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A complex valued approach to the solutions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear Telegraph equation via fixed point method

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ABSTRACT

This paper involves complex valued versions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear Telegraph equation. Under various suitable assumptions the results are established in the setting of complex valued double controlled metric space. Thereafter, by making consequent use of the fixed point method, short and simple proofs are obtained for solutions of Riemann-Liouville integral, complex valued Atangana-Baleanu integral operator and non-linear Telegraph equation.

1. Introduction

The mathematical terms 'Integral Equations' and 'Fractional Calculus' are introduced in more than a 100 years back. These 100 years seems like a really long time, but predominantly, these topics are extensively gain new structures and are applied to all branches of sciences as well as engineering sciences.

Very recently, Atangana-Baleanu [1] has focused on new fractional derivative targeting non-local and non-singular kernal; thereafter they applied these new structures to fractional heat transfer. This notion is a beautiful mixture of non-local and non-singular kernal and so on. We can see many generalizations/usefulness of Atangana-Baleanu fractional derivatives in the literature of *fractional calculus* such as AB_C type [1], AB_{RL} type [2], AB derivatives via MHD channel flow [3], AB derivatives via Chaotic burst dynamics [4] and so on (see for example [5–20], [42–56]). These huge generalizations provide undoubtedly a very important increase in the pertinent research topics. Based on these generalizations, many authors showed their interest in Atangana-Baleanu fractional derivatives. Here we recollecting the definition of the AB-fractional integral.

$$
{}^{AB}_{s}I^{\lambda}_{t}f(t)=\frac{1-\lambda}{\mathcal{B}(\lambda)}f(t)+\frac{\lambda}{\mathcal{B}(\lambda)\Gamma(\lambda)}\int_{s}^{t}f(h)(t-h)^{\lambda-1}dh,
$$

where $0<\lambda\le 1$, $0<t<s$ and note that the normalization function $\mathcal{B}(0)$ and $\mathcal{B}(1)$, both are equal to 1.

A bunch of scientists has tackled Riemann-Liouville integral equations and recently identified a new technique, so called 'fixed point approach'. This novel approach have promising the existence of solution of Riemann-Liouville integral equations. (see for example [14,21–25]). The mirror side of this section is as below.

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Nonlinear analysis is a predominant discipline where fixed point theory achieve an influential role in the fractional calculus, boundary value problems. The literature on *fixed point theory* speaks to two variants of arguments, *fixed point theory in flavour of metric spaces* and *topological problems via fixed point theory*, but only topological problems via fixed point theory is of particular interest to topologists and theoretical computer scientists. The investigations in fixed point theory in flavour of metric spaces is of great importance in computing, bio-informatics and computational biology. Because of this solid relation of fixed point theory and branches of sciences, a big process has been recorded in the last century in the theory of fixed point.

In mathematical analysis, a metric space is a set along with a metric on the set. The metric is a function that characterizes an idea of distance between any two elements of the set, which are normally termed as points. Metric spaces may be a special kind of topological space but its generalizations/extensions is always produces exciting, high quality research outputs that addresses fundamental questions in mathematical-multidisciplinary phenomena. Extensions of metric space could encompass a careful comparative analysis of balls or connected components, but also local compactness and the one-point compactification and neighborhoods will likely continue to use for *problem finding analysis* leading to novel mathematical-multidisciplinary insights.

Very recently, we can see various generalized metric spaces such as, an extended b-metric space [26], controlled metric type space [27], double controlled metric type space [28], complex valued metric space [29] and complex valued b-metric space [30]. Many authors showed their attention in order to obtain fixed point theorems in such spaces (see for example [31–40]).

2. Notations and results on complex valued double controlled metric type space

Let $\mathbb{C}$ be the set of complex numbers and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Now, define a partial order $\preceq$ on $\mathbb{C}$ as below:
$\alpha_1 \preceq \alpha_2 \Leftrightarrow Re(\alpha_1) \le Re(\alpha_2),\ Im(\alpha_1) \le Im(\alpha_2)$.
Thus, $\alpha_1 \preceq \alpha_2$ if meet one of the following condition.

(A) $Re(\alpha_1) = Re(\alpha_2)$ and $Im(\alpha_1) = Im(\alpha_2)$,
(B) $Re(\alpha_1) < Re(\alpha_2)$ and $Im(\alpha_1) = Im(\alpha_2)$,
(C) $Re(\alpha_1) = Re(\alpha_2)$ and $Im(\alpha_1) < Im(\alpha_2)$,
(D) $Re(\alpha_1) < Re(\alpha_2)$ and $Im(\alpha_1) < Im(\alpha_2)$.

We can list the notation as, $\alpha_1 \precnsim \alpha_2$ if $\alpha_1 \ne \alpha_2$ and one of (B),(C) and (D) is satisfied; by following the same pattern, we can write $\alpha_1 \prec \alpha_2$ if only (D) satisfied. It follows that

(1) $0 \preceq \alpha_1 \precnsim \alpha_2$ implies $|\alpha_1| < |\alpha_2|$,
(2) $\alpha_1 \preceq \alpha_2$ and $\alpha_2 \prec \alpha_3 \Rightarrow \alpha_1 \prec \alpha_3$,
(3) $0 \preceq \alpha_1 \preceq \alpha_2$ implies $|\alpha_1| \le |\alpha_2|$,
(4) if $p, q \in \mathbb{R}, 0 \le p \le q$ and $\alpha_1 \preceq \alpha_2$, then $p\alpha_1 \preceq q\alpha_2$ for all $\alpha_1, \alpha_2 \in \mathbb{C}$.

We introduce following definition.

Definition 2.1. Let S be a non-empty set and functions $\wp\ \ell$: $S \times S \to [1, \infty)$. Define a function $d_{\mathbb{C}} : S \times S \to \mathbb{C}$ where $\mathbb{C}$ be the set of all complex numbers is called a distance function $d_{\mathbb{C}}$ on S. We deal with following:

$\mathcal{D}_1$: $\forall_{\rho,q \in S}\ d_{\mathbb{C}}(p, q) = 0$ if and only if $p = q$,
$\mathcal{D}_2$: $\forall_{p,q \in S}\ d_{\mathbb{C}}(p, q) = d_{\mathbb{C}}(q, p)$,
$\mathcal{D}_3$: $\forall_{p,q,r \in S}\ d_{\mathbb{C}}(p, q) \preceq \wp(p, q)[d_{\mathbb{C}}(p, r) + d_{\mathbb{C}}(r, q)]$ (so called, an extended triangle inequality),
$\mathcal{D}_4$: $\forall_{p,q,r \in S}\ d_{\mathbb{C}}(p, q) \preceq \wp(p, r)d_{\mathbb{C}}(p, r) + \wp(r, q)d_{\mathbb{C}}(r, q)$ (so called, controlled triangle inequality),
$\mathcal{D}_5$: $\forall_{p,q,r \in S}\ d_{\mathbb{C}}(p, q) \preceq \wp(p, r)d_{\mathbb{C}}(p, r) + \ell(r, q)d_{\mathbb{C}}(r, q)$ (so called, double controlled triangle inequality).

$d_{\mathbb{C}}$ is called,

1. Complex valued extended metric if $d_{\mathbb{C}}$ satisfies $\mathcal{D}_1$ through $\mathcal{D}_3$.
2. Complex valued controlled metric type if $d_{\mathbb{C}}$ satisfies $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_4$.
3. Complex valued double controlled metric type if $d_{\mathbb{C}}$ satisfies $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_5$.

The pair $(S, d_{\mathbb{C}})$ is called a complex valued extended metric/complex valued controlled metric type/complex valued double controlled metric type space if $d_{\mathbb{C}}$ is a complex valued extended metric/complex valued controlled metric type/complex valued double controlled metric type on S.

Example 2.1. Let $S = \{1, 2, 3\}$. Define $d_{\mathbb{C}} : S \times S \to \mathbb{C}$ by

$$d_{\mathbb{C}}(1, 2) = d_{\mathbb{C}}(2, 1) = 2 + 4i;\ d_{\mathbb{C}}(2, 3) = d_{\mathbb{C}}(3, 2) = i;$$
$$d_{\mathbb{C}}(1, 3) = 1 - i = d_{\mathbb{C}}(3, 1); d_{\mathbb{C}}(1, 1) = d_{\mathbb{C}}(2, 2) = d_{\mathbb{C}}(3, 3) = 0.$$

Define $\wp\ \ell$: $S \times S \to [1, \infty)$ by

$$\wp(1, 2) = \wp(2, 1) = 1;\ \wp(2, 3) = \wp(3, 2) = \frac{8}{7};\ \wp(3, 1) = \wp(1, 3) = \frac{3}{2}.$$
$$\ell(1, 2) = \ell(2, 1) = \frac{7}{6};\ \ell(2, 3) = \ell(3, 2) = \frac{9}{2};\ \ell(3, 1) = \ell(1, 3) = 1.$$

Note that,

$$|d_{\mathbb{C}}(1, 2)| \succ \wp(1, 3)|d_{\mathbb{C}}(1, 3)| + \wp(3, 2)|d_{\mathbb{C}}(3, 2)|.$$

Thus, $d_{\mathbb{C}}$ is neither a complex valued extended metric nor complex valued controlled metric type for the function $\wp$ but $d_{\mathbb{C}}$ is double controlled metric type.

