7. (a) To find k, we use $\iint f(x,y)dxdy = 1$, thus there is

$$1 = \iint f(x,y)dxdy = \int_0^2 \int_x^3 kxy^2 dydx = \int_0^2 kx\frac{1}{3}(3^3 - x^3)dx$$
$$= \frac{9}{2}kx^2 \mid_0^2 - \frac{k}{15}x^5 \mid_0^2 = \frac{238}{15}k,$$

hence, $k = 15/238$.

(b) The joint CDF of X and Y is

$$F(x,y) = \int_0^x \int_u^y kuv^2 dvdu = \int_0^x \frac{k}{3}uv^3\Big|_u^y du = \int_0^x \frac{ku}{3}(y^3 - u^3)du$$
$$= \left(\frac{ku^2y^3}{6} - \frac{ku^5}{15}\right)\Big|_0^x = kx^2y^3/6 - kx^5/15.$$

8. (a) Let region $R = \{(x,y)|0 \le x \le y\} \cap \{(x,y)|x + y \le 3\}$, then

$$P(X + Y \le 3) = \iint_R f(x,y)dxdy = \int_0^{1.5} \int_x^{3-x} 2e^{-x-y}dydx = \int_0^{1.5} 2e^{-x}(e^{-x} - e^{x-3})dx$$
$$= \int_0^{1.5} 2e^{-2x}dx - 2e^{-3} \times 1.5 = 1 - e^{-3} - 3e^{-3} = 1 - 4e^{-3}.$$

(b) The marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_x^{\infty} 2e^{-x-y}dy = 2e^{-x}[-e^{-y}] \mid_x^{\infty} = 2e^{-2x} \quad \text{for} \quad x \ge 0,$$

and $f_X(x) = 0$ for $x < 0$.

The marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y 2e^{-x-y}dx = 2e^{-y}[-e^{-x}] \mid_0^y = 2e^{-y}[1-e^{-y}] \quad \text{for} \quad y \ge 0,$$

and $f_Y(y) = 0$ for $y < 0$.

1. (a) The marginal PMF of X is $p_X(0) = 0.06 + 0.04 + 0.2 = 0.3$, $p_X(1) = 0.08 + 0.3 + 0.06 = 0.44$, $p_X(2) = 0.1 + 0.14 + 0.02 = 0.26$.

   The marginal PMF of Y is $p_Y(0) = 0.06+0.08+0.1 = 0.24$, $p_Y(1) = 0.04+0.3+0.14 = 0.48$, $p_Y(2) = 0.2 + 0.06 + 0.02 = 0.28$. X and Y are not independent, for example $P(X = 0, Y = 0) = 0.06 \neq p_X(0)p_Y(0)$.

   (b) The conditional PMF $p_{Y|X=0}(y) = P(X = 0, Y = y)/p_X(0)$. That is, the first row of the table divides $p_X(0)$, thus $p_{Y|X=0}(0) = 0.06/0.3 = 0.2$, $p_{Y|X=0}(1) = 0.133$, and $p_{Y|X=0}(2) = 0.667$.

   By the same reason, we have

   $p_{Y|X=1}(0) = 0.1818$, $p_{Y|X=1}(1) = 0.6818$, and $p_{Y|X=1}(2) = 0.1364$.

   $p_{Y|X=2}(0) = 0.3846$, $p_{Y|X=2}(1) = 0.5385$, and $p_{Y|X=2}(2) = 0.0769$.

   The conditional PMF of Y depends on the value of X, thus X and Y are not independent.

(c) $E(Y|X = 1) = 0p_{Y|X=1}(0)+1p_{Y|X=1}(1)+2p_{Y|X=1}(2) = 0.9546$, and $E(Y^2|X = 1) = 0^2p_{Y|X=1}(0) + 1^2p_{Y|X=1}(1) + 2^2p_{Y|X=1}(2) = 1.2274$, Thus, $\text{Var}(Y|X = 1) = E(Y^2|X = 1) - E(Y|X = 1)^2 = 1.2274 - 0.9546^2 = 0.3161$.

10. (a) By the definition of Binomial distribution, it is clear that given $X = x$, $Y \sim \text{Bin}(x, 0.6)$. The joint PMF can be calculated using the formula

$$p_{X,Y}(x, y) = \binom{x}{y} 0.6^y 0.4^{x-y} p_X(x), \text{ with } 0 \leq y \leq x \leq 4.$$

According to this formula and the given marginal distribution of X, we have

$$p_{X,Y}(0, 0) = 0.1, \ p_{X,Y}(1, 0) = 0.08, \ p_{X,Y}(1, 1) = 0.12,$$

$$p_{X,Y}(2, 0) = 0.048, \ p_{X,Y}(2, 1) = 0.144, \ p_{X,Y}(2, 2) = 0.108$$

$$p_{X,Y}(3, 0) = 0.016, \ p_{X,Y}(3, 1) = 0.072, \ p_{X,Y}(3, 2) = 0.108,$$

$$p_{X,Y}(3, 3) = 0.054, \ p_{X,Y}(4, 0) = 0.00384, \ p_{X,Y}(4, 1) = 0.02304,$$

and

$$p_{X,Y}(4, 2) = 0.05184, \ p_{X,Y}(4, 3) = 0.05184, \ p_{X,Y}(4, 4) = 0.01944.$$

(b) Given $X = x$, $Y \sim \text{Bin}(x, 0.6)$, thus the regression function $E(Y|X = x) = 0.6x$.

