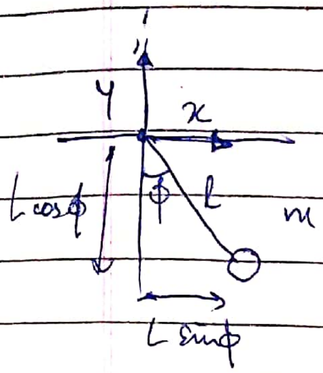
Assignment 2

2017 1089
Abhigyan Ghosh

1. Simple Pendulum



Langrangian:

$$x = L\sin\phi$$
$$y = -L\cos\phi$$
$$\frac{dx}{dt} = L\cos\phi \cdot \dot\phi$$
$$\frac{dy}{dt} = L\sin\phi \cdot \dot\phi$$

$$K = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{1}{2}mL^2\dot\phi^2$$

$$U = -mgL\cos\phi.$$
$$L = K - U = \frac{1}{2}mL^2\dot\phi^2 + mgL\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) = \frac{d}{dt}\left(mL^2\dot\phi\right) = mL^2\ddot\phi$$

$$\frac{\partial L}{\partial\phi} = mgL\sin\phi$$

$$\Rightarrow mL^2\ddot\phi = mgL\sin\phi$$
$$\Rightarrow \boxed{\ddot\phi = \frac{g}{L}\sin\phi}$$

Hamiltonian:

$$H = K + U = \frac{1}{2}mL^2\dot\phi^2 + mgL\cos\phi =$$

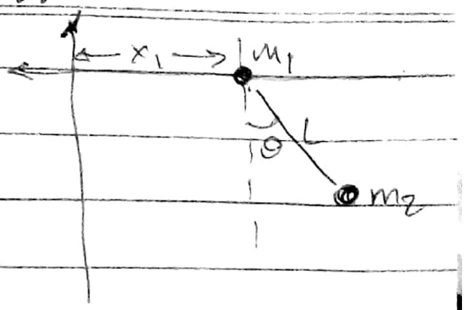
$$p = \frac{\partial L}{\partial\dot\phi} = mL^2\dot\phi$$

$$H = \frac{1}{2}mL^2\dot\phi^2 + mgl\cos\phi = \frac{p^2}{2mL^2} + mgl\cos\phi.$$

$$\frac{\partial H}{\partial\phi} = -mgl\sin\phi \qquad \frac{dp}{dt} = mL^2\ddot\phi$$

$$\Rightarrow \boxed{\ddot\phi = \frac{g}{L}\sin\phi.}$$

~~Bead~~ Sliding Pendulum



L: $x = x_1 + l\sin\theta$

$y = -l\cos\theta$.

$\dot{x} = \dot{x}_1\, l\cos\theta\,\dot{\theta}$

$\dot{y} = l\sin\theta\,\dot{\theta}$

$U = -mgl\cos\theta$.

$K = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}ml^2\sin^2\dot{\theta} + \dot{x}_1 l\cos\theta\,\dot{\theta}m + \frac{1}{2}ml^2\cos^2\dot{\theta}$

$= m\dot{x}_1^2 + \frac{1}{2}ml^2\dot{\theta}^2 + \dot{x}_1 l\cos\theta\,\dot{\theta}m$

$L = K - U = m\dot{x}_1^2 + \frac{1}{2}ml^2\dot{\theta}^2 + \dot{x}_1 l\cos\theta\,\dot{\theta}m + mgl\cos\theta$

$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + \dot{x}_1 l\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} + \ddot{x}_1 L\cos\theta - m\dot{x}_1 l\sin\theta\,\dot{\theta}$

$\frac{\partial L}{\partial\theta} = -m\dot{x}_1 l\sin\theta\,\dot{\theta} - mgl\sin\theta$.

According to Langrange's Equation

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$

$\Rightarrow ml^2\ddot{\theta} + m\ddot{x}_1 L\cos\theta = -mgl\sin\theta$.

$\Rightarrow \boxed{ml^2\ddot{\theta} + m\ddot{x}_1 l\cos\theta + mgl\sin\theta = 0}$

H: $H = K + U = \frac{1}{2}ml^2\dot{\theta}^2 + m\dot{x}_1 l\cos\theta\,\dot{\theta} + m\dot{x}_1^2 - mgl\cos\theta$

$p = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + m\dot{x}_1 l\cos\theta$

$\frac{dp}{dt} = ml^2\ddot{\theta} + m\ddot{x}_1 l\cos\theta - m\dot{x}_1 l\sin\theta\,\dot{\theta}$

$$\frac{\partial H}{\partial \theta} = -m\dot{x}_1 l \sin\theta\,\dot{\theta} + mgl\sin\theta.$$