Definition 2.2. Let $(S, d_{\mathbb{C}})$ be a complex valued double controlled metric type space (in short, $d_{\mathbb{C}}$-metric space) and let $\{p_n\}$ be a sequence in S and $p \in S$.

For every $c \in \mathbb{C}$, and $0 \prec s$, there exist $N \in \mathbb{N}$ such that,

1. $\forall_{n>N}\ d_{\mathbb{C}}(p_n, p) \prec c$, then $\{p_n\}$ is said to be convergent, $\{p_n\}$ converges to p, and p is the limit point of $\{p_n\}$. We call this notation by $\lim_{n\to\infty} p_n = p$ or $\{p_n\} \to p$ as $n \to \infty$.
2. $\forall_{n>N}\ d_{\mathbb{C}}(p_n, p_{n+m}) \prec c$, here $m \in \mathbb{N}$, then $\{p_n\}$ is called Cauchy sequence.
3. If every Cauchy sequence in S is convergent, then $(S, d_{\mathbb{C}})$ is said to be a complete $d_{\mathbb{C}}$-metric space.

Lemma 2.1. *Let $(S, d_{\mathbb{C}})$ be a $d_{\mathbb{C}}$-metric space and let $\{p_n\}$ be a sequence in S. Then $\{p_n\}$ converges to p iff $|d_{\mathbb{C}}(p_n, p)| \to 0$ as $n \to \infty$.*

Lemma 2.2. *Let $(S, d_{\mathbb{C}})$ be a $d_{\mathbb{C}}$-metric space and let $\{p_n\}$ be a sequence in S. Then $\{p_n\}$ is a Cauchy sequence if and only if $|d_{\mathbb{C}}(p_n, p_{n+m})| \to 0$ as $n, m \to \infty$, where $n, m \in \mathbb{N}$.*

Remark 2.1. If for all $p, q, r \in S$, $\wp(p, r) = \ell(r, q) = b \ge 1$, then $d_{\mathbb{C}}$-metric space becomes complex valued b-metric space.

Theorem 2.1. *Let $(S, d_{\mathbb{C}})$ be a complete $d_{\mathbb{C}}$-metric space and let the mappings $\mathcal{O}, \mathcal{H} : S \to S$ satisfy,*

$$a^b d_{\mathbb{C}}(\mathcal{O}p, \mathcal{H}q) \preceq \lambda d_{\mathbb{C}}(p, q) + \mu \frac{d_{\mathbb{C}}(p, \mathcal{O}p) d_{\mathbb{C}}(q, \mathcal{H}q)}{1 + d_{\mathbb{C}}(p, q)}, \tag{2.1}$$

for all $p,\ q \in S$ whenever $0 \prec d_{\mathbb{C}}(p, q)$ where $a,\ b \ge 1$ and $\lambda,\ \mu$ are non-negative real numbers with $\lambda + \mu < 1$. For $p_0 \in S$, choose $p_n = \mathcal{H}^n p_0$. Assume that,

$$\sup_{m\ge1} \lim_{i\to\infty} \frac{\wp(p_{i+1}, p_{i+2})}{\wp(p_i, p_{i+1})} \ell(p_{i+1}, p_m) < \frac{1}{A};\ \text{where } A = \frac{\lambda}{a^b - \mu}. \tag{2.2}$$

In addition, for each $p \in S$ suppose that $\lim_{n\to\infty}\wp(p,\ p_n)$ and $\lim_{n\to\infty}\ell(p_n,\ p)$ exist and are finite. Then $\mathcal{O}, \mathcal{H}$ have a unique common fixed point.

Proof. Let p_0 be an arbitrary point in S. Consider the sequence $p_n = \mathcal{H}^n p_0$ in S that satisfies the hypothesis of the theorem and define as below:

$$p_{2n+1} = \mathcal{O} p_{2n}.$$
$$p_{2n+2} = \mathcal{H} p_{2n+1},\ \ n = 0, 1, 2, 3 \dots$$

Consider,

$$\begin{aligned} a^b d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}) &= a^b d_{\mathbb{C}}(\mathcal{O}p_{2n}, \mathcal{H}p_{2n+1}) \\ &\preceq \lambda d_{\mathbb{C}}(p_{2n}, p_{2n+1}) + \mu \frac{d_{\mathbb{C}}(p_{2n}, \mathcal{O}p_{2n}) d_{\mathbb{C}}(p_{2n+1}, \mathcal{H}p_{2n+1})}{1 + d_{\mathbb{C}}(p_{2n}, p_{2n+1})} \\ &\preceq \lambda d_{\mathbb{C}}(p_{2n}, p_{2n+1}) + \mu \frac{d_{\mathbb{C}}(p_{2n}, p_{2n+1}) d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})}{1 + d_{\mathbb{C}}(p_{2n}, p_{2n+1})} \\ &\preceq \lambda d_{\mathbb{C}}(p_{2n}, p_{2n+1}) + \mu d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}). \end{aligned} \tag{2.3}$$

This gives, $d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}) \preceq \dfrac{\lambda}{a^b - \mu} d_{\mathbb{C}}(p_{2n}, p_{2n+1})$.

Similarly, $d_{\mathbb{C}}(p_{2n+2}, p_{2n+3}) \preceq \dfrac{\lambda}{a^b - \mu} d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$.

Now with $A = \frac{\lambda}{a^b - \mu} < 1$, we get,

$$\begin{aligned} |d_{\mathbb{C}}(p_n, p_{n+1})| &\preceq A|d_{\mathbb{C}}(p_{n-1}, p_n)| \\ &\preceq A^2 |d_{\mathbb{C}}(p_{n-2}, p_{n-1})| \\ &\ \vdots \\ &\preceq A^n |d_{\mathbb{C}}(p_0, p_1)|. \end{aligned} \tag{2.4}$$

For any $m > n$ where $m, n \in \mathbb{N}$.

$$\begin{aligned} |d_{\mathbb{C}}(p_n, p_m)| &\preceq \wp(p_n, p_{n+1})|d_{\mathbb{C}}(p_n, p_{n+1})| + \ell(p_{n+1}, p_m)|d_{\mathbb{C}}(p_{n+1}, p_m)| \\ &\preceq \wp(p_n, p_{n+1})|d_{\mathbb{C}}(p_n, p_{n+1})| + \ell(p_{n+1}, p_m)[\wp(p_{n+1}, p_{n+2})|d_{\mathbb{C}}(p_{n+1}, p_{n+2})| + \ell(p_{n+2}, p_m)|d_{\mathbb{C}}(p_{n+2}, p_m)|] \\ &\preceq \wp(p_n, p_{n+1})|d_{\mathbb{C}}(p_n, p_{n+1})| + \ell(p_{n+1}, p_m)\wp(p_{n+1}, p_{n+2})|d_{\mathbb{C}}(p_{n+1}, p_{n+2})| + \ell(p_{n+1}, p_m)\ell(p_{n+2}, p_m)|d_{\mathbb{C}}(p_{n+2}, p_m)| \\ &\ \vdots \\ &\preceq \wp(p_n, p_{n+1})|d_{\mathbb{C}}(p_n, p_{n+1})| + \Sigma_{i=n+1}^{m-2}\left(\Pi_{j=n+1}^{i} \ell(p_j, p_m)\right)\wp(p_i, p_{i+1})|d_{\mathbb{C}}(p_i, p_{i+1})| + \Pi_{k=n+1}^{m-1}\ell(p_k, p_m)|d_{\mathbb{C}}(p_{m-1}, p_m)| \\ &\preceq \wp(p_n, p_{n+1})A^n|d_{\mathbb{C}}(p_0, p_1)| + \Sigma_{i=n+1}^{m-2}\left(\Pi_{j=n+1}^{i} \ell(p_j, p_m)\right)\wp(p_i, p_{i+1})A^i|d_{\mathbb{C}}(p_0, p_1)| + \Pi_{i=n+1}^{m-1}\ell(p_i, p_m)A^{m-1}|d_{\mathbb{C}}(p_0, p_1)| \\ &\preceq \wp(p_n, p_{n+1})A^n|d_{\mathbb{C}}(p_0, p_1)| + \Sigma_{i=n+1}^{m-2}\left(\Pi_{j=n+1}^{i} \ell(p_j, p_m)\right)\wp(p_i, p_{i+1})A^i|d_{\mathbb{C}}(p_0, p_1)| \\ &\quad + \Pi_{i=n+1}^{m-1}\ell(p_i, p_m)A^{m-1}\wp(p_{m-1}, p_m)|d_{\mathbb{C}}(p_0, p_1)| \\ &\preceq \wp(p_n, p_{n+1})A^n|d_{\mathbb{C}}(p_0, p_1)| + \Sigma_{i=n+1}^{m-1}\left(\Pi_{j=n+1}^{i} \ell(p_j, p_m)\right)\wp(p_i, p_{i+1})A^i|d_{\mathbb{C}}(p_0, p_1)| \\ &\preceq \wp(p_n, p_{n+1})A^n|d_{\mathbb{C}}(p_0, p_1)| + \Sigma_{i=n+1}^{m-1}\left(\Pi_{j=0}^{i} \ell(p_j, p_m)\right)\wp(p_i, p_{i+1})A^i|d_{\mathbb{C}}(p_0, p_1)|. \end{aligned} \tag{2.5}$$

