NEGATIVE BINOMIAL

2018年3月28日 12:23

Why do we call it negative binomial?

we have

$$\binom{x-1}{r-1} p^r (1-p)^{x-r}$$

now

$$\binom{x-1}{r-1} = \frac{\overbrace{(x-1)(x-2)(x-3)\cdots(x-r+1)}^{x-r \text{ terms.}}}{(r-1)!}$$

$$= \frac{(-x+1)(-x+2)(-x+3)\cdots(-x+r-1)\,(-1)^{x-r}}{(r-1)!}$$

$$= \frac{(-x+r-1)!}{(r-1)!\,(-x)!}(-1)^{x-r}$$

although factorial of negative number does not really make sense

$$= \binom{-x+r-1}{r-1} (-1)^{r-1}$$

So our probability mass function can be written as

$$\binom{-x+r-1}{r-1} p^r (p-1)^{x-r}$$

this binomial has a negative value on the top.

Therefore it is called the negative binomial coefficient.

And so this distribution is called the negative binomial.

On the mean & Variance of Hypergeometric.

Hypergeometric

$$E[X] = \sum_{x=0 \vee (n-M_2)}^{n \wedge M_1} x \frac{\binom{M_1}{x}\binom{M_2}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=1 \vee (n-M_2)}^{n \wedge M_1} \frac{x \dfrac{M_1!}{x!(M_1-x)!} \dfrac{M_2!}{(n-x)!(M_2-n+x)!}}{\dfrac{N!}{n!(N-n)!}}$$

because when $x=0$ ...

$$= \sum_{x=1 \vee (n-M_2)}^{n \wedge M_1} \frac{\dfrac{M_1 (M_1-1)!}{(x-1)!(M_1-x)!} \dfrac{M_2!}{(n-1-[x-1])!(M_2-n+x)!}}{\dfrac{N (N-1)!}{n (n-1)! (N-n)!}}$$

$$= \sum_{x=1 \vee (n-M_2)}^{n \wedge M_1} \frac{M_1 \binom{M_1-1}{x-1} \binom{M_2}{n-1-[x-1]}}{\frac{N}{n} \binom{N-1}{n-1}}$$

$$= \frac{M_1 n}{N} \sum_{x=1 \vee (n-M_2)}^{n \wedge M_1} \frac{\binom{M_1-1}{x-1} \binom{M_2}{n-1-[x-1]}}{\binom{N-1}{}}$$

$$N \qquad x=1 \vee (n-M_2) \qquad \frac{\qquad}{\binom{N-1}{n-1}}$$

$$= \frac{M_1 n}{N} \sum_{t=0 \vee (n-1-M_2)}^{(n-1)\wedge(M_1-1)} \frac{\binom{M_1-1}{t}\binom{M_2}{n-1-t}}{\binom{N-1}{n-1}}$$

$$= \frac{M_1 n}{N}$$

now for the variance.

$$E\left[X(X-1)\right] = \sum_{x=0 \vee (n-M_2)}^{n \wedge M_1} x(x-1) \frac{\binom{M_1}{x}\binom{M_2}{n-x}}{\binom{N}{n}}$$

$$= \sum_{x=2 \vee (n-M_1)}^{n \wedge M_1} \frac{\frac{M_1!}{(x-2)!\,(M_1-x)!}\binom{M_2}{n-x}}{\binom{N}{n}}$$

$$= M_1(M_1-1) \sum_{x=2 \vee (n-M_2)}^{n \wedge M_1} \frac{\frac{(M_1-2)!}{(x-2)!\,(M_1-x)!}\binom{M_2}{n-x}}{\binom{N}{n}}$$

$$\binom{N}{n}$$

$$= M_1(M_1-1) \sum_{x=2\vee(n-M_2)}^{n\wedge M_1} \frac{\binom{M_1-2}{x-2}\binom{M_2}{n-x}}{\dfrac{N!}{n!\,(N-n)!}}$$

$$= M_1(M_1-1) \sum_{x=2\vee(n-M_2)}^{n\wedge M_1} \frac{\binom{M_1-2}{x-2}\binom{M_2}{[n-2]-[x-2]}}{\dfrac{N(N-1)\,(N-2)!}{n(n-1)\,(n-2)!\,\,(N-n)!}}$$

