1 Introduction

These brief notes include major definitions and theorems of the graph theory lecture held by Prof. Maria Axenovich at KIT in the winter term 2013/14. We neither prove nor motivate the results and definitions. You can look up the proofs of the theorems in the book “Graph Theory” by Reinhard Diestel [4]. A free version of the book is available at `http://diestel-graph-theory.com`.

Conventions:

- $G = (V, E)$ is an arbitrary (undirected, simple) graph
- $n := |V|$ is its number of vertices
- $m := |E|$ is its number of edges

2 Notations

notation	definition	meaning
$\binom{V}{k}$, V finite set, k integer	$\{S \subseteq V : \|S\| = k\}$	the set of all k-element subsets of V
V^2, V finite set	$\{(u,v) : u, v \in V\}$	the set of all ordered pairs of elements in V
$[n]$, n integer	$\{1, \ldots, n\}$	the set of the first n positive integers
$\mathbb{N}$	$1, 2, \ldots$	the natural numbers, not including 0
2^S, S finite set	$\{T : T \subseteq S\}$	the power set of S, i.e., the set of all subsets of S
$S \triangle T$, S, T finite sets	$(S \cup T) \setminus (S \cap T)$	the symmetric difference of sets S and T, i.e., the set of elements that appear in exactly one of S or T
$A \dot{\cup} B$, A, B disjoint sets	$A \cup B$	the disjoint union of A and B

3 Preliminaries

Definition. A *graph* G is an ordered pair (V, E), where V is a finite set and $E \subseteq \binom{V}{2}$ is a set of pairs of elements in V.

graph, G

- The set V is called the set of *vertices* and E is called the set of *edges* of G.

vertex, edge

- The edge $e = \{u, v\} \in \binom{V}{2}$ is also denoted by $e = uv$.
- If $e = uv \in E$ is an edge of G, then u is called *adjacent* to v and u is called *incident* to e.

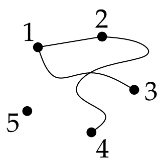
adjacent, incident

- If e_1 and e_2 are two edges of G, then e_1 and e_2 are called *adjacent* if $e_1 \cap e_2 \neq \emptyset$, i.e., the two edges are incident to the same vertex in G.

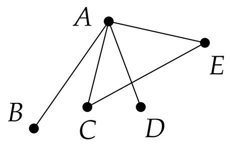
We can visualize graphs $G = (V, E)$ using pictures. For each vertex $v \in V$ we draw a point (or small disc) in the plane. And for each edge $uv \in E$ we draw a continuous curve starting and ending in the point/disc for u and v, respectively.

Several examples of graphs and their corresponding pictures follow:

$V = [5]$, $E = \{12, 13, 24\}$



$$V = \{A, B, C, D, E\},$$
$$E = \{AB, AC, AD, AE, CE\}$$



Definition (Graph variants)**.**

- A *directed graph* is a pair $G = (V, A)$ where V is a finite set and $E \subseteq V^2$. The edges of a directed graph are also called *arcs*.

directed graph
arc

- A *multigraph* is a pair $G = (V, E)$ where V is a finite set and E is a multiset of elements from $\binom{V}{1} \cup \binom{V}{2}$, i.e., we also allow loops and multiedges.

multigraph

- A *hypergraph* is a pair $H = (X, E)$ where X is a finite set and $E \subseteq 2^X \setminus \{\emptyset\}$.

hypergraph

Definition. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we say that G_1 and G_2 are *isomorphic*, denoted by $G_1 \simeq G_2$, if there exists a bijection $\phi : V_1 \to V_2$ with $xy \in E_1$ if and only if $\phi(x)\phi(y) \in E_2$. Loosely speaking, G_1 and G_2 are isomorphic if they are the same up to renaming of vertices.

isomorphic, $\simeq$

When making structural comments, we do not normally distinguish between isomorphic graphs. Hence, we usually write $G_1 = G_2$ instead of $G_1 \simeq G_2$ whenever vertices

$=$

are indistinguishable. Then we use the informal expression *unlabeled graph* (or just *graph* when it is clear from the context) to mean an isomorphism class of graphs.

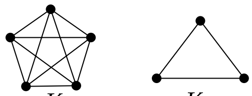
unlabeled graph

Important graphs and graph classes

Definition. For all natural numbers n we define:

- the *complete graph* K_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \binom{[n]}{2}\big)$. Complete graphs correspond to *cliques*.

  complete graph, K_n

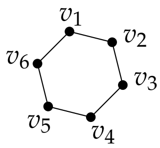
  

  K_5 K_3

- for $n \geq 3$, the *cycle* C_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\} \cup \big\{n, 1\big\}\big)$. The *length of a cycle* is its number of edges. We write $C_n = 12\ldots n1$. The cycle of length 3 is also called a *triangle*.

  cycle, C_n

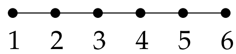
  triangle

  $C_6 = v_1v_2v_3v_4v_5v_6v_1$

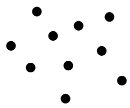
- the *path* P_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \big\{\{i, i+1\} : i = 1, \ldots, n-1\big\}\big)$. The vertices 1 and n are called the *endpoints* or *ends* of the path. The *length of a path* is its number of edges. We write $P_n = 12\ldots n$.

  path, P_n

- the *empty graph* E_n on n vertices as the (unlabeled) graph isomorphic to $\big([n], \emptyset\big)$. Empty graphs correspond to independent sets.

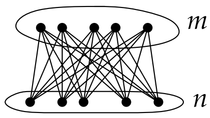
  empty graph, E_n

  E_{10}

- for $m \geq 1$, the *complete bipartite graph* $K_{m,n}$ on $n+m$ vertices as the (unlabeled) graph isomorphic to $(A \cup B, \{xy : x \in A, y \in B\})$, where $|A| = m$ and $|B| = n$, $A \cap B = \emptyset$.

  complete bipartite graph, $K_{m,n}$

  $K_{m,n}$

- for $r \geq 2$, a *complete r-partite* graph as an (unlabeled) graph isomorphic to

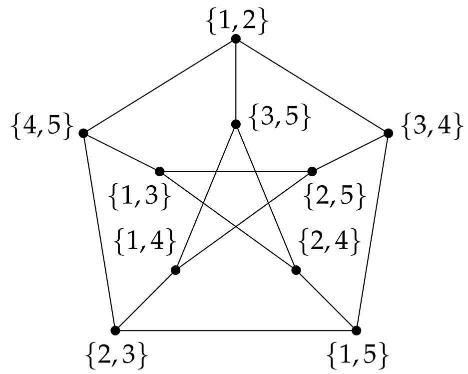
  complete r-partite

$$\left(A_1\dot\cup\cdots\dot\cup A_r, \{xy : x \in A_i, y \in A_j, i \neq j\}\right),$$

  where $A_1, \dots, A_r$ are non-empty finite sets. In particular, the complete bipartite graph $K_{m,n}$ is a complete 2-partite graph.

- the *Petersen graph* as the (unlabeled) graph isomorphic to $\left(\binom{[5]}{2}, \{\{S,T\} : S, T \in \binom{[5]}{2}, S \cap T = \emptyset\}\right)$.

  Petersen graph



- for a natural number k, $k \leq n$, the *Kneser graph* $K(n,k)$ as the (unlabeled) graph isomorphic to

  Kneser graph, $K(n,k)$

$$\left(\binom{[n]}{k}, \left\{\{S,T\} : S, T \in \binom{[n]}{k}, S \cap T = \emptyset\right\}\right).$$

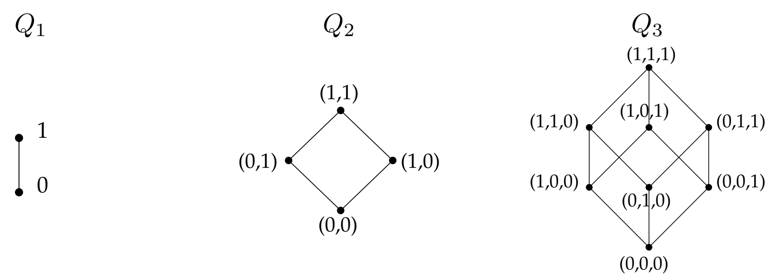
  Note that $K(5,2)$ is the Petersen graph.

- the *n-dimensional hypercube* Q_n as the (unlabeled) graph isomorphic to

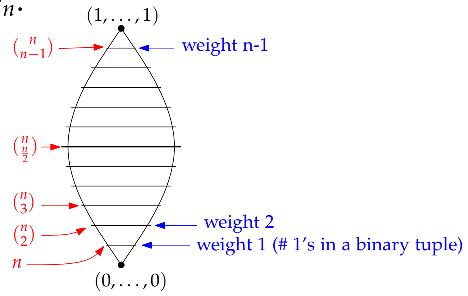
  hypercube, Q_n

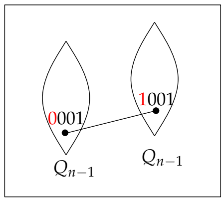
$$\left(2^{[n]}, \{\{S,T\} : S, T \in 2^{[n]}, |S \triangle T| = 1\}\right).$$

  Vertices are labeled either by corresponding sets or binary indicators vectors. For example the vertex $\{1,3,4\}$ in Q_6 is coded by $(1,0,1,1,0,0,0)$.



Q_n:



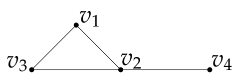


Basic graph parameters and degrees

Definition. Let $G = (V, E)$ be a graph. We define the following parameters of G.

- The graph G is *non-trivial* if it contains at least one edge, i.e., $E \neq \emptyset$. Equivalently, G is non-trivial if G is not an empty graph. non-trivial
- The *order of* G, denoted by $|G|$, is the number of vertices of G, i.e., $|G| = |V|$. order, $|G|$
- The *size of* G, denoted by $\|G\|$, is the number of edges of G, i.e., $\|G\| = |E|$. Note that if the order of G is n, then the size of G is between 0 and $\binom{n}{2}$. size, $\|G\|$
- Let $S \subseteq V$. The *neighbourhood of* S, denoted by $N(S)$, is the set of vertices in V that have an adjacent vertex in S. The elements of $N(S)$ are called *neighbours* of S. Instead of $N(\{v\})$ for $v \in V$ we usually write $N(v)$. neighbourhood, $N(v)$ neighbour
- If the vertices of G are labeled $v_1, \ldots, v_n$, then there is an $n \times n$ matrix A with entries in $\{0, 1\}$, which is called the *adjacency matrix* and is defined as follows: adjacency matrix

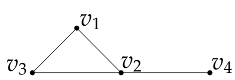
$$v_i v_j \in E \qquad \Leftrightarrow \qquad A[i, j] = 1$$



$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A graph and its adjacency matrix.

- The *degree* of a vertex v of G, denoted by $d(v)$ or $\deg(v)$, is the number of edges incident to v. degree, $d(v)$



$\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 2$, $\deg(v_4) = 1$

- A vertex of degree 1 in G is called a *leaf*, and a vertex of degree 0 in G is called an *isolated vertex*.

  leaf
  isolated vertex

- The *degree sequence* of G is the multiset of degrees of vertices of G, e.g. in the example above the degree sequence is $\{1, 2, 2, 3\}$.

  degree sequence

- The *minimum degree of* G, denoted by $\delta(G)$, is the smallest vertex degree in G (it is 1 in the example).

  minimum degree, $\delta(G)$

- The *maximum degree of* G, denoted by $\Delta(G)$, is the highest vertex degree in G (it is 3 in the example).

  maximum degree, $\Delta(G)$

- The graph G is called k*-regular* for a natural number k if all vertices have degree k. Graphs that are 3-regular are also called *cubic*.

  regular
  cubic

- The *average degree of* G is defined as $d(G) = \left(\sum_{v \in V} \deg(v)\right)/|V|$. Clearly, we have $\delta(G) \leq d(G) \leq \Delta(G)$ with equality if and only if G is k-regular for some k.

  average degree, $d(G)$

Lemma 1 (Handshake Lemma, 1.2.1)**.** For every graph $G = (V, E)$ we have

$$2|E| = \sum_{v \in V} d(v).$$

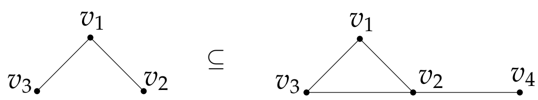
Corollary 2. The sum of all vertex degrees is even and therefore the number of vertices with odd degree is even.

Subgraphs

Definition.

- A graph $H = (V', E')$ is a *subgraph of* G, denoted by $H \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If H is a subgraph of G, then G is called a *supergraph of* H, denoted by $G \supseteq H$. In particular, $G_1 = G_2$ if and only if $G_1 \subseteq G_2$ and $G_1 \supseteq G_2$.

