2×2 Matrices. The **determinant** of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\det A = ad - bc$.

The matrix A is invertible if and only if $\det A \neq 0$.

The determinant $\det A$ is the area of the parallelogram spanned by the columns of A.

3×3 Matrices. The **determinant** of a 3×3 matrix $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} a & b & c \\ r & s & t \\ x & y & z \end{bmatrix}$ is given by

$$\det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = asz - aty + btx - brz + cry - csx$$

From 21a: The matrix A is invertible if and only if $\det A \neq 0$ (columns don't span a plane or line) and $\det A$ is the volume of the parallelepiped spanned by the columns of A.

Note: We get a term for every **pattern** of the 3×3 chessboard. The sign depends on if there are an even or odd number of **up-crossings** in the pattern.

Patterns. An $n \times n$ **pattern** for is an arrangement of n rooks on an $n \times n$ chessboard so that no rook can take any other one. That is, there is exactly one rook in each row and column.

In an $n \times n$ pattern π an **up-crossing** occurs every time a rook is north-east of another. The number of up-crossings of a pattern π is denoted $|\pi|$. The **sign** of a pattern π is $\operatorname{sgn}(\pi) = (-1)^{|\pi|}$.

Example. Find all of the 2×2 patterns along with their signs. Do the same for 4×4 patterns.

Example. How many $n \times n$ patterns are there?

Determinants (Pattern Formula). The **determinant** of an $n \times n$ matrix A is

$$\det(A) = \sum_{\text{patterns } \pi} \operatorname{sgn}(\pi) \operatorname{prod}(\pi)$$

where $\operatorname{prod}(\pi)$ is the product of the entries of A corresponding to rook positions in π.

We'll see later that $\det(A)$ is the (n-dimensional) volume of the parallelepiped spanned by the columns of A.

Example. Find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 7 \end{bmatrix}$$

Permutations. A **permutation** of size n is a rearrangement of $\{1, 2, \ldots, n\}$. There are $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ permutations of size n.

A permutation is another way of looking at a pattern: A pattern has a rook in the (i, j)-spot if and only if the permutation has j in the i-th spot.

The **sign** of a permutation is the sign of the corresponding pattern. The number of up-crossings in a pattern is the same as the number of transpositions needed to put the permutation back in order.

Determinants (Permutation Formula). The **determinant** of an $n \times n$ matrix A with entries a_{ij} is

$$\det(A) = \sum_{\text{permutations } \pi} \operatorname{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}.$$

Laplace Expansion. Let A be an $n \times n$ matrix and pick a column j. For each entry a_{ij} in that column denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the j-th column and i-th rows from A. Then

$$\det A = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj} \det(A_{nj})$$

There is a similar formula for the Laplace Expansion about a row.

Note: The Laplace Expansion is just collecting terms in the pattern formula together based off of the position of the rook in the j-th column.

Example. Find the determinants of the following matrices using whichever method you like.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$