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On the number of positions in chess without promotion

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Abstract The number of different legal positions in chess is usually estimated to be between 10^{40} and 10^{50}. Within this range, the best upper bound 10^{46} is some orders of magnitude bigger than the estimate 5×10^{42} made by Claude Shannon in his seminal 1950 paper, which is usually considered by computer scientists to be a better approximation. We improve Shannon's estimate and show that the number of positions where any number of chessmen may have been captured but no promotion has occured is bounded from above by 2×10^{40}. The actual number should be quite a bit smaller than that and outline possible ways towards improving our result.

Keywords Shannon's number · Chess · State space

1 Introduction and statement of results

1.1 Introduction

The state space of a game is the set of states that can be reached by starting from a valid initial position and following the rules—it is widely used as a basic measure of a game's complexity or of the difficulty in getting an algorithm to play in a reasonable fashion. The state space is considered one of the most fundamental characteristics in game complexity and understood for a variety of games. The problem of estimating the state space of chess is mentioned in Claude Shannon's seminal 1950 paper *Programming a Computer for Playing Chess* Shannon (1950), where the following guess is made.

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> The number of possible positions, of the general order of $64!/32!(8!)^2(2!)^2$, or roughly 10^{43}, [...] (Shannon, Shannon (1950))

The number is known as Shannon's number: it counts the number of ways to arrange all chessmen (henceforth simply called men) taking into account that no two men can occupy the same square and that furthermore any two identical men of the same color are indistinguishable. This number does not consider the possibility that *not* all the men need to be on the board (some might have been already captured) and and it also does not account for the rule of *promotion* whereby a pawn must be promoted to a more powerful figure if it advances to the end of the file (column of the chessboard). However, it also accounts for all sorts of illegal positions that can never possibly occur. This combination of factors makes it difficult to say whether Shannon's argument over– or underestimates the actual state space.

1.2 Known results

As for definite upper bounds on the state space of chess, Allis Allis (1994) proved an upper bound of 5×10^{52}, estimates the true number to be around 10^{50} and does comment on 10^{43} as being 'in our opinion too low an estimate'. Yet it seems that many people consider Shannon's number to be a realistic estimate and it is widely quoted as such; for example, Schaeffer et al. Schaeffer et al. (1991) in a 1991 paper on Checkers as well as in a 2005 paper Schaeffer et al. (2005) estimate the number to be around 10^{44} and only when the full solution of Checkers was announced in *Nature* Schaeffer (2007) was the estimate corrected to a more conservative 'somewhere in the 10^{40} to 10^{50} range'. Chinchalkar (1996) proved the upper bound 10^{46} on the size of the state space.

The purpose of this short paper is to give some evidence that the number of legal positions could be be much smaller than is commonly assumed. We give a simple argument demonstrating that in the absence of promotions, Shannon's argument actually *overestimates* the number of possible games. Hopefully will spark interest in the problem itself, a full attack on which seems to require a much broader approach. Several ways in which the approach could potentially be improved can be found at the end of the paper.

1.3 Main result

The main statement of this paper bounds a number strictly larger than Shannon's estimate: we estimate all legal positions that can occur in games which feature no obtrusive force (chessmen obtained via the rule of promotion). Recall that Shannon's estimate only included those legal positions without obtrusive force where no piece has been captured. We show that our number of interest is actually smaller than Shannon's number: this shows that in his estimate the number of illegal positions erroneously counted far outweighs the number of positions with not all pieces being on the board.

Theorem *The number of legal positions without obtrusive force in which an arbitrary number of pieces may have already been captured is smaller than* 2×10^{40}.

The proof is based on distinguishing between 1180980 different cases, a precise result on arranging bishops under additional constraints and an easy bound on the size of that part of the state space where no piece has been captured yet—only basic rules of chess are being used.

1.4 Possible improvement

Our argument does not yet incorporate the rather intricate pawn structure: if a pawn is not on its initial file, it must have captured another piece – intricate pawn structures require many chessmen to have been captured, which reduces the count of posiitons. Additionally, they can only move to an adjacent file in a diagonal fashion and the pawn on a2 can never be in d3, which further reduces complexity.

Conjecture *The number of legal positions in chess without obtrusive force in which an arbitrary number of pieces may have already been captured is smaller than* 10^{35}.

This estimate seems to be smaller than common estimates but, given our argument below, it seems likely. A possible but nontrivial way to shed some light on the truth of the conjecture could be to use Monte Carlo simulations, create random arrangements following the construction from our proof and see how many of those are actually legal. Understanding to which extent the rule of promotion increases the combinatorial complexity seems challenging (for example, in order for a promotion to be possible at all, at least one pawn has to have been captured). Nonetheless, we are inclined to believe that the total number of legal positions in chess should also be quite a bit smaller than commonly accepted and consider an upper bound of 10^{39} to be very likely.

