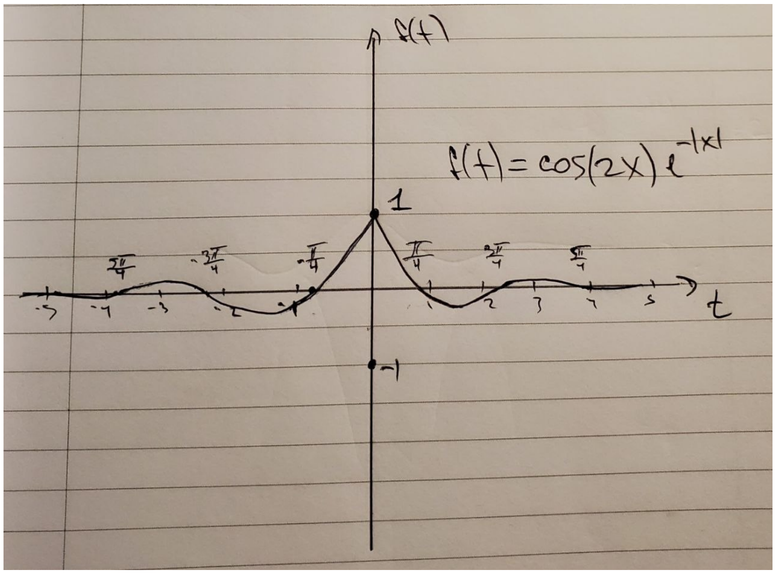
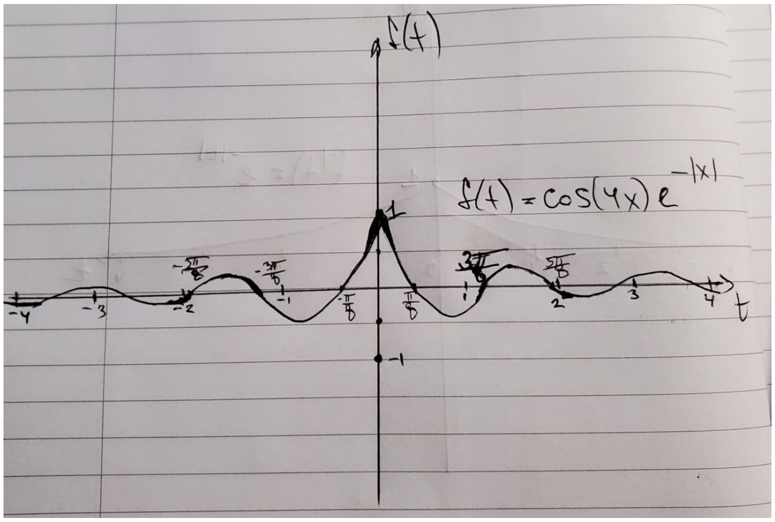


Figure 4: Sketch of $f_4(t), a = 0$



Support

The function $\cos(at)e^{-|t|} = 0 \iff cos(ax) = 0$. The function $cos(ax) = 0$ when $ax = \frac{\pi}{2} + \pi n, n \in \mathbb{N}$. Even though the function equals to 0 at these points, support includes all the points where the function is non-zero and their limit points, even if the function is zero at those points. Thus, the support of $f_4(t)$ is the set of all real numbers:

$$supp(f_4(t)) = \mathbb{R}$$

Stable

Check $L^1(f_4(t))$:

$$L^1(f_4(t)) = \int_{\mathbb{R}} |f_4(t)|dt = \int_{\mathbb{R}} |\cos(at)e^{-|t|}|dt$$

Since the function $f_4(t)$ is even, we have:

$$\int_{\mathbb{R}} |f_4(t)|dt = 2\int_0^{+\infty} f_4(t)dt = 2\int_0^{+\infty} \cos(at)e^{-|t|}dt$$

Note that we integrate over positive values, $t > 0$. Thus, we can expand modulus:

$$\int_{\mathbb{R}} f_4(t)dt = 2\int_0^{+\infty} \cos(at)e^{-t}dt \tag{2.1}$$

Let us solve $\int_0^{+\infty} \cos(at)e^{-t}dt$. We can solve this integral by integrating by parts twice. Let $f = e^{-t}, f' = -e^{-t}, g = \frac{1}{a}\sin(at), g' = \cos(at)$. We have:

$$\int_0^{+\infty} \cos(at)e^{-t}dt = e^{-t}\frac{\sin(at)}{a} - \int_0^{+\infty} (-e^{-t})(\frac{1}{a}\sin(at))dt$$