Let

$$\mathcal{S}_l = \Sigma_{i=0}^{l}\left(\Pi_{j=0}^{i}\ell(p_j, p_m)\right)\wp(p_i, p_{i+1})A^i.$$

Hence from above inequality (2.5),

$$|d_{\mathbb{C}}(p_n, p_m)| \preceq |d_{\mathbb{C}}(p_0, p_1)|[A^n \wp(p_n, p_{n+1}) + (\mathcal{S}_{m-1} - \mathcal{S}_n)]. \tag{2.6}$$

By using ratio test and (2.2), we will get that the limit of the real number sequence $\{\mathcal{S}_n\}$ exists, and so $\{\mathcal{S}_n\}$ is Cauchy. Indeed the ratio test is applied to the term $z_i = (\Pi_{j=0}^{i}\ell(p_j, p_m))\wp(p_i, p_{i+1})$.

Letting $n, m \to \infty$ in (2.5) gives

$$\lim_{n,m\to\infty} |d_{\mathbb{C}}(p_n, p_m)| = 0. \tag{2.7}$$

So the sequence $\{p_n\}$ is Cauchy. Since (X, d) is a complete $d_{\mathbb{C}}$-metric space, there exists some $u \in X$ such that

$$\lim_{n\to\infty} |d_{\mathbb{C}}(p_n, u)| = 0. \tag{2.8}$$

By using triangle inequality, we have,

$$d_{\mathbb{C}}(u, p_{n+1}) \preceq \wp(u, p_n) d_{\mathbb{C}}(u, p_n) + \ell(p_n, p_{n+1}) d_{\mathbb{C}}(p_n, p_{n+1}). \tag{2.9}$$

Using (2.7) and (2.9), we can get,

$$\lim_{n\to\infty} |d_{\mathbb{C}}(u, p_{n+1})| = 0. \tag{2.10}$$

We claim that $u = \mathcal{O}u$, otherwise $0 \prec d_{\mathbb{C}}(u, \mathcal{O}u)$ and we would then have

$$\begin{aligned}|d_{\mathbb{C}}(u,\mathcal{O}u)| &\preceq \wp(u,p_{n+2})|d_{\mathbb{C}}(u,p_{n+2})| + \ell(p_{n+2},\mathcal{O}u)|d_{\mathbb{C}}(p_{n+2},\mathcal{O}u)|\\ &\preceq \wp(u,p_{n+2})|d_{\mathbb{C}}(u,p_{n+2})| + \ell(p_{n+2},\mathcal{O}u)|d_{\mathbb{C}}(\mathcal{H}p_{n+1},\mathcal{O}u)|\\ &\preceq \wp(u,p_{n+2})|d_{\mathbb{C}}(u,p_{n+2})| + \ell(p_{n+2},\mathcal{O}u)a^b|d_{\mathbb{C}}(\mathcal{O}u,\mathcal{H}p_{n+1})|\\ &\preceq \wp(u,p_{n+2})|d_{\mathbb{C}}(u,p_{n+2})| + \ell(p_{n+2},\mathcal{O}u)\left(\lambda|d_{\mathbb{C}}(u,p_{n+1})| + \frac{\mu|d_{\mathbb{C}}(u,\mathcal{O}u)||d_{\mathbb{C}}(p_{n+1},\mathcal{H}p_{n+1})|}{1+|d_{\mathbb{C}}(u,p_{n+1})|}\right)\\ &\preceq \wp(u,p_{n+2})|d_{\mathbb{C}}(u,p_{n+2})| + \ell(p_{n+2},\mathcal{O}u)\left(\lambda|d_{\mathbb{C}}(u,p_{n+1})| + \frac{\mu|d_{\mathbb{C}}(u,\mathcal{O}u)||d_{\mathbb{C}}(p_{n+1},p_{n+2})|}{1+|d_{\mathbb{C}}(u,p_{n+1})|}\right).\end{aligned} \tag{2.11}$$

From (2.7), (2.8) and (2.9), we get,
$|d_{\mathbb{C}}(u, \mathcal{O}u)| = 0$, a contradiction. Thus $u = \mathcal{O}u$. Similarly, we can prove $u = \mathcal{H}u$.
Now we will prove that $\mathcal{O}$ and $\mathcal{H}$ have a unique common fixed point.
Let u and u^* are two common fixed points of $\mathcal{O}$ and $\mathcal{H}$. So that $u \neq u^*$.
Then $0 \prec d_{\mathbb{C}}(u, u^*)$. Then from (2.1),

$$\begin{aligned}a^b d_{\mathbb{C}}(u,u^*) &= a^b d_{\mathbb{C}}(\mathcal{O}u,\mathcal{H}u^*)\\ &\preceq \lambda d_{\mathbb{C}}(u,u^*) + \mu\frac{d_{\mathbb{C}}(u,\mathcal{O}u)d_{\mathbb{C}}(u^*,\mathcal{H}u^*)}{1+d_{\mathbb{C}}(u,u^*)}\\ &\preceq \lambda d_{\mathbb{C}}(u,u^*)\\ &\Rightarrow (a^b-\lambda)d_{\mathbb{C}}(u,u^*) \preceq 0.\\ &\Rightarrow (a^b-\lambda)|d_{\mathbb{C}}(u,u^*)| \preceq 0.\end{aligned} \tag{2.12}$$

This implies, $d_{\mathbb{C}}(u, u^*) = 0$, a contradiction. Thus $u = u^*$. Hence $\mathcal{O}$ and $\mathcal{H}$ have unique common fixed point.
If we assume that two mappings $\mathcal{O}, \mathcal{H}$ are equal, *i.e.*, $\mathcal{O} = \mathcal{H}$ then the above Theorem 2.1 reduces to below corollary.

Corollary 2.1. *Let $(S, d_{\mathbb{C}})$ be a complete complex double controlled metric type space and let the mappings $\mathcal{O} : S \to S$ satisfy,*

$$a^b d_{\mathbb{C}}(\mathcal{O}p,\mathcal{O}q) \preceq \lambda d_{\mathbb{C}}(p,q) + \mu\frac{d_{\mathbb{C}}(p,\mathcal{O}p)d_{\mathbb{C}}(q,\mathcal{O}q)}{1+d_{\mathbb{C}}(p,q)}, \tag{2.13}$$

for all $p, q \in S$ whenever $0 \prec d_{\mathbb{C}}(p, q)$ where $a, b \geq 1$ and λ, μ are non-negative real numbers with $\lambda + \mu < 1$. For $p_0 \in S$, choose $p_n = \mathcal{O}^n p_0$. Assume that,

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\wp(p_{i+1},p_{i+2})}{\wp(p_i,p_{i+1})}\ell(p_{i+1},p_m) < \frac{a^b-\mu}{\lambda}. \tag{2.14}$$

In addition, for each $p \in S$ suppose that $\lim_{n\to\infty}\wp(p, p_n)$ and $\lim_{n\to\infty}\ell(p_n, p)$ exist and are finite. Then $\mathcal{O}$ have a unique common fixed point.

If we take $\mathcal{O} = \mathcal{H}$ and $a = b = 1$; $\mu = 0$ then the above Theorem 2.1 reduces to below corollary.

Corollary 2.2. *Let $(S, d_{\mathbb{C}})$ be a complete complex double controlled metric type space and let the mappings $\mathcal{O} : S \to S$ satisfy,*

$$d_{\mathbb{C}}(\mathcal{O}p,\mathcal{O}q) \preceq \lambda d_{\mathbb{C}}(p,q), \tag{2.15}$$

for all $p, q \in S$ whenever $0 \prec d_{\mathbb{C}}(p, q)$ where λ is a non-negative real numbers with $\lambda < 1$. For $p_0 \in S$, choose $p_n = \mathcal{O}^n p_0$. Assume that,

$$\sup_{m\geq 1}\lim_{i\to\infty}\frac{\wp(p_{i+1},p_{i+2})}{\wp(p_i,p_{i+1})}\ell(p_{i+1},p_m) < \frac{1}{\lambda}. \tag{2.16}$$

In addition, for each $p \in S$ suppose that $\lim_{n\to\infty}\wp(p, p_n)$ and $\lim_{n\to\infty}\ell(p_n, p)$ exist and are finite. Then $\mathcal{O}$ have a unique common fixed point.

