



The extremal spheres theorem

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ABSTRACT

Consider a polygon P and all neighboring circles (circles going through three consecutive vertices of P). We say that a neighboring circle is extremal if it is empty (no vertices of P inside) or full (no vertices of P outside). It is well known that for any convex polygon there exist at least two empty and at least two full circles, i.e. at least four extremal circles. In 1990 Schatteman considered a generalization of this theorem for convex polytopes in d -dimensional Euclidean space. Namely, he claimed that there exist at least $2d$ extremal neighboring spheres for generic polytopes. His proof is based on the Bruggesser–Mani shelling method.

In this paper, we show that there are certain gaps in Schatteman's proof. We also show that using the Bruggesser–Mani–Schatteman method it is possible to prove that there are at least $d + 1$ extremal neighboring spheres. However, the existence problem of $2d$ extremal neighboring spheres is still open.

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1. Introduction

In 1990 Schatteman [16] published a four-vertex type theorem for polyhedrons. We must emphasize that this was a unique generalization of the four-vertex theorem for higher dimensions.

Consider a d -dimensional simplicial polytope P in d -dimensional Euclidean space. We call this polytope generic if it has no $d + 2$ cospherical vertices and is not a d -dimensional simplex. From now on, we consider only generic simplicial polytopes. Each $(d - 2)$ -dimensional face uniquely defines a neighboring sphere going through the vertices of two facets sharing this $(d - 2)$ -dimensional face. A neighboring sphere is called empty if it does not contain other vertices of P and it is called full if all other vertices of P are inside of it. We call an empty or full neighboring sphere extremal. Schatteman [16, Theorem 2, p. 232] proves the following claim.

Claim 1.1 (Schatteman, 1990). *For any convex d -dimensional polytope P there are at least d different $(d - 2)$ -dimensional faces defining empty neighboring spheres and at least d different $(d - 2)$ -dimensional faces defining full neighboring spheres.*

This result for $d = 2$ is well known (see Section 2). Actually, it immediately follows from Theorem 3.1 in Section 3 of our paper. Unfortunately, already for $d = 3$ the proof by Schatteman contains a crucial gap (see Section 6). At the moment we do not know if the claim is correct.

In Section 7, using Bruggesser–Mani–Schatteman's method, we show that there are at least two different $(d - 2)$ -dimensional faces defining empty neighboring spheres and at least two different $(d - 2)$ -dimensional faces defining full neighboring spheres. Moreover, we prove that there are at least $d + 1$ extremal (full and empty) neighboring spheres.

In his paper, Schatteman did not necessarily consider simplicial and generic polytopes. The lack of these conditions can only add some minor technical difficulties (for instance, the neighboring sphere is not well defined by a non-simplicial

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($d - 2$)-dimensional face and Delaunay triangulation is not determined without the generic assumption), which do not affect the main arguments. So in this paper we consider only simplicial and generic polytopes.

2. Four-vertex type theorems for polygons

We define an *oval* as a convex smooth closed plane curve. The classical four-vertex theorem by Mukhopadhyaya [11] published in 1909 says the following: *The curvature function on an oval has at least four local extrema (vertices)*. It is well known that any continuous function on a compact set has at least two (local) extrema: a maximum and a minimum. It turns out that the curvature function has at least four local extrema. The paper was noticed, and generalizations of the result appeared almost immediately. In 1912, Kneser [8] showed that convexity is not a necessary condition and proved the four-vertex theorem for a simple closed plane curve.

The famous book [4] by Blaschke (first published in 1916), together with other generalizations, contains a “relative” version of the four-vertex theorem. Here we preserve the formulation and notation from [4]. *Let C_1 and C_2 be two (positively oriented) convex closed curves, and let do_1 and do_2 be arc elements at points with parallel (and codirected) support lines. Then the ratio do_1/do_2 has at least four extrema*. In the case where C_2 is a circle, this theorem becomes the theorem on four vertices of an oval.

