Classification of Finite Groups

How to make every group into a special friend of yours?

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Once there was a group. Then we wanted to know them all.
-David

There are a lot of groups and it is impossible to say much about groups in general. But mathematicians are persistent and some classification results have started creeping in.

Theorem 1 (Structure Theorem for fg Modules over PIDs) *Let R be a principal ideal domain and M a finitely generated R-module. Then*

$$M \cong \frac{R}{d_1 R} \oplus \cdots \oplus \frac{R}{d_k R} \oplus R \oplus \cdots \oplus R$$

for some $d_1, \cdots, d_k \in R$ with $d_1 | \cdots | d_k$.

In case $R = \mathbb{Z}$ and M a finite $\mathbb{Z}$-module, this specialises to the following theorem.

Theorem 2 (Structure Theorem for Finite Abelian Groups) *Let G be a finite abelian group. Then*

$$G \cong C_{d_1} \times \cdots \times C_{d_k}$$

for some integers $d_1, \cdots, d_k$ such that $d_1 | \cdots | d_k$ where C_i denotes the cyclic group of order i.

This is a complete classification of all finite abelian groups and although imposing commutativity is an incredibly strong assumption, the result sheds some hope at the complete classification.

At this point we should stop for a moment and think about what is meant by a classification. There is no formal notion of that concept after all. Informally, what we would like is a complete description of all finite groups in terms of an object that is better understood and more easily used for computations.

In that respect, Theorem 2 is indeed the classification of all finite abelian groups. It states that every abelian group can be built from elementary building blocks (cyclic groups) using some construction (a direct product). Both cyclic groups and direct products are extremely well-understood and this is why we are extremely happy with this result.

In an attempt of generalising the approach to the general case, we are guided by trying to answer the following two questions:

Question 1 *What are the building blocks in the general case?*

Question 2 *How do we build more complicated groups of elementary ones?*

Let G be a group. If $N \lhd G$ is a normal subgroup of G, then in many ways the study of G can be reduced to the study of N and the study of G/N.

This is made precise with the notion of the composition series: a composition series is a subnormal series

$$1 = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

where H_{i-1} is maximal normal in H_i. The composition series is not necessarily unique and does not necessarily exist (for example for $\mathbb{Z}$). However, it does exits for all finite groups and we have a Jordan-Hölder theorem which tells us that it is in a sense unique.

Theorem 3 (Jordan-Hölder) *Any two composition series have the same length and composition factors up to permutation and isomorphism.*

The Jordan-Hölder theorem does something striking for us. Regardless of the composition series we write down, it will have the same composition factors with the same multiplicities.

There is a great resemblance to the factorisation of integers in the product of primes - every integer is a product of primes and the factorisation is unique up to the permutation of factors. What is the analogy for primes in the setting of groups? Those are the groups that cannot be decomposed further, *ie* the groups without any non-trivial normal subgroups.

Definition 1 *Let G be a group whose only normal subgroups are G and 1. Then G is called a simple group.*

This is a good definition to use for the answer to Question 1. We can now rephrase the questions we had earlier.

Question 3 *Can we classify all finite simple groups?*

Question 4 *In what ways do simple groups fit together to form finite groups?*

For a naive observer, it is not clear whether the classification of finite simple groups should be easy, hard or impossible. It turns out that it is possible, but only slightly so. The complete classification has been found by joint contributions of hundreds of mathematicians, mostly in the years between 1955 and 2004 and the proof spans tens of thousands of pages. This classification has been called the greatest intellectual achievement of humanity.

We present the classification in all its glory:

Theorem 4 (Classification of Finite Simple Groups) *Let G be a finite simple group. Then G is isomorphic to one of the following groups:*

- *A cyclic group C_p for some prime number p*
- *An alternating group A_n for some $n \geq 5$*
- *A group of Lie type*
- *One of 27 sporadic groups.*

While this monumental first question has been settled, the second question remains wide open. If we return to the analogy with factorisation into prime numbers, one could say that we now know what all the primes are. For any set of primes, we can multiply them together to obtain an integer. However, the situation is much more delicate with groups. For some set of finite simple groups, there are many ways of putting them together to obtain a larger group.

Formally, assume that we know the composition factors of the group G, but not the composition series $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ itself. H_1 is one of the composition factors and we start with it. Pick another composition factor C and we are looking for the group H_2 with a normal subgroup H_1 and a quotient $H_2/H_1 = C$. Then we have H_2, pick another composition factor C' and try to extend H_2 by C' ... In the end, we obtain a group whose composition series has prescribed factors. The complicated part here is that at each step, the extension is not unique and we obtain a lot of non-isomorphic groups with the same composition series.

Trying to understand extensions is an active area of study and it will likely remain so for a very long time.

Let's focus on one step at a time. Start with a groups N and H (for group classification purposes it suffices to take H to be a simple group, but we can also study the problem more generally). Then we would like to find solutions to the extension problem, ie find all groups G that fit in the short exact sequence

$$1 \to N \to G \to H \to 1.$$

Two extensions are equivalent if there is an isomorphism (equivalently an homomorphism) such that the extension diagram commutes.

Note that this is actually more subtle than just requiring that the groups are isomorphic. There can be non-isomorphic extensions with isomorphic groups G in the middle. For example, there are 8 inequivalent extensions of the form $0 \to C_2 \to G \to C_2 \times C_2 \to 0$ but only 5 isomorphism classes of groups of appropriate order.

There is no mathematical theory that would deal with all cases of the extension problem, but there are solutions in some very special cases.

Abelian Extensions The first special case alluded to above is when N, H, G are all required to be abelian - so they are $\mathbb{Z}$-modules. In this case, the isomorphism classes of extensions are given by the $\mathrm{Ext}^1_{\mathbb{Z}}(H, N)$. This is a non-trivial fact proven in Chapter 3 of Weibel's book.

Central Extensions Another special case of the above is when $N \leq Z(G)$ is contained in the centre of G. Such extensions of H by N are called central and there is a good theory about their classification. The isomorphism classes of central extensions are in bijection with $H^2(H, N)$, where H acts trivially on N. This is explained in the Group Cohomology by Brown.

As suggested by these examples, group cohomology is an active area of research that plays a prominent role in our understanding of the extension problem. Solving it in its entirety is widely considered hopeless, but this has never stopped us from trying.