- **Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: nonempty set V of vertices (sommets) and set $E \subseteq \binom{V}{2} \cup \binom{V}{1}$ of edges (arrêtes). It is simple if $E \subseteq \binom{V}{2}$. The elements $e \in E$ are **endpoints** of e. If $\{u, v\} \in e$, u is **adjacent** to v. $e = \{v\}$ $(v \in V)$ is a **loop**.

- **Digraph (or directed graph)** (graphe dirigé) $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: nonempty set V of vertices (sommets) and set $E \subseteq V \times V$ of directed edges. It is simple if it is irreflexive, i.e. $(v, v) \notin E$ for any $v \in V$.

- **Weighted graph** (graphe à poids) $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \mathbf{f})$: graph (V, E) and map $f : E \longrightarrow A$ (usually, $A = \mathbb{R}^+$). If $e \in E$, the value of $f(e)$ is the weight of e.

- **Multigraph $\mathbf{G} = (\mathbf{V}, \mathbf{E}, \epsilon)$**: nonempty set V, set E disjoint of V and map $\epsilon : E \longrightarrow \binom{V}{2} \cup \binom{V}{1}$ which maps any edge to its set of endpoints.

- **Complete graph $\mathbf{K_n}$**: graph whose set of vertices is $V_n = \{v_1, ..., v_n\}$ and whose set of edges is $\binom{V_n}{2}$. In other words, all vertices are connected with each other through edges.

- **Path $\mathbf{P_n}$**: graph whose set of vertices is V_n and whose set of edges is $\{\{i, i+1\} \mid i \in V_n \setminus \{n\}\}$. A worm with vertices.

- **Cyclic graph $\mathbf{C_n}$**: graph whose set of vertices is V_n and whose set of edges is $\{\{i, i+1\} \mid i \in V_n \setminus \{n\}\} \cup \{\{n, 1\}\}$. It looks like a path that closes up.

- **Complete bipartite graph $\mathbf{K_{m,n}}$**: graph whose set of vertices is $V \cup U$ (V, U disjoint sets of cardinality n and m respectively) and whose set of edges is $\{\{u, v\} \mid u \in U, v \in V\}$.

- **Bipartite graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: V can be partitioned into two nonempty sets S and T (called **parts** of V) such that $E \subseteq \{\{s, t\} \mid s \in S, t \in T\}$.

- Stable marriage proves the following: $G = (V, E)$ bipartite $(|V| = n)$, S and T parts. Then $\exists$ matching from S into T, i.e. $\exists E' \subseteq E$ s.t. each $s \in S$ is the endpoint of exactly one vertex in E' and each $t \in T$ is the endpoint of at most one vertex in E' $\iff$ for any $k \leq n$, $S' \subseteq S$ $(|S'| = k)$, the set $E(S') = \{t \in T \mid \exists s \in S', \exists e \in E : \{s, t\} \in e\}$ has at least k elements.

- $\deg_G(v) := \#\{e \in E \mid v \in e\}$, $\sum_{v \in V} \deg_G(v) = 2|E|$

- $G = (V, E)$, $V = \{v_1, ..., v_n\}$. A **score of** G is a sequence $(\deg_G(v_1), ..., \deg_G(v_n))$.

- A sequence $D \in \mathbb{N}$ is **graphic** if there exists a graph G of order n such that D is its sequence of degrees.

- $G = (V, E)$, $G' = (V', E')$ graphs. A bijection $f : V \longrightarrow V'$ is an **isomorphism** (i.e. G, G' are isomorphic) if the condition $(\{x, y\} \in E) \iff (\{f(x), f(y)\} \in E')$ is satisfied for $x, y \in V$.

- **Subgraph of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$.

- **Induced subgraph** (sous-graphe induit) **of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' = E \cap \left(\binom{V'}{2} \cup \binom{V'}{1}\right)$. In other words, $E(G')$ is made of the edges of G whose endpoints are in $V(G')$.

- **Spanning subgraph** (graphe couvrant) **of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: graph $G' = (V', E')$ such that $V' = V$ and $\forall v \in V, \exists e \in E' : v \in e$.

- **Path of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$** (of length t): subgraph P of G that is isomorphic to the path P_t. P can be identified with a sequence $(v_0, e_1, v_1, ..., e_t, v_t)$ where $v_0, ..., v_t$ are mutually distinct vertices of G and $e_{i+1} = \{v_i, v_{i+1}\} \in E$ for $i \in \{0, ..., t-1\}$. P is a path from v_0 to v_t and we denote $\|P\|$ the length of the path P.

