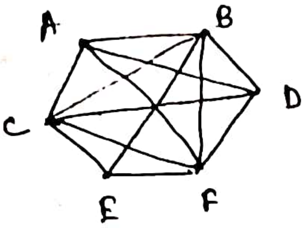
Some specific Types of graph :

Complete graph :

A simple graph in which there is exactly one edge between each pair of distinct vertices is known as a complete graph and denoted by K_n (for n vertices)



: K_6

Δ : K_3

: K_4

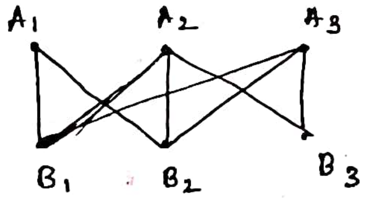
. . : K_2

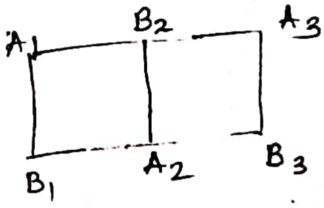
. K_1

Remember that A complete graph is always a regular graph.

Bipartite graph :

Let $A = \{A_1, A_2, A_3, \dots, A_n\}$ and $B = \{B_1, B_2, \dots, B_n\}$ be two sets of vertices. Then a graph obtained by edges joining one vertex of A with one vertex of B and not joining two vertices of the same sets will be called a bipartite graph. Note that bipartite graph can not have a self loop.

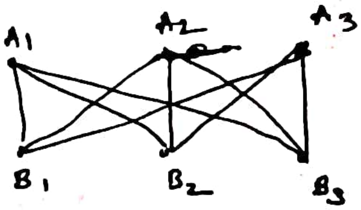




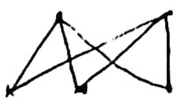
Complete Bipartite graph : If in a bipartite graph, if each vertex of vertex set A is joined by an edge with each vertex of the vertex set B, then that bipartite graph is called a complete bipartite graph. it is denoted by $K_{m,n}$

$m \rightarrow$ vertices of A

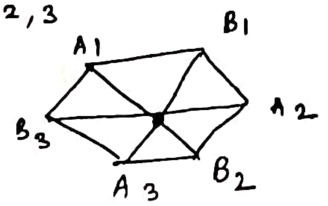
$n \rightarrow$ vertices of B



$K_{3,3}$



$K_{2,3}$



Note that in a complete bipartite graph $K_{m,n}$ total no. of edges are mn.

Th: A complete graph with n vertices consists of $\frac{n(n-1)}{2}$ number of edges. (W. B. U.T – 08, 10, 12)

⇒ A complete graph is a simple graph in which there is exactly one edge between each pair of distinct vertices. Let G be a complete graph with n vertices and e edges. A complete graph has no loop and no parallel. edges, so from each vertex there are $(n-1)$ incident edges and so the degree of each edge is $(n-1)$. So for n vertices, the total degree will be $n(n-1)$.

Now by handshaking theorem, the sum of degrees of all vertices in G is twice the number of edges in the graph.

So, $n(n-1) = 2e \Rightarrow e = \frac{n(n-1)}{2}$

Hence the th. is proved.

Th: A bipartite graph with n vertices has at most $\frac{n^2}{4}$ edges.

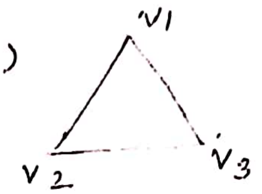
⇒ Let the n vertices in a bipartite graph, the vertex set A consists of m vertices and the vertex set B consists of $(n-m)$ vertices. Since each vertex of set A is connected with each vertex of set B, so total number of edges in this bipartite graph is $m(n-m) = f(m)$, say. To get maximum of $f(m)$, we get

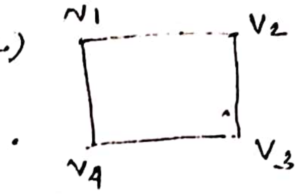
$$f'(m) = 0 \Rightarrow m = n/2, \quad f''(n/2) = -2 < 0$$

So, f is maximum at $n/2$ and $f_{max}(n/2) = \frac{n^2}{4}$

Ex: Examine which of the following graphs are bipartite ?

a) 

b) 

c) 

⇒ a) Let one vertex set $V_1 = \{v_6\}$ and another $V_2 = \{v_1, v_2, v_3, v_4, v_5\}$
it is clear that the graph is drawn by taking one element of V_1 with another element of V_2. So it is a bipartite graph. It is also a complete bipartite graph.

b) If we take any vertex set $V_1 = \{v_1\}$ and another vertex set $V_2 = \{v_2, v_3\}$ then there is an edge connecting v_2 and v_3. So it is not a bipartite graph.

c) Let $V_1 = \{v_1, v_3\}$ and $V_2 = \{v_2, v_4\}$ Then the graph is drawn by connecting one vertex from V_1 with one vertex of V_2. Also since v_1 is not connected v_3 and v_2 is not connected with v_4. so it is a bipartite.

