ETC3420-5342 Tutorial Weeks 10 & 11 - Solutions

Part I: Building Blocks

Question 1: List, in words, the three technical properties which a copula function must satisfy to ensure that it correctly captures the properties expected of a joint distribution function.

Solution: Three technical properties a copula function must satisfy are:

- A copula function is 0 if any argument is 0.
- The marginals of a copula are uniform CDFs.
- A copula function assumes positive "density".

Question 2: An investor purchases three 5-year bonds from different companies within the same industry sector. The probability that an individual bond defaults within the first year is 10

- Using a Gumbel copula with parameter $\alpha = 2$, calculate the probability that all three bonds default within the first year.

  Solution: Let T_i be the time until default of bond i where $i = 1, 2, 3$. We want to calculate the joint probability

  $$\mathbb{P}(T_1 < 1, T_2 < 1, T_3 < 1) = C[u_1, u_2, u_3],$$

  where $u_i = \mathbb{P}[T_i < 1] = 0.1$ for $i = 1, 2, 3$. Using the Gumbel copula with parameter

$\alpha = 2$, we have

$$
\begin{aligned}
\mathbb{P}(T_1 < 1, T_2 < 1, T_3 < 1) &= C[u_1, u_2, u_3] \\
&= \exp\Bigg(- \Big((-\log(u_1))^2 + (-\log(u_2))^2 + (-\log(u_3))^2 \Big)^{\frac{1}{2}} \Bigg) \\
&= \exp\Big(- (3(-\log(0.1))^2)^{\frac{1}{2}} \Big) \\
&= 0.0185.
\end{aligned}
$$

- Discuss the suitability of the Gumbel copula in this situation. **Solution:** The Gumbel copula exhibits (non-zero) upper-tail dependence, the degree of which can be varied by adjusting the single parameter. However, it exhibits no lower tail dependence. Hence, the Gumbel copula is appropriate if we believe that the three investments are likely to behave similarly as the term approaches five years but not at early durations. This is unlikely to be the case though. If one bond defaults early on, then it may be indicative of problems in the industry sector or the economy and so the other investments may also be likely to default early on. If we believe the performance of investments issued by companies within the industry are much more closely associated (eg subject to the same systemic and operational risk factors), then a copula that exhibits both lower and upper tail dependence, such as the Students t copula, may be more appropriate.

Part II: Example Demonstrations

Question 3: For the Clayton copula:

- Determine whether the generator function

$$\Psi(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$$

  is valid.

  Solution:

  - $\Psi(0) = \lim_{t \to 0} \frac{1}{\alpha}(t^{-\alpha} - 1) = \frac{1}{\alpha} \lim_{t \to 0}(\frac{1}{t^{\alpha}} - 1) = \infty$
  - $\Psi(1) = \frac{1}{\alpha}(1^{-\alpha} - 1) = 0$
  - In the range $0 < t < 1$, t^{α} takes increasing values (starting at 1), so $\Psi(t)$ decreases over $0 < t < 1$.

- Determine the inverse generator function.
  Solution: By letting
  $$x = \Psi(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$$
  and solve for t, one obtains
  $$t = \Psi^{-1}(x) = (1 + \alpha x)^{-\frac{1}{\alpha}}$$
- Derive the Clayton copula function in the bivariate case.
  Solution:
  $$\begin{aligned} C[u, v] &= \Psi^{-1}\left(\frac{1}{\alpha}(u^{-\alpha} - 1) + \frac{1}{\alpha}(v^{-\alpha} - 1)\right) \\ &= \left(1 + \alpha\left(\frac{1}{\alpha}(u^{-\alpha} - 1) + \frac{1}{\alpha}(v^{-\alpha} - 1)\right)\right)^{-\frac{1}{\alpha}} \\ &= \left(1 + (u^{-\alpha} - 1) + (v^{-\alpha} - 1)\right)^{-\frac{1}{\alpha}} \\ &= \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-\frac{1}{\alpha}} \end{aligned}$$

