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A NOTE ON SWAN MODULES

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1. INTRODUCTION

Swan modules determine a canonical subgroup of the locally free classgroup of a group ring. In fact, this subgroup is contained in the kernel group, i.e. Swan modules are free over maximal orders. In this note we consider Swan modules over certain non-maximal orders. Motivation for studying such Swan modules is provided by Galois module structure problem in Taylor[10] where a certain generalised Swan module is shown to be the obstruction to the freeness of ring of integers over its associated order.

Let G be a group of order n. We set $\Sigma = \sum_{G} = \sum_{g \in G} g$, the sum of all group elements in the group ring ZG. For each integer s relatively prime to n, we define a Swan module

$$<s, \Sigma> = s \cdot ZG + \Sigma \cdot ZG$$

a two-sided locally free ZG-ideal. Swan modules were introduced in Swan[7]. We remark that some authors use a different definition of Swan module, for instance see Gruenburg and Linnell[4].

Each Swan module determines a class of $Cl\,(ZG)$, the locally free classgroup of ZG. It is well known that the set of all Swan classes form a finite subgroup $T\,(ZG)$ of $Cl\,(ZG)$, called the Swan subgroup of $Cl\,(ZG)$ (cf. Curtis and Reiner[2].) The locally free classgroup $Cl\,(ZG)$ is defined as the subgroup of elements of zero rank in $K_0\,(ZG)$, the Grothendieck group of finitely generated, locally free ZG-modules.

We now present the Galois module structure problem discussed in Taylor[10]. For a number field M, we write $\mathcal{O}_M$ for its ring of integers and for any ring R we write $(a, b)\,R$ for the right ideal $aR + bR$. Let K be a quadratic imaginary number field with discriminant less than -4. Moreover, assume that prime 2 splits in K/Q. Let $\mathscr{P} = \lambda\mathcal{O}_K$ denote a non-ramified, principal prime ideal of $\mathcal{O}_K$, where $\lambda \equiv \pm 1 \bmod 4\mathcal{O}_K$. We fix positive integers $r > m$ and let N (respectively, L) denote K ray classifield mod $4\mathscr{P}^{m+r}$ (respectively, $4\mathscr{P}^{r}$). Let $\Gamma = Gal\,(N/L)$ and $\mathscr{A} = \{x \in L\Gamma : \mathcal{O}_N \cdot x \subseteq \mathcal{O}_N\}$, the associated order of the extension N/L in $L\Gamma$.

For $s \in \mathbf{Z}$ with $(s, \lambda)\,\mathcal{O}_K = \mathcal{O}_K$, we define a locally free $\mathscr{H}$-ideal, $I_s = (s, \lambda^{-m} \sum_{\Gamma})\,\mathscr{H}$. We shall call I_s an elliptic Swan module (the use of 'elliptic' would become clear in Section 3). Taylor[10] showed that $\mathcal{O}_N$ is a free $\mathscr{H}$-module if, and only if the elliptic Swan module I_2 is $\mathscr{H}$-free.

In Srivastav[6] it is shown that $I_s = (s, \sum_{\Gamma})\,\mathscr{H}$ and therefore, it is obtained from the Swan module $(s, \sum_{\Gamma})\,\mathbf{Z}\Gamma$ by an extension of rings. Thus, if $\mathscr{P}$ splits in $K/\mathbf{Q}$ then Γ is cyclic so that $T(\mathbf{Z}\Gamma) = 0$ (cf. Swan[7]). In that case $(2, \sum_{\Gamma})\,\mathbf{Z}\Gamma$ is $\mathbf{Z}\Gamma$-free (since Γ is abelian, the Eichler condition is satisfied) and so I_2 is $\mathscr{H}$-free. Taylor[9] had shown that $\mathcal{O}_N$ is $\mathscr{H}$-free if $\mathscr{P}$ splits in $K/\mathbf{Q}$ without using Swan modules. So in the sequel we assume that $\mathscr{P}$ is inert in $K/\mathbf{Q}$ and set λ equal to an odd rational prime p. In this case Γ is a non-cyclic group of order p^{2m}.