2 Counting bishops

Our argument is based on the fact that Shannon's way of looking at things is not optimal: both bishops and pawns cannot reach all 64 possible positions. Since bishops are color-locked, every bishop can only occupy 32 squares. Since promotion does not occur, a pawn can only reach 48 squares (it cannot move backwards and if it were to reach the last line, promotion would occur). We will combine these two facts by selecting a subset A containing the 16 squares that can never be reached by the pawns and study all the possible arrangements of bishops with a prescribed number in A. Let us fix notation and denote the number of white pawns, knights, bishops, rooks and queens by p_w, k_w, b_w, r_w, q_w, respectively, and use the subscript $\cdot_b$ for black pieces. Note that for a position to be legal, the presence of both kings on the board is required.

Lemma 1 *Let A be a fixed subset of the chessboard made up of precisely 8 white and 8 black squares and for any* $0 \leq i \leq 4$ *let* $f_i(b_w, b_b)$ *denote the number of ways* *that b_w white bishops and b_b black bishops can be arranged on the board such that precisely i of the bishops are contained in A. Then*

$f_0(\cdot,\cdot)$	*0*	*1*	*2*
0	1	48	576
1	48	2256	26496
2	576	26496	304704

$f_1(\cdot,\cdot)$	*0*	*1*	*2*
0	0	16	384
1	16	1536	27264
2	384	27264	423936

$f_2(\cdot,\cdot)$	*0*	*1*	*2*
0	*0*	*0*	64
1	*0*	240	8832
2	64	8832	209280

$f_3(\cdot,\cdot)$	*0*	*1*	*2*
0	0	0	0
1	0	0	896
2	0	896	43008

$f_4(\cdot,\cdot)$	*0*	*1*	*2*
0	0	0	0
1	0	0	0
2	0	0	3136

Proof The proof is elementary. The rule of chess being exploited is that two bishops of the same color must occupy squares of different color. We describe three special cases, the rest can be carried out analogously.

$f_2(1, 1)$ requires both a white and a black bishop inside A. There are no color restrictions and therefore $f_2(1, 1) = 16 * 15 = 240$. $f_1(2, 1)$ describes the number of ways two white and one black bishop can be arranged in such a way that precisely one of the bishops is contained in A. Suppose the bishop in A is white. Then there 16 possible ways to place it. The black bishop outside has to be of the opposite color and is thus restricted to one of 24 fields. The black bishop can occupy any of the remaining 47 unoccupied fields outside of A. Conversely, if the bishop in A is black, there are 16 possible fields to place it on while the remaining white bishops have $f_0(2, 0) = 24*24$ possible ways to be arranged. Therefore

$$f_1(2, 1) = 16 * 24 * 47 + 16 * 24 * 24 = 27264.$$

The most involved case is $f_2(2, 2)$. There are three cases: two white bishops in A, two black bishops in A and one bishop of either color in A. The first two cases are symmetric and consist of $8 * 8 * 24 * 24$ cases each. We distinguish two subcases in the third case: both bishops in A being on the same color or on different color. The first case has $16 * 7$ ways to place the two bishops in A and since the two remaining bishops outside are also on the same color (which is fixed by placing the first one), there are $24 * 23$ remaining ways to place these, giving us $16 * 7 * 24 * 23$ cases in total. Arguing analogously in the second subcase, there are $16 * 8 * 24 * 24$ ways to place these bishops. □

The combinatorially most complicated scenario is certainly when no man has been captured since then all of them can be arranged in a variety of ways; there is a very short argument showing that this case is actually relatively simple; this allows us to assume in the remainder of the argument that at most 31 men are on the board.

Lemma 2 *The state space of chess with no pieces captured is of size less than* 10^{20}.

Proof The only way for a pawn to change the file in which it started is to capture a piece. Since no piece has been captured, all the pawns are in their initial file which allows for efficient encoding. For each file, there are 15 possible arrangements of the two pawns yielding 15^8 possible pawn arrangements. Ignoring pawns, there are 48 squares to distribute 16 pieces on, some of which are indistinguishable, yielding the Shannon-type argument

$$15^8 \binom{48}{16} \frac{1}{2^6} \sim 9 \times 10^{19}.$$

□

3 The proof

The proof proceeds by a rather large case distinction; every particular case has a simple combinatorial structure—the improvement that we achieve over Shannon (which is large since summing over all cases still yields an improved result) comes from a particular way of setting up each particular case.