In 1932, Bose [5] published a remarkable version of the four-vertex theorem in a global sense. While in the classical four-vertex theorem the extrema are defined “locally”, here they are defined “globally”. Let G be an oval such that no four points lie on a circle. We denote by s_- and s_+ (resp., t_- and t_+) the number of its circles of curvature (resp., the circles which are tangent to G at exactly three points) lying inside ($-$) and outside ($+$) the oval G , respectively (the *curvature circle* of G at a point p is tangent to G at p and has radius $1/k_G(p)$, where $k_G(p)$ is the curvature of G at p). In this notation, we have the relation $s_- - t_- = s_+ - t_+ = 2$. If we define *vertices* as the points of tangency of the oval G with its circles of curvature lying entirely inside or outside G , then these formulas imply that the oval G has at least four vertices. It is worth mentioning that this fact was proved by Kneser [9] ten years before Bose.

Since then, publications related to the four-vertex theorem did not halt and their number considerably increased throughout recent years (see [2,3], etc.) to a large extent due to papers and talks by Arnold. In the above papers, various versions of the four-vertex theorem for plane curves and convex curves in \mathbb{R}^d , and their special points (vertices) are considered: critical points of the curvature function, flattening points, inflection points, zeros of higher derivatives, etc. The paper by Umehara [21] and the book [15] by Pak contain long lists of papers devoted to these topics. Several interesting results were obtained by Tabachnikov [19,20] also in this direction.

It is interesting to note that the first discrete analog of the four-vertex theorem arose almost 100 years before its smooth version. In 1813, the splendid paper by Cauchy on rigidity of convex polyhedra used the following lemma: *Let M_1 and M_2 be convex n -gons with sides and angles a_i, α_i and b_i, β_i , respectively. Assume that $a_i = b_i$ for all $i = 1, \dots, n$. Then either $\alpha_i = \beta_i$, or the quantities $\alpha_i - \beta_i$ change sign for $i = 1, \dots, n$ at least four times*.

In Aleksandrov's book [1], the proof of uniqueness of a convex polyhedron with given normals and areas of faces involves a lemma where the angles in the Cauchy lemma are replaced by the sides. We present a version of it, which is somewhat less general than the original one. *Let M_1 and M_2 be two convex polygons on the plane that have respectively parallel sides. Assume that no parallel translation puts one of them inside the other. Then when we pass along M_1 (as well as along M_2), the difference of the lengths of the corresponding parallel sides changes the sign at least four times*.

We easily see the resemblance between the above relative four-vertex theorem for ovals (apparently belonging to Blaschke) and the Cauchy and Aleksandrov lemmas. Furthermore, approximating ovals by polygons, we can easily prove the Blaschke theorem with the help of any of these lemmas.

The Cauchy and Aleksandrov lemmas easily imply four-vertex theorems for a polygon:

Corollary of the Cauchy lemma. *Let M be an equilateral convex polygon. Then at least two angles of M do not exceed the neighboring angles, and at least two angles of M are not less than the neighboring angles.*

Corollary of the Aleksandrov lemma. *Let all angles of a polygon M be pairwise equal. Then at least two sides of M do not exceed their neighboring sides, and at least two sides of M are not less than their neighboring sides.*

In applications, the *curvature radius* at a vertex of a polygon is usually calculated as follows. Consider a polygon M with vertices A_1, \dots, A_n . Each vertex A_i has two neighbors: A_{i-1} and A_{i+1} . We define the curvature radius of M at A_i as follows: $R_i(M)$ is equal to the circumradius of $\triangle A_{i-1}A_iA_{i+1}$.

Theorem 2.1 ([12]). *Assume that M is a convex polygon and that for each vertex A_i of M , the circumcenter of $\triangle A_{i-1}A_iA_{i+1}$ lies inside the angle $\angle A_{i-1}A_iA_{i+1}$. Then the theorem on four local extrema holds true for the (cyclic) sequence of the numbers $R_1(M), R_2(M), \dots, R_n(M)$, i.e., at least two of the numbers do not exceed the neighboring ones, and at least two of the numbers are not less than the neighboring ones.*

The generalization of this theorem for the case of non-convex polygons was given by Sedykh [17,18]. Furthermore, this theorem generalizes the four-vertex theorems following from the Cauchy and Aleksandrov lemmas.