Ex: Suppose G is a graph with 5 vertices. Find the maximum no of edges if i) G is simple graph.
ii) G is multigraph.

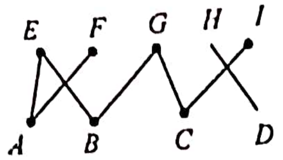
⇒ a) Since there are no loop and parallel edge in G.
so the no of possible ways of selecting two vertices from v is $5c_2 = 10$.

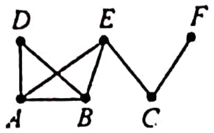
b) for multigraph, it can have finite or infinite no of edges i.e. no exact no. of edges.

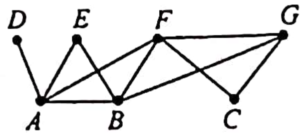
Ex: If a simple regular graph has n vertices and 24 edges. Find all possible values of n.

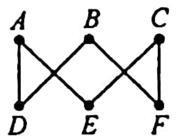
⇒ let deg of each vertex = m
so by hand shaking th. $mn = 48$ and max no of of edges is $\frac{n(n-1)}{2} \geq 24$ i.e. $n(n-1) \geq 48$
so $n = \frac{48}{m}$ and $n(n-1) \geq 48$
if $m = 1$, $n = 48$ and which satisfies $n(n-1) \geq 48$
and so on. so $m = 1(1)8$. and possible values of n is 8, 12, 16, 24, 48.

10. Examine which of the following graphs are bipartite:

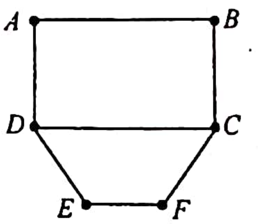
✓ (a)


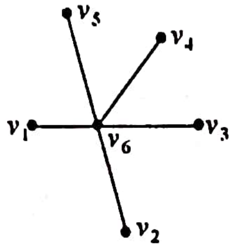
✗ (b)


✗ (c)


✓ (d)


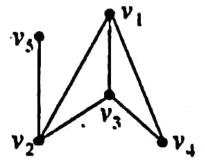
11. Examine the following graphs are bipartite or not.

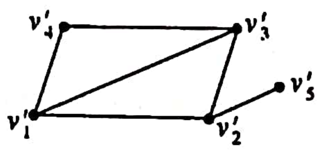
✗


✓


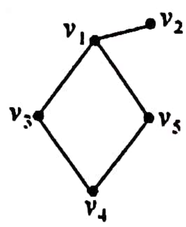
(a) (b)

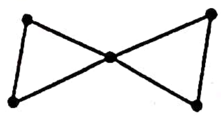
12. Prove that the following pairs of graphs are isomorphic graph:



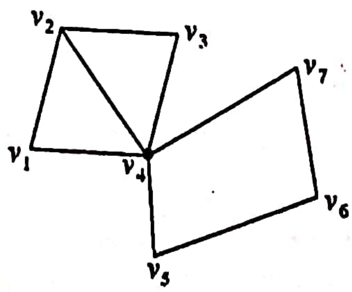


13. Prove that the following pair of graphs are not isomorphic:





14. Find all simple paths form v_1 to v_6 of the following graph; hence find the length the shortest path from v_1 to v_6:



$\{\cancel{7}, 6, 4, 3, 3, 3, 2, 1\}$

$\downarrow$

$\{\cancel{5}, 3, 2, 2, 2, 1, 0\}$

$\downarrow$

$\{\cancel{2}, 1, 1, 1, 0, 0\}$

$\downarrow$

$\{0, 0, 1, 0, 0\}$

$\downarrow$

$\{\cancel{1}, 0, 0, 0, 0\}$

$\{-1, 0, 0, 0\}$

$\{6, 3, 5, 3, 2\}$

$\downarrow$

$\{\cancel{6}, 5, 3, 3, 2\}$

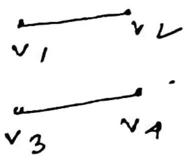
$\{\cancel{1}, 1, 1, 1\} \rightarrow \{0, 1, 1\}$

