

Asset Pricing Theory

Zubin

December 31, 2025

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1 Expected Returns and Risk

1.1 Holding Period Return

The holding period return measures the total return earned on an investment over a specific period. It consists of two components:

1. **Income:** Dividends or interest received.
2. **Capital Gain/Loss:** The change in the price of the asset.

The general formula for Holding Period Return (HPR) is:

$$R = \frac{P_1 - P_0 + D_1}{P_0}$$

Where:

- P_0 : Initial Price (Purchase Price)
- P_1 : Price at the end of the period
- D_1 : Dividend paid during the period

Example: You buy a stock for \$50 (P_0). After one year, the price is \$55 (P_1) and it paid a dividend of \$1 (D_1).

$$R = \frac{55 - 50 + 1}{50} = \frac{6}{50} = 0.12 \text{ or } 12\%$$

1.2 Expected Return and Risk

In reality, future prices (P_1) and dividends (D_1) are unknown. Therefore, the return is a **random variable** characterized by possible outcomes and their probabilities.

- **Expected Return** ($E[R]$): The probability-weighted average of all possible returns. It is used as the discount rate for present value calculations.
- **Risk:** The uncertainty of future returns (both good and bad outcomes). It is typically measured by the **Variance** (σ^2) or **Standard Deviation** (σ).

1.3 Comparison: Risk-Free vs. Risky Investment

Consider an investor with \$100 and two investment choices:

1.3.1 Investment 1: Risk-Free

Guaranteed payoff of \$105 after 1 year.

- **Return:** $\frac{105-100}{100} = 5\%$

- **Risk:** Since there is no uncertainty, Risk (σ) = 0.

1.3.2 Investment 2: Risky

Payoff depends on probabilities:

- 60% probability of \$150 (Return: $\frac{150-100}{100} = 50\%$)
- 40% probability of \$80 (Return: $\frac{80-100}{100} = -20\%$)

Calculating Expected Return:

$$E[R] = (0.6 \times 0.50) + (0.4 \times -0.20) = 0.30 - 0.08 = 0.22 \text{ or } 22\%$$

Calculating Risk (Variance and Standard Deviation): Variance is the weighted average of squared deviations from the expected return.

$$\sigma^2 = \sum P_i(r_i - E[R])^2$$

$$\sigma^2 = 0.6(0.50 - 0.22)^2 + 0.4(-0.20 - 0.22)^2$$

$$\sigma^2 = 0.6(0.28)^2 + 0.4(-0.42)^2 = 0.6(0.0784) + 0.4(0.1764)$$

$$\sigma^2 = 0.04704 + 0.07056 = 0.1176$$

$$\text{Standard Deviation } (\sigma) = \sqrt{0.1176} \approx 34.29\%$$

1.3.3 Conclusion

Investment 2 offers a higher expected return (22% vs 5%) but comes with significantly higher risk (34.29% vs 0%). Whether the additional 17% return is sufficient compensation for the risk depends on the investor's **utility** (attitude toward risk) and **asset pricing models**.

2 Diversification and Risk

2.1 The Concept of Diversification

Investing in a single stock is risky; if the company fails, the investor loses their entire investment. To mitigate this, investors form a **portfolio** (a collection of stocks).

- **Diversification:** The strategy of investing in multiple assets to spread risk.
- **Mechanism:** In a portfolio, the poor performance of some stocks is likely to be offset by the good performance of others. Therefore, the risk (variance) of a portfolio is typically lower than the risk of a single stock.

2.2 Types of Risk

As the number of stocks in a portfolio increases, the portfolio variance (risk) decreases. This relationship reveals two distinct types of risk:

1. Diversifiable Risk (Unsystematic Risk):

- The portion of risk that can be eliminated by adding more stocks to the portfolio.
- It is represented by the gap between the total risk curve and the minimum risk floor.
- This risk becomes negligible with a portfolio of about 50 to 60 stocks.

2. Systematic Risk:

- The portion of risk that **cannot** be eliminated, regardless of how many stocks are added.
- It represents the inherent uncertainty of the market that affects all stocks.

2.3 Compensation for Risk

A fundamental principle is that investors should **not** expect compensation (higher returns) for holding diversifiable risk. Since this risk can be eliminated for free through diversification, the market does not reward it. Investors are compensated only for holding systematic risk.

