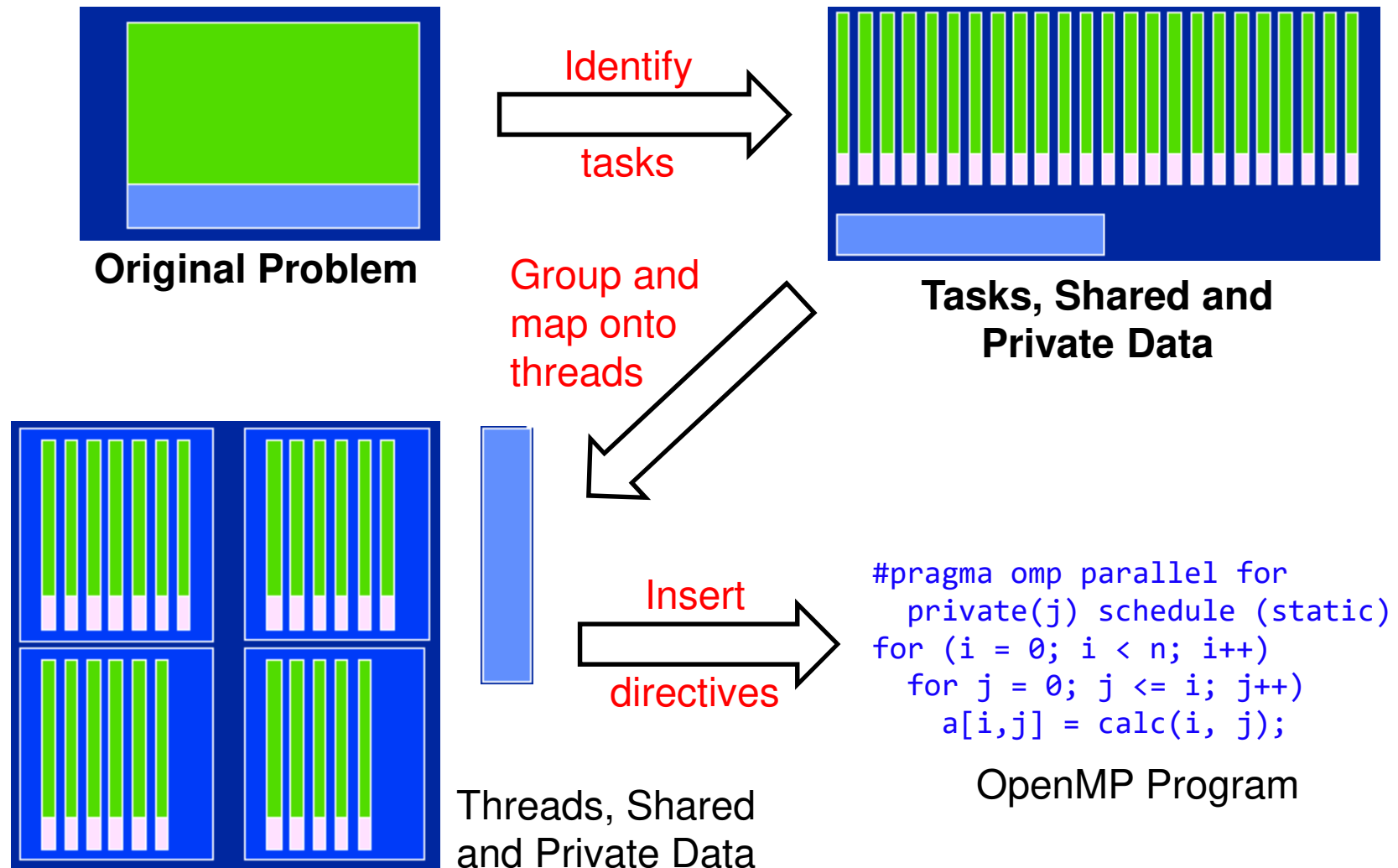




# Loop Parallelism

CS121 Parallel Computing  
Fall 2022

# Shared memory algorithm design





# Design considerations

- Break program into tasks, consisting of statements that must be executed in order.
  - Use data dependence analysis.
- Map independent tasks to different processors.
  - Mapping needs to consider load balancing, e.g. static vs dynamic, block vs cyclic work assignment.
- Variable specification
  - Shared vs. private vs. reduction
  - Shared variables cause cache coherence traffic and much lower performance.
  - Private and reduction variables don't need synchronization (except possibly at end of a loop).
- Dimension mapping, e.g. row-wise vs column-wise.
  - Matching mapping to access pattern improves cache locality.

