6.4-6

$$\hat{\theta}_1 = \hat{\mu} = \bar{\mathbf{X}} = \frac{1}{n}\Sigma_{i=1}^{n} X_i \approx 33.4267$$

$$\hat{\theta}_2 = \hat{\sigma}^2 = \frac{1}{n}\Sigma_{i=1}^{n}(X_i - \bar{\mathbf{X}})^2 \approx 5.0980$$

6.4-10

(a) $\bar{\mathbf{X}} = \frac{1}{p}$, so $\tilde{p} = \frac{1}{\bar{\mathbf{X}}} = \frac{n}{\Sigma_{i=1}^{n} X_i}$;

(b) $\tilde{p}$ equals the number of success, n, divided by the number of total trials, $\Sigma_{i=1}^{n} X_i$, for example $X_i = 5$ means that the the researcher tried 5 times to get his wanted result once. $\frac{n}{\Sigma_{i=1}^{n} X_i}$ makes sense because it presents the ratio between success numbers and total numbers of experiments ;

(c) $\bar{\mathbf{X}} = \frac{1}{n}\Sigma_{i=1}^{n} X_i = \frac{252}{20} = 12.6$, thus $\tilde{p} = \frac{20}{252} \approx 0.0794$

6.5-1

$$\begin{aligned}
\Sigma_{i=1}^{n}(y_i - \hat{y}_i) &= \Sigma_{i=1}^{n}[y_i - (\hat{\alpha} + \hat{\beta}x_i]\\
&= \Sigma[y_i - (\bar{y} - \hat{\beta}\bar{x} + \hat{\beta}x_i)]\\
&= \Sigma_{i=1}^{n}(y_i - \bar{y}) - \hat{\beta}\Sigma_{i=1}^{n}(x_i - \bar{x})\\
&= 0 - 0\\
&= 0
\end{aligned}$$

6.5-2

(a)

$$\begin{aligned}
L(\beta, \sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_i - \beta x_i)^2}{2\sigma^2}\right]\\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{\Sigma_{i=1}^{n}(y_i - \beta x_i)^2}{2\sigma^2}\right]
\end{aligned}$$

Maximizing L or equivalently maximizing

$$lnL(\beta, \sigma^2) = \frac{n}{2} ln(2\pi\sigma^2) + \frac{\Sigma_{i=1}^{n}(y_i - \beta x_i)^2}{2\sigma^2}$$

$$\begin{aligned}
\frac{\partial lnL}{\partial \beta} &= 2\Sigma_{i=1}^{n}(y_i - \beta x_i)(-x_i) = 0\\
\hat{\beta} &= \frac{\Sigma_{i=1}^{n} x_i y_i}{\Sigma_{i=1}^{n} x_i^2}
\end{aligned}$$

which is a point of minimum since $\frac{\partial^2 lnL}{\partial \beta^2} = 2\Sigma_{i=1}^{n} x_i^2 > 0$.

$$\begin{aligned}
\frac{\partial lnL}{\partial \sigma^2} &= \frac{n}{2\sigma^2} - \frac{\Sigma_{i=1}^{n}(y_i - \beta x_i)^2}{2\sigma^4} = 0\\
\hat{\sigma^2} &= \frac{1}{n}\Sigma_{i=1}^{n}(y_i - \hat{\beta}x_i)^2.
\end{aligned}$$

(b) $\hat{\beta} = \frac{\Sigma_{i=1}^n x_i Y_i}{\Sigma_{i=1}^n x_i^2}$ Since $\hat{\beta}$ is a linear combination of independent normal random variables, it is normal with mean and variance

$$\begin{aligned} E(\hat{\beta}) &= \frac{\Sigma_{i=1}^n x_i E(Y_i)}{\Sigma_{i=1}^n x_i^2} \\ &= \frac{\Sigma_{i=1}^n x_i (\beta x_i)}{\Sigma_{i=1}^n x_i^2} \\ &= \frac{\beta \Sigma_{i=1}^n x_i^2}{\Sigma_{i=1}^n x_i^2} \\ &= \beta \\ Var(\hat{\beta}) &= \frac{\Sigma_{i=1}^n x_i^2 Var(Y_i)}{(\Sigma_{i=1}^n x_i^2)^2} \\ &= \frac{\Sigma_{i=1}^n x_i^2 (\sigma^2)}{(\Sigma_{i=1}^n x_i^2)^2} \\ &= \frac{\sigma^2}{\Sigma_{i=1}^n x_i^2} \end{aligned}$$

Also

$$\begin{aligned} \Sigma_{i=1}^n (Y_i - \beta x_i)^2 &= \Sigma_{i=1}^n [Y_i - \hat{\beta} x_i + (\hat{\beta} - \beta) x_i]^2 \\ &= \Sigma_{i=1}^n (Y_i - \hat{\beta} x_i)^2 + (\hat{\beta} - \beta)^2 \Sigma_{i=1}^n x_i^2. \end{aligned}$$

