Chapter 2

Random Variables

2.1 INTRODUCTION

In this chapter, the concept of a random variable is introduced. The main purpose of using a random variable is so that we can define certain probability functions that make it both convenient and easy to compute the probabilities of various events.

2.2 RANDOM VARIABLES

A. Definitions:

Consider a random experiment with sample space S. A *random variable* $X(\zeta)$ is a single-valued real function that assigns a real number called the *value* of $X(\zeta)$ to each sample point ζ of S. Often, we use a single letter X for this function in place of $X(\zeta)$ and use r.v. to denote the random variable.

Note that the terminology used here is traditional. Clearly a random variable is not a variable at all in the usual sense, and it is a function.

The sample space S is termed the *domain* of the r.v. X, and the collection of all numbers [values of $X(\zeta)$] is termed the *range* of the r.v. X. Thus the range of X is a certain subset of the set of all real numbers (Fig. 2-1).

Note that two or more different sample points might give the same value of $X(\zeta)$, but two different numbers in the range cannot be assigned to the same sample point.

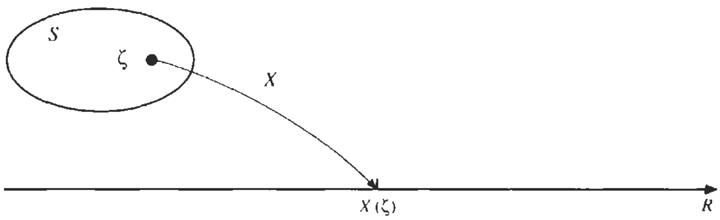

Fig. 2-1 Random variable X as a function.

EXAMPLE 2.1 In the experiment of tossing a coin once (Example 1.1), we might define the r.v. X as (Fig. 2-2)

$$X(H) = 1 \qquad X(T) = 0$$

Note that we could also define another r.v., say Y or Z, with

$$Y(H) = 0, \; Y(T) = 1 \quad \text{or} \quad Z(H) = 0, \; Z(T) = 0$$

B. Events Defined by Random Variables:

If X is a r.v. and x is a fixed real number, we can define the event $(X = x)$ as

$$(X = x) = \{\zeta: X(\zeta) = x\} \tag{2.1}$$

Similarly, for fixed numbers x, x_1, and x_2, we can define the following events:

$$\begin{aligned} (X \le x) &= \{\zeta: X(\zeta) \le x\} \\ (X > x) &= \{\zeta: X(\zeta) > x\} \\ (x_1 < X \le x_2) &= \{\zeta: x_1 < X(\zeta) \le x_2\} \end{aligned} \tag{2.2}$$

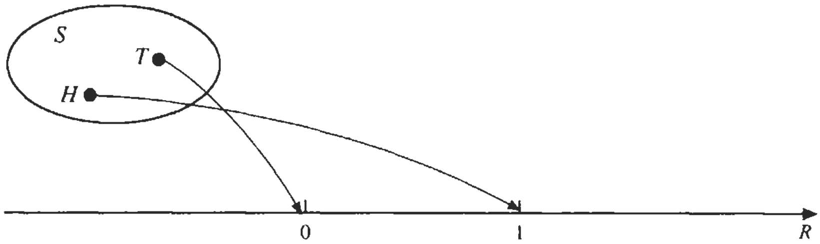

Fig. 2-2 One random variable associated with coin tossing.

These events have probabilities that are denoted by

$$\begin{aligned} P(X = x) &= P\{\zeta: X(\zeta) = x\} \\ P(X \le x) &= P\{\zeta: X(\zeta) \le x\} \\ P(X > x) &= P\{\zeta: X(\zeta) > x\} \\ P(x_1 < X \le x_2) &= P\{\zeta: x_1 < X(\zeta) \le x_2\} \end{aligned} \tag{2.3}$$

EXAMPLE 2.2 In the experiment of tossing a fair coin three times (Prob. 1.1), the sample space S_1 consists of eight equally likely sample points $S_1 = \{HHH, \dots, TTT\}$. If X is the r.v. giving the number of heads obtained, find (*a*) $P(X = 2)$; (*b*) $P(X < 2)$.

(*a*) Let $A \subset S_1$ be the event defined by $X = 2$. Then, from Prob. 1.1, we have

$$A = (X = 2) = \{\zeta: X(\zeta) = 2\} = \{HHT, HTH, THH\}$$

Since the sample points are equally likely, we have

$$P(X = 2) = P(A) = \tfrac{3}{8}$$

(*b*) Let $B \subset S_1$ be the event defined by $X < 2$. Then

$$B = (X < 2) = \{\zeta: X(\zeta) < 2\} = \{HTT, THT, TTH, TTT\}$$

and $$P(X < 2) = P(B) = \tfrac{4}{8} = \tfrac{1}{2}$$

2.3 DISTRIBUTION FUNCTIONS

A. Definition:

The *distribution function* [or *cumulative distribution function* (cdf)] of X is the function defined by

$$F_X(x) = P(X \le x) \qquad -\infty < x < \infty \tag{2.4}$$

Most of the information about a random experiment described by the r.v. X is determined by the behavior of $F_X(x)$.

B. Properties of $F_X(x)$:

Several properties of $F_X(x)$ follow directly from its definition (*2.4*).

1. $0 \le F_X(x) \le 1$ (2.5)
2. $F_X(x_1) \le F_X(x_2) \qquad \text{if } x_1 < x_2$ (2.6)
3. $\lim\limits_{x \to \infty} F_X(x) = F_X(\infty) = 1$ (2.7)
4. $\lim\limits_{x \to -\infty} F_X(x) = F_X(-\infty) = 0$ (2.8)
5. $\lim\limits_{x \to a^+} F_X(x) = F_X(a^+) = F_X(a) \qquad a^+ = \lim\limits_{0 < \varepsilon \to 0} a + \varepsilon$ (2.9)

Property 1 follows because $F_X(x)$ is a probability. Property 2 shows that $F_X(x)$ is a nondecreasing function (Prob. 2.5). Properties 3 and 4 follow from Eqs. (*1.22*) and (*1.26*):

$$\lim_{x \to \infty} P(X \le x) = P(X \le \infty) = P(S) = 1$$

$$\lim_{x \to -\infty} P(X \le x) = P(X \le -\infty) = P(\varnothing) = 0$$

Property 5 indicates that $F_X(x)$ is *continuous on the right*. This is the consequence of the definition (*2.4*).

Table 2.1

x	$(X \le x)$	$F_X(x)$
-1	$\varnothing$	0
0	(TTT)	$\frac{1}{8}$
1	$(TTT, TTH, THT, HTT\}$	$\frac{4}{8} = \frac{1}{2}$
2	$\{TTT, TTH, THT, HTT, HHT, HTH, THH\}$	$\frac{7}{8}$
3	S	1
4	S	1

EXAMPLE 2.3 Consider the r.v. X defined in Example 2.2. Find and sketch the cdf $F_X(x)$ of X.

Table 2.1 gives $F_X(x) = P(X \le x)$ for $x = -1, 0, 1, 2, 3, 4$. Since the value of X must be an integer, the value of $F_X(x)$ for noninteger values of x must be the same as the value of $F_X(x)$ for the nearest smaller integer value of x. The $F_X(x)$ is sketched in Fig. 2-3. Note that $F_X(x)$ has jumps at $x = 0, 1, 2, 3$, and that at each jump the upper value is the correct value for $F_X(x)$.

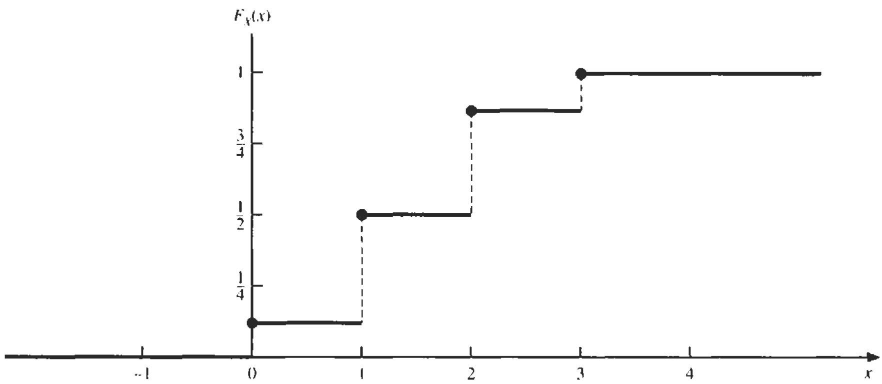

Fig. 2-3

C. Determination of Probabilities from the Distribution Function:

From definition (*2.4*), we can compute other probabilities, such as $P(a < X \le b)$, $P(X > a)$, and $P(X < b)$ (Prob. 2.6):

$$P(a < X \le b) = F_X(b) - F_X(a) \tag{2.10}$$

$$P(X > a) = 1 - F_X(a) \tag{2.11}$$

$$P(X < b) = F_X(b^-) \qquad b^- = \lim_{0 < \varepsilon \to 0} b - \varepsilon \tag{2.12}$$

2.4 DISCRETE RANDOM VARIABLES AND PROBABILITY MASS FUNCTIONS

A. Definition:

Let X be a r.v. with cdf $F_X(x)$. If $F_X(x)$ changes values only in jumps (at most a countable number of them) and is constant between jumps—that is, $F_X(x)$ is a staircase function (see Fig. 2-3)— then X is called a *discrete* random variable. Alternatively, X is a discrete r.v. only if its range contains a finite or countably infinite number of points. The r.v. X in Example 2.3 is an example of a discrete r.v.