# Data dependence analysis

- Let S1 and S2 be two statements in a sequential execution of a program.
- S1 and S2 are independent if running them in different orders produces the same result. Otherwise they're dependent.
- Suppose S1 occurs before S2. They can have the following dependencies.
  - $S1 \rightarrow^T S2$  denotes true dependence (RAW), i.e. S1 writes to a location that is read by S2.
  - $S1 \rightarrow^A S2$  denotes anti dependence (WAR), i.e. S1 reads a location written by S2.
  - $S1 \rightarrow^O S2$  denotes output dependence (WAW), i.e. S1 writes to the same location written by S2.

```
S1: x = 2;  
S2: z = 3;  
S3: y = x;  
S4: y = x + z;  
S5: z = 6;
```

## □ Dependences

- $S1 \rightarrow^T S3$
- $S1 \rightarrow^T S4$
- $S2 \rightarrow^T S4$
- $S2 \rightarrow^O S5$
- $S3 \rightarrow^O S4$
- $S4 \rightarrow^A S5$



# Parallelizing loops

- When parallelizing a program, must ensure dependent statements run in the same order in the sequential and parallel programs.
  - Guarantees the parallel and sequential programs behave the same way.
  - A parallel program may run correctly without satisfying this condition, but it's not guaranteed.
  - The ordering requirement is transitive. I.e. if  $S1 \rightarrow^* S2$  and  $S2 \rightarrow^* S3$ , then  $S1$  must run before  $S3$  in any parallelization.
- Dependent statements cannot on run on different processors, since we can't enforce the order of execution (interleaving) on different processors.
- Independent statements can run on different processors if they haven't been ordered by transitivity.



# Parallelizing loops

- Goal is to identify all independent statements, to maximize parallelism.
- Focus on parallelizing loops, since these are common in shared memory programs and are the main performance hotspots.
- Notation
  - Let  $S$  denote a statement in the source program.
  - Given a nested loop with iteration variables  $i, j, \dots$ , let  $S[i, j, \dots]$  denote a statement in loop iteration  $[i, j, \dots]$ .

# Loop dependence analysis

- Loop-carried dependence
  - Dependence exists across different iterations of loop.
- Loop-independent dependence
  - Dependence exists within the same iteration of loop.

```
for (i=1; i<=n; i++) {  
    S1: a[i] = a[i-1] + 1;  
    S2: b[i] = a[i];  
}
```

$S1[i] \rightarrow T S1[i+1]$   
- loop-carried dependence  
 $S1[i] \rightarrow T S2[i]$   
- loop-independent dependence

```
for (i=1; i<=n; i++)  
    for (j=1; j<=n; j++)  
        S3: a[i][j] = a[i][j-1] + 1;
```

$S3[i,j] \rightarrow T S3[i,j+1]$   
- loop-carried dependence in j loop  
- no loop-carried dependence in i loop

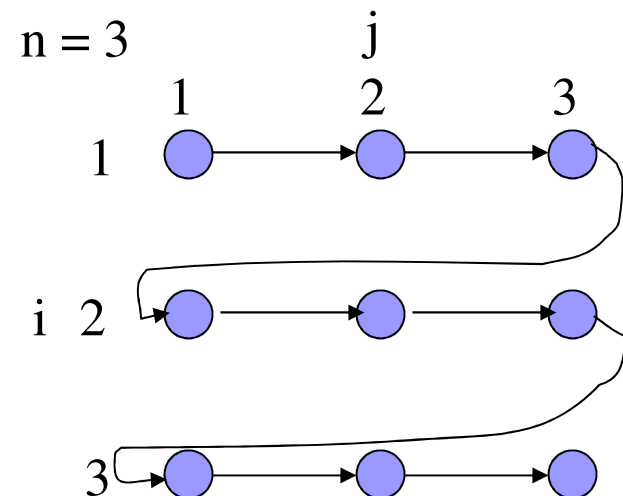
```
for (i=1; i<=n; i++)  
    for (j=1; j<=n; j++)  
        S4: a[i][j] = a[i-1][j] + 1;
```

$S4[i,j] \rightarrow T S4[i+1,j]$   
- loop-carried dependence in i loop  
- no loop-carried dependence in j loop