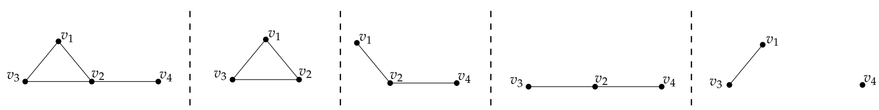
  subgraph, $\subseteq$
  supergraph, $\supseteq$



- A subgraph H of G is called an *induced subgraph* of G if for every two vertices $u, v \in V(H)$ we have $uv \in E(H) \Leftrightarrow uv \in E(G)$. In the example above H is not an induced subgraph of G. Every induced subgraph of G can be obtained by deleting vertices (and all incident edges) from G.

  induced subgraph

  Examples:

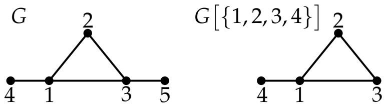


- Every induced subgraph of G is uniquely defined by its vertex set. We write

$G[X]$ for the induced subgraph of G on vertex set X, i.e., $G[X] = \big(X, \{xy : x, y \in X,\ xy \in E(G)\}\big)$. Then $G[X]$ is called the *subgraph of G induced by the vertex set $X \subseteq V(G)$*.

$G[X]$

Example:



- If H and G are two graphs, then an *(induced) copy* of H in G is an (induced) subgraph of G that is isomorphic to H.

  copy

- A subgraph $H = (V', E')$ of $G = (V, E)$ is called a *spanning subgraph* of G if $V' = V$.

  spanning subgraph

- A graph $G = (V, E)$ is called *bipartite* if there exists natural numbers m, n such that G is isomorphic to a subgraph of $K_{m,n}$. In this case, the vertex set can be written as $V = A \dot{\cup} B$ such that $E \subseteq \{ab \mid a \in A, b \in B\}$. The sets A and B are called *partite sets of G*.

  bipartite

  partite sets

- A *cycle (path, clique)* in G is a subgraph H of G that is a cycle (path, complete graph).

  clique

- An *independent set* in G is an induced subgraph H of G that is an empty graph.

  independent set

- A *walk* (of length k) is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*.

  walk

  closed walk

- Let $A, B \subseteq V$, $A \cap B = \emptyset$. A path P in G is called an *A-B-path* if $P = v_1 \dots v_k$, $V(P) \cap A = \{v_1\}$ and $V(P) \cap B = \{v_k\}$. When $A = \{a\}$ and $B = \{b\}$, we simply call P an a-b-path. If G contains an a-b-path we say that the vertices a and b are *linked by a path*.

  A-B-path

- Two paths P, P' in G are called *independent* if every vertex contained in both P and P' (if any) is an endpoint of P and P'. I.e., P and P' can share only endpoints.

  independent paths

- A graph G is called *connected* if any two vertices are linked by a path.

  connected

- A subgraph H of G is *maximal*, respectively *minimal*, with respect to some property if there is no supergraph, respectively subgraph, of H with that property.

  maximal, minimal

- A maximal connected subgraph of G is called a *connected component* of G.

  component

- A graph G is called *acyclic* if G does not have any cycle. Acyclic graphs are also called *forests*.

  acyclic

  forest

- A graph G is called a *tree* if G is connected and acyclic.

  tree

Proposition 3. If a graph G has minimum degree $\delta(G) \geq 2$, then G has a path of length $\delta(G)$ and a cycle with at least $\delta(G)+1$ vertices.

Proposition 4. If a graph has a u-v-walk, then it has a u-v-path.

Proposition 5. If a graph has a closed walk of odd length, then it contains an odd cycle.

Proposition 6. If a graph has a closed walk with a non-repeated edge, then the graph contains a cycle.

Proposition 7. A graph is bipartite if and only if it has no cycles of odd length.

Definition. An *Eulerian tour of G* is a closed walk containing all edges of G, each exactly once.

Eulerian tour

Theorem 8 (Eulerian Tour Condition, 1.8.1)**.** A connected graph has an Eulerian tour if and only if every vertex has even degree.

Lemma 9. Every tree on at least two vertices has a leaf.

Lemma 10. A tree of order $n \geq 1$ has exactly $n-1$ edges.

Lemma 11. Every connected graph contains a spanning tree.

Lemma 12. A connected graph on $n \geq 1$ vertices and $n-1$ edges is a tree.

Lemma 13. The vertices of every connected graph can be ordered $(v_1, \ldots, v_n)$ so that for every $i \in \{1, \ldots, n\}$ the graph $G\big[\{v_1, \ldots, v_i\}\big]$ is connected.

Operations on graphs

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, $U \subseteq V$ be a subset of vertices of G and $F \subseteq \binom{V}{2}$ be a subset of pairs of vertices of G. Then we define

- $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. Note that $G, G' \subseteq G \cup G'$ and $G \cap G' \subseteq G, G'$. Sometimes, we also write $G + G'$ for $G \cup G'$.

$G \cup G', G \cap G'$

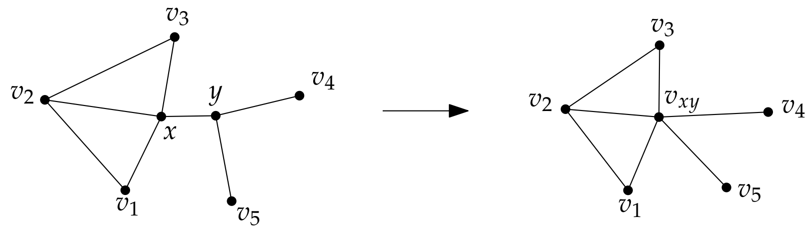
- $G - U := G[V \setminus U]$, $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$. If $U = \{u\}$ or $F = \{e\}$ then we simply write $G - u$, $G - e$ and $G + e$ for $G - U$, $G - F$ and $G + F$, respectively.

$G - U$, $G - F$, $G + F$

- For an edge $e = xy$ in G we define $G \circ e$ as the graph obtained from G by identifying x and y and removing (if necessary) loops and multiple edges. We say that $G \circ e$ arises from G by *contracting the edge e*.

$G \circ e$

contract

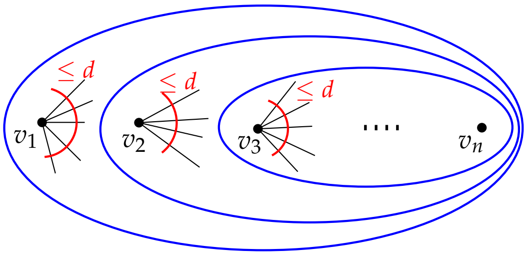


- The *complement of* G, denoted by $\overline{G}$ or G^C, is defined as the graph $(V, \binom{V}{2} \setminus E)$. In particular, $G + \overline{G}$ is a complete graph, and $\overline{G} = (G + \overline{G}) - E$. complement, $\overline{G}$

More graph parameters

Definition. Let $G = (V, E)$ be any graph.

- The *girth of* G, denoted by $g(G)$, is the length of a shortest cycle in G. If G is acyclic, its girth is said to be ∞. girth, $g(G)$
- The *circumference* of G is the length of a longest cycle in G. If G is acyclic, its circumference is said to be 0. circumference
- The graph G is called *Hamiltonian* if G has a spanning cycle, i.e., there is a cycle in G that contains every vertex of G. In other words, G is Hamiltonian if and only if its circumference is $|V|$. Hamiltonian
- The graph G is called *traceable* if G has a spanning path, i.e., there is a path in G that contains every vertex of G. traceable
- For two vertices u and v in G, the *distance between* u *and* v, denoted by $d(u, v)$, is the length of a shortest u-v-path in G. If no such path exists, $d(u, v)$ is said to be ∞. distance, $d(u, v)$
- The *diameter of* G, denoted by $\operatorname{diam}(G)$, is the maximum distance among all pairs of vertices in G, i.e. diameter, $\operatorname{diam}(G)$
$$\operatorname{diam}(G) = \max_{u,v \in V} d(u, v).$$
- The *radius of* G, denoted by $\operatorname{rad}(G)$, is defined as radius, $\operatorname{rad}(G)$
$$\operatorname{rad}(G) = \min_{u \in V} \max_{v \in V} d(u, v).$$
- If there is a vertex ordering $v_1, \ldots, v_n$ of G and a $d \in \mathbb{N}$ such that
$$|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \leq d,$$
for all $i \in [n - 1]$ then G is called d-*degenerate*. The minimum d for which G is d-degenerate is called the *degeneracy* of G. d-degenerate degeneracy



We remark that the 1-degenerate graphs are precisely the forests.

- A *proper k-edge colouring* is an assignment $c'\colon E \to [k]$ of colours in $[k]$ to edges such that no two adjacent edges receive the same colour. The *chromatic index of G*, or *edge-chromatic number*, is the minimal k such that G has a k-edge colouring. It is denoted by $\chi'(G)$. proper edge colouring chromatic index, $\chi'(G)$
- A *proper k-vertex colouring* is an assignment $c\colon V \to [k]$ of colours in $[k]$ to vertices such that no two adjacent vertices receive the same colour. The *chromatic number of G* is the minimal k such that G has a k-vertex colouring. It is denoted by $\chi(G)$. proper vertex colouring chromatic number, $\chi(G)$

Proposition 14. For any graph $G = (V, E)$ the following are equivalent:

(i) G is a tree, that is, G is connected and acyclic.

(ii) G is connected, but for any edge $e \in E$ in G the graph $G - e$ is not connected.

(iii) G is acyclic, but for any edge $e \notin E$ not in G the graph $G + e$ has a cycle.

(iv) G is connected and 1-degenerate.

(v) G is connected and $|E| = |V| - 1$.

(vi) G is acyclic and $|E| = |V| - 1$.

(vii) G is connected and every non-trivial subgraph of G has a vertex of degree at most 1.

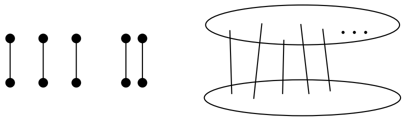
(viii) Any two vertices are joined by a unique path in G.

4 Matchings

Definition.

- A *matching* (*independent edge set*) is a vertex-disjoint union of edges, i.e., the union of pairwise non-adjacent edges.

matching



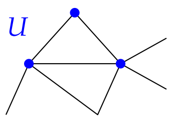
- A *matching in* G is a subgraph of G isomorphic to a matching. We denote the size of the largest matching in G by $\nu(G)$.

$\nu(G)$

- A *vertex cover in* G is a set of vertices $U \subseteq V$ such that each edge in E is incident to at least one vertex in U. We denote the size of the smallest vertex cover in G by $\tau(G)$.

vertex cover

$\tau(G)$



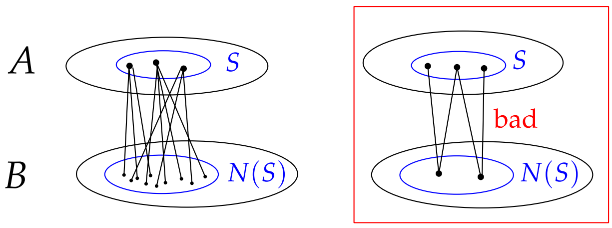
- A *k-factor of* G is a k-regular spanning subgraph of G.

k-factor

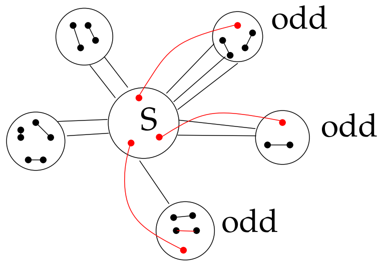
- A *1-factor of* G is also called a *perfect matching* since it is a matching of largest possible size in a graph of order $|V|$. Clearly, G can only contain a perfect matching if $|V|$ is even.

perfect matching

Theorem 15 (Hall's Marriage Theorem, 2.1.2)**.** Let G be a bipartite graph with partite sets A and B. Then G has a matching containing all vertices of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.



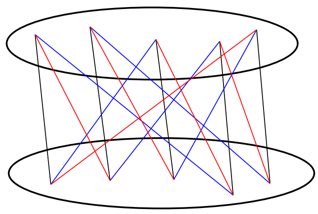
Theorem 16 (Tutte's Theorem, 2.2.1)**.** For $S \subseteq V$ define $q(S)$ to be the number of odd components of $G - S$, i.e., the number of connected components of $G - S$ consisting of an odd number of vertices. A graph G has a perfect matching if and only if $q(S) \leq |S|$ for all $S \subseteq V$.



$|S| \geq$ odd components of $G - S$

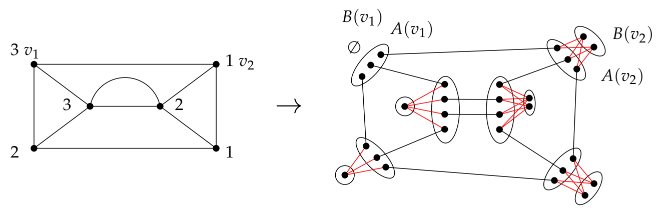
Corollary 17.

- Let G be bipartite with partite sets A and B such that $|N(S)| \geq |S| - d$ for all $S \subseteq A$, and a fixed positive integer d. Then G contains a matching of size at least $|A| - d$.
- A k-regular bipartite graph has a perfect matching.
- A k-regular bipartite graph has a proper k-edge coloring.

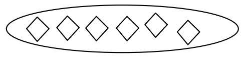


Definition. Let $G = (V, E)$ be any graph.

