Introduction to Graph Theory | Homework 1

Anton Zaitsev | 0230981826

November 27, 2023

Exercise 1

(i)

Let $G = (V, E), v = \{1, 2\}, E = \{(1, 2)\}$. Then $V' = \{(1, 2)\}, E' = \emptyset$.

Clearly, G' is Hamiltonian, while G is not Eulerian (no edge from $v_2 = 2$ to $v_1 = 1$). Thus, if G' is Hamiltonian, it does not imply that G is Eulerian.

(ii)

Since G is Eulerian, then we visit each edge exactly once. We also know that $V' = E =>$ we visit each vertex of G' exactly once. Let us call the walk where we visit each vertex of G' as W. To prove that W is Hamiltonian cycle, we need to prove that the walk W is closed. G has an Eulerian cycle: $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$, where $v_n = v_0$. Thus, we have a vertex in G' $v'_n = (v_{n-1}, v_n) = (v_{n-1}, v_0)$ and a vertex $v'_0 = (v_0, v_1)$. Since $(v_{n-1, v_0}) \cap (v_0, v_1) \neq \emptyset$, it belongs to E'.

Thus, W is closed and it is a Hamiltonian cycle and G' is Hamiltonian.

Exercise 2

Since K_n is a complete graph, there are n^{n-2} spanning trees, each with $(n-1)$ edges. Thus, we have $(n-1) \cdot n^{n-2}$ total number of edges contained in spanning trees.

For a complete graph with n verticies, there are $\binom{n}{2}$ pairs that must be connected by an edge. Thus, we have that there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges in K_n.

Let us define t as the number of spanning trees of K_n. Each edge of K_n is contained in t number of spanning trees. Thus, we have $t \cdot \frac{n(n-1)}{2}$ edges contained in the spanning trees.

Thus, we get

$$(n-1) \cdot n^{n-2} = t \cdot \frac{n(n-1)}{2}$$

$$t = 2n^{n-3}$$

If we delete an edge from a complete graph, we remove t number of trees containing this edge. Thus, when we remove an edge e from a complete graph K_n we get

$$n^{n-2} - t(\text{total number of trees minus trees that contained edge e}) =$$

$$= n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$$

$n^{n-3}(n-2)$ is the number of spanning trees of graph $K_n - e$.

Exercise 3

Let us define $e = (u, v), w(e) = w(u) + w(v)$.

By contradiction, assume T is not unique. Then, $\exists T_1, T_2, T_1 \neq T_2$, both distinct minimum spanning trees. Since $T_1 \neq T_2$, there must exist at least one edge that belongs to either $E(T_1)$ or $E(T_2)$.

Consider such an edge of minimum weight and, without loss of generality, assume it belongs to $E(T_1)$. Define it as e_1.

Also, since $T_1 \neq T_2, \exists e_2 \in E(T_2), e_2 \notin E(T_1)$. We know that $w(e_1) < w(e_2)$. Then, we have that $T_m = T_2 \cup \{e_1\} \setminus \{e_2\}$ a spanning tree, whose total weight is less than of T_2. However, this is a contradiction, since we chose T_2 as minimum spanning tree.

Thus, T is unique.

Exercise 4

First, since G contains no cycle and adding an edge $e, e \notin E(G)$, to graph G, by Proposition 3.4(4) we have that G is a tree.

Second, since isomorphism $\phi : G \to G$ preserves structural properties, we have that $\forall v \in V(G), deg_G(v) = 1 : deg_{\phi(G)}(\phi(v)) = 1$ (all end verticies in G are also end verticies in $\phi(G)$).

Third, let us define $T_0 = G$ an initial graph (tree) and $T_0' = \phi(T_0)$. We have that the set of verticies with degree 1 is the same in T_0 and T_0'. Let us define V_0 and V_0' as such sets and let us remove these sets from T_0 and T_0', respectively: define $T_1 = T_0 \setminus V_0, T_1' = T_0' \setminus V_0'$. Since T_0 and T_0' are trees and V_0 and V_0' are sets of end verticies, by Lemma 3.2 we have that T_1 and T_1' are also trees.

Next, let us again remove the sets of end verticies from T_1 and T_1' and define $T_2 = T_1 \setminus V_1$ and $T_2' = T_1' \setminus V_1'$. Again, T_2 and T_2' are trees.

We iterative removal step unti we are left with either 1 or 2 verticies. Let us say that we arrive at these conditions at step n.

1. In case when $|V_n| = 1$: since there is only one vertex left, the isomorphism $\phi : G \to G$ maps vertex v_n to itself: $\phi(v_n) = v_n$. Thus, $\exists v \in V$ s.t. $\phi(v) = v$.
2. In case when $|V_n| = 2$: since T_n is still a tree, there exists a last edge connecting two last verticies in T_n, let us call it e_n. Since isomorphism preserves edges, we have that $\phi(e_n) = e_n$. Thus, $\exists e \in E$ s.t. $\phi(e) = e$