Asset Pricing Theory

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1 Expected Returns and Risk

1.1 Holding Period Return

The holding period return measures the total return earned on an investment over a specific period. It consists of two components:

1. **Income:** Dividends or interest received.
2. **Capital Gain/Loss:** The change in the price of the asset.

The general formula for Holding Period Return (HPR) is:

$$R = \frac{P_1 - P_0 + D_1}{P_0}$$

Where:

- P_0: Initial Price (Purchase Price)
- P_1: Price at the end of the period
- D_1: Dividend paid during the period

Example: You buy a stock for \$50 (P_0). After one year, the price is \$55 (P_1) and it paid a dividend of \$1 (D_1).

$$R = \frac{55 - 50 + 1}{50} = \frac{6}{50} = 0.12 \text{ or } 12\%$$

1.2 Expected Return and Risk

In reality, future prices (P_1) and dividends (D_1) are unknown. Therefore, the return is a **random variable** characterized by possible outcomes and their probabilities.

- **Expected Return ($E[R]$):** The probability-weighted average of all possible returns. It is used as the discount rate for present value calculations.
- **Risk:** The uncertainty of future returns (both good and bad outcomes). It is typically measured by the **Variance** (σ^2) or **Standard Deviation** (σ).

1.3 Comparison: Risk-Free vs. Risky Investment

Consider an investor with \$100 and two investment choices:

1.3.1 Investment 1: Risk-Free

Guaranteed payoff of \$105 after 1 year.

- **Return:** $\frac{105-100}{100} = 5\%$

- **Risk:** Since there is no uncertainty, Risk $(\sigma) = 0$.

1.3.2 Investment 2: Risky

Payoff depends on probabilities:

- 60% probability of \$150 (Return: $\frac{150-100}{100} = 50\%$)
- 40% probability of \$80 (Return: $\frac{80-100}{100} = -20\%$)

Calculating Expected Return:

$$E[R] = (0.6 \times 0.50) + (0.4 \times -0.20) = 0.30 - 0.08 = 0.22 \text{ or } 22\%$$

Calculating Risk (Variance and Standard Deviation): Variance is the weighted average of squared deviations from the expected return.

$$\sigma^2 = \sum P_i(r_i - E[R])^2$$

$$\sigma^2 = 0.6(0.50 - 0.22)^2 + 0.4(-0.20 - 0.22)^2$$

$$\sigma^2 = 0.6(0.28)^2 + 0.4(-0.42)^2 = 0.6(0.0784) + 0.4(0.1764)$$

$$\sigma^2 = 0.04704 + 0.07056 = 0.1176$$

$$\text{Standard Deviation } (\sigma) = \sqrt{0.1176} \approx 34.29\%$$

1.3.3 Conclusion

Investment 2 offers a higher expected return (22% vs 5%) but comes with significantly higher risk (34.29% vs 0%). Whether the additional 17% return is sufficient compensation for the risk depends on the investor's **utility** (attitude toward risk) and **asset pricing models**.

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2 Diversification and Risk

2.1 The Concept of Diversification

Investing in a single stock is risky; if the company fails, the investor loses their entire investment. To mitigate this, investors form a **portfolio** (a collection of stocks).

- **Diversification:** The strategy of investing in multiple assets to spread risk.
- **Mechanism:** In a portfolio, the poor performance of some stocks is likely to be offset by the good performance of others. Therefore, the risk (variance) of a portfolio is typically lower than the risk of a single stock.

2.2 Types of Risk

As the number of stocks in a portfolio increases, the portfolio variance (risk) decreases. This relationship reveals two distinct types of risk:

1. **Diversifiable Risk (Unsystematic Risk):**
   - The portion of risk that can be eliminated by adding more stocks to the portfolio.
   - It is represented by the gap between the total risk curve and the minimum risk floor.
   - This risk becomes negligible with a portfolio of about 50 to 60 stocks.
2. **Systematic Risk:**
   - The portion of risk that **cannot** be eliminated, regardless of how many stocks are added.
   - It represents the inherent uncertainty of the market that affects all stocks.

