8.6-2

(a)

$$\frac{L(4)}{L(16)} = \frac{(1/2\sqrt{2\pi})^n \exp\left[-\Sigma x_i^2/8\right]}{(1/4\sqrt{2\pi})^n \exp\left[-\Sigma x_i^2/32\right]}$$
$$= 2^n \exp\left[-3\Sigma x_i^2/32\right] \leq k$$
$$-\frac{3}{32}\sum_{i=1}^{n} x_i^2 \leq \ln k - \ln 2^n$$
$$\sum_{i=1}^{n} x_i^2 \geq -\left(\frac{32}{3}\right)(\ln k - \ln 2^n) = c$$

(b)

$$0.05 = P\left(\sum_{i=1}^{15} X_i^2 \geq c; \sigma^2 = 4\right)$$
$$= P\left(\frac{\sum_{i=1}^{15} X_i^2}{4} \geq \frac{c}{4}; \sigma^2 = 4\right)$$
$$\text{Thus } \frac{c}{4} = \chi^2_{0.05}(15) = 25 \text{ and } c = 100$$

(c)

$$\beta = P\left(\sum_{i=1}^{15} X_i^2 < 100; \sigma^2 = 16\right)$$
$$= P\left(\frac{\sum_{i=1}^{15} X_i^2}{16} < \frac{100}{16} = 6.25\right) \approx 0.025$$

8.6-4

(a)

$$\frac{L(0.9)}{L(0.8)} = \frac{(0.9)^{\Sigma x_i}(0.1)^{n-\Sigma x_i}}{(0.8)^{\sum x_i}(0.2)^{n-\Sigma x_i}} \leq k$$
$$\left[\left(\frac{9}{8}\right)\left(\frac{2}{1}\right)\right]^{\sum_1^n x_i}\left[\frac{1}{2}\right]^n \leq k$$
$$\left(\sum_{i=1}^{n} x_i\right)\ln(9/4) \leq \ln k + n\ln 2$$
$$y = \sum_{i=1}^{n} x_i \leq \frac{\ln k + n\ln 2}{\ln(9/4)} = c$$

Recall that the distribution of the sum of Bernoulli trials, Y, is $b(n;p)$.

(b)

$$0.10 = P[Y \leq n(0.85); p = 0.9]$$
$$= P\left[\frac{Y - n(0.9)}{\sqrt{n(0.9)(0.1)}} \leq \frac{n(0.85) - n(0.9)}{\sqrt{n(0.9)(0.1)}}; p = 0.9\right]$$

It is true, approximately, that $\frac{n(-0.05)}{\sqrt{n}(0.3)} = -1.282$. That refers $n = 59.17$ or $n = 60$.

(c)

$$\beta = P[Y > n(0.85) = 51; p = 0.8] = P\left[\frac{Y - 60(0.8)}{\sqrt{60(0.8)(0.2)}} > \frac{51-48}{\sqrt{9.6}}; p = 0.8\right]$$
$$\approx P(Z \geq 0.97) = 0.166$$

(d) Let $Y \sim b(n,p)$. To find a UMP test of the simple null hypothesis $H_0 : p = 0.9$ against the one-sided alternative hypothesis $H_1 : p < 0.9$, consider, with $p_1 < 0.9$,we have

$$\frac{L(0.9)}{L(p_1)} = \frac{\begin{pmatrix} n \\ y \end{pmatrix} 0.9^y (1-0.9)^{n-y}}{\begin{pmatrix} n \\ y \end{pmatrix} p_1^y (1-p_1)^{n-y}} \leq k$$

$$\left[\tfrac{0.9(1-p_1)}{p_1(1-0.9)}\right]^y \left[\tfrac{1-0.9}{1-p_1}\right]^n \leq k$$

$$y \ln\left[\tfrac{0.9(1-p_1)}{p_1(1-0.9)}\right] \leq \ln k - n \ln\left[\tfrac{1-0.9}{1-p_1}\right]$$

Since $p_1 < 0.9$, we have $0.9(1-p_1) > (1-0.9)p_1$. Thus $\ln\left[\tfrac{0.9(1-p_1)}{p_1(1-0.9)}\right] > 0$. It follows that

$$y \leq n\{\frac{\ln k - n\ln[(1-0.9)/(1-p_1)]}{n\ln[0.9(1-p_1)/p_1(1-0.9)]}\} = nc$$

Thus the statement is true.

8.6-6

(a)

$$0.05 = P\left(\frac{\overline{X}-80}{3/4} \geq \frac{c_1-80}{3/4}\right)$$
$$= 1 - \Phi\left(\frac{c_1-80}{3/4}\right)$$

Thus

$$\frac{c_1-80}{3/4} = 1.645$$
$$c_1 = 81.234$$

Similarly,

$$\frac{c_2-80}{3/4} = -1.645$$
$$c_2 = 78.766$$
$$\frac{c_3}{3/4} = 1.96$$
$$c_3 = 1.47$$

(b)

$$K_1(\mu) = 1 - \Phi([81.234-\mu]/[3/4])$$
$$K_2(\mu) = \Phi([78.766-\mu]/[3/4]$$
$$K_3(\mu) = 1 - \Phi([81.47-\mu]/[3/4]) + \Phi([78.53-\mu]/[3/4])$$

