

Graphics Tutorial 1 notesh

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January 5, 2019

$$\text{Rotation Matrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\text{Scaling matrix} = \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. X is a random variable with PDF given by

$$f(n) = \begin{cases} cx^2 & \text{if } x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Constant c

Summation of PDF over domain adds up to 1.

Or,

$$\int_{-\infty}^{\infty} cx^2 dx = 1$$

Since the function returns 0 everywhere except at $[-1, 1]$, we just calculate

$$\begin{aligned} \int_{-1}^1 cx^2 dx &= 1 \\ c \times \frac{x^3}{3} \Big|_{-1}^1 &= 1 \\ c \times \frac{2}{3} &= 1 \\ c &= 1.5 \text{ (Answer)} \end{aligned}$$

(b) $E[X]$ and $Var(X)$ $E[X]$ is

$$\begin{aligned}\int_{-1}^1 xcx^2 dx &= 1.5 \times \int_{-1}^1 x^3 dx \\ &= 1.5t \times \frac{x^4}{4} \Big|_{-1}^1 \\ &= 0\end{aligned}$$

Now, $Var(X) = E[x^2] - (E[X])^2$, or

$$\begin{aligned}Var(X) &= 1.5 \times \int_{-1}^1 x^4 dx - 0 \\ &= 1.5 \times \frac{x^5}{5} \Big|_{-1}^1 \\ &= 1.5 \times 0.4 \\ &= 0.6\end{aligned}$$

(c) $P(X \geq \frac{1}{2})$

Since the function given is a PDF, to get the $P(X \geq \frac{1}{2})$, all we need to do is integrate $f(x)$ from $\frac{1}{2}$ to 1

Or,

$$\begin{aligned}P\left(X \geq \frac{1}{2}\right) &= \int_{\frac{1}{2}}^1 cx^2 dx \\ &= 1.5 \times \frac{x^3}{3} \Big|_{\frac{1}{2}}^1 \\ &= 1.5 \times \frac{7}{24} \\ &= 0.4375\end{aligned}$$

2. Given, the CDF is:

$$F(x) = \frac{x^3 + k}{40} \quad x = 1, 2, 3$$

(a) Value of k

Since $F(x)$ is a CDF, value of $F(3) = 1$

or

$$\begin{aligned}\frac{27+k}{40} &= 1 \\ k &= 13 \quad (\text{Q.E.D})\end{aligned}$$

(b) Find the probability distribution of X

This can be obtained by simple subtraction, answer is

$$\begin{aligned}P(X=1) &= \frac{1+13}{40} \\ &= \frac{7}{20} \\ P(X=2) &= \frac{21-14}{40} \\ &= \frac{7}{40} \\ P(X=3) &= \frac{40-21}{40} \\ &= \frac{19}{40}\end{aligned}$$

(c) Given $Var(X) = \frac{259}{320}$, calculate $Var(4X - 5)$

$$\begin{aligned}\sigma_{ax+b}^2 &= a^2 \times \sigma_x^2 \\ \sigma_{4x+5}^2 &= 16 \times \frac{259}{320} \\ &= \frac{259}{20} \\ &= 12.95\end{aligned}$$

3.

$$\begin{aligned}
 P(\text{First 6 in 2nd throw} \mid \text{First 6 on even throw}) &= \frac{P(\text{2nd throw}) \cap P(\text{even throw})}{P(\text{even throw})} \\
 &= \frac{\frac{5}{6} \times \frac{1}{6}}{\frac{1}{6} \times \left[\sum \frac{5}{6} + \left(\frac{5}{6}\right)^3 + \dots \right]} \\
 &= \frac{\frac{5}{36}}{\frac{1}{6} \times \frac{30}{11}} \\
 &= \frac{\frac{5}{30}}{\frac{11}{11}} \\
 &= \frac{5}{6} \times \frac{11}{30} \\
 &= \frac{11}{36} \text{ (Answer)}
 \end{aligned}$$

4. By Bayes' Theorem,

$$\begin{aligned}
 P(0 \text{ sent} \mid 0 \text{ received}) &= \frac{P(0 \text{ received} \mid 0 \text{ sent}) \times P(0 \text{ sent})}{P(0 \text{ received})} \\
 &= \frac{\frac{4}{5} \times \frac{2}{3}}{P(0 \text{ received} \mid 0 \text{ sent}) \times P(0 \text{ sent}) + P(0 \text{ received} \mid 1 \text{ sent}) \times P(1 \text{ sent})} \\
 &= \frac{\frac{8}{15}}{\frac{8}{15} \times \frac{1}{18}} \\
 &= \frac{\frac{8}{15}}{\frac{8}{90}} \\
 &= \frac{48}{53} \text{ (Ans)}
 \end{aligned}$$