2.3.1 Arbitrage Example

Consider a scenario with a risk-free rate of 5% and four stocks (A, B, C, D) that have **no systematic risk** (only diversifiable risk).

- Expected Returns: A (12%), B (15%), C (17%), D (20%).

If an investor forms an equally weighted portfolio (\$25 in each):

$$E[R_p] = 0.25(12\%) + 0.25(15\%) + 0.25(17\%) + 0.25(20\%) = 16\%$$

Analysis: Since the stocks have no systematic risk, a diversified portfolio eliminates the diversifiable risk, resulting in a total risk of zero. This creates a risk-free asset yielding 16%, while the market risk-free rate is only 5%. Investors would borrow at 5% and invest in this portfolio to earn a guaranteed profit (Arbitrage). This demand would drive up stock prices, reducing returns until they equal the risk-free rate (5%).

Conclusion: In the long run, investors are not compensated for diversifiable risk.

3 The Capital Asset Pricing Model (CAPM)

3.1 The CAPM Formula

The Capital Asset Pricing Model (CAPM) defines the mathematical relationship between expected return and risk. The formula is written as:

$$E[r_i] = r_f + \beta_i(E[r_m] - r_f)$$

Where:

- $E[r_i]$: Expected return of risky asset i .
- r_f : Risk-free rate of return.
- β_i : Beta of the risky asset i .
- $E[r_m]$: Expected return of the market portfolio.

In plain English, the expected return of a risky asset over and above the risk-free rate equals the sensitivity of its return with respect to the market (β) times the expected market risk premium.

3.2 Market Risk Premium

The term $(E[r_m] - r_f)$ is the **Expected Market Risk Premium**. It represents the return in excess of the risk-free rate that an investor expects to earn from holding the market portfolio. It is the compensation for holding the risk of the market portfolio.

3.3 Beta (β) and Systematic Risk

Since the market portfolio is mean-variance efficient, all diversifiable (idiosyncratic) risk has been diversified away. The risk that remains is **Systematic Risk**.

- Beta is a measure of systematic risk.
- Investors are compensated **only** for holding systematic risk, not diversifiable risk.

3.3.1 Calculating Beta

Beta is calculated as the covariance between the risky asset's returns and the market portfolio's returns, divided by the variance of the market portfolio's returns:

$$\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} = \frac{Cov(r_i, r_m)}{\sigma_m^2}$$

3.3.2 Interpreting Beta Values

- $\beta = 1$: The beta of the market portfolio is 1.

- $\beta > 1$: The asset is riskier than the market portfolio.
 - $\beta < 1$: The asset is less risky than the market portfolio.
 - $\beta = 0$: The risk-free asset has a beta of 0.
 - $\beta < 0$: A negative beta means the asset's return moves in the opposite direction to the market's return.
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4 Calculating CAPM Beta

We now apply the CAPM formula to calculate the betas of risky assets.

4.1 Scenario Setup

Let's revisit our three risky asset case (X, Y, Z) and one risk-free asset ($R_f = 5\%$). In previous examples, we found that some portfolios had negative weights, which cannot be a market portfolio. To solve this, we adjust the covariances slightly to ensure positive weights.

- **Asset X:** $E[R] = 10\%$, $\sigma = 7\%$
- **Asset Y:** $E[R] = 20\%$, $\sigma = 10\%$
- **Asset Z:** $E[R] = 15\%$, $\sigma = 12\%$

Table 1: Expected Return and Risk

	Expected Return	Risk
Investment X	10%	7%
Investment Y	20%	10%
Investment Z	15%	12%

Covariances:

- $Cov(X, Y) = 0.0032$
- $Cov(X, Z) = 0.0013$
- $Cov(Y, Z) = 0.0054$

Table 2: Covariance Matrix

Covariance	Investment X	Investment Y	Investment Z
Investment X	0.0049	0.0032	0.0013
Investment Y	0.0032	0.0100	0.0054
Investment Z	0.0013	0.0054	0.0144

4.2 The Market Portfolio

Using Solver in Excel, the weights for the Mean-Variance Efficient (Market) Portfolio are found to be: $w_X = 0.0546$, $w_Y = 0.8424$, $w_Z = 0.1030$. Since all weights are positive, this is a valid market portfolio.