Question 4: For the Frank copula:

- Determine whether the generator function
  $$\Psi(t) = -\log\left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\right)$$
  is valid.
  Solution:
  - $\Psi(0) = -\log\left(\frac{e^0 - 1}{e^{-\alpha} - 1}\right) = +\infty$
  - $\Psi(1) = -\log\left(\frac{e^{-\alpha} - 1}{e^{-\alpha} - 1}\right) = 0$

- In $0 < t < 1$, $e^{-\alpha t}$ is decreasing, $\frac{e^{-\alpha}-1}{e^{-\alpha}-1}$ is increasing, and therefore $\Psi(t)$ is decreasing over $0 < t < 1$.

- Determine the inverse generator function.
Solution: By letting

$$x = \Psi(t) = -\log\left(\frac{e^{-\alpha t}-1}{e^{-\alpha}-1}\right)$$

and solve for t, one obtains

$$t = \Psi^{-1}(x) = -\frac{1}{\alpha}\log\Big(1 + (e^{-\alpha}-1)e^{-x}\Big)$$

- Derive the Frank copula function.
Solution:

$$\begin{aligned}
C[u,v] &= \Psi^{-1}\left(\left(-\log\Big(\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}\Big)\right) + \left(-\log\Big(\frac{e^{-\alpha v}-1}{e^{-\alpha}-1}\Big)\right)\right)\\
&= \Psi^{-1}\left(-\log\left(\Big(\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}\Big)\Big(\frac{e^{-\alpha v}-1}{e^{-\alpha}-1}\Big)\right)\right)\\
&= -\frac{1}{\alpha}\log\left(1 + (e^{-\alpha}-1)\Big(\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}\Big)\Big(\frac{e^{-\alpha v}-1}{e^{-\alpha}-1}\Big)\right)\\
&= -\frac{1}{\alpha}\log\left(1 + \frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{e^{-\alpha}-1}\right)
\end{aligned}$$

Part III: Student Practice

Question 5: For Clayton copula:

(a) Derive the coefficient of lower and upper tail dependence in the case where the parameter $\alpha > 0$.
Solution: The coefficient of lower tail dependence is defined as:

$$\lambda_L = \lim_{u\to 0^+}\frac{C[u,u]}{u}.$$

Substituting in for the Clayton copula formula:

$$\lambda_L = \lim_{u \to 0^+} \frac{(u^{-\alpha} + u^{-\alpha} - 1)^{-\frac{1}{\alpha}}}{u}$$

$$= \lim_{u \to 0^+} \frac{(2u^{-\alpha} - 1)^{-\frac{1}{\alpha}}}{u}$$

$$= \lim_{u \to 0^+} (2u^{-\alpha} - 1)^{-\frac{1}{\alpha}} (u^{\alpha})^{-\frac{1}{\alpha}}$$

$$= \lim_{u \to 0^+} (2 - u^{\alpha})^{-\frac{1}{\alpha}}$$

$$= 2^{-\frac{1}{\alpha}}$$

The coefficient of upper tail dependence is defined as:

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C[u, u]}{1 - u}.$$

Substituting in for the Clayton copula formula:

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + (2u^{-\alpha} - 1)^{-\frac{1}{\alpha}}}{1 - u}$$

$$= \lim_{u \to 1^-} \frac{-2 - \frac{1}{\alpha}(2u^{-\alpha} - 1)^{-\frac{1}{\alpha} - 1}(-2\alpha u^{-\alpha - 1})}{-1}$$

$$= \lim_{u \to 1^-} (2 - 2(2u^{-\alpha} - 1)^{-\frac{1}{\alpha} - 1} u^{-\alpha - 1})$$

$$= 0.$$

(b) Comment on how the value of the parameter α affects the degree of lower tail dependence.

