6.8-1

(a) $\mathbf{Y} \sim Poisson(n\theta)$ which means $f_Y(y) = \frac{e^{-n\theta}(n\theta)^y}{y!}$

$$f(\theta|Y) \propto \theta^{\alpha+y-1}e^{-(n+\frac{1}{\beta})\theta}$$

Which is a $\Gamma(\alpha+y, \frac{1}{n+\frac{1}{\beta}})$ distribution.

(b) $w(y) = E(\theta|\mathbf{Y}=y] = \frac{\alpha+y}{n+\frac{1}{\beta}}$

(c)

$$\frac{\alpha+y}{n+\frac{1}{\beta}} = \frac{n}{n+\frac{1}{\beta}}\frac{y}{n} + \frac{\frac{1}{\beta}}{n+\frac{1}{\beta}}\alpha\beta$$

6.8-2

(a)

$$g(\tau|x_1, x_2, \cdots, x_m) \propto \tau^{\alpha n+\alpha_0-1}e^{-(\Sigma_{i=1}^n x_i+\frac{1}{\theta_0})\tau}$$

which is $\Gamma(n\alpha+\alpha_0, \frac{\theta_0}{1+\theta_0\Sigma_{i=1}^n x_i})$

(b)

$$E(\tau|x_1, x_2, \cdots, x_n) = (n\alpha+\alpha_0)\frac{\theta_0}{1+\theta_0\Sigma_{i=1}^n x_i} = \frac{\alpha_0\theta_0}{1+\theta_0 n\bar{\mathbf{X}}} + \frac{\alpha n\theta_0}{1+\theta_0 n\bar{\mathbf{X}}}$$

(c) The posterior distribution is $\Gamma(30+10, 1/[0.5+10\bar{x}]$. Select a and b so that $P(a < \tau < b) = 0.95 = \int_a^b \frac{(0.5+10\bar{x})^{40}}{\Gamma(40)}u^{40-1}e^{-u(0.5+10\bar{x})}du$. Making a change of variable $z = u(0.5+10\bar{x})$.

$$\int_a^b \frac{(0.5+10\bar{x})^{40}}{\Gamma(40)}u^{40-1}e^{-u(0.5+10\bar{x})}du = \int_{a(0.5+10\bar{x}}{)}^{b(0.5+10\bar{x})}) \frac{1}{\Gamma(40)}z^{39}e^{-z}dz$$

Let $v_{0.025}$ and $v_{0.975}$ be the quantiles for the $\Gamma(40, 1)$ distribution. Then

$$a = \frac{v_{0.025}}{0.5+10\bar{x}}$$

$$b = \frac{v_{0.975}}{0.5+10\bar{x}}$$

6.8-3

(a)

$$\begin{aligned} w(y) &= \frac{15+y}{50} \\ E[(\theta-w(Y))^2] &= E[(\theta-\frac{3}{10}-\frac{Y}{50})^2] \\ &= (\theta-\frac{3}{10})^2 - \frac{1}{25}(\theta-\frac{3}{10})E(Y) + \frac{1}{50^2}E(Y^2) \\ &= \frac{37}{250}\theta^2 - \frac{57}{250}\theta + \frac{9}{100} \end{aligned}$$

(b) In order to compare the bayesian loss function and $\frac{\theta(1-\theta)}{30}$, define

$$G(\theta) = \frac{37}{250}\theta^2 - \frac{57}{250}\theta + \frac{9}{100} - \frac{\theta(1-\theta)}{30} \approx 0.1813x^2 - 0.26133x + 0.09$$

when $\theta \in [0.569, 0.872]$ $G(\theta) < 0$.

6.8-4

$$(3\theta)^n(x_1x_2\cdots x_n)^2e^{-\theta\Sigma x_i^3}\theta^{4-1}e^{-16\theta} \propto \theta^{n+3}e^{-(16+\Sigma x_i^3)\theta}$$

which is $\Gamma(n+16, \frac{1}{16+\Sigma x_i^3})$. Thus

$$E(\theta|x_1, x_2, \cdots, x_n) = \frac{n+16}{\Sigma x_i^3 + 16}$$

6.8-5 According to Example 6.8-3, the posterior pdf of θ is a normal distribution. Also the penalty function $E|\theta - w(Y)|$ is minimized by $w(y) = median(\theta)$. Thirdly,for any normal distribution $\sim N(\mu, \sigma^2)$, it is symmetric around the point $x = \mu$, which is at the same time the mode, the median and the mean of the distribution. Thus

$$w(y) = (\frac{\sigma_0^2}{\sigma_0^2 + \sigma^2/n})y + (\frac{\sigma^2/n}{\sigma_0^2 + \sigma^2/n})\theta_0$$

6.8-6

(a)

$$\begin{aligned}
g(y|\theta) &= \frac{ny^{n-1}}{\theta^n}, \quad 0 < y < \theta \\
g(y|\theta)h(\theta) &\propto \frac{1}{\theta^{n+\beta+1}}, \quad max(\alpha, y) < \theta < \infty \\
k(\theta|y) &= \frac{c}{\theta^{n+\beta+1}}, \quad max(\alpha, y) < \theta < \infty \\
c &= [max(\alpha, y)]^{n+\beta} \quad by \int_{max(\alpha,y)}^{\infty} k(\theta|y) = 1 \\
E(\theta|y) &= (\frac{n+\beta}{n+\beta-1})max(\alpha, y) = w(y) \\
w(Y) &= (\frac{n+\beta}{n+\beta-1})max(\alpha, Y)
\end{aligned}$$