# Iteration-space traversal graph

- Iteration-space traversal graph (ITG) is a line graph showing the order of traversal in the iteration space.
- Node in ITG is a point in the iteration space, i.e. a particular iteration.
- Directed edge in ITG gives the next iteration that will be executed after the current iteration.

```
for (i=1; i<=n; i++)  
  for (j=1; j<=n; j++)  
    S3: a[i][j] = a[i][j-1] + 1;
```





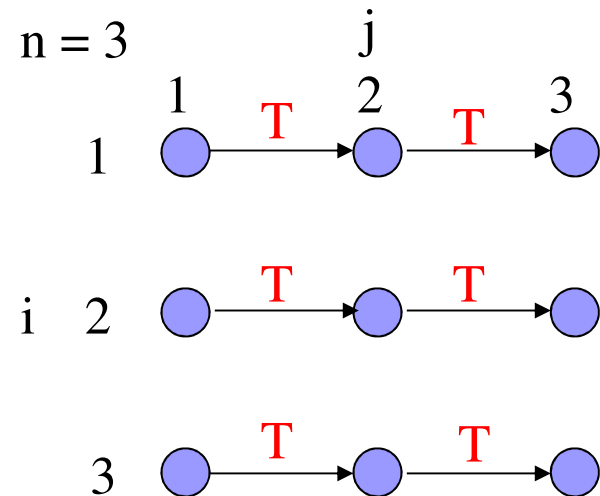
# Loop-carried dependence graph

- Given the ITG, can determine the dependence between different loops.
- Loop-carried Dependence Graph (LDG) shows the loop-carried true/anti/output dependence relationships.
- Node in LDG is a point in the iteration space.
- Directed edge in LDG is the dependence.
- LDG helps identify parts of the loop that can be done in parallel.
  - Different connected components can be done in parallel.
  - Each connected component must be done sequentially.

```
for (i=1; i<=n; i++)  
  for (j=1; j<=n; j++)  
    S3: a[i][j] = a[i][j-1] + 1;
```

$S3[i,j] \rightarrow T S3[i,j+1]$

- loop-carried dependence in j loop
- no loop-carried dependence in i loop



# Example 1

```

for (i=1; i<=n; i++)
  for (j=1; j<=n; j++) {
    S1: a[i][j] = b[i][j] + c[i][j];
    S2: b[i][j] = a[i][j-1] * d[i][j];
  }

```

True dependences

$S1[i,j] \rightarrow T S2[i,j+1]$

- loop-carried  
dependence

Anti dependences

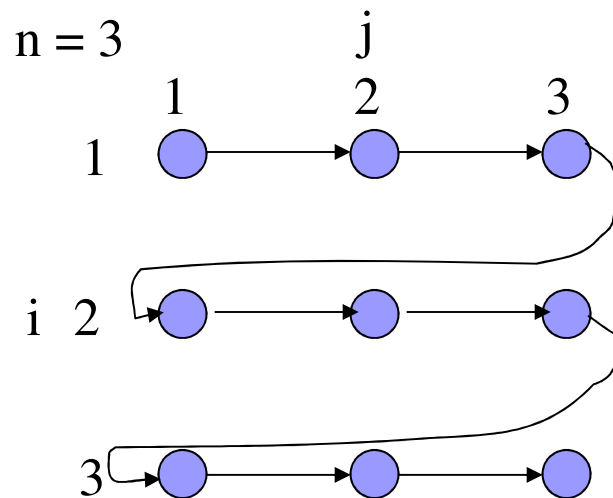
$S1[i,j] \rightarrow A S2[i,j]$

- loop-independent  
dependence

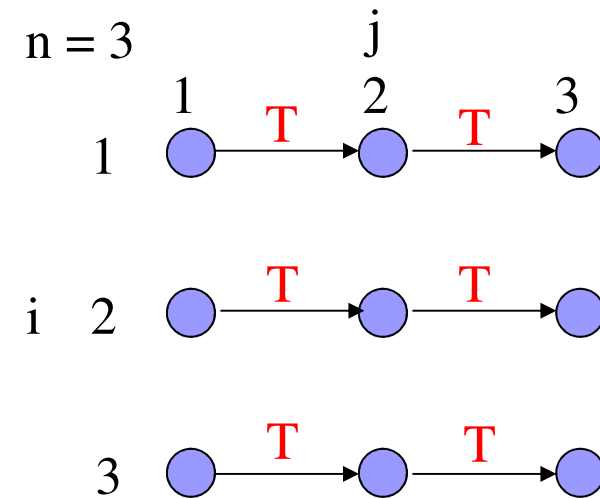
Output dependences

None

## ■ ITG



## ■ LDG





# Example 1

- Task Identification

- n parallel tasks - one task per iteration of i loop.