Now integrate $\int_0^{+\infty}(-e^{-t})(\frac{1}{a}\sin(at))dt$ by parts again: $f = -e^{-t}, f' = e^{-t}, g = -\frac{1}{a^2}\cos(at), g' = \frac{1}{a}\sin(at)$. We get:

$$\int_0^{+\infty} (-e^{-t})(\frac{1}{a}\sin(at))dt = -e^{-t}(-\frac{1}{a^2}\cos(at)) - \int_0^{+\infty} e^{-t}(-\frac{1}{a^2}\cos(at))dt =$$

$$= \frac{1}{a^2}e^{-t}\cos(at) + \frac{1}{a^2}\int_0^{+\infty} e^{-t}\cos(at)dt$$

Now going back to the integral $\int_0^{+\infty}\cos(at)e^{-t}$, combining everything together we have:

$$\int_0^{+\infty} \cos(at)e^{-t}dt = e^{-t}\frac{\sin(at)}{a} - \frac{1}{a^2}e^{-t}\cos(at) - \frac{1}{a^2}\int_0^{+\infty} e^{-t}\cos(at)dt =$$

$$= (\frac{a^2+1}{a^2})\int_0^{+\infty} e^{-t}\cos(at)dt = \frac{ae^{-t}\sin(at) - e^{-t}\cos(at)}{a^2}$$

$$(\frac{a^2+1}{a^2})\int_0^{+\infty} e^{-t}\cos(at)dt = \frac{e^{-t}(a\sin(at) - \cos(at))}{a^2}$$

$$\int_0^{+\infty} e^{-t}\cos(at)dt = \frac{a\sin(at) - \cos(at)}{e^t(a^2+1)}$$

Now, substitute this fraction to the equation 2.1 and evaluate it:

$$L^1(f_4(t)) = 2\int_0^{+\infty} \cos(at)e^{-t}dt = \lim_{l\to+\infty} \frac{2(a\sin(at) - \cos(at))}{e^t(a^2+1)}|_{t=0}^{t=l} = \frac{1}{a^2+1} \tag{2.2}$$

Thus, we proved that the function $f_4(t)$ is stable.

Finite Energy

Function $f_4^2(t)$ is also even, so we can also evaluate the integral $\forall t > 0$ from 0 to $+\infty$ and multiply it by 2:

$$L^2(f_4(t)) = \int_{\mathbb{R}} |f_4^2(t)|dt = 2\int_0^{+\infty} (|e^{-t}\cos(at)|)^2 dt = 2\int_0^{+\infty} e^{-2t}\cos^2(at)dt$$

We can use trigonometric identity $cos^2(at) = \frac{cos(2at)+1}{2}$:

$$\int_{\mathbb{R}} f_4^2(t)dt = 2\int_0^{+\infty} e^{-2t}\frac{1}{2}(\cos(2at)+1)dt = \int_0^{+\infty} e^{-2t}\cos(2at)dt + \int_0^{+\infty} e^{-2t}dt$$

We can notice that the integral $\int_0^{+\infty} e^{-2t}\cos(2at)dt$ is almost the same as in the equation 2.1. The main difference here is multiplier 2 inside $\cos(2at)$. After integrating this integral twice, we would get the same answer $\frac{1}{a^2+1}$ but the denominator would be multiplied by 2:

$$\int_0^{+\infty} e^{-2t}\cos(2at)dt = \frac{1}{2a^2+2} \tag{2.3}$$

Now let us solve $\int_0^{+\infty} e^{-2t}dt$. This integral can be solved using basic integral rules.

$$\int_0^{+\infty} e^{-2t}dt = -\frac{1}{2}\lim_{l\to+\infty} e^{-2t}|_{t=0}^{t=l} = \frac{1}{2}$$

Thus, the function $f_4(t)$ is of finite energy:

$$L^2(f_4(t)) = \frac{1}{2a^2+2} + \frac{1}{2} = \frac{a^2+2}{2a^2+2}$$

Fourier Transform

Since the function $f_4(t)$ is even, the Fourier Transform of $f_4(t)$ is real. Thus:

$$F(f_4) = \int_{\mathbb{R}} f_4(t)\cos(vt)dt = \int_{\mathbb{R}} e^{-|t|}\cos(at)\cos(vt)dt$$