- For all functions $f : V \to \mathbb{N} \cup \{0\}$ an *f-factor of G* is a spanning subgraph H of G such that $\deg_H(v) = f(v)$ for all $v \in V$. f-factor
- Let $f : V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \leq \deg(v)$ for all $v \in V$. We can construct the auxiliary graph $T(G, f)$ by replacing each vertex v with vertex sets $A(v) \cup B(v)$ such that $|A(v)| = \deg(v)$ and $|B(v)| = \deg(v) - f(v)$. For adjacent vertices u and v we place an edge between $A(u)$ and $A(v)$ such that the edges between the A-sets are independent. We also insert a complete bipartite graph between $A(v)$ and $B(v)$ for each vertex v. $T(G, f)$



- Let H be a graph. An *H-factor of G* is a spanning subgraph of G that is a vertex-disjoint union of copies of H, i.e., a set of copies of H in G whose vertex sets form a partition of V. H-factor

$H =$  $G =$ 

Lemma 18. Let $f\colon V \to \mathbb{N} \cup \{0\}$ be a function with $f(v) \leq \deg(v)$ for all $v \in V$. Then G has an f-factor if and only if $T(G, f)$ has a 1-factor.

Theorem 19 (König's Theorem, 2.1.1)**.** Let G be bipartite. Then $\nu(G) = \tau(G)$, i.e., the size of a largest matching is the same as the size of a smallest vertex cover.

Theorem 20 (Hajnal and Szemerédi)**.** If G satisfies $\delta(G) \geq (1 - 1/k)n$, where k is a divisor of n, then G has a K_k-factor.

Theorem 21 (Alon and Yuster)**.** Let H be a graph. If G satisfies

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n,$$

then G contains at least $\big(1 - o(1)\big) \cdot n/|V(H)|$ vertex-disjoint copies of H.

5 Connectivity

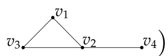
Definition.

- For a natural number $k \geq 1$, a graph G is called k*-connected* if $|V(G)| \geq k+1$ and for any set U of $k-1$ vertices in G the graph $G - U$ is connected. In particular, K_n is $(n-1)$-connected.

  k-connected

- The maximum k for which G is k-connected is called the *connectivity of* G, denoted by $\kappa(G)$.

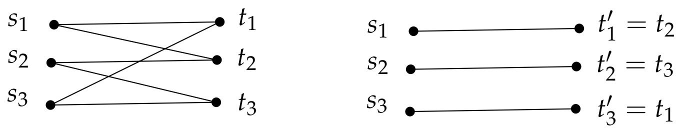
  connectivity, $\kappa(G)$

  $\kappa($ 

$) = 1,\ \kappa(C_n) = 2,\ \kappa(K_{n,m}) = \min\{m, n\}.$

- For a natural number $k \geq 1$, a graph G is called k*-linked* if for any $2k$ distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ there are vertex-disjoint s_i-t_i-paths, $i = 1, \ldots, k$.

  k-linked

- For a graph $G = (V, E)$ a set $X \subseteq V \cup E$ of vertices and edges of G is called a *cut set* of G if $G - X$ has more connected components than G. If a cut set consists of a single vertex v, then v is called a *cut vertex* of G; if it consists of a single edge e, then e is called a *cut edge or bridge* of G.

  cut set
  cut vertex
  cut edge, bridge

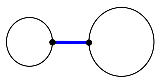
- For a natural number $\ell \geq 1$, a graph G is called ℓ*-edge-connected* if G is non-trivial and for any set $F \subseteq E$ of fewer than ℓ edges in G the graph $G - F$ is connected.

  ℓ-edge-connected

- The *edge-connectivity of* G is the maximum ℓ such that G is ℓ-edge-connected. It is denoted by $\kappa'(G)$ or $\lambda(G)$.

  edge-connectivity, $\kappa'(G)$

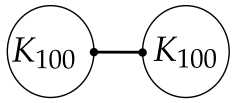
$$G \text{ non-trivial tree} \Rightarrow \lambda(G) = 1,\ G \text{ cycle} \Rightarrow \lambda(G) = 2.$$



Clearly, for every $k, \ell \geq 2$, if a graph is k-connected, k-linked or ℓ-edge-connected, then it is also $(k-1)$-connected, $(k-1)$-linked or $(\ell-1)$-edge-connected, respectively. Moreover, for a non-trivial graph is it equivalent to be 1-connected, 1-linked, 1-edge-connected, or connected.

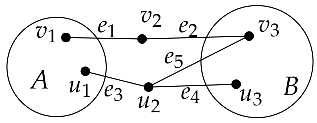
Lemma 22. For any connected, non-trivial graph G we have

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$



A graph G with $\kappa(G), \lambda(G) \ll \delta(G)$.

Definition. For a subset X of vertices and edges of G and two vertex sets A, B in G we say that X *separates A and B* if each A-B-path contains an element of X. separate



Some sets separating A and B: $\{e_1, e_4, e_5\}$, $\{e_1, u_2\}$, $\{u_1, u_3, v_3\}$

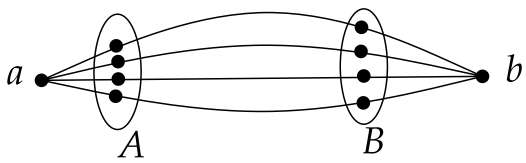
Note that if X separates A and B, then necessarily $A \cap B \subseteq X$.

Theorem 23 (Menger's Theorem, 3.3.1)**.** For any graph G and any two vertex sets $A, B \subseteq V(G)$ we have

$$\min \#\text{vertices separating } A \text{ and } B = \max \#\text{independent } A\text{-}B\text{-paths}.$$

Corollary 24. If a, b are vertices of G, $\{a, b\} \notin E(G)$, then

$$\min \#\text{vertices separating } a \text{ and } b \ = \ \max \#\text{independent } a\text{-}b\text{-paths}$$

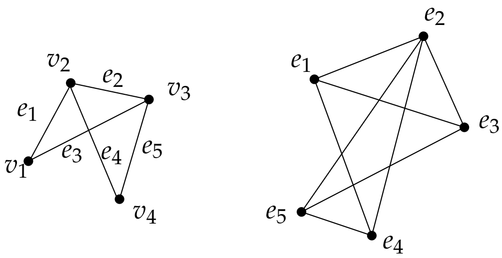


Theorem 25 (Global Version of Menger's Theorem, 3.3.6)**.** A graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths.

Note that Menger's Theorem implies that if G is k-linked, then G is k-connected. Moreover, Bollobás and Thomason proved in 1996 that if G is $22k$-connected, then G is k-linked.

Definition. For a graph $G = (V, E)$ the *line graph* $L(G)$ of G is the graph $L(G) = (E, E')$, where line graph $L(G)$

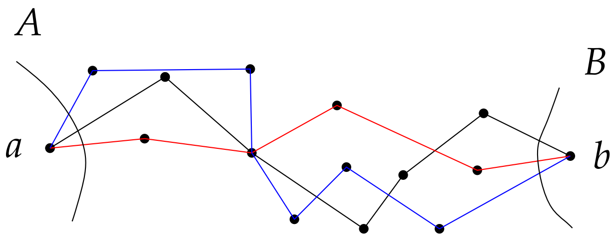
$$E' = \left\{ \{e_1, e_2\} \in \binom{E}{2} : e_1 \text{ adjacent to } e_2 \text{ in } G \right\}.$$



A graph and its line graph.

Corollary 26. If a, b are vertices of G, then

$$\min \#\text{edges separating } a \text{ and } b = \max \#\text{edge-disjoint } a\text{-}b\text{-paths}$$



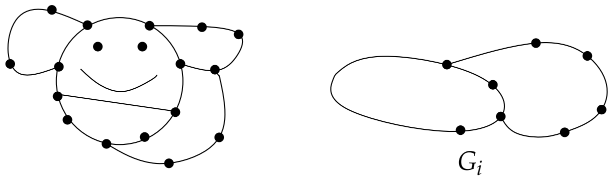
Moreover, a graph is k-edge-connected if and only if there are k edge-disjoint paths between any two vertices.

Definition. Given a graph H, we call a path P an H*-path* if P is non-trivial (has length at least one) and meets H exactly in its ends. In particular, the edge of any H-path of length 1 is never an edge of H. H-path

An *ear* of H is a non-trivial non-seperating path P in H whose internal vertices have degree 2 and whose ends have degree at least 3 each. In particular, if P is an ear of H, then P is an H'-path for the graph H' obtained from H by removing all edges and internal vertices of P. Conversely, if both ends of an H-path P lie in the same connected component of H, then P is an ear of $H + P$. ear

An *ear-decomposition* of a graph G is a sequence $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k$ of graphs, such that ear-decomposition

- G_0 is a cycle,
- for each $i = 1, \ldots, k$ the graph G_i arises from G_{i-1} by adding a G_{i-1}-path P_i, i.e., P_i is an ear of G_i, and
- $G_k = G$.

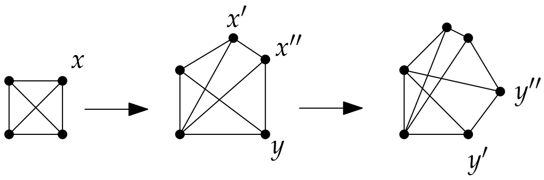


Theorem 27 (3.1.1)**.** A graph is 2-connected if and only if it has an ear-decomposition.

Lemma 28. If G is 3-connected, then there exists an edge e of G such that $G \circ e$ is also 3-connected.

Theorem 29 (Tutte, 3.2.3)**.** A graph G is 3-connected if and only if there exists a sequence of graphs $G_0, G_1, \dots, G_k$, such that

- $G_0 = K_4$,
- for each $i = 1, \dots, k$ the graph G_i has two adjacent vertices x', x'' of degree at least 3, so that $G_{i-1} = G_i \circ x'x''$, and
- $G_k = G$.

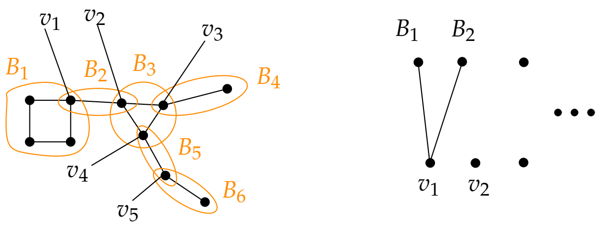


Definition. Let G be a graph. A maximal connected subgraph of G without a cut vertex is called a *block* of G. In particular, the blocks of G are exactly the bridges and the maximal 2-connected subgraphs of G.

block

The *block-cut-vertex graph or block graph* of G is a bipartite graph H whose partite sets are the *blocks* of G and the cut vertices of G, respectively. There is an edge between a block B and a cut vertex a if and only if $a \in B$, i.e., the block contains the cut vertex.

block-cut-vertex graph



The leaves of this graph are called *block leaves*.

block leaf

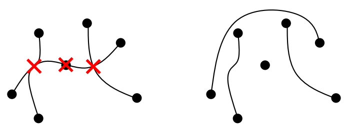
Theorem 30. The block-cut-vertex graph of a connected graph is a tree.

6 Planar graphs

This section deals with graph drawings. We restrict ourselves to graph drawings in the plane $\mathbb{R}^2$. It is also feasible to consider graph drawings in other topological spaces, such as the torus.

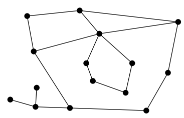
Definition.

- The *straight line segment* between $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$ is the set $\{p+\lambda(q-p) : 0 \le \lambda \le 1\}$. straight line segment
- A *homeomorphism* is a continuous function that has a continuous inverse function. homeomorphism
- Two sets $A \in \mathbb{R}^2$ and $B \in \mathbb{R}^2$ are said to be *homeomorphic* if there is a homeomorphism $f\colon A \to B$. homeomorphic
- A *polygon* is a union of finitely many line segments that is homeomorphic to the circle $S^1 := \{x \in \mathbb{R}^2 : \|x\| = 1\}$. polygon
- An *arc* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the closed unit interval $[0,1]$. The images of 0 and 1 under such a homeomorphism are the *endpoints of the arc*. If P is an arc with endpoints p and q, then P *links* them and runs *between* them. The set $P \setminus \{p,q\}$ is the *interior of* P, denoted by $\mathring{P}$. arc endpoint of arc interior of arc
- Let $O \subseteq \mathbb{R}^2$ be an open set. Being linked by an arc in O is an equivalence relation on O. The corresponding equivalence classes are the *regions of* O. A closed set $X \subseteq \mathbb{R}^2$ is said to *separate* O if $O \setminus X$ has more regions than O. The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \setminus X$. Note that if X is closed, its frontier lies in X, while if X is open, its frontier lies in $\mathbb{R}^2 \setminus X$. region separate frontier
- A *plane graph* is a pair (V,E) of finite sets with the following properties (the elements of V are again called *vertices*, those in E *edges*): plane graph
  1. $V \subseteq \mathbb{R}^2$;
  2. every $e \in E$ is an arc between two vertices;
  3. different edges have different sets of endpoints;
  4. the interior of an edge contains no vertex and no point of any other edge.

