2024

COMPUTER SCIENCE AND ENGINEERING

Paper : CSC-901

(Mathematical Foundations of Computer Science)

Full Marks : 70

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question nos.*** **1 & 2** and ***any four*** questions from the rest.

Any question asking to prove a statement must be proved formally. Proving with an example will not be marked. All steps of your calculation should be shown clearly; any non-trivial step jump will deduct marks.

1. Answer ***any five*** questions from the following : 2×5

(a) Consider a fair dice is rolled 5 times. What is the probability that a six did not appear in any of the rolls?

(b) Find the coefficient of $x^{12}y^{14}$ in the expansion of $(2x-3y)^{20}$.

(c) Suppose a basket contains eight red balls and four white balls. If we draw two balls from the basket without replacement, what is the probability that both drawn balls are red?

(d) Consider a Bernoulli process for a biased coin toss where the head occurs with a probability of 0.7. What is the expected value of such a process?

(e) Consider a set of vectors V such that each vector v in V is of the form $(a, b, 2)$, where a and b are integers. Does V form a vector space? If yes, formally prove it. Otherwise give a counter-example.

(f) Show that the vectors A = (2 2 1), B = (1 3 2) and C = (3 5 3) are not linearly independent.

(g) Calculate the number of 3-digit integers that are even and do not have any repeated digits.

2. Answer ***any five*** questions from the following :

(a) Prove that the eigenvalues of a Hermitian matrix must be real. 4

(b) How many solutions does the following equation have?

$x_1+x_2+x_3 = 10$, when x_1, x_2, x_3 are integers. 4

(c) A committee of 5 is to be selected from a group of six men and nine women. If the selection is made random, what is the probability that the committee consists of three men and two women? 4

(d) If E is a subset of F, show that F^c is a subset of E^c (F^c implies a complement of the set F). 4

(e) You are playing a match where your opponent can be from Group A or B. The percentage of total players in Group A is 30%, and that in Group B is 70%, and your probability of winning against a player from Group A is 0.4, while that against a player from Group B is 0.6. What is your probability of winning the match if your opponent is selected randomly? 4

(f) Show that if A and B are invertible matrices, then $(A.B)^{-1} = B^{-1} A^{-1}$. 4

(g) Compute the total number of 5-letter words such that (i) the first and last letters are vowels (ii) the first and last letters are consonants. 2+2

3. Suppose there are n people at a party. Find the probability that all persons have distinct birthdays when (i) $n = 10$, (ii) $n = 20$. 10

4. Show that the expectation value of a geometric random variable with parameter p is $1/p$. 10

5. Consider the following matrix :

$A = 1/\sqrt{2}([[1,1], [1,-1]])$. The rows of A are, therefore, $[1/\sqrt{2}, 1/\sqrt{2}]$ and $[1/\sqrt{2}, -1/\sqrt{2}]$. Calculate e^A (exponential of A). 10

6. N different cards with numbers 1, 2, ..., N are shuffled randomly. A card is picked randomly and you are asked to guess the card.

(a) What is the expectation of a correct guess if you do not remember any previous cards you have seen (i.e., guess without memory)?

(b) What is the expectation of a correct guess when you remember all the previous cards that you have seen and use that information to guess the next card? 4+6

7. Show that if X is a random variable and a and b are constants, then,

(i) $E[aX + b] = aE[X] + b$

(ii) $\mathrm{Var}[aX + b] = a^2\mathrm{Var}[X]$. 5+5