(c) By the law of total expectation, $E(Y) = \sum_x E(Y|X = x)p_X(x) = 0.6 \sum_x xp_X(x) = 0.6 \times [0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.25 + 4 \times 0.15] = 1.29$

11. (a) The marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (x + y)dy = x + \frac{1}{2}, \quad \text{for} \quad 0 < x < 1.$$

Thus, the conditional PDF of Y given $X = x$ is

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+1/2}, \quad \text{for} \quad 0 < x < 1, 0 < y < 1.$$

Hence,

$$\begin{aligned} P(0.3 < Y < 0.5|X = x) &= \int_{0.3}^{0.5} f_{Y|X=x}(y)dy = \int_{0.3}^{0.5} \frac{x+y}{x+1/2}dy \\ &= \frac{0.2x + 0.08}{x+1/2}. \end{aligned}$$

(b) By (4.3.16),

$$\begin{aligned} P(0.3 < Y < 0.5) &= \int_0^1 P(0.3 < Y < 0.5|X = x)f_X(x)dx \\ &= \int_0^1 \frac{0.2x + 0.08}{x+1/2}(x+1/2)dx = 0.18. \end{aligned}$$

15. The conditional PDF of X given $Y = y$ is

$$f_{X|Y=y}(x) = \frac{1}{y}e^{-x/y} \quad \text{for } x > 0,$$

and $f_{X|Y=y}(x) = 0$ otherwise. Thus, $f_{X|Y=y}(x)$ depends on y. According to Proposition 4.3-2 (4), X and Y are not independent.

16. (a) The regression function is

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy = \int_0^{\infty} yxe^{-xy}dy = \frac{1}{x}\int_0^{\infty} ze^{-z}dz = \frac{1}{x}\Gamma(2) = \frac{1}{x},$$

where we used the variable transformation $z = xy$ and the definition of Gamma function.

Clearly, $E(Y|X = 5.1) = 1/5.1 = 0.1961$.

(b) By the law of total expectation,

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x)f_X(x)dx = \int_5^6 \frac{1}{x}\frac{1}{\log 6 - \log 5}\frac{1}{x}dx = \frac{1/5 - 1/6}{\log 6 - \log 5}.$$

1. The price the person pays is $P = \min\{X, Y\}$. So

$$\begin{aligned}E(P) &= \min(150,150)\times 0.25 + \min(150,135)\times 0.05 + \min(150,120)\times 0.05\\ &\quad + \min(135,150)\times 0.05 + \min(135,135)\times 0.2 + \min(135,120)\times 0.1\\ &\quad + \min(120,150)\times 0.05 + \min(120,135)\times 0.1 + \min(120,120)\times 0.15 = 132,\end{aligned}$$

and

$$\begin{aligned}E(P^2) &= \min(150,150)^2\times 0.25 + \min(150,135)^2\times 0.05 + \min(150,120)^2\times 0.05\\ &\quad + \min(135,150)^2\times 0.05 + \min(135,135)^2\times 0.2 + \min(135,120)^2\times 0.1\\ &\quad + \min(120,150)^2\times 0.05 + \min(120,135)^2\times 0.1 + \min(120,120)^2\times 0.15\\ &= 17572.5.\end{aligned}$$

Therefore $\mathrm{Var}(P) = E(P^2) - E(P)^2 = 17572.5 - 132^2 = 148.5$.

2. (a) Let X and Y be the times components A and B fail, respectively, so the system fails at time $T = \max\{X, Y\}$. Then, the CDF of T is $F_T(t) = 0$ if $t \notin [0,1]$; if $t \in [0,1]$,

$$\begin{aligned}F_T(t) &= P(T \le t) = P(\max\{X,Y\} \le T) = P(X \le T, Y \le T)\\ &= P(X \le t)P(Y \le t) = t^2,\end{aligned}$$

where we used the independence of X and Y, and since X and Y are uniform(0,1) random variables, $P(X \le t) = P(Y \le t) = t$.
Thus, the PDF of T $f_T(t) = 2t$ for $t \in [0,1]$ and $f_T(t) = 0$ otherwise.