- Grouping / mapping

- Static block as different iterations have same work.

- OpenMP

```
#pragma omp parallel for private(j) schedule(static)
for (i=1; i<=n; i++)
    for (j=1; j<=n; j++) {
        S1: a[i][j] = b[i][j] + c[i][j];
        S2: b[i][j] = a[i][j-1] * d[i][j];
    }
```

# Example 2

```
for (i=1; i<=n; i++)  
  S1: a[i] = a[i-2];
```

True dependences

$S1[i] \rightarrow T S1[i+2]$

- loop-carried dependence

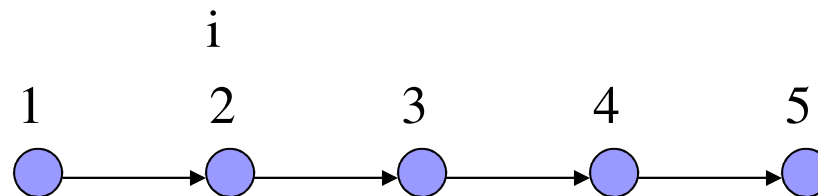
Anti dependences

None

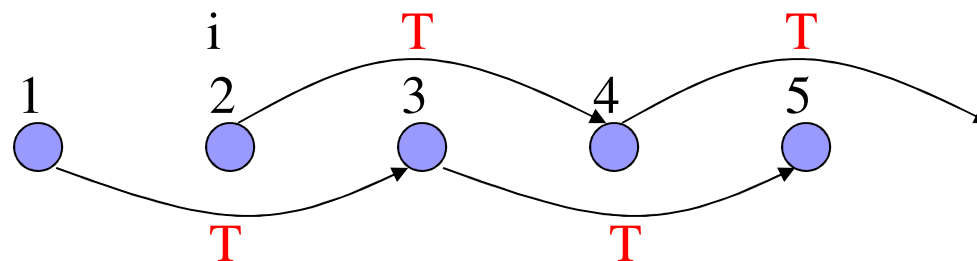
Output dependences

None

■ ITG



■ LDG





# Example 2

- Task Identification
  - There are opportunities when some dependences are missing.
  - Can divide the for loop into two parallel tasks, one with even iterations and another with odd iterations.

- Grouping / mapping
  - One task per thread
- OpenMP

```
#pragma omp parallel sections
private(i)
{
    #pragma omp section
    for (i=1; i<=n; i+=2)
        a[i] = a[i-2];
    #pragma omp section
    for (i=2; i<=n; i+=2)
        a[i] = a[i-2];
}
```

# Example 3

```
for (i=1; i<=n; i++)  
  S1: a[i] = a[i-1] + b[i]*c[i] + d[i];
```

True dependences

$S1[i] \rightarrow T S1[i+1]$

- loop-carried  
dependence

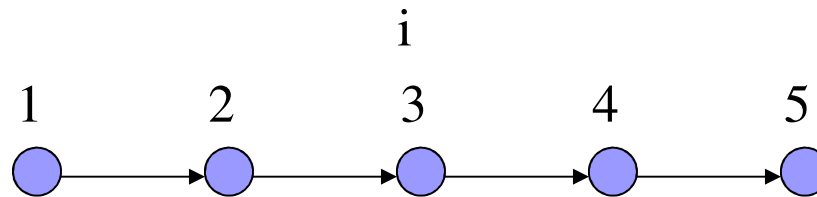
Anti dependences

None

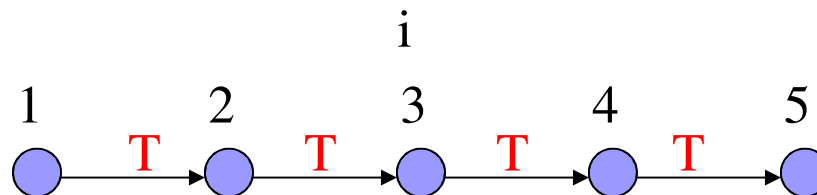
Output dependences

None

■ ITG



■ LDG



Must respect loop-carried dependence

# Example 3

## ■ Task Identification

- Any opportunities for parallelism? Loop-carried dependence  $S[i] \rightarrow T S[i+1]$  must be respected.
- But no loop-carried dependence in  $b[i]*c[i]+d[i]$  part.
- Loop fission
  - Distribute into two separate loops .