(b) With $n = 4, \quad \alpha = 1, \quad \beta = 2,$

$$k(\theta|y) = \frac{[max(1, y)]^6(6)}{\theta^7}, \quad max(1, y) < \theta.$$

Since $Loss = |\theta - w(Y)|$, $w(Y) = median$

$$\frac{1}{2} = \int_{max(1,y)}^{w(y)} \frac{[max(1, y)]^6(6)}{\theta^7}dy.$$

So,

$$w(Y) = 2^{1/6}max(1, Y)$$

6.8-7

$$\begin{aligned}
\frac{\sigma^2}{n}\frac{d\sigma^2}{d\sigma^2 + \sigma^2/n} &= \frac{2\sigma^2}{3n} \\
\frac{d\sigma^2}{d\sigma^2 + \sigma^2/n} &= \frac{2}{3} \\
d &= \frac{2}{n}
\end{aligned}$$

6.8-8

$$L = (\frac{1}{\sqrt{2\pi\sigma^2}})^n exp{-}\Sigma_{i=1}^n[y_i - \alpha - \beta(x_i - \bar{x})]^2/2\sigma^2$$

THe summation in the exponent can be written as follows:

$$\Sigma_{i=1}^n[y_i - \alpha - \beta(x_i - barx)]^2 = n(\hat{\alpha} - \alpha)^2 + (\hat{\beta} - \beta)^2\Sigma_{i=1}^n(x_i - \bar{x})^2 + \Sigma_{i=1}^n[y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})]^2$$

So the posterior of α is

$$
\begin{aligned}
k_1(\alpha|\mathbf{x},\mathbf{y}) &\propto (\frac{1}{\sqrt{2\pi\sigma_1^2}})exp[\frac{-(\alpha-\alpha_1)^2}{2\sigma_1^2} - \frac{n(\alpha-\hat{\alpha})^2}{2\sigma^2}] \\
&\propto exp[-\frac{(\alpha^2 - 2\alpha\alpha_1)}{2\sigma_1^2} - \frac{(n\alpha^2 - 2n\hat{\alpha}\alpha)}{2\sigma^2}] \\
&\propto exp[-(\frac{1}{2\sigma^2} + \frac{n}{2\sigma^2})\{\alpha - \frac{\alpha_1/(2\sigma_1^2) + n\hat{\alpha}/(2\sigma^2)}{1/(2\sigma_1^2 + n/(2\sigma^2)}\}^2]
\end{aligned}
$$

This is normal with posterior mean equal to

$$
\frac{\frac{\alpha_1}{\sigma_1^2} + \frac{n\hat{\alpha}}{\sigma^2}}{\frac{1}{\sigma_1^2} + \frac{n}{\sigma^2}}. \quad (1)
$$

$$
\begin{aligned}
k_2(\beta|\mathbf{x},\mathbf{y}) \; & exp[-\frac{(\beta-\beta_1)^2\Sigma_{i=1}^n(x_1\bar{x})^2}{2\sigma^2} - \frac{(\beta-\beta_0)^2}{2\sigma_0^2}] \\
&\propto exp[-(\frac{\Sigma_{i=1}^n(x_i-\bar{x})^2}{2\sigma^2} + \frac{1}{2\sigma_0^2})\beta^2 + 2(\frac{\hat{\beta}\Sigma_{i=1}^n(x_i-\bar{x})^2}{2\sigma^2} + \frac{\beta}{2\sigma_0^2})\beta] \\
&\propto exp[-(\frac{\Sigma_{i=1}^n(x_i-\bar{x})^2}{2\sigma^2} + \frac{1}{2\sigma_0^2})\{\beta - \frac{\hat{\beta}\Sigma(x_i-\bar{x})^2/\sigma^2 + \beta_0/\sigma_0^2}{\Sigma(x_i-\bar{x})^2/\sigma^2 + 1/\sigma_0^2}\}]
\end{aligned}
$$

This is normal with posterior mean

$$
\frac{\hat{\beta}\Sigma(x_i-\bar{x})^2/\sigma^2 + \beta_0/\sigma_0^2}{\Sigma(x_i-\bar{x})^2/\sigma^2 + 1/\sigma_0^2} \quad (2)
$$

Hence the posterior mean of $\alpha + \beta(x - \bar{x})$ is

$$
\frac{\frac{\alpha_1}{\sigma_1^2} + \frac{n\hat{\alpha}}{\sigma^2}}{\frac{1}{\sigma_1^2} + \frac{n}{\sigma^2}} + \frac{\hat{\beta}\Sigma(x_i-\bar{x})^2/\sigma^2 + \beta_0/\sigma_0^2}{\Sigma(x_i-\bar{x})^2/\sigma^2 + 1/\sigma_0^2}(x-\bar{x})
$$