A circle C passing through certain vertices of a polygon M is said to be *empty* (respectively, *full*) if all the remaining vertices of M lie outside (respectively, inside) C . The circle C is *extremal* if C is empty or full.

Theorem 2.2 (Folklore). Let $M = A_1 \dots A_n$ be a convex n -gon, $n > 3$, no four vertices of which lie on one circle. Then at least two of the n circles $C_i(M)$ (the circumcircle of $\triangle A_{i-1}A_iA_{i+1}$, $i = 1, \dots, n$, are empty and at least two of them are full, i.e., there are at least four extremal circles.

(S.E. Rukshin told one of the authors that this result for many years has been included in the list of problems for training for mathematical competitions and is well known to St. Petersburg school students attending mathematical circles.)

Theorem 2.2 was also generalized for the case of non-convex polygons by Sedykh [17,18].

It is easy to see a direct generalization of the Bose theorem for polygons.

We denote by s_- and s_+ the numbers of empty and full circles among the circles $C_i(M)$, and we denote by t_- and t_+ the numbers of empty and full circles passing through three pairwise non-neighboring vertices of M , respectively. Then, as before, we have $s_- - t_- = s_+ - t_+ = 2$.

This fact was suggested by Musin as a problem for the All-Russia mathematics competition of high-school students in 1998. It will be proved in the next section by means of elementary planar geometry methods.

One more generalization of the Bose theorem is given in [23], where one considers the case of an *equilateral* polygon, which is not necessarily convex.

Sedykh [17] proved a theorem on four support planes for weakly convex polygonal lines in \mathbb{R}^3 : *If any two neighboring vertices of a polygon M lie on an empty circle, then at least four of the circles $C_i(M)$ are extremal.* It is clear that convex and equilateral polygons satisfy this condition. Furthermore, Sedykh constructed examples of polygons showing that his theorem is wrong without this assumption (see [18]).

This short survey cannot be considered as complete. For more details see [10,13–15,21].

3. The Bose theorem for polygons

Here we provide an elementary proof for the two-dimensional case of the extremal spheres theorem. The proof of all statements from this section is the same for empty and full circles, so without loss of generality, we consider only empty circles.

To be consistent with [15] we call a circle through three vertices of a polygon *neighboring* if these three vertices are consecutive, *disjoint* if there are no adjacent vertices among these three and *intermediate* in all other cases.

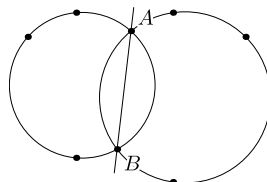
Theorem 3.1. Let $Q \in \mathbb{R}^2$ be generic convex polygon with n vertices and $n \geq 4$. Denote by s_+ , t_+ and u_+ the number of full circles that are neighboring, disjoint and intermediate, respectively. Similarly, denote by s_- , t_- and u_- the number of empty circles that are neighboring, disjoint and intermediate, respectively. Then

$$s_+ - t_+ = s_- - t_- = 2,$$

$$s_+ + t_+ + u_+ = s_- + t_- + u_- = n - 2.$$

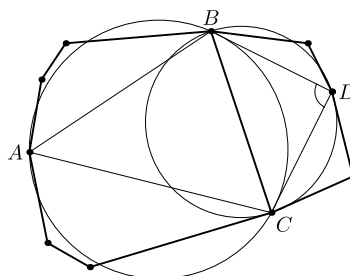
Proof. Let us prove that triangles with empty circumcircles form a triangulation of Q . The proof is based on two lemmas.