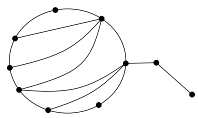
  

  A plane graph (V,E) defines a graph G on V in a natural way. As long as no confusion can arise, we shall use the name G of this abstract graph also for the plane graph (V, E), or for the point set $V \cup \bigcup E$.

- For any plane graph G, the set $\mathbb{R}^2 \setminus G$ is open; its regions are the *faces of* G. faces, $F(G)$
- The face of G corresponding to the unbounded region is the *outer face* of G; the other faces are its *inner faces*. The set of all faces is denoted by $F(G)$. outer face inner face
- The subgraph of G whose point set is the frontier of a face f is said to *bound* f and is called its *boundary*; we denote it by $G[f]$. boundary of f, $G[f]$
- Let G be a plane graph. If one cannot add an edge to form a plane graph $G' \supsetneq G$ with $V(G') = V(G)$, then G is called *maximally plane*. If every face in $F(G)$ (including the outer face) is bounded by a triangle in G, then G is called a *plane triangulation*. maximally plane triangulation
- A *planar embedding* of an abstract graph $G = (V, E)$ is an isomorphism between G and a plane graph G'. The latter is called a *drawing* of G. We shall not distinguish notational between the vertices of G and G'. A graph $G = (V, E)$ is *planar* if it has a planar embedding. planar embedding planar graph



- A graph $G = (V, E)$ is *outerplanar* if it has a plane embedding such that the boundary of the outer face contains all of the vertices V. outerplanar graph

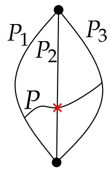


Theorem 31 (Fáry's Theorem)**.** Every planar graph has a plane embedding with straight line segments as edges.

Lemma 32 (Jordan Curve Theorem for Polygons, 4.1.1)**.** Let $P \subseteq \mathbb{R}^2$ be a polygon. Then $\mathbb{R}^2 \setminus P$ has exactly two regions. One of the regions is unbounded, the other is bounded. Each of the two regions has P as frontier.

Lemma 33. Let P_1, P_2 and P_3 be internally disjoint arcs that have the same endpoints. Then

1. $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively.
2. Let P be an arc from the interior of P_1 to the interior of P_3 whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ containing the interior of P_2. Then P contains a points of P_2.



Lemma 34. Let G be a plane graph and e be an edge of G. Then the following hold.

- The frontier X of a face of G either contains e or is disjoint from the interior of e.
- If e is on a cycle in G, then e is on the frontier of exactly two faces.
- If e is on no cycle in G, then e is on the frontier of exactly one face.

Lemma 35. A plane graph is maximally plane if and only if it is a triangulation.

Theorem 36 (Euler's Formula, 4.2.9)**.** Let G be a connected plane graph with v vertices, e edges and f faces. Then

$$v - e + f = 2.$$

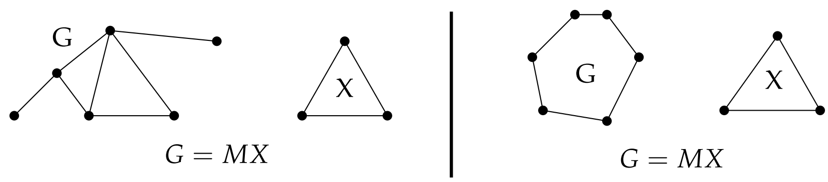
Corollary 37. Let $G = (V, E)$ be a plane graph. Then

- $|E| \leq 3|V| - 6$ with equality exactly if G is a plane triangulation.
- $|E| \leq 2|V| - 4$ if no face in $F(G)$ is bounded by a triangle.

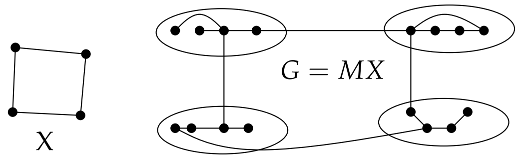
Lemma 38 (Pick's Formula)**.** Let P be a polygon with corners on the grid $\mathbb{Z}^2$, let A be its area, I be the number of grid points strictly inside of P and B be the number of grid points on the boundary of P. Then $A = I + B/2 - 1$.

Definition. Let G and X be two graphs.

- We say that G *is an* MX, denoted by $G = MX$, if $V(G)$ can be partitioned as $\{V_x \mid x \in V(X)\}$ such that $G[V_x]$ is connected for every $x \in V(X)$ and there is an V_x–V_y edge in G if and only if $xy \in E(X)$. MX, $G = MX$
- We say that X *is a minor of* G if $H = MX$ for some subgraph H of G. minor, $X \preccurlyeq G$



Alternatively, X is a minor of G if and only if X can be obtained from G by successive vertex deletions, edge deletions and edge contractions.



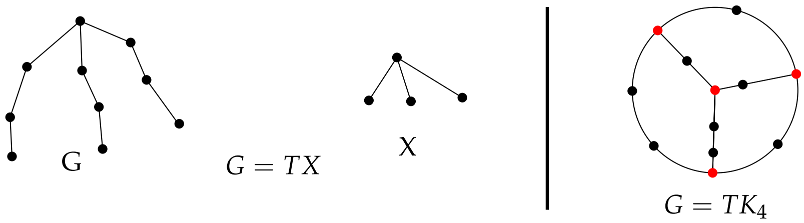
- The graph G is a *single-edge subdivision of* X if $V(G) = V(X) \cup \{v\}$ and $E(G) = E(X) - xy + xv + vy$ for some edge $xy \in E(X)$ and $v \notin V(X)$. We say that G *is a* TX, denoted by $G = TX$, if G can be obtained from X by a series of single-edge subdivisions.

  subdivision

  TX, $G = TX$

- We say that X *is a topological minor of* G, if $H = TX$ for some subgraph H of G.

  topological minor



Theorem 39 (Kuratowski's Theorem, 4.4.6)**.** A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as topological minors.

Definition.

- Let X be a set and $\leq \subseteq X^2$ be a relation on X, i.e., $\leq$ is a subset of all ordered pairs of elements in X. Then $\leq$ is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order is *total* if $x \leq y$ or $y \leq x$ for every $x, y \in X$.

  partial order
  total order

- Let $\leq$ be a partial order on a set X. The pair $(X, \leq)$ is called a *poset* (partially ordered set). If $\leq$ is clear from context, the set X itself is called a poset. The *poset dimension of* $(X, \leq)$ is the smallest number d such that there are total orders $R_1, \ldots, R_d$ on X with $\leq \, = R_1 \cap \cdots \cap R_d$.

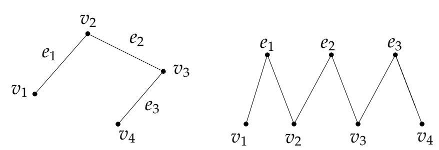
  poset

  poset dimension, $\dim(X, \leq)$

$$\dim(\substack{\bullet\\ \bullet}) = 1,\ \dim(\underset{x}{\bullet}\ \ \underset{y}{\bullet}) = 2 \text{ since } \underset{x}{\bullet}\ \ \underset{y}{\bullet} = \substack{\bullet x\\ \bullet y} \cap \substack{\bullet y\\ \bullet x}$$

- The *incidence poset* $(V \cup E, \leq)$ on a graph $G = (V, E)$ is given by $v \leq e$ if and only if e is incident to v for all $v \in V$ and $e \in E$.

  incidence poset



Theorem 40 (Schnyder)**.** Let G be a graph and P be its incidence poset. Then G is planar if and only if $\dim(P) \leq 3$.

Theorem 41 (5-Color Theorem, 5.1.2)**.** Every planar graph is 5-colorable.

The more well-known 4-coloring theorem is much harder to prove. Interestingly, it is one of the first theorems that has been proved using computer assistance. The computer-generated proof uses an enormous case distinction. Some mathematicians have philosophical problems with this approach since the resulting proof cannot be easily verified by humans. A shorter proof is still outstanding.

Theorem 42 (4-Color Theorem, 5.1.1)**.** Every planar graph is 4-colorable.

Definition.

- Let $L(v) \subseteq \mathbb{N}$ be a list of colors for each vertex $v \in V$. We say that G is *L-list-colorable* if there is coloring $c\colon V \to \mathbb{N}$ such that $c(v) \in L(v)$ for each $v \in V$ and adjacent vertices receive different colors. L-list-colorable
- Let $k \in \mathbb{N}$. We say that G is *k-list-colorable or k-choosable* if G is L-list-colorable for each list L with $|L(v)| = k$ for all $v \in V$. k-list-colorable
- The *choosability*, denoted by $\mathrm{ch}(G)$, is the smallest k such that G is k-choosable. choosability, $\mathrm{ch}(G)$
- The *edge choosability*, denoted by $\mathrm{ch}'(G)$, is defined analogously. edge choosability, $\mathrm{ch}'(G)$

Theorem 43 (Thomassen's 5-List Color Theorem, 5.4.2)**.** Every planar graph is 5-choosable.

7 Colorings

Lemma 44 (Greedy estimate for the chromatic number)**.**
Let G be a graph. Then $\chi(G) \leq \Delta(G) + 1$.

Theorem 45 (Brook's Theorem, 5.2.4)**.** Let G be a connected graph.
Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle.

Definition.

- The *clique number* $\omega(G)$ *of* G is the largest order of a clique in G. clique number, $\omega(G)$
- The *co-clique number* $\alpha(G)$ *of* G is the largest order of an independent set in G. co-clique number, $\alpha(G)$
- A graph G is called *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. For example, bipartite graphs are perfect with $\chi = \omega = 2$. perfect graph

Lemma 46 (Small Coloring Results)**.**

- $\chi(G) \geq \max\{\omega(G), n/\alpha(G)\}$ since each color class is an empty induced subgraph and $\chi(K_k) = k$.
- $\|G\| \geq \binom{\chi(G)}{2} \Leftrightarrow \chi(G) \leq 1/2 + \sqrt{2\|G\| + 1/4}$ since there must be at least one edge between any two color classes.
- The chromatic number $\chi(G)$ of G is at most one more than the length of a longest directed path in any orientation of G.

Theorem 47 (Lovász' Perfect Graph Theorem, 5.5.4)**.** A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Theorem 48 (Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour & Thomas, 5.5.3)**.** A graph G is perfect if and only if it does not contain an odd cycle on at least 5 vertices (an *odd hole*) or the complement of an odd hole as an induced subgraph.

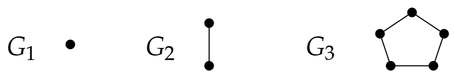
Definition. For an integer $k \geq 1$ we define k*-constructible* graphs recursively as follows: k-constructible

- K_k is k-constructible.
- If G is k-constructible and $x, y \in V(G)$ are non-adjacent, then also $(G + xy)/xy$ is k-constructible.
- If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$, $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible.

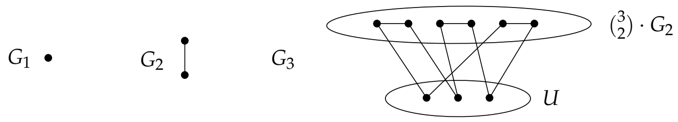
Theorem 49 (Hajós Theorem, 5.2.6)**.** Let G be a graph and $k \geq 1$ be an integer. Then $\chi(G) \geq k$ if and only if G has a k-constructible subgraph.

Example (Mycielski's Construction)**.**
We can construct a family $\big(G_k = (V_k, E_k)\big)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows:

- G_1 is the single-vertex graph, G_2 is the single-edge graph, i.e., $G_1 = K_1$ and $G_2 = K_2$.
- $V_{k+1} := V_k \cup U \cup \{w\}$ where $V_k \cap \big(U \cup \{w\}\big) = \emptyset$, $V_k = \{v_1, \dots, v_n\}$ and $U = \{u_1, \dots, u_n\}$.
- $E_{k+1} := E_k \cup \big\{wu_i : i = 1, \dots, k\big\} \cup \bigcup_{i=1}^{n} \big\{u_i v : v \in N_{G_k}(v_i)\big\}$.



Example (Tutte's Construction)**.** We can construct a family $(G_k)_{k\in\mathbb{N}}$ of triangle-free graphs with $\chi(G_k) = k$ as follows: G_1 is the single-vertex graph. To get from G_k to G_{k+1}, take an independent set U of size $k(|G_k| - 1) + 1$ and $\binom{|U|}{|G_k|}$ vertex-disjoint copies of G_k. For each subset of size $|G_k|$ in U then introduce a perfect matching to exactly one of the copies of G_k.



Theorem 50 (König's Theorem, 5.3.1)**.**
Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.

Theorem 51 (Vizing's Theorem, 5.3.2)**.**
Let G be a graph. Then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.

Lemma 52. We have $\mathrm{ch}(K_{n,n}) \geq c \cdot \log(n)$ for some constant $c > 0$. In particular,

$$\mathrm{ch}\left(K_{\binom{2k-1}{k},\binom{2k-1}{k}}\right) \geq c \cdot k.$$

Theorem 53 (Galvin's Theorem, 5.4.4)**.**
Let G be a bipartite graph. Then $\mathrm{ch}'(G) = \chi'(G)$.