(b) $E(T) = \int t f_T(t)dt = \int_0^1 2t^2 dt = 2/3$ and

$$E(T^2) = \int t^2 f_T(t)dt = \int_0^1 2t^3 dt = 1/2,$$

thus

$$\mathrm{Var}(T) = E(T^2) - E(T)^2 = 1/2 - (2/3)^2 = 1/18.$$

5. (a) Let X be the height of a randomly selected segment, then X is a uniform(35.5, 36.5) random variable. Thus $E(X) = (35.5 + 36.5)/2 = 36$, and $\text{Var}(X) = (35.5 - 36.5)^2/12 = 1/12$.

   (b) Let H_1 be the height of tower 1, then $H_1 = X_1 + \cdots + X_{30}$. Thus, $E(H_1) = E(X_1) + \cdots + E(X_{30}) = 36 \times 30 = 1080$, and $\text{Var}(H_1) = \text{Var}(X_1) + \cdots + \text{Var}(X_{30}) = 30/12 = 2.5$.

   (c) Let $Y_1, \cdots, Y_{30}$ be the heights of the segments used in tower 2, and let H_2 be the height of tower 2, then $H_2 = Y_1 + \cdots + Y_{30}$. As in part (b), we can find $E(H_2) = 1080$ and $\text{Var}(H_2) = 2.5$. Let D be the difference of the heights of the two towers, then $D = H_1 - H_2$. It makes sense to assume that the concrete segments are independent, thus H_1 and H_2 are independent. Then, $E(D) = E(H_1) - E(H_2) = 0$ and $\text{Var}(D) = \text{Var}(H_1 - H_2) = \text{Var}(H_1) + \text{Var}(H_2) = 5$.

8. The marginal PDF of X was derived in Example 4.3-9 as $f_X(x) = 12x(1-x)^2$ for $0 \le x \le 1$ and zero otherwise. Then,

$$E(X) = \int x f_X(x)dx = \int_0^1 12x^2(1-x)^2 dx = \frac{2}{5}.$$

By the symmetry of the joint PDF, the marginal PDF of Y is the same as that of X. It follows that $E(Y) = E(X) = 2/5$. We calculate

$$\begin{aligned} E(XY) &= \iint xyf(x,y)dxdy = \int_0^1 dx \int_0^{1-x} 24x^2y^2 dxdy = 8\int_0^1 x^2(1-x)^3 dx \\ &= \frac{2}{15}. \end{aligned}$$

Hence,

$$\text{Cov}(X,Y) = E(XY) - E(Y)E(X) = \frac{2}{15} - \frac{2}{5}\frac{2}{5} = -\frac{2}{75}.$$

9. $\text{Cov}(X,Y) = \text{Cov}(X, 9.3 + 1.5X + \epsilon) = 1.5\text{Cov}(X,X) + \text{Cov}(X,\epsilon) = 1.5\sigma_X^2 = 1.5 \times 9 = 13.5$, $\text{Cov}(\epsilon, Y) = \text{Cov}(\epsilon, 9.3 + 1.5X + \epsilon) = 1.5\text{Cov}(\epsilon, X) + \text{Cov}(\epsilon,\epsilon) = \sigma_\epsilon^2 = 16$.

12. The joint PMF is

		y	
	$P(x,y)$	1	2
	1	0.132	0.068
x	2	0.24	0.06
	3	0.33	0.17

Thus,

$$E(C) = E(2\sqrt{X} + 3Y^2) = \sum_{x=1}^{3}\sum_{y=1}^{2}(2\sqrt{x} + 3y^2)p(x,y) = (2 \times \sqrt{1} + 3 \times 1^2) \times 0.132$$

$$+ (2 \times \sqrt{1} + 3 \times 2^2) \times 0.068 + (2 \times \sqrt{2} + 3 \times 1^2) \times 0.24$$
$$+ (2 \times \sqrt{2} + 3 \times 2^2) \times 0.06 + (2 \times \sqrt{3} + 3 \times 1^2) \times 0.33$$
$$+ (2 \times \sqrt{3} + 3 \times 2^2) \times 0.17 = 8.662579,$$

and

$$E(C^2) = E[(2\sqrt{X} + 3Y^2)^2] = \sum_{x=1}^{3}\sum_{y=1}^{2}(2\sqrt{x} + 3y^2)^2 p(x,y) = (2 \times \sqrt{1} + 3 \times 1^2)^2 \times 0.132$$
$$+ (2 \times \sqrt{1} + 3 \times 2^2)^2 \times 0.068 + (2 \times \sqrt{2} + 3 \times 1^2)^2 \times 0.24$$
$$+ (2 \times \sqrt{2} + 3 \times 2^2)^2 \times 0.06 + (2 \times \sqrt{3} + 3 \times 1^2)^2 \times 0.33$$
$$+ (2 \times \sqrt{3} + 3 \times 2^2)^2 \times 0.17 = 92.41633.$$

Hence, $\mathrm{Var}(C) = E(C^2) - E(C)^2 = 92.41633 - 8.662579^2 = 17.37606$.

