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Paving the Chessboard

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A recursion for determining exact numbers $\mu(m, n)$ of monomer–dimer configurations on $m \times n$ rectangular boards is established. For large m and n close approximations to $\mu(m, n)$ are obtained. The methods may be extended to the case of a given fixed number of dimers.

1. INTRODUCTION

Given a chessboard, or, more generally, an $m \times n$ rectangle of squares, and a sufficient supply of dominoes ("dimers") which can occupy two adjacent squares; in how many ways may the board be paved? If mn is even, complete coverings by $mn/2$ dominoes are possible, and the numbers of such "pure" dimer configurations have been determined by M. E. Fisher and P. W. Kastelyn (cf. Graph Theory and Crystal Physics by P. W. Kastelyn, Chap. 2 of [3]). In this article we shall be concerned with incomplete pavings or, equivalently, with covers consisting of dominoes *and* single squares ("monomers"). No formula for the numbers $\mu(m, n)$ of all possible monomer–dimer ("MD") configurations is known, and generating functions are also lacking.

Our attempt will be based on a recursive procedure which is established in Section 2 and speeded up in Section 3 such that all $\mu(m, n)$ up to $\mu(33, 16)$ could be calculated by computer. This enables us to approximate the $\mu(m, n)$ for fixed $n \leqslant 16$ in terms of dominant eigenvalues of transfer matrices (Section 4) and also to study the behavior of $\mu(m, n)$ for large (m, n) in Section 5. Finally, in Section 6, we show how the methods may be extended to the paving problem 26 in [3, Chap. 1]: determine the numbers $\mu(m, n, k)$ of MD-configurations with a fixed amount k of dimers. So far we know numerical values up to $\mu(20, 10, k)$ for all possible $k \leqslant mn/2$, but we have to leave a detailed analysis of the behavior of $\mu(m, n, k)$ to future efforts.

For the extensive literature on various aspects of the monomer–dimer

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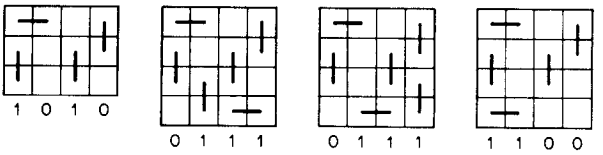

FIGURE 1

problem the reader is referred to [3, 4]; we merely quote some articles, with which there is a little overlap.

2. TRANSFER MATRICES

The monomer–dimer configurations in an $m \times n$ board may be classified according to the patterns in the bottom (m-th) row: empty (monomer) positions are translated into bits 0, whereas squares which are occupied (by half a dimer) are marked with bits 1. For instance, the bottom row of the first diagram in Fig. 1 "reads" 1010 which is the binary representation of ten.

Let v_k $(k = 0, 1, \ldots, 2^n - 1)$ be the number of different MD-configurations which belong to the binary number k in the last row. Then

$$\mu(m, n) = \sum_{0}^{N-1} v_k, \qquad N = 2^n. \tag{1}$$

By adding a further row an $(m + 1) \times n$ rectangle is obtained. Any MD-configuration of this extended board may be constructed from one of the $m \times n$ board by either leaving the new row empty or by inserting one or several non-overlapping dimers of the following two types. If position (m, j) is empty a vertical dimer on $(m, j) - (m + 1, j)$ is possible, and if $j < n$ a horizontal dimer may be placed onto $(m + 1, j) - (m + 1, j + 1)$. It is clear that a given new MD-cover stems from a unique old MD-configuration which is recovered by crossing out all dimers which lie wholly or partially in the new $(m + 1)$-th row.

For fixed n let a_{ij} be the number of ways in which an i-pattern in the m-th row can be extended such that the new $(m + 1)$-th row reads j. In Fig. 1 the first configuration $(i = 10)$ is extended twice to $j = 7$ and once to $j = 12$. No more possibilities exist, so we have $a_{10,7} = 2$ and $a_{10,12} = 1$ (for $n = 4$).