## ■ Code after loop fission

```
for (i=1; i<=n; i++) {  
    S1: temp[i] = b[i]*c[i] + d[i];  
}  
  
for (i=1; i<=n; i++) {  
    S2: a[i] = a[i-1] + temp[i];  
}
```

No dependences in first loop,  
so can be parallelized.

In second loop

True dependences:

$S2[i] \rightarrow T S2[i+1]$

- loop-carried dependence

Anti dependences None

Output dependences None

# Example 3

## □ OpenMP

```
#pragma omp parallel for schedule(static)
for (i=1; i<=n; i++) {
    temp[i] = b[i]*c[i] + d[i];
}
for (i=1; i<=n; i++) {
    a[i] = a[i-1] + temp[i];
}
```

- Note array temp[] introduces storage overhead.

## □ Better OpenMP solution

```
#pragma omp parallel for ordered private(t) schedule(static,1)
for (i=1; i<=n; i++) {
    t = b[i]*c[i] + d[i]; /* one copy of t per thread */
    #pragma omp ordered
    a[i] = a[i-1] + t; }
```

- ordered statement enforces a[i] assignment ordering.
- With k threads, uses k extra storage.
- Typically  $k \ll n$  so we save space.



# Example 4

```
for (i=1; i<=n; i++) {  
  S1: a[i] = b[i+1]*a[i];  
  S2: b[i] = b[i]*coef;  
  S3: c[i] = 0.5*(c[i] + a[i]);  
  S4: d[i] = d[i-1] + d[i];  
}
```

## True dependences

$S1[i] \rightarrow_T S3[i]$

-loop-independent dependence

$S4[i] \rightarrow_T S4[i+1]$

-loop-carried dependence

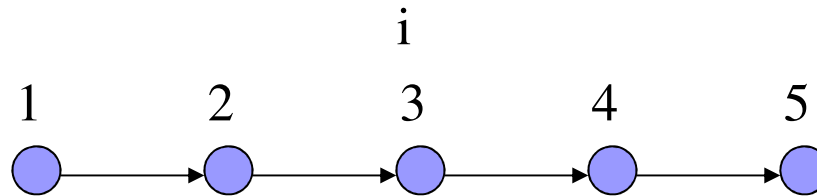
## Anti dependences

$S1[i] \rightarrow_A S2[i+1]$

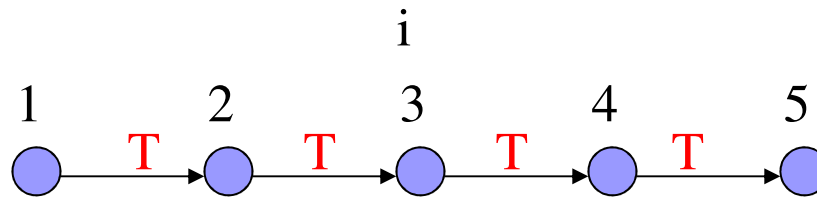
-loop-carried dependence

Output dependences None

## ITG



## LDG



Must respect loop-carried dependence

# Example 4

## ■ Task Identification

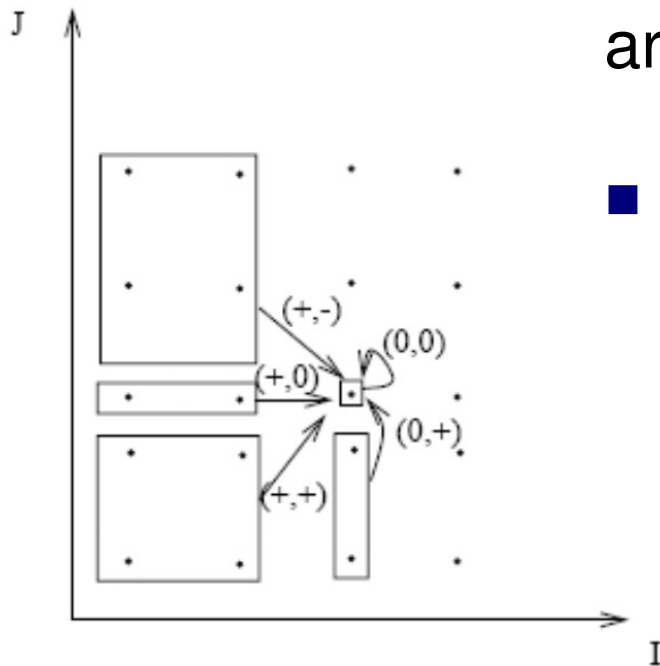
- S4 has no dependences with other statements, so can distribute into two separate loops (loop fission).
- This gives two parallel tasks, a loop containing S1, S2 and S3 and a loop containing S4.