We can use trigonometric identity: $\cos(at)\cos(vt) = \frac{1}{2}[cos(at+vt) + cos(at-vt)]$. Thus, we get:

$$F(f_4) = \frac{1}{2}\int_{\mathbb{R}} e^{-|t|}[cos(t(a+v)) + cos(t(a-v))]dt$$

Function $e^{-|t|}[cos(t(a+v)) + cos(t(a-v))]$ is even, symmetric with respect to the ordinate, thus:

$$F(f_4) = \int_0^{+\infty} e^{-t}[cos(t(a+v)) + cos(t(a-v))]dt =$$

$$= \int_0^{+\infty} e^{-t} cos(t(a+v))dt + \int_0^{+\infty} e^{-t} cos(t(a-v))dt$$

Next, we need to evaluate each integral. Again, the main difference between these two integrals and the ones in 2.2 and 2.3 is what inside the cosine function. We can simply skip the computation part and move to the summation of evaluated integrals:

$$\int_0^{+\infty} e^{-t} cos(t(a+v))dt = \frac{1}{(a+v)^2+1}$$

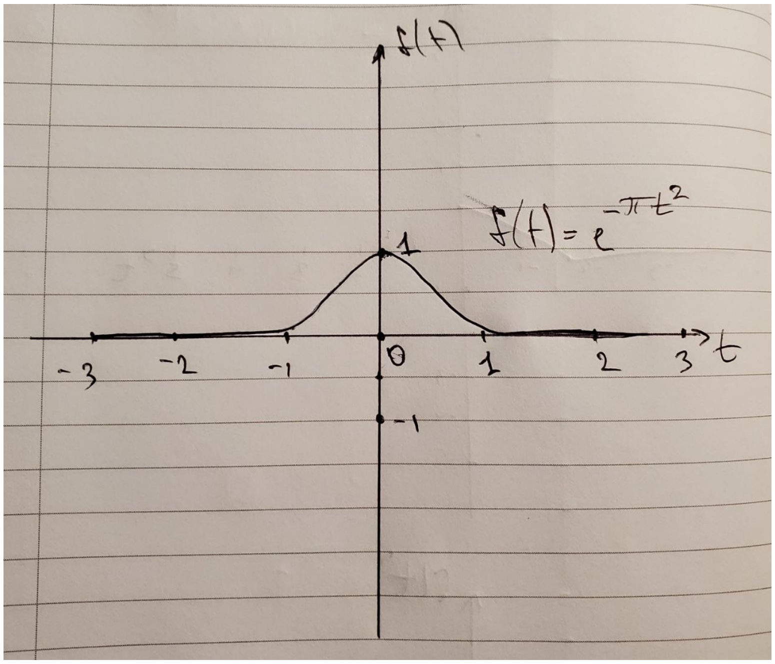
$$\int_0^{+\infty} e^{-t} cos(t(a-v))dt = \frac{1}{(a-v)^2+1}$$

$$F(f_4) = \frac{1}{(a+v)^2+1} + \frac{1}{(a-v)^2+1} = \frac{2(a^2+v^2+1)}{((a+v)^2+1)((a-v)^2+1)}$$

Signal $f_5(t)$

$$f_5(t) = e^{-\pi t^2}$$

Figure 5: Sketch of $f_5(t)$



Support

The function $f_5(t) \neq 0 \forall t \in \mathbb{R}$, thus:

$$supp(f_5(t)) = \mathbb{R}$$

Stable

To verify $f_5(t)$ is stable, compute $L^1(f_5(t))$:

$$L^1(f_5(t)) = \int_{\mathbb{R}} |e^{-\pi t^2}| dt \tag{3.1}$$

Since the exponent function is always positive, we have:

$$L^1(f_5(t)) = \int_{\mathbb{R}} |e^{-\pi t^2}| dt = \int_{\mathbb{R}} e^{-\pi t^2} dt$$

To find the value of this integral, we can switch to polar coordinates. First, let us define integral I:

$$I = \int_{\mathbb{R}} e^{-\pi t^2} dt = \int_{\mathbb{R}} e^{-\pi x^2} dx = \int_{\mathbb{R}} e^{-\pi y^2} dy$$

$$I^2 = \int_{\mathbb{R}} e^{-\pi x^2} dx \cdot \int_{\mathbb{R}} e^{-\pi y^2} dy = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\pi (x^2+y^2)} dx dy$$

