• Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 7 \\ 4 & 0 & 3 \end{bmatrix}$

→

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 7 \\ 4 & 0 & 3 \end{bmatrix}$$

The characterestic equation is,

$$|A - \lambda I| = 0$$

$$\therefore \text{ or, } \begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 7 \\ 4 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\text{or, } \begin{bmatrix} 1-\lambda & 0 & 0 \\ 5 & 2-\lambda & 7 \\ 4 & 0 & 3-\lambda \end{bmatrix} = 0$$

$$\text{or } (1-\lambda)\left[(2-\lambda)(3-\lambda) - 0\right] = 0$$

$$\text{or } (1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2) = 0$$

$$\text{or, } 6 - 2\lambda - 3\lambda + \lambda^2 - 6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 = 0$$

$$\text{or } 6 - 5\lambda - 6\lambda + 6\lambda^2 - \lambda^3 = 0$$

$$\text{or } -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\text{or } \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \text{[Multiplying both sides by -1]}$$

$$\text{or } \lambda^2(\lambda - 1) - 5\lambda(\lambda - 1) + 6(\lambda - 1) = 0$$

$$\text{or } (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\text{or } (\lambda - 1)(\lambda^2 - 2\lambda - 3\lambda + 6) = 0$$

$$\text{or } (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Therefore, $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$

Thus, the eigenvalues of the given 3x3 matrix are, 1, 2 and 3.

- Examine whether the following set of vectors are linearly independent or dependent : $\{(2,-3,1), (3,-1,5), (1,-4,3)\}$

$\rightarrow$

The set of given vectors are $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$

Now we assume that,

$$a \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} + c \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we have,

$$2a + 3b + c = 0 \quad \text{—— (i)}$$
$$-3a - b - 4c = 0 \quad \text{—— (ii)}$$
$$a + 5b + 3c = 0 \quad \text{—— (iii)}$$

Solving equation (i) and (iii) we have

$$\begin{array}{r} 2a + 3b + c = 0 \\ 2a + 10b + 6c = 0 \\ \hline 7b + 5c = 0 \end{array} \quad \text{—— (iv)}$$

Solving equation (ii) and (iii) we have.

$$\begin{array}{r} -3a - b - 4c = 0 \\ 3a + 15b + 9c = 0 \\ \hline 14b + 5c = 0 \end{array} \quad \text{—— (v)}$$

Solving equation (iv) and (v) we have,

$$\begin{array}{r} 14b + 5c = 0 \\ 7b + 5c = 0 \\ \hline 7b \quad = 0 \end{array}$$

$\therefore b = 0$

Putting the value of b in eq^n (IV) we have

$$7 \times 0 + 5c = 0$$

$$5c = 0$$

$$c = 0$$

Putting the values of b and c in eq^n (1) we have,

$$2a + 3 \times 0 + 0 = 0$$

$$2a = 0$$

$$a = 0$$

Since $a = b = c = 0$ we can say that the given vectors $\{(2,-3,1), (3,-1,5), (1,-4,3)\}$ are linearly independent.

- Diagonalize the following 3×3 matrix. $A = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

$\rightarrow$

The first step is to find the eigen values of matrix A. So we calculate the characteristic polynomial solving the determinant of the matrix.

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 0 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)(4-\lambda) - 1\} + 2(-1) = 0$$

$$\Rightarrow (2-\lambda)\{8 - 2\lambda - 4\lambda + \lambda^2 - 1\} \div 2 = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 6\lambda + 7) - 2 = 0$$

$$\Rightarrow 2\lambda^2 - 12\lambda + 14 - \lambda^3 + 6\lambda^2 - 7\lambda - 2 = 0$$

$$\Rightarrow -\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 7\lambda(\lambda-1) + 12(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 7\lambda + 12) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 4\lambda - 3\lambda + 12) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-3)(\lambda-4) = 0$$

Therefore $\lambda = 1$, $\lambda = 3$ and $\lambda = 4$

Now finding the eigenvector of each eigenvalue.

Calculating the eigenvector the corresponds to the eigenvalue 1.

$$(A - I)v = 0$$

$$\begin{bmatrix} 2-1 & 0 & 2 \\ -1 & 2-1 & 1 \\ 0 & 1 & 4-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + 2z &= 0 \\ -x + y + z &= 0 \\ y + 3z &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} x &= -2z \\ y &= -3z \end{aligned}$$

$$\therefore \; v_1 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Now we calculate the eigenvector associated with the eigenvalue 3.

$$(A - 3I)v = 0$$

$$\begin{bmatrix} 2-3 & 0 & 2 \\ -1 & 2-3 & 1 \\ 0 & 1 & 4-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -x + 2z &= 0 \\ -x - y + z &= 0 \\ y + z &= 0 \end{aligned}\right\} \longrightarrow \begin{aligned} x &= 2z \\ y &= -z \end{aligned}$$

$$\therefore \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

finally we calculate the eigenvector associated with eigenvalue 4

$$(A - 4I)v = 0$$

$$\begin{bmatrix} 2-4 & 0 & 2 \\ -1 & 2-4 & 1 \\ 0 & 1 & 4-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + 2z &= 0 \\ -x - 2y + z &= 0 \\ y &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} x &= z \\ y &= 0 \end{aligned}$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

we construct matrix P, formed by the eigenvectors of the matrix,

$$P = \begin{bmatrix} -2 & 2 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad [D = P^{-1} A P]$$

As all the eigenvalues are different from each other which means that matrix A is diagonalizable.

