http://www.hnagata.net/archives/496 p\t

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I will show the Poisson Limit Theorem

original from link above.

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$p = \frac{\lambda}{n}$ and we are letting $n \to \infty$

p also becomes small accordingly

and λ is kept moderate.

$$= \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \underbrace{\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{n^k}}$$

$$= \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$= \frac{\lambda^k}{k!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)$$

now $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$, $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^k = 1$, and

$$\lim_{n\to\infty}\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) = 1$$

therefore $\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$

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this the proof you often see in textbooks. there are two issues.

one minor problem is with $k=0$. But this can be solved easily.

The bigger problem is that under this limit $n \gg k$ (n is much larger than k. we want this approximation to hold even for k close to n.

The proof below addresses this problem. But before we start, we need a sub result.

lemma: $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{\sqrt{n}} = 1$

<u>proof</u> put $m = \sqrt{n}$.

$$\lim_{m\to\infty} \left(1 - \frac{\lambda}{m^2}\right)^m = \lim_{m\to\infty} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i}}$$

note we are assuming n is a square of an integer. This is ok to do. You may want to think why.

$$= 1 + \lim_{m\to\infty} \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i}}$$

$$= 1 + \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i-1}}$$

look at the terms of the summation.

$$\left| (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i-1}} \right| = \binom{m}{i} \frac{\lambda^i}{m^{2i-1}} = \frac{m!}{i!(m-i)!} \frac{\lambda^i}{m^{2i-1}}$$

$$\leq m\cdot(m-1)\cdots(m-i+1) \frac{\lambda^i}{m^{2i-1}} \leq m^i \frac{\lambda^i}{m^{2i-1}} = \frac{\lambda^i}{m^{i-1}}$$

a series with these terms are absolutely convergent.

$\uparrow$ google.

Since the terms of our series is bounded by the terms of this series, our series is also absolutely convergent.

therefore $\lim_{m\to\infty} \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i-1}}$ is finite

therefore $\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{\lambda^i}{m^{2i-1}}$ is 0

$$\therefore \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{\sqrt{n}} = 1 \quad \blacksquare$$

$$\therefore \lim_{n\to\infty}\left(1-\frac{i}{n}\right)=1 \quad \blacksquare$$

Now to the proof of PLT.

we will divide it in to three cases

1st case : $K=0$

$$\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{n-k}=\lim_{n\to\infty}(1-p)^n=\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n=e^{-\lambda}$$

$$=e^{-\lambda}\frac{\lambda^0}{0!}$$

good

无穷可分分布的中心
极限定理

(1) (2) (3) → check

speed rate

Lindeberg

Feller μ σ^2

ignore

ignore.

case 2: $1 \le k \le \sqrt{n}$

$$\left(1-\frac{\lambda}{n}\right)^k \le \left(1-\frac{\lambda}{n}\right)^0 = 1$$

$$\left(1-\frac{\lambda}{n}\right)^k \ge \left(1-\frac{\lambda}{n}\right)^{\sqrt{n}} \to 1 \text{ as } n \to \infty$$

therefore $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^k = 1$ (sandwitching)

now for $0 \le i \le k$,

$1-\frac{i}{n} \le 1$ and

$1-\frac{i}{n} \ge 1-\frac{\sqrt{n}}{n} = 1-\frac{1}{\sqrt{n}} \to 1$ as $n \to \infty$

therefore again by the sandwitching

$\lim_{n\to\infty} 1-\frac{i}{n} = 1$ therefore

$$\lim_{n\to\infty} \prod_{i=0}^{k-1} \left(1-\frac{i}{n}\right) = 1.$$

so as in above, we get what we want. good.

case 3: lastly $k > \sqrt{n}$

now $\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^k < \infty$

$\lim_{n\to\infty} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) < \infty$

$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda} < \infty$

and since $\frac{\lambda^{\sqrt{n}}}{\sqrt{n}!} \to 0$ as $n \to \infty$

and $k > \sqrt{n}$

$\frac{\lambda^k}{k!} \to 0$ as $n \to \infty$.

therefore the binomial probability in the limit, for $k > \sqrt{n}$ is 0.

on the other hand, for the Poisson,

$e^{-\lambda} \frac{\lambda^k}{k!}$ also goes to 0 as

$\sqrt{n} \to \infty$ for $k > \sqrt{n}$.

$\sqrt{n} \to \infty$ for $k > \sqrt{n}$.

Therefore, again, the two probabilities

good

match.

$\blacksquare$