- **Cycle (or circuit) C of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$** (of length t): subgraph isomorphic to the cycle C_t where $t \geq 3$. C can be identified with a sequence $(v_0, e_1, v_1, ..., e_{t-1}, v_{t-1}, e_t, v_0)$ where $v_0, ..., v_t$ are mutually distinct vertices of G and where $e_t = \{v_{t-1}, v_0\} \in E$ and $\{v_i, v_{i+1}\} \in E$ for $i \in \{0, ..., t-1\}$.

- **Connected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$** (graphe connexe): graph such that for any $u \neq v \in V(G)$, there is a path from u to v.

- $u \sim v$: there is a walk from u to v. The class $u/\sim$ is a connected induced subgraph of G and the elements of $\{u/\sim \mid u \in V\}$ are the **connected components of G**. In particular, G is connected if and only if it has exactly one connected component.

- **distance associated with G**: function $d_G : V^2 \longrightarrow \mathbb{R}^+$ that associates to any $(u, v) \in V^2$ the length of a shortest path from u to v.

- A simple graph is bipartite if and only if it contains no cycle of odd length.

- **Adjacency matrix of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: matrix $A_G \in \mathbb{N}_n^n$ defined by setting $(A_G)_{ij} = 1$ if $\{v_i, v_j\} \in E$ and $(A_G)_{ij} = 0$ otherwise, for every $i, j \in \{1, ..., n\}$. The element $(A_G^k)_{i,j}$ is equal to the number of walks of length k from v_i to v_j.

- Let G be connected. The distance d_G satisfies: $d_G(v_i, v_j) = \min\{k \geq 0 \mid (A_G^k)_{i,j} \neq 0\}$, $1 \leq i \neq j \leq n$. Moreover, any shortest walk from between two vertices of G is a path.

- **Eulerian tour of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: closed walk $(v_0, e_1, v_1, ..., e_t, v_0)$ in which edges are mutually distinct and where $\{e_1, ..., e_t\} = E$. A graph is eulerian if it admits an eulerian tour. More generally, a tour is a walk in which no edge is repeated.

- A simple graph is eulerian if and only if it is connected and each vertex has an even degree.

- **Tour of a digraph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$**: sequence $(v_0, e_1, v_1, ..., e_t, v_t)$ where $e_i = (v_{i-1}, v_i) \in E$ for $i \in \{1, ..., t\}$ and e_i distinct. The tour is eulerian if $E = \{e_1, ..., e_t\}$.

- Let $v_0 \in V$. The **outer degree of $\mathbf{v_0}$** denoted $\deg_-(v_0)$ and **inner degree of $\mathbf{v_0}$** denoted $\deg_+(v_0)$ are $\deg_-(v_0) = |\{u \mid (v_0, u) \in E\}|$ and $\deg_+(v_0) = |\{u \mid (u, v_0) \in E\}|$.

- **Symmetrization of G**: undirected graph $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{\{u, v\} \mid (u, v) \in E \text{ or } (v, u) \in E\}$

- A digraph is eulerian if and only if its symmetrization is connected and $\deg_-(v) = \deg_+(v)$ for any $v \in V$.

- **Hamiltonian cycle of G**: cycle that contains all vertices of G. A graph is hamiltonian if it contains a hamiltonian cycle.

- $\mathbf{G - S}$: graph obtained from G by removing the vertices in S and their adjacent edges. We write $V(G-S) = V \setminus S$ and $E(G-S) = \{\{u,v\} \in E \mid u \in S \text{ or } v \in S\}$. For a vertex s, we write $G - s$ (not $G - \{s\}$!).

- Let $G = (V,E)$ a graph, $S \subseteq V$. If G is hamiltonian then the number of connected components of $G - S$ is at most $|S|$.

- Let $G = (V,E)$ be a simple graph with $|V| \geq 3$. If the degree of any vertex is at least $\frac{|V|}{2}$, then G is hamiltonian.

- Let $G = (V,E)$ be a simple graph with $|V| \geq 3$ and which contains two distinct vertices x and y whose sum of degrees is at least $|V|$. Then G is hamiltonian if and only if so is $G' := (V, E \cup \{x,y\})$.

- Let $G = (V,E)$ be a simple graph. Construct a sequence $G_0, G_1, ...$ in such a way that $G_0 = G$ and that for any $i \geq 0$, $G_{i+1} = (V, E_{i+1})$ where $E_{i+1} = E_i \cup \{\{u,v\}\}$ for some $u, v \in V$ such that $\{u,v\} \notin E_i$ and $\deg_{G_i}(u) + \deg_{G_i}(v) \geq |V|$. Such a sequence eventually stops and the last graph of the sequence is called a **closure of G**. The closure is unique.