5. (a) Sum of a PDF over its given range is 1
Given function:

$$f(x) = \begin{cases} cx^2 & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^3 cx^2 \, dx = 1$$

$$c \times \left. \frac{x^3}{3} \right|_0^3 = 1$$

$$c \times 9 = 1$$

$$c = \frac{1}{9}$$

(b) $P(1 < X < 2)$

$$\begin{aligned} P(1 < X < 2) &= \frac{1}{9} \times \int_1^2 x^2 \, dx \\ &= \frac{1}{9} \times \left. \frac{x^3}{3} \right|_1^2 \\ &= \frac{1}{9} \times \left[\frac{8}{3} - \frac{1}{3} \right] \\ &= \frac{7}{27} \end{aligned}$$

6. (a)

$$c \times \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} \, dx = 1$$

$$c \times \tan^{-1}(x) \Big|_{-\infty}^{\infty} = 1$$

$$c \times \Pi = 1$$

$$c = \frac{1}{\Pi}$$

(b)

$$x \in \left(-1, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, 1\right)$$

$$\begin{aligned} P\left(\frac{1}{3} < x^2 < 1\right) &= \frac{1}{\Pi} \times \left[\int_{-1}^{-\sqrt{\frac{1}{3}}} \frac{1}{1+x^2} dx + \int_{\sqrt{\frac{1}{3}}}^1 \frac{1}{1+x^2} dx \right] \\ &= \frac{1}{\Pi} \times \left[\tan^{-1}(x) \Big|_{-1}^{-\sqrt{\frac{1}{3}}} + \tan^{-1}(x) \Big|_{\sqrt{\frac{1}{3}}}^1 \right] \\ &= \frac{1}{\Pi} \times \left[-\frac{\Pi}{6} + \frac{\Pi}{4} + \frac{\Pi}{4} - \frac{\Pi}{6} \right] \\ &= \frac{1}{\Pi} \times \frac{\Pi}{6} \\ &= \frac{1}{6} \end{aligned}$$

7. (a) For values of $x > 0$

$$\begin{aligned} \int_0^x f(x) dx &= F(x), \quad F(x) = 1 - e^{-2x} \\ f(x) &= \frac{dF(x)}{dx} \\ f(x) &= 2e^{-2x} \end{aligned}$$

for values of $x < 0$, $f(x) = 0$

Answer:

$$\text{PDF } f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $P(X > 2)$, we do $F(\infty) - F(2)$

$$\begin{aligned} F(\infty) - F(2) &= 1 - (1 - e^{-4}) \\ &= e^{-4} \quad (\text{Ans}) \end{aligned}$$

(c) $P(-3 < X \leq 4) = P(X \leq 4)$

$$\begin{aligned} &= F(4) \\ &= 1 - e^{-8} \quad (\text{Ans}) \end{aligned}$$

8. (a)

$$F1(-\infty) = 0$$

$$F1(\infty) = 1$$

$$\frac{d}{dx}F1(x) \geq 0 \forall x \in domain$$

\Rightarrow the function is a CDF

(b)

$$F1(-\infty) \neq 0$$

$$F1(\infty) = 1$$

\Rightarrow the function is not a CDF

9. B = black, O = orange, W = white

Assumption: the balls are picked out one by one, so the following combinations are all possible and distinct

possible outcomes = BB(+4), WW(-2), OO(0), BW(1), WB(1), BO(2), OB(2), WO(-1), OW(-1)

Outcome is +4:

$$\begin{aligned} BB &= \frac{4}{14} \times \frac{3}{13} \\ &= \frac{6}{91} \end{aligned}$$

Outcome is -2:

$$\begin{aligned} WW &= \frac{8}{14} \times \frac{7}{13} \\ &= \frac{28}{91} \end{aligned}$$

Outcome is 0:

$$\begin{aligned} OO &= \frac{2}{14} \times \frac{1}{13} \\ &= \frac{1}{91} \end{aligned}$$

Outcome is +1:

$$\begin{aligned}\text{BW and WB} &= 2 \times \frac{4}{14} \times \frac{8}{13} \\ &= \frac{16}{91} \\ &= \frac{32}{91}\end{aligned}$$

Outcome is +2:

$$\begin{aligned}\text{BO and OB} &= 2 \times \frac{4}{14} \times \frac{2}{13} \\ &= \frac{4}{91} \\ &= \frac{8}{91}\end{aligned}$$

Outcome is -1:

$$\begin{aligned}\text{WO and OW} &= 2 \times \frac{8}{14} \times \frac{2}{13} \\ &= \frac{8}{91} \\ &= \frac{16}{91}\end{aligned}$$

10.

$$\begin{aligned}P(X = 0) &= \frac{\binom{5}{2} \times 3!}{5!} \\&= \frac{1}{2}\end{aligned}$$

similarly,

$$\begin{aligned}P(X = 1) &= \frac{\binom{5}{3} \times 2!}{5!} \\&= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}P(X = 2) &= \frac{\binom{5}{4} \times 2! \times 1!}{5!} \\&= \frac{1}{12}\end{aligned}$$

$$\begin{aligned}P(X = 3) &= \frac{\binom{5}{5} \times 3!}{5!} \\&= \frac{1}{20}\end{aligned}$$

$$\begin{aligned}P(X = 4) &= \frac{4!}{5!} \\&= \frac{1}{5}\end{aligned}$$

11. Assume X is the number of defective items in the sample
or, possible values of $X = 0, 1, 2, 3$

$$\begin{aligned}E(X) &= \sum x f(x) \\&= f(1) + 2 \times f(2) + 3 \times f(3) \\&= \frac{4 \times 16 \times 15}{20 \times 19 \times 18} + 2 \times \frac{4 \times 3 \times 16}{20 \times 19 \times 18} + 3 \times \frac{4 \times 3 \times 2}{20 \times 19 \times 18} \\&= \frac{1416}{6840} \\&= 0.207 \quad (\text{Ans})\end{aligned}$$