2.3 Compensation for Risk

A fundamental principle is that investors should **not** expect compensation (higher returns) for holding diversifiable risk. Since this risk can be eliminated for free through diversification, the market does not reward it. Investors are compensated only for holding systematic risk.

2.3.1 Arbitrage Example

Consider a scenario with a risk-free rate of 5% and four stocks (A, B, C, D) that have **no systematic risk** (only diversifiable risk).

- Expected Returns: A (12%), B (15%), C (17%), D (20%).

If an investor forms an equally weighted portfolio ($25 in each):

$$E[R_p] = 0.25(12\%) + 0.25(15\%) + 0.25(17\%) + 0.25(20\%) = 16\%$$

Analysis: Since the stocks have no systematic risk, a diversified portfolio eliminates the diversifiable risk, resulting in a total risk of zero. This creates a risk-free asset yielding 16%, while the market risk-free rate is only 5%. Investors would borrow at 5% and invest in this portfolio to earn a guaranteed profit (Arbitrage). This demand would drive up stock prices, reducing returns until they equal the risk-free rate (5%).

Conclusion: In the long run, investors are not compensated for diversifiable risk.

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3 The Capital Asset Pricing Model (CAPM)

3.1 The CAPM Formula

The Capital Asset Pricing Model (CAPM) defines the mathematical relationship between expected return and risk. The formula is written as:

$$E[r_i] = r_f + \beta_i(E[r_m] - r_f)$$

Where:

- $E[r_i]$: Expected return of risky asset i.
- r_f: Risk-free rate of return.
- β_i: Beta of the risky asset i.
- $E[r_m]$: Expected return of the market portfolio.

In plain English, the expected return of a risky asset over and above the risk-free rate equals the sensitivity of its return with respect to the market (β) times the expected market risk premium.

3.2 Market Risk Premium

The term $(E[r_m] - r_f)$ is the **Expected Market Risk Premium**. It represents the return in excess of the risk-free rate that an investor expects to earn from holding the market portfolio. It is the compensation for holding the risk of the market portfolio.

3.3 Beta (β) and Systematic Risk

Since the market portfolio is mean-variance efficient, all diversifiable (idiosyncratic) risk has been diversified away. The risk that remains is **Systematic Risk**.

- Beta is a measure of systematic risk.
- Investors are compensated **only** for holding systematic risk, not diversifiable risk.

3.3.1 Calculating Beta

Beta is calculated as the covariance between the risky asset's returns and the market portfolio's returns, divided by the variance of the market portfolio's returns:

$$\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} = \frac{Cov(r_i, r_m)}{\sigma_m^2}$$

3.3.2 Interpreting Beta Values

- $\beta = 1$: The beta of the market portfolio is 1.

- $\beta > 1$: The asset is riskier than the market portfolio.
- $\beta < 1$: The asset is less risky than the market portfolio.
- $\beta = 0$: The risk-free asset has a beta of 0.
- $\beta < 0$: A negative beta means the asset's return moves in the opposite direction to the market's return.

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4 Calculating CAPM Beta

We now apply the CAPM formula to calculate the betas of risky assets.

4.1 Scenario Setup

Let's revisit our three risky asset case (X, Y, Z) and one risk-free asset ($R_f = 5\%$). In previous examples, we found that some portfolios had negative weights, which cannot be a market portfolio. To solve this, we adjust the covariances slightly to ensure positive weights.

- **Asset X:** $E[R] = 10\%$, $\sigma = 7\%$
- **Asset Y:** $E[R] = 20\%$, $\sigma = 10\%$
- **Asset Z:** $E[R] = 15\%$, $\sigma = 12\%$

Table 1: Expected Return and Risk

	Expected Return	Risk
Investment X	10%	7%
Investment Y	20%	10%
Investment Z	15%	12%

Covariances:

- $Cov(X, Y) = 0.0032$
- $Cov(X, Z) = 0.0013$
- $Cov(Y, Z) = 0.0054$

Table 2: Covariance Matrix

Covariance	Investment X	Investment Y	Investment Z
Investment X	0.0049	0.0032	0.0013
Investment Y	0.0032	0.0100	0.0054
Investment Z	0.0013	0.0054	0.0144

4.2 The Market Portfolio

Using Solver in Excel, the weights for the Mean-Variance Efficient (Market) Portfolio are found to be: $w_X = 0.0546$, $w_Y = 0.8424$, $w_Z = 0.1030$. Since all weights are positive, this is a valid market portfolio.

