6.9-1

$$\begin{aligned}
k(x,\theta) &= \frac{\theta^x e^{-\theta}}{x!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}} \quad x=0,1,\cdots, \quad \theta>0 \\
k_1(x) &= \int_0^\infty \frac{\theta^x e^{-\theta}}{x!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}d\theta \\
&= \frac{1}{x!\Gamma(\alpha)\beta^\alpha}\int_0^\infty \theta^{\alpha+x-1}e^{-\frac{1+\beta}{\beta}\theta}d\theta \\
&= \frac{1}{x!\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+x)(\frac{\beta}{1+\beta})^{\alpha+x} \\
&= \frac{\Gamma(\alpha+x)\beta^x}{\Gamma(\alpha)x!(1+\beta)^{\alpha+x}}, \quad x=0,1,2,\cdots
\end{aligned}$$

6.9-2

$$\begin{aligned}
k(x,\theta) &= \binom{n}{x}\theta^x(1-\theta)^{n-x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad x=0,1,\cdots,n, \quad 0<\theta<1. \\
k_1(x) &= \int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta \\
&= \frac{n!}{x!(n-x)!}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1}d\theta \\
&= \frac{n!\Gamma(\alpha+\beta)\Gamma(x+\alpha)\Gamma(n-x+\beta)}{x!(n-x)!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}, \quad x=0,1,2,\cdots,n
\end{aligned}$$

6.9-3

$$\begin{aligned}
k(x,\theta) &= \frac{\Gamma(\alpha+\beta}{\Gamma(\alpha)\Gamma(\beta)}\theta^\alpha(1-\theta)^{x+\beta-2} \quad \alpha,\beta>0 \quad x=1,2,3\cdots \\
k_1(x) &= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^\alpha(1-\theta)^{x+\beta-2}d\theta \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^\alpha(1-\theta)^{x+\beta-2}d\theta \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)\Gamma(\beta+x-1)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+x)} \quad x=1,2,3,\cdots
\end{aligned}$$

6.9-4

$$\begin{aligned}
k_1(x) &= \int_0^\infty \theta\tau x^{\tau-1}e^{-\theta x^\tau}\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}d\theta \quad 0<x<\infty \\
&= \frac{\tau x^{\tau-1}}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty \theta^\alpha e^{-(x^\tau+1/\beta)\theta}d\theta \\
&= \frac{\tau x^{\tau-1}}{\Gamma(\alpha)\beta^\alpha}\frac{\Gamma(\alpha+1)}{(x^\tau+1/\beta)^{\alpha+1}}, \quad 0<x<\infty \\
&= \frac{\alpha\beta\tau x^{\tau-1}}{(\beta x^\tau+1)^{\alpha+1}}, \quad 0<x<\infty.
\end{aligned}$$

6.9-5

(a)

$$\begin{aligned}
k(x_1, x_2, \cdots, x_n) &\propto \int_0^\infty (\frac{1}{\theta})^{n+1} e^{-\frac{1}{\theta}\Sigma x_i} d\theta, \quad 0 < x_i < \infty \\
&= \int_0^\infty z^{n+1} e^{-(\Sigma x_i)z} \frac{1}{z^2} dz \quad z = \frac{1}{\theta} \\
&= \int_0^\infty z^{n-1} e^{-(\Sigma x_i)z} dz \\
&= \frac{\Gamma(n)}{(\Sigma x_i)^n} \\
g(\theta | x_1, x_2, \cdots, x_n) &= f(x_1, x_2, \cdots, x_n | \theta) h(\theta) / k(x_1, x_2, \cdots, x_n) \\
&= (\frac{1}{\theta})^{n+1} e^{-\frac{1}{\theta}\Sigma x_i} \frac{(\Sigma x_i)^n}{\Gamma(n)}
\end{aligned}$$

(b) Denote $z = \frac{1}{\theta}$, $-\frac{1}{z^2} dz = d\theta$

$$g(z | x_1, x_2, \cdots, x_n) = \frac{(\Sigma x_i)^n}{\Gamma(n)} z^{n+1} e^{-z\Sigma x_i} \frac{1}{z^2} = \frac{(\Sigma x_i)^n}{\Gamma(n)} z^{n-1} e^{-z\Sigma x_i}$$

which shows that $z \sim \Gamma(n, 1/\Sigma x_i) = \Gamma(n, 1/y)$

(c) $z \sim \Gamma(n, 1/\Sigma x_i) = \Gamma(n, 1/y)$, thus $2yz \sim \chi^2(2n)$ The $1 - \alpha$ confidence interval could be constructed as

$$\frac{2y}{\chi^2_{\alpha/2}(2n)} < \theta < \frac{2y}{\chi^2_{1-\alpha/2}(2n)}$$

6.9-6

$$g(\theta_1, \theta_2 | x_1 = 3, x_2 = 7) \propto (\frac{1}{\pi})^2 \frac{\theta_2^2}{[\theta_2^2 + (3 - \theta_1)^2][\theta_2^2 + (7 - \theta_1)^2]}.$$

Without considering the normalized constant term.

$$h(\theta_1, \theta_2) = (\frac{1}{\pi})^2 \frac{\theta_2^2}{[\theta_2^2 + (3 - \theta_1)^2][\theta_2^2 + (7 - \theta_1)^2]} \quad (1)$$

To maximize $h(\theta_1, \theta_2)$, try to solve $\frac{\partial h}{\partial \theta_1} = 0$, $\frac{\partial h}{\partial \theta_2} = 0$ and check whether the corresponding hessian matrix is negative definite.

$$(\theta_1 = 5, \quad \theta_2 = 2) \quad or$$
$$\theta_2 = \sqrt{-(\theta_1 - 3)(\theta_1 - 7)}$$

7.1-2

(a) $[77.272, 92.728]$

(b) $[79.12, 90.88]$

(c) $[80.065, 89.935]$

(d) $[81.154, 88.846]$

7.1-4

(a) $\bar{X} = 56.8$

(b) $[56.8 - 1/96(2/\sqrt{10}), 56.8 + 1/96(2/\sqrt{10})] = [55.56, 58.04]$

(c) $P(X < 52) = P(Z < \frac{52-56.8}{2}) = P(Z < -2.4) = 0.0082 \quad Z \sim N(0, 1)$