B. Probability Mass Functions:

Suppose that the jumps in $F_X(x)$ of a discrete r.v. X occur at the points $x_1, x_2, \ldots$, where the sequence may be either finite or countably infinite, and we assume $x_i < x_j$ if $i < j$.

Then
$$F_X(x_i) - F_X(x_{i-1}) = P(X \le x_i) - P(X \le x_{i-1}) = P(X = x_i) \tag{2.13}$$

Let
$$p_X(x) = P(X = x) \tag{2.14}$$

The function $p_X(x)$ is called the *probability mass function* (pmf) of the discrete r.v. X.

Properties of $p_X(x)$:

1. $0 \le p_X(x_k) \le 1 \qquad k = 1, 2, \ldots$ (2.15)
2. $p_X(x) = 0 \qquad \text{if } x \ne x_k \ (k = 1, 2, \ldots)$ (2.16)
3. $\sum_k p_X(x_k) = 1$ (2.17)

The cdf $F_X(x)$ of a discrete r.v. X can be obtained by

$$F_X(x) = P(X \le x) = \sum_{x_k \le x} p_X(x_k) \tag{2.18}$$

2.5 CONTINUOUS RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS

A. Definition:

Let X be a r.v. with cdf $F_X(x)$. If $F_X(x)$ is continuous and also has a derivative $dF_X(x)/dx$ which exists everywhere except at possibly a finite number of points and is piecewise continuous, then X is called a *continuous* random variable. Alternatively, X is a continuous r.v. only if its range contains an interval (either finite or infinite) of real numbers. Thus, if X is a continuous r.v., then (Prob. 2.18)

$$P(X = x) = 0 \tag{2.19}$$

Note that this is an example of an event with probability 0 that is not necessarily the impossible event $\varnothing$.

In most applications, the r.v. is either discrete or continuous. But if the cdf $F_X(x)$ of a r.v. X possesses features of both discrete and continuous r.v.'s, then the r.v. X is called the *mixed* r.v. (Prob. 2.10).

B. Probability Density Functions:

Let
$$f_X(x) = \frac{dF_X(x)}{dx} \tag{2.20}$$

The function $f_X(x)$ is called the *probability density function* (pdf) of the continuous r.v. X.

Properties of $f_X(x)$:

1. $f_X(x) \geq 0$ (2.21)

2. $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (2.22)

3. $f_X(x)$ is piecewise continuous.

4. $\displaystyle P(a < X \leq b) = \int_a^b f_X(x)\,dx$ (2.23)

The cdf $F_X(x)$ of a continuous r.v. X can be obtained by

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi \tag{2.24}$$

By Eq. (2.19), if X is a continuous r.v., then

$$\begin{aligned} P(a < X \leq b) &= P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b) \\ &= \int_a^b f_X(x)\,dx = F_X(b) - F_X(a) \end{aligned} \tag{2.25}$$

2.6 MEAN AND VARIANCE

A. Mean:

The *mean* (or *expected value*) of a r.v. X, denoted by μ_X or $E(X)$, is defined by

$$\mu_X = E(X) = \begin{cases} \displaystyle\sum_k x_k\, p_X(x_k) & X\text{: discrete} \\ \displaystyle\int_{-\infty}^{\infty} x f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.26}$$

B. Moment:

The nth *moment* of a r.v. X is defined by

$$E(X^n) = \begin{cases} \displaystyle\sum_k x_k{}^n p_X(x_k) & X\text{: discrete} \\ \displaystyle\int_{-\infty}^{\infty} x^n f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.27}$$

Note that the mean of X is the first moment of X.

C. Variance:

The *variance* of a r.v. X, denoted by $\sigma_X{}^2$ or $\mathrm{Var}(X)$, is defined by

$$\sigma_X{}^2 = \mathrm{Var}(X) = E\{[X - E(X)]^2\} \tag{2.28}$$

Thus,

$$\sigma_X{}^2 = \begin{cases} \displaystyle\sum_k (x_k - \mu_X)^2 p_X(x_k) & X\text{: discrete} \\ \displaystyle\int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.29}$$

Note from definition (2.28) that

$$\operatorname{Var}(X) \geq 0 \tag{2.30}$$

The *standard deviation* of a r.v. X, denoted by σ_X, is the positive square root of Var(X).

Expanding the right-hand side of Eq. (2.28), we can obtain the following relation:

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 \tag{2.31}$$

which is a useful formula for determining the variance.

2.7 SOME SPECIAL DISTRIBUTIONS

In this section we present some important special distributions.

A. Bernoulli Distribution:

A r.v. X is called a *Bernoulli* r.v. with parameter p if its pmf is given by

$$p_X(k) = P(X = k) = p^k(1 - p)^{1-k} \qquad k = 0, 1 \tag{2.32}$$

where $0 \leq p \leq 1$. By Eq. (2.18), the cdf $F_X(x)$ of the Bernoulli r.v. X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{2.33}$$

Figure 2-4 illustrates a Bernoulli distribution.

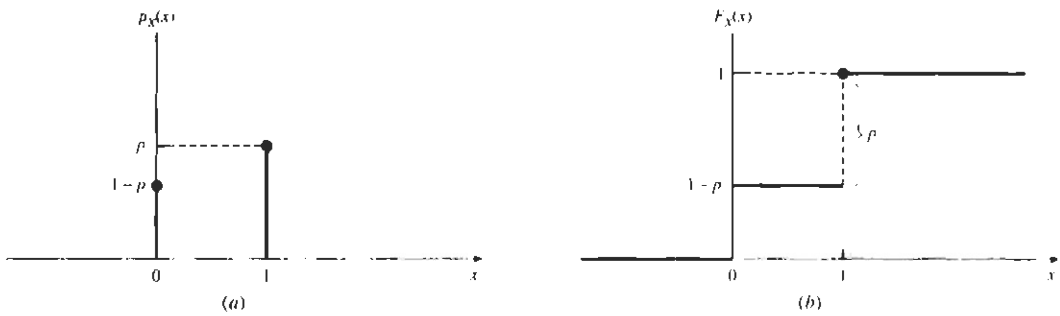

Fig. 2-4 Bernoulli distribution.

The mean and variance of the Bernoulli r.v. X are

$$\mu_X = E(X) = p \tag{2.34}$$

$$\sigma_X^{\,2} = \operatorname{Var}(X) = p(1 - p) \tag{2.35}$$

A Bernoulli r.v. X is associated with some experiment where an outcome can be classified as either a "success" or a "failure," and the probability of a success is p and the probability of a failure is $1 - p$. Such experiments are often called *Bernoulli trials* (Prob. 1.61).

B. Binomial Distribution:

A r.v. X is called a *binomial* r.v. with parameters (n, p) if its pmf is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n \tag{2.36}$$

where $0 \leq p \leq 1$ and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which is known as the binomial coefficient. The corresponding cdf of X is

$$F_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \qquad n \leq x < n+1 \tag{2.37}$$

Figure 2-5 illustrates the binomial distribution for $n = 6$ and $p = 0.6$.

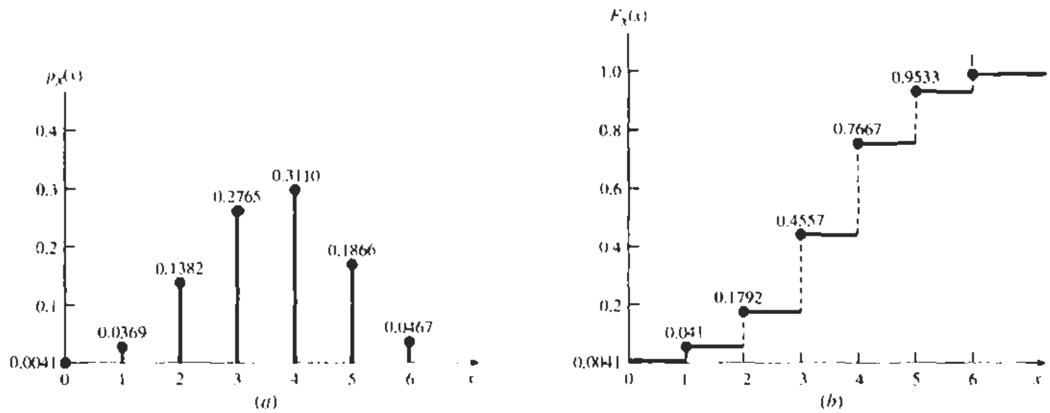

Fig. 2-5 Binomial distribution with $n = 6$, $p = 0.6$.

The mean and variance of the binomial r.v. X are (Prob. 2.28)

$$\mu_X = E(X) = np \tag{2.38}$$

$$\sigma_X{}^2 = \operatorname{Var}(X) = np(1-p) \tag{2.39}$$

A binomial r.v. X is associated with some experiments in which n independent Bernoulli trials are performed and X represents the number of successes that occur in the n trials. Note that a Bernoulli r.v. is just a binomial r.v. with paramcters $(1, p)$.

C. Poisson Distribution:

A r.v. X is called a *Poisson* r.v. with parameter $\lambda\ (>0)$ if its pmf is given by

$$p_X(k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots \tag{2.40}$$

The corresponding cdf of X is

$$F_X(x) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \qquad n \leq x < n+1 \tag{2.41}$$

Figurc 2-6 illustrates the Poisson distribution for $\lambda = 3$.