Lemma 3.1. Triangles with empty circumcircles do not intersect



Proof. If the respective circumcircles ω_1, ω_2 corresponding to each triangle do not intersect, then the statement of this lemma is obvious. So we assume they intersect in points A and B . All vertices of the first triangle lie on the arc of ω_1 outside of ω_2 and all vertices of the second triangle lie on the arc of ω_2 outside of ω_1 . But, these two arcs lie in different half-planes with respect to line AB . Thus the triangles do not intersect. \square

Lemma 3.2. If triangle ABC has an empty circumcircle and BC is a diagonal of Q , then there exists a triangle BCD with an empty circumcircle.



Proof. Consider all angles $\angle BFC$ such that F is a vertex of Q and is in a different half-plane than A with respect to BC . Suppose $\angle BDC$ is the maximal angle in this set (it is unique due to the generic assumption). Then obviously triangle BCD is the triangle required. \square

From these two lemmas it follows that triangles with empty circles form a triangulation.

The total number of triangles in a triangulation of a polygon with n vertices is always $n - 2$, so the second part of the theorem is proved.

Each triangle corresponding to a neighboring circle has two edges of Q as its own edges. Each triangle corresponding to an intermediate circle has one edge of Q as its own edge. Each triangle corresponding to a disjoint circle has no edges of Q as its own edges. Hence, $2s_- + u_- = n$. We subtract $s_- + t_- + u_- = n - 2$ from this equality and obtain that $s_- - t_- = 2$. \square

From this theorem we can obtain the theorem on extremal circles. We have that $s_- - t_- = 2$ and therefore $s_- \geq 2$. Analogously, $s_+ \geq 2$ proving that each generic convex polygon possesses at least two full and at least two empty neighboring circles.

The proof of this theorem has a very reasonable explanation in terms of Delaunay triangulations which we will consider in the next section.

4. Constructing Delaunay triangulations by lifting to a spherical paraboloid

Suppose S is a set of points in \mathbb{R}^d , the affine dimension of S is d and there are no $d + 2$ cospherical points in S . We define a Delaunay (upper Delaunay) simplex as a simplex, whose circumsphere contains no (all) points of S . A Delaunay (upper Delaunay) triangulation of S is a triangulation of S consisting of all Delaunay (upper Delaunay) simplices [7]. In this section, we show how these triangulations can be constructed (and simultaneously prove that they exist) for the set of vertices of a d -dimensional convex generic simplicial polytope.

Consider the set S of all vertices of the d -dimensional polytope P . Let $DT(S)$ denote a Delaunay triangulation of the set S and $UDT(S)$ be an upper Delaunay triangulation of S .

Here we use a method of constructing $DT(S)$ and $UDT(S)$ by lifting all the points of S to a spherical paraboloid. The idea of this construction belongs to Voronoi [22]. We define a function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ such that for $X(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, its image $f(X) = (x_1, x_2, \dots, x_d, x_1^2 + x_2^2 + \dots + x_d^2)$. Then, for all points $X_1, X_2, \dots, X_n \in S$, we consider the set S' of $f(X_1), f(X_2), \dots, f(X_n)$. Because of the generic assumption on P there are no $d + 2$ points of S' on the same d -dimensional hyperplane. Let $CH(S')$ be the convex hull of the set S' . Obviously, $CH(S')$ is a simplicial polytope and each $f(X_i)$ is a vertex of this polytope. For each facet F_j of $CH(S')$, consider the exterior normal of this facet n_j . We then divide all facets into two groups subject to the following rule: if the last coordinate of n_j is negative, we place F_j in the first group and if it is positive, we place F_j in the second group (since P is simplicial, it can never be equal to 0).

Lemma 4.1. *Projections of all facets of the first group to \mathbb{R}^d give $DT(S)$. Projections of all facets of the second group to \mathbb{R}^d give $UDT(S)$.*