Since $Y_i \sim N(\beta x, \sigma^2)$ and $\hat{\beta} \sim N(\beta, \frac{\sigma^2}{\Sigma_{i=1}^n x_i^2})$,

$$\frac{\Sigma_{i=1}^n (Y_i - \beta x_i)^2}{\sigma^2} \sim \chi^2(n) \quad and \quad \frac{(\hat{\beta} - \beta)^2 \Sigma_{i=1}^n x_i^2}{\sigma^2} \sim \chi^2(1).$$

By Theorem 9.3-1, $\Sigma_{i=1}^n (Y_i - \hat{\beta} x_i)^2 \sim \chi^2(n-1)$.

6.5-6

(a)

$$\begin{aligned} \hat{\alpha} &= \frac{395}{15} = 26.333 \\ \hat{\beta} &= \frac{9292 - (346)(395)/15}{8338 - (346)^2/15} \approx 0.506 \\ \hat{y} &= 26.333 + 0.506(x - \frac{346}{15}) \\ &= 0.506x + 14.657; \end{aligned}$$

(b)

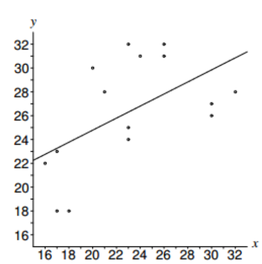

Figure 1: ACT natural science (y) versus ACT social science (x) scores

(c)

$$\hat{\alpha} = 26.33, \quad \hat{\beta} = 0.506$$
$$n\hat{\sigma}^2 = 10705 - \frac{395^2}{15} - 0.506(9292) + 0.506(346)(395)/15$$
$$= 211.886$$
$$\hat{\sigma}^2 = \frac{211.886}{15} = 14.126$$

6.7-2 The distribution of Y is Poisson with mean $n\lambda$. Thus, since $y = \Sigma_{i=1}^n x_i$,

$$P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n | Y = y) = \frac{(\lambda^{\Sigma x_i} e^{-n\lambda})/(x_1!x_2!\cdots x_n!)}{(n\lambda)^y e^{-n\lambda}/y!}$$
$$= \frac{y!}{x_1!x_2!\cdots x_n! n^y},$$

which does not depend on λ

6.7-4

(a) $f(x;\theta) = e^{(\theta-1)lnx + ln\theta}, \quad 0 < x < 1, \quad 0 < x < \infty;$
so $K(x) = lnx$ and thus

$$Y = \Sigma_{i=1}^n lnX_i = ln(X_1X_2\cdots X_n)$$

is a sufficient statistics for θ.

(b)

$$L(\theta) = \theta^n (x_1x_2\cdots x_n)^{\theta-1}$$
$$lnL(\theta) = nln\theta + (\theta-1)ln(x_1x_2\cdots x_n)$$
$$\frac{dlnL(\theta)}{d\theta} = \frac{n}{\theta} + ln(x_1x_2\cdots x_n) = 0.$$

Hence $\hat{\theta} = -n/ln(X_1X_2\cdots X_n)$, which is a function of Y.

(c) Since $\hat{\theta}$ is single valued function of Y with a single valued inverse, knowing the value of $\hat{\theta}$ is equivalent to knowing the value Y, and hence it is sufficient.

6.7-6

(a)

$$f(x_1, x_2, \cdots, x_n) = \frac{(x_1x_2\cdots x_n)^{\alpha-1} e^{-\Sigma x_i/\theta}}{[\Gamma(\alpha)]^n \theta^{\alpha n}}$$
$$= (\frac{e^{-\Sigma x_i/\theta}}{\theta^{\alpha n}})(\frac{(x_1x_2\cdots x_n)^{\alpha-1}}{[\Gamma(\alpha)]^n})$$

The second factor is free of θ. The first factor is a function of the x_is through $\Sigma_{i=1}^n x_i$ only, so $\Sigma_{i=1}^n x_i$ is a sufficient statistic for θ.

(b)

$$lnL(\theta) = ln(x_1x_2\cdots x_n)^{\alpha-1} - \Sigma_{i=1}^n x_i/\theta - ln[\Gamma(\alpha)]^n - \alpha n ln\theta$$
$$\frac{dlnL(\theta)}{d\theta} = \Sigma_{i=1}^n x_i/\theta^2 - \alpha n/\theta = 0$$
$$\alpha n\theta = \Sigma_{i=1}^n x_i$$
$$\hat{\theta} = \frac{1}{\alpha n}\Sigma_{i=1}^n X_i.$$

$Y = \Sigma_{i=1}^n X_i$ has a gamma distribution with parameters αn and θ. Hence

$$E(\hat{\theta}) = \frac{1}{\alpha n}(\alpha n\theta) = \theta.$$