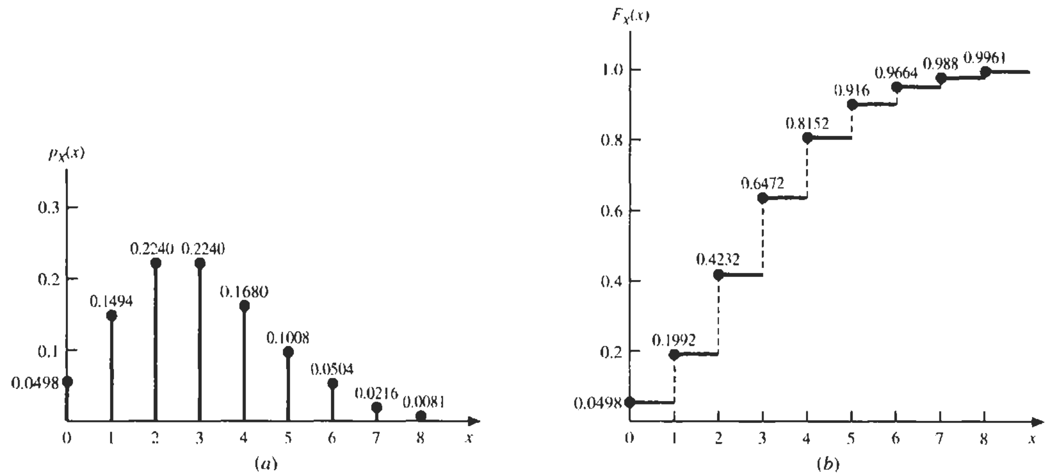

Fig. 2-6 Poisson distribution with $\lambda = 3$.

The mean and variance of the Poisson r.v. X are (Prob. 2.29)

$$\mu_X = E(X) = \lambda \tag{2.42}$$

$$\sigma_X{}^2 = \text{Var}(X) = \lambda \tag{2.43}$$

The Poisson r.v. has a tremendous range of applications in diverse areas because it may be used as an approximation for a binomial r.v. with parameters (n, p) when n is large and p is small enough so that np is of a moderate size (Prob. 2.40).

Some examples of Poisson r.v.'s include

1. The number of telephone calls arriving at a switching center during various intervals of time
2. The number of misprints on a page of a book
3. The number of customers entering a bank during various intervals of time

D. Uniform Distribution:

A r.v. X is called a *uniform* r.v. over (a, b) if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \tag{2.44}$$

The corresponding cdf of X is

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases} \tag{2.45}$$

Figure 2-7 illustrates a uniform distribution.

The mean and variance of the uniform r.v. X are (Prob. 2.31)

$$\mu_X = E(X) = \frac{a+b}{2} \tag{2.46}$$

$$\sigma_X{}^2 = \text{Var}(X) = \frac{(b-a)^2}{12} \tag{2.47}$$

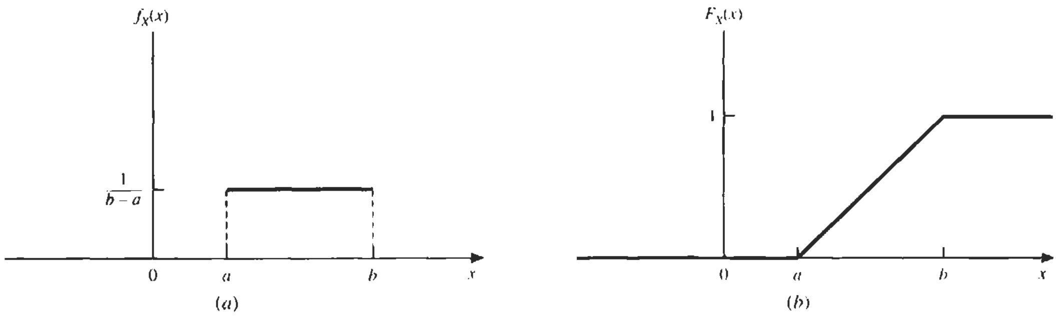

Fig. 2-7 Uniform distribution over (a, b).

A uniform r.v. X is often used where we have no prior knowledge of the actual pdf and all continuous values in some range seem equally likely (Prob. 2.69).

E. Exponential Distribution:

A r.v. X is called an *exponential* r.v. with parameter λ (>0) if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.48}$$

which is sketched in Fig. 2-8(a). The corresponding cdf of X is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{2.49}$$

which is sketched in Fig. 2-8(b).

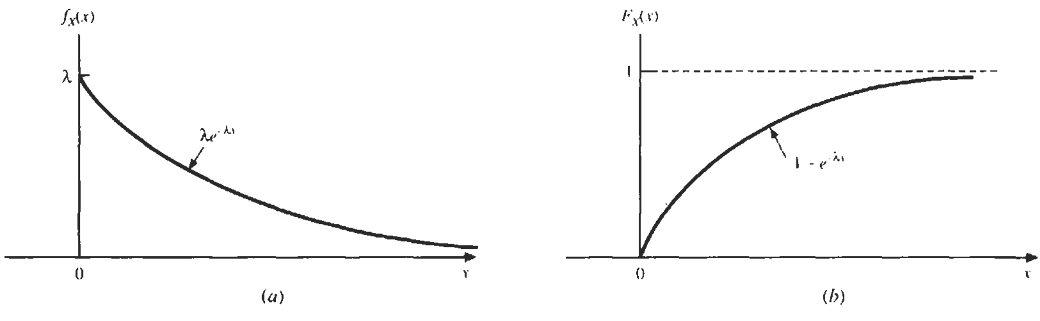

Fig. 2-8 Exponential distribution.

The mean and variance of the exponential r.v. X are (Prob. 2.32)

$$\mu_X = E(X) = \frac{1}{\lambda} \tag{2.50}$$

$$\sigma_X{}^2 = \mathrm{Var}(X) = \frac{1}{\lambda^2} \tag{2.51}$$

The most interesting property of the exponential distribution is its "memoryless" property. By this we mean that if the lifetime of an item is exponentially distributed, then an item which has been in use for some hours is as good as a new item with regard to the amount of time remaining until the item fails. The exponential distribution is the only distribution which possesses this property (Prob. 2.53).

F. Normal (or Gaussian) Distribution:

A r.v. X is called a *normal* (or *gaussian*) r.v. if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \tag{2.52}$$

The corresponding cdf of X is

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-(\xi-\mu)^2/(2\sigma^2)}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-\xi^2/2}\, d\xi \tag{2.53}$$

This integral cannot be evaluated in a closed form and must be evaluated numerically. It is convenient to use the function $\Phi(z)$, defined as

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi \tag{2.54}$$

to help us to evaluate the value of $F_X(x)$. Then Eq. (2.53) can be written as

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{2.55}$$

Note that

$$\Phi(-z) = 1 - \Phi(z) \tag{2.56}$$

The function $\Phi(z)$ is tabulated in Table A (Appendix A). Figure 2-9 illustrates a normal distribution.

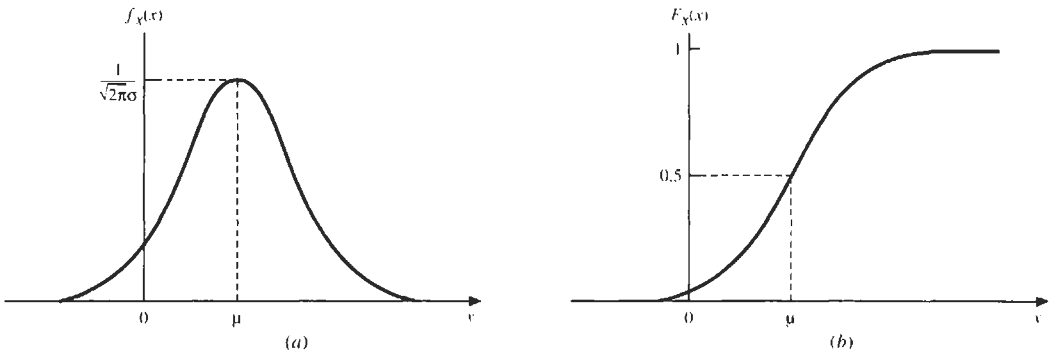

Fig. 2-9 Normal distribution.

The mean and variance of the normal r.v. X are (Prob. 2.33)

$$\mu_X = E(X) = \mu \tag{2.57}$$
$$\sigma_X{}^2 = \text{Var}(X) = \sigma^2 \tag{2.58}$$

We shall use the notation $N(\mu;\, \sigma^2)$ to denote that X is normal with mean μ and variance σ^2. A normal r.v. Z with zero mean and unit variance—that is, $Z = N(0;\, 1)$—is called a *standard normal* r.v. Note that the cdf of the standard normal r.v. is given by Eq. (2.54). The normal r.v. is probably the most important type of continuous r.v. It has played a significant role in the study of random phenomena in nature. Many naturally occurring random phenomena are approximately normal. Another reason for the importance of the normal r.v. is a remarkable theorem called the *central limit theorem*. This theorem states that the sum of a large number of independent r.v.'s, under certain conditions, can be approximated by a normal r.v. (see Sec. 4.8C).

