FURTHER REVIEW PROBLEMS FOR THE FINAL EXAM

1. Consider the initial-value problem

$$y'' - 4y' + 3y = e^t,\ y(0) = 1,\ y'(0) = 2.$$

We take the Laplace transform $\mathcal{L}y = Y$.

Then

$$\mathcal{L}y' = s\mathcal{L}y - y(0) = sY - 1,$$
$$\mathcal{L}y'' = s^2\mathcal{L}y - sy(0) - y'(0) = s^2Y - s - 2.$$

Then the Laplace transform of the original equation is

$$s^2Y - s - 2 - 4(sY - 1) + 3Y = \frac{1}{s-1},$$

so

$$(s^2 - 4s + 3)Y = s + 6 + \frac{1}{s-1}.$$

Thus

$$Y = \frac{s+6}{(s-1)(s-3)} + \frac{1}{(s-1)^2(s-3)}.$$

Let

$$\frac{s+6}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3}.$$

Then $A = -7/2$, $B = 9/2$.

Let

$$\frac{1}{(s-1)^2(s-3)} = \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{s-3} = \frac{C(s-1)(s-3) + D(s-3) + E(s-1)^2}{(s-1)^2(s-3)}.$$

Then $E = 1/4$, $D = -1/2$, and $C = -1/4$.

Then

$$Y = -15/4\frac{1}{s-1} + 19/4\frac{1}{s-3} + (-1/2)\frac{1}{(s-1)^2}.$$

Now we have to undo the Laplace transform, i.e. compute the inverse Laplace transform: find the function y such that $\mathcal{L}y = Y$. We write this as $y = \mathcal{L}^{-1}Y$.

We get

$$y = -15/4e^t + 19/4e^{3t} + (-1/2)te^t.$$

2. Consider the initial-value problem

$$y'' - 4y = \sin(2t),\ y(0) = 0,\ y'(0) = 0.$$

Then

$$s^2Y - 4Y = \frac{2}{s^2+4},$$

so

$$Y = \frac{2}{(s^2+4)(s^2-4)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s-2i} + \frac{D}{s+2i}.$$

Then $A = \frac{1}{16}$, $B = -\frac{1}{16}$, $C = \frac{i}{16}$, $D = -\frac{i}{16}$ and the solution is

$$y = \mathcal{L}^{-1}Y = \frac{1}{16}e^{2t} - \frac{1}{16}e^{-2t} - \frac{1}{8}\sin(2t).$$

3. Consider the system of equations

$$x' = y$$
$$y' = x.$$

The system (coefficient) matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Its eigenvalues are the roots of the characteristic polynomial $\lambda^2 + 0\lambda + 1 = 0$, where the coefficient of λ is the trace of A, given by the sum of values on the main diagonal: $0 + 0 = 0$, while the 1 in the equation is the determinant of A, $\det A = 0 \cdot 0 - 1(-1) = 1$.

The solutions of $\lambda^2 + 1 = 0$ (the eigenvalues of the matrix) are

$$\lambda_{1,2} = \pm i$$

and we next compute the eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = i$ the eigenvectors are solutions of the eigenvector equation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The system $v_2 = iv_1$, $-v_1 = iv_2$ has an infinity of solutions, among which a particularly simple one is

$$v_1 = -i,\ v_2 = 1.$$

For $\lambda_2 = -i$ the eigenvectors are solutions of the eigenvector equation

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The system $u_2 = -iu_1$, $-u_1 = -iu_2$ has an infinity of solutions, among which a particularly simple one is

$$u_1 = i,\ u_2 = 1.$$

Note that the two eigenvectors we got are conjugate to each other.

The matrix exponential is

$$e^{tA} = CDC^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -ie^{it} & ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix} \frac{1}{-2i} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$
$$= \frac{1}{-2i} \begin{pmatrix} -ie^{it} - ie^{-it} & -e^{it} + e^{-it} \\ e^{it} - e^{-it} & -ie^{it} - ie^{-it} \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

by Euler's formula.

The solution of the equation can then be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$