Standard Deviation of the Market Portfolio

$$SD_P = \sqrt{0.0546^2 \times 0.07^2 + 0.8424^2 \times 0.10^2 + 0.1030^2 \times 0.12^2 + 2 \times 0.0546 \times 0.8424 \times 0.0032 + 2 \times 0.0546 \times 0.1030 \times 0.0013 + 2 \times 0.8424 \times 0.1030 \times 0.0054}$$

$$SD_P = 9.22\%$$

Market Expected Return ($E[R_m]$):

$$E[R_m] = (0.0546 \times 10\%) + (0.8424 \times 20\%) + (0.1030 \times 15\%) = 18.94\%$$

Market Standard Deviation (σ_m): Calculated using the portfolio variance formula (sum of weighted variances and covariances):

$$\sigma_m = 9.22\% \implies \sigma_m^2 = 0.0922^2 \approx 0.0085$$

4.3 Calculating Covariance with the Market

To find Beta, we calculate the covariance of each asset's returns with the market portfolio's returns. The covariance of X with the market is the weighted sum of the covariance of X with each component of the market.

$$Cov(R_X, R_m) = w_X Var(X) + w_Y Cov(X, Y) + w_Z Cov(X, Z)$$

Note that $Cov(X, X)$ is simply the variance of X (0.07^2).

$$Cov(R_X, R_m) = 0.0546(0.07^2) + 0.8424(0.0032) + 0.1030(0.0013) \approx 0.0031$$

Similarly, $Cov(R_Y, R_m) = 0.0092$ and $Cov(R_Z, R_m) = 0.0061$.

4.4 Calculating Beta

Using $\beta_i = \frac{Cov(R_i, R_m)}{\sigma_m^2}$ (where $\sigma_m^2 = 0.0922^2$):

- $\beta_X = \frac{0.0031}{0.0085} = 0.36$
- $\beta_Y = \frac{0.0092}{0.0085} = 1.08$

- $\beta_Z = \frac{0.0061}{0.0085} = 0.72$

Conclusion: Asset Y has the highest expected return (20%) and highest beta (1.08), while X has the lowest return (10%) and lowest beta (0.36). This aligns with CAPM: higher systematic risk (beta) yields higher expected returns.

5 Multifactor Models

5.1 Motivation

So far, we have assumed that only one variable, the market portfolio, affects expected returns (as in CAPM). However, this may not be realistic for several reasons:

1. **Other Single Factors:** It could be that another single factor, such as interest rates or the business cycle, affects expected returns for all risky assets.
2. **Multiple Factors:** It is possible that a number of macroeconomic factors simultaneously determine expected returns.

5.2 Decomposition of Returns

The return of any asset can be decomposed into two parts: an **expected** part and an **unanticipated** part. The unanticipated part consists of shocks attributable to risk factors and firm-specific shocks.

$$r_i = E[r_i] + \beta_i F + e_i$$

Where:

- $E[r_i]$: Expected return of asset i .
- F : Unanticipated shock to the risk factor ($E[F] = 0$).
- β_i : Sensitivity of asset i 's returns to the risk factor.
- e_i : Unanticipated firm-specific shock ($E[e_i] = 0$).

Example (Single Factor): Consider Microsoft stock (r_M) where the single risk factor is GDP growth.

$$r_M = E[r_M] + \beta_M (GDP_{actual} - GDP_{expected}) + e_M$$

Here, $(GDP_{actual} - GDP_{expected})$ is the unanticipated shock to the risk factor.

5.3 Multifactor Model (K-Factors)

We can extend the single index model to a model with K factors.

$$r_i = E[r_i] + \beta_{i1}F_1 + \beta_{i2}F_2 + \cdots + \beta_{iK}F_K + e_i$$

Each F_j represents the unanticipated shock to risk factor j , and β_{ij} is the sensitivity to that factor.

5.3.1 Two-Factor Example

Consider a model with two risk factors: **GDP Growth** and **Inflation**.

$$r_M = E[r_M] + \beta_{M,1}(GDP_{act} - GDP_{exp}) + \beta_{M,2}(Inf_{act} - Inf_{exp}) + e_M$$

Numerical Example:

- Microsoft Expected Return ($E[r_M]$) = 10%
- GDP: Expected = 4%, Actual = 5%. Sensitivity (β_1) = 1.
- Inflation: Expected = 6%, Actual = 7%. Sensitivity (β_2) = 0.4.

$$r_M = 10\% + 1(5\% - 4\%) + 0.4(7\% - 6\%) + e_M = 11.4\% + e_M$$