## ■ OpenMP

```
#pragma omp parallel sections
private(i)
{
    #pragma omp section
    for (i=0; i<=n; i++) {
        S1: a[i] = b[i+1]*a[i];
        S2: b[i] = b[i]*coef;
        S3: c[i] = 0.5*(c[i] + a[i]);
    }
    #pragma omp section
    for (i=0; i<n; i++) {
        S4: d[i] = d[i-1] + d[i];
    }
}
```

# Distance and direction vectors

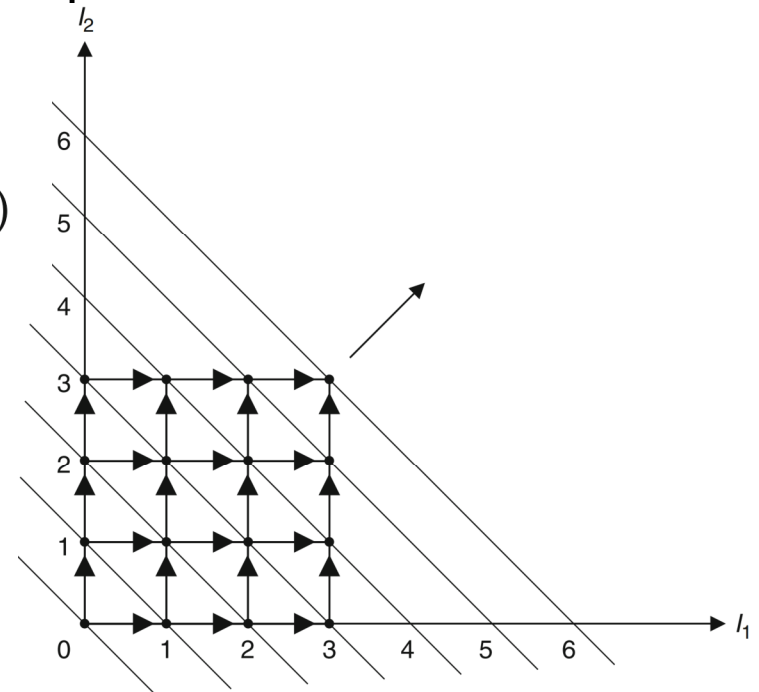
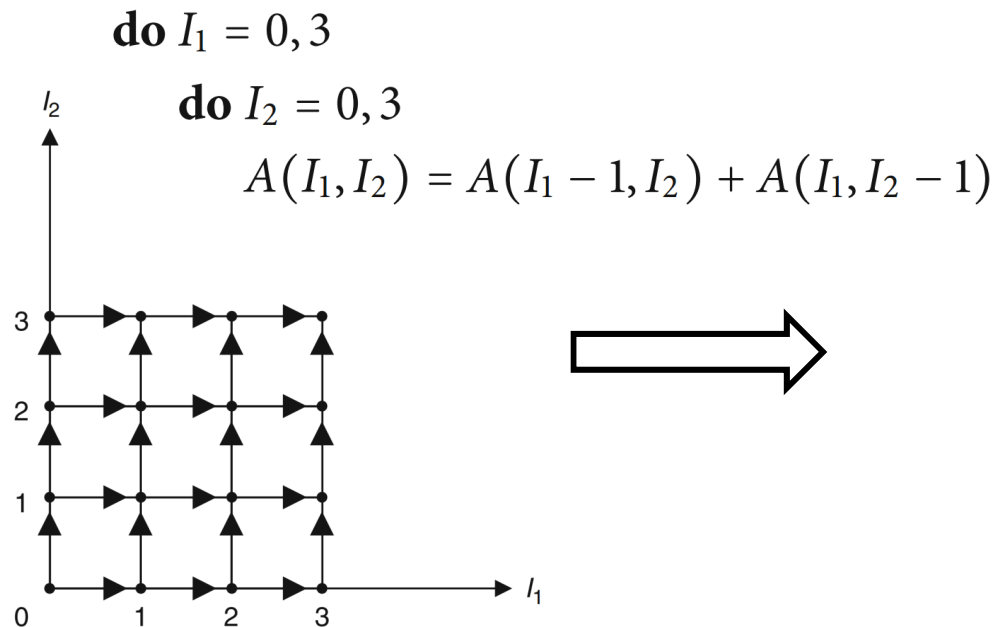
- Let  $T_1$  and  $T_2$  be iterations s.t.  $T_1 \rightarrow^* T_2$ .
- Distance vector from  $T_1$  to  $T_2$  is  $T_2 - T_1$ .
- Direction vector from  $T_1$  to  $T_2$  is  $\text{sign}(T_2 - T_1)$ .
- Consider a nested loop over  $(i,j)$  ( $j$  is the inner loop).