Theorem 2.2. *Let $(S, d_{\mathbb{C}})$ be a complete complex double controlled metric type space and let the mappings $\mathcal{O}, \mathcal{H} : S \to S$ satisfy*

$$d_{\mathbb{C}}(\mathcal{O}p,\mathcal{H}q) \preceq (\Theta(p)-\Theta(\mathcal{O}p))\yen(p,q), \tag{2.17}$$

where, $\yen(p,q) = \max\left\{d_{\mathbb{C}}(p,q), d_{\mathbb{C}}(p,\mathcal{O}p), d_{\mathbb{C}}(q,\mathcal{H}q), \frac{d_{\mathbb{C}}(p,\mathcal{O}p)+d_{\mathbb{C}}(q,\mathcal{H}q)}{2}\right\}$ and $\Theta : S \to \mathbb{R}$ is bounded from below $\{\inf\Theta(S)\} > -\infty$. For $p_0 \in S$, choose $p_n = \mathcal{H}^n p_0$.

Assume that $\sup_{m\geq 1}\lim_{i\to\infty}\frac{\wp(p_{i+1},p_{i+2})}{\wp(p_i,p_{i+1})}\ell(p_{i+1},p_m) < \frac{1}{\hbar}$; where $\hbar \in (0, 1)$.

In addition for each $p \in S$, suppose that $\lim_{n\to\infty}\wp(p, p_n), \lim_{n\to\infty}\ell(p_n, p)$ exist and are finite. Then $\mathcal{O}, \mathcal{H}$ have a unique common fixed point.

Proof. Let p_0 be an arbitrary point in S. Consider the sequence $\{p_n = \mathcal{H}^n p_0\}$ in S that satisfies the hypothesis of the theorem and define

$p_{2n+1} = \mathcal{O}p_{2n}$.
$p_{2n+2} = \mathcal{H}p_{2n+1}$; $n = 0, 1, 2, 3 \ldots$

Consider,

$$\begin{aligned}d_{\mathbb{C}}(p_{2n+1},p_{2n+2}) &= d_{\mathbb{C}}(\mathcal{O}p_{2n},\mathcal{H}p_{2n+1})\\ &\preceq (\Theta(p_{2n})-\Theta(\mathcal{O}p_{2n}))\yen(p_{2n},p_{2n+1}),\end{aligned} \tag{2.18}$$

where,

$$\begin{aligned}
¥(p_{2n}, p_{2n+1}) &= \max\left\{d_{\mathbb{C}}(p_{2n}, p_{2n+1}), d_{\mathbb{C}}(p_{2n}, \mathcal{O}p_{2n}), d_{\mathbb{C}}(p_{2n+1}, \mathcal{H}p_{2n+1}), \frac{d_{\mathbb{C}}(p_{2n}, \mathcal{O}p_{2n}) + d_{\mathbb{C}}(p_{2n+1}, \mathcal{H}p_{2n+1})}{2}\right\} \\
&= \max\left\{d_{\mathbb{C}}(p_{2n}, p_{2n+1}), d_{\mathbb{C}}(p_{2n}, p_{2n+1}), d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), \frac{d_{\mathbb{C}}(p_{2n}, p_{2n+1}) + d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})}{2}\right\} \\
&= \max\{d_{\mathbb{C}}(p_{2n}, p_{2n+1}), d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})\}. \qquad (2.19)
\end{aligned}$$

If $¥(p_{2n}, p_{2n+1}) = d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$, from (2.18),

$d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}) \preceq (\Theta(p_{2n}) - \Theta(\mathcal{O}p_{2n}))d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$, a contradiction.

Hence $¥(p_{2n}, p_{2n+1}) = d_{\mathbb{C}}(p_{2n}, p_{2n+1})$.
From (2.18),

$$\begin{aligned}
d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}) &\preceq (\Theta(p_{2n}) - \Theta(\mathcal{O}p_{2n}))d_{\mathbb{C}}(p_{2n}, p_{2n+1}) \\
&\preceq (\Theta(p_{2n}) - \Theta(p_{2n+1}))d_{\mathbb{C}}(p_{2n}, p_{2n+1}). \qquad (2.20)
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_{\mathbb{C}}(p_{2n+2}, p_{2n+3}) &= d_{\mathbb{C}}(\mathcal{O}p_{2n+1}, \mathcal{H}p_{2n+2}) \\
&\preceq (\Theta(p_{2n+1}) - \Theta(\mathcal{O}p_{2n+1}))¥(p_{2n+1}, p_{2n+2}), \qquad (2.21)
\end{aligned}$$

where,

$$\begin{aligned}
¥(p_{2n+1}, p_{2n+2}) &= \max\left\{d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), d_{\mathbb{C}}(p_{2n+1}, \mathcal{O}p_{2n+1}), d_{\mathbb{C}}(p_{2n+2}, \mathcal{H}p_{2n+2}), \frac{d_{\mathbb{C}}(p_{2n+1}, \mathcal{O}p_{2n+1}) + d_{\mathbb{C}}(p_{2n+2}, \mathcal{H}p_{2n+2})}{2}\right\} \\
&= \max\left\{d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), d_{\mathbb{C}}(p_{2n+2}, p_{2n+3}), \frac{d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}) + d_{\mathbb{C}}(p_{2n+2}, p_{2n+3})}{2}\right\} \\
&= \max\{d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), d_{\mathbb{C}}(p_{2n+2}, p_{2n+3})\}. \qquad (2.22)
\end{aligned}$$

From (2.21),

$d_{\mathbb{C}}(p_{2n+2}, p_{2n+3}) \preceq (\Theta(p_{2n+1}) - \Theta(\mathcal{O}p_{2n+2}))d_{\mathbb{C}}(p_{2n+2}, p_{2n+3})$, a contradiction.

Hence $¥(p_{2n+1}, p_{2n+2}) = d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$. From (2.21),

$d_{\mathbb{C}}(p_{2n+2}, p_{2n+3}) \preceq (\Theta(p_{2n+1}) - \Theta(\mathcal{O}p_{2n+2}))d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$.

Hence we can conclude that

$d_{\mathbb{C}}(p_n, p_{n+1}) \preceq (\Theta(p_{n-1}) - \Theta(p_n))d_{\mathbb{C}}(p_{n-1}, p_n)$; for each $n \in \mathbb{N}$.

So we have,

$$\frac{d_{\mathbb{C}}(p_n, p_{n+1})}{d_{\mathbb{C}}(p_{n-1}, p_n)} \preceq (\Theta(p_{n-1}) - \Theta(p_n)).$$

$$\Rightarrow \frac{|d_{\mathbb{C}}(p_n, p_{n+1})|}{|d_{\mathbb{C}}(p_{n-1}, p_n)|} \preceq (\Theta(p_{n-1}) - \Theta(p_n)), \text{ for each } n \in \mathbb{N}.$$

Thus the sequence $\{\Theta(p_n)\}$ is necessarily positive and non-increasing. Hence it converges to some $g \geq 0$.
On the other hand for each $n \in \mathbb{N}$, we have,

$$\begin{aligned}
\Sigma_{k=1}^{n} \frac{|d_{\mathbb{C}}(p_k, p_{k+1})|}{|d_{\mathbb{C}}(p_{k-1}, p_k)|} &\preceq \Sigma_{k=1}^{n}(\Theta(p_{k-1}) - \Theta(p_k)) \\
&= (\Theta(p_0) - \Theta(p_1)) + (\Theta(p_1) - \Theta(p_2)) + \ldots + (\Theta(p_{n-1}) - \Theta(p_n)) \\
&= (\Theta(p_0) - \Theta(p_n)) \\
&\to \Theta(p_0) - g < \infty \text{ as } n \to \infty, \qquad (2.23)
\end{aligned}$$

which yields, $\Sigma_{k=1}^{n} \frac{|d_{\mathbb{C}}(p_k, p_{k+1})|}{|d_{\mathbb{C}}(p_{k-1}, p_k)|} < \infty$.