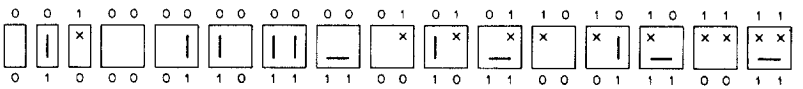

FIGURE 2

The simplest cases $n = 1$ and $n = 2$ are covered completely in Fig. 2. All possible combinations of old and new bottom rows are displayed; the crosses ($\times$) mark positions which were already occupied before the transition.

From Fig. 2 the "transfer" matrices for $n = 1$ and $n = 2$ are read off easily.

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Higher $2^n \times 2^n$ transfer matrices A_n are constructed recursively. The entries in Fig. 3 are the last bits of the subscript i (in a_{ij}) and the last two bits of j.

A new vertical dimer may be placed if $i = \cdots 0$ and $j = \cdots 1$. A new horizontal dimer requires $j = \cdot\ \cdot\ 11$. Let

$$i' = [i/2], \qquad i'' = [i/4], \qquad j' = [j/2], \qquad j'' = [j/4], \tag{3}$$

(cutting off the last bit (last two bits) from i, j yields i', j' (i'', j'')).

Now consider, for instance, the case $i = \cdots 0$, $j = \cdot\ \cdot\ 11$. A new vertical dimer may complement the $a_{i'j'}$ transitions in respect to the first $n-1$ bits (columns), and a horizontal dimer may be added to the $a_{i''j''}$ transitions in respect to the first $n-2$ bits. Hence $a_{ij} = a_{i'j'} + a_{i''j''}$ in this case. All other i,j-combinations are easier, but the reader should check the resulting complete set of identities using Fig. 3:

	$j = 4k$	$j = 4k+1$	$j = 4k+2$	$j = 4k+3$
$i = 2k$	$a_{ij} = a_{i'j'}$	$a_{ij} = a_{i'j'}$	$a_{ij} = a_{i'j'}$	$a_{ij} = a_{i'j'} + a_{i''j''}$
$i = 2k+1$	$a_{ij} = a_{i'j'}$	$a_{ij} = 0$	$a_{ij} = a_{i'j'}$	$a_{ij} = a_{i''j''}$

(4)

(If $j = 4k$ then $a_{ij} = a_{i''j''}$ is also true.) The matrices A_n—even A_1 and A_2—may now be obtained by setting $a_{00} \leftarrow 1$ and using the recursions

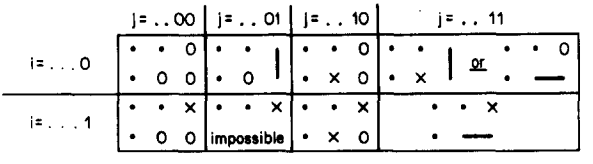

FIGURE 3

(4)—both rowwise and columnwise progressions through A_n will succeed. The reader is invited to check the case $n = 3$: note that the left half of A_n always consists of two copies of A_{n-1}.

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

It follows from (4) that in all A_n

$$a_{i0} = 1 \text{ for all } i, \qquad a_{0j} \geqslant 1 \text{ for all } j, \qquad A_n^2 > 0 \text{ for all } n. \tag{6}$$

($A_n^2 > 0$ means $\sum a_{ik} a_{kj} > 0$, but $\sum a_{ik} a_{kj} \geqslant a_{i0} a_{0j} \geqslant 1$.)

