Introduction to Graph Theory | Homework 2

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Exercise 1

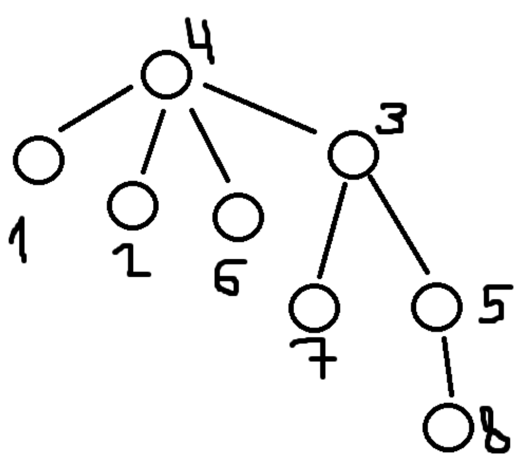
Let $P = (4, 4, 4, 3, 3, 5)$ a Prüfer sequence, $G = (V, E)$ labelled tree that admits P as a Prüfer sequence. Let us reconstruct G using P.

- Input: $P = (4, 4, 4, 3, 3, 5), n = len(P) + 2 = 8, D = (1, 2, 3, 4, 5, 6, 7, 8), E = \emptyset, V = D = (1, 2, 3, 4, 5, 6, 7, 8)$
- Algorithm:
  - for i in D:
    - if $len(D) = 2$:
      - add edge to E consisting of last two elements in D.
      - return E.
    - else:
      - find smallest element in D which is not in P and first element in P and add an edge to E consisting of these 2 elements.
      - delete these elements from D and P, respectively.

1. Iteration 1:
   - $E = \{(1, 4)\}$
   - $P = (4, 4, 3, 3, 5), D = (2, 3, 4, 5, 6, 7, 8)$
2. Iteration 2:
   - $E = \{(1, 4), (2, 4)\}$
   - $P = (4, 3, 3, 5), D = (3, 4, 5, 6, 7, 8)$
3. Iteration 3:
   - $E = \{(1, 4), (2, 4), (6, 4)\}$
   - $P = (3, 3, 5), D = (3, 4, 5, 7, 8)$
4. Iteration 4:
   - $E = \{(1, 4), (2, 4), (6, 4), (4, 3)\}$
   - $P = (3, 5), D = (3, 5, 7, 8)$
5. Iteration 5:
   - $E = \{(1, 4), (2, 4), (6, 4), (4, 3), (7, 3)\}$
   - $P = (5), D = (3, 5, 8)$
6. Iteration 6:
   - $E = \{(1, 4), (2, 4), (6, 4), (4, 3), (7, 3), (3, 5)\}$
   - $P = \emptyset, D = (5, 8)$
7. Iteration 7:
   - $E = \{(1, 4), (2, 4), (6, 4), (4, 3), (7, 3), (3, 5), (5, 8)\}$

Output: $V = \{1, 2, 3, 4, 5, 7, 8\}, E = \{(1, 4), (2, 4), (6, 4), (4, 3), (7, 3), (3, 5), (5, 8)\}$

Figure 1: Graph of G



Graph of G reconstructed from its Prüfer sequence.

Exercise 2

Since G' is a tree, there exists only one simple path between s and t. Let us call W - the path from s to t. Now, let us consider an edge e' in path W such that its capacity is minimal: $\min_{e \in E(W)} C(e) = C(e')$. Since W is simple, the cut (S, T) must intersect this path. In this case, the minimal cut would intersect path W exactly and only at edge e'. Thus, we get

$$\sum_{\{e|e^- \in S, e^+ \in T\}} C(e) = C(e') = \min_{e \in E(W)} C(e)$$

Notice that the capacity function C maps values to $\mathbb{R}_0^+$, meaning that the minimal value of an edge $e \in E$ is 0. Thus, if there exists an edge e', such that $C(e') = 0$, then, by applying the previous reasoning, we have

$$\sum_{\{e|e^- \in S, e^+ \in T\}} C(e) = C(e') = 0$$

Exercise 3

For clarification, let us say that all verticies in V are unique: $\forall x, y \in V, x \neq y$. Let us only consider $p \geq 2$, since for $p = 1$ we only have 1 edge connecting vertex 0 with vertex 1 and there is no cycle possible, and if $p = 0$ we dont have any verticies.

