The LASSO

Least Absolute Shrinkage and Selection Operator

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Overview

Motivation for the LASSO

Ordinary Least Squares Estimates

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- **Prediction Accuracy:**
  - OLS estimates often have low bias but large variance
  - This can be improved by shrinking or setting some coefficients to 0
- **Interpretability**
  - Large number of predictors lead to less interpretable models
  - We would rather determine a smaller subset of features exhibiting the strongest effects

Subset Selection and Ridge Regression

Subset Selection

- Retains only a subset of the variables
- Discards the rest from the model
  - **More Interpretability:** Fewer predictors in the model
  - **Low Prediction Accuracy:** Extremely variable since it is a discrete process

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Ridge Regression

- Shrinks the regression coefficients by imposing a penalty on their size
- Ridge coefficients minimize a penalized residual sum of squares
  - **More Stability:** Continuous process and doesn't suffer as much from high variability
  - **Less Interpretability:** Does not set any coefficients to 0

The LASSO

- Regression analysis method that is able to achieve both of these goals:
  - **Variable Selection** $\rightarrow$ More Interpretable Models
  - **Regularization** $\rightarrow$ Enhanced Prediction Accuracy
- Popularized in 1996 by Robert Tibshirani
- Shrinks some coefficients and sets others to 0
- Falls somewhere between ridge regression and subset selection : enjoys some of the properties of each

Model

Model Assumptions

- Suppose we have data $(x_i, y_i), i = 1, 2, \ldots, N$, where $x_i = (x_{i_1}, \ldots, x_{i_p})^T$ are the predictor variables and y_i are the responses
- Observations are independent or $y_i's$ are conditionally independent given the $x_{ij}'s$
- $x_{ij}'s$ are standardized so that

$$\sum_{i=1}^{N} \frac{x_{ij}}{N} = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \frac{x_{ij}^2}{N} = 1$$

- The design matrix need not be of full rank

Model: Constraint Form

- The lasso estimate is defined by:

$$\hat{\beta}_{lasso} = \arg\min_{\beta} \sum_{i=1}^{N} (y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij})^2 \qquad subject \qquad to \qquad \sum_j |\beta_j| \leq t \qquad (1)$$

- Quadratic programming problem with linear inequality constraints
- **The tuning parameter t** controls the amount of shrinkage that is applied to the estimates
- For all t, the solution for : $\beta_0 = \hat{\beta}_0 = \bar{y}$

Model: Penalized Form

- The lasso problem can be framed in the equivalent **Lagrangian form:**

$$\hat{\beta}_{lasso} = \arg\min_{\beta}\{\frac{1}{2}\sum_{i=1}^{N}(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij})^2 + \lambda\sum_{j=1}^{p}|\beta_j|\} \tag{2}$$

- **The tuning parameter λ** controls the amount of penalty:
  - $\lambda = 0 \rightarrow$ no parameters are eliminated, hence, $\hat{\beta}_{lasso} = \hat{\beta}_{OLS}$
  - As λ increases $\rightarrow$ more and more coefficients are set to zero and eliminated
  - $\lambda = \infty \rightarrow$ all coefficients are eliminated.

- λ has a one-to-one correspondence with t

Sparsity of LASSO

Orthonormal Design Case

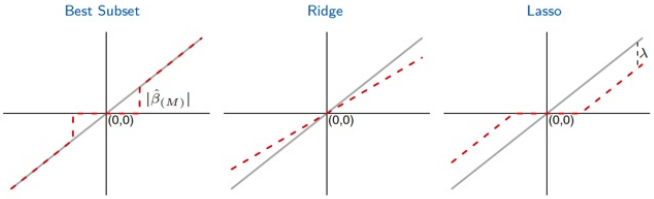
- Let X be the $n \times p$ design matrix with ij^{th} entry x_{ij}
- Suppose $X^T X = I$, the identity matrix
- The solutions to equation (1) are shown to be:

$$\hat{\beta}_j = \textit{sign}(\hat{\beta}_j^o)(|\hat{\beta}_j^o| - \lambda)_+ \tag{3}$$

where,

- $\hat{\beta}_j^{o\prime} s$ are the OLS estimates
- λ is a constant chosen by the corresponding technique
- sign denotes the sign of its argument (± 1)
- x_+ denotes the "positive part" of x

Geometric form of the functions



- Best subset selection of size M drops all variables with coefficients smaller than the M^{th} largest
- Ridge regression scales the coefficient by a constant factor
- LASSO translates by a constant factor, truncating at 0

Constraint regions

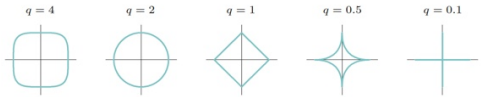
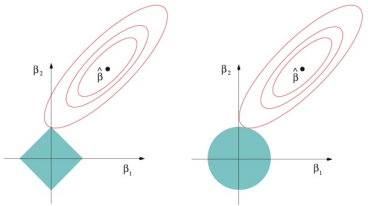


FIGURE 3.12. *Contours of constant value of* $\sum_j |\beta_j|^q$ *for given values of* q.

