a) $P(x|BG) = N(\mu_1, \sigma_1^2)$ ~~BG if $x_{ij} < \theta$, else FG~~

$P(x|FG) = N(\mu_2, \sigma_2^2)$

now, $\sigma_1 = \sigma_2$ and $P(BG) = P(FG)$

$$P(x|BG) \cdot P(BG) \gtrless P(x|FG) \cdot P(FG)$$

$$P(x|BG) > P(x|FG)$$

or $$N(\mu_1, \sigma^2) > N(\mu_2, \sigma^2)$$

or $$\cancel{\frac{1}{\sqrt{2\pi}\,\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma}\right)^2} > \cancel{\frac{1}{\sqrt{2\pi}\,\sigma}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma}\right)^2}$$

ln:

$$-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma^2} > -\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma^2}$$

or $$(x-\mu_1)^2 < (x-\mu_2)^2$$

or $$(x-\mu_1)^2 - (x-\mu_2)^2 < 0$$

or $$(x-\mu_1-x+\mu_2)(x-\mu_1+x-\mu_2)^2 < 0$$

or $$(\mu_2-\mu_1)(2x-\mu_1-\mu_2) < 0.$$

$\Rightarrow$ $2x - \mu_1 - \mu_2 < 0$

or $$x < \frac{\mu_1+\mu_2}{2} \quad (\mu_1 < \mu_2)$$

$\therefore$ optimal θ : $\theta^* = \frac{\mu_1+\mu_2}{2}$

c. $P(BG) = 4\, P(FG)$ $\mu_1 = 100$, $\mu_2 = 200$ $\sigma_1 = \sigma_2$

$$P(X \mid BG) \cdot \overset{4}{\cancel{P(BG)}} > P(X \mid FG) \cdot \cancel{P(FG)}$$

$$4 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\theta - \mu_1}{\sigma}\right)^2} > \frac{1}{\cancel{\sqrt{2\pi\sigma^2}}} e^{-\frac{1}{2}\left(\frac{\theta - \mu_2}{\sigma}\right)^2}$$

$$\ln 4 - \frac{1}{2\sigma^2}(\theta - \mu_1)^2 > -\frac{1}{2\sigma^2}(\theta - \mu_2)^2$$

$$\frac{1}{2\sigma^2}\left[(\theta - \mu_1)^2 - (\theta - \mu_2)^2\right] < \ln 4$$

$$(\theta - \mu_1 + \theta - \mu_2)(\cancel{\theta} - \mu_1 - \cancel{\theta} + \mu_2) < 2\sigma^2 \ln 4$$

$$(2\theta - 300)(100) < 2\sigma^2 \ln 4$$

$$100(\theta - 150) < \sigma^2 \ln 4$$

$$\theta < \left(\frac{\sigma}{10}\right)^2 \ln 4 + 150$$

$\therefore$ Optimum $\theta^* = \left(\frac{\sigma}{10}\right)^2 \ln 4 + 150$

6. Relaxing conditions for $\theta^* = \frac{\mu_1+\mu_2}{2}$

$$N(\mu_1, \sigma_1^2, \theta^*)\ P(BG) = N(\mu_2, \sigma_2^2, \theta^*)\ P(FG)$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} P(BG) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2} . P(FG).$$

$$\ln P(BG) - \ln \sigma_1 - \frac{(x-\mu_1)^2}{2\sigma_1^2} = \ln P(FG) - \ln \sigma_2 - \frac{(x-\mu_2)^2}{2\sigma_2^2}$$

$x = \theta$.

$$\ln \frac{P(BG)}{P(FG)} + \ln \frac{\sigma_2}{\sigma_1} = \frac{(\theta-\mu_1)^2}{2\sigma_1^2} \quad \frac{(\theta-\mu_2)^2}{2\sigma_2^2}$$

or $$2 \ln \frac{P(BG)}{P(FG)} + \ln \frac{\sigma_2^2}{\sigma_1^2} = \left[\frac{\theta-\mu_1}{\sigma_1}\right]^2 - \left[\frac{\theta-\mu_2}{\sigma_2}\right]^2$$

$$= \left(\frac{\mu_2-\mu_1}{2\sigma_1}\right)^2 - \left[\frac{\mu_1-\mu_2}{2\sigma_2}\right]^2$$

$$= (\mu_2-\mu_1)^2 \left[\frac{1}{4\sigma_1^2} - \frac{1}{4\sigma_2^2}\right]$$

$$2 \ln \frac{P(BG)}{P(FG)} + \ln \frac{\sigma_2^2}{\sigma_1^2} = \left(\frac{\mu_2-\mu_1}{2}\right)^2 \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)$$