- The following direction vectors are possible
  - $(+, +)$ ,  $(+, 0)$ ,  $(+, -)$ ,  $(0, +)$ ,  $(0, 0)$ .
- The following directions are not possible.
  - $(0, -)$ ,  $(-, +)$ ,  $(-, 0)$ ,  $(-, -)$ .
  - **Ex** Direction vector  $(-, +)$  would mean, e.g. iteration  $(i, j)$  depends on iteration  $(i+1, j-1)$ . But  $(i, j)$  occurs before  $(i+1, j-1)$ .

# Loop skewing

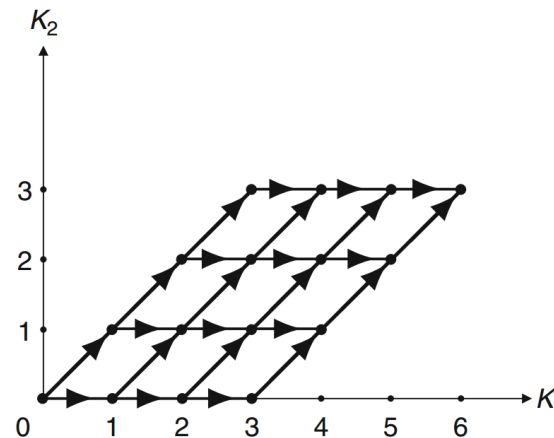
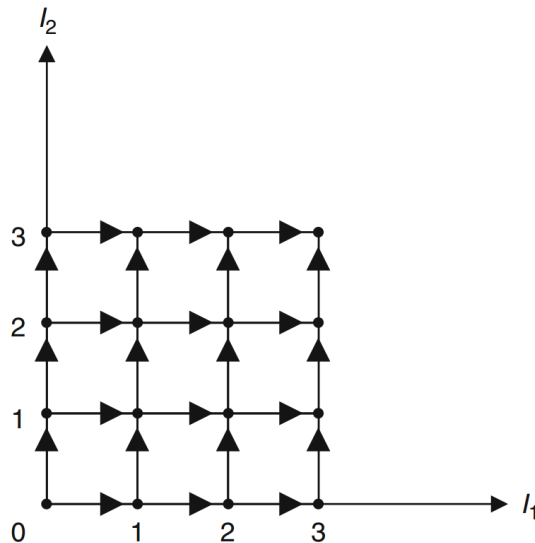
- If a loop has dependencies that prevent it being parallelized, we can try to transform (skew) the loop indices to create parallelizable loops.
- **Ex** Create new loop indices, the outer loop across the diagonal lines, the inner loop along the diagonal lines.
  - Inner loop iterations can be done in parallel.



Source: Unimodular Transformations, Utpal Banerjee

# Unimodular transformations

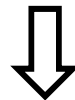
- Can define new indices using a linear transformation.
  - We use unimodular linear transformations, where the matrix has determinant -1 or 1.
  - Unimodularity ensures all iterations in original loop are performed in the transformed loop.
- Ex Let  $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ , and  $(K_1, K_2) = (I_1, I_2)U = (I_1 + I_2, I_2)$ .
  - Let  $K_1, K_2$  be the outer and inner loops, resp. Then for each  $K_1$  iteration, can run all the  $K_2$  iterations in parallel.