Girth To find the girth of G we can follow the following algorithm:

1. Create first vertex X_1 with $x_i = 0, i = 1, ..., p$
2. Create a second vertex X_2 with $x_1 = 1, x_i = 0, i = 2, ...p$. Notice there is an edge $e_{12} = (X_1, X_2)$ since $||X_1 - X_2|| = 1$
3. Create a third vertex X_3 with $x_1, x_2 = 1, x_i = 0, i = 3, ...p$. If we tried to connect X_3 with X_1 to create a cycle with length 3, then we would not be able to connect X_3 with X_2, since the distance between them would be > 1, thus no cycle of length 3 possible. We are required to create an intermediate vertex X_4 that would connect X_3 with X_0 to create a cycle.
4. Create a fourth vertex X_4 with $x_2 = 1, x_1 = 0, x_i = 0, i = 3, ..., p$. Now there exists a set of edges $E = (X_1, X_2), (X_2, X_3), (X_3, X_4), (X_4, X_1)$, i.e. a cycle of length 4, which is minimal, thus it is a girth of G.

Circumference For $p \geq 2$: Consider the verticies $(x_1, \ldots, x_p)$, where $x_i \in \{0, 1\}, i = 1, \ldots, p$. Each vertex represents a binary string of length p. The maximum number of elements in V can be created by changing the value of each x_i sequentially, i.e. start with $(0, 0, \ldots, 0)$ and change the value of x_1, then x_2, and so on, until you reach $(1, 1, \ldots, 1)$. The problem can be restated in the following way: how many ordered verticies of length p can be formed where vertex' elements take values in the set $\{0, 1\}$? This is exactly a number of permutations with repetition, which equals to

$$\max |V| = |\{0, 1\}|^p = 2^p$$

Let us prove by induction that the circumference equals to the maximum number of verticies in graph G, which is 2^p.

1. For $p = 2$ the circumference is: $2^2 = 4$, which is true.
2. For $p = 3$ the circumference is: $2^3 = 8$, which is also true.
3. Let us say that this holds for $n - 1 = p$ and prove it for $n = p + 1$: Let $n = p + 1$. Let us split the graph into 2 subgraphs: first subgraph, call it G_0, contains all verticies that end with 0 and the second subgraph, call it G_1, contains all verticies that end with 1. Note that for G_0 and G_1, the distances between two vertices are the same as in G, since we can disregard the last coefficient of each vertex. We have that the number of verticies in $G_0 = G_1 = \frac{2^n}{2} = 2^{n-1}$. Since the statement holds for $n - 1 = p$, we have that the length of the longest cycle in $G_0 = G_1 = 2^p$. Now, let us take two verticies in G_0 which are connected by an edge and remove this edge. Without loss of generality, say the first vertex is $V_0^0 = \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ v_{n-1} \\ 0 \end{bmatrix}$ and

the second vertex is $V_1^0 = \begin{bmatrix} 1 \\ v_2 \\ \vdots \\ v_{n-1} \\ 0 \end{bmatrix}$, $v_i = \{0, 1\}, i = 2, ..., n-1$. Since we removed 1 edge from the cycle, we subtracted 1 from the circumference in G_0. Next, let us take two verticies in G_1 whose distances with V_0^0 and V_1^0 equal to 1 and which are connected by an edge and remove this edge. Again, without loss of generality, take verticies $V_0^1 = \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ v_{n-1} \\ 1 \end{bmatrix}$ and $V_1^1 = \begin{bmatrix} 1 \\ v_2 \\ \vdots \\ v_{n-1} \\ 1 \end{bmatrix}$, $v_i = \{0, 1\}, i = 2, ..., n-1$. Again, since we removed 1 edge from the cycle, we subtracted 1 from the circumference in G_1. Now, we connect V_0^0 with V_0^1 and V_1^0 with V_1^1 (added 2 new edges), since their distances are 1, to obtain graph G. By connecting these verticies we obtain a cycle in G which circumference equals to:

$$(2^{n-1} - 1) + (2^{n-1} - 1) + 2 = 2 \cdot 2^{n-1} = 2^n = 2^{p+1}$$

Which proves the statement.