5.4-22

(a)

$$\begin{aligned} E(e^{tY}) &= E(e^{tX_1}e^{tX_2}) \\ &= E(e^{tX_1})E(e^{tX_2}) \\ \frac{1}{(1-2t)^{r/2}} &= \frac{1}{(1-2t)^{r_1/2}}E(e^{tX_2}) \\ E(e^{tX_2}) &= \frac{1}{(1-2t)^{(r-r_1)/2}} \end{aligned}$$

(b) X_2 is $\chi^2(r-r_1)$.

5.6-6

(a)

$$\begin{aligned} \mu &= \int_0^2 x(1-x/2)dx = [\frac{x^2}{2} - \frac{x^3}{6}]_0^2 = \frac{2}{3} \\ \sigma^2 &= \int_0^2 x^2(1-x/2)dx - (\frac{2}{3})^2 \\ &= \frac{2}{9} \end{aligned}$$

(b) Denote $Z = \frac{\bar{X}-\frac{2}{3}}{\sqrt{\frac{2}{9}/18}}$

$$P(\frac{2}{3} \le \bar{X} \le \frac{5}{6}) \approx P(0 \le Z \le 1.5) = 0.4332.$$

5.8-6

$$\begin{aligned} P(75 < \bar{X} < 85) &= (75-80 < \bar{X}-80 < 85-80) \\ &= P(|X-80| < 5) \ge 1 - \frac{60/15}{5^2} = 0.84 \end{aligned}$$

6.3-6

(a)

$$\begin{aligned} F(x) &= x, \quad 0 < x < 1 \\ g_1(\omega) &= n[1-\omega]^{n-1}, \quad 0 < \omega < 1 \\ g_n(\omega) &= n\omega^{n-1}, \quad 0 < \omega < 1 \end{aligned}$$

(b)

$$\begin{aligned} E(W_1) &= \int_0^1 \omega n[1-\omega]^{n-1}d\omega \\ &= [-\omega(1-\omega)^n - \frac{1}{n+1}(1-\omega)^{n+1}]_0^1 = \frac{1}{n+1} \\ E(W_2) &= \int_0^1 \omega n\omega^{n-1}d\omega = [\frac{n}{n+1}\omega^{n+1}]_0^1 = \frac{n}{n+1} \end{aligned}$$

(c) Lew $\omega = \omega_r$. The pdf of W_r is

$$\begin{aligned} g_r(\omega) &= \frac{n!}{(r-1)!(n-r)!}\omega^{r-1}(1-\omega)^{n-r} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)}\omega^{r-1}(1-\omega)^{n-r} \end{aligned}$$

Thus W_r has a beta distribution with $\alpha = r,\ \beta = n-r$

6.3-8

(a)

$$E(W_r^2) = \int_0^1 \omega^2 \frac{n!}{(r-1)!(n-r)!}\omega^{r-1}(1-\omega)^{n-r}$$
$$= \frac{r(r+1)}{(n+1)(n+2)} \int_0^1 \frac{(n+2)!}{(r+1)!(n-r)!}\omega^{r+1}(1-\omega)^{n-r}d\omega$$
$$= \frac{r(r+1)}{(n+1)(n+2)}$$

(b) $Var(W_r) = \frac{r(r+1)}{(n+1)(n+2)} - \frac{r^2}{(n+1)^2} = \frac{r(n-r+1)}{(n+2)(n+1)^2}$

6.4-2 The likelihood function is

$$L(\theta) = [\frac{1}{2\pi\theta}]^{n/2} exp[-\Sigma_{i=1}^n (x_i-\mu)^2/(2\theta)], \quad 0 < \theta < \infty.$$

The logarithm of the likelihood function is

$$lnL(\theta) = \frac{-n}{2}(ln(2\pi)) - \frac{n}{2}(ln(\theta)) - \frac{1}{2\theta}\Sigma_{i=1}^n (x_i-\mu)^2$$

Setting the first derivative equal to zero and solving the θ yields

$$\frac{dlnL(\theta)}{d\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\Sigma_{i=1}^n (x_i-\mu)^2 = 0$$
$$\theta = \frac{1}{n}\Sigma_{i=1}^n (x_i-\mu)^2$$

Thus

$$\hat{\theta} = \frac{1}{n}\Sigma_{i=1}^n (X_i-\mu)^2$$

. To see that $\hat{\theta}$ is an unbiased estimator of θ, note that

$$E(\hat{\theta}) = E(\frac{\sigma^2}{n}\Sigma_{i=1}^n \frac{(X_i-\mu)^2}{\sigma^2}) = \frac{\sigma^2}{n} n = \sigma^2.$$