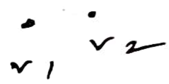
$\downarrow$

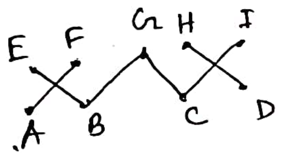
$\{\cancel{1}, 1, 0\}$

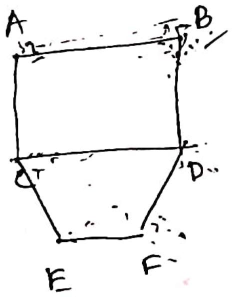
$\{0, 0\}$

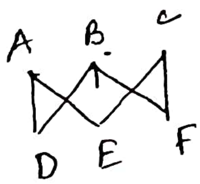
: K5



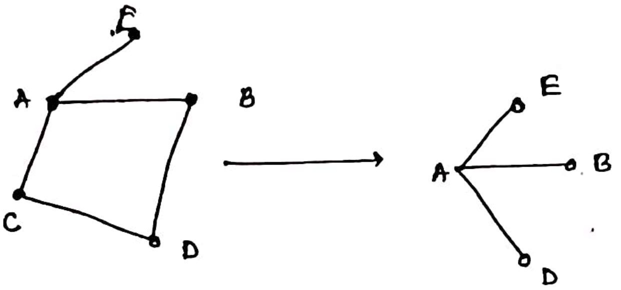


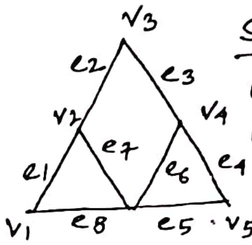






Subgraph : Let $G = (V, E)$ be a graph with vertex set V and edge E. A graph $G_1(V_1, E_1)$ with vertex set V_1 and edge set E_1 is called subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$ and each edge of G_1 has the same end vertices in G_1 as in G.





Subgraphs

$G_1 = \{\{v_3, v_4\}, \{e_3\}\}$

$G_2 = \{\{v_4, v_5, v_6\}, \{e_4, e_5, e_6\}\}$

$G_3 = \{\{v_4, v_5, v_6, v_1\}, \{e_5, e_6\}\}$

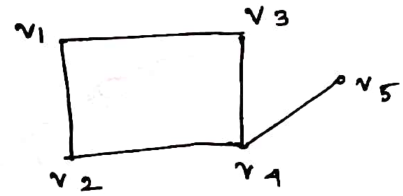
Note that Every graph is a subgraph of itself.

A single vertex of a graph may be considered as a subgraph.

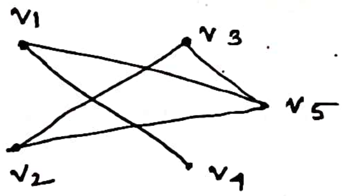
Complement of a graph :

A graph $\overline{G}$ is said to be the complement of a simple graph G if G and $\overline{G}$ have the same vertices and if for any vertices (u, v), the edge joining (u, v) is an edge of G, it will not be an edge of $\overline{G}$ and vice versa.

G :

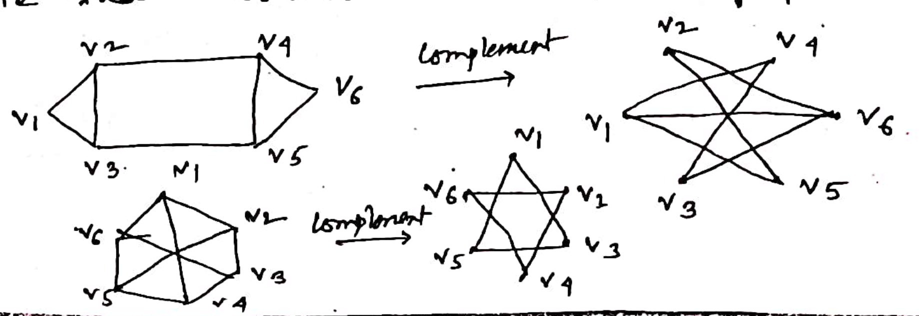


$\overline{G}$:



G and $\overline{G}$ have both vertices v_1, v_2, v_3, v_4, v_5. v_1v_3 is and edge of G but not of $\overline{G}$. Again v_1v_4 is not an edge of G but an edge of $\overline{G}$ etc. So, G and $\overline{G}$ are complement of each other.