Next, we note that the $\mathbf{Z}$-order $\mathbf{Z}\Gamma + \mathbf{Z}\,(p^{-m} \sum_{\Gamma})$ is contained in $\mathscr{H}$. More generally, we let G be any abstract group of order p^k. For each integer j, $0 \leqslant j < k$ we consider the $\mathbf{Z}$-order $\Lambda_j = \mathbf{Z}G + \mathbf{Z}\,(p^{-j}\,\Sigma)$. The Swan subgroup $T(\mathbf{Z}G)$ maps, by an extension of rings, onto a subgroup $T(\Lambda_j)$, determined by Swan modules $(s, p^{-j}\,\Sigma)\,\Lambda_j$ with $p \nmid s$, in the locally free classgroup $Cl\,(\Lambda_j)$. From Taylor's theorem[8] that $T(\mathbf{Z}G)$ is a cyclic group of order p^{k-1}, we are able to show that $T(\Lambda_j)$ is a cyclic group of order p^{k-1-j} (cf. Theorem 2).

As a corollary, set $G = \Gamma$ and $m = 1$ to show that I_2 is $\mathscr{H}$-free. This led the author to state.

Theorem 1—The elliptic Swan module I_2 is a principal ideal of the associated order $\mathscr{H}$.

We[6] used transcendental means to show that for $p \equiv \pm 1 \bmod 8$ the elliptic Swan module is, indeed, $\mathscr{H}$-free. The hypothesis $p \equiv \pm 1 \bmod 8$ was introduced in Srivastav[6] since in that case we could use the Lubin-Tate formal group law of the Fueter model in describing the local Galois module structure (cf. remark on page 173 of Cassou-Noguès and Taylor[1]). We shall employ the technical device of relative Lubin-Tate formal groups in removing this hypothesis in Section 3 to complete the proof of Theorem 1.

2. The Swan Subgroup

We keep the notation of Section 1. Let G be a group of order n. For each positive integer f that divides n, we define a $\mathbf{Z}$-order

$$\Lambda_{(f)} = \mathbf{Z}G + \mathbf{Z} \cdot f^{-1}\,\Sigma \qquad \ldots(2.1)$$

in the group algebra $\mathbf{Q}G$. We should note that $\Lambda_{(f)}$ is $\mathbf{Z}$-torsion free and finitely generated as a $\mathbf{Z}$-module.

Let us fix f. For each integer s relatively prime to n, we define a $\Lambda(f)$-ideal

$$<s, f^{-1}\Sigma>(f) = (s, f^{-1}\Sigma)\,\Lambda(f). \qquad \ldots(2.2)$$

We call this ideal a Swan module of $\Lambda(f)$ in the view of the following,

Lemma 1—With the above notation,

$$<s, f^{-1}\Sigma>(f) = (s, \Sigma)\,\Lambda(f).$$

PROOF : Clearly, $<s, f^{-1}\Sigma>(f) \supseteq (s, \Sigma)\,\Lambda(f)$. It suffices to show the equality locally at each prime q. If $q \mid s$, then $f \in \mathbf{Z}_q^x$ and we obtain the desired equality. On the other hand, if $q \nmid s$, then $s \in \mathbf{Z}_q^x$ and both ideals equal $\Lambda(f)_q$.

Thus $<s, f^{-1}\Sigma>(f)$ is a locally free $\Lambda(f)$-ideal obtained from the usual Swan module $<s, \Sigma>$ by extension of rings. It, therefore, determines a class, $<s, \Sigma>(f)$ in $Cl(\Lambda(f))$. We denote the set of all swan classes in $Cl(\Lambda(f))$ by $T(\Lambda(f))$. The inclusion

$$i : \mathbf{Z}G \hookrightarrow \Lambda(f)$$

induces a surjective homomorphism

$$i_* : Cl(\mathbf{Z}G) \to Cl(\Lambda(f))$$

such that $T(\Lambda(f)) = i_*(T(\mathbf{Z}G))$. Hence $T(\Lambda(f))$ is a subgroup of $Cl(\Lambda(f))$ and we shall call $T(\Lambda(f))$, the Swan subgroup of $Cl(\Lambda(f))$.