We are ready to calculate the vectors $\mathbf{v} = (v_0, v_1, \ldots, v_{N-1})$ for any given n. If $\mathbf{v}^{(m)}$ belongs to the bottom row of an $m \times n$ board, the transition to $\mathbf{v}^{(m+1)}$ is governed by A_n:

$$\mathbf{v}^{(m+1)} = \mathbf{v}^{(m)} A_n \tag{7}$$

or, if $\mathbf{v}$ were written as a column vector, $\mathbf{v}^{(m+1)} = A_n^T \mathbf{v}^{(m)}$. A starting vector $\mathbf{v}^{(1)}$ may be established from an easy analysis of $1 \times n$ boards, but it is even simpler to postulate an $0 \times n$ board which admits the empty set of dimers only. The corresponding vector $\mathbf{v}^{(0)}$ has therefore just one non-zero component $v_l^{(0)} = 1$. But l must be equal to a binary $111 \cdots 1$, that is, $l = 2^n - 1 = N - 1$, since a zero bit in position k would imply the possibility of a vertical dimer in $(0,j) - (1,j)$. Hence,

$$\mathbf{v}^{(0)} = (0, 0, \ldots, 0, 1), \qquad [v_{N-1}^{(0)} = 1,\ v_k^{(0)} = 0 \text{ if } k \neq N - 1], \tag{8}$$

and repeated applications of (7) yield

$$\mathbf{v}^{(m)} = \mathbf{v}^{(0)} A_n^m. \tag{9}$$

3. Algorithms

The first computational procedure for calculating the numbers $\mu(m, n)$ is based directly on (7) and (8). For a given n establish the matrix A_n from $a_{00} \leftarrow 1$ and the recursion (4). Then set up $\mathbf{v}^{(0)}$ (8) and multiply successively by A_n. The $\mu(m, n)$ are the sums of the components of the resulting vectors

TABLE I

	n							
m	1	2	3	4	5	6	7	8
1	1	2	3	5	8	13	21	34
2		7	22	71	228	733	2356	7573
3			131	823	5096	31687	196785	1222550
4				10012	120465	1453535	17525619	211351945
5					2810694	65805403	1539222016	36012826776
6						2989126727	135658637925	6158217253688
7							11945257052321	1052091957273408
8							Chessboard:	179788343101980135

$\mathbf{v}^{(m)}$; because of $a_{i0} = 1$ (6) they are also the first components $v_0^{(m+1)}$ of the next vector $\mathbf{v}^{(m+1)}$.

Table I contains the $\mu(m, n)$ up to the size of a chessboard. Since the procedure arrives at $\mu(m, n)$ and $\mu(n, m)$ in different ways, the self-evident symmetry $\mu(m, n) = \mu(n, m)$ provides a check on the method and on the computer program. The largest figure in Table I has more than 16 digits which is the double precision accuracy of our Siemens 7760 computer. Therefore we repeated the calculation in terms of residues mod 10^{12} so that even the last digits could be presented accurately.

Unfortunately, the (Fortran) computation times and the storage requirements were $O(4^n)$. So, although it took only 20 sec CPU-time to produce Table I, we were unable to cope with $n > 9$ on account of limited main memory. But a larger table was essential for the study of asymptotic behavior (Section 5), and a more efficient method had to be found. Our second algorithm "MD" is indeed superior in three respects:

(i) Only the vector $\mathbf{v}$ of length $N = 2^n$ is kept in memory.

(ii) The computation times are $O(n2^n)$.

(iii) Only additions (no multiplications) occur.

Algorithm MD recovered Table I in one-third of a second, and by a one-time expense of 11 min CPU-time a table of $\mu(m, n)$ for all $n \leqslant 16$ and $m \leqslant 33$ was generated.

The idea of MD is to break up the transition $m \times n \to (m + 1) \times n$ into n stages as shown in Fig. 4 for the case $n = 4$.

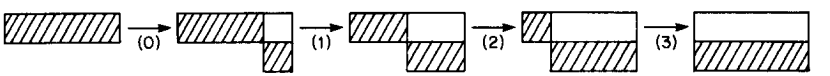

FIGURE 4

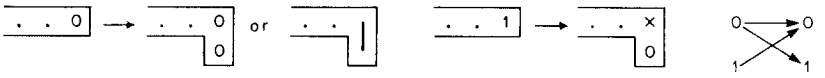

FIGURE 5

The intermediate vectors v refer to the bit combinations in the intermediate "bottom rows" which are the shaded areas in Fig. 4. Obviously, the first transition (0) is different from the rest; there we have the three possibilities in Fig. 5.