2.8 CONDITIONAL DISTRIBUTIONS

In Sec. 1.6 the conditional probability of an event A given event B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

The conditional cdf $F_X(x \mid B)$ of a r.v. X given event B is defined by

$$F_X(x \mid B) = P(X \le x \mid B) = \frac{P\{(X \le x) \cap B\}}{P(B)} \tag{2.59}$$

The conditional cdf $F_X(x \mid B)$ has the same properties as $F_X(x)$. (See Prob. 1.37 and Sec. 2.3.) In particular,

$$F_X(-\infty \mid B) = 0 \qquad F_X(\infty \mid B) = 1 \tag{2.60}$$

$$P(a < X \le b \mid B) = F_X(b \mid B) - F_X(a \mid B) \tag{2.61}$$

If X is a discrete r.v., then the conditional pmf $p_X(x_k \mid B)$ is defined by

$$p_X(x_k \mid B) = P(X = x_k \mid B) = \frac{P\{(X = x_k) \cap B\}}{P(B)} \tag{2.62}$$

If X is a continuous r.v., then the conditional pdf $f_X(x \mid B)$ is defined by

$$f_X(x \mid B) = \frac{dF_X(x \mid B)}{dx} \tag{2.63}$$

Solved Problems

RANDOM VARIABLES

2.1. Consider the experiment of throwing a fair die. Let X be the r.v. which assigns 1 if the number that appears is even and 0 if the number that appears is odd.

(a) What is the range of X?

(b) Find $P(X = 1)$ and $P(X = 0)$.

The sample space S on which X is defined consists of 6 points which are equally likely:

$$S = \{1, 2, 3, 4, 5, 6\}$$

(a) The range of X is $R_X = \{0, 1\}$.

(b) $(X = 1) = \{2, 4, 6\}$. Thus, $P(X = 1) = \frac{3}{6} = \frac{1}{2}$. Similarly, $(X = 0) = \{1, 3, 5\}$, and $P(X = 0) = \frac{1}{2}$.

2.2. Consider the experiment of tossing a coin three times (Prob. 1.1). Let X be the r.v. giving the number of heads obtained. We assume that the tosses are independent and the probability of a head is p.

(a) What is the range of X?

(b) Find the probabilities $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$.

The sample space S on which X is defined consists of eight sample points (Prob. 1.1):

$$S = \{HHH, HHT, \ldots, TTT\}$$

(a) The range of X is $R_X = \{0, 1, 2, 3\}$.

(b) If $P(H) = p$, then $P(T) = 1 - p$. Since the tosses are independent, we have

$$P(X = 0) = P[\{TTT\}] = (1 - p)^3$$
$$P(X = 1) = P[\{HTT\}] + P[\{THT\}] + P[\{TTH\}] = 3(1 - p)^2 p$$
$$P(X = 2) = P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}] = 3(1 - p)p^2$$
$$P(X = 3) = P[\{HHH\}] = p^3$$

2.3. An information source generates symbols at random from a four-letter alphabet $\{a, b, c, d\}$ with probabilities $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{4}$, and $P(c) = P(d) = \frac{1}{8}$. A coding scheme encodes these symbols into binary codes as follows:

a	0
b	10
c	110
d	111

Let X be the r.v. denoting the length of the code, that is, the number of binary symbols (bits).

(a) What is the range of X?

(b) Assuming that the generations of symbols are independent, find the probabilities $P(X = 1)$, $P(X = 2)$, $P(X = 3)$, and $P(X > 3)$.

(a) The range of X is $R_X = \{1, 2, 3\}$.

(b) $P(X = 1) = P[\{a\}] = P(a) = \frac{1}{2}$
$P(X = 2) = P[\{b\}] = P(b) = \frac{1}{4}$
$P(X = 3) = P[\{c, d\}] = P(c) + P(d) = \frac{1}{4}$
$P(X > 3) = P(\varnothing) = 0$

2.4. Consider the experiment of throwing a dart onto a circular plate with unit radius. Let X be the r.v. representing the distance of the point where the dart lands from the origin of the plate. Assume that the dart always lands on the plate and that the dart is equally likely to land anywhere on the plate.

(a) What is the range of X?

(b) Find (i) $P(X < a)$ and (ii) $P(a < X < b)$, where $a < b \le 1$.

(a) The range of X is $R_X = \{x: 0 \le x < 1\}$.

(b) (i) $(X < a)$ denotes that the point is inside the circle of radius a. Since the dart is equally likely to fall anywhere on the plate, we have (Fig. 2-10)

$$P(X < a) = \frac{\pi a^2}{\pi 1^2} = a^2$$

(ii) $(a < X < b)$ denotes the event that the point is inside the annular ring with inner radius a and outer radius b. Thus, from Fig. 2-10, we have

$$P(a < X < b) = \frac{\pi(b^2 - a^2)}{\pi 1^2} = b^2 - a^2$$

DISTRIBUTION FUNCTION

2.5. Verify Eq. (2.6).

Let $x_1 < x_2$. Then $(X \le x_1)$ is a subset of $(X \le x_2)$; that is, $(X \le x_1) \subset (X \le x_2)$. Then, by Eq. (1.27), we have

$$P(X \le x_1) \le P(X \le x_2) \qquad \text{or} \qquad F_X(x_1) \le F_X(x_2)$$

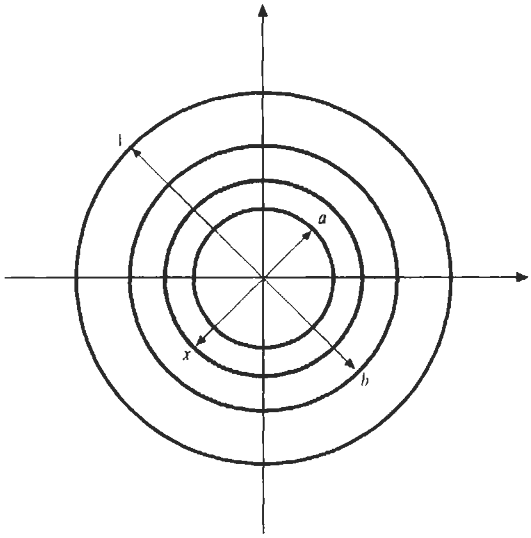

Fig. 2-10

2.6. Verify (*a*) Eq. (*2.10*); (*b*) Eq. (*2.11*); (*c*) Eq. (*2.12*).

(*a*) Since $(X \le b) = (X \le a) \cup (a < X \le b)$ and $(X \le a) \cap (a < X \le b) = \varnothing$, we have

$$P(X \le b) = P(X \le a) + P(a < X \le b)$$

or
$$F_X(b) = F_X(a) + P(a < X \le b)$$

Thus,
$$P(a < X \le b) = F_X(b) - F_X(a)$$

(*b*) Since $(X \le a) \cup (X > a) = S$ and $(X \le a) \cap (X > a) = \varnothing$, we have

$$P(X \le a) + P(X > a) = P(S) = 1$$

Thus,
$$P(X > a) = 1 - P(X \le a) = 1 - F_X(a)$$

(*c*) Now
$$\begin{aligned} P(X < b) &= P[\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} X \le b - \varepsilon] = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} P(X \le b - \varepsilon) \\ &= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} F_X(b - \varepsilon) = F_X(b^-) \end{aligned}$$

2.7. Show that

(*a*) $P(a \le X \le b) = P(X = a) + F_X(b) - F_X(a)$ (*2.64*)

(*b*) $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$ (*2.65*)

(*c*) $P(a \le X < b) = P(X = a) + F_X(b) - F_X(a) - P(X = b)$ (*2.66*)

(*a*) Using Eqs. (*1.23*) and (*2.10*), we have

$$\begin{aligned} P(a \le X \le b) &= P[(X = a) \cup (a < X \le b)] \\ &= P(X = a) + P(a < X \le b) \\ &= P(X = a) + F_X(b) - F_X(a) \end{aligned}$$

(*b*) We have

$$\begin{aligned} P(a < X \le b) &= P[(a < X < b) \cup (X = b)] \\ &= P(a < X < b) + P(X = b) \end{aligned}$$

Again using Eq. (2.10), we obtain

$$P(a < X < b) = P(a < X \le b) - P(X = b)$$
$$= F_X(b) - F_X(a) - P(X = b)$$

(c) Similarly,

$$P(a \le X \le b) = P[(a \le X < b) \cup (X = b)]$$
$$= P(a \le X < b) + P(X = b)$$

Using Eq. (2.64), we obtain

$$P(a \le X < b) = P(a \le X \le b) - P(X = b)$$
$$= P(X = a) + F_X(b) - F_X(a) - P(X = b)$$

2.8. Let X be the r.v. defined in Prob. 2.3.

(a) Sketch the cdf $F_X(x)$ of X and specify the type of X.

(b) Find (i) $P(X \le 1)$, (ii) $P(1 < X \le 2)$, (iii) $P(X > 1)$, and (iv) $P(1 \le X \le 2)$.

(a) From the result of Prob. 2.3 and Eq. (2.18), we have

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

which is sketched in Fig. 2-11. The r.v. X is a discrete r.v.

(b) (i) We see that

$$P(X \le 1) = F_X(1) = \tfrac{1}{2}$$

(ii) By Eq. (2.10),

$$P(1 < X \le 2) = F_X(2) - F_X(1) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

(iii) By Eq. (2.11),

$$P(X > 1) = 1 - F_X(1) = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

(iv) By Eq. (2.64),

$$P(1 \le X \le 2) = P(X = 1) + F_X(2) - F_X(1) = \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{3}{4}$$

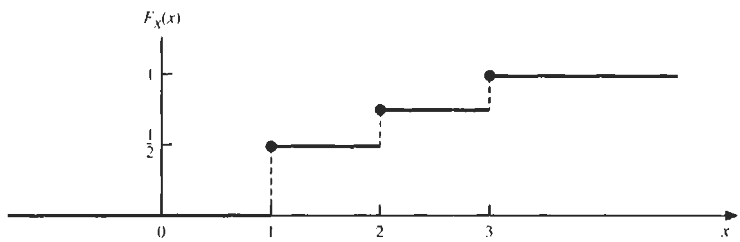

Fig. 2-11

2.9. Sketch the cdf $F_X(x)$ of the r.v. X defined in Prob. 2.4 and specify the type of X.

From the result of Prob. 2.4, we have

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

which is sketched in Fig. 2-12. The r.v. X is a continuous r.v.