12. For errors, at least three bits should be flipped
Therefore, probability

$$\begin{aligned}
&= flip(3) + flip(4) + flip(5) \\
&= \binom{5}{3} \times 0.8^2 \times 0.2^3 + \binom{5}{4} \times 0.8 \times 0.2^4 + \binom{5}{5} \times 0.2^5 \\
&= 0.0512 + 0.0064 + 0.00032 \\
&= 0.05792 \quad (\text{Ans})
\end{aligned}$$

13.

$$\begin{aligned}
P(X = 1) &= \frac{\binom{5}{1} \times 9!}{10!} \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
P(X = 2) &= \frac{\binom{5}{1} \times \binom{5}{1} \times 8!}{10!} \\
&= \frac{5}{18}
\end{aligned}$$

$$\begin{aligned}
P(X = 3) &= \frac{(2! \times \binom{5}{2}) \times \binom{5}{1} \times 7!}{10!} \\
&= \frac{5}{36}
\end{aligned}$$

$$\begin{aligned}
P(X = 4) &= \frac{(3! \times \binom{5}{3}) \times \binom{5}{1} \times 6!}{10!} \\
&= \frac{5}{84}
\end{aligned}$$

$$P(X = 5) = \frac{(4! \times \binom{5}{4}) \times \binom{5}{1} \times 5!}{10!}$$

$$= \frac{5}{252}$$

$$P(X = 6) = \frac{(5! \times \binom{5}{5}) \times \binom{5}{1} \times 4!}{10!}$$

$$= \frac{1}{252}$$

$$P(X \geq 7) = 0$$

14.

$$\frac{\text{Number of ways to have 3 men in each team}}{\text{All possible combinations}}$$

$$= \frac{\binom{6}{3} \times \binom{6}{3}}{\binom{12}{6}}$$

$$= \frac{400}{924}$$

$$= 0.4329$$

15. E = probability that Elvis' twin was male

A = probability that Elvis' twin was identical = $\frac{2}{19}$

B = probability that Elvis' twin was fraternal = $1 - \frac{2}{19} = \frac{17}{19}$

Now,

$$P(M|A) = 1$$

$$P(M|B) = 0.5$$

$$P(A|M) = \frac{P(M|A) \times P(A)}{P(M|A) \times P(A) + P(B) \times P(M|B)}$$

$$= \frac{1 \times \frac{2}{19}}{1 \times \frac{2}{19} + \frac{1}{2} \times \frac{17}{19}}$$

$$= \frac{4}{21} \quad (\text{Ans})$$

16.

$$f_{xy}(x, y) = \begin{cases} 2 & \text{if } x + y \leq 1, x, y > 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow 0 < x, y < 1$$

$$\sigma_{x,y} = E(x, y) - \mu_x \mu_y$$

$$\begin{aligned} E(x, y) &= \int_0^1 \int_0^y xyf(x, y)dydx \\ &= 2 \int_0^1 x \left[\frac{y^2}{2} \right] dx \\ &= \int_0^1 x(x^2 - 2x + 1)dx \\ &= \int_0^1 (x^3 - 2x^2 + x)dx \\ &= \left. \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right|_0^1 \\ &= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \mu_x &= \int_x xg(x)dx \\ &= \int_x x \int_y f(x,y) dy dx \\ &= \int_0^1 x \int_0^{1-x} 2 dy dx \\ &= 2 \times \int_0^1 x(1-x)dx \\ &= 2 \times \left. \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 \\ &= 2 \times \frac{1}{6} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned}
\mu_y &= \int_y y h(y) dy \\
&= \int_y y \int_x f(x,y) dx dy \\
&= \int_0^1 y \int_0^{1-y} 2 dx dy \\
&= 2 \times \int_0^1 y(1-y) dy \\
&= 2 \times \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\sigma_{xy} &= E(x, y) - \mu_x \mu_y \\
&= \frac{1}{12} - \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{12} - \frac{1}{9} \\
&= \frac{-3}{108} \\
&= \frac{-1}{36} \quad (\text{Ans})
\end{aligned}$$

17. X = Probability of getting head and getting tail in a coin is equal, = $\frac{1}{2}$

Assumption of ideal conditions:

Coin toss is unaffected by surrounding conditions and is perfectly unbiased.

Ans: $P(X) = 1$

Assumption of non ideal conditions:

Coin toss is affected by surrounding conditions and there is bias. So probabilities are not equal

Ans: $P(X) = 0$