# Extracting parallelism

- Transformed loop must be legal. Also, we want it to be parallelizable.
- **Thm** Let  $U$  be a unimodular transformation. If  $vU$  is legal for all distance vectors  $v$ , then  $U$  is a legal transformation.
- **Ex** For  $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $(0,1)U = (1,1)$ ,  $(1,0)U = (1,0)$ .
  - New direction vectors are  $(+,+)$  and  $(+,0)$ , so the loop with the new loop indices is legal.
- **Thm** Suppose all the direction vectors for a loop are  $+$  in the  $i$ 'th coordinate, for some  $i$ . Then all loops deeper than level  $i$  can be run in parallel.
- **Ex** For the above  $U$ , direction vectors are all  $+$  in  $K_1$  coordinate, so the  $K_2$  loop can be parallelized.

$$A[I_1, I_2] = A[I_1-1, I_2] + A[I_1, I_2-1]$$



$$A[K_1, K_2] = A[K_1-1, K_2-1] + A[K_1-1, K_2]$$

# Data accesses after skewing

- Since  $(K_1, K_2) = (I_1, I_2)U$ , then  $(I_1, I_2) = (K_1, K_2)U^{-1}$ .
- So in iteration  $(K_1, K_2)$  of the transformed loop, we access data from iteration  $(K_1, K_2)U^{-1}$  of the original loop.
- **Ex** For  $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $U^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ , and  $(K_1, K_2)U^{-1} = (K_1 - K_2, K_2) = (I_1, I_2)$ .  
So in iteration  $(K_1, K_2)$ , we do

$$\begin{aligned} A[I_1, I_2] &= A[K_1 - K_2, K_2] = \\ &A[K_1 - K_2 - 1, K_2] + A[K_1 - K_2, K_2 - 1] \end{aligned}$$

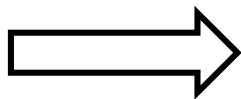
# Loop bounds after skewing

- We have  $I_1 = K_1 - K_2$ ,  $I_2 = K_2$ . Also,  $0 \leq I_1 \leq 3$  and  $0 \leq I_2 \leq 3$ .
- So  $0 \leq K_1 - K_2 \leq 3$  and  $0 \leq K_2 \leq 3$ .
- Since  $0 \leq K_1 - 3 \leq 3$ , then  $0 \leq K_1 \leq 6$ .
- Since  $K_1 - K_2 \leq 3$ , then  $K_1 - 3 \leq K_2$ . Also,  $0 \leq K_2$ . So  $K_2 \geq \max(0, K_1 - 3)$ .
- Likewise,  $K_2 \leq \min(3, K_1)$ .
- In general, the bounds can be computed using the Fourier-Motzkin method.
- Altogether, we have the following. The  $K_1$  loop is sequential, but the  $K_2$  loop can be run in parallel.

**do**  $I_1 = 0, 3$

**do**  $I_2 = 0, 3$

$A(I_1, I_2) = A(I_1 - 1, I_2) + A(I_1, I_2 - 1)$



**do**  $K_1 = 0, 6$

**do**  $K_2 = \max(0, K_1 - 3), \min(3, K_1)$

$A(K_1 - K_2, K_2) = A(K_1 - K_2 - 1, K_2)$   
       $+ A(K_1 - K_2, K_2 - 1)$





# Algorithmic analysis

- Sometimes there is no way to restructure a loop to increase parallelism.
- We can try to restructure the algorithm to eliminate dependences and improve parallelism.
- Need to understand the purpose of the algorithm and how it is used.
- For example, some algorithms are nondeterministic or calculate an approximation (e.g., Gauss-Seidel iteration).
  - In this case, restructuring the algorithm or ignoring some dependences may still give a valid result.

# Fibonacci numbers

- Fibonacci numbers

- $F_1 = F_2 = 1.$

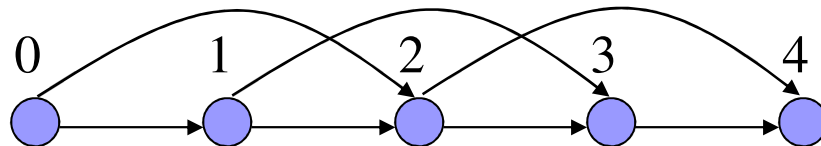
- $F_n = F_{n-1} + F_{n-2},$  for  $n > 2.$

- $1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$

- Computing  $F_n$  sequentially takes  $O(n)$  time.

- Can we compute  $F_n$  in parallel?

- Looking at the LDG, it seems there's no parallelism available.



# Fibonacci numbers in parallel

- A simple identity.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

- Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

- By repeatedly applying the identity, we get

$$A^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix}$$

- So if we can quickly compute  $A^n$  in parallel, we can compute  $F_{n+2}$ .
  - Can compute  $F_n$  in  $O(\log n)$  time with  $n$  processors.