Proof. Consider a facet F from the first group. Suppose the equation of the d -dimensional hyperplane containing F is $x_{d+1} = a_1x_1 + \dots + a_dx_d$. Consider the intersection of this hyperplane with the paraboloid $x_{d+1} = x_1^2 + \dots + x_d^2$. Thus, $a_1x_1 + \dots + a_dx_d = x_1^2 + \dots + x_d^2$ and $(x_1 - \frac{a_1}{2})^2 + \dots + (x_d - \frac{a_d}{2})^2 = \frac{a_1^2}{4} + \dots + \frac{a_d^2}{4}$. Hence the projection of the intersection of this hyperplane with the paraboloid is a sphere. Now we notice that F is from the first group so for all vertices of S' on the paraboloid we have that $x_{d+1} \geq a_1x_1 + \dots + a_dx_d$. So, for their projections $(x_1 - \frac{a_1}{2})^2 + \dots + (x_d - \frac{a_d}{2})^2 \geq \frac{a_1^2}{4} + \dots + \frac{a_d^2}{4}$. Thus all the vertices are outside of the circumsphere of the projection of F , so the projection of F is a Delaunay simplex.

The proof is exactly the same for $UDT(S)$. \square

Notice that because of convexity $DT(S)$ and $UDT(S)$ are also triangulations of P . So further we will call them Delaunay triangulation of P and upper Delaunay triangulation of P .

5. Different types of “ears” and connections between them

Definition 5.1. Consider a convex generic simplicial polytope P and its Delaunay triangulation $DT(P)$ (upper Delaunay triangulation $UDT(P)$). We say that a simplex $S \in DT(P)$ ($S \in UDT(P)$) is a D-ear (UD-ear) if at least two facets of S are on the boundary of P .

There is a direct connection between ears and extremal spheres: the sphere circumscribed around any ear is extremal and the simplex on the vertices of P inscribed in the extremal sphere is an ear.

In his paper, Schatteman uses the concept of shellability. Let us give a formal definition of a polytopal complex and its shelling (see also [24]).

A *polytopal complex* C in \mathbb{R}^d is a collection of polytopes in \mathbb{R}^d such that (1) the empty set is in C , (2) for any polytope $T \in C$ every face of T is also in C , (3) the intersection of any two polytopes in C is a face of both.

The dimension of C is the largest dimension of a polytope in C . Inclusion-maximal faces of a complex are called facets. If all facets are of the same dimension then we call C pure.

Every 0-dimensional complex is called shellable, its shelling is any ordering of facets. Let C be a pure d -dimensional polytopal complex. C is called *shellable* if there exists an ordering (shelling) of its facets (F_1, F_2, \dots, F_m) such that $\forall s : 2 \leq s \leq m; (\bigcup_{t=1}^{s-1} F_t) \cap F_s$ is a beginning of a shelling of ∂F_s . If complex C is a d -cell then each partial union $\bigcup_{t=1}^s F_t$, $1 \leq s \leq m$, of its shelling is homeomorphic to a d -dimensional disk (if it is a d -sphere, then $\bigcup_{t=1}^s F_t$ is homeomorphic to a disk for all $s < m$ and to a sphere for $s = m$).

We say that facet F of a polytope P is *visible* from point A if A and P are in different open half-spaces with respect to the hyperplane through F . In the classical paper by Bruggesser and Mani [6] it was proved that the complex of facets of some polytope visible from some point is shellable, and the method of constructing a shelling is given. Connect this point with some internal point of the polytope and the order by which hyperplanes determined by the facets of the polytope intersect this segment (starting from interior of the polytope) is the order of corresponding simplices in a shelling.

This method can be applied to the simplicial complex of $DT(P)$. We use the method of lifting to a spherical paraboloid from the previous section. Suppose we obtain a $(d+1)$ -dimensional polytope P' from this procedure. Then we know that the lower facets (facets for which the last coordinates of normals are negative) correspond to simplices in $DT(P)$. In fact, the simplicial complex of $DT(P)$ and the complex of lower facets are isomorphic. Obviously, there exists a sufficiently low point in \mathbb{R}^{d+1} such that the set of facets visible from it is exactly the set of lower facets. Thus the following lemma is true:

Lemma 5.1 ([16], Lemma 1, p. 237). *The Delaunay (upper Delaunay) simplicial complex is shellable.*

In his paper, Schatteman proves the following lemma:

Lemma 5.2 ([16], Lemma 2, p. 237). *The last simplex in the Delaunay (upper Delaunay) shelling is a D-ear (UD-ear).*

Proof. Consider the last simplex T_m from the Delaunay shelling. $(\bigcup_{t=1}^{m-1} T_t), (\bigcup_{t=1}^m T_t) = P$ and T_m are homeomorphic to a d -dimensional disk. Hence $\partial T_m \cap \partial P$ must be homeomorphic to a $(d-1)$ -dimensional disk. But this common bound contains all vertices of T_m . Thus it contains at least two facets of T_m , so T_m is an ear.