1.c)State and prove the handshaking theorem. [June 2017, 5 marks]

Handshaking problem :

If G is a (p,q) graph with $V(G)=\{V_1....V_p\}$ and $d_i=d_G(V_i)$, $1\leq i\leq p$, then

$$2q = \sum_{i=1}^{p} d_i$$

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x \text{ is an endpoint of } e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

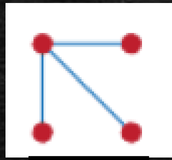
$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

1.d)Define the following symbols : i) δ (G)[June 2017,1 mark]

Minimum vertex degree of a graph is δ (G). It is min{$d_G(x)$:x∈ V(G)} .

δ (G) is a non-negative integer.

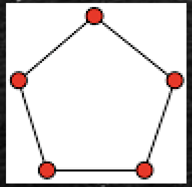


G

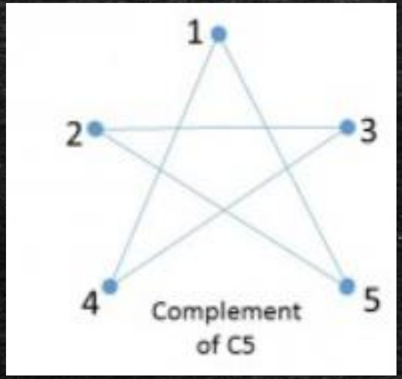
δ (G)=1

2.a)What is meant by complement of a graph ? Find the complement of the C5 graph (i.e. C5). [June 2017, 3 marks]

Complement of a graph : Let graph G=(V,E) be a (p,q) graph. Complement of the graph is a graph $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy : xy \notin E(G),\ x, y \in V(G)\}$.

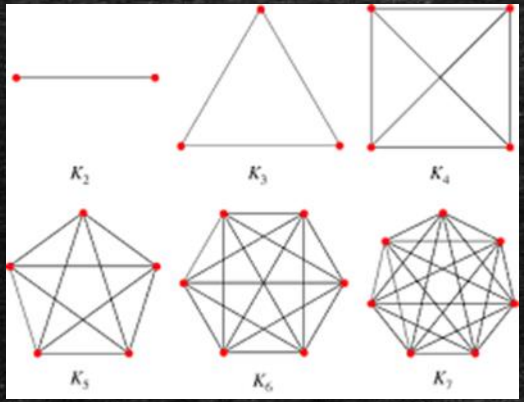


C_5



2.b) What is a complete graph ? [June 2017, 2 marks]

Complete graph : Graph in which any two vertices are adjacent, i.e. each vertex is joined to every other vertex by a vertex. A complete graph on n vertices is represented by K_n.



5.c)Define isomorphism. Determine whether the following pair of graphs are isomorphic : [June 2017, 3 marks]

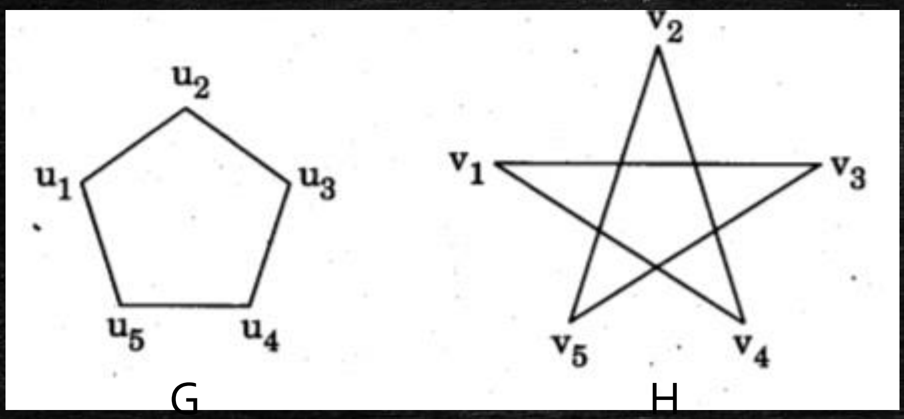
Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

ii) xy ∈ E(G) if and only if f(x) f(y) ∈ E(H)

If not they are called non-isomorphic graphs.



To check for isomorphism check the following :

1. Number of vertices

Number of vertices in G=5

Number of vertices in H=5

2. Number of edges

Number of edges in G=5

Number of edges in H=5

3. Degree sequence

Degree sequence of G : <2,2,2,2,2>

Degree of sequence of H : <2,2,2,2,2>

The above shows that degree sequence of two graphs is the same.

f(u1)=v1, f(u2)=v2, f(u3)=v3, f(u4)=v4, f(u5)=v5

From the above checks, we can conclude that the two graphs are isomorphic.

3.c)What do you mean by isomorphic graphs ?[June 2016, 2 marks]

Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

ii) xy ∈ E(G) if and only if f(x) f(y) ∈ E(H)

If not they are called non-isomorphic graphs.

To check if two graphs check for these conditions :

1. Count the number of vertices – must be equal
2. Count the number of edges – must be equal
3. Degree sequence – must be same
4. Number of cycles – must be same
5. Max degree vertex and min degree vertex
6. Peculiarity of adjacent vertices

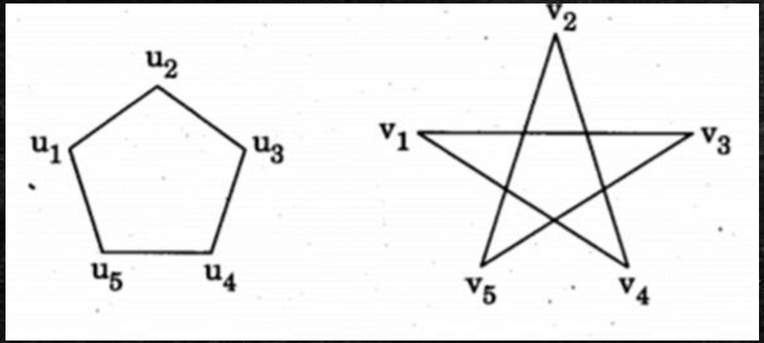
To check for isomorphism check the following :

1. Number of vertices

Number of vertices in G=5

Number of vertices in H=5

Consider the two graphs:



2. Number of edges

Number of edges in G=5

Number of edges in H=5

3. Degree sequence

Degree sequence of G : <2,2,2,2,2>

Degree of sequence of H : <2,2,2,2,2>

The above shows that degree sequence of two graphs is the same.

From the above checks, we can conclude that the two graphs are isomorphic.

4.a) State Handshaking Theorem.[June 2016,3 marks]

If G is a (p,q) graph with $V(G)=\{V_1....V_p\}$ and $d_i=d_G(V_i)$, $1\le i\le p$, then $2q = \sum_{i=1}^{p} d_i$

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x$ is an endpoint of $e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

4.b) A non-directed graph G has 8 edges. Find the number of vertices, if the degree of each vertex in G is 2. [June 2016, 3 marks]

According to the formula,

$$2q = \sum_{i=1}^{p} d_i$$

q=8

Sum of degree of all vertices < = 2 * no. of edges . [According to Handshaking theorem]

Let n be number of vertices in graph.
==> 2*n = 2*8
==> 2n=16

==> n=8

1.b)Prove that the complement of $\overline{G}$ is G.[December 2016,5 marks]

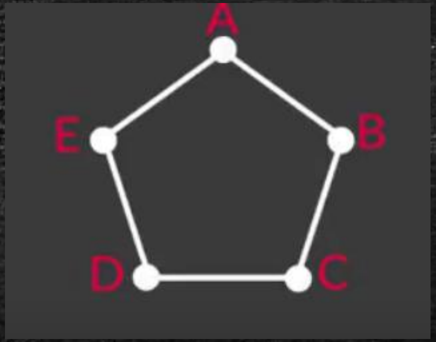
Let graph G=(V,E) be a (p,q) graph. Complement of the graph is a graph V($\overline{G}$) = V(G) and E($\overline{G}$)= {xy : xy ∉ E (G), x, y ∈V (G)}.

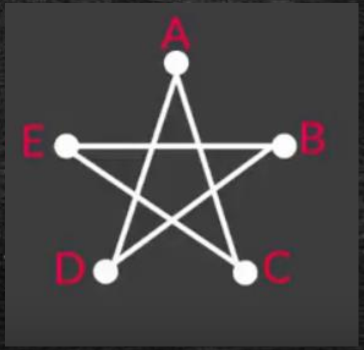
From the above definition, we can say that complement of a graph $\overline{G}$ has,

V( )=V(G) and E($\overline{G}$)= {xy : xy ∉ E (G), x, y ∈V (G)}.
Complement of $\overline{G}$ is G
(V($\overline{G}$))’=V(G) and (E$\overline{G}$))’={xy : xy ∉ E ($\overline{G}$), x, y ∈V (G)}=E(G)

Hence proved.

Example :





G

$\overline{G}$

V(G)={A,B,C,D,E}

V($\overline{G}$)=V(G)={A,B,C,D,E}

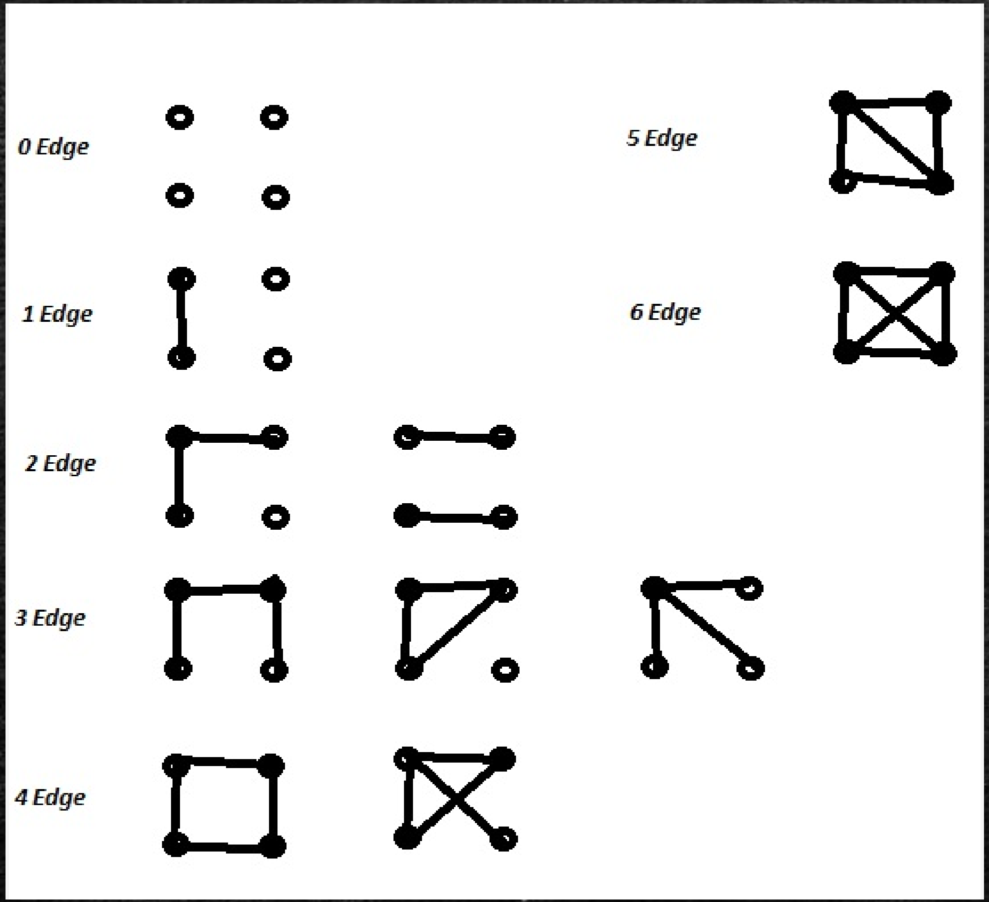
E(G)={AE,AB,BC,CD,DE}

E($\overline{G}$)={AD,AC,BE,BD,EC}

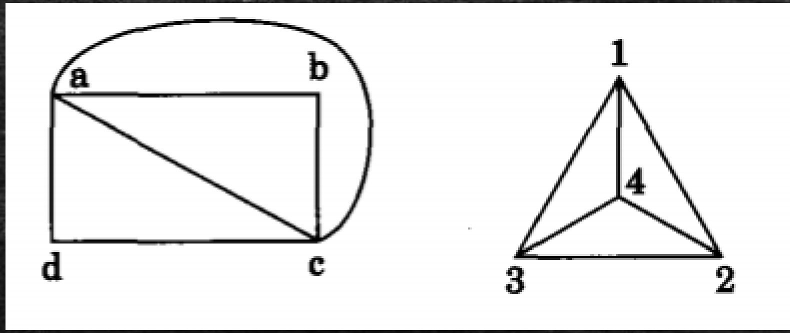
Similarly :Complement of $\overline{G}$ is G.

Hence proved.

1.c)Draw at least 3 non-isomorphic graphs on 4 vertices.[December 2016,5 marks]



1.c)Determine whether the following graphs are isomorphic. If yes, justify your answer. [December 2016,December 2010,4 marks]



Number of vertices in G= 4
Number of vertices in H=4

Number of edges in G=6
Number of edges in H=6

Degree sequence of G : {4,4,2,2}
Degree sequence of H : {3,3,3,3}

Degree sequences of graph G and H are different, therefore the two graphs are non-isomorphic.