Since the first $n-1$ bits are not involved, the (0)-transition of Fig. 5 is controlled by an $N \times N$ matrix $A_n^{(0)}$ which is described by

$$a^{(0)}_{2k,2k} = a^{(0)}_{2k,2k+1} = a^{(0)}_{2k+1,2k} = 1. \qquad a^{(0)}_{ij} = 0 \text{ for all other } (i,j). \quad (10)$$

The remaining partial transfer matrices $A_n^{(r)}$ $(r = 1, 2, \ldots, n-1)$ are read off from Fig. 6.

In the rth step the bottom row changes occur only between subscripts i, j whose last $r-1$ bits and first $n-(r+1)$ bits are equal. So let

$$\begin{aligned} &i \equiv j \pmod{2^{r-1}}, \qquad \lfloor i/2^{r+1} \rfloor = \lfloor j/2^{r+1} \rfloor, \\ &I \equiv \lfloor i/2^{r-1} \rfloor \pmod 4, \; J \equiv \lfloor j/2^{r-1} \rfloor \pmod 4, \qquad 0 \leqslant I, J < 4. \end{aligned} \quad (11)$$

I and J are the two bits of i and j in the arrow diagram of Fig. 6 from which we read off the non-zero elements of the rth partial transfer matrix $A_n^{(r)}$ as follows:

$$a^{(r)}_{ij} = 1 \quad \text{if } (I,J) = (0,0), (0,2), (0,3), (1,1), (1,3), (2,0), (2,3), (3,1). \quad (12)$$

The product $\prod_0^{n-1} A_n^{(r)}$ is the complete transfer matrix A_n as exemplified below in the case $n = 2$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} = A_2. \quad (13)$$

The formal statement of the Algorithm MD (below) can be readily translated into computer languages. Since some of these do not permit zero subscripts, the v_j $(1 \leqslant j \leqslant N)$ in the algorithm correspond to the v_{j-1} $(0 \leqslant j-1 \leqslant N-1)$ in the above derivations. n is assumed to be fixed, and all $\mu(m, n)$ are returned in Step 9. Step 1 is (8), Step 3 expresses (10), and Steps 5–7 apply (11) and (12). There the counters $t = 2^{r-1}$ and $s = 2^{r+1}$ are employed. These are initialized in Step 4 for $r = 1$ and updated $(r \leftarrow r+1)$ in

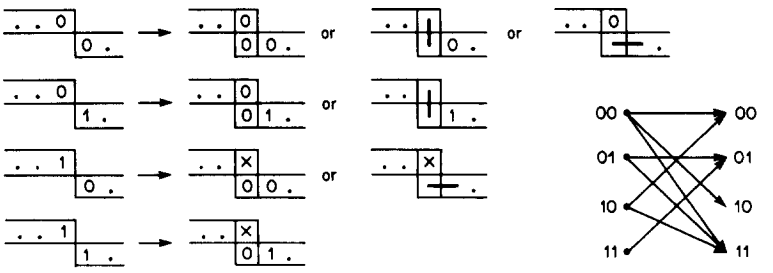

FIGURE 6

Step 8. In Step 5 the $n-(r+1)$ front bits are varied, and Step 6 prepares the progression of the $r-1$ tail bits during the $t=2^{r-1}$ performances of Step 7.

Algorithm MD (Input n, $N=2^n$ and M (maximal m).)