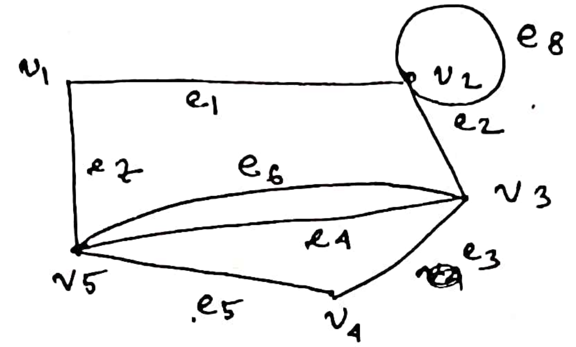
Note that $G \cup \overline{G}$ is a complete graph.



B.Sc

Walk (or chain): A walk is defined as a finite seqⁿ of adjacent vertices and edges, begining and ending with vertices, such that each edge is incident to its preceding and following vertices.

G:



In the above figure the seqⁿs $v_5, e_5, v_4, e_3, v_3, e_2, v_2$ and $v_1, e_1, v_2, e_2, v_3, e_6, v_5, e_4, v_3$ are called $v_5 - v_2$ and $v_1 - v_3$ walk respectively.

Length of a walk:

The number of edges in a walk is called the length of a walk. The length of the walk $v_5 - v_2$ is 3 and $v_1 - v_3$ is 4
Repeated edges also counted

closed and open walk: If a walk begins and ends at the same vertex then the walk is called closed walk. A walk which is not closed is called an open walk.

The walk $v_1, e_1, v_2, e_2, v_3, e_4, v_5, e_7, v_1$ is closed.

The walk $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ is open

Path: An open walk in which no vertex apear more than once is called a path (except possibly initial and ~~first~~ terminal vertices)
$v_1, e_1, v_2, e_2, v_3, e_3, v_4$ is a path.

Note :

a) The starting and terminal vertices in a walk may be same.

b) A vertex may appear twice or more in a walk but not in path.

c) A self loop can be included in a walk.

d) No edge can appear more than once in walk, path, and circuit.

e) Path is always an open walk but an open walk need not be a path.

f) Circuit is always a closed walk but a closed walk need not be a circuit.

g) Every walk is a subgraph of a graph.

h) ~~A walk~~ If In a walk vertices are not repeated, then its edges can not be repeated.

Trail ~~and path~~ : A walk with no repeated edges is called trail. ~~and a walk with no repeated vertices~~ $v_1 e_7 v_5 e_4 v_3$ is trail but $v_1 e_7 v_5 e_4 v_3 e_3 v_4 e_5 v_5 e_4 v_3$ is not.

Circuit : A non trivial closed trail from a vertex u to itself is called circuit. Thus in a graph a circuit is a closed trail of length 3 or more. $v_1 e_7 v_5 e_4 v_3 e_2 v_2 e_1 v_1$ is circuit, $v_2 e_8 v_2 e_1 v_1 e_7 v_5 e_6 v_3 e_2 v_2$

Cycle : A circuit that does not contain any repetition of vertices except the starting vertex and the terminal vertex is called cycle. $v_5 e_5 v_4 e_3 v_3 e_4 v_5 \rightarrow$ 3 cycle

	Vertices	Edges	
Walks	Repetition allowed	Repetition allowed	
Trails	Repetition allowed	No Repetition of edges	
Paths	No repetition of vertices except possibly starting and terminal vertices	No repetition of edges	
Circuits	Repetition allowed	No repetition of edges	A nontrivial closed trail.
cycles	No repetition of vertices except starting and terminal vertices	No repetion of edges	

Isomorphic graph :

Two graphs $G(V, E)$ and $G'(V', E')$ are said to be isomorphic if $\exists$ a mapping $f : V \to V'$ for which

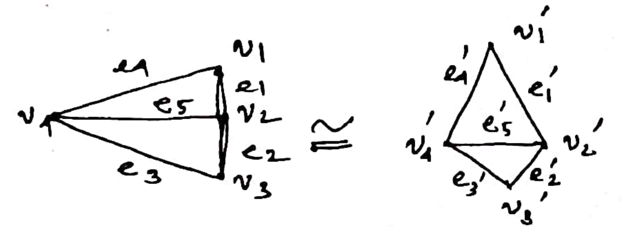
i) There is one and only one edge joining two vertices v_1', v_2' of $G'(V', E')$ corresponding two vertices v_1, v_2 of $G(V, E)$

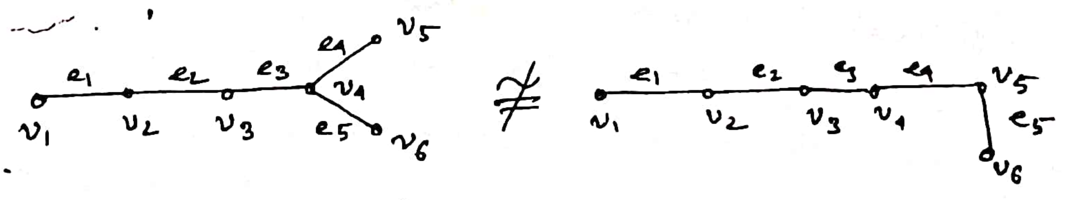
ii) There is one and only one vertex in $G'(V', E')$ corresponding to a vertex in $G(V, E)$.

iii) The incidence relationship between vertex and edges on both G and G' is preserved.