Solution: As α increases, $-\frac{1}{\alpha}$ increases and hence $2^{-\frac{1}{\alpha}}$ increases. So the higher the value of the parameter α, the higher the degree of lower tail dependence.

Question 6: Let X and Y be two random variables representing the future lifetimes of two 40-year old individuals. The two lives are married.

You are given that: $\mathbb{P}(X \leq 25) = 0.17831$ and $\mathbb{P}(Y \leq 25) = 0.11086$.

(a) Calculate the joint probability that both lives will die by the age of 65 using:

i. the Gumbel copula with $\alpha = 5$,

ii. the Clayton copula with $\alpha = 5$,

iii. the Frank copula with $\alpha = 5$.

Solution: The joint probability is given by

$$\mathbb{P}(X \leq 25, Y \leq 25) = C[\mathbb{P}(X \leq 25), \mathbb{P}(Y \leq 25)] = C[u, v]$$

where u, v are given.

i. For Gumbel copula:

$$\begin{aligned} C[u, v] &= \exp\left(-\left((-\log(u))^{\alpha} + (-\log(v))^{\alpha} \right)^{\frac{1}{\alpha}} \right) \\ &= \exp\left(-\left((-\log(0.17831))^{5} + (-\log(0.11086))^{5} \right)^{\frac{1}{5}} \right) \\ &= 0.0986 \end{aligned}$$

ii. For Clayton copula:

$$\begin{aligned} C[u, v] &= \left(u^{-\alpha} + v^{-\alpha} - 1 \right)^{-\frac{1}{\alpha}} \\ &= \left(0.17831^{-5} + 0.11086^{-5} - 1 \right)^{-\frac{1}{5}} \\ &= 0.1089 \end{aligned}$$

iii. For Frank copula:

$$C[u,v] = -\frac{1}{\alpha}\log\left(1 + \frac{\left(e^{-\alpha u}-1\right)\left(e^{-\alpha v}-1\right)}{\left(e^{-\alpha}-1\right)}\right)$$

$$= -\frac{1}{5}\log\left(1 + \frac{\left(e^{-5\times 0.17831}-1\right)\left(e^{-5\times 0.11086}-1\right)}{\left(e^{-5}-1\right)}\right)$$

$$= 0.0583$$

(b) Comment on the results as well as on which copula you think is most appropriate to use for modelling joint life expectancy.

Solution: The Clayton copula gives the highest probability of both lives dying within 25 years. This is because the Clayton copula exhibits lower tail dependence. This means that if one life does not survive for long (i.e., dies early), there is a high probability that the other life will not survive for long (i.e., will also die early).

The Gumbel copula gives the lowest probability of both lives dying within 25 years. This is because the Gumbel copula exhibits upper tail dependence. This means that if one life survives for a long time, there is a high probability that the other life will also survive for a long time.

Studies also suggest that if one member of a married couple dies, this can precipitate the death of the other member (broken heart syndrome). On this basis, we might choose to use a copula function where there is a degree of positive interdependence throughout, e.g., the co-monotonic (or minimum) copula.

Although we used the same parameter $\alpha = 5$ in each of the three copula functions, the effect of this parameter on the calculation will vary depending on the copula, and so the results are not directly comparable.

Question 7: You are considering whether there is a link between heights and weights, and have gathered some pairs of data:

(172cm, 68kg), (182cm, 70kg), (158cm, 75kg), (150cm, 60kg), (174cm, 65kg)

Calculate Spearmans rho for this dataset.

Solution: Ranked data:

Height	Rank	Weight	Rank	d_i	d_i^2
150_- cm	1	60kg	1	0	0
150_+ cm	2	75kg	5	-3	9
158cm	3	68kg	3	0	0
174cm	4	65kg	2	2	4
182cm	5	70kg	4	1	1

Spearman's rho:

$$s\rho = 1 - \frac{6}{T(T^2-1)}\sum_{i=1}^{T} d_i^2 = 1 - \frac{6}{5(5^2-1)} \times 14 = 0.3.$$