Accordingly, we have,

$$\lim_{k\to\infty} \frac{|d_{\mathbb{C}}(p_k, p_{k+1})|}{|d_{\mathbb{C}}(p_{k-1}, p_k)|} = 0. \qquad (2.24)$$

From (2.24), there exist $\rho \in (0, 1)$ such that

$$\frac{|d_{\mathbb{C}}(p_k, p_{k+1})|}{|d_{\mathbb{C}}(p_{k-1}, p_k)|} \preceq \rho \text{ for all } k \geq k_0. \qquad (2.25)$$

It yields that,

$$|d_{\mathbb{C}}(p_k, p_{k+1})| \preceq \rho|d_{\mathbb{C}}(p_{k-1}, p_k)| \text{ for all } k \geq k_0. \qquad (2.26)$$

By following the same pattern as followed in above theorem, we obtain that the sequence $\{p_n\}$ converges to some $a \in S$.

$$i.e., \lim_{n\to\infty} |d_{\mathbb{C}}(p_n, a)| = 0. \tag{2.27}$$

Consider,

$$|d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})| \preceq \wp(p_{2n+1}, a)|d_{\mathbb{C}}(p_{2n+1}, a)| + \ell(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})|.$$

Letting $n \to \infty$, we get

$$|d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})| \to 0. \tag{2.28}$$

We claim that $a = \mathcal{O}a$. Suppose $a \neq \mathcal{O}a$, *i.e.*, $0 \prec d_{\mathbb{C}}(a, \mathcal{O}a)$.
Consider,

$$\begin{aligned} |d_{\mathbb{C}}(a, \mathcal{O}a)| &\preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)|d_{\mathbb{C}}(p_{2n+2}, \mathcal{O}a)| \\ &\preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)|d_{\mathbb{C}}(\mathcal{O}a, \mathcal{H}p_{2n+1})| \\ &\preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)(\Theta(a) - \Theta(\mathcal{O}a))¥(a, p_{2n+1}), \end{aligned} \tag{2.29}$$

where,

$$\begin{aligned} ¥(a, p_{2n+1}) &= \max\left\{d_{\mathbb{C}}(a, p_{2n+1}), d_{\mathbb{C}}(a, \mathcal{O}a), d_{\mathbb{C}}(p_{2n+1}, \mathcal{H}p_{2n+1}), \frac{d_{\mathbb{C}}(a, \mathcal{O}a) + d_{\mathbb{C}}(p_{2n+1}, \mathcal{H}p_{2n+1})}{2}\right\} \\ &= \max\left\{d_{\mathbb{C}}(a, p_{2n+1}), d_{\mathbb{C}}(a, \mathcal{O}a), d_{\mathbb{C}}(p_{2n+1}, p_{2n+2}), \frac{d_{\mathbb{C}}(a, \mathcal{O}a) + d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})}{2}\right\} \\ &= \max\{d_{\mathbb{C}}(a, p_{2n+1}), d_{\mathbb{C}}(a, \mathcal{O}a), d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})\}. \end{aligned}$$

If $¥(a, p_{2n+1}) = d_{\mathbb{C}}(a, \mathcal{O}a)$, from (2.29) we have

$$|d_{\mathbb{C}}(a, \mathcal{O}a)| \preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)(\Theta(a) - \Theta(\mathcal{O}a))|d_{\mathbb{C}}(a, \mathcal{O}a)|.$$

i.e., $|d_{\mathbb{C}}(a, \mathcal{O}a)| = 0$, a contradiction and hence $a = \mathcal{O}a$.
If $¥(a, p_{2n+1}) = d_{\mathbb{C}}(a, p_{2n+1})$, from (2.29) we have

$$|d_{\mathbb{C}}(a, \mathcal{O}a)| \preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)(\Theta(a) - \Theta(\mathcal{O}a))|d_{\mathbb{C}}(a, p_{2n+1})|.$$

Taking limit as $n \to \infty$, we get $|d_{\mathbb{C}}(a, \mathcal{O}a)| = 0$, a contradiction. Hence $a = \mathcal{O}a$.
If $¥(a, p_{2n+1}) = d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})$, from (2.29) we have

$$|d_{\mathbb{C}}(a, \mathcal{O}a)| \preceq \wp(a, p_{2n+2})|d_{\mathbb{C}}(a, p_{2n+2})| + \ell(p_{2n+2}, \mathcal{O}a)(\Theta(a) - \Theta(\mathcal{O}a))|d_{\mathbb{C}}(p_{2n+1}, p_{2n+2})|.$$

Letting $n \to \infty$ and using (2.28), we get $|d_{\mathbb{C}}(a, \mathcal{O}a)| = 0$, a contradiction. Hence $a = \mathcal{O}a$.
Similarly, we can prove $a = \mathcal{H}a$. Now we will prove that $\mathcal{O}$ and $\mathcal{H}$ have a unique common fixed point. Let a and a^* are two common fixed points of $\mathcal{O}$ and $\mathcal{H}$. So that $a \neq a^*$, *i.e.*, $0 \prec d_{\mathbb{C}}(a, a^*)$.
Consider,

$$\begin{aligned} d_{\mathbb{C}}(a, a^*) &= d_{\mathbb{C}}(\mathcal{O}a, \mathcal{H}a^*) \\ &\preceq (\Theta(a) - \Theta(\mathcal{O}a))¥(a, a^*), \end{aligned} \tag{2.30}$$

where,

$$\begin{aligned} ¥(a, a^*) &= \max\left\{d_{\mathbb{C}}(a, a^*), d_{\mathbb{C}}(a, \mathcal{O}a), d_{\mathbb{C}}(a^*, \mathcal{H}a^*), \frac{d_{\mathbb{C}}(a, \mathcal{O}a) + d_{\mathbb{C}}(a^*, \mathcal{H}a^*)}{2}\right\} \\ &= \max\left\{d_{\mathbb{C}}(a, a^*), d_{\mathbb{C}}(a, a), d_{\mathbb{C}}(a^*, a^*), \frac{d_{\mathbb{C}}(a, a) + d_{\mathbb{C}}(a^*, a^*)}{2}\right\} \\ &= d_{\mathbb{C}}(a, a^*). \end{aligned}$$

From (2.30),

$$d_{\mathbb{C}}(a, a^*) \preceq (\Theta(a) - \Theta(\mathcal{O}a))|d_{\mathbb{C}}(a, a^*)|, \text{ a contradiction.}$$

Hence $|d_{\mathbb{C}}(a, a^*)| = 0$, *i.e.*, $a = a^*$.
Thus, $\mathcal{O}$ and $\mathcal{H}$ have a unique common fixed point.

Example 2.2. Let $S = \{1, 2, 3\}$ and define $d_{\mathbb{C}} : S \times S \to \mathbb{C}$ by

$$d_{\mathbb{C}}(1, 2) = d_{\mathbb{C}}(2, 1) = 2 + 4i;\ \ d_{\mathbb{C}}(2, 3) = d_{\mathbb{C}}(3, 2) = i;$$

$$d_{\mathbb{C}}(1, 3) = d_{\mathbb{C}}(3, 1) = 1 - i;\ \ d_{\mathbb{C}}(1, 1) = d_{\mathbb{C}}(2, 2) = d_{\mathbb{C}}(3, 3) = 0$$

and define $\wp$, ℓ: $S \times S \to [1, \infty)$ by

$$\wp(1, 2) = \wp(2, 1) = 1;\ \ \wp(2, 3) = \wp(3, 2) = \frac{8}{7};\ \ \wp(3, 1) = \wp(1, 3) = \frac{3}{2}$$

$\ell(1,2) = \ell(2,1) = \frac{7}{6};\ \ \ell(2,3) = \ell(3,2) = \frac{9}{2};\ \ \ell(3,1) = \ell(1,3) = 1.$

Then $(S, d_{\mathbb{C}})$ is a complete complex double controlled metric type space.

Define the mapping $\mathcal{O}, \mathcal{H} : S \to S$ by $\mathcal{O}1 = \mathcal{O}3 = 1;\ \ \mathcal{O}2 = 3$ and $\mathcal{H}1 = \mathcal{H}2 = 1;\ \ \mathcal{H}3 = 2$ respectively.
Define Θ: $S \to [0, \infty)$ as $\Theta(1) = 4; \Theta(2) = 2; \Theta(3) = 1$. Thus for all $p(\neq 1) \in S$, we have,
Case. 1: If $p = 2, q = 3$.
$d_{\mathbb{C}}(\mathcal{O}2, \mathcal{H}3) = d_{\mathbb{C}}(3, 2) = i$
Now consider,

$$\begin{aligned}
(\Theta(2) - \Theta(\mathcal{O}2))¥(2,3) &= (2-1)\max\left\{d_{\mathbb{C}}(2,3), d_{\mathbb{C}}(2,\mathcal{O}2), d_{\mathbb{C}}(3,\mathcal{H}3), \frac{d_{\mathbb{C}}(2,\mathcal{O}2) + d_{\mathbb{C}}(3,\mathcal{H}3)}{2}\right\} \\
&= \max\left\{d_{\mathbb{C}}(2,3), d_{\mathbb{C}}(2,3), d_{\mathbb{C}}(3,2), \frac{d_{\mathbb{C}}(2,3) + d_{\mathbb{C}}(3,2)}{2}\right\} \\
&= d_{\mathbb{C}}(2,3) \\
&= i.
\end{aligned}$$