- Let $G = (V,E)$ be a simple graph with $|V| \geq 3$. Then G is hamiltonian if and only if so is its closure. Moreover, if the closure of G is a complete graph then G is hamiltonian.

- Let $G = (V,E)$ be a simple graph with $|V| \geq 3$. If for any pair of non adjacent vertices x and y, the sum of the degrees of x and y is at least $|V|$, then G is hamiltonian.

- **Tree**: simple undirected graph which is connected and contains no cycle. A **forest** is a graph whose connected components are trees. A 1-degree vertex is called an **end-vertex** or a **leaf**.

- Let $G = (V,E)$ be a simple graph with $|V| \geq 2$. If G is a tree, it has at least two end-vertices. Moreover, for any end-vertex v of G, the graph $G - v$ is a tree if and only if G is a tree.

- Let $G = (V,E)$ be a graph. If $e \in E$, the graph $G - e$ is defined by $V(G-e) = V$ and $E(G-e) = E \setminus \{e\}$. If $e \in \binom{V}{2}$, the graph $G + e$ is defined by $V(G+e) = V$ and $E(G+e) = E \cup \{e\}$.

- **Rooted tree**: pair (T,v) where T is a tree and v is a distinguished vertex of T called the **root**. Let $e = \{x,y\}$ be an edge of (T,v). We say that x is the **father** of y if it lies on the unique path from v to y. Otherwise, x is y's **son**.

- **Planted tree** $(\mathbf{T}, \mathbf{v}, \preceq)$: rooted tree (T,v) with a total order $\preceq_u$ on the set of the sons of any vertex u of T.

- Let $G = (V,E)$ be a simple graph. Then the following are equivalent:

  (1) G is a tree.

  (2) For every two distinct vertices of G there is a unique path joining them.

  (3) G is connected and for every $e \in E$ the graph $G - e$ is not connected.

  (4) G contains no cycle and for any $e \in \binom{V}{2} \setminus E$ the graph $G + e$ has a cycle.

  (5) G is connected and $|V| = |E| + 1$ (Euler's formula).

- Two rooted trees (T,v), (T',v') are isomorphic if there is a graph isomorphism $f : T \longrightarrow T'$ such that $f(v) = v'$.

- Two planted trees $(T, v, \preceq)$, $(T', v', \preceq')$ are isomorphic if there is a rooted tree isomorphism $f : (T,v) \longrightarrow (T',v')$ that satisfies $f(a) \preceq_{f(u)} f(b) \iff a \preceq_u b$.

- $T = (T, v, \preceq)$ a planted tree [or $T = (T,v)$ rooted]. Define $L : V(T) \longrightarrow \{0,1\}$. If u is an end-vertex then $L(u) := 01$. If u is a vertex with sons $a_1 \preceq_u ... \preceq_u a_l$ then $L(u) := 0L(a_1)...L(a_l)1$. [If u is a vertex with sons $a_1, ..., a_l$ and $L(a_1) \leq ... \leq L(a_l)$, then $L(u) := 0L(a_1)...L(a_l)1$.] The **code of** $(\mathbf{T}, \mathbf{v}, \preceq)$, denoted by $c(T, v, \preceq)$, is the string $L(v)$.

- Two trees are isomorphic iff their codes are equal.

- **Lexicographic order** on $\{0,1\}^*$ ($\{0,1\}^*$ being the set of strings over $\{0,1\}$): total order such that for $a, b \in \{0,1\}^*$, if a is a prefix of b then $a \leq b$. Now, let $j = \min\{i \mid a_i \neq b_i\}$. If $a_j = 0$ and $b_j = 1$, then $a \leq b$.

- $G = (V,E)$ connected. The **eccentricity** of $v \in V$ is $\mathrm{ecc}_G(v) = \max\{d_G(u,v) \mid u \in V\}$. The **center** of G is $C(G) = \{v \in V \mid \mathrm{ecc}_G(v) \text{ is minimum}\}$.

- For $T = (V,E)$ a tree, $|C(T)| \leq 2$. If $C(T) = \{x,y\}$, then $\{x,y\} \in E$. Moreover, $(V, E \setminus \{x,y\})$ is a forest with two connected components $V_1 \ni x_1$, $V_2 \ni x_2$. With T_1, T_2 the subgraphs of T induced by V_1 and V_2, $|C(T_1)| = |C(T_2)| = 1$.

- If $C(T) = \{v\}$, then the code of T is $c(T) := c(T,v)$. If $C(T) = \{x_1, x_2\}$ ($\{x_1,x_2\} \in E$) and $c(T_1, x_1) \leq c(T_2, x_2)$ then $c(T) := c(T, x_1)$.