Switch to polar coordinates:

$$\int_0^{2\pi} \int_0^{\mathbb{R}} e^{-\pi r^2} r dr d\theta$$

Let us compute $\int_0^{\mathbb{R}} e^{-\pi r^2} r dr$. We can compute it using u-substitution: $u = r^2$, $\frac{du}{dt} = 2r, dr = \frac{du}{2r}$. Thus, we have:

$$\int_0^{\mathbb{R}} e^{-\pi r^2} r dr = \int_0^{\mathbb{R}} e^{-\pi u} r \frac{du}{2r} = \frac{1}{2} \int_0^{\mathbb{R}} e^{-\pi u} du =$$

$$= -\frac{1}{2\pi}\lim_{l\to+\infty} e^{-\pi u}|_{u=0}^{u=l} = \frac{1}{2\pi}$$

Next, find $\int_0^{2\pi} d\theta$:

$$\int_0^{2\pi} d\theta = \theta|_{\theta=0}^{\theta=2\pi} = 2\pi$$

Thus, we have:

$$I^2 = \frac{1}{2\pi}\cdot 2\pi = 1$$

$$I = \pm 1$$

Since $\int_{\mathbb{R}} e^{-\pi t^2} dt > 0$:

$$L^1(f_5(t)) = 1$$

Thus, $f_5(t)$ is stable.

Finite Energy

To see if $f_5(t)$ is of finite energy, evaluate $L^2(f_5(t))$:

$$L^2(f_5(t)) = \int_{\mathbb{R}} |e^{-\pi t^2}|^2 dt = \int_{\mathbb{R}} e^{-2\pi t^2} dt$$

We can solve it the same way we solved 3.1. Since this integral is similar to $L^1(f_5(t))$, we can notice that when we switch to polar coordinates, 2 from the power of the exponent will go to the denominator, thus:

$$\int_0^{\mathbb{R}} e^{-2\pi r^2} r dr = \frac{1}{4\pi}$$

Then, we can finish the integral evaluation:

$$I^2 = \frac{1}{4\pi}\int_0^{2\pi} d\theta = \frac{1}{4\pi}\theta|_{\theta=0}^{\theta=2\pi} = \frac{1}{2}$$

$$I = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Hence the signal $f_5(t)$ is of finite energy:

$$L^2(f_5(t)) = \frac{\sqrt{2}}{2}$$

Fourier Transform

$$F(f_5) = \int_{\mathbb{R}} e^{-\pi t^2} e^{-ivt} dt = \int_{\mathbb{R}} e^{-(\pi t^2 + ivt)} dt$$

We can notice that $e^{-(\pi t^2+ivt)}$ is almost in the form that can be converted to polar coordinates. We need to convert $(\pi t^2 + ivt)$ to the form $(at+y)^2$, so that we can move e^y from the integral (as a constant) and switch $e^{(at)^2}$ to the polar coordinates (here a and y are just some constants). In fact, we have:

$$(\sqrt{\pi}t + \frac{iv}{2\sqrt{\pi}})^2 = \pi t^2 + ivt - \frac{v^2}{4\pi}$$

$$(\sqrt{\pi}t + \frac{iv}{2\sqrt{\pi}})^2 + \frac{v^2}{4\pi} = \pi t^2 + ivt$$

Thus:

$$F(f_5) = \int_{\mathbb{R}} e^{-(\pi t^2+ivt)} dt = \int_{\mathbb{R}} e^{-((\sqrt{\pi}t+\frac{iv}{2\sqrt{\pi}})^2+\frac{v^2}{4\pi})} dt = \int_{\mathbb{R}} e^{-(\sqrt{\pi}t+\frac{iv}{2\sqrt{\pi}})^2-\frac{v^2}{4\pi}} dt =$$

$$= e^{-\frac{v^2}{4\pi}} \int_{\mathbb{R}} e^{-(\sqrt{\pi}t+\frac{iv}{2\sqrt{\pi}})^2} dt$$

Now we can compute $\int_{\mathbb{R}} e^{-(\sqrt{\pi}t+\frac{iv}{2\sqrt{\pi}})^2} dt$ using u-substitution. Let $u = \sqrt{\pi}t + \frac{iv}{2\sqrt{\pi}}, \frac{du}{dt} = \sqrt{\pi}, dt = \frac{du}{\sqrt{\pi}}$. Thus:

$$\int_{\mathbb{R}} e^{-(\sqrt{\pi}t+\frac{iv}{2\sqrt{\pi}})^2} dt = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du$$

By switching to polar coordinates, we will get that $\int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}$ (Check: $\int_0^{\mathbb{R}} e^{-\pi r^2} r dr = \frac{1}{2}, \int_0^{2\pi} d\theta = 2\pi, I^2 = \pi, I = \sqrt{\pi}$). Hence, the Fourier Transform of $f_5(t)$ is:

$$F(f_5) = e^{-\frac{v^2}{4\pi}} \cdot \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = e^{-\frac{v^2}{4\pi}}$$