1. For $j \leftarrow 1, 2, \ldots, N-1$ set $v_j \leftarrow 0$. Set $v_N \leftarrow 1$.
2. For $m \leftarrow 1, 2, \ldots, M$ carry out Steps 3–9.
3. For $j \leftarrow 1, 3, 5, \ldots$ while $j \leqslant N$ set $x \leftarrow v_j$, $v_j \leftarrow x + v_{j+1}$, $v_{j+1} \leftarrow x$.
4. If $n = 1$ go to 9; otherwise initialize $t \leftarrow 1$ and $s \leftarrow 4$.
5. For $j \leftarrow 1, 1+s, 1+2s, \ldots$ while $j \leqslant N$ carry out Step 6 (incl. 7).
6. Initialize $i \leftarrow j-1$, $i' \leftarrow i+t$, $i'' \leftarrow i'+t$, $i''' \leftarrow i''+t$.

Thereafter perform Step 7 exactly t times.

7. Set $i \leftarrow i+1$, $i' \leftarrow i'+1$, $i'' \leftarrow i''+1$, $i''' \leftarrow i'''+1$, then set $x \leftarrow v_i$, $x' \leftarrow v_{i'}$, $x'' \leftarrow v_{i''}$, $x''' \leftarrow v_{i'''}$, and finally set $v_i \leftarrow x + x''$, $v_{i'} \leftarrow x' + x'''$, $v_{i''} \leftarrow x$, $v_{i'''} \leftarrow x + x' + x''$.
8. Set $t \leftarrow t+t$, $s \leftarrow s+s$, and if $s \leqslant N$ go back to Step 5.
9. Output $\mu(m, n) \leftarrow \prod_1^N v_j$.

In our Fortran version of Algorithm MD we replaced $v_N \leftarrow 1$ in Step 1 with $v_N \leftarrow 16^{-65}$ and obtained all $\mu(m, n)$ $(n \leqslant 16, m \leqslant M = 33)$ scaled by this factor, so that even $\mu(33, 16) = 0.3186472524 \times 10^{149}$ did not cause floating point overflow.

Although factorization has yielded sufficient numerical data for the study of asymptotic behavior in Section 5, we shall revert to the original transfer matrices A_n in the next section.

4. EIGENVALUES

The recursion (7) is also a system of linear difference equations with constant coefficients for the unknowns $v_k^{(m)}$. If A_n has $N=2^n$ distinct eigen-

values, solutions of the form

$$c_{n0}\lambda_{n0}^m + c_{n1}\lambda_{n1}^m + \cdots + c_{n,N-1}\lambda_{n,N-1}^m \tag{14}$$

exist for the $v_k^{(m)}$ and for their sum $\mu(m, n)$. Table II contains all c and λ for $n = 1, 2$ and 3. The λ are the zeros of the listed characteristic polynomials $P(\lambda)$ of the A_n, and the c were calculated by using N known $\mu(m, n)$ as initial conditions in (14).

Values c_{nk} and λ_{nk} from Table II inserted into (14) yield closed form representations of all $\mu(m, 1)$, $\mu(m, 2)$ and $\mu(m, 3)$. At least in these cases the eigenvalues are indeed single roots of the $P(\lambda)$. For $n = 1$ the Fibonacci numbers in the first row of Table I are confirmed. Here and also in the case $n = 2$ it is sufficient to round the first term in (14) to the nearest integer—for $n = 2$ the eigenvalue $\lambda_{21} = -1$ does not spoil this effect since $c_{21} = 0$. However, since $|\lambda_{31}| > 1$ and $c_{31} \neq 0$, the same cannot be expected if $n = 3$. Generally we can at most hope that

$$\mu(m, n) = c_n\lambda_n^m + o(\lambda_n^m), \tag{15}$$

where $c_n = c_{n0}$ and $\lambda_n = \lambda_{n0}$ is the absolute largest eigenvalue. Then $c_n\lambda_n^m$ is an approximation to $\mu(m, n)$ whose *relative* error tends to zero as $m \to \infty$.