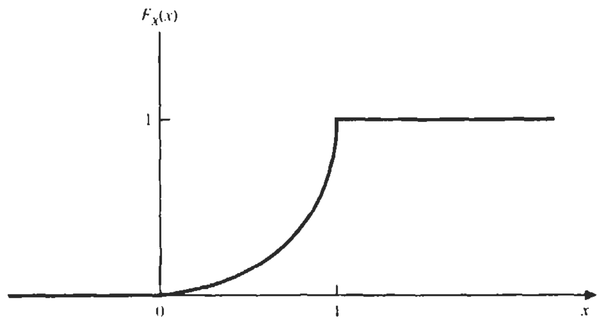

Fig. 2-12

2.10. Consider the function given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x + \frac{1}{2} & 0 \le x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}$$

(*a*) Sketch $F(x)$ and show that $F(x)$ has the properties of a cdf discussed in Sec. 2.3B.

(*b*) If X is the r.v. whose cdf is given by $F(x)$, find (i) $P(X \le \frac{1}{4})$, (ii) $P(0 < X \le \frac{1}{4})$, (iii) $P(X = 0)$, and (iv) $P(0 \le X \le \frac{1}{4})$.

(*c*) Specify the type of X.

(*a*) The function $F(x)$ is sketched in Fig. 2-13. From Fig. 2-13, we see that $0 \le F(x) \le 1$ and $F(x)$ is a nondecreasing function, $F(-\infty) = 0$, $F(\infty) = 1$, $F(0) = \frac{1}{2}$, and $F(x)$ is continuous on the right. Thus, $F(x)$ satisfies all the properties [Eqs. (2.5) to (2.9)] required of a cdf.

(*b*) (i) We have

$$P(X \le \tfrac{1}{4}) = F(\tfrac{1}{4}) = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

(ii) By Eq. (*2.10*),

$$P(0 < X \le \tfrac{1}{4}) = F(\tfrac{1}{4}) - F(0) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

(iii) By Eq. (*2.12*),

$$P(X = 0) = P(X \le 0) - P(X < 0) = F(0) - F(0^-) = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

(iv) By Eq. (*2.64*),

$$P(0 \le X \le \tfrac{1}{4}) = P(X = 0) + F(\tfrac{1}{4}) - F(0) = \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{3}{4}$$

(*c*) The r.v. X is a mixed r.v.

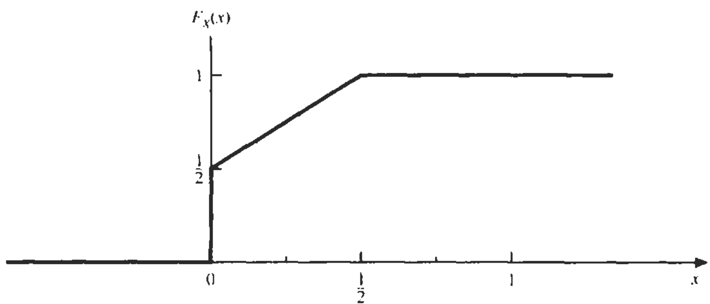

Fig. 2-13

2.11. Find the values of constants a and b such that

$$F(x) = \begin{cases} 1 - ae^{-x/b} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is a valid cdf.

To satisfy property 1 of $F_X(x)$ $[0 \leq F_X(x) \leq 1]$, we must have $0 \leq a \leq 1$ and $b > 0$. Since $b > 0$, property 3 of $F_X(x)$ $[F_X(\infty) = 1]$ is satisfied. It is seen that property 4 of $F_X(x)$ $[F_X(-\infty) = 0]$ is also satisfied. For $0 \leq a \leq 1$ and $b > 0$, $F(x)$ is sketched in Fig. 2-14. From Fig. 2-14, we see that $F(x)$ is a nondecreasing function and continuous on the right, and properties 2 and 5 of $F_X(x)$ are satisfied. Hence, we conclude that $F(x)$ given is a valid cdf if $0 \leq a \leq 1$ and $b > 0$. Note that if $a = 0$, then the r.v. X is a discrete r.v.; if $a = 1$, then X is a continuous r.v.; and if $0 < a < 1$, then X is a mixed r.v.

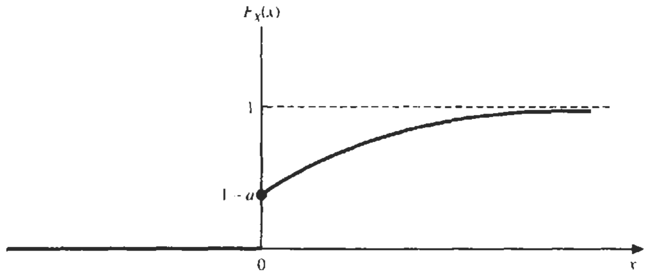

Fig. 2-14

DISCRETE RANDOM VARIABLES AND PMF'S

2.12. Suppose a discrete r.v. X has the following pmfs:

$$p_X(1) = \tfrac{1}{2} \qquad p_X(2) = \tfrac{1}{4} \qquad p_X(3) = \tfrac{1}{8} \qquad p_X(4) = \tfrac{1}{8}$$

(a) Find and sketch the cdf $F_X(x)$ of the r.v. X.

(b) Find (i) $P(X \leq 1)$, (ii) $P(1 < X \leq 3)$, (iii) $P(1 \leq X \leq 3)$.

(a) By Eq. (2.18), we obtain

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \frac{7}{8} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

which is sketched in Fig. 2-15.

(b) (i) By Eq. (2.12), we see that

$$P(X < 1) = F_X(1^-) = 0$$

(ii) By Eq. (2.10),

$$P(1 < X \leq 3) = F_X(3) - F_X(1) = \tfrac{7}{8} - \tfrac{1}{2} = \tfrac{3}{8}$$

(iii) By Eq. (2.64),

$$P(1 \leq X \leq 3) = P(X = 1) + F_X(3) - F_X(1) = \tfrac{1}{2} + \tfrac{7}{8} - \tfrac{1}{2} = \tfrac{7}{8}$$

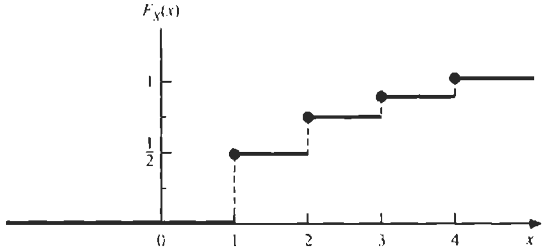

Fig. 2-15

2.13. (*a*) Verify that the function $p(x)$ defined by

$$p(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

is a pmf of a discrete r.v. X.

(*b*) Find (i) $P(X = 2)$, (ii) $P(X \le 2)$, (iii) $P(X \ge 1)$.

(*a*) It is clear that $0 \le p(x) \le 1$ and

$$\sum_{i=0}^{\infty} p(i) = \frac{3}{4} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{3}{4} \frac{1}{1 - \frac{1}{4}} = 1$$

Thus, $p(x)$ satisfies all properties of the pmf [Eqs. (*2.15*) to (*2.17*)] of a discrete r.v. X.

(*b*) (i) By definition (*2.14*),

$$P(X = 2) = p(2) = \tfrac{3}{4}\left(\tfrac{1}{4}\right)^2 = \tfrac{3}{64}$$

(ii) By Eq. (*2.18*),

$$P(X \le 2) = \sum_{i=0}^{2} p(i) = \tfrac{3}{4}\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = \tfrac{63}{64}$$

(iii) By Eq. (*1.25*),

$$P(X \ge 1) = 1 - P(X = 0) = 1 - p(0) = 1 - \tfrac{3}{4} = \tfrac{1}{4}$$

2.14. Consider the experiment of tossing an honest coin repeatedly (Prob. 1.35). Let the r.v. X denote the number of tosses required until the first head appears.

(*a*) Find and sketch the pmf $p_X(x)$ and the cdf $F_X(x)$ of X.

(*b*) Find (i) $P(1 < X \le 4)$, (ii) $P(X > 4)$.

(*a*) From the result of Prob. 1.35, the pmf of X is given by

$$p_X(x) = p_X(k) = P(X = k) = \left(\tfrac{1}{2}\right)^k \qquad k = 1, 2, \ldots$$

Then by Eq. (*2.18*),

$$F_X(x) = P(X \le x) = \sum_{k=1}^{m \le x} p_X(k) = \sum_{k=1}^{m \le x} \left(\tfrac{1}{2}\right)^k$$

or

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \vdots & \vdots \\ 1 - \left(\frac{1}{2}\right)^n & n \le x < n + 1 \\ \vdots & \vdots \end{cases}$$

These functions are sketched in Fig. 2-16.