The proof for the case of upper Delaunay is the same. \square

Definition 5.2. The last simplex of some shelling of $DT(P)$ ($UDT(P)$) obtained by the method of Bruggesser and Mani for P' is called a BMD-ear or Delaunay BM-ear (BMUD-ear or upper Delaunay BM-ear).

6. The status of Schatteman's theorem

Here we briefly observe Schatteman's proof of his theorem and show that there are gaps that cannot be filled by his ideas.

In order to prove the theorem, Schatteman considered two cases: (1) each Delaunay simplex is a D-ear, (2) there is a Delaunay simplex which is not a D-ear.

The proof for the second case was based on the following claim:

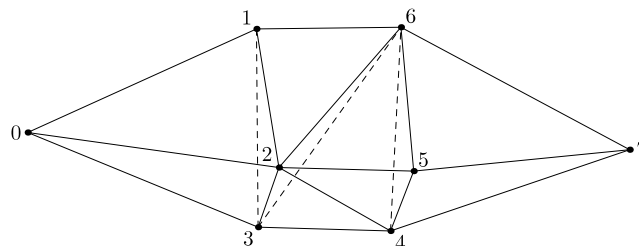
Claim 6.1. *If there exists a Delaunay simplex which is not a BMD-ear then there are at least d BMD-ears.*

(Although this claim was not formulated in his paper, Schatteman tries to prove it and uses it in the proof of his main theorem.)

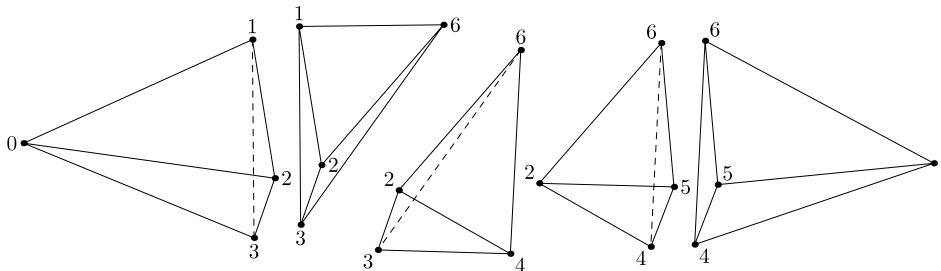
We show that the proof of this claim is wrong. Moreover, we give a counterexample in dimension three. We think that this counterexample can be generalized for higher dimensions and consequently the claim is completely false.

His proof was the following. He lifted vertices of P to a paraboloid and constructed the convex hull P' of the set of lifted vertices. Then, for all lower facets of P' , he constructed a polyhedron E defined by the intersection of open half-spaces determined by the facets and not containing P' . He considered the vertex of E and tried to find a shelling by the Bruggesser–Mani method using a line connecting this vertex and the interior of P' . The problem here is that the Bruggesser–Mani method allows us to find a shelling only for the set of facets visible from a given point. From the chosen point in Schatteman's proof, there can be some upper facets that are visible. Hence his method cannot give us a shelling of the complex of lower facets which correspond to Delaunay cells.

Actually there are even configurations that contradict this claim. Here is an example of the three-dimensional polytope that has only two BMD-ears.



As it can be seen from the Delaunay triangulation of this polytope, it has five D-ears. However only two of them are BMD-ears — $(0, 1, 2, 3)$ and $(4, 5, 6, 7)$.



Thus, the existence problem of $2d$ extremal neighboring spheres is still open.