Proof Let A be the first and the last row, which is the set of 16 squares, where no pawn can ever be. The proof proceeds by the following sequence of case distinctions.

1. The first distinction is into cases $0 \leq k \leq 4$, where k is the total number of both black and white bishops contained in A.
2. The second distinction is into the 9 possible values the tuple $(b_w, b_b) \in \{0, 1, 2\}^2$ can assume.
3. The number of ways b_w and b_b bishops can be placed on the board with precisely k of them in A has already been computed in Lemma 1 and is $f_k(b_w, b_b)$.
4. We add p_w white pawns. On the board, there are already $b_w + b_b$ bishops, $b_w + b_b - k$ outside of A. This leaves $48 - (b_w + b_b - k)$ fields on which the p_w pawns can be placed.
5. Now we add the remaining men: since there are no promotions, the p_b black pawns are spread over 48 fields, where p_w are already occupied by white pawns and $b_w + b_b - k$ are occupied by bishops.
6. There are $64 - b_w - b_b - p_w - p_b$ fields on which to place r_w white rooks.
7. There are $64 - b_w - b_b - p_w - p_b - r_w$ fields on which to place r_b black rooks.
8. There are $64 - b_w - b_b - p_w - p_b - r_w - r_b$ fields on which to place k_w white knights.
8. There are $64 - b_w - b_b - p_w - p_b - r_w - r_b - k_w$ fields on which to place k_r black knights.
9. There are $64 - b_w - b_b - p_w - p_b - r_w - r_b - k_w - k_b$ fields on which to place q_w white queen.
10. There are $64 - b_w - b_b - p_w - p_b - r_w - r_b - k_w - k_b - q_w$ fields on which to place q_b black queen.

11. A position is only valid if both kings are on the board. There are $64 - b_w - b_b - p_w - p_b - r_w - r_b - k_w - k_b - q_w - q_b$ fields on which to place one of the kings; the remaining number of free fields is one smaller and that gives the number of ways to place the remaining king. All these numbers can now be multiplied to yield the number of ways to arrange that particular collection of men.

Each special case reduces to a product of $f_k(b_w, b_b)$ and several binomial coefficients and can be directly computed. Summing over all possibilities (taking roughly 20 second on a standard desktop computer) yields that there are

$$1.5236 \cdots \times 10^{40} \qquad \text{arrangements of this type}$$

to which, of course, the remaining $\sim 9 \times 10^{19}$ cases from Lemma 2 need to be added. $\square$

4 Concluding remarks

4.1 Further improvements

The argument given here is far from optimal; indeed, the various ways in which it is suboptimal suggests very strongly that the true number of legal chess positions without obtrusive force should still be much, much smaller. A natural way to proceed would be to generalize the reasoning given in Lemma 2 and exploit the fact that a pawn can only move in a very restricted manner. Indeed, since pawns can only change files by capturing another man, any time a pawn moves to a different file the combinatorial complexity of the setup is reduced. Additionally, pawns can only change files diagonally and the pawn starting out at, say, a2, can never be found in d3. If this behavior could be suitably encoded in a combinatorial way, then it seems likely that exploiting it would lead to rather drastic improvements.

Even more challenging would be to understand the interplay between the pawns and the (color-locked) bishops and to try to understand all possible arrangements of these 20 chessmen. A complete understanding of that situation would still leave room for various illegal scenarios (for example a king being threatened in such a way that he must have also been threatened at a previous point in time), nonetheless it seems that this would account for the bulk of the combinatorial complexity and give rise to a rather reasonable estimate for the number of positions without obtrusive force reachable from the initial position.

4.2 Monte-Carlo estimates

As already mentioned in the introduction, one possible way of getting a rough estimate on the true number of legal positions in game without obtrusive force can be achieved using a Monte Carlo approach: generate a large number of random arrangements following the combinatorial approach from the proof and see how many of these are actually legal. Checking whether an arrangement is legal seems nontrivial on many levels, however, it might be feasible to check merely whether certain simplified rules are violated (i.e. whether the number of pawns outside of their file is compatible with the number of chessmen already captured) thereby providing an upper bound. A theoretical difficulty is that Monte Carlo methods become unstable if the phenomenon one tries to observe happens very rarely; if this difficulty were to occur (i.e. only 1 in, say, 10^8 randomly generated position is legal) it would nonetheless indicate just how much smaller than 10^{41} the number of legal positions in a game without obtrusive force actually is.

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