The validity of (15) follows from theorems of Perron and Frobenius (1907): if $A \geqslant 0$—that is, $a_{ij} \geqslant 0$ for all (i,j)—and if $A^m > 0$ for some mth power of A, then A has a single real positive eigenvalue which is larger than the absolute values of all other eigenvalues of A. For proofs compare, e.g., [1, Chap. XIII, pp. 2 – 5]. Our transfer matrices A_n satisfy $A_n \geqslant 0$ and $A_n^2 > 0$ (6). Hence such "dominant" eigenvalues $\lambda_n = \lambda_{n0}$ always exist, and

TABLE II

$n = 1$: $P(\lambda) = \lambda^2 - \lambda - 1$, $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$, $c = \frac{1}{10}(5 \pm \sqrt{5})$			
$\lambda_{10} =$ 1.61803399	$c_{10} = 0.72360680$	$\lambda_{11} = -0.61803399$	$c_{11} = 0.27639320$
$n = 2$: $P(\lambda) = \lambda^4 - 2\lambda^3 - 4\lambda^2 + 1$			
$\lambda_{20} =$ 3.21431974	$c_{20} = 0.66459138$	$\lambda_{21} = -1.00000000$	$c_{21} = 0.00000000$
$\lambda_{22} = -0.67513087$	$c_{22} = 0.25597190$	$\lambda_{23} =$ 0.46081113	$c_{23} = 0.07943672$
$n = 3$: $P(\lambda) = \lambda^8 - 3\lambda^7 - 19\lambda^6 - 10\lambda^5 + 24\lambda^4 + 10\lambda^3 - 11\lambda^2 - \lambda + 1$			
$\lambda_{30} =$ 6.21207025	$c_{30} = 0.55127307$	$\lambda_{31} = -1.91893741$	$c_{31} = 0.13648023$
$\lambda_{32} = -1.61803399$	$c_{32} = 0.00000000$	$\lambda_{33} = -1.00000000$	$c_{33} = 0.20000000$
$\lambda_{34} =$ 0.69820434	$c_{34} = 0.02795346$	$\lambda_{35} =$ 0.61803399	$c_{35} = 0.00000000$
$\lambda_{36} =$ 0.35098368	$c_{36} = 0.06734138$	$\lambda_{37} = -0.34232086$	$c_{37} = 0.01695185$

TABLE III

n	λ_n	c_n	n	λ_n	c_n
1	1.61803398875	0.723606797750	9	331.538472708	0.210456309951
2	3.21431974338	0.664591384526	10	643.256037661	0.179080547672
3	6.21207025015	0.551273069782	11	1248.05523827	0.152381536659
4	12.0590973632	0.472589376767	12	2421.49593305	0.129663285372
5	23.3953686210	0.401247346080	13	4698.22358279	0.110331996876
6	45.3925690297	0.341642486417	14	9115.56552005	0.093882792404
7	88.0712113358	0.290654158532	15	17686.1601594	0.079885967663
8	170.877157631	0.247334196318	16	34314.9594501	0.067975906400

(15) holds even if $P(\lambda) = 0$ should yield multiple eigenvalues below λ_n for some $n > 3$. Numerically the dominant λ_n and their coefficients c_n are determined from

$$\lambda_n = \lim_{m\to\infty} \mu(m+1, n)/\mu(m, n); \qquad c_n = \lim_{m\to\infty} \mu(m, n)/\lambda_n^m. \tag{16}$$

(16) follows from (15), and this evaluation of λ_n is called the "power method" in numerical analysis. The 12-digit representations in Table III were reached in all cases $n \leqslant 16$ before $m = 30$.

5. Asymptotic Behavior

With the notation $\bar{x} = \ln x$ (15) transforms into

$$\ln \mu(m, n) = \bar{\mu}(m, n) = \bar{c}_n + m\bar{\lambda}_n + o(1). \tag{17}$$

