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two ways to parametrize Gamma distribution.

$$\frac{1}{\Gamma(k)\,\theta^{k}}\, x^{k-1} e^{-x/\theta} \quad \text{or} \quad \frac{\lambda^{k}}{\Gamma(k)}\, x^{k-1} e^{-\lambda x}$$

in the first parametrization.

k is called the shape parameter

and θ is called the scale parameter.

under the second parametrization,

again k is called the shape parameter

while λ is called the rate parameter.

if you consider the case when $k=1$,

we get the exponential distribution.

also note that the mean and variance under the first parametrization is

$$k\theta, \quad k\theta^{2}$$

the θ, the larger the

so larger the θ, the larger the variance, so θ can be considered to be a scale parameter.

in the second parametrization, the mean and the variance is

$$\frac{k}{\lambda}, \quad \frac{k}{\lambda^2}$$

so larger the λ, smaller the variance.
also smaller the mean.

So on average, you can fit more of these r.v.s in a fixed region. So we may say there is a higher rate.

Gamma function.

it is a generalization of the factorial.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

it can also be defined for complex numbers, but let's just say the input is a positive real number.

Properties.

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt = \int_0^\infty t^z (-e^{-t})'\, dt$$

$$= \left[-t^z e^{-t}\right]_0^\infty + z\int_0^\infty t^{z-1} e^{-t}\, dt$$

$$= -0 + 0 + z\Gamma(z) = z\Gamma(z)$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1.$$

So for integer z, $\Gamma(z+1) = z!$

also

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$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} \, dt$$

$t^{1/2} = s \qquad \frac{ds}{dt} = \frac{1}{2} t^{-1/2} \qquad ds \, 2t^{1/2} = dt$

$$= \int_0^\infty s^{-1} e^{-s^2} 2s \, ds = \int_0^\infty 2e^{-s^2} ds$$

now you remember

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds = 1$$

again change of variable

with $\frac{s}{\sqrt{2}} = t \qquad \frac{dt}{ds} = \frac{1}{\sqrt{2}}$

$\sqrt{2} \, dt = ds$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} = 1$$

$$\int_0^\infty \frac{1}{\sqrt{\pi}} e^{-t^2} = 1/2$$

$$\int_0 \sqrt{\pi}\, e \qquad 1/2$$

$$\int_0^\infty e^{-t^2} = \frac{\sqrt{\pi}}{2}$$

So $\Gamma(1/2) = \sqrt{\pi}$.

When you calculate moments you use this Gamma function together with change of variables.

first let's see how

$\frac{\lambda^k}{\Gamma(k)}$ came out.

$$\int_0^\infty x^{k-1} e^{-\lambda x}\, dx$$

$$\int_0^\infty x^{k-1} e^{-\lambda x} \, dx$$

$$\lambda x = t \qquad \frac{dt}{dx} = \lambda \qquad dx = \frac{dt}{\lambda}$$

$$\frac{1}{\lambda} \int_0^\infty \frac{t^{k-1}}{\lambda^{k-1}} e^{-t} \, dt$$

$$= \frac{1}{\lambda^k} \int_0^\infty t^{k-1} e^{-t} \, dt = \frac{\Gamma(k)}{\lambda^k}.$$

now calculating $E(X)$ is really a similar idea.

$$\frac{\lambda^k}{\Gamma(k)} \int_0^\infty x \, x^{k-1} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^k e^{-\lambda x} \, dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{t^k}{\lambda^k} e^{-t} \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda \Gamma(k)} \int_0^\infty t^k e^{-t} dt$$

$$= \frac{1}{\lambda \Gamma(k)} \Gamma(k+1) = \frac{k \cancel{\Gamma(k)}}{\lambda \cancel{\Gamma(k)}} = \frac{k}{\lambda}.$$

$E[X^2]$ is again, really the same thing.

$$\frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^2 x^{k-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k+1} e^{-\lambda x} dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty \frac{t^{k+1}}{\lambda^{k+1}} e^{-t} \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda^2 \Gamma(k)} \int_0^\infty t^{k+1} e^{-t} \, dt$$

$$= \frac{\Gamma(k+2)}{\lambda^2 \Gamma(k)} = \frac{(k+1) k \Gamma(k)}{\lambda^2 \Gamma(k)}$$

$$= \frac{(k+1) k}{\lambda^2}$$

so the variance is

$$\frac{k^2+k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

the calculation for the other form of parametrization is form of parametization is

similar.