

Graph Theory Homework 1

November 2023

1. Exercise 1:

Let $G = (V, E)$ a simple undirected graph. Let $G' = (V', E')$ the simple undirected graph satisfying $V' = E$ and $E' = \{\{e_1, e_2\} \mid e_1, e_2 \in E, e_1 \neq e_2, e_1 \cap e_2 \neq \emptyset\}$. Prove or disprove the following assertions.

- (i) G' is hamiltonian implies that G is eulerian,
- (ii) G is eulerian implies that G' is hamiltonian.

2. Exercise 2:

Let $K_n = (V, E)$ the complete graph. Let $e \in E$. Determine the number of spanning trees of $K_n - e$ and prove that your statement holds.

3. Exercise 3:

Let $G = (V, E, w)$ a simple undirected weighted graph with $w(v) \neq w(u)$ for all $u, v \in V$ such that $u \neq v$. Prove that if T is a minimum spanning tree of G , then it is unique.

4. Exercise 4:

Let $G = (V, E)$ a simple undirected graph such that G contains no cycle and $G + e$ contains a cycle for any $e \in \{\{v, u\} \mid v, u \in V, v \neq u \text{ and } e \notin E\}$. Let $\varphi : G \rightarrow G$ an isomorphism (for graphs). Show that the following holds

$$\exists v \in V \text{ with } \varphi(v) = v \quad \text{or} \quad \exists e \in E \text{ with } \varphi(e) = e.$$