Alternatively two graphs $G(V, E)$ and $G'(V', E')$ are said to be isomorphic if $\exists$ an one one and onto mapping $f : V \to V'$ s.t there correspond an edge between $f(v_1)$ and $f(v_2)$ in G' iff there is an edge between v_1 and v_2 in G.

When such a funn exists, G and G' are called isomorphic graphs and written as $G \cong G'$.





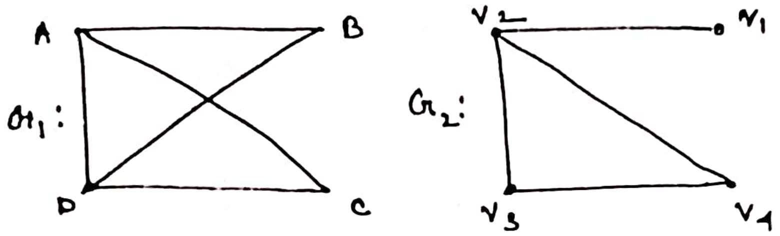
Left graph has a vertex of degree 3, but in the graph right side, it has no such vertex. So they are not isomorphic

** A sub graph H of a graph G is called component of G if

a) Any two vertices of H are connected in H and

b) H is not properly contained in any connected subgraph of G.

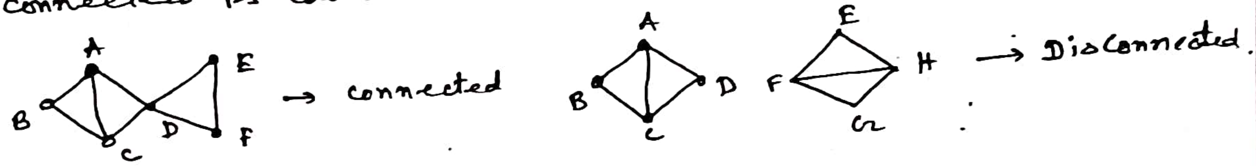
Ex: W. B. U. T; 2014



G_1 has 4 vertices and 5 edges, but G_2 has 4 vertices and 4 edges. Since they have different edges, so they cannot be isomorphic

B.Sc
Connected and disconnected graphs:

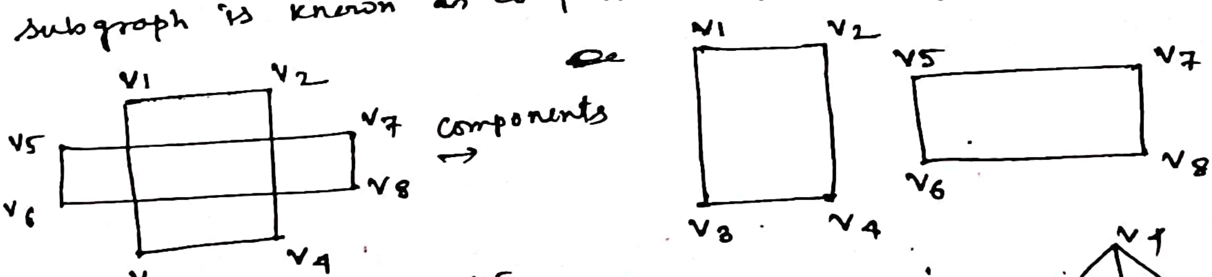
A graph G is said to be connected if there is some path from any vertex to any other vertex. A graph which is not connected is called disconnected.

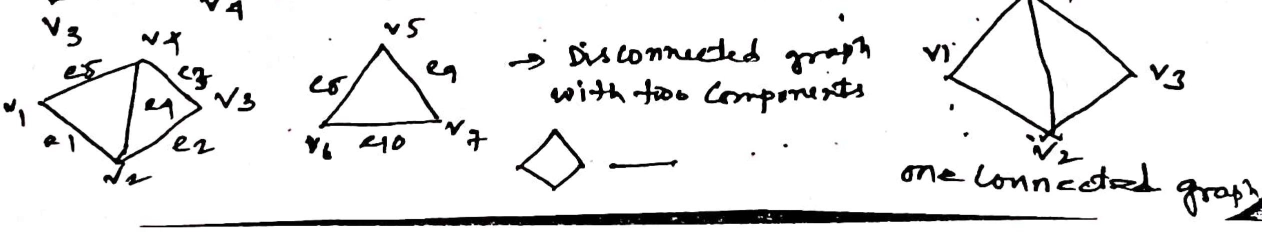


Note that a disconnected graph is composed of two or more connected graphs. A complete graph is a connected graph.