For convenience we shall write $\Lambda = \Lambda(1) = \mathbf{Z}G$. Let ϵ be the augmentation map of $\mathbf{Q}G$ and we denote its restriction on $\Lambda(f)$ by ϵ_f. Now, we consider a fiber diagram

$$\begin{array}{ccc} \Lambda(f) & \xrightarrow{\epsilon_f} & \mathbf{Z} \\ {\scriptstyle\theta_f}\downarrow & & \downarrow{\scriptstyle\phi_f} \\ \dfrac{\Lambda(f)}{(f^{-1}\Sigma)} & \xrightarrow{\bar{\epsilon}_f} & \dfrac{\mathbf{Z}}{f^{-1}n\mathbf{Z}} \end{array} \qquad \ldots(2.3)$$

where $\bar{\epsilon}_f$ is induced by ϵ_f and ϕ_f, θ_f are quotient maps.

The inclusion $i : \Lambda \hookrightarrow \Lambda(f)$ induces an isomorphism

$$i' : \frac{\Lambda}{(\Sigma)} \xrightarrow{\sim} \frac{\Lambda(f)}{(f^{-1}\Sigma)}. \qquad \ldots(2.4)$$

The fact that $\Lambda/(\Sigma)$ is a finitely generated free $\mathbf{Z}$-module shows that $\Lambda(f)/(f^{-1}\Sigma)$ is also a $\mathbf{Z}$-order. For $f = 1$, the fiber diagram (2.3) was considered by Ullom[11]. This fiber diagram allows us to study the relation between the K-theory of $\Lambda(f)$ and that of Z, $\Lambda(f)/(f^{-1}\Sigma)$ and $Z/f^{-1}nZ$. In particular, there is an exact Mayer-Vietoris sequence of Reiner and Ullom[5].

$$K_1(Z) \oplus K_1\left(\frac{\Lambda(f)}{(f^{-1}\Sigma)}\right) \to K_1\left(\frac{Z}{f^{-1}nZ}\right) \xrightarrow{\partial_f} D(\Lambda(f)) \to 0 \quad \ldots(2.5)$$

where $D(\Lambda(f))$ is the kernel group of $\Lambda(f)$. We recall that $D(\Lambda(f))$ is a subgroup of $Cl(\Lambda(f))$ and for any ring R, $K_1(R) = \dfrac{GL(R)}{GL'(R)}$, where the general linear group $GL(R) = \varinjlim GL_n(R)$ and $GL'(R) =$ the commutator subgroup of $GL(R)$. Moreover, in (2.5) $K_1(Z)$ (respectively, $K_1(\frac{Z}{f^{-1}nZ})$) may be identified with $Z^x = \{\pm 1\}$ (respectively, $(\frac{Z}{f^{-1}nZ})^x$) via the determinant map.

For $f = 1$, in (2.7) of Ullom[11] it is shown that $\partial_1(s \bmod nZ) = [s, \Sigma]$. In exactly the same manner we obtain the following :

Proposition 1—The connecting homomorphism ∂_f in (2.5) is given by

$$\partial_f(s \bmod f^{-1}nZ) = [s, \Sigma](f).$$

The exact sequence (2.5) may now be rewritten as

$$Z^x \oplus K_1\left(\frac{\Lambda(f)}{(f^{-1}\Sigma)}\right) \to \left(\frac{Z}{f^{-1}nZ}\right)^x \to T(\Lambda(f)) \to 0. \quad \ldots(2.6)$$

Next, we note the commutative diagram

$$\begin{array}{ccc} Z \oplus \dfrac{\Lambda}{(\Sigma)} & \xrightarrow{(\phi_1,\, -\bar{\epsilon}_1)} & \dfrac{Z}{nZ} \\ (id, i')\downarrow & & \downarrow \pi_f \\ Z \oplus \dfrac{\Lambda(f)}{(f^{-1}\Sigma)} & \xrightarrow{(\phi_f,\, -\bar{\epsilon}_f)} & \dfrac{Z}{f^{-1}nZ} \end{array} \quad \ldots(2.7)$$

where π_f is the quotient map.