Standard Deviation of the Market Portfolio

$$SD_P = \sqrt{\begin{array}{l} 0.0546^2 \times 0.07^2 + 0.8424^2 \times 0.10^2 + 0.1030^2 \times 0.12^2 \\ + 2 \times 0.0546 \times 0.8424 \times 0.0032 \\ + 2 \times 0.0546 \times 0.1030 \times 0.0013 \\ + 2 \times 0.8424 \times 0.1030 \times 0.0054 \end{array}}$$

$$SD_P = 9.22\%$$

Market Expected Return ($E[R_m]$):

$$E[R_m] = (0.0546 \times 10\%) + (0.8424 \times 20\%) + (0.1030 \times 15\%) = 18.94\%$$

Market Standard Deviation (σ_m): Calculated using the portfolio variance formula (sum of weighted variances and covariances):

$$\sigma_m = 9.22\% \implies \sigma_m^2 = 0.0922^2 \approx 0.0085$$

4.3 Calculating Covariance with the Market

To find Beta, we calculate the covariance of each asset's returns with the market portfolio's returns. The covariance of X with the market is the weighted sum of the covariance of X with each component of the market.

$$Cov(R_X, R_m) = w_X Var(X) + w_Y Cov(X, Y) + w_Z Cov(X, Z)$$

Note that $Cov(X, X)$ is simply the variance of X (0.07^2).

$$Cov(R_X, R_m) = 0.0546(0.07^2) + 0.8424(0.0032) + 0.1030(0.0013) \approx 0.0031$$

Similarly, $Cov(R_Y, R_m) = 0.0092$ and $Cov(R_Z, R_m) = 0.0061$.

4.4 Calculating Beta

Using $\beta_i = \frac{Cov(R_i, R_m)}{\sigma_m^2}$ (where $\sigma_m^2 = 0.0922^2$):

- $\beta_X = \frac{0.0031}{0.0085} = 0.36$
- $\beta_Y = \frac{0.0092}{0.0085} = 1.08$

- $\beta_Z = \frac{0.0061}{0.0085} = 0.72$

Conclusion: Asset Y has the highest expected return (20%) and highest beta (1.08), while X has the lowest return (10%) and lowest beta (0.36). This aligns with CAPM: higher systematic risk (beta) yields higher expected returns.

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5 Multifactor Models

5.1 Motivation

So far, we have assumed that only one variable, the market portfolio, affects expected returns (as in CAPM). However, this may not be realistic for several reasons:

1. **Other Single Factors:** It could be that another single factor, such as interest rates or the business cycle, affects expected returns for all risky assets.
2. **Multiple Factors:** It is possible that a number of macroeconomic factors simultaneously determine expected returns.

5.2 Decomposition of Returns

The return of any asset can be decomposed into two parts: an **expected** part and an **unanticipated** part. The unanticipated part consists of shocks attributable to risk factors and firm-specific shocks.

$$r_i = E[r_i] + \beta_i F + e_i$$

Where:

- $E[r_i]$: Expected return of asset i.
- F: Unanticipated shock to the risk factor ($E[F] = 0$).
- β_i: Sensitivity of asset i's returns to the risk factor.
- e_i: Unanticipated firm-specific shock ($E[e_i] = 0$).

Example (Single Factor): Consider Microsoft stock (r_M) where the single risk factor is GDP growth.

$$r_M = E[r_M] + \beta_M(GDP_{actual} - GDP_{expected}) + e_M$$

Here, $(GDP_{actual} - GDP_{expected})$ is the unanticipated shock to the risk factor.

