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HOMEWORK 1

Exercise 1. Let $Y \sim \mathcal{N}_n(\mu, \Sigma)$, with $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ being positive definite. Let $A \in \mathbb{R}^{n\times n}$. Show that the following holds true :

1. $\mathbb{E}[Y^T AY] = \mu^T A\mu + \mathrm{tr}(A\Sigma)$, where tr is the trace of the matrix.
2. Assume that A is symmetric. Show : $\mathrm{cov}(Y, Y^T AY) = 2\Sigma A\mu \in \mathbb{R}^n$.

Hint : Use the decomposition $Y = \mu + \Sigma^{\frac{1}{2}} Z$, where $Z \sim \mathcal{N}_n(0, I_n)$. We recall that $\Sigma^{\frac{1}{2}} = PD^{\frac{1}{2}}P^T$, where P is an orthogonal matrix and D is a diagonal matrix such that $\Sigma = PDP^T$.

Exercise 2. Let $Y \sim \mathcal{N}_n(0, \Sigma)$ with $\Sigma \in \mathbb{R}^{n\times n}$ being positive definite, as in previous exercise. Show that, for $A, B \in \mathbb{R}^{n\times n}$, it holds that

$$\mathrm{cov}(Y^T AY, Y^T BY) = \mathrm{tr}(A\Sigma B\Sigma) + \mathrm{tr}(A^T \Sigma B\Sigma).$$

Exercise 3. Show that the estimator $\widehat{\sigma}^2$ is unbiased (i.e. $\mathbb{E}[\widehat{\sigma}^2] = \sigma^2$), where

$$\widehat{\sigma}^2 = \frac{1}{n-k} \left\| Y - X\widehat{b} \right\|_2^2 \qquad \text{and} \qquad \widehat{b} = (X^T X)^{-1} X^T Y.$$

Exercise 4. (Numerical exercise to solve with R or Python).
In an experiment on the analysis of the link between the beats per minute under stress and the age of a sample of 10 men, the following data have been collected :

Beats per minute	200	195	200	190	188	180	185	180	163	170
Age	10	20	21	25	29	30	31	40	45	50

1. Build the linear regression model $Y_i = b_0 + b_1 t_i + \epsilon_i$ and estimate b_0 and b_1.
2. Verify the null hypothesis $H_0 : b_1 = 0$ against the alternative $H_1 : b_1 \neq 0$ with a significance level $\alpha = 0.05$.