6.4-4

For a Poisson distribution,

$$f(x_1, \ldots, x_n \mid \lambda) = \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdots \frac{e^{-\lambda}\lambda^{x_n}}{x_n!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{x_1! \cdots x_n!}$$
$$\ln f = -n\lambda + (\ln\lambda)\sum x_i - \ln\left(\prod x_i!\right)$$
$$\frac{d(\ln f)}{\lambda} = -n + \frac{\sum x_i}{\lambda} = 0$$
$$\hat{\lambda} = \frac{\sum x_i}{n}.$$

Figure 1: The MLE of poission distribution

(a) $\hat{x} = 5516/98 = 56.2857; \quad s^2 = 5452/97 = 56.2062$

(b) $\hat{\lambda} = \hat{x} = 5516/98 = 56.2857;$

(c) Yes;

(d) $\hat{x}$ is better than s^2 because

$$Var(\hat{X}) \approx \frac{56.2857}{98} = 0.5473 < 65.8956 \approx Var(S^2).$$

6.4-8

(a)

$$L(\theta) = (\frac{1}{\theta^n})(\prod_{i=1}^{n} x_i)^{(1/\theta-1)}, \quad 0 < \theta < \infty$$

$$lnL(\theta) = -nln\theta + (\frac{1}{\theta} - 1)ln\prod_{i=1}^{n} x_i$$

$$\frac{dlnL(\theta)}{d\theta} = \frac{-n}{\theta} - \frac{1}{\theta^2}ln\prod_{i=1}^{n} x_i = 0$$

$$\hat{\theta} = -\frac{1}{n}ln\prod_{i=1}^{n} x_i$$

$$= -\frac{1}{n}\Sigma_{i=1}^{n} lnx_i.$$

(b) We first find $E(lnX)$:

$$E(lnX) = \int_0^1 lnx(\frac{1}{\theta})x^{1/\theta-1}dx.$$

Using the integration by parts

$$E(lnX) = \lim_{\alpha\to 0}[x^{1/\theta}lnx - \theta x^{1/\theta}]_{\alpha}^{1} = -\theta$$

Thus

$$E(\hat{\theta}) = \frac{-1}{n}\Sigma_{i=1}^{n}(-\theta) = -\theta$$

6.4-12

(a) $E(\bar{X}) = E(Y)/n = np/n = p$;

(b) $Var(\bar{X}) = Var(Y)/n^2 = np(1-p)/n^2 = p(1-p)/n$;

(c)

$$\begin{aligned} E[\bar{X}(1-\bar{X})/n] &= [E(\bar{X}) - E(\bar{X}^2)]/n \\ &= p - [p^2 + p(1-p)/n]/n \\ &= (1-1/n)p(1-p)/n = (n-1)p(1-p)/n^2; \end{aligned}$$

(d) From Part(c), the constant $c = 1/(n-1)$

6.4-18

$$L = (\frac{1}{2\pi\sigma^2})^{kn/2}exp\{-\Sigma_{i=1}^{n}\Sigma_{j=1}^{k}[x_{ij} - c - d(j - \frac{k+1}{2})]^2/(2\sigma^2)\}.$$

To find $\hat{c}$ and $\hat{d}$ try to minimize

$$S = \Sigma_{i=1}^{n}\Sigma_{j=1}^{k}[x_{ij} - c - d(j - \frac{k+1}{2})]^2$$

. We have

$$\frac{\partial S}{\partial c} = \Sigma_{i=1}^{n}\Sigma_{j=1}^{k}2[x_{ij} - c - d(j - \frac{k+1}{2})](-1) = 0.$$

Since

$$\Sigma_{j=1}^{k}d(j - \frac{k+1}{2}) = 0, \quad \hat{c} = \frac{\Sigma_{i=1}^{n}\Sigma_{j=1}^{k}x_{ij}}{kn} = \bar{x}.$$

$$\frac{\partial S}{\partial d} = \Sigma_{i=1}^{n}\Sigma_{j=1}^{k}2[x_{ij} - c - d(j - \frac{k+1}{2})][-(j - \frac{k+1}{2})] = 0$$

Thus

$$\hat{d} = \frac{\Sigma_{i=1}^{n}\Sigma_{j=1}^{k}(x_{ij} - \bar{x})(j - \{k+1\}/2)}{\Sigma_{i=1}^{n}\Sigma_{j=1}^{k}(j - \{k+1\}/2)^2}$$