According to Hamiltonian's Equation

$$\frac{\partial H}{\partial \theta} = -\frac{dp}{dt}$$

$$\Rightarrow \boxed{ml^2\ddot{\theta} + m\ddot{x}_1 l\cos\theta + mgl\sin\theta = 0}$$

To calculate for x_1

$$\frac{\partial L}{\partial \dot{x}_1} = 2m\dot{x}_1 + ml\cos\theta\,\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 2m\ddot{x}_1 - ml\sin\theta(\dot{\theta})^2 + ml\cos\theta\,\ddot{\theta}$$

$$\frac{\partial L}{\partial x_1} = 0.$$

According to Lagrangian Equation

$$\frac{\partial L}{\partial x_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)$$

$$\Rightarrow \boxed{2m\ddot{x}_1 + ml\cos\theta\,\ddot{\theta} = ml\sin\theta(\dot{\theta})^2}$$

$$\frac{\partial H}{\partial x_1} = 0 \qquad p = \frac{\partial L}{\partial \dot{x}_1} = 2m\dot{x}_1 + ml\cos\theta\,\dot{\theta}$$

$$\frac{dp}{dt} = 2m\ddot{x}_1 + ml\cos\theta\,\ddot{\theta} - ml\sin\theta(\dot{\theta})^2$$

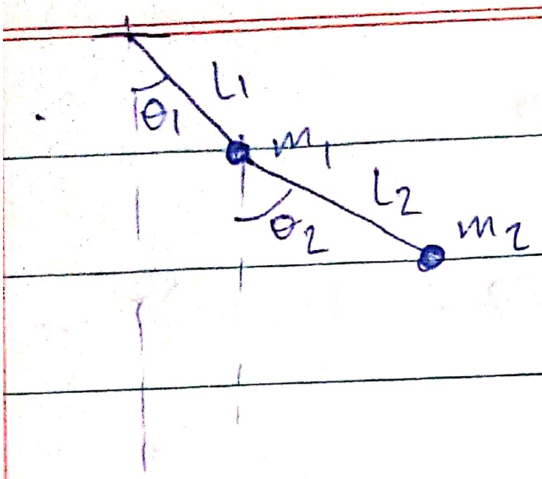
According to Hamiltonian Equation

$$\frac{\partial H}{\partial x_1} = -\frac{dp}{dt}$$

$$\Rightarrow \boxed{2m\ddot{x}_1 + ml\cos\theta\,\ddot{\theta} = ml\sin\theta(\dot{\theta})^2}$$

Date:__.__.__

Double Pendulum.



$$x_1 = l_1 \sin\theta_1$$

$$y_1 = -l \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = -l_2 \cos\theta_2 - l_1 \cos\theta_1$$

$$\dot{x}_1 = l_1 \cos\theta_1 \, \dot{\theta}_1 \qquad \dot{x}_2 = \dot{\theta}_1 l_1 \cos\theta_1 + l_2 \cos\theta_2 \dot{\theta}_2$$

$$\dot{y}_1 = l_1 \sin\theta_1 \, \dot{\theta}_1 \qquad \dot{y}_2 = \dot{\theta}_2 l_2 \sin\theta_2 + l_1 \sin\theta_1 \dot{\theta}_1$$

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 .$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$U = - m_1 g L_1 \cos\theta_1 - m_2 g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2$$

$$= -(m_1 + m_2) L_1 \cos\theta_1 g - m_2 g l_2 \cos\theta_2 .$$

$$L = K - U$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$= (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2 .$$

~~Applying Lagrange's Equations of motion, we get~~

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

Applying Lagrange's Equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1}$$

$$\Rightarrow \boxed{(m_1 + m_2)\left[l_1 \ddot{\theta}_1 + g \sin\theta_1\right] + m_2 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2}$$

$$\Rightarrow \boxed{l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin\theta_2 = 0}$$

$H = K + U$

$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$
$- (m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$

$\frac{dp_1}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$
$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$

$\frac{dp_2}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$
$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$

$\frac{\partial H}{\partial \theta_1} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1$

$\frac{\partial H}{\partial \theta_2} = - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2$

Applying Hamilton's Equantions of motion

$\frac{\partial H}{\partial \theta_1} = - \frac{dp_1}{dt}$

$\Rightarrow \boxed{(m_1 + m_2)\left[l_1 \ddot{\theta}_1 + g \sin\theta_1\right] + m_2 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] = 0}$

$\frac{\partial H}{\partial \theta_2} = \frac{-dp_2}{dt}$

$\Rightarrow \boxed{l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1 \dot{\theta}_1^2 \sin(\dot{\theta}_1 - \theta_2) + g \sin\theta_2 = 0}$