B.Sc
Components of graph: In a graph, a connected subgraph of it will be called the component of G if it is not contained in any bigger connected subgraph of G. Maximal connected subgraph of G is a Component of G.

In the case of a disconnected graph, it may consists of two or more connected subgraphs. Each of these connected subgraph is known as components of the graph.





Some theorems :

Q: Prove that the minimum number of edge in a connected graph with n vertices is $(n-1)$.

⇒ Suppose x = minimum number of edges in a connected graph.

Now we have to prove that $x \geq n-1$. We apply mathematical induction on x.

If $x = 0$ then $n = 1$. Also $x \geq n-1$ is true.

Let it is true for $x = 1, 2, .. k$. So we are to prove it is true for $x = k+1$

Consider G be a graph with n vertices and $k+1$ edges. Let e be an edge of G. Then the graph $(G-e)$ contains n vertices and k edges.

Case I : If the graph $(G-e)$ connected then by our hypothesis $k \geq n-1$. So, $k+1 \geq n \geq n-1$.

Case II : If it is disconnected then it will have two connected components. Suppose the components have k_1 and k_2 no of edges and n_1, n_2 number of vertices respectively so $k = k_1 + k_2$ and $n = n_1 + n_2$. and by our hypothesis $k_1 \geq n_1 - 1$ and $k_2 \geq n_2 - 1$. Then $k_1 + k_2 \geq n_1 + n_2 - 2$

ie $k \geq n-2$ ie $k+1 \geq n-1$

That means the result is true for $x = k+1$.

Ex : Justify wheather it is possible or not to draw a graph with 12 vertices i) having 9 edges. ii) having 13 edges iii) having 70 edges.

⇒ i) The min no of edges with $n = 12$ vertices $= 12 - 1 = 11$. Since $9 < 11$, so the graph is not possible.

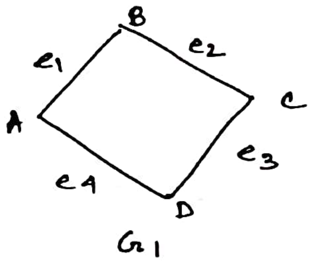
ii) $13 > 11$ so graph possible

iii) Max no of edges with 12 vertices $= \frac{12(12-1)}{2} = 66$. Since $70 > 66$, so graph is not possible.

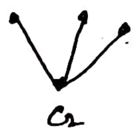
~~Euler line~~ Euler circuit and Euler graph :

~~A trail in a graph~~

A path of the graph G is called an Eulerian path if every edge of G appears exactly once in the path. Similarly, a circuit which includes every edge of the graph G exactly once is called an Eulerian circuit. A graph containing an Eulerian circuit is called an Eulerian graph.



: $A e_1 B e_2 C e_3 D e_4$ is an Euler circuit, so this graph is an Euler graph.

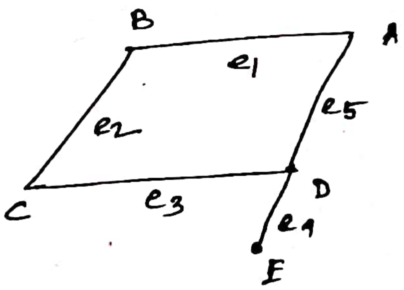


: Has no Eulerian path. So it is not Euler graph.

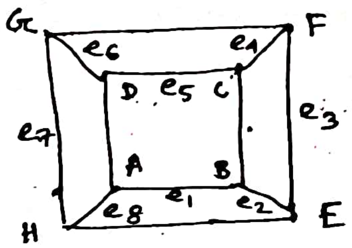
<u>Th</u>: A non empty connected graph G is Eulerian iff its vertexes are of even degree.

Hamiltonian graph : BSC

A path of the graph G is called a Hamiltonian path if every vertex of G appears exactly once. Similarly, a circuit which includes every vertex of the graph G exactly once is called a Hamiltonian circuit. A graph containing a Hamiltonian circuit is called a Hamiltonian graph.