5.3 Multifactor Model (K-Factors)

We can extend the single index model to a model with K factors.

$$r_i = E[r_i] + \beta_{i1}F_1 + \beta_{i2}F_2 + \cdots + \beta_{iK}F_K + e_i$$

Each F_j represents the unanticipated shock to risk factor j, and β_{ij} is the sensitivity to that factor.

5.3.1 Two-Factor Example

Consider a model with two risk factors: **GDP Growth** and **Inflation**.

$$r_M = E[r_M] + \beta_{M,1}(GDP_{act} - GDP_{exp}) + \beta_{M,2}(Inf_{act} - Inf_{exp}) + e_M$$

Numerical Example:

- Microsoft Expected Return ($E[r_M]$) = 10%
- GDP: Expected = 4%, Actual = 5%. Sensitivity (β_1) = 1.
- Inflation: Expected = 6%, Actual = 7%. Sensitivity (β_2) = 0.4.

$$r_M = 10\% + 1(5\% - 4\%) + 0.4(7\% - 6\%) + e_M = 11.4\% + e_M$$

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6 Arbitrage Pricing Theory (APT)

6.1 Learning Outcomes

After this section, you will be able to:

- Understand the Arbitrage Pricing Theory (APT).
- Use the APT to calculate expected returns.

6.2 The Arbitrage Pricing Theory Model

The APT is the most general asset pricing model. It relates expected returns and risk through factor betas. It posits that the expected return of asset i, $E[r_i]$, is given by:

$$E[r_i] = r_f + \beta_{i,1}RP_1 + \beta_{i,2}RP_2 + \cdots + \beta_{i,K}RP_K$$

Where:

- $E[r_i]$: Expected return of asset i.
- r_f: Risk-free rate.
- $\beta_{i,j}$: Sensitivity of asset i's returns to factor j.
- RP_j: Risk premium for factor j (excess expected return over the risk-free rate).

6.2.1 Comparison with CAPM

- **CAPM:** A specific case where the market portfolio is the only risk factor.
- **APT:** Allows for multiple risk factors. Like CAPM, it is a theoretical model.

6.3 The Law of One Price and Arbitrage

APT relies on the **Law of One Price**, which states that assets with the same future payoff must have the same current price.

- **Arbitrage Opportunity:** A chance to make infinite profits with no risk and no net investment.
- If prices violate the Law of One Price, investors will exploit the arbitrage opportunity until prices correct.

6.4 Determining Risk Premiums: An Example

How do we determine the risk premium for each factor? We use **Factor Portfolios**. A factor portfolio tracks a specific source of macroeconomic risk ($\beta = 1$ for that factor) and is uncorrelated with others ($\beta = 0$ for others).

Example Setup:

- Risk-free rate (r_f) = 3%.
- **Portfolio G (GDP Factor):** $\beta_{GDP} = 1, \beta_{Inf} = 0, E[R_G] = 10\%$.
- **Portfolio I (Inflation Factor):** $\beta_{GDP} = 0, \beta_{Inf} = 1, E[R_I] = 13\%$.
- **Microsoft:** $\beta_{GDP} = 1, \beta_{Inf} = 0.4$.

Replicating Portfolio Q: We form a portfolio Q to mimic Microsoft's sensitivities: Weight in G (w_G) = 1.0; Weight in I (w_I) = 0.4; Weight in Risk-free (w_{rf}) = $1 - 1.0 - 0.4 = -0.4$.

$$E[R_Q] = (1 \times 10\%) + (0.4 \times 13\%) + (-0.4 \times 3\%) = 14\%$$

To prevent arbitrage, Microsoft must also have an expected return of 14%.

Verification using APT Formula:

$$E[r_M] = 3\% + 1(10\% - 3\%) + 0.4(13\% - 3\%) = 14\%$$

6.5 Security Market Plane and Relative Pricing

With two factors, the relationship is a **Security Market Plane** (sensitivities on two axes, expected return on the third). Assets that do not lie on the SMP present arbitrage opportunities.

Relative Pricing: APT determines the price of an asset relative to the factor portfolios. It does not determine if the factor portfolios themselves are correctly priced (absolute pricing).