(b) (i) By Eq. (2.10),

$$P(1 < X \leq 4) = F_X(4) - F_X(1) = \tfrac{15}{16} - \tfrac{1}{2} = \tfrac{7}{16}$$

(ii) By Eq. (1.25),

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \tfrac{15}{16} = \tfrac{1}{16}$$

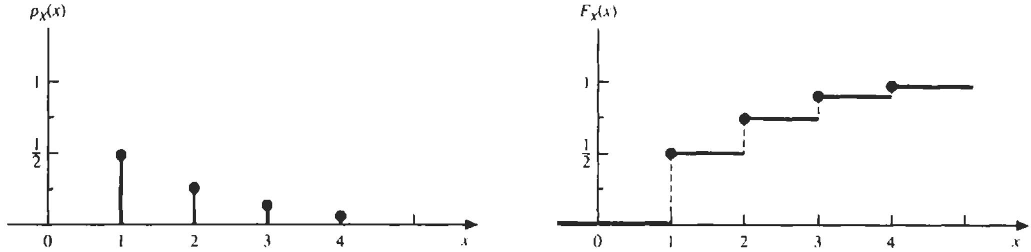

Fig. 2-16

2.15. Consider a sequence of Bernoulli trials with probability p of success. This sequence is observed until the first success occurs. Let the r.v. X denote the trial number on which this first success occurs. Then the pmf of X is given by

$$p_X(x) = P(X = x) = (1 - p)^{x-1}p \qquad x = 1, 2, \ldots \tag{2.67}$$

because there must be $x - 1$ failures before the first success occurs on trial x. The r.v. X defined by Eq. (2.67) is called a *geometric* r.v. with parameter p.

(a) Show that $p_X(x)$ given by Eq. (2.67) satisfies Eq. (2.17).

(b) Find the cdf $F_X(x)$ of X.

(a) Recall that for a geometric series, the sum is given by

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1 \tag{2.68}$$

Thus,

$$\sum_x p_X(x) = \sum_{i=1}^{\infty} p_X(i) = \sum_{i=1}^{\infty} (1-p)^{i-1}p = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

(b) Using Eq. (2.68), we obtain

$$P(X > k) = \sum_{i=k+1}^{\infty} (1-p)^{i-1}p = \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k \tag{2.69}$$

Thus,
$$P(X \leq k) = 1 - P(X > k) = 1 - (1-p)^k \tag{2.70}$$

and
$$F_X(x) = P(X \leq x) = 1 - (1-p)^x \qquad x = 1, 2, \ldots \tag{2.71}$$

Note that the r.v. X of Prob. 2.14 is the geometric r.v. with $p = \frac{1}{2}$.

2.16. Let X be a binomial r.v. with parameters (n, p).

(a) Show that $p_X(x)$ given by Eq. (2.36) satisfies Eq. (2.17).

(b) Find $P(X > 1)$ if $n = 6$ and $p = 0.1$.

(a) Recall that the binomial expansion formula is given by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{2.72}$$

Thus, by Eq. (2.36),

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1^n = 1$$

(b) Now

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{6}{0}(0.1)^0(0.9)^6 - \binom{6}{1}(0.1)^1(0.9)^5 \\ &= 1 - (0.9)^6 - 6(0.1)(0.9)^5 \approx 0.114 \end{aligned}$$

2.17. Let X be a Poisson r.v. with parameter λ.

(a) Show that $p_X(x)$ given by Eq. (2.40) satisfies Eq. (2.17).

(b) Find $P(X > 2)$ with $\lambda = 4$.

(a) By Eq. (2.40),

$$\sum_{k=0}^{\infty} p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

(b) With $\lambda = 4$, we have

$$p_X(k) = e^{-4} \frac{4^k}{k!}$$

and

$$P(X \le 2) = \sum_{k=0}^{2} p_X(k) = e^{-4}(1 + 4 + 8) \approx 0.238$$

Thus,

$$P(X > 2) = 1 - P(X \le 2) \approx 1 - 0.238 = 0.762$$

CONTINUOUS RANDOM VARIABLES AND PDF'S

2.18. Verify Eq. (2.19).

From Eqs. (1.27) and (2.10), we have

$$P(X = x) \le P(x - \varepsilon < X \le x) = F_X(x) - F_X(x - \varepsilon)$$

for any $\varepsilon \ge 0$. As $F_X(x)$ is continuous, the right-hand side of the above expression approaches 0 as $\varepsilon \to 0$. Thus, $P(X = x) = 0$.

2.19. The pdf of a continuous r.v. X is given by

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{2}{3} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the corresponding cdf $F_X(x)$ and sketch $f_X(x)$ and $F_X(x)$.

By Eq. (2.24), the cdf of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\int_0^x \tfrac{1}{3}\, d\xi = \frac{x}{3} & 0 \le x < 1 \\ \displaystyle\int_0^1 \tfrac{1}{3}\, d\xi + \int_1^x \tfrac{2}{3}\, d\xi = \tfrac{2}{3}x - \tfrac{1}{3} & 1 \le x < 2 \\ \displaystyle\int_0^1 \tfrac{1}{3}\, d\xi + \int_1^2 \tfrac{2}{3}\, d\xi = 1 & 2 \le x \end{cases}$$

The functions $f_X(x)$ and $F_X(x)$ are sketched in Fig. 2-17.

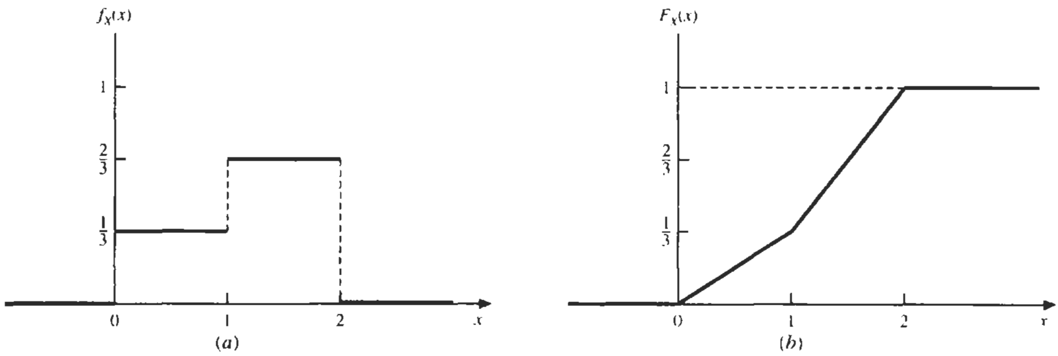

Fig. 2-17

2.20. Let X be a continuous r.v. X with pdf

$$f_X(x) = \begin{cases} kx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Determine the value of k and sketch $f_X(x)$.

(b) Find and sketch the corresponding cdf $F_X(x)$.

(c) Find $P(\frac{1}{4} < X \le 2)$.

(a) By Eq. (2.21), we must have $k > 0$, and by Eq. (2.22),

$$\int_0^1 kx\, dx = \frac{k}{2} = 1$$

Thus, $k = 2$ and

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

which is sketched in Fig. 2-18(a).

(b) By Eq. (2.24), the cdf of X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\int_0^x 2\xi\, d\xi = x^2 & 0 \le x < 1 \\ \displaystyle\int_0^1 2\xi\, d\xi = 1 & 1 \le x \end{cases}$$

which is sketched in Fig. 2-18(b).

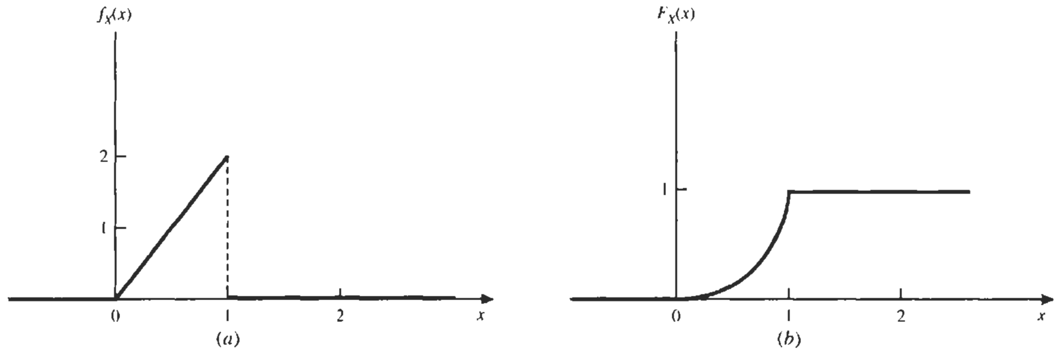

Fig. 2-18

(c) By Eq. (2.25),

$$P(\tfrac{1}{4} < X \le 2) = F_X(2) - F_X(\tfrac{1}{4}) = 1 - (\tfrac{1}{4})^2 = \tfrac{15}{16}$$

2.21. Show that the pdf of a normal r.v. X given by Eq. (2.52) satisfies Eq. (2.22).

From Eq. (2.52),

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

Let $y = (x-\mu)/(\sqrt{2}\,\sigma)$. Then $dx = \sqrt{2}\,\sigma\,dy$ and

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

Let
$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = I$$

Then
$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Letting $x = r\cos\theta$ and $y = r\sin\theta$ (that is, using polar coordinates), we have

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = 2\pi\int_0^{\infty} e^{-r^2} r\,dr = \pi$$

Thus,
$$I = \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi} \tag{2.73}$$

and
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

2.22. Consider a function

$$f(x) = \frac{1}{\sqrt{\pi}}\,e^{(-x^2+x-a)} \qquad -\infty < x < \infty$$

Find the value of a such that $f(x)$ is a pdf of a continuous r.v. X.

$$f(x) = \frac{1}{\sqrt{\pi}} e^{(-x^2+x-a)} = \frac{1}{\sqrt{\pi}} e^{-(x^2-x+1/4+a-1/4)}$$
$$= \left[\frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\right] e^{-(a-1/4)}$$

If $f(x)$ is a pdf of a continuous r.v. X, then by Eq. (*2.22*), we must have

$$\int_{-\infty}^{\infty} f(x)\, dx = e^{-(a-1/4)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\, dx = 1$$

Now by Eq. (*2.52*), the pdf of $N(\frac{1}{2}; \frac{1}{2})$ is $\dfrac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\, dx = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\, dx = e^{-(a-1/4)} = 1$$

from which we obtain $a = \frac{1}{4}$.