It is easily seen that $d_{\mathbb{C}}(\mathcal{O}2, \mathcal{H}3) \preceq (\Theta(2) - \Theta(\mathcal{O}2))¥(2,3)$.
Case. 2: If $p = 2, q = 1$.
$d_{\mathbb{C}}(\mathcal{O}2, \mathcal{H}1) = d_{\mathbb{C}}(3, 1) = 1 - i$
Now consider,

$$\begin{aligned}
(\Theta(2) - \Theta(\mathcal{O}2))¥(2,1) &= (2-1)\max\left\{d_{\mathbb{C}}(2,1), d_{\mathbb{C}}(2,\mathcal{O}2), d_{\mathbb{C}}(1,\mathcal{H}1), \frac{d_{\mathbb{C}}(2,\mathcal{O}2) + d_{\mathbb{C}}(1,\mathcal{H}1)}{2}\right\} \\
&= \max\left\{d_{\mathbb{C}}(2,1), d_{\mathbb{C}}(2,3), d_{\mathbb{C}}(1,1), \frac{d_{\mathbb{C}}(2,3) + d_{\mathbb{C}}(1,1)}{2}\right\} \\
&= \max\left\{2 + 4i, i, \frac{i}{2}\right\} \\
&= 2 + 4i.
\end{aligned}$$

Thus $d_{\mathbb{C}}(\mathcal{O}2, \mathcal{H}1) \preceq (\Theta(2) - \Theta(\mathcal{O}2))¥(2,1)$.
Case. 3: If $p = 3, q = 2$.
$d_{\mathbb{C}}(\mathcal{O}3, \mathcal{H}2) = d_{\mathbb{C}}(1, 1) = 0$ and $(\Theta(3) - \Theta(\mathcal{O}3))¥(3,2) = 0$.
Thus, $d_{\mathbb{C}}(\mathcal{O}3, \mathcal{H}2) \preceq (\Theta(3) - \Theta(\mathcal{O}3))¥(3,2)$.
Case. 4: If $p = 3, q = 1$.
$d_{\mathbb{C}}(\mathcal{O}3, \mathcal{H}1) = d_{\mathbb{C}}(1, 1) = 0$ and $(\Theta(3) - \Theta(\mathcal{O}3))¥(3,1) = 0$.
Thus, $d_{\mathbb{C}}(\mathcal{O}3, \mathcal{H}1) \preceq (\Theta(3) - \Theta(\mathcal{O}3))¥(3,1)$.
Hence the conditions managed by the Theorem 2.2 are fulfilled, thereby we can conclude that $\mathcal{O}$ has a common fixed point. Moreover it is unique.
i.e., 1 is a unique common fixed point of $\mathcal{O}$ and $\mathcal{H}$

3. Existence and unique solution of Reimann-Lioville fractional integrals

Let $S = \mathcal{C}([0,1], \mathbb{R})$ be the set of all continuous functions from [0,1] into $\mathbb{R}$ and $d_{\mathbb{C}} : S \times S \to \mathbb{C}$ is defined as follows:

$$d_{\mathbb{C}}(p, q) = |p(\kappa) - q(\kappa)| + i|p(\kappa) - q(\kappa)|,$$

for all $p(\kappa), q(\kappa) \in S$ and $\kappa \in [0,1]$. And also define $\wp, \ell$: $S \times S \to [1, \infty)$ by $\wp(p(\kappa), q(\kappa)) = 2p(\kappa) + 3q(\kappa) + 5$ and $\ell(p(\kappa), q(\kappa)) = p(\kappa) + 4q(\kappa) + 2$ respectively. Then $(S, d_{\mathbb{C}})$ is complete complex valued double controlled metric type space.
In this section we will investigate the existence and unique solution of Reimann-Lioville (or RL).

$${}^{RL}_{c}\mathcal{I}^{\upsilon}_{\kappa} p(\kappa) = \frac{1}{\Gamma(\upsilon)} \int_{c}^{\kappa} (\kappa - \varsigma)^{\upsilon - 1} p(\varsigma) d\varsigma;\ \ \Re(\upsilon) > 0, \tag{3.31}$$

where $\upsilon \in \mathbb{C}$, $p(\kappa) \in S$ and $\kappa, \varsigma \in [0,1]$ which is the fractional integral. Define the operator $\mathcal{O} : S \to S$ by

$$\mathcal{O}p(\kappa) = \frac{1}{\Gamma(\upsilon)} \int_{c}^{\kappa} (\kappa - \varsigma)^{\upsilon - 1} p(\varsigma) d\varsigma. \tag{3.32}$$

Now we will prove that Reimann-Lioville fractional integral equation has a unique solution if the following condition hold.

$$\frac{1}{\Gamma(\upsilon + 1)} \frac{(\kappa - \varsigma)^{\upsilon - 1}(\kappa - c)^{\upsilon}}{|(\kappa - \varsigma)^{\upsilon - 1}|} < \sigma;\quad \text{where}\ \ 0 < \sigma < 1.$$

Consider,

$$|\mathcal{O}p(\kappa) - \mathcal{O}q(\kappa)| + i|\mathcal{O}p(\kappa) - \mathcal{O}q(\kappa)|$$

$$
\begin{aligned}
&= \left|\frac{1}{\Gamma(\upsilon)}\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}p(\varsigma)d\varsigma - \frac{1}{\Gamma(\upsilon)}\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}q(\varsigma)d\varsigma\right| \\
&\quad + i\left|\frac{1}{\Gamma(\upsilon)}\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}p(\varsigma)d\varsigma - \frac{1}{\Gamma(\upsilon)}\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}q(\varsigma)d\varsigma\right| \\
&\preceq \frac{1}{\Gamma(\upsilon)}\left|\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}d\varsigma\right|(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq \frac{1}{\Gamma(\upsilon)}\int_c^{\kappa}|(\kappa-\varsigma)^{\upsilon-1}|d\varsigma(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq \frac{1}{\Gamma(\upsilon)}\frac{(\kappa-\varsigma)^{\upsilon-1}}{|(\kappa-\varsigma)^{\upsilon-1}|}\int_c^{\kappa}(\kappa-\varsigma)^{\upsilon-1}d\varsigma(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\quad \text{put } z=\kappa-\varsigma \\
&\preceq -\frac{1}{\Gamma(\upsilon)}\frac{(\kappa-\varsigma)^{\upsilon-1}}{|(\kappa-\varsigma)^{\upsilon-1}|}\int_{\kappa-c}^{0}(z)^{\upsilon-1}dz(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq -\frac{1}{\Gamma(\upsilon)}\frac{(\kappa-\varsigma)^{\upsilon-1}}{|(\kappa-\varsigma)^{\upsilon-1}|}\left[\frac{z^{\upsilon}}{\upsilon}\right]_{\kappa-c}^{0}(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq \frac{1}{\Gamma(\upsilon)}\frac{(\kappa-\varsigma)^{\upsilon-1}}{|(\kappa-\varsigma)^{\upsilon-1}|}\frac{(\kappa-c)^{\upsilon}}{\upsilon}(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq \frac{1}{\Gamma(\upsilon+1)}\frac{(\kappa-\varsigma)^{\upsilon-1}(\kappa-c)^{\upsilon}}{|(\kappa-\varsigma)^{\upsilon-1}|}(|p(\kappa)-q(\kappa)|+i|p(\kappa)-q(\kappa)|) \\
&\preceq \frac{1}{\Gamma(\upsilon+1)}\frac{(\kappa-\varsigma)^{\upsilon-1}(\kappa-c)^{\upsilon}}{|(\kappa-\varsigma)^{\upsilon-1}|}d_{\mathbb{C}}(p(\kappa),q(\kappa)).
\end{aligned}
\tag{3.33}
$$

Thus,

$$d_{\mathbb{C}}(\mathcal{O}p(\kappa),\mathcal{O}q(\kappa)) \preceq \frac{1}{\Gamma(\upsilon+1)}\frac{(\kappa-\varsigma)^{\upsilon-1}(\kappa-c)^{\upsilon}}{|(\kappa-\varsigma)^{\upsilon-1}|}d_{\mathbb{C}}(p(\kappa),q(\kappa)).$$

This implies, $d_{\mathbb{C}}(\mathcal{O}p,\mathcal{O}q) \preceq \sigma d_{\mathbb{C}}(p,q)$.

Thus all the conditions of the Corollary 2.2 satisfied. Hence $\mathcal{O}$ has a unique fixed point which yields that Reimann-Lioville fractional integral equation has a unique solution.

4. Existence and unique solution of complex valued atangana-Baleanu fractional integral operator

Let S be the set of all complex analytic functions and $d_{\mathbb{C}}: S\times S\to\mathbb{C}$ is defined as follows:

$$d_{\mathbb{C}}(p,q) = |p(\kappa)-q(\kappa)| + i|p(\kappa)-q(\kappa)|,$$

for all $p(\kappa), q(\kappa)\in S$. And also define $\wp, \ell: S\times S\to[1,\infty)$ by $\wp(p(\kappa),q(\kappa)) = 2p(\kappa)+3q(\kappa)+5$ and $\ell(p(\kappa),q(\kappa)) = p(\kappa)+4q(\kappa)+2$ respectively. Then $(S,d_{\mathbb{C}})$ is complete complex valued double controlled metric type space.