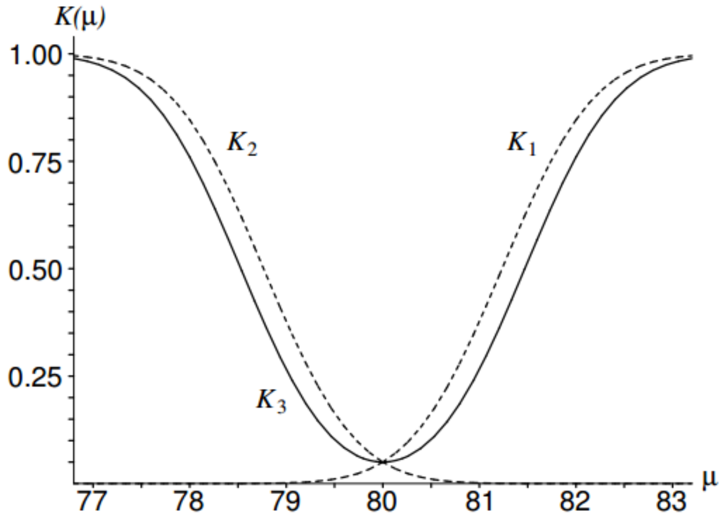

Figure 1: Three power functions

8.6-8 Let

$$L(\theta)=\theta^{n} \prod_{i=1}^{n}\left(1-x_{i}\right)^{\theta-1}$$

thus

$$L\left(\theta_{0}\right)=\theta_{0}^{n} \prod_{i=1}^{n}\left(1-x_{i}\right)^{\theta_{0}-1}=1$$

because $\theta_0 = 1$. So when $\theta > 1$, a best critical region is

$$\frac{L\left(\theta_{0}\right)}{L(\theta)}=\frac{1}{\theta^{n} \prod_{i=1}^{n}\left(1-x_{i}\right)^{\theta-1}} \leq k$$

That is,

$$\theta^{n}\left[\prod_{i=1}^{n}\left(1-x_{i}\right)\right]^{\theta-1} \geq \frac{1}{k}$$

Or

$$\prod_{i=1}^{n}\left(1-x_{i}\right) \geq\left(\frac{1}{k \theta^{n}}\right)^{1 /(\theta-1)}=c$$

This is true for all $\theta > 1$; so it is a uniformly most powerful test.
8.7-8 In $\Omega, \widehat{\mu}=\overline{x}$. Thus,

$$\lambda=\frac{\left(1 / \theta_{0}\right)^{n} \exp \left[-\sum_{1}^{n} x_{i} / \theta_{0}\right]}{(1 / \overline{x})^{n} \exp \left[-\sum_{1}^{n} x_{i} / \overline{x}\right]} \leq k$$
$$\left(\frac{\overline{x}}{\theta_{0}}\right)^{n} \exp \left[-n\left(\overline{x} / \theta_{0}-1\right)\right] \leq k$$

Plotting λ as a function of $\omega=\bar{x}/\theta_0$, we see that $\lambda = 0$ when $\bar{x}/\theta_0 = 0$, it has a maximum when $\bar{x}/\theta_0 = 1$, and it approaches 0 as $\bar{x}/\theta_0$ becomes large. Thus $\lambda \leq k$ when $\bar{x} \leq c_1$ or $\bar{x} \geq c_2$. Since the distribution of $\frac{2}{\theta_0}\sum_{i=1}^{n} X_i$ is $\chi^2(2n)$ when H_0 is true, we could let the critical region be such that we reject H_0 if

$$\frac{2}{\theta_{0}} \sum_{i=1}^{n} X_{i} \leq \chi_{1-\alpha / 2}^{2}(2 n) \quad \text { or } \quad \frac{2}{\theta_{0}} \sum_{i=1}^{n} X_{i} \geq \chi_{\alpha / 2}^{2}(2 n)$$

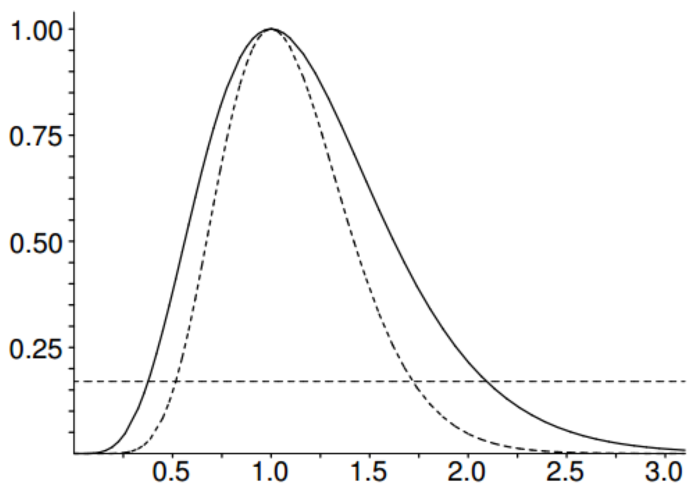

Figure 2: Likelihood functions: solid, $n = 5$; dotted, $n = 10$