1.d)What is an undirected graph ? Prove that an undirected graph has even number vertices of odd degree.[December 2016, 4 marks]

Undirected graph G is a finite non-empty set V together with set E containing pairs of points of V. V is called the vertex set and E is the edge set of G. In undirected graph, E(G) will be symmetric on V(G). If (u,v) is there, then (v,u) will be there.

Any graph can only have an even number of odd vertices.

Consider a (p,q) graph with {x1,x2,.....xt} is a set of odd vertices and {xt+1,.....xp} is a set of even vertices.

Let $d_G(x_i)=2c_i+1$ $1 \le i \le t$ and $d_G(x_i)=2r_i$ $t+1 \le i \le p$

Then Theorem 1 says that $2q = \sum_{1}^{p} d_G(x_i)$

$$\Rightarrow 2q = \sum_{1}^{t}(2c_i + 1) + \sum_{t+1}^{p}(2r_i) = 2(c_1 + c_2 + \Lambda + c_t) + t + 2(r_{t+1} + \Lambda + r_p),$$

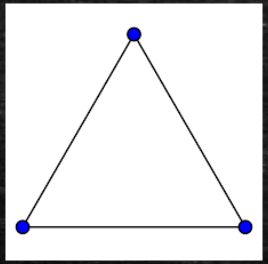
which shows that t is even.

2.a)Define n-regular graph. Show for which value of n the following graphs are regular : (i) K_n (ii) Q_n [December 2016, 5 marks]

It is a graph in which each vertex has the same degree. It is said to be regular graph degree of regularity r. G is an r-regular graph where $0 \le r \le (p-1)$.

i) K_n

K_n is a regular graph with n=3.



The degree of each vertex is 2. So, K3 is regular graph.

K_n for n>3 it is (n–1)-regular.

2.c)How many edges does a complete graph of 5 vertices have ? [December 2016,2 marks]

Number of edges in a complete graph of n vertices = n(n-1)/2

In the above question,

number of vertices ,n =5

Number of edges = (n(n-1))/2

= (5(5-1))/2

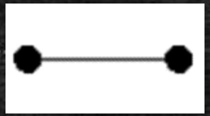
=(5*4)/2

=10

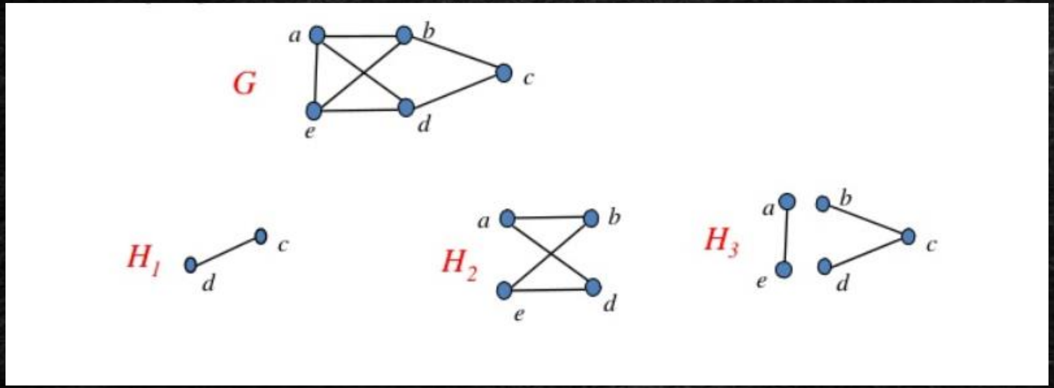
3.b)Define a graph and a subgraph. Show that for a subgraph H of a graph G Δ (H) ≤ Δ (G). [December 2016S, 5 marks]

A graph is a set of the form {(x,f(x)): x is a domain of function f}.

Example :



Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) ⊆ V (G) and E (H) ⊆ E (G).



3.a)Show that for a subgraph H of a graph G $\Delta(H) \leq \Delta(G)$. [December 2014, December 2011,June 2010,December 2010, 5marks]

Let $x \in V(H)$ such that $d_H(x) = H(\Delta)$

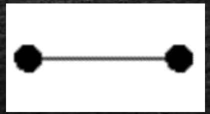
Then, $N_H(x) \subseteq N_G(x)$. Thus,

$\Delta(H) = |N_H(x)| \leq |N_G(x)| \leq \Delta(G)$

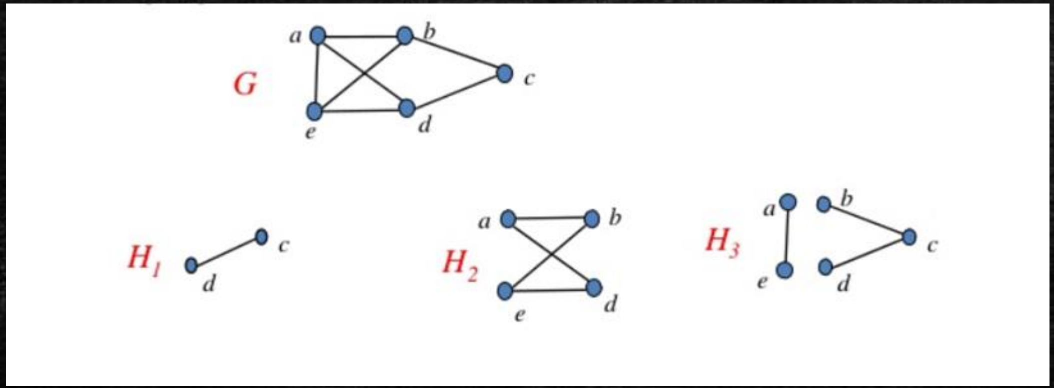
2.d)Define Graph and Subgraph. Give an example of a subgraph H of a graph G with $\delta(G) < \delta(H)$ and $\Delta(H) \le \Delta(G)$.[June 2015,4 marks]

A graph is a set of the form {(x,f(x)): x is a domain of function f}.

Example :



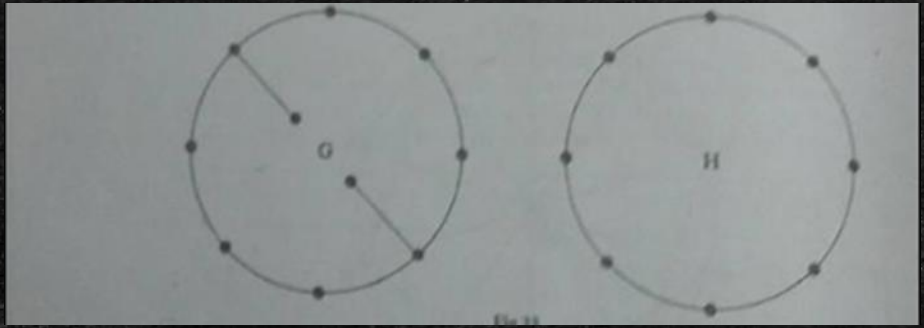
Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) ⊆ V (G) and E (H) ⊆ E (G).



$\delta(G)=1<2=\delta(H)$

$\Delta(H)=2<3=\Delta(G)$

Diagram :

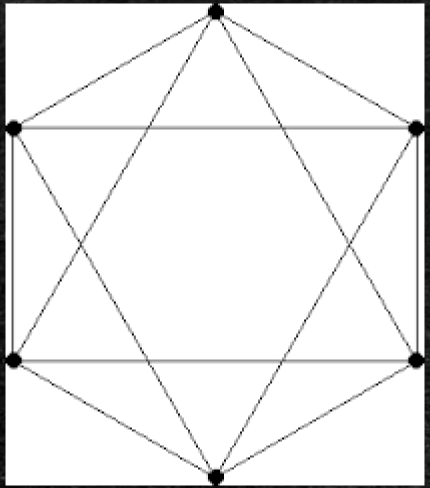


1.a)Define regular graph. Find the number of edges of a 4-regular graph with 6 vertices.[December 2015,3 marks]

It is a graph in which each vertex has the same degree. It is said to be regular graph degree of regularity r. G is an r-regular graph where $0 \le r \le (p-1)$.

K_n is a regular graph with degree of regularity (n-1) when n > 3.

4-regular graph with 6 vertices:



Number of edges =12

3.c)Define isomorphic graph. Give an example of the same.[December 2015, 2 marks]

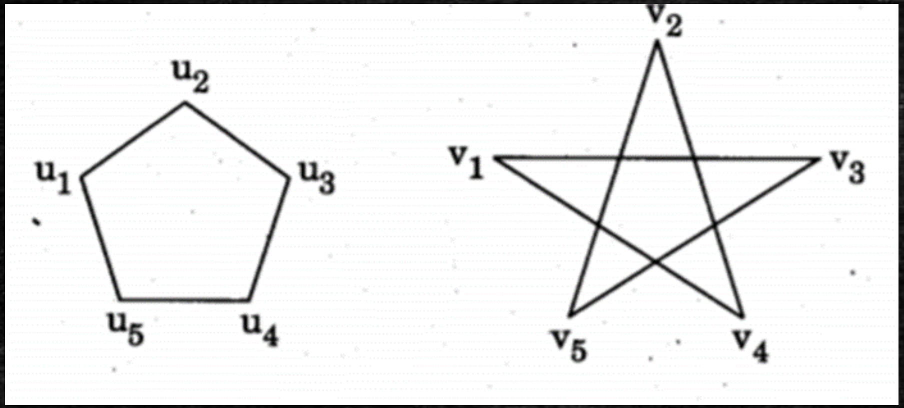
Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

ii) xy ∈ E(G) if and only if f(x) f(y) ∈ E(H)

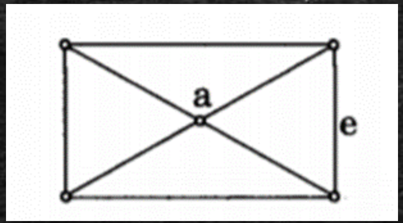
If not they are called non-isomorphic graphs.



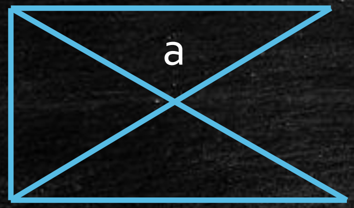
Both are (5, 5)-graphs. Degree sequence of both the graphs is <2,2,2,2,2>. Both these graphs have a copy of C_5. Therefore, both these graphs are isomorphic.

4.b)For the following graph G, draw subgraphs 3 (i) G - e (ii) G - a . [December 2015,3 marks]

Graph G:



i) G-e

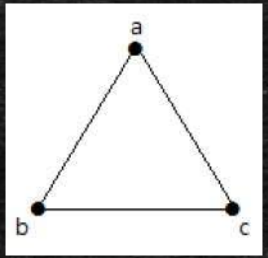


ii)

Define : (i) Simple graph (ii) Finite and infinite graph (iii) Isolated vertex (iv) Subgraph[June 2014, 4 marks]

i) Simple graph :

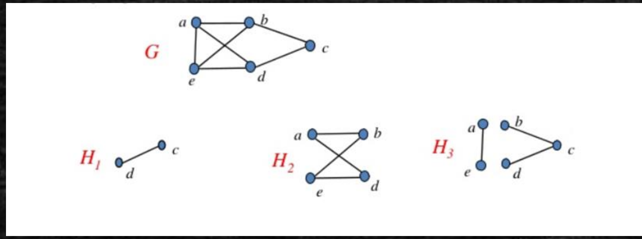
Undirected graph that has no loops or multiple edges is called a simple graph. When an edge joins a vertex to itself is called a loop. Two or more edges that joins the same vertices are parallel or multiple edges.



ii) Finite and infinite graph : A graph with a finite number of vertices and edges is called a finite graph. A graph with a finite number of nodes and edges.

iii)Isolated vertex : Vertex with degree zero is called an isolated vertex.

iv) Subgraph : Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) ⊆ V (G) and E (H) ⊆ E (G).



1.f)How many edges are there in a graph with 10 vertices each of degree 6 ? [June 2014,3 marks]

According to Handshaking theorem,

$$2q = \sum_{i=1}^{p} d_i$$

q: number of edges

p: number of vertices

d_i: degree of vertex I

In the above question : p=10, d(i)=6

2q=10*6=60

q=30

3.b)Define Isomorphism of two graphs. Find whether the given graphs are isomorphic or not.[June 2014,5 marks]

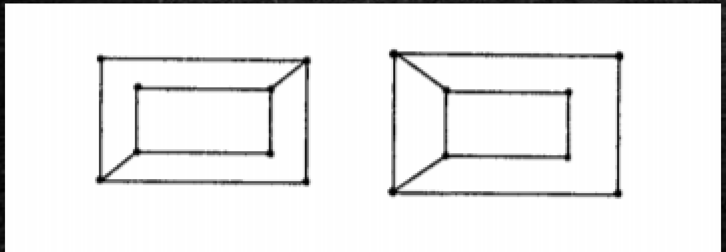
Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

ii) xy ∈ E(G) if and only if f(x) f(y) ∈ E(H)

If not they are called non-isomorphic graphs.



