2018年4月13日　　15:00

$$E\left[(X-\mu_X)(Y-\mu_Y)\right]^2 \leq E\left[(X-\mu_X)^2\right]E\left[(Y-\mu_Y)^2\right]$$

this is what we want to show.

$$E\left[\left\{(X-\mu_X) - t\,E\left[(X-\mu_X)(Y-\mu_Y)\right](Y-\mu_Y)\right\}^2\right] \geq 0$$

$$E\Big[(X-\mu_X)^2 - 2t\,(X-\mu_X)(Y-\mu_Y)\,E\left[(X-\mu_X)(Y-\mu_Y)\right] + t^2 (Y-\mu_Y)^2\, E\left[(X-\mu_X)(Y-\mu_Y)\right]^2\Big] \geq 0$$

$$E\left[(X-\mu_X)^2\right] - 2t\,E\left[(X-\mu_X)(Y-\mu_Y)\right]^2 + t^2\,E\left[(Y-\mu_X)^2\right]E\left[(X-\mu_X)(Y-\mu_Y)\right]^2 \geq 0$$

this means that this quadratic of

t has 0 or one root.

the roots of

$ax^2+bx+c=0$ are

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

so $b^2-4ac \leq 0$

so

$$4E[(X-\mu_X)(Y-\mu_Y)]^4$$

$$-4E[(X-\mu_X)^2]E[(Y-\mu_Y)^2]$$

$$E[(X-\mu_X)(Y-\mu_Y)]^2 \leq 0$$

so

$$\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]^2 \leq \mathbb{E}[(X-\mu_X)^2]$$
$$\mathbb{E}[(Y-\mu_Y)^2]$$

if $\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]^2$

$$= \mathbb{E}[(X-\mu_X)^2]\,\mathbb{E}[(Y-\mu_Y)^2]$$

at $t = -\frac{b}{2a} = \frac{1}{\mathbb{E}[(Y-\mu_Y)^2]}$,

the quadratic would be 0.

i.e. plugging into the very first inequality.

$$E\left[\left\{(X-\mu_X) - (Y-\mu_Y)K\right\}^2\right] = 0$$

where $K = \dfrac{E[(X-\mu_X)(Y-\mu_Y)]}{E[(Y-\mu_Y)^2]}$

this means

$X-\mu_X$ is equal to a

constant times $(Y-\mu_Y)$.

On the other hand, let's assume

$(X-\mu_X) = K(Y-\mu_Y)$ for some

constant K.

then

$$= \left[(X-\mu_X)(Y-\mu_Y)\right]^2$$

$$E\left[(X-\mu_X)(Y-\mu_Y)\right]^2$$

$$= K^2 E\left[(Y-\mu_Y)^2\right]^2$$

while $E\left((X-\mu_X)^2\right) E\left((Y-\mu_Y)^2\right)$

$$= K^2 E\left[(Y-\mu_Y)^2\right]^2$$

therefore

$$E\left((X-\mu_X)(Y-\mu_Y)\right)^2$$

$$= E\left((X-\mu_X)^2\right] E\left((Y-\mu_Y)^2\right)$$

in other words

$$E\left[(X-\mu_X)(Y-\mu_Y)\right]^2$$

$$= E\left((X-\mu_X)^2\right) E\left((Y-\mu_Y)^2\right)$$

a contrast

$= E[$

iff $X - \mu_X$ is a constant multiplied to $Y - \mu_X$

Maybe it might be easier to think of the above with X & Y switched.