7. Theorems on the number of ears

In this section we prove results pertaining to the number of ears that can be obtained using Schatteman's method.

Theorem 7.1. *For each generic convex simplicial d -dimensional polytope there exist at least two Delaunay and at least two upper Delaunay BM-ears.*

Proof. We consider a $(d+1)$ -dimensional polytope P' that is the result of lifting P to a spherical paraboloid. As in the previous sections, the lower facets of P' give $DT(P)$ and the upper facets give $UDT(P)$. We define the unbounded polyhedron P_+ as the intersection of all upper half-spaces with all facets of P' . Analogously, we define P_- as the intersection of all lower half-spaces with all facets of P' . Obviously the facet of P_- defined by the lower facet of P' is the last in some Bruggesser–Mani shelling and corresponds to a BMD-ear of P . Similarly, the facet of P_+ defined by the upper facet of P' is the last in some Bruggesser–Mani shelling and it corresponds to a BMUD-ear of P . Let us now show that there exist at least two facets of P_- defined by lower facets of P' (for P_+ the proof is the same). We take any point inside P' and the vertical ray from this point to $x_{d+1} = -\infty$. The last hyperplane intersecting this ray is a facet of P_- and it cannot be a hyperplane defined by some upper facet of P' . Thus we already have one BMD-ear. Now let F be the facet of P' defining the facet of P_- observed earlier. Let us take a point inside P' such that the vertical ray from this point $x_{d+1} = -\infty$ intersects only F out of all facets of P' . Then the last hyperplane intersecting this ray is not a hyperplane defined by F and it gives us one more BMD-ear. \square

Theorem 7.2. *For each convex generic simplicial d -dimensional polytope P there exist at least $d + 1$ BM-ears.*

Proof. As in the proof of the previous theorem, we are interested in facets of P_+ defined by upper facets of P' and facets of P_- defined by lower facets of P' .

For each facet of P' consider a hyperplane parallel to this facet and passing through the origin. Consider the unbounded polyhedron T_+ given by the intersection of all upper half-spaces defined by all such hyperplanes.

Lemma 7.1. *For each facet of T_+ there is an unbounded facet of P_+ parallel to it.*

Proof. Suppose f is a facet of T_+ . There is a ray r in f which is strictly in the upper half-space for all other hyperplanes. Hyperplane f corresponds to one or two facets of P' (there cannot be more than two parallel facets for one polytope). Consider the highest of these facets and the ray $r' = r$ (equality is given with respect to parallel translations) lying on it. It is obvious that, for a hyperplane defined by some facet h of P' , the unbounded part of r' (subray r'_h of r') lies in the upper half-space, i.e. only a segment of this ray can be in a lower half-space. So, $\hat{r} = \bigcap_h r'_h$ is a ray which is in our hyperplane and is in an upper half-space for all hyperplanes defined by facets of P' . This means that f defines an unbounded facet of P_+ . \square

We can define T_- as the intersection of lower half-spaces. The same lemma connecting facets of T_- and P_- is true by the same arguments. T_+ and T_- are symmetric with respect to the origin and have at least $d + 1$ facets. By this lemma and its analogue for T_- , for each of these facets there is an unbounded facet of P_+ and P_- parallel to it. Thus, if a facet of P' defining a facet of T_+ were to be a lower face, then we have a BM-ear from P_- , if it were to be an upper face, then we have a BM-ear from P_+ , and if there were to be two parallel facets defining this facet of T_+ , then we have BM-ears from both P_- and P_+ . So for each facet of T_+ we have at least one corresponding BM-ear. It means that there exist at least $d + 1$ BM-ears. \square

8. Remarks

Here we want to make a remark on non-convex polytopes. The algorithm of Section 4 allows us to construct Delaunay triangulations of sets of points. These triangulations are always subdivisions of the convex hull of the points into simplices. Hence using the ideas of Section 5 (and further Section 7) we can obtain only extremal spheres of the convex hulls but not of the actual polytopes in the case where these polytopes are not convex. For a non-convex polytope there is no direct connection between its subsimplices and facets of P' constructed by means of lifting to a paraboloid, since facets of P' correspond to subsimplices of the convex hull. Maybe for some special cases (for instance there exists a triangulation of

the polytope which is a subset of the Delaunay triangulation of the set of points) partial results can be achieved but in the general case the proofs in Sections 5 and 7 do not work.