: The path $A e_1 B e_2 C e_3 D e_4 E$ is Hamiltonian path



: Circuit $A e_1 B e_2 E e_3 F e_4 C e_5 D e_6 G e_7 A$ which connected every vertex once, except the starting and finishing vertex A. This is a Hamiltonian circuit.

~~Some Theorems~~

Ex: Give an example of a graph, which is Hamiltonian but not Eulerian.

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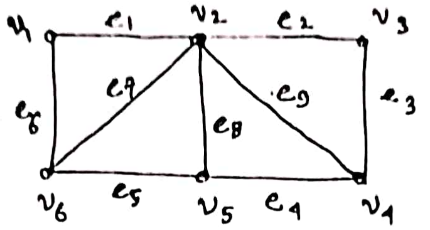
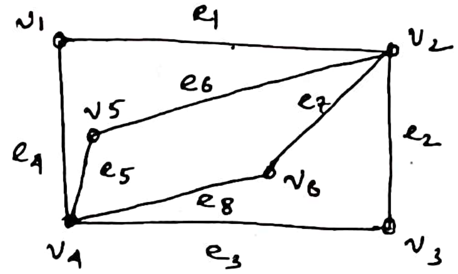


Hamiltonian circuit is $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_1$. Since the degree of each vertex is not even, the graph is not Eulerian.

Ex: Give an example of a graph, which is not Hamiltonian, but Eulerian.

⇒



The graph is Eulerian, since the degree of each vertex is even. It does not contain Hamiltonian circuit.

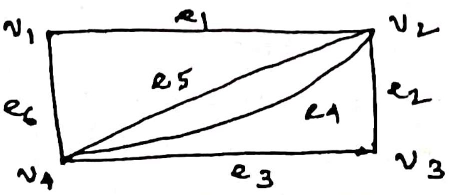
Note: 1) Dirac's th: Let G be a graph with $m \geq 3$ vertices and if $\deg(v) \geq \frac{m}{2}$ for every vertex v in G then G is a Hamiltonian graph.

2) Ore's th: If G be a graph with $m \geq 3$ vertices s.t for all non-adjacent vertices u and v, $\deg(u) + \deg(v) \geq m$, then G is Hamiltonian.

3) A disconnected graph can't have Hamiltonian graph.

Ex: Give an example of a graph which has both Hamiltonian and Eulerian circuits.

⇒



Hamiltonian circuit is $\{v_1, v_2, v_3, v_4\}$

Eulerian circuit is $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

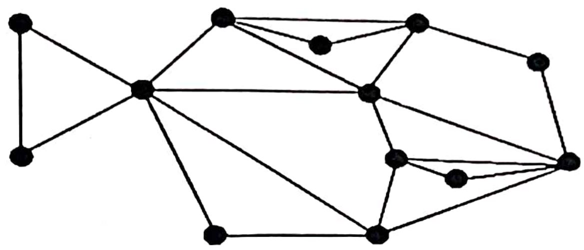


Fig. 37. An Eulerian graph.

For example, the connected graph of Figure 37 in which every point has even degree has an Eulerian trail, and the set of lines can be partitioned into cycles.

Corollary (1) :

Let G be a connected graph with exactly $2n$ odd points, $n \geq 1$, then the set of lines of G can be partitioned into n open trails.

Corollary (2) :

Let G be a connected graph with exactly two odd points. Then G has an open trail containing all the points and lines of G (which begins at one of the odd points and ends at the other).

Problem 1.18. *A non empty connected graph G is Eulerian if and only if its vertices are all of even degree.*

Proof. Let G be Eulerian.

Then G has an Eulerian trail which begins and ends at u, say.

If we travel along the trail then each time we visit a vertex we use two edges, one in and one out.

This is also true for the start vertex because we also ends there.

Since an Eulerian trial uses every edge once, each occurrence of v represents a contribution of 2 to its degree.

Thus deg(v) is even.

Conversely, suppose that G is connected and every vertex is even.

We construct an Eulerian trail. We begin a trail T_1 at any edge e. We extend T_1 by adding an edge after the other.

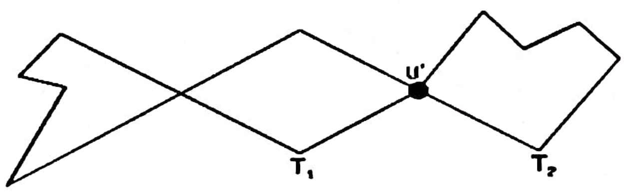
If T_1 is not closed at any step, say T_1 begins at u but ends at $v \neq u$, then only an odd number of the edges incident on v appear in T_1.

Hence we can extend T_1 by another edge incident on v.