From (2.7) using the functoriality of K_1 and the exactness of (2.6) together with Proposition 1 and Lemma 1 we obtain a commutative diagram with exact rows

$$\begin{array}{ccccc} Z^x \oplus K_1\left(\dfrac{\Lambda}{(\Sigma)}\right) & \longrightarrow & \left(\dfrac{Z}{nZ}\right)^x & \xrightarrow{\partial_1} & T(\Lambda) \to 0 \\ \left(id, i'_*\right)\downarrow & & \pi_f \downarrow & & i_* \downarrow \\ Z^x \oplus K_1\left(\dfrac{\Lambda(f)}{(f^{-1}\Sigma)}\right) & \longrightarrow & \left(\dfrac{Z}{f^{-1}nZ}\right)^x & \xrightarrow{\partial_f} & T(\Lambda(f)) \to 0. \end{array} \quad \ldots(2.8)$$

In particular, we have

$$\operatorname{Ker}(\partial_f) = \pi_f(\operatorname{Ker}(\partial_1)). \qquad \ldots(2.9)$$

Thus, if $T(\Lambda)$ is explicitly known then $T(\Lambda_{(f)})$ can also be calculated explicitly. For instance, if G is a cyclic group then we known that $T(\Lambda)$ is trivial so that $T(\Lambda_{(f)})$ is also trivial. As another example, we consider the case of p-groups. We remark that only in this section p may equal 2.

Let G be a non-cyclic p-group so that $n = p^k$. It is known that $T(\Lambda)$ is trivial for a dihedral 2-group and that $|T(\Lambda)| = 2$ for a generalized quaternion 2-group or a semidihedral 2-group (cf. Taylor[8]). Ullom conjectured that for a non-cyclic p-group which is not one of these types of 2-groups mentioned above

$$|T(ZG)| = \begin{cases} p^{k-1}, & p \text{ odd} \\ 2^{k-2}, & p = 2. \end{cases}$$

Taylor used Fröhlich's hom description for $Cl(\Lambda)$, and introduced a modified version of the p-adic logarithm to give a remarkable proof of Ullom's conjecture [cf. (2.5) of Taylor[8]]. Taylor's theorem can now be generalised to describe $T(\Lambda_{(f)})$.

For convenience we now write Λ_j for $\Lambda_{(p^j)}$ where $0 \leqslant j \leqslant k$. Using (2.8) and so (2.9) we deduce from Taylor's theorem [(2.5) of Taylor[8]].

Theorem 2—Let G be a non-cyclic p-group of order p^k. If $p = 2$, assume that G is not a generalised quaternion, dihedral or semidihedral group. Let $0 \leqslant j < k$ (if $p = 2$, $0 \leqslant j < k - 1$).

(i) If $p \neq 2$, then $T(\Lambda_j)$ is a cyclic group of order p^{k-j-1} with $[1+p, \Sigma]_{(p^j)}$ as a generator.

(ii) If $p = 2$, then $T(\Lambda_j)$ is a cyclic group of order 2^{k-j-2} with $[5, \Sigma]_{(2^j)}$ as a generator.

Now let us return to the Galois module structure problem of Section 1. We let p be an odd prime, inert in K/Q and set $G = \Gamma$, then $|G| = p^{2m}$. We also note that $\Lambda_m \subseteq \mathscr{H}$. In case $m = 1$, we conclude that $T(\Lambda_m) = 0$. Moreover, if $m = 2$ and p^2 is a Weiferich square, i.e. $2^{p-1} \equiv 1 \bmod p^2$, then $[2, \Sigma]_{(p^2)} = 0$. Thus we have,

Corollary 1—Let p be inert in K/Q. If either $m = 1$ or $m = 2$ and p^2 is a Weiferich squre then I_2 is a principal $\mathscr{H}$-ideal.