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References

- [1] A.D. Aleksandrov, *Convex Polyhedra*, GITTL, Moscow, 1950.
- [2] V.I. Arnold, Topological invariants of plane curves and caustics, in: Rutgers Univ. Lect. Series, vol. 5, Amer. Math. Soc, Providence, 1994.
- [3] V.I. Arnold, Geometry of spherical curves and the quaternionic algebra, *Uspekhi. Mat. Nauk* 50 (1) (1995) 3–68.
- [4] W. Blaschke, *Kreis und Kugel*, Nauka, Moscow, 1967.
- [5] R.C. Bose, On the number of circles of curvature perfectly enclosing or perfectly enclosed by a closed oval, *Math. Ann.* 35 (1932) 16–24.
- [6] H. Bruggesser, P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.* 29 (1971) 197–205.
- [7] B.N. Delaunay, Sur la sphere vide. A la memorie de Georges Voronoi, On a simple ball, *Izv. Akad. Nauk SSSR, Otd. Mat. Est. Nauk* 6 (1934) 793–800.
- [8] A. Kneser, Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie, *Festschrift H. Weber*, 1912, pp. 170–180.
- [9] H. Kneser, Neuer Beweis des Vierscheitelsatzes, *Christiaan Huygens* 2 (1922/23) 315–318.
- [10] W.J. Mogilski, The four-vertex theorem, The evolute, and the decomposition of polygons, [arXiv:0906.2388](https://arxiv.org/abs/0906.2388).
- [11] S. Mukhopadhyaya, New methods in the geometry of a plane arc — I, Cyclic and sextactic points, *Bull. Calcutta Math. Soc.* 1 (1909) 31–37.
- [12] O.R. Musin, A four-vertex theorem for a polygon, *Kvant* (2) (1997) 11–13.
- [13] O.R. Musin, Chebyshev systems and zeros of a function on a convex curve, *Proc. Steklov Inst. Math.* 221 (1998) 236–246.
- [14] O.R. Musin, Curvature extrema and four-vertex theorems for polygons and polyhedra, *J. Math. Sci.* 119 (2) (2004) 268–277.
- [15] I. Pak, Lectures on discrete and polyhedral geometry, <http://www.math.umn.edu/pak/book.htm>.
- [16] A. Schatteman, A four-vertex theorem for polygons and its generalization to polytopes, *Geom. Dedicata* 34 (1990) 229–242.
- [17] V.D. Sedykh, Theorem on four support vertices of a polygonal line, *Funktsional Anal. Prilozhen* 30 (3) (1996) 88–90.
- [18] V.D. Sedykh, Discrete versions of the four-vertex theorem, in: *Topics in Singularity Theory*, AMS, 1997, pp. 197–207.
- [19] S. Tabachnikov, Around four vertices, *Russian Math. Surveys* 45 (1) (1990) 229–230.
- [20] S. Tabachnikov, The four-vertex theorem revisited — two variations on the old theme, *AMM* 102 (10) (1995) 912–916.
- [21] M. Umehara, A unified approach to the four vertex theorems. I, *Amer. Math. Soc. Transl. Ser. 2* 190 (1999) 185–228.
- [22] G.F. Voronoi, Nouvelles applications des parametres continus a la theorie des formes quadratiques, *J. Reine Angew. Math.* 34 (1908) 198–287.
- [23] B. Wegner, Bose's vertex theorem for simply closed polygons, *Math. Pannon.* 6 (1995) 121–132.
- [24] G.M. Ziegler, Lectures on polytopes, in: *Graduate Texts in Mathematics*, vol. 152, Springer-Verlag, New York, 1995.