Thus we can continue to extend T_1 until T_1 returns to its initial vertex u.

i.e., until T_1 is closed.

If T_1 includes all the edges of G then T_1 is an Eulerian trail.



Suppose T_1 does not include all edges of G.

Consider the graph H obtained by deleting all edges of T_1 from G.

H may not be connected, but each vertex of H has even degree since T_1 contains an even number of the edges incident on any vertex.

Since G is connected, there is an edge e' of H which has an end point u' in T_1.

We construct a trail T_2 in H beginning at u' and using e'. Since all vertices in H have even degree.

We can continue to extent T_2 until T_2 returns to u' as shown in Figure.

We can clearly put T_1 and T_2 together to form a larger closed trail in G.

We continue this process until all the edges of G are used.

We finally obtain an Eulerian trail, and so G is Eulerian.

Theorem 1.18. *A connected graph G has an Eulerian trail if and only if it has at most two odd vertices.*

i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof. Suppose G has an Eulerian trail which is not closed. Since each vertex in the middle of the trail is associated with two edges and since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and the other vertices must be even.

Conversely, suppose that G is connected with atmost two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

The leaves us to treat the case when G has two odd vertices (G cannot have just one odd vertex since in any graph there is an even number of vertices with odd degree).

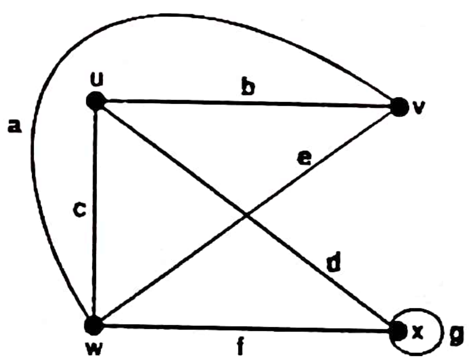
Corollary (1) :

A directed multigraph G has an Euler path if and only if it is unilaterally connected and the in degree of each vertex is equal to its out degree with the possible exception of two vertices, for which it may be that the in degree of is larger than its out degree and the in degree of the other is oneless than its out degree.

Corollary (2) :

A directed multigraph G has an Euler circuit if and only if G is unilaterally connected and the indegree of every vertex in G is equal to its out degree.

Problem 1.84. *Show that the graph shown in Figure has no Eulerian circuit but has a Eulerian trail.*



The graph of Figure 36(*a*) has an Euler path but no Euler circuit. Note that two vertices A and B are of odd degrees 1 and 3 respectively. That means AB can be used to either arrive at vertex A or leave vertex A but not for both.

Thus an Euler path can be found if we start either from vertex A or from B.

ABCDEB and BCDEBA are two Euler paths. Starting from any vertex no Euler circuit can be found.

The graph of Figure 36(*b*) has both Euler circuit and Euler path. ABDEGFDCA is an Euler path and circuit. Note that all vertices of even degree.

No Euler path and circuit is possible in Figure 36(*c*).

Ncte that all vertices are not even degree and more than two vertices are of odd degree.

The existence of Euler path and circuit depends on the degree of vertices.

Note : To determine whether a graph G has an Euler circuit, we note the following points :

(*i*) List the degree of all vertices in the graph.

(*ii*) If any value is zero, the graph is not connected and hence it cannot have Euler path or Euler circuit.

(*iii*) If all the degrees are even, then G has both Euler path and Euler circuit.

(*iv*) If exactly two vertices are odd degree, then G has Euler path but no Euler circuit.

Theorem 1.17. *The following statements are equivalent for a connected graph G :*

(*i*) *G is Eulerian*

(*ii*) *Every point of G has even degree*

(*iii*) *The set of lines of G be partitioned into cycles.*

Proof. (*i*) implies (*ii*)

Let T be an Eulerian trail in G.

Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

(*ii*) implies (*iii*)

Since G is connected and non trivial, every point has degree at least 2, so G contains a cycle Z.

The removal of the lines of Z results in a spanning subgraph G_1 in which every point still has even degree.

If G_1 has no lines, then (*iii*) already holds ; otherwise, repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even, etc.

When a totally disconnected graph G_n is obtained, we have a partition of the lines of G into *n* cycles.

(*iii*) implies (*i*)

Let Z_1 be one of the cycles of this partition.

If G consists only of this cycle, then G is obviously Eulerian.

Otherwise, there is another cycle Z_2 with a point *v* in common with Z_1.

The walk beginning at *v* and consisting of the cycles Z_1 and Z_2 in succession is a closed trail containing the lines of these two cycles.

By continuing this process, we can construct a closed trail containing all lines of G.

Hence G is Eulerian.