Let c be a fixed complex number and p be a complex function which is analytic on an open star-domain R centered at c. The complex valued Atangana-Baleanu integral ${}_{c}^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\upsilon}p(\kappa)$ is defined for any $\upsilon\in\mathbb{C}$ and $\kappa\in\mathbb{R}\setminus\{c\}$ by

$${}_{c}^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\upsilon}p(\kappa) = \frac{1-\upsilon}{\mathfrak{B}(\upsilon)}p(\kappa) + \frac{\upsilon}{\mathfrak{B}(\upsilon)}\,{}_{c}^{\mathcal{RL}}\mathcal{I}_{\kappa}^{\upsilon}p(\kappa). \tag{4.34}$$

Very recently, Arran Fernandez [41] has introduced complex valued form of Atangana-Baleanu integral as below:

$${}_{c}^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\upsilon}p(\kappa) = \frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa} + \frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)p(\varsigma)d\kappa. \tag{4.35}$$

where, $\kappa,\varsigma\in\mathbb{R}\setminus\{c\};\ \upsilon\in\mathbb{C}\setminus\mathbb{N}$ and $\mathbb{H}$ is the Hankel contour as defined in [29].

Define the operator $\mathcal{O}_{\mathcal{AB}}: S\to S$ by,

$$\mathcal{O}_{\mathcal{AB}}\ p(\kappa) = \frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa} + \frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)p(\varsigma)d\kappa. \tag{4.36}$$

Now we will prove that the complex valued Atangana-Baleanu integral has a unique solution if the following condition hold.

$$\frac{1}{2\pi\mathfrak{B}(\upsilon)}\left[\frac{|\upsilon-1|(\kappa-\varsigma)\ln(|\kappa-\varsigma|)}{|\kappa-\varsigma|} - \Gamma(1-\upsilon)(\varsigma-\kappa)^{\upsilon}\right] < e;\ \text{ where } 0<e<1.$$

Consider,

$$
\begin{aligned}
&|\mathcal{O}_{\mathcal{AB}}p(\kappa)-\mathcal{O}_{\mathcal{AB}}q(\kappa)| + i|\mathcal{O}_{\mathcal{AB}}p(\kappa)-\mathcal{O}_{\mathcal{AB}}q(\kappa)| \\
&= \left|\frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)p(\varsigma)d\kappa - \frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)q(\varsigma)d\kappa\right| \\
&\quad + i\left(\left|\frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)p(\varsigma)d\kappa - \frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)q(\varsigma)d\kappa\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\left(\frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)d\kappa\right)(p(\varsigma)-q(\varsigma))\right| \\
&\qquad + i\left|\left(\frac{1}{2\pi i\mathfrak{B}(\upsilon)}\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)d\kappa\right)(p(\varsigma)-q(\varsigma))\right| \\
&\preceq \frac{1}{|2\pi i\mathfrak{B}(\upsilon)|}\left|\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)d\kappa\right||p(\varsigma)-q(\varsigma)| \\
&\qquad + i\left(\frac{1}{|2\pi i\mathfrak{B}(\upsilon)|}\left|\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)d\kappa\right||p(\varsigma)-q(\varsigma)|\right) \\
&\preceq \frac{1}{|2\pi i\mathfrak{B}(\upsilon)|}\left|\int_{\mathbb{H}}\left(\frac{1-\upsilon}{\varsigma-\kappa}+\frac{\upsilon\Gamma(1-\upsilon)}{(\varsigma-\kappa)^{1-\upsilon}}\right)d\kappa\right|[|p(\varsigma)-q(\varsigma)|+i(|p(\varsigma)-q(\varsigma)|)] \\
&\preceq \frac{1}{2\pi\mathfrak{B}(\upsilon)}\left[\frac{|\upsilon-1|(\kappa-\varsigma)\ln(|\kappa-\varsigma|)}{|\kappa-\varsigma|}-\Gamma(1-\upsilon)(\varsigma-\kappa)^{\upsilon}\right]d_{\mathbb{C}}(p,q) \\
&\preceq e d_{\mathbb{C}}(p(\kappa),q(\kappa)).
\end{aligned}
\tag{4.37}
$$

This means, $d_{\mathbb{C}}(\mathcal{O}_{\mathcal{AB}}p, \mathcal{O}_{\mathcal{AB}}q) \preceq e d_{\mathbb{C}}(p, q)$. Thus all the conditions of Corollary 2.2 satisfied. Hence $\mathcal{O}_{\mathcal{AB}}$ has a unique fixed point, which concludes that complex valued Atangana-Baleanu fractional integral has a unique solution.

5. Solution to the Telegraph equation via fixed point approach

Thompson's mathematical phenomena for the signal conduction via cables was concerning to Fourier's equation for heat conduction in a wire. In telegraph's equation characterizing the variation of voltage $\mathfrak{V}$ along an electrical cable as function of time and position.

$$\mathfrak{V}_{zz}+(\theta+\gamma)\mathfrak{V}_z+\theta\gamma\mathfrak{V}=\mathcal{C}^2\mathfrak{V}_{pp}, \tag{5.38}$$

where $\mathcal{C}^2=\frac{1}{\mathcal{LC}}$, $\theta=\frac{\mathfrak{G}}{\mathcal{C}}$, $\gamma=\frac{\mathcal{R}}{\mathcal{L}}$ which consists of a resistor of resistance $\mathcal{R}$, a coil of inductance $\mathcal{L}$, a resistor of conductance $\mathfrak{G}$, or a capasitor of capacitance $\mathcal{C}$.

In 1893, the physicist oliver Heavyside showed that if $\frac{\mathcal{R}}{\mathcal{L}}$ could be made equal to $\frac{\mathfrak{G}}{\mathcal{C}}$ (*or* $\mathcal{RC}=\mathfrak{G}\mathcal{L}$) a constant velocity of propagation would result and the attenuation would be minimized.

This equation is a special case of nonlinear Cauchy problem as follow.

$$\frac{\partial^2\mathfrak{V}}{\partial z^2}+\frac{\partial}{\partial z}(g(p,z,\mathfrak{V}))=\frac{\partial^2\mathfrak{V}}{\partial p^2}+\frac{\partial}{\partial p}(g(p,z,\mathfrak{V}))+\Upsilon(p,z,\mathfrak{V}),$$

which, $(p,z)\in\varsigma:=\{(p,z): p+z\geq 0,\ \ p-z\geq 0\}$. This Eq. (5.38) is equivalent to the following 2D Volterra integral equation as,

$$\mathfrak{V}(S,\mathcal{O})=\int_0^{\mathcal{O}}\int_0^{S}\mathcal{K}(S',z',\mathfrak{V}(S',z'))dS'dz'+\int_0^{\mathcal{O}}\mathcal{N}(S',z',\mathfrak{V}(S',z'))dz'+\int_0^{S}\mathcal{M}(S',z',\mathfrak{V}(S',z'))dS'+\mathcal{R}(S,\mathcal{O}), \tag{5.39}$$

where $p'=p+z$ and $z'=p-z$.

Common form of 2D Volterra integral equation can be written as

$$\mathfrak{V}(\alpha,\beta)=f(\alpha,\beta)+\int_0^{\beta}\int_0^{\alpha}\Lambda_1(p,q,\mathfrak{V}(p,q))dpdq+\varrho\int_0^{\beta}\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))dq+\vartheta\int_0^{\alpha}\Lambda_3(\beta,p,\mathfrak{V}(p,\beta))dp, \tag{5.40}$$

where $\alpha,\beta,p,q\in[0,1]$, $\mathfrak{V}\in S=C([0,1]\times[0,1])$ and $\varrho,\vartheta\in R$ and $f:[0,1]\times[0,1]\to\mathbb{R}^2$; $\Lambda_1,\Lambda_2,\Lambda_3:[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$.

Let $S=C([0,1]^2,\mathbb{C})$ be the set of all continuous functions from $[0,1]^2$ into $\mathbb{C}$, and $d_{\mathbb{C}}: S\times S\to\mathbb{C}$ is defined as follows:

$$d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))=|\mathfrak{V}(\alpha,\beta)-\mathfrak{U}(\alpha,\beta)|+i|\mathfrak{V}(\alpha,\beta)-\mathfrak{U}(\alpha,\beta)|,$$

for all $\mathfrak{V},\mathfrak{U}\in S$. And also define ϱ,ϑ: $S\times S\to[1,\infty)$ by

$\varrho(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))=2\mathfrak{V}(\alpha,\beta)+3\mathfrak{U}(\alpha,\beta)+2$ and

$\vartheta(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))=5\mathfrak{V}(\alpha,\beta)+3\mathfrak{U}(\alpha,\beta)+2.$

Then $(S,d_{\mathbb{C}})$ is clearly $d_{\mathbb{C}}$-metric space.