$$= M_1(M_1-1)\left(\sum_{x=2\vee(n-M_2)}^{n\wedge M_1} \frac{\binom{M_1-2}{x-2}\binom{M_2}{[n-2]-[x-2]}}{\binom{N-2}{n-2}} \right)$$

$$\times \left[\frac{N(N-1)}{n(n-1)}\right]^{-1}$$

$$= \frac{n(n-1)\,\,M_1(M_1-1)}{N(N-1)} \sum_{s=0\vee(n-2-M_2)}^{(n-2)\wedge(M_1-2)} \frac{\binom{M_1-2}{s}\binom{M_2}{[n-2]-s}}{\binom{N-2}{n-2}}$$

$$= \frac{n(n-1) \quad M_1(M_1-1)}{N(N-1)} = E(X^2) - E(X)$$

$$E(X^2) = \frac{n(n-1) \quad M_1(M_1-1)}{N(N-1)} + \frac{M_1 n}{N}$$

$$V(X) = \frac{n(n-1)M_1(M_1-1) + M_1 n(N-1)}{N(N-1)} - \frac{M_1^2 n^2}{N^2}$$

$$= \frac{N n(n-1) M_1(M_1-1) + N M_1 n(N-1) - M_1^2 n^2 (N-1)}{N^2(N-1)}$$

$$= \frac{M_1 n \left[N(n-1)(M_1-1) + N(N-1) - M_1 n(N-1) \right]}{N^2(N-1)}$$

$$= M_1 n \left[\cancel{N n M_1} - Nn - NM_1 + \cancel{N} \right.$$
$$\left. N^2 \quad \cancel{N} \quad - \cancel{M_1 n N} + M_1 n \right]$$

$$= M_1 n \Big(\ldots$$

$$\frac{+ N^2 - \cancel{N} - \cancel{M_1 n N} + M_1 n \Big)}{N^2(N-1)}$$

$$= \frac{M_1 n \left[N^2 - Nn - NM_1 + M_1 n \right]}{N^2(N-1)}$$

$$= \frac{M_1 n (N - M_1)(N - n)}{N^2(N-1)} \qquad \blacksquare$$

another way

$$V(X) = E[X(X-1)] + E(X) - E(X)^2$$

$$= E[X(X-1)] + E(X)\big(1 - E(X)\big)$$

$$= \frac{n(n-1) M_1(M_1-1)}{N(N-1)} + \frac{M_1 n}{N}\left(1 - \frac{M_1 n}{N}\right)$$

$$= \frac{\quad}{N(N-1)}$$

$$= \frac{n(n-1)\,M_1(M_1-1)}{N(N-1)} + \frac{M_1 n\,(N - M_1 n)}{N^2}$$

$$= \frac{N n(n-1)M_1(M_1-1) + M_1 n\,(N - M_1 n)(N-1)}{N^2(N-1)}$$

$$= \frac{M_1 n\left[N(n-1)(M_1-1) + (N - M_1 n)(N-1)\right]}{N^2(N-1)}$$

$$= \frac{M_1 n\left[\cancel{N n M_1} - Nn - NM_1 \cancel{+N} + N^2 \cancel{-N} \cancel{-M_1 n N} + M_1 n\right]}{N^2(N-1)}$$

$$= \frac{M_1 n\left(N^2 - Nn - NM_1 + M_1 n\right)}{N^2(N-1)}$$

$$N^2(N-1)$$

$$= \frac{M_1 n \,(N-n)(N-M_1)}{N^2(N-1)} \qquad \blacksquare$$

so not really different.

intuition

$E(X)$: n times $\frac{M_1}{N}$.

$V(X)$: $n\left(\frac{N-n}{N-1}\right)$ times $\frac{M_1}{N}\,\frac{(N-M_1)}{N}$

$\uparrow$

this part is not n

Variance should be smaller and smaller for later draws.

indeed $\frac{N-n}{} < 1$.

indeed $\quad \dfrac{N-n}{N-1} < 1.$