Number of vertices in first graph = 8

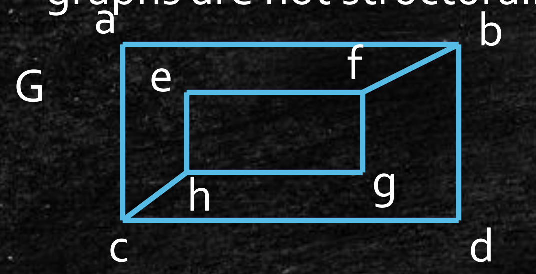
Number of vertices in second graph = 8

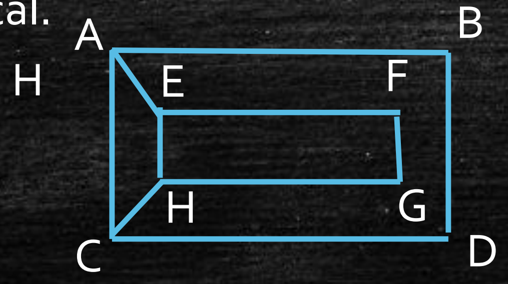
Number of edges in first graph = 10

Number of edges in second graph = 10

Degree sequence of both the graphs is : <3,3,3,3,2,2,2,2>

These conditions satisfies but still the graphs are non-isomorphic. This is because the two graphs are not structurally identical.





In graph G, A is a vertex of degree 2, which must corresponds to either B, D, F or G in H.

Each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

Therefore, these are not isomorphic.

5.b)State and prove Handshaking Theorem.[June 2014, 5 marks]

Handshaking problem :

If G is a (p,q) graph with $V(G)=\{V_1....V_p\}$ and $d_i=d_G(V_i)$, $1\le i\le p$, then $$2q = \sum_{i=1}^{p} d_i$$

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x$ is an endpoint of $e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i \,. \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

1.d)State and prove Handshaking Theorem.[December 2014,December 2010, 4 marks]

Handshaking problem :

If G is a (p,q) graph with $V(G)=\{V_1....V_p\}$ and $d_i=d_G(V_i)$, $1 \le i \le p$, then $$2q = \sum_{i=1}^{p} d_i$$

Proof: Consider the set $S = \{(x, e): x \in V(G), e \in E(G), x \text{ is an endpoint of } e\}$. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

3.a)Show that for a subgraph H of a graph G $\Delta(H) \le \Delta(G)$. [December 2014, December 2011,June 2010,December 2010, 5marks]

Let $x \in V(H)$ such that $d_H(x) = H(\Delta)$

Then, $N_H(x) \subseteq N_G(x)$. Thus,

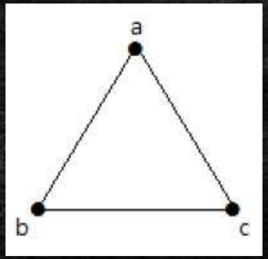
$\Delta(H) = |N_H(x)| \le |N_G(x)| \le \Delta(G)$

1.a)Define : 4 (i) Graph (ii) Simple Graph (iii) null graph (iv) connected Graph [December 2013,4 marks]

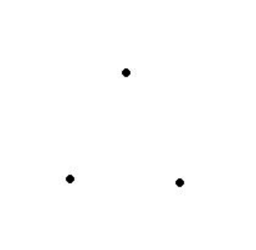
i) Graph :It is a set of the form {(x,f(x)): x is a domain of function f}.

ii) Simple graph :

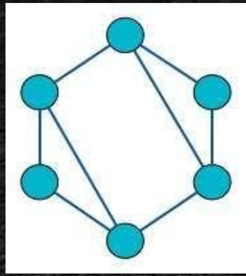
Undirected graph that has no loops or multiple edges is called a simple graph. When an edge joins a vertex to itself is called a loop. Two or more edges that joins the same vertices are parallel or multiple edges.



iii) Null graph : A graph with isolated vertices and no edges is called a null graph. It is also known as an empty graph.

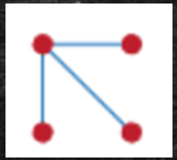


iv) Connected graph : A graph is connected when there is a path between every pair of vertices. In a connected graph, there are no unreachable vertices.



1.d)Define δ (G) and Δ (G) for a graph G.[December 2013,2 marks]

$\delta(G)$ is called the minimum vertex degree of G. It is $\min\{d_G(x) : x \in V(G)\}$. It is a non-negative integer.

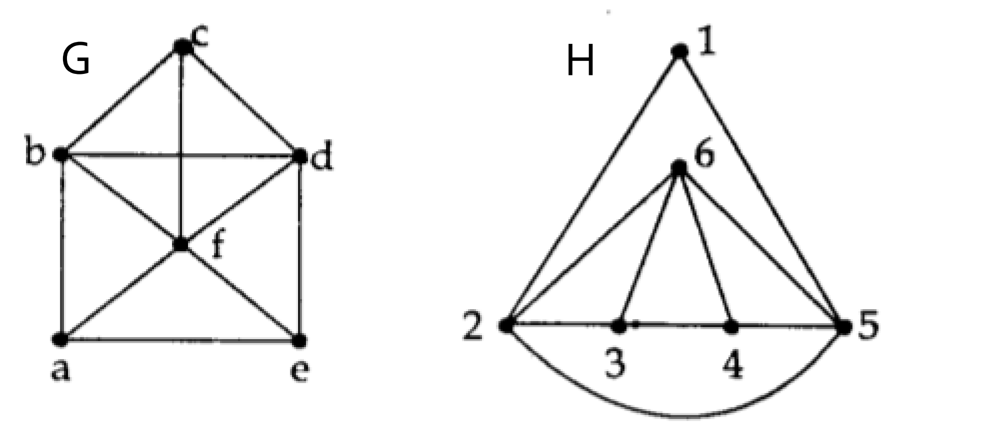


$\delta(G)=1$

$\Delta(G)$ is called the maximum vertex degree of G. It is $\max\{d_G(x) : x \in V(G)\}$. It is a non-negative integer.

$\Delta(G)=3$

4.b)Are the following graphs isomorphic ? If Yes or No justify.[December 2013,June 2010,4 marks]



Number of vertices in G = 6

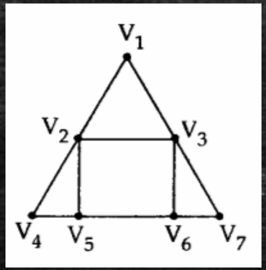
Number of vertices in H = 6

Number of edges in G = 11

Number of edges in H = 10

The two graphs have different number of edges. Therefore, the two graphs are not isomorphic.

1.d)Find the degree of each vertex in the given graph.[June 2012,4 marks]



Degree of each vertex in the above graph is :

d(v1)=2

d(v2)=4

d(v3)=4

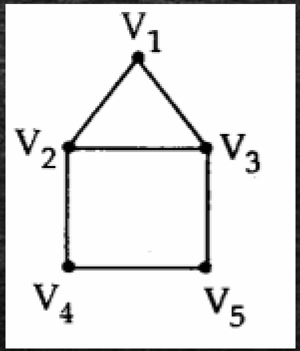
d(v4)=2

d(v5)=3

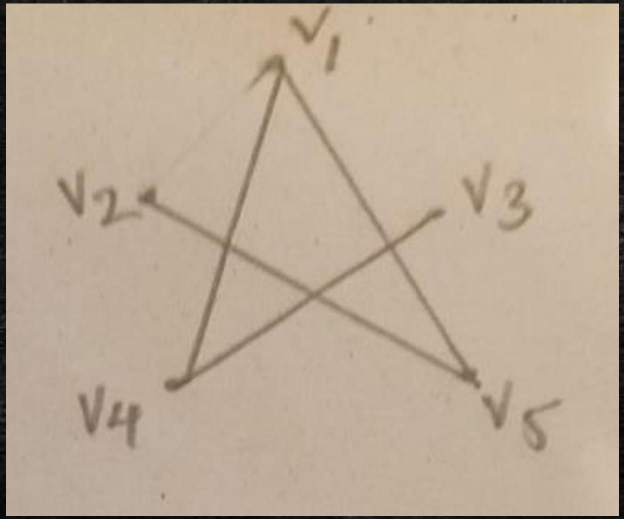
d(v6)=

d(v7)=2

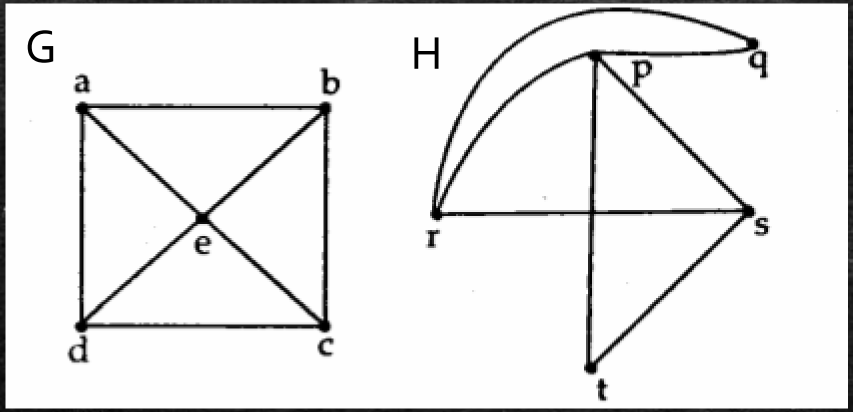
1.e)What is the complement of the given graph.[June 2012,4 marks]



Complement of the above graph :



2.a)Determine whether the graphs are isomorphic. [June 2012,5 marks]



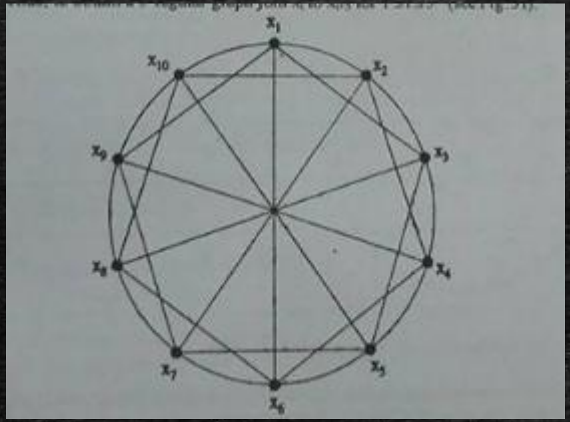
V(G)=5 V(H)=5

E(G)=8 E(H)=7

Number of edges is not the same in G and H.

Therefore, the graphs are not isomorphic.

1.b)Construct a 5-regular graph on 10 vertices.[December 2012,June 2010,3 marks]



1.b)A graph G is said to be **self complementary** if it is isomorphic to its complement G . Show that for a self complementary (p, q)-graph G, either p or (p – 1) is divisible by 4.[June 2011,4 marks]

Suppose G is a (p, q)-graph.

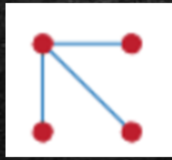
Then E (G) ∪ E($\overline{G}$) = {the set of all pairs of vertices in V (G)}.

Thus, q+$\overline{q}$ =(p(p-1))/2

If the graph G is self complementary, then q = $\overline{q}$. Thus, p (p –1) = 2q + 2q = 4q, that is 4 divides p (p –1). Since only one of p or (p –1) is even, this means either p or (p – 1) is divisible by 4.

1.c)Define minimum vertex degree of G (δ (G))and maximum vertex degree of G (Δ (G)).[June 2011,3 marks]

$\delta(G)$ is called the minimum vertex degree of G. It is $\min\{d_G(x) : x \in V(G)\}$. It is a non-negative integer.



$\delta(G)=1$

Δ (G) is called the maximum vertex degree of G. It is $\max\{d_G(x) : x \in V(G)\}$. It is a non-negative integer.

$\Delta(G)=3$

5.a)Can a simple graph exist with 15 vertices, with each of degree five ? Justify your answer.[June 2011,3 marks]

A corollary in graph theory states that “Any graph can only have an even number of odd vertices”. This is because of the handshaking problem.

$$2q = \sum_{i=1}^{p} d_i$$

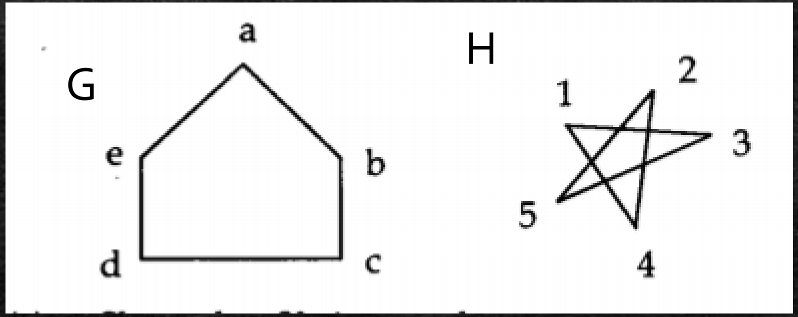
According to the question,

Number of vertices, p=15

Degree of each vertices, $D(V_i)=5$

This graph has 15 odd vertices which is odd, so the above graph cannot exist.