So the corresponding diagonal matrix is,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- How to diagonalize a matrix?

→

step 1: find the eigenvalues of the matrix

step 2: Calculate the eigenvector associated with each eigenvalue

step 3: Form matrix P, whose calumn are the eigen vector of the matrix

step 4: Verify that the matrix can be diagonalized

step 5: Form diagonal matrix D, whose elements are all 0 except those on the main diagonal, which are the eigenvalues found in step 1.

Properties of linear transformation:

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

i) T preserves the zero vector

$$T(0\vec{x}) = 0\,T(\vec{x}) \qquad \text{Hence } T(\vec{0}) = 0$$

ii) T preserves the negative of a vector

$$T((-1)\vec{x}) = (-1)T(\vec{x}). \quad \text{Hence } T(-\vec{x}) = -T(\vec{x}).$$

iii) T preserves linear combinations.

$$T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$$

$$\therefore T(\vec{x}_1 + \vec{x}_2 + \cdots + \vec{x}_k) = T(\vec{x}_1) + T(\vec{x}_2) + \cdots + T(\vec{x}_k)$$

• Prove that $T(0) = 0$

→

Let any $\alpha \in \mathbb{R}^n$ such that $T(\alpha) \in \mathbb{R}^m$

$$T(\alpha) + 0 = T(\alpha)$$

$$\text{or } T(\alpha) + 0 = T(\alpha + 0)$$

$$\text{or } T(\alpha) + 0 = T(\alpha) + T(0) \qquad [\because T(x_1 + x_2) = T(x_1) + T(x_2)]$$

By Cancellation law

$$0 = T(0) \qquad \text{[Hence Proved]}$$

• Prove that $T(-\alpha) = -T(\alpha)$

→

From the property we have,

$$T(0) = 0$$

$$\text{or } T(\alpha + (-\alpha)) = 0$$

$\Rightarrow T(\alpha) + T(-\alpha) = 0$ [T is linear transformation]

$T(\alpha)$ is inverse element of $T(-\alpha)$ in $\mathbb{R}^m$

Adding both side by $-T(\alpha)$ we have,

$\Rightarrow T(\alpha) + T(-\alpha) - T(\alpha) = -T(\alpha)$

$\Rightarrow T(-\alpha) = -T(\alpha)$ [Hence proved]

- Prove that $T(\alpha - \beta) = T(\alpha) - T(\beta)$

$\rightarrow$

$$T(\alpha - \beta) = T(\alpha + (-\beta))$$

$\Rightarrow T(\alpha - \beta) = T(\alpha) + T(-\beta)$ $[\because T(x_1 + x_2) = T(x_1) + T(x_2)]$

$\Rightarrow T(\alpha - \beta) = T(\alpha) - T(\beta)$ $[\because T(-x_1) = -T(x_1)]$

[Hence proved]

<u>Example 1:</u>

Consider the mapping $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T[x_1, x_2, x_3]$ $= [x_1 - x_2, x_1 + x_3]$

$\rightarrow$

Let $x = [x_1, x_2, x_3]$ and $y = [y_1, y_2, y_3]$ where $v_1, v_2 \in \mathbb{R}^3$

Then, $T[x + y] = T[x_1 + y_1, x_2 + y_2, y_3 + y_3]$

$\Rightarrow T[x + y] = T[x_1, x_2, x_3]$ [Assuming $x_1 = (x_1 + y_1)$, $x_2 = (x_2 + y_2)$, $x_3 = (x_3 + y_3)$]

$\Rightarrow T[x + y] = [[(x_1 + y_1) - (x_2 + y_2)] + [(x_1 + y_1) + (x_3 + y_3)]]$

[By defination]

Now,

$$T[x+y] = [(x_1 - x_2,\ x_1 + x_3) + (y_1 - y_2,\ y_1 + y_3)]$$

$$T[x+y] = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)]$$

$$T[x+y] = T(x) + T(y) \qquad \left[\because \begin{array}{l} x = [x_1, x_2, x_3] \\ y = [y_1, y_2, y_3] \end{array}\right]$$

Example 2:

Consider the mapping $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by,

$$T[x_1, x_2, x_3] = [x_1, x_2, 0], \quad [x_1, x_2, x_3] \in \mathbb{R}^3$$

$\rightarrow$

Let $x = [x_1, x_2, x_3] \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}^3$

Then,

$$T(\alpha x) = T(\alpha [x_1, x_2, x_3])$$

$$T(\alpha x) = T([\alpha x_1, \alpha x_2, \alpha x_3])$$

$$T(\alpha x) = [\alpha x_1, \alpha x_2, 0]$$

$$T(\alpha x) = \alpha [x_1, x_2, 0]$$

$$T(\alpha x) = \alpha T([x_1, x_2, x_3])$$

$$T(\alpha x) = \alpha T(x)$$