An example of Wieferich square is 1093^2.

3. Elliptic Swan Modules

As in Taylor[10] and Srivastav[6] we consider the lattice $\Omega = \mathcal{O}_K$ in $\mathbf{C}$. We fix a primitive 4-division point ψ of $\mathbf{C}/\Omega$ such that 2ψ has annihilator 2Ω. We set a complex number

$$t = \frac{12\,\mathscr{P}(2\psi)}{\mathscr{P}(\psi) - \mathscr{P}(2\psi)}$$

where $\mathscr{P}$ is the usual Weierstrass $\mathscr{P}$-function for Ω.

Let

$$\epsilon : y^2 = 4x^3 + tx^2 + 4x$$

be an elliptic curve with the identity of the group law at the origin $O = (0:0:1)$. We know[1] (Chapter XI) that $t^2 - 2^6$ is a unit in $K(4)$, the K ray classifield mod 4 $\mathcal{O}_K$. Moreover, the discriminant of $\mathscr{E}$ is $4(t^2 - 2^6)$. Thus $\mathscr{E}$ has good reduction at all odd primes. There is an isomorphism [cf. (4.10) Srivastav[6]] called the Fueter model

$$\xi : \mathbf{C}/\Omega \xrightarrow{\sim} \epsilon$$

given by

$$\xi(z) = \begin{cases} (T(z) : T_1(z) : 1), & z \neq 2\psi \\ (0 \quad : 1 \quad : 0), & z = 2\psi \end{cases}$$

where T and T_1 are two elliptic functions for Ω. We set

$$D(z) = \frac{T(z)}{T_1(z)}$$

an elliptic function for Ω.

Let $E = \dfrac{\mathcal{O}_K}{p^m\,\mathcal{O}_K}$, a finite ring. Then as in Taylor[10], the Galois group Γ and the group of p^m-division points of $\mathbf{C}/\Omega$ are both rank one free E-modules. We write both the E-actions exponentially as in Srivastav[6].

Let γ be an E-generator of Γ and α a primitive p^m-division point of $\mathbf{C}/\Omega$. In Srivastav[6] we defined the resolvent element ρ associated with α and γ by

$$\rho = p^{-m} \sum_{e \in E} \frac{D(\alpha^e + \psi)}{D(p^m\psi)}\, \gamma^{[e]}.$$

From (5.7) of Srivastav[6] we know that $p^m\,\rho \in \mathcal{O}_L\,\Gamma$.

Moreover, in (8.1) of Srivastav[6] we showed that if $p \equiv \pm 1 \bmod 8$, then $\rho \in \mathscr{H}$. We show

Theorem 3—The resolvent element ρ lies in the associated order $\mathscr{H}$.

From (3.6) and (5.17) of Srivastav[6] we obtain Theorem 1 as a consequence of

Theorem 3. In order to prove Theorem 3 we need to look at the formal group associated with the Fueter model in some detail.

We fix an embedding of $\bar{Q}$, a fixed algebraic closure of Q, in $\bar{Q}_p$, a fixed algebraic closure of Q_p, so that it corresponds to $\mathscr{P}$ for K. We write M' for the closure in $\bar{Q}_p$ of a field $M \subseteq \bar{Q}$. We note that $K(4)' = K'$ and $t \in K(4)$. We denote by P the maximal ideal of the ring of integers of $\bar{Q}_p$.

Let $\mathscr{E}'$ denote the elliptic curve $\mathscr{E}$ of the Fueter model (2.3) considered locally at P. This local elliptic curve $\mathscr{E}'$ admits complex multiplication and has good reduction modulo P. Let $\mathscr{E}_0'$ denote the kernel of reduction of $\mathscr{E}'$ modulo P. For convenience we write (x, y) for a point on $\mathscr{E}'$ with projective coordinates $(x : y : 1)$.