- The constraint region for ridge regression is the disk $\beta_1^2 + \beta_2^2 \leq t$
- The constraint region for LASSO is the diamond $|\beta_1| + |\beta_2| \leq t$
- $q = 1$**:** the smallest value of q giving a convex region which is convenient for optimization
- $q > 1$**:** $|\beta_j|^q$ is differentiable at 0, hence, doesn't share the ability of lasso ($q = 1$) for setting coefficients exactly to 0

General Case



- **Residual sum of squares** $\rightarrow$ elliptical contours centered at the full least square estimates
- **LASSO solution** $\rightarrow$ first place where contours touch the diamond; if solution occurs at a corner, it corresponds to a zero solution
- **Ridge solution** $\rightarrow$ No corners for the contours to hit; zero solutions will rarely result
- $\boldsymbol{p} > 2$ $\rightarrow$ the diamond becomes a rhomboid; more opportunities for the estimated parameters to be zero

Illustration

Prostrate Cancer Data

- Interested in the level of prostate-specific antigen (PSA), elevated in men who have prostate cancer
- $y_i = \log(PSA), x_{ij}$ measurements on a man and his prostate
- **Data:** n = 97 men with prostate cancer and p = 8 clinical predictors
- What if we want to derive a linear model using only a few of the 8 predictors to predict the level of PSA?

Model Evaluation

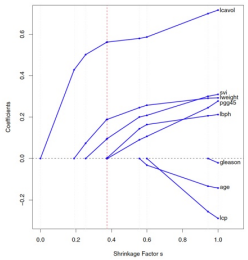


TABLE 3.3. *Estimated coefficients and test error results, for different subset and shrinkage methods applied to the prostate data. The blank entries correspond to variables omitted.*

Term	LS	Best Subset	Ridge	Lasso	PCR	PLS
Intercept	2.465	2.477	2.452	2.468	2.497	2.452
lcavol	0.680	0.740	0.420	0.533	0.543	0.419
lweight	0.263	0.316	0.238	0.169	0.289	0.344
age	−0.141		−0.046		−0.152	−0.026
lbph	0.210		0.162	0.002	0.214	0.220
svi	0.305		0.227	0.094	0.315	0.243
lcp	−0.288		0.000		−0.051	0.079
gleason	−0.021		0.040		0.232	0.011
pgg45	0.267		0.133		−0.056	0.084
Test Error	0.521	0.492	0.492	0.479	0.449	0.528
Std Error	0.179	0.143	0.165	0.164	0.105	0.152

- Shrinkage factor $s = t/\sum_j |\hat{\beta}_j^o|$
- Vertical line is drawn at $s = 0.36$, optimal value chosen by cross-validation
- LASSO profiles hit zero, while those for ridge would not
- Lasso has the lowest test error. Hence, performs better than Subset, Ridge

Comparison between Subset Selection, Ridge, LASSO

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Better Prediction Accuracy

More Interpretable Models

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Better Prediction Accuracy *More Interpretable Models*

- **Small number of large effects:** Subset selection does best here, the lasso not quite as well and ridge regression does quite poorly

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- **Small to moderate number of moderate-sized effects:** Lasso does best, followed by ridge regression and then subset selection

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- **Small to moderate number of moderate-sized effects:** Lasso does best, followed by ridge regression and then subset selection
- **Large number of small effects:** Ridge regression does best by a good margin, followed by the lasso and then subset selection

Summary

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- We need better techniques to handle high-dimensional data with better prediction accuracy and more interpretability
- LASSO makes use of the L_1 penalty to set some coefficents exactly to 0 and gives us sparse models
- Some Applications of LASSO include:
  - **Tree-based methods:** Rather than prune a large tree, use the LASSO idea to shrink it
  - **Multivariate Adaptive Regression Splines:** Special Lasso-type algorithm to grow and prune a MARS model

References

- **Regression Shrinkage and Selection via the Lasso**, R. Tibshirani. Journal of the Royal Statistical Society. Series B (Methodological) (1996) (Methodological), Vol. 58, No. 1(1996), pp. 267-288

- **Regression shrinkage and selection via the lasso: a retrospective**, R. Tibshirani. Journal of the Royal Statistical Society Series B, 2011, vol. 73, issue 3, 273-282

- **The Elements of Statistical Learning**, T. Hastie, R. Tibshirani, and J. Friedman. Springer Series in Statistics Springer New York Inc., New York, NY, USA, (2001)

Questions?