2.23. A r.v. X is called a *Rayleigh* r.v. if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.74}$$

(*a*) Determine the corresponding cdf $F_X(x)$.

(*b*) Sketch $f_X(x)$ and $F_X(x)$ for $\sigma = 1$.

(*a*) By Eq. (*2.24*), the cdf of X is

$$F_X(x) = \int_0^x \frac{\xi}{\sigma^2} e^{-\xi^2/(2\sigma^2)}\, d\xi \qquad x \geq 0$$

Let $y = \xi^2/(2\sigma^2)$. Then $dy = (1/\sigma^2)\xi\, d\xi$, and

$$F_X(x) = \int_0^{x^2/(2\sigma^2)} e^{-y}\, dy = 1 - e^{-x^2/(2\sigma^2)} \tag{2.75}$$

(*b*) With $\sigma = 1$, we have

$$f_X(x) = \begin{cases} xe^{\ x^2/2} & x > 0 \\ 0 & x < 0 \end{cases}$$

and

$$F_X(x) = \begin{cases} 1 - e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

These functions are sketched in Fig. 2-19.

2.24. A r.v. X is called a *gamma* r.v. with parameter (α, λ) $(\alpha > 0$ and $\lambda > 0)$ if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.76}$$

where $\Gamma(\alpha)$ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx \qquad \alpha > 0 \tag{2.77}$$

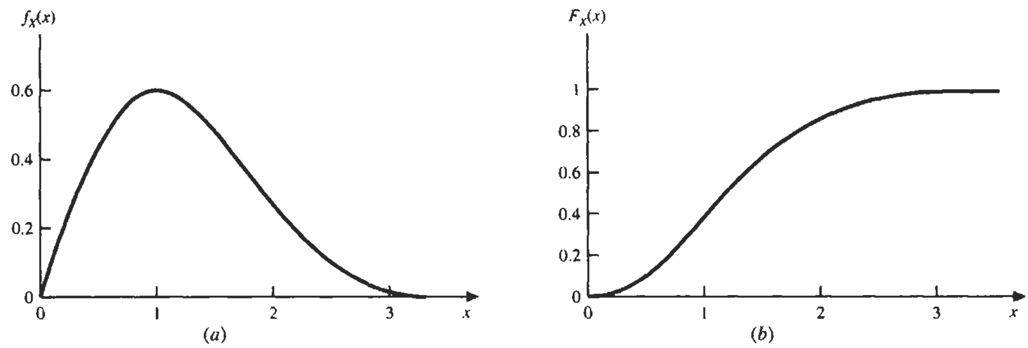

Fig. 2-19 Rayleigh distribution with $\sigma = 1$.

(a) Show that the gamma function has the following properties:

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \qquad \alpha > 0$ (2.78)
2. $\Gamma(k + 1) = k! \qquad k\ (\geq 0)$: integer (2.79)
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (2.80)

(b) Show that the pdf given by Eq. (2.76) satisfies Eq. (2.22).

(c) Plot $f_X(x)$ for $(\alpha, \lambda) = (1, 1), (2, 1)$, and $(5, 2)$.

(a) Integrating Eq. (2.77) by parts ($u = x^{\alpha-1}$, $dv = e^{-x}\,dx$), we obtain

$$\Gamma(\alpha) = -e^{-x}x^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2}\,dx$$

$$= (\alpha - 1)\int_0^\infty e^{-x}x^{\alpha-2}\,dx = (\alpha-1)\Gamma(\alpha-1) \tag{2.81}$$

Replacing α by $\alpha + 1$ in Eq. (2.81), we get Eq. (2.78).

Next, by applying Eq. (2.78) repeatedly using an integral value of α, say $\alpha = k$, we obtain

$$\Gamma(k+1) = k\Gamma(k) = k(k-1)\Gamma(k-1) = k(k-1)\cdots(2)\Gamma(1)$$

Since

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$$

it follows that $\Gamma(k + 1) = k!$. Finally, by Eq. (2.77),

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}\,dx$$

Let $y = x^{1/2}$. Then $dy = \frac{1}{2}x^{-1/2}\,dx$, and

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2}\,dy = \int_{-\infty}^\infty e^{-y^2}\,dy = \sqrt{\pi}$$

in view of Eq. (2.73).

(b) Now

$$\int_{-\infty}^\infty f_X(x)\,dx = \int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda x}x^{\alpha-1}\,dx$$

Let $y = \lambda x$. Then $dy = \lambda\,dx$ and

$$\int_{-\infty}^\infty f_X(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha}\int_0^\infty e^{-y}y^{\alpha-1}\,dy = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha}\Gamma(\alpha) = 1$$

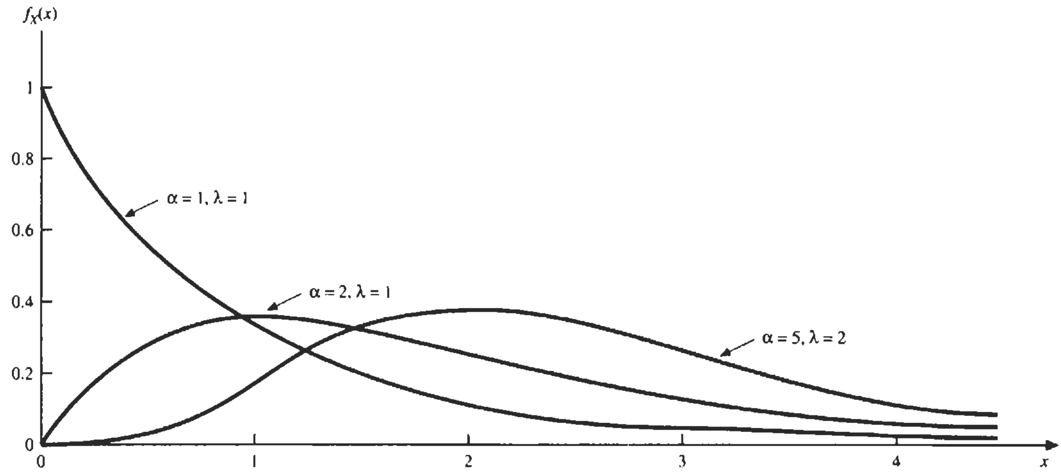

Fig. 2-20 Gamma distributions for selected values of α and λ.

(c) The pdf's $f_X(x)$ with $(\alpha, \lambda) = (1, 1), (2, 1)$, and $(5, 2)$ are plotted in Fig. 2-20.

Note that when $\alpha = 1$, the gamma r.v. becomes an exponential r.v. with parameter λ [Eq. (2.48)].

MEAN AND VARIANCE

2.25. Consider a discrete r.v. X whose pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot $p_X(x)$ and find the mean and variance of X.

(b) Repeat (a) if the pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -2, 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The pmf $p_X(x)$ is plotted in Fig. 2-21(a). By Eq. (2.26), the mean of X is

$$\mu_X = E(X) = \tfrac{1}{3}(-1 + 0 + 1) = 0$$

By Eq. (2.29), the variance of X is

$$\sigma_X{}^2 = \mathrm{Var}(X) = E[(X - \mu_X)^2] = E(X^2) = \tfrac{1}{3}[(-1)^2 + (0)^2 + (1)^2] = \tfrac{2}{3}$$

(b) The pmf $p_X(x)$ is plotted in Fig. 2-21(b). Again by Eqs. (2.26) and (2.29), we obtain

$$\mu_X = E(X) = \tfrac{1}{3}(-2 + 0 + 2) = 0$$

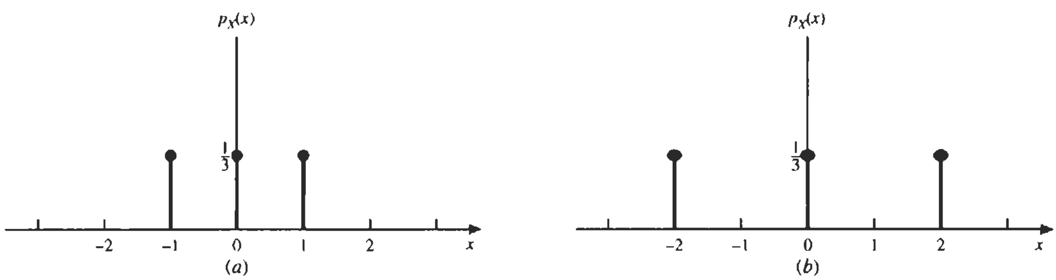

Fig. 2-21

$$\sigma_X{}^2 = \text{Var}(X) = \tfrac{1}{3}[(-2)^2 + (0)^2 + (2)^2] = \tfrac{8}{3}$$

Note that the variance of X is a measure of the spread of a distribution about its mean.