We know[1] (Chapter X) that there is a Lubin-Tate formal group law F' defined over $\mathcal{O}_{K'}$, for a uniformizer $p' \in \{\pm p\}$, where the parameter

$$t = \frac{2x}{y} \text{ on } F' \qquad \ldots(3.1)$$

is associated with the point (x, y) on $\mathscr{E}_0'$. Therefore, for a positive integer s and a primitive p^s-division point α_s of C/Ω, $(T(\alpha_s), T_1(\alpha_s)) \in \mathscr{E}_0'$ and the associated parameter $2D(\alpha_s)$ on F' is a primitive p^s-division point for F'.

Next, we note that $\dfrac{D(\alpha_s)}{D(\psi)} \in K(4\mathscr{P}^s)$ (cf. (5.6) of Srivastav[6]) and in addition, $\left[K'\left(\dfrac{D(\alpha_s)}{D(\psi)}\right) : K'\right] = [K(4\mathscr{P}^s)' : K']$. Thus we obtain

$$K(4\mathscr{P}^s)' = K'\left(\frac{D(\alpha_s)}{D(\psi)}\right). \qquad \ldots(3.2)$$

Now from (5.5) of Srivastav[6] and (6.8). Chapter IX of Cassou-Noguès and Taylor[1] we infer that $D(\psi) \in K(8)$ and $D^2(\psi) = (t+8)^{-1} \in K(4)$. This shows that $K'(D(\psi))/K'$ is an unramified extension of degree d, where $d/2$. Henceforth, we write K_n' for the unique unramified extension of K' of degree n so that

$$K_d' = K'(D(\psi)). \qquad \ldots(3.3)$$

In view of (3.2) we have

$$K_d'(D(\alpha_s)) = K(4\mathscr{P}^s)'(D(\psi)). \qquad \ldots(3.4)$$

From local classfield theory as in Taylor[10] there is a relative Lubin-Tate formal group law F'' on $\mathcal{O}_{K'}$, for a uniformizer p'' such that

$$K(4\mathscr{P}^s)' = K'(\omega_s) \quad \text{...(3.5)}$$

where ω_s is a primitive p^s-division point for F''. Combining (3.4) and (3.5) we obtain

$$K_d'(D(\alpha_s)) = K_d'(\omega_s). \quad \text{...(3.6)}$$

The finite ring E acts on p^m-division points of $\mathbf{C}/\Omega$ and also on p^m-division points of F''.

Proposition 2—Let α be a primitive p^m-division point of $\mathbf{C}/\Omega$. Then there exists

(i) a formal power series $\theta(X) \in \mathcal{O}_{K_d'}[[X]]$

and

(ii) a primitive p^m-division point ω of F'' such that

$$2D(\alpha^e) = \theta(\omega_{[e]}) \ \forall \ e \in E.$$

PROOF : We view both F' and F'' as relative Lubin-Tate formal group laws over $\mathcal{O}_{K_d'}$. From (1.2), Chapter I of de Shalit[3] we see that F' (respectively, F'') is a relative Lubin-Tate formal group law for $(p')^d$ (respectively, $(p'')^d$). For each positive integer s, let α_s (respectively, ω_s) be a primitive p^s-division point of $\mathbf{C}/\Omega$ (respectively, F'').