5.b)Are the following graphs are isomorphic ? 4 If Yes or No Justify.[June 2011,4 marks]



Number of vertices in G=5

Number of vertices in H=5

Number of edges in G=5

Number of edges in H=5

Degree sequence of G : <2,2,2,2,2>

Degree Sequence of H : <2,2,2,2,2>

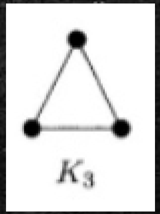
f(a)=2 f(b)=4 f(c)=1 f(d)=3 f(e)=5

This shows that the two graphs are isomorphic.

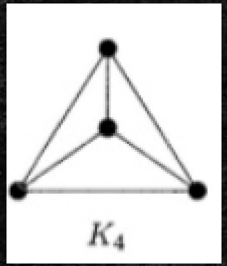
4.a)Define the concept of a complete graph. Draw complete graph each for the case when number of vertices is given by : n=3, n=4.[June 2010,3 marks]

Complete graph : Graph in which any two vertices are adjacent, i.e. each vertex is joined to every other vertex by a vertex. A complete graph on n vertices is represented by K_n.

n=3



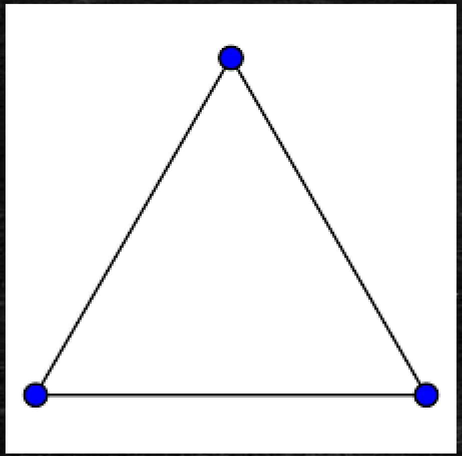
n=4



1.c)Define r-regular graph. Give an example of 3-regular graph.[December 2010,3 marks]

It is a graph in which each vertex has the same degree. It is said to be regular graph degree of regularity r. G is an r-regular graph where 0≤r≤(p-1).

K_n is a regular graph with degree of regularity (n-1) when n > 3.



1.b)Show that the sum of the degrees of all vertices of a graph is twice the number of edges in the graph.[June 2009,3 marks]

Sum of the degrees of al vertices of a graph is twice the number of edges in the graph. This is called handshaking problem.

Proof: Consider the set S = {(x, e): x $\in$ V(G), e $\in$ E(G), x is an endpoint of e}. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge e $\in$ E (G), we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

1.c)Define isomorphism of graphs. Determine whether the graphs are isomorphic.

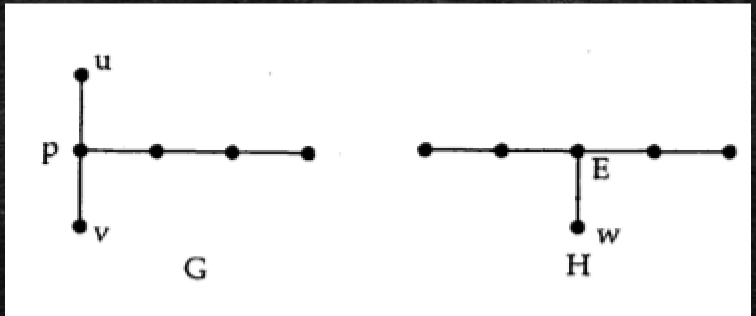
Let G=(V(G),E(G)) and H=(V(H),E(H)) be two graphs. Let us map a function f: V(G)-> V(H).

Then two graphs are said to be isomorphic, if

i) F is one-one and onto, and

ii) xy ∈ E(G) if and only if f(x) f(y) ∈ E(H)

If not they are called non-isomorphic graphs.

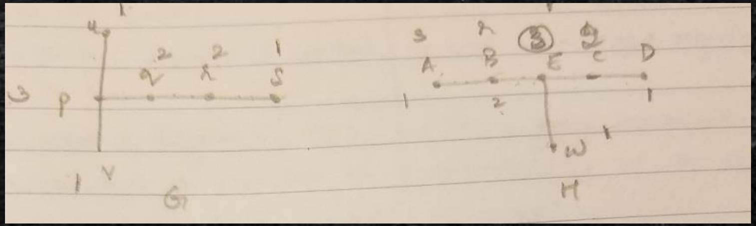


Number of vertices in G=6

Number of vertices in H=6

Number of edges in G=5

Number of edges in H=5

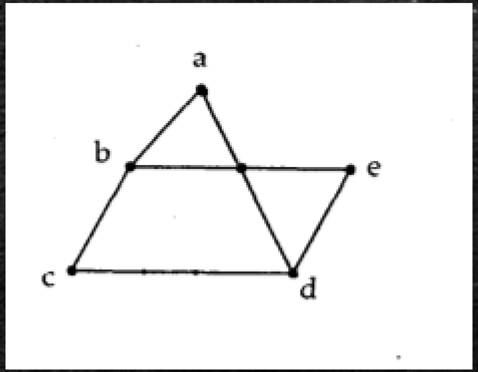


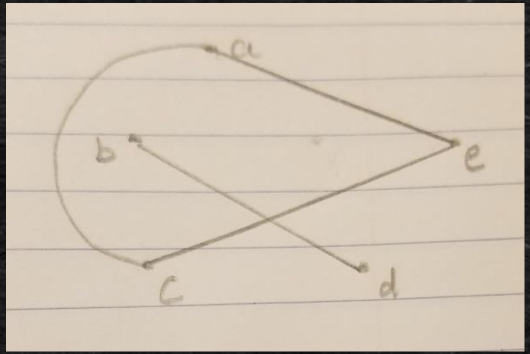
Degree sequence of G : <3,2,2,1,1,1>

Degree sequence of H : <3,2,2,1,1,1>

The two graphs are not isomorphic. This is because in graph G, vertex p with degree 3 is adjacent to two vetices of degree 1 (u,v) and a vertex with degree 2 (q). This is not the case in graph H(vertex with degree 3 is adjacent to two vertices with degree 2 and a vertex with degree 1).

1.f) What is the complement of the given graph?[June 2009,3 marks]





3.b)How many vertices will the following graphs have if they contain :[June 2009,4 marks]

i) 16 edges and all vertices of degree 2.

Sum of all degrees of vertices=2* number of edges

Let number of vertices be n.

$2 * n= 2*16$

$n=16$

ii) 21 edges, 3 vertices of degree 4 and the other vertices of degree 3

Let n be the number of vertices.

$(3*4) +(n*3)=2*21$

$12+3n=42$

$30=3n$

$n=10$

1.b)The number of vertices of odd degree in a graph is always even.[December 2009,3 marks]

Any graph can only have an even number of odd vertices.

Consider a (p,q) graph with {x1,x2,.....xt} is a set of odd vertices and {xt+1,.....xp} is a set of even vertices.

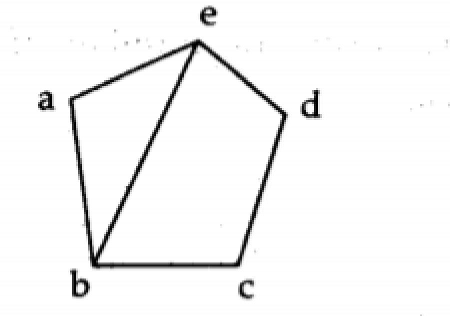
Let $d_G(x_i)=2c_i+1 \quad 1\le i\le t$ and $d_G(x_i)=2r_i \quad t+1\le i\le p$

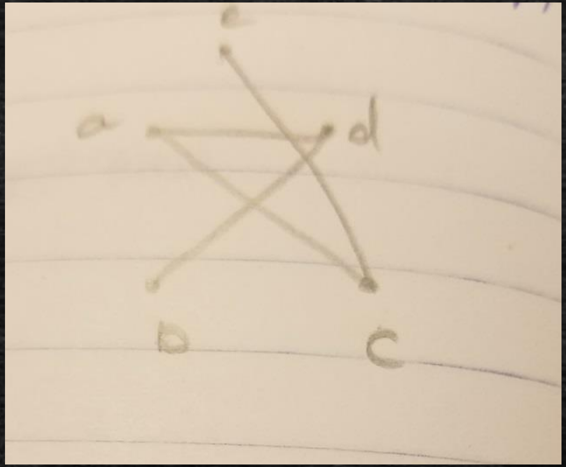
Then Theorem 1 says that $2q=\sum_{1}^{p} d_G(x_i)$

$$\Rightarrow 2q=\sum_{1}^{t}(2c_i+1)+\sum_{t+1}^{p}(2r_i) = 2(c_1+c_2+\Lambda+c_t)+t+2(r_{t+1}+\Lambda+r_p),$$

which shows that t is even.

1.c)What is the complement of the given graph?[December 2009,2 marks]





4.b)What is the largest number of vertices in a graph with 35 edges if all vertices are of degree at least 3 ?[December 2009,5 marks]

Maximum degree of a graph >= Sum of degree of individual vertices

2E >= deg(V1) + deg(V2) + … + deg(Vn)

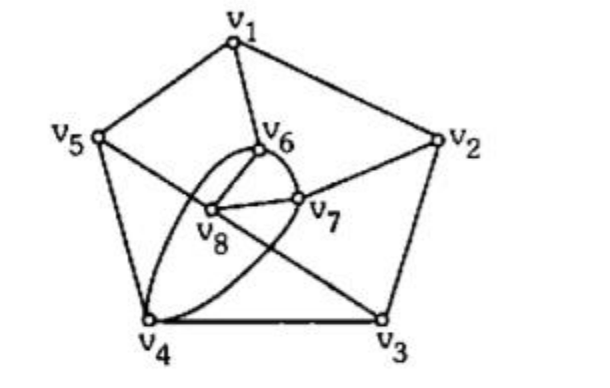
2 * 35 >= 3 + 3 + … + 3 …(I),

70 >= 3n

23.33 >= n or 23 >= n

1.a)Consider the graph below :[June 2008,2 mark]

i) Find δ (G) and , Δ (G)

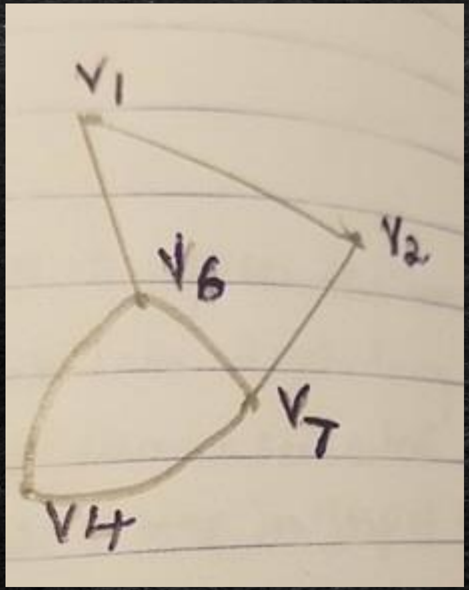


Degree of v1 = 3 Degree of v2 = 3 Degree of v3 = 3 Degree of v4 = 4 Degree of v5 = 3

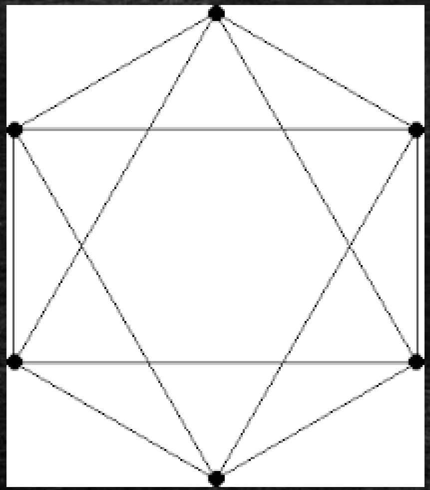
Degree of v6 = 4 Degree of v7 = 4 Degree of v8 = 4

From the above diagram, δ (G) = 3 Δ (G) = 4

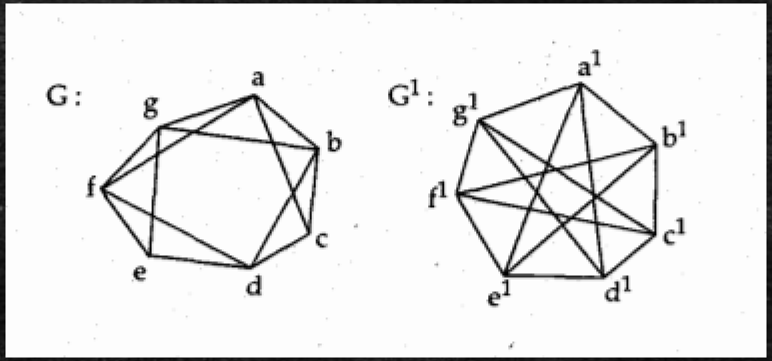
ii) Draw the subgraph induced by the set {v1,v6, v4, v7,v2}



3.c)Draw a 4-regular graph on 6 vertices.[June 2008,2 marks]



1.a)Show that the graphs G and G’ are isomorphic.[December 2008,4 marks]



Number of vertices in G=7

Number of edges= 13

Number of vertices in G’= 7

Number of edges= 14

Therefore both the graphs are not isomorphic.