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7 Arbitrage and Factor Sensitivities

7.1 Learning Outcomes

After this section, you will be able to:

- Identify arbitrage opportunities.
- Solve for factor sensitivities given information on asset returns and unanticipated shocks to risk factors.

7.2 Identifying Arbitrage Opportunities: An Example

Consider a world with two time periods ($t = 0$ and $t = 1$) and two equally likely states in $t = 1$: **Good** and **Bad** (50% probability each).

Stock Data:

- **Microsoft:** $P_0 = 25$. $P_1(\text{Good}) = 40$, $P_1(\text{Bad}) = 20$.
- **Intel:** $P_0 = 20$. $P_1(\text{Good}) = 60$, $P_1(\text{Bad}) = 0$.
- **Coke:** $P_0 = 22.5$. $P_1(\text{Good}) = 55$, $P_1(\text{Bad}) = 15$.

Is there an arbitrage opportunity? Yes. We can construct a zero-cost portfolio with a guaranteed positive payoff.

- **Strategy:** Buy 1 share of Coke, Short 0.5 share of Microsoft, Short 0.5 share of Intel.
- **Cost at $t = 0$:**

$$(1 \times 22.5) - (0.5 \times 25) - (0.5 \times 20) = 22.5 - 12.5 - 10 = 0$$

- **Payoff at $t = 1$ (Good State):**

$$(1 \times 55) - (0.5 \times 40) - (0.5 \times 60) = 55 - 20 - 30 = 5$$

- **Payoff at $t = 1$ (Bad State):**

$$(1 \times 15) - (0.5 \times 20) - (0.5 \times 0) = 15 - 10 - 0 = 5$$

Result: The portfolio costs nothing to set up and guarantees a payoff of 5 regardless of the state. This is an arbitrage opportunity.

7.3 Using APT to Determine Correct Prices

APT relies on relative pricing. Assuming Microsoft and Intel are correctly priced, we can determine the correct price of Coke (which implies Coke is currently mispriced at 22.5).

7.3.1 Step 1: Identify Risk Factors

Assume a single risk factor: the **Business Cycle**.

- Value in Good State = 1.
- Value in Bad State = 0.
- Expected Value $E[F] = 0.5(1) + 0.5(0) = 0.5$.

Unanticipated Shocks ($F - E[F]$):

- Good State: $1 - 0.5 = 0.5$.
- Bad State: $0 - 0.5 = -0.5$.

7.3.2 Step 2: Determine Factor Sensitivities (β) and Expected Returns

We use the single factor model equation:

$$r = E[r] + \beta(F_{\text{shock}})$$

(Note: We assume no firm-specific risk for simplicity in this example).

Microsoft Analysis:

- Return (Good): $\frac{40}{25} - 1 = 60\% = 0.60$.
- Return (Bad): $\frac{20}{25} - 1 = -20\% = -0.20$.

System of equations:

$$\begin{aligned} 0.60 &= E[r_M] + \beta_M(0.5) \\ -0.20 &= E[r_M] + \beta_M(-0.5) \end{aligned}$$

Solving this system:

- Summing the equations: $0.40 = 2E[r_M] \implies E[r_M] = 20\%$.
- Subtracting the second from the first: $0.80 = \beta_M(1) \implies \beta_M = 0.8$.

Intel Analysis:

- Return (Good): $\frac{60}{20} - 1 = 200\% = 2.00$.
- Return (Bad): $\frac{0}{20} - 1 = -100\% = -1.00$.

System of equations:

$$\begin{aligned} 2.00 &= E[r_I] + \beta_I(0.5) \\ -1.00 &= E[r_I] + \beta_I(-0.5) \end{aligned}$$

Solving this system:

- Summing the equations: $1.00 = 2E[r_I] \implies E[r_I] = 50\%$.

- Subtracting the second from the first: $3.00 = \beta_I(1) \implies \beta_I = 3$.

Note: In the next section, we will use these values to compute the risk premium and the arbitrage-free price of Coke.