We know from (1.8), Chapter I of de Shalit[3] that $K'(D(\alpha_s))$ (respectively, $K_d'(\omega_s)$) is the classifield for K' to the subgroup $\langle(p')^d\rangle \cdot (1 + \mathscr{P}^s)$ (respectively, $\langle(p'')^d\rangle \cdot (1 + \mathscr{P}^s)$) of $(K')^*$. From (3.6) we deduce that

$$\langle(p)'^d\rangle \cdot (1 + \mathscr{P}^s) = \langle(p'')^d\rangle \cdot (1 + \mathscr{P}^s). \quad \text{...(3.7)}$$

Since (3.7) holds for each positive integer s we must have

$$(p')^d = (p'')^d. \quad \text{...(3.8)}$$

From (3.8) we conclude that F' and F'' are both relative Lubin-Tate formal group law for $(p')^d$. Hence by (1.5), Chapter I of de Shalit[3], F' and F'' are isomorphic formal group laws over $\mathcal{O}_{K_d'}$ and there exists a formal power series $\theta(X) \in \mathcal{O}_{K_d'}[[x]]$ such that

$$\theta((F''(X, Y)) = F'(\theta(X), \theta(Y)) \quad \text{...(3.9)}$$

and

$$\theta([a]_{F''}(X)) = [a]_{F'}(\theta(X)) \ \forall \ a \in \mathcal{O}_{K'}. \quad \text{...(3.10)}$$

In view of (3.10) there exists a primitive p^m-division point ω of F'' such that

$$2D(\alpha) = \theta(\omega). \quad \text{...(3.11)}$$

Moreover, applying (3.10) on (3.11) we obtain

$$2D(\alpha^e) = \theta(\omega^{[e]}) \ \forall \ e \in E$$

proving the proposition.

Remark : In the case that $d = 1$, from (3.8) we obtain that $p' = p''$ and then $\theta(X) = X$. Indeed, this is the case for $p \equiv \pm 1 \bmod 8$ as seen in (8.2) of Srivastav[6], which is

$$2D(\alpha^e) = \omega^{[e]} \ \forall \ e \in E. \qquad \ldots(3.12)$$

We also note that if $d = 2$ then $p' = -p''$.

Now we can prove Theorem 3.

Proof of Theorem 3—As in the proof of (8.1) of Srivastav[6] it suffices to show that $\rho \in \mathscr{H}_q$ whenever q is a prime of $\mathcal{O}_L$ such that $q \mid \mathscr{F}$. For such a prime q of $\mathcal{O}_L$ we fix the embedding of $\bar{\mathbf{Q}}$ in $\bar{\mathbf{Q}}_p$ so that it corresponds to q_N over N where q_N is the unique prime of $\mathcal{O}_N$ with $q = q_N \cap \mathcal{O}_L$.

We write $\mathscr{H}'$ for $\mathscr{H}_q$. From (3.3) and (3.6) of Srivastav[6] we note that

$$\mathscr{H}' = \mathcal{O}_{L'} \cdot 1_\Gamma + \sum_{i=0}^{p^{2m}-2} \mathcal{O}_{L'} \cdot \sigma_i \qquad \ldots(3.13)$$

where

$$\sigma_i = p^{-m} \cdot \sum_{e \in E} (\omega^{[e]})^i \, \gamma^{[e]} - 1_\Gamma) \in \mathscr{H}' \ \forall i \geqslant 0.$$

We set

$$L'' = L'(D(\psi)) \qquad \ldots(3.14)$$

and

$$\mathscr{H}'' = \mathcal{O}_{L''} \cdot 1_\Gamma + \sum_{i=0}^{p^{2m}-2} \mathcal{O}_{L''} \cdot \sigma_i \qquad \ldots(3.15)$$

an $\mathcal{O}_{L''}$-order in $L''\Gamma$. Since $\{1_\Gamma, \sigma_0, \sigma_1, \ldots, \sigma_{p^{2m}-2}\}$ forms an L'-basis of $L'\Gamma$ we deduce that

$$\mathscr{H}' = \mathscr{H}'' \cap L'\Gamma. \qquad \ldots(3.16)$$

Thus in order to prove Theorem 3 it suffices to show that $\rho \in \mathscr{H}''$ since $\rho \in L\Gamma$. We show that $\rho \in \mathscr{H}''$ by proceeding exactly as in the proof of (8.1) of Srivastav[6] and using Proposition 2 instead of (3.12).

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