From (17) we conjecture that there is a function $\bar{f}(m, n)$ which approximates $\bar{\mu}(m, n)$ closely if *both* m and n are large, and that this $\bar{f}(m, n)$ ought to be linear in m for fixed n, linear in n for fixed m and symmetrical in m, n. Therefore it should be of the form

$$\bar{f}(m, n) = \bar{b} + \bar{c}(m + n) + \bar{\lambda}mn. \tag{18}$$

Removing the logarithms yields the conjecture

$$\mu(m, n) \approx f(m, n) = b\, c^{m+n}\lambda^{mn}. \tag{19}$$

Numerical evidence that (19) provides a true asymptotic formula for $m, n \to \infty$ is very strong: λ_{n+1}/λ_n converges fast to the number λ in (20) below. Both c_{n+1}/c_n and λ_n/λ^n tend to the same number c (20), and c_n/c^n

converges to b (20). Our 33×16-table of $\mu(m, n)$ enables us to state these three constants quite accurately:

$$b = 0.89991794880, \quad c = 0.850911720748, \quad \lambda = 1.940215351483. \quad (20)$$

In respect to a possible proof of (19) we gratefully acknowledge the help of a referee who has pointed out that $\lim(\bar{\lambda}_n/n)$ is known to exist: this follows, for instance, from the Theorem 8,6 by Heilmann and Lieb in [4] (case $\mu = 0$). In the meantime we have also obtained a simpler proof in our specific case of rectangular boards so that the leading term λ^{mn} in (19) is confirmed. Nevertheless, a proof of $\lim(\mu(m, n)/f(m, n)) = 1$ as $m, n \to \infty$ requires $|\bar{\mu}(m, n) - \bar{f}(m, n)| < \varepsilon$ if both m and n are large enough. But decreasing errors in (17) for fixed m (or, symmetrically, for fixed n) do not imply $o(1)$ if m, n tend to infinity simultaneously (consider, for instance, $\mathrm{Min}(m/n, n/m)$ as a hypothetical error function and $m = n \to \infty$). Ostrowski's bound on subdominant (second largest) eigenvalues of non-negative matrices in [5] is too weak to replace $o(1)$ in (17) with a more specific bound that could carry a proof of $\bar{\mu}(m, n) - \bar{f}(m, n) = o(1)$.

So at present we have to be content with listing relative accuracies of (15) (c_n, λ_n from Table III) and (19) (b, c, λ from (20)). A digit 7, say, in Table IV means that $|\mu(m, n)$-approximation$|$/approximation is below 10^{-7}. Bold digits **0**, **1**,... are substitutes for 10, 11,....

TABLE IV

	Relative errors of (15) $n = 1234567890123456$		Relative errors of (19) $n = 1234567890123456$
$m = 1$	0100000000000000	$m = 1$	0100000000000000
$m = 2$	1111111111111111	$m = 2$	1111111111111111
$m = 3$	1222222221111111	$m = 3$	0112222221111111
$m = 4$	2322222222222222	$m = 4$	0122222222222222
$m = 5$	2333333333333333	$m = 5$	0122233333333333
$m = 6$	2433333333333333	$m = 6$	0122333333333333
$m = 7$	3544444444444444	$m = 7$	0122334444444444
$m = 8$	3545444444444444	$m = 8$	0122334454444444
$m = 9$	4655555555555555	$m = 9$	0122334555555555
$m = 10$	4756565555555555	$m = 10$	0112334455655555
$m = 11$	5766666666666666	$m = 11$	0112334456666666
$m = 12$	5867677777777777	$m = 12$	0112334455667777
$m = 13$	5977777777777777	$m = 13$	0112334455677777
$m = 14$	6978888888888888	$m = 14$	0112334455677788
$m = 15$	6088888888888888	$m = 15$	0112334455677888
$m = 16$	7189999999999999	$m = 16$	0112334455677888
$m = 17$	7190999999999999	$m = 17$	0112334455677889
$m = 18$	7290000000000000	$m = 18$	0112334455677889

Using Table I (or Table II for $m \leqslant 3$ or $n \leqslant 3$) if $m, n \leqslant 8$, and employing (15) if $\mathrm{Min}(m, n) \leqslant 16$ will yield at least five significant digits of $\mu(m, n)$ in all known cases. With our 33×16-table an overall precision of nine decimals—at least in the logarithms $\bar{\mu}(m, n)$—is obtained for all (m, n) assuming that (19) is a valid asymptotic expression. Since we have no real doubts about this, we state the approximate number of monomer–dimer configurations for a GO-board: it is $\mu(19, 19) = 0.159823713 \times 10^{102}$.