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8 Calculating Arbitrage-Free Prices and APT Drawbacks

8.1 Learning Outcomes

After this section, you will be able to:

- Solve for risk premiums given factor sensitivities.
- Determine the arbitrage-free price of an asset given the prices of other risky assets.
- Discuss a key drawback of the APT regarding relative pricing.

8.2 Step 2: Solving for Risk Premiums

Continuing from the previous example, we have the expected returns and betas for Microsoft and Intel (assuming they are correctly priced):

- **Microsoft:** $E[r_M] = 20\% = 0.20$, $\beta_M = 0.8$.
- **Intel:** $E[r_I] = 50\% = 0.50$, $\beta_I = 3$.

Using the APT equation $E[r] = r_f + \beta(RP)$, we set up a system of two equations with two unknowns (r_f and RP):

$$\begin{aligned} 0.20 &= r_f + 0.8(RP) \\ 0.50 &= r_f + 3(RP) \end{aligned}$$

Solving the system: Subtracting the first equation from the second:

$$0.30 = 2.2(RP) \implies RP = \frac{0.30}{2.2} \approx 0.1364 \text{ or } 13.64\%$$

Substituting RP back into the first equation:

$$r_f = 0.20 - 0.8(0.1364) = 0.20 - 0.1091 = 0.0909 \text{ or } 9.09\%$$

So, the implied risk-free rate is 9.09% and the factor risk premium is 13.64%.

8.3 Step 3: Determining the Arbitrage-Free Price of Coke

Now we determine the correct price (P) for Coke using the APT parameters derived above.

- **Coke's Payoffs:** Good State = 55, Bad State = 15.
- **APT Requirement:** $E[r_C] = 0.0909 + 0.1364\beta_C$.

We express Coke's expected return and beta in terms of its unknown price P.

- Return (Good): $\frac{55}{P} - 1$.
- Return (Bad): $\frac{15}{P} - 1$.

Using the single factor model decomposition (shocks: Good $+0.5$, Bad -0.5):

$$\frac{55}{P} - 1 = E[r_C] + 0.5\beta_C$$
$$\frac{15}{P} - 1 = E[r_C] - 0.5\beta_C$$

Solving for $E[r_C]$ and β_C:

- Summing: $\frac{70}{P} - 2 = 2E[r_C] \implies E[r_C] = \frac{35}{P} - 1$.
- Subtracting: $\frac{40}{P} = \beta_C$.

Substitute these into the APT equation:

$$\left(\frac{35}{P} - 1\right) = 0.0909 + 0.1364\left(\frac{40}{P}\right)$$

$$\frac{35}{P} - 1 = 0.0909 + \frac{5.456}{P}$$

$$\frac{35 - 5.456}{P} = 1.0909$$

$$P = \frac{29.544}{1.0909} \approx 27.08$$

The arbitrage-free price of Coke is **27.08**.

8.4 Verifying the Arbitrage Portfolio

If Coke is priced at 27.08, let's check the portfolio constructed earlier (Buy 1 Coke, Short 0.5 Microsoft, Short 0.5 Intel).

- **Cost:** $27.08 - 0.5(25) - 0.5(20) = 27.08 - 12.5 - 10 = 4.58$.
- **Payoff:** Always 5 (Risk-free).
- **Return:** $\frac{5}{4.58} - 1 \approx 0.0917 \approx 9.1\%$.

This matches the implied risk-free rate (9.09%), confirming no arbitrage exists at this price.

8.5 Drawbacks of APT: Relative Pricing

APT relies on the assumption that the benchmark assets are correctly priced. If we choose different pairs of assets as our "correct" benchmarks, we get different results.

- **Assuming Intel & Coke are correct:** $r_f = 63.64\%$, $RP = -4.55\%$.
- **Assuming Microsoft & Coke are correct:** $r_f = -9.09\%$, $RP = 36.36\%$.

Conclusion:

- APT is about relative pricing. It ensures internal consistency but does not guarantee that the entire market is priced correctly in absolute terms.
- Depending on the reference assets, implied parameters (r_f, RP) can be unrealistic (e.g., negative risk-free rates).