Let us define the integral operator $\mathcal{O}:S\to S$ by

$$\mathcal{O}\mathfrak{V}(\alpha,\beta)=f(\alpha,\beta)+\int_0^{\beta}\int_0^{\alpha}\Lambda_1(p,q,\mathfrak{V}(p,q))dpdq+\varrho\int_0^{\beta}\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))dq+\vartheta\int_0^{\alpha}\Lambda_3(\beta,p,\mathfrak{V}(p,\beta))dp. \tag{5.41}$$

Theorem 5.1. *Presume that the below prospectives are gratified:*

1. *Let* $\Lambda_1,\Lambda_2,\Lambda_3:[0,1]\times[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$ *be continuous satisfying the Lipschitz condition that is there exists* $\mathfrak{Q}_1,\mathfrak{Q}_2$ *and* $\mathfrak{Q}_3$ *such that*

   $|\Lambda_1(p,q,\mathfrak{U}_1(p,q))-\Lambda_1(p,q,\mathfrak{U}_2(p,q))|\preceq\mathfrak{Q}_1|\mathfrak{U}_1-\mathfrak{U}_2|$

   $|\Lambda_2(p,q,\mathfrak{U}_1(p,q))-\Lambda_2(p,q,\mathfrak{U}_2(p,q))|\preceq\mathfrak{Q}_2|\mathfrak{U}_1-\mathfrak{U}_2|$

   $|\Lambda_3(p,q,\mathfrak{U}_1(p,q))-\Lambda_3(p,q,\mathfrak{U}_2(p,q))|\preceq\mathfrak{Q}_3|\mathfrak{U}_1-\mathfrak{U}_2|$

   where, $\mathfrak{U}_1,\mathfrak{U}_2\in\mathbb{R}^2$.
2. $\mathfrak{Q}_1+|\varrho|\mathfrak{Q}_2+|\vartheta|\mathfrak{Q}_3<\Upsilon$; where $\Upsilon\in(0,1)$.

Then $\mathcal{O}$ *has a unique fixed point.*

Proof. By applying (5.41) gives,

$$
\begin{aligned}
&|\mathcal{O}(\mathfrak{V}(\alpha,\beta))-\mathcal{O}(\mathfrak{U}(\alpha,\beta))|+i|\mathcal{O}(\mathfrak{V}(\alpha,\beta))-\mathcal{O}(\mathfrak{U}(\alpha,\beta))|\\
&=|f(\alpha,\beta)+\int_0^\beta\int_0^\alpha \Lambda_1(p,q,\mathfrak{V}(p,q))dpdq+\varrho\int_0^\beta \Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))dq+\vartheta\int_0^\alpha \Lambda_3(\beta,p,\mathfrak{V}(\beta,p))dp\\
&\quad-[f(\alpha,\beta)+\int_0^\beta\int_0^\alpha \Lambda_1(p,q,\mathfrak{U}(p,q))dpdq+\varrho\int_0^\beta \Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q))dq+\vartheta\int_0^\alpha \Lambda_3(\beta,p,\mathfrak{U}(\beta,p))dp]|\\
&\quad+i|f(\alpha,\beta)+\int_0^\beta\int_0^\alpha \Lambda_1(p,q,\mathfrak{V}(p,q))dpdq+\varrho\int_0^\beta \Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))dq+\vartheta\int_0^\alpha \Lambda_3(\beta,p,\mathfrak{V}(\beta,p))dp\\
&\quad-[f(\alpha,\beta)+\int_0^\beta\int_0^\alpha \Lambda_1(p,q,\mathfrak{U}(p,q))dpdq+\varrho\int_0^\beta \Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q))dq+\vartheta\int_0^\alpha \Lambda_3(\beta,p,\mathfrak{U}(\beta,p))dp]|\\
&=|\int_0^\beta\int_0^\alpha \Big(\Lambda_1(p,q,\mathfrak{V}(p,q))-\Lambda_1(p,q,\mathfrak{U}(p,q))\Big)dpdq+\varrho\int_0^\beta (\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))-\Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q)))dq\\
&\quad+\vartheta\int_0^\alpha (\Lambda_3(\beta,p,\mathfrak{V}(\beta,p))-\Lambda_3(\beta,p,\mathfrak{U}(\beta,p)))dp|+i|\int_0^\beta\int_0^\alpha \Big(\Lambda_1(p,q,\mathfrak{V}(p,q))-\Lambda_1(p,q,\mathfrak{U}(p,q))\Big)dpdq\\
&\quad+\varrho\int_0^\beta (\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))-\Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q)))dq+\vartheta\int_0^\alpha (\Lambda_3(\beta,p,\mathfrak{V}(\beta,p))-\Lambda_3(\beta,p,\mathfrak{U}(\beta,p)))dp|\\
&\preceq\int_0^\beta\int_0^\alpha \Big|\Big(\Lambda_1(p,q,\mathfrak{V}(p,q))-\Lambda_1(p,q,\mathfrak{U}(p,q))\Big)\Big|dpdq+|\varrho|\int_0^\beta |(\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))-\Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q)))|dq\\
&\quad+|\vartheta|\int_0^\alpha |(\Lambda_3(\beta,p,\mathfrak{V}(\beta,p))-\Lambda_3(\beta,p,\mathfrak{U}(\beta,p)))|dp+i\Big[\int_0^\beta\int_0^\alpha \Big|\Big(\Lambda_1(p,q,\mathfrak{V}(p,q))-\Lambda_1(p,q,\mathfrak{U}(p,q))\Big)\Big|dpdq\\
&\quad+|\varrho|\int_0^\beta |(\Lambda_2(\alpha,q,\mathfrak{V}(\alpha,q))-\Lambda_2(\alpha,q,\mathfrak{U}(\alpha,q)))|dq+|\vartheta|\int_0^\alpha |(\Lambda_3(\beta,p,\mathfrak{V}(\beta,p))-\Lambda_3(\beta,p,\mathfrak{U}(\beta,p)))|dp\Big]\\
&\preceq\int_0^\beta\int_0^\alpha \mathfrak{Q}_1|\mathfrak{V}(p,q)-\mathfrak{U}(p,q)|dpdq+|\varrho|\int_0^\beta \mathfrak{Q}_2|\mathfrak{V}(\alpha,q)-\mathfrak{U}(\alpha,q)|dq\\
&\quad+|\vartheta|\int_0^\alpha \mathfrak{Q}_3|\mathfrak{V}(p,\beta)-\mathfrak{U}(p,\beta)|dp+i\Big[\int_0^\beta\int_0^\alpha \mathfrak{Q}_1|\mathfrak{V}(p,q)-\mathfrak{U}(p,q)|dpdq\\
&\quad+|\varrho|\int_0^\beta \mathfrak{Q}_2|\mathfrak{V}(\alpha,q)-\mathfrak{U}(\alpha,q)|dq+|\vartheta|\int_0^\alpha \mathfrak{Q}_3|\mathfrak{V}(p,\beta)-\mathfrak{U}(p,\beta)|dp\Big]\\
&\preceq\mathfrak{Q}_1[|\mathfrak{V}(p,q)-\mathfrak{U}(p,q)|+i|\mathfrak{V}(p,q)-\mathfrak{U}(p,q)|]\\
&\quad+\mathfrak{Q}_2|\varrho|[|\mathfrak{V}(\alpha,q)-\mathfrak{U}(\alpha,q)|+i|\mathfrak{V}(\alpha,q)-\mathfrak{U}(\alpha,q)|]\\
&\quad+\mathfrak{Q}_3|\vartheta|[|\mathfrak{V}(p,\beta)-\mathfrak{U}(p,\beta)|+i|\mathfrak{V}(p,\beta)-\mathfrak{U}(p,\beta)|]\\
&\preceq\mathfrak{Q}_1 d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))+\mathfrak{Q}_2|\varrho|d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))+\mathfrak{Q}_3|\vartheta|d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))\\
&=(\mathfrak{Q}_1+|\varrho|\mathfrak{Q}_2+|\vartheta|\mathfrak{Q}_3)d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta))\\
&<\Upsilon d_{\mathbb{C}}(\mathfrak{V}(\alpha,\beta),\mathfrak{U}(\alpha,\beta)).
\end{aligned}
$$

Hence the conditions managed by the Corollary 2.2 are fulfilled, thereby we can conclude that $\mathcal{O}$ has a fixed point which means that (5.40) has a unique solution.

6. Conclusion

Complex valued versions of Riemann-Liouville integral, Atangana-Baleanu integral operator and non-linear Telegraph equation are introduced. Moreover, we introduced complex valued double controlled metric type space (in short, $d_{\mathbb{C}}$-metric space). Thereafter, by making consequent use of the fixed point method, we proposed short and simple proofs are obtained for solutions of Riemann-Liouville integral, complex valued Atangana-Baleanu integral operator and non-linear Telegraph equation.

A few words about possible extensions of the preceding conclusions:

- Fixed point method for solutions of time fractional telegraph equation in complex valued double controlled metric type space.
- Solutions for Atangana-Baleanu integral operator via collocation-type method in complex valued double controlled metric type space.
- Solutions for Riemann-Liouville integral equations via Guo-Gupta-Suzuki-Ćirić type results in complex valued double controlled metric type space.

Declaration of Competing Interest

The authors declare that they have no competing interests to influence the work reported in this paper.

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