6. Extensions

The methods in this paper may be extended to some related enumeration problems. For instance, folding the $m \times n$ rectangles into cylinders will lead to only minor changes. More interesting is the enumeration of MD-covers with a *given* number k of dimers (and $mn - 2k$ monomers) for which we go back to Section 3. There the downsloping arrows in the diagrams of Figs. 5 and 6 belong to transitions in which one new dimer is created. Correspondingly let $a^{(0)}_{2k,2k+1}$ in (10) be equated to a variable y (instead of 1) and let $a^{(r)}_{ij} = y$ in (11) for the dimer producing pairs $(I, J) = (0, 2), (0, 3), (1, 3)$ and $(2, 3)$. Then, for $n = 2$, (13) changes to

$$\begin{pmatrix} 1 & y & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & y & y \\ 0 & 1 & 0 & y \\ 1 & 0 & 0 & y \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & y & y & y+y^2 \\ 1 & 0 & y & y \\ 1 & y & 0 & y \\ 1 & 0 & 0 & y \end{pmatrix} = A_2(y). \quad (21)$$

Replacing A_n in (7) with these variable transfer matrices $A_n(y)$ produces vectors $\mathbf{v}^{(m)}(y)$ whose components $v^{(m)}_j(y)$ are generating polynomials: it is easy to see that their kth coefficients count the MD-configurations (of bottom row j) which possess exactly k dimers. Consequently, the sums $G(y) = \sum_j v^{(m)}_j(y)$ are generating functions for the desired total numbers $\mu(m, n, k)$ of MD-covers with k dimers.

Numerically $G(y)$ may be studied by a slight modification of Algorithm MD: just write $v_{j+1} \leftarrow xy$ in Step 3 and $v_{i''} \leftarrow xy$, $v_{i'''} \leftarrow (x + x' + x'')y$ in Step 7. However, for exact values of $\mu(m, n, k)$ it is necessary to carry the coefficients of all polynomials $v^{(m)}_j(y)$ separately. Such a more elaborate extension of Algorithm MD yielded a table of $\mu(m, n, k)$ for $1 \leqslant n \leqslant 10$, $1 \leqslant m \leqslant 20$ and all $0 \leqslant k \leqslant mn/2$ in less than 4 min CPU-time. The symmetry $\mu(m, n, k) = \mu(n, m, k)$ and the identity $\mu(m, n) = \sum_k \mu(m, n, k)$ provided computational checks, and the Fisher/Kastelyn results for $k = mn/2$ appeared as special cases. The overall behavior of $\mu(m, n, k)$ which is of interest to theoretical physicists will be studied in a forthcoming thesis

TABLE V

k	$\mu(8,8,k)$	k	$\mu(8,8,k)$	k	$\mu(8,8,k)$
0	1	11	25993868765368	22	20736624703789616
1	112	12	113179382211247	23	12159927883366932
2	5924	13	416207377096444	24	5641564870710449
3	196916	14	1294376782034344	25	2029257121240220
4	4618099	15	3404176551015420	26	551328493544558
5	81324796	16	7561681603209033	27	109304288135224
6	1117512434	17	14151572553740496	28	15081025826348
7	12293405172	18	22230618775639430	29	1351689196944
8	110214209813	19	29164625906707456	30	70575683312
9	815627047768	20	31744505655966719	31	1760337760
10	5028332871572	21	28430924549581392	32	12988816
				Total: $\mu(8,8) = 179788343101980135$	

(K. D. Kohrt). We shall be content to end this exercise by going back to its title. In Table V the numbers of chessboard pavings with k dimers ($0 \leqslant k \leqslant 32$) and $64\text{-}2k$ monomers are listed.

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