1. For the exponential(λ) distribution, $\mu = 1/\lambda$. Letting $\bar{X} = 1/\lambda$, we can solve for the method of moment estimator for λ as $\hat{\lambda} = 1/\bar{X}$. It is not unbiased estimator because $E(1/\bar{X}) \neq 1/E(\bar{X})$.

3. For gamma(α, β) distribution, we have $\mu = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$. Thus, $\beta = \sigma^2/\mu$, and $\alpha = \mu^2/\sigma^2$. We get an estimator of $\hat{\alpha} = \bar{X}^2/S^2$ and $\hat{\beta} = S^2/\bar{X}$. For the given problem, $\hat{\alpha} = 113.5^2/1205.55 = 10.686$ and $\hat{\beta} = 1205.55/113.5 = 10.622$.

5. (a) Since $X \sim \text{Bin}(n, p)$, $E(X) = np$. Thus, we can estimate p by $\hat{p} = X/n$. It is unbiased because $E(\hat{p}) = E(X)/n = np/n = p$.

7. (a) There are $X+5$ helmets and the last one has flaw, among the rest $X+4$ helmets, there are 4 with flaw and X flawless, thus, we have the probability

$$P(X=x) = \binom{x+4}{4} p^5 (1-p)^x.$$

Therefore, the log-likelihood function is

$$\mathcal{L}(p) = \log \binom{X+4}{4} + 5\log p + X \log(1-p).$$

Setting the first derivative of the log-likelihood function to zero yields the equation

$$\frac{5}{p} - \frac{X}{1-p} = 0.$$

Solving this equation yields the MLE $\hat{p} = 5/(5+X)$.

(b) The distribution of X is easily identified as Negative binomial with $r = 5$ and parameter p (compare to formula (3.4.15)). Thus, $E(X) = r/p = 5/p$. In method of moment estimation, set $X = 5/p$, and we can solve for the estimator $\hat{p} = 5/X$.

(c) If $X = 47$, the MLE (a) gives $\hat{p} = 5/(5+47) = 0.096$ and the method of moment formula in (b) gives $\hat{p} = 5/47 = 0.106$.

9. (a) To get the moments estimator for θ, solve the equation $\hat{P} = E(P)$, that is $\hat{P} = \theta/(1+\theta)$, and we have the estimator

$$\hat{\theta} = \frac{\hat{P}}{1-\hat{P}}.$$

(b) For the given data, the estimate of θ is $\hat{\theta} = 0.202$.

1. (a) $\text{Bias}(\hat{\theta}_1) = E(\hat{\theta}_1) - \theta = 2E(\bar{X}) - \theta = 2E(X) - \theta = 2 \times \theta/2 - \theta = 0$. The bias for $\hat{\theta}_2$ is $\text{Bias}(\hat{\theta}_2) = E(\hat{\theta}_2) - \theta = n\theta/(n+1) - \theta = -\theta/(n+1)$. Thus, $\hat{\theta}_1$ is unbiased while $\hat{\theta}_2$ is biased.

(b) For $\hat{\theta}_1$, we have

$$\text{MSE}(\hat{\theta}_1) = \text{Var}(\hat{\theta}_1) + \text{Bias}(\hat{\theta}_1)^2 = \text{Var}(2\bar{X}) = 4\text{Var}(\bar{X}) = 4\frac{\sigma^2}{n} = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}.$$

For $\hat{\theta}_2$,

$$\begin{aligned}\text{MSE}(\hat{\theta}_2) = \text{Var}(\hat{\theta}_2) + \text{Bias}(\hat{\theta}_2)^2 &= \frac{n}{(n+1)^2(n+2)}\theta^2 + \left(-\frac{\theta}{n+1}\right)^2 \\ &= \frac{2\theta^2}{(n+1)(n+2)}.\end{aligned}$$

(c) When $n = 5$ and true value of θ is 10, we have $\text{MSE}(\hat{\theta}_1) = 10^2/(3 \times 5) = 6.67$, while $\text{MSE}(\hat{\theta}_2) = 2 \times 10^2/[((5+1)(5+2)] = 4.76$. According to the MSE selection criterion, $\hat{\theta}_2$ is preferable.

2. From the distributions of $X_1, \cdots, X_{10}$ and $Y_1, \cdots, Y_{10}$, we have $E(\bar{X}) = E(\bar{Y}) = \mu$, $\text{Var}(\bar{X}) = \sigma^2/10$, and $\text{Var}(\bar{Y}) = 4\sigma^2/10$. $\bar{X}$ and $\bar{Y}$ are also independent. Thus,

(a) For any $0 \le \alpha \le 1$, $E(\hat{\mu}) = E(\alpha\bar{X} + (1-\alpha)\bar{Y}) = \alpha E(\bar{X}) + (1-\alpha)E(\bar{Y}) = \alpha\mu + (1-\alpha)\mu = \mu$. Thus, $\hat{\mu}$ is unbiased for μ.

(b) Since $\hat{\mu}$ is unbiased,

$$\begin{aligned}\text{MSE}(\hat{\mu}) = \text{Var}(\hat{\mu}) = \text{Var}(\alpha\bar{X} + (1-\alpha)\bar{Y}) &= \alpha^2\text{Var}(\bar{X}) + (1-\alpha)^2\text{Var}(\bar{Y}) \\ &= \alpha^2\frac{\sigma^2}{10} + (1-\alpha)^2\frac{4\sigma^2}{10} = (5\alpha^2 - 8\alpha + 4)\frac{\sigma^2}{10}\end{aligned}$$

(c) The estimator $0.5\bar{X} + 0.5\bar{Y}$ corresponds to $\hat{\mu}$ with $\alpha = 0.5$. The MSE is

$$\text{MSE}(0.5\bar{X} + 0.5\bar{Y}) = (5 \times 0.5^2 - 8 \times 0.5 + 4)\frac{\sigma^2}{10} = 1.25\frac{\sigma^2}{10}.$$

Since $\text{MSE}(\bar{X}) = \text{Var}(\bar{X}) = \sigma^2/10 < \text{MSE}(0.5\bar{X} + 0.5\bar{Y})$, $\bar{X}$ is a preferable estimator.