3. (a) Using the R commands

x = c(12.8, 12.9, 12.9, 13.6, 14.5, 14.6, 15.1, 17.5, 19.5, 20.8)
y = c(5.5, 6.2, 6.3, 7.0, 7.8, 8.3, 7.1, 10.0, 10.8, 11.0)
var(x); var(y); cov(x,y); cor(x,y),

we get $S_X^2 = 8.268$, $S_Y^2 = 3.907$, $S_{X,Y} = 5.46$, and $r_{X,Y} = 0.9607$.

(b) If the distances had been given in inches, S_X^2, S_Y^2, and $S_{X,Y}$ would be changed by a factor of 12^2, but $r_{X,Y}$ would be the same.

7. (a) From the marginal distributions, we calculate

$$E(X) = \int x f_X(x)dx = \int_0^{0.5} 24x^2(1-2x)dx = 3\int_0^1 t^2(1-t)dt = 3\frac{2!1!}{4!} = \frac{1}{4},$$
$$E(X^2) = \int x^2 f_X(x)dx = \int_0^{0.5} 24x^3(1-2x)dx = \frac{3}{2}\int_0^1 t^3(1-t)dt = \frac{3}{2}\frac{3!1!}{5!} = \frac{3}{40},$$
$$E(Y) = \int y f_Y(y)dy = \int_0^1 3y(1-y)^2dx = 3\frac{1!2!}{4!} = \frac{1}{4},$$

and

$$E(Y^2) = \int y^2 f_Y(y)dy = \int_0^1 3y^2(1-y)^2dx = 3\frac{2!2!}{5!} = \frac{1}{10}.$$

Thus, $\sigma_X^2 = E(X^2) - E(X)^2 = 3/40 - 1/16 = 1/80$, and $\sigma_Y^2 = E(Y^2) - E(Y)^2 = 1/10 - 1/16 = 3/80$. Further

$$E(XY) = \iint xyf(x,y)dxdy = \int_0^{0.5}\int_0^{1-2x} 24x^2ydydx = \int_0^{0.5} 12x^2(1-2x)^2dx$$
$$= \frac{3}{2}\int_0^1 t^2(1-t)^2dt = \frac{3}{2}\frac{2!2!}{5!} = \frac{1}{20},$$

therefore, $\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/20 - 1/4 \times 1/4 = -1/80$.

(b) The linear correlation coefficient is

$$\rho_{X,Y} = \frac{\sigma_{X,Y}}{\sigma_X\sigma_Y} = \frac{-1/80}{\sqrt{1/80}\sqrt{3/80}} = -\frac{\sqrt{3}}{3}.$$

(c) Given $X = x$, we have the conditional PDF of Y is $f_{Y|X=x}(y) = 0$ if $y \notin [0, 1-2x]$, otherwise

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{24x}{24x(1-2x)} = \frac{1}{1-2x}.$$

Therefore, given $X = x$, Y is uniformly distributed on $[0, 1-2x]$, hence the regression function of Y and X is

$$E(Y|X=x) = \frac{1}{2(1-2x)}.$$

The dependence between X and Y is not linear, thus it is not appropriate to use $\rho_{X,Y}$.

9. (a) It is seen that $f(x)$ is an even function on $[-1,1]$, thus $xf(x)$ and $x^3f(x)$ are odd functions on $[-1,1]$. Then $E(X) = \int_{-1}^1 xf(x)dx = 0$, by the same reason $E(X^3) = 0$. Hence,

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = E(X^3) - E(X)E(Y) = 0.$$

(b) When the value of X is given as x, the value of Y is known as x^2. Thus $E(Y|X=x) = x^2$, without any calculation.

(c) The dependence between X and Y is not linear, thus it is not appropriate to use $\rho_{X,Y}$.

2. The difference of the average maximum penetration between the two types is estimated as $0.49 - 0.36 = 0.13$ and the estimated standard error of $\bar{X}_1 - \bar{X}_2$ is calculated as

$$S_{\bar{X}_1-\bar{X}_2} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = \sqrt{\frac{0.19^2}{48} + \frac{0.16^2}{42}} = 0.0369.$$

5. The standard error is $S_{\hat{\theta}} = S_{2\bar{X}} = 2S_{\bar{X}} = 2\theta/\sqrt{12n}$. $\hat{\theta}$ is unbiased because $E(\hat{\theta}) = E(2\bar{X}) = 2E(\bar{X}) = 2 \times \theta/2 = \theta$.