- **Spanning tree of G**: spanning subgraph of G that is a tree.

- **Binet-Cauchy**: If $A \in \mathbb{R}^n_m$, $B \in \mathbb{R}^m_n$, then $\det(AB) = \sum_S \det(A_S)\det(B_S)$, where S runs through $\binom{\{1,\ldots,m\}}{n}$ and A_S is obtained by retaining only columns whose index is in S and B_S is obtained from B by retaining only the rows whose index is in S.

- A_{ij}: matrix whose row i and column j are deleted.

- $\overline{A}$: matrix whose first row is removed.

- $G = (V,E)$ simple graph, $V = \{1, ..., n\}$, $E = \{e_1, ..., e_m\}$. The **Laplace matrix of G** is the matrix $Q \in \mathbb{R}^n_n$ defined by

$$q_{ij} = \begin{cases} \deg_G(i) & \text{if } 1 \leq i = j \leq n, \\ -1 & \text{if } 1 \leq i \neq j \leq n \text{ and } \{i,j\} \in E, \\ 0 & \text{if } 1 \leq i \neq j \leq n \text{ and } \{i,j\} \notin E. \end{cases}$$

- **Orientation of G**: directed graph $\overrightarrow{G} = (V, \{\overrightarrow{e_1}, ..., \overrightarrow{e_m}\})$ where for all $i \in \{1, ..., n\}$, $\overrightarrow{e_i} \in \{(x,y),(y,x)\}$ if $\overrightarrow{e_i} = \{x, y\}$.

- **Incidence matrix of the orientation** $\overrightarrow{G}$: matrix $D_{\overrightarrow{G}} \in \mathbb{R}^n_m$ defined by

$$d_{ik} = \begin{cases} -1 & \text{if } \overrightarrow{e_k} = (i,l) \text{ for some } l \in \{1,...,n\}, \\ 1 & \text{if } \overrightarrow{e_k} = (l,i) \text{ for some } l \in \{1,...,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The sum of the rows of $D_{\overrightarrow{G}}$ is the null vector.

- If G is a simple and undirected graph, its number of spanning trees is equal to $\det(Q_{11})$ where Q is its Laplace matrix. If G a simple graph with n vertices: $DD^T = Q$, $\overline{DD}^T = Q_{11}$.

- $T = (V, N)$ a simple graph with $|V| = n \geq 2$ and $|E| = n-1$, $\overrightarrow{T}$ an orientation of T. Then $\det(\overline{D}_{\overrightarrow{T}}) \in \{-1, 0, 1\}$. Moreover, $\det(\overline{D}_{\overrightarrow{T}}) \neq 0$ iff T is a tree.

- G a simple graph, $\overrightarrow{G}$ an orientation of G. The number of spanning trees of G is equal to the number of $(n-1) \times (n-1)$ submatrices of $\overline{D}_{\overrightarrow{G}}$ with a nonzero determinant.

- **Planar graph G**: simple graph which can be drawn in the Euclidean space so that no two different edges cross in a point that is not a vertex (it is called a **planar drawing** of G).

- **Arc** (in the plane): subset of $\mathbb{R}^2$ which has the form $\gamma([0,1])$ where $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ is a continuous injective map.

- A couple (b, α) is a **drawing of** $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ such that b is an injective map from $\mathbb{R}^2$ to $\mathbb{R}^2$ with $v \mapsto b(v)$ and α is a map that assigns an arc $\alpha(e)$ to each $e \in E$, such that for $v \in V$ and $e \in E$, $b(v)$ belongs to $\alpha(e) = \gamma_{\alpha(e)}([0,1])$ iff $b(v) \in \{\gamma_{\alpha(e)}(0), \gamma_{\alpha(e)}(1)\}$ and e is adjacent to v.

- **Planar drawing of G**: drawing (b, α) of G that satisfies $\gamma_{\alpha(e)}(]0,1[) \cap \gamma_{\alpha(e')}(]0,1[) = \varnothing$, $e \neq e' \in E$. A **topological planar graph** is a planar graph with a planar drawing.

- For $G = (V, E)$ a topological, connected and planar graph: $|V| - |E| + f = 2$ where f is the number of G's faces.

- **Jordan curve**: set k of the form $\gamma([0,1])$ where $\gamma([0,1]) \longrightarrow \mathbb{R}^2$ is continuous, injective on $]0,1[$ and such that $\gamma(0) = \gamma(1)$.

- If k is a Jordan curve, then $\mathbb{R}^2 \setminus k$ has exactly two connected components. One of them is bounded and called the **interior of k** and the other unbounded and called the **exterior of k**. The curve k is the boundary of each of the components.

- G a topological planar graph, C the union of the arcs that correspond to G's edges. Let $\equiv$ be the equivalence defined on $\mathbb{R}^2 \setminus C$ by setting $x \equiv y$ if for every Jordan curve k that represents a cycle in G (such a Jordan curve will be called a cycle of G), the elements x and y are in the same connected component of $\mathbb{R}^2 \setminus k$. A face of G is a class of $\equiv$.

- **Platonic solid**: convex polyhedron whose faces are isomorphic convex regular polygons. The only Platonic solids are: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

- **Vertex cut** (or separating set) **of G**: subset $S \subseteq V$ such that $G - S$ has more that one connected component. The **connectivity of G** is the minimum cardinality of S such that $G - S$ is disconneced or has only one vertex. A graph G is **k-connected** if its connectivity is at least k. In other words, G is $k-$connected iff it has $\geq k+1$ vertices and for any $S \subseteq V$ with cardinality at most $k-1$, $G - S$ is connected.

- **Subdivision of G**: graph $G\%e$ obtained by adding a new vertex on an edge $e = \{u, v\}$ of G. We have $V(G\%e) = V(G) \cup \{z\}$ for $z \notin V(G)$ and $E(G\%e) = (E(G) \setminus \{\{u,v\}\}) \cup \{\{u,z\},\{z,v\}\}$.

- G a $2-$connected planar graph. Then in every planar drawing of G, every face is the interior/exterior of a cycle of G.

- **Maximal planar graph** $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: graph such that for any $e \in \binom{V}{2} \setminus E$, $G + e$ is not planar.

- Every simple planar graph has a vertex of degree ≤ 5. Every simple triangle free planar graph has a vertex of degree ≤ 3.

- $G = (V, E)$ a topological and planar graph and $\forall i \geq 1$, $f_i := \big|\{F \mid F \text{ is a face bounded by a cycle } C_i\}\big|$ and $n_i := \big|\{v \in V \mid \deg_G(v) = i\}\big|$: $\sum(6-i)n_i = 12 + 2\sum_{j\geq 3}(j-3)f_j$.

- A graph is planar iff it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or K_5 (which are not planar).

- **Map**: drawing of a planar multigraph. Let $G = (V, E, \varepsilon)$ a multigraph, $\varepsilon : V \longrightarrow \binom{V}{2} \cup \binom{V}{1}$. A $k-$**coloring** is a map $c_\_ : V \longrightarrow \{1, ..., k\}$. It is proper if $\forall v \neq v' \in V$, $V(\{v, v'\} \in \varepsilon(V) \implies c_v \neq c_{v'})$.

- The **dual** of a topological planar multigraph $G = (V, E, \varepsilon)$ with a set of faces $\mathcal{F}$ is the multigraph $(\mathcal{F}, E, \varepsilon')$ such that $\varepsilon'(e) = \{F, F'\}$ if e is adjacent to F and F'. A map admits a $k-$coloring iff its dual graph admits one.

- **Chromatic number** $\chi(\mathbf{G})$: minimum k such that G admits a $k-$coloring. G is $k-$chromatic if $\chi(G) = k$. If G is a planar multigraph then $\chi(G) \leq 5$. $\chi(G) = 1$ iff $E(G) = \varnothing$, $\chi(G) = 2$ iff G bipartite. A simple G is critical if $\chi(H) < \chi(G)$ for all proper subgraphs H of G. A $k-$critical graph is simple, critical and $k-$chromatic. For any G, $\chi(G) \leq \max\{\deg_G(v) + 1 \mid v \in V\}$.

- If G is $k-$critical, then $\delta := \min\{\deg_G(v) \mid v \in V(G)\} \geq k-1$. Every $k-$chromatic graph has at least k vertices of degree at least $k-1$.

- $S \subseteq V$ is a **clique** if the subgraph $G[S]$ of G induced by S is complete. In a critical graph, no vertex cut is a clique.

- For G critical: if G has ≥ 3 vertices then it is 2-connected and if $\{u, v\}$ is a cut then $\{u, v\} \notin E(G)$.

- G $k-$critical with vertex cut $\{u, v\}$: $\deg_G(u) + \deg_G(v) \geq 3k-5$. Also, there are two components G_1 and G_2 of $\{u, v\}$ such that G_i is of type i for $i \in \{1, 2\}$ and such that $G = G_1 \cup G_2$. Moreover, the graphs $G_1 + \{u, v\}$ and $G_2 \cdot \{u, v\}$ are $k-$critical ($G_2 \cdot \{u, v\}$ is obtained by identifying u and v).