2.26. Let a r.v. X denote the outcome of throwing a fair die. Find the mean and variance of X.

Since the die is fair, the pmf of X is

$$p_X(x) = p_X(k) = \tfrac{1}{6} \qquad k = 1, 2, \ldots, 6$$

By Eqs. (2.26) and (2.29), the mean and variance of X are

$$\mu_X = E(X) = \tfrac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \tfrac{7}{2} = 3.5$$

$$\sigma_X{}^2 = \tfrac{1}{6}[(1 - \tfrac{7}{2})^2 + (2 - \tfrac{7}{2})^2 + (3 - \tfrac{7}{2})^2 + (4 - \tfrac{7}{2})^2 + (5 - \tfrac{7}{2})^2 + (6 - \tfrac{7}{2})^2] = \tfrac{35}{12}$$

Alternatively, the variance of X can be found as follows:

$$E(X^2) = \tfrac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \tfrac{91}{6}$$

Hence, by Eq. (2.31),

$$\sigma_X{}^2 = E(X^2) - [E(X)]^2 = \tfrac{91}{6} - (\tfrac{7}{2})^2 = \tfrac{35}{12}$$

2.27. Find the mean and variance of the geometric r.v. X defined by Eq. (2.67) (Prob. 2.15).

To find the mean and variance of a geometric r.v. X, we need the following results about the sum of a geometric series and its first and second derivatives. Let

$$g(r) = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad |r| < 1 \tag{2.82}$$

Then

$$g'(r) = \frac{dg(r)}{dr} = \sum_{n=1}^{\infty} anr^{n-1} = \frac{a}{(1-r)^2} \tag{2.83}$$

$$g''(r) = \frac{d^2g(r)}{dr^2} = \sum_{n=2}^{\infty} an(n-1)r^{n-2} = \frac{2a}{(1-r)^3} \tag{2.84}$$

By Eqs. (2.26) and (2.67), and letting $q = 1 - p$, the mean of X is given by

$$\mu_X = E(X) = \sum_{x=1}^{\infty} xq^{x-1}p = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \tag{2.85}$$

where Eq. (2.83) is used with $a = p$ and $r = q$.

To find the variance of X, we first find $E[X(X-1)]$. Now,

$$\begin{aligned} E[X(X-1)] &= \sum_{x=1}^{\infty} x(x-1)q^{x-1}p = \sum_{x=2}^{\infty} pqx(x-1)q^{x-2} \\ &= \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2} = \frac{2(1-p)}{p^2} \end{aligned} \tag{2.86}$$

where Eq. (2.84) is used with $a = pq$ and $r = q$.

Since $E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$, we have

$$E(X^2) = E[X(X-1)] + E(X) = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2-p}{p^2} \tag{2.87}$$

Then by Eq. (2.31), the variance of X is

$$\sigma_X{}^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{2.88}$$

2.28. Let X be a binomial r.v. with parameters (n, p). Verify Eqs. (2.38) and (2.39).

By Eqs. (2.26) and (2.36), and letting $q = 1 - p$, we have

$$E(X) = \sum_{k=0}^{n} k p_X(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!\,(k-1)!} p^{k-1} q^{n-k}$$

Letting $i = k - 1$ and using Eq. (2.72), we obtain

$$E(X) = np \sum_{i=0}^{n-1} \frac{(n-1)!}{(n-1-i)!\,i!} p^i q^{n-1-i}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i q^{n-1-i}$$

$$= np(p+q)^{n-1} = np(1)^{n-1} = np$$

Next,

$$E[X(X-1)] = \sum_{k=0}^{n} k(k-1) p_X(k) = \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k(k-1) \frac{n!}{(n-k)!\,k!} p^k q^{n-k}$$

$$= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!\,(k-2)!} p^{k-2} q^{n-k}$$

Similarly, letting $i = k - 2$ and using Eq. (2.72), we obtain

$$E[X(X-1)] = n(n-1)p^2 \sum_{i=0}^{n-2} \frac{(n-2)!}{(n-2-i)!\,i!} p^i q^{n-2-i}$$

$$= n(n-1)p^2 \sum_{i=0}^{n-2} \binom{n-2}{i} p^i q^{n-2-i}$$

$$= n(n-1)p^2(p+q)^{n-2} = n(n-1)p^2$$

Thus,

$$E(X^2) = E[X(X-1)] + E(X) = n(n-1)p^2 + np \tag{2.89}$$

and by Eq. (2.31),

$$\sigma_X{}^2 = \mathrm{Var}(x) = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

2.29. Let X be a Poisson r.v. with parameter λ. Verify Eqs. (2.42) and (2.43).

By Eqs. (2.26) and (2.40),

$$E(X) = \sum_{k=0}^{\infty} k p_X(k) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = 0 + \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Next,

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!} = \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$$

$$= \lambda^2 e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

Thus,

$$E(X^2) = E[X(X-1)] + E(X) = \lambda^2 + \lambda \tag{2.90}$$

and by Eq. (2.31),

$$\sigma_X{}^2 = \operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

2.30. Find the mean and variance of the r.v. X of Prob. 2.20.

From Prob. 2.20, the pdf of X is

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

By Eq. (2.26), the mean of X is

$$\mu_X = E(X) = \int_0^1 x(2x)\,dx = 2\,\frac{x^3}{3}\bigg|_0^1 = \tfrac{2}{3}$$

By Eq. (2.27), we have

$$E(X^2) = \int_0^1 x^2(2x)\,dx = 2\,\frac{x^4}{4}\bigg|_0^1 = \tfrac{1}{2}$$

Thus, by Eq. (2.31), the variance of X is

$$\sigma_X{}^2 = \operatorname{Var}(X) = E(X^2) - [E(X)]^2 = \tfrac{1}{2} - (\tfrac{2}{3})^2 = \tfrac{1}{18}$$

2.31. Let X be a uniform r.v. over (a, b). Verify Eqs. (2.46) and (2.47).

By Eqs. (2.44) and (2.26), the mean of X is

$$\mu_X = E(X) = \int_a^b x\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\bigg|_a^b = \frac{1}{2}\,\frac{b^2-a^2}{b-a} = \tfrac{1}{2}(b+a)$$

By Eq. (2.27), we have

$$E(X^2) = \int_a^b x^2\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^3}{3}\bigg|_a^b = \tfrac{1}{3}(b^2 + ab + a^2) \tag{2.91}$$

Thus, by Eq. (2.31), the variance of X is

$$\begin{aligned}\sigma_X{}^2 &= \operatorname{Var}(X) = E(X^2) - [E(X)]^2 \\ &= \tfrac{1}{3}(b^2 + ab + a^2) - \tfrac{1}{4}(b+a)^2 = \tfrac{1}{12}(b-a)^2\end{aligned}$$

2.32. Let X be an exponential r.v. X with parameter λ. Verify Eqs. (2.50) and (2.51).

By Eqs. (2.48) and (2.26), the mean of X is

$$\mu_X = E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

Integrating by parts ($u = x$, $dv = \lambda e^{-\lambda x}\,dx$) yields

$$E(X) = -xe^{-\lambda x}\bigg|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

Next, by Eq. (2.27),

$$E(X^2) = \int_0^\infty x^2\lambda e^{-\lambda x}\,dx$$

Again integrating by parts ($u = x^2$, $dv = \lambda e^{-\lambda x}\,dx$), we obtain

$$E(X^2) = -x^2e^{-\lambda x}\bigg|_0^\infty + 2\int_0^\infty xe^{-\lambda x}\,dx = \frac{2}{\lambda^2} \tag{2.92}$$

Thus, by Eq. (2.31), the variance of X is

$$\sigma_X{}^2 = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

2.33. Let $X = N(\mu; \sigma^2)$. Verify Eqs. (2.57) and (2.58).

Using Eqs. (2.52) and (2.26), we have

$$\mu_X = E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

Writing x as $(x - \mu) + \mu$, we have

$$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x - \mu) e^{-(x-\mu)^2/(2\sigma^2)}\, dx + \mu \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

Letting $y = x - \mu$ in the first integral, we obtain

$$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y e^{-y^2/(2\sigma^2)}\, dy + \mu \int_{-\infty}^{\infty} f_X(x)\, dx$$

The first integral is zero, since its integrand is an odd function. Thus, by the property of pdf Eq. (2.22), we get

$$\mu_X = E(X) = \mu$$

Next, by Eq. (2.29),

$$\sigma_X{}^2 = E[(X - \mu)^2] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

From Eqs. (2.22) and (2.52), we have

$$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sigma\sqrt{2\pi}$$

Differentiating with respect to σ, we obtain

$$\int_{-\infty}^{\infty} \frac{(x - \mu)^2}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sqrt{2\pi}$$

Multiplying both sides by $\sigma^2/\sqrt{2\pi}$, we have

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sigma^2$$

Thus, $$\sigma_X{}^2 = \operatorname{Var}(X) = \sigma^2$$

2.34. Find the mean and variance of a Rayleigh r.v. defined by Eq. (2.74) (Prob. 2.23).

Using Eqs. (2.74) and (2.26), we have

$$\mu_X = E(X) = \int_0^{\infty} x \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}\, dx = \frac{1}{\sigma^2} \int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\, dx$$

Now the variance of $N(0; \sigma^2)$ is given by

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x^2 e^{-x^2/(2\sigma^2)}\, dx = \sigma^2$$

Since the integrand is an even function, we have

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\,dx = \tfrac{1}{2}\sigma^2$$

or

$$\int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\,dx = \tfrac{1}{2}\sqrt{2\pi}\,\sigma^3 = \sqrt{\frac{\pi}{2}}\,\sigma^3$$

Then

$$\mu_X = E(X) = \frac{1}{\sigma^2}\sqrt{\frac{\pi}{2}}\,\sigma^3 = \sqrt{\frac{\pi}{2}}\,\sigma \tag{2.93}$$

Next,

$$E(X^2) = \int_0^{\infty} x^2 \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}\,dx = \frac{1}{\sigma^2}\int_0^{\infty} x^3 e^{-x^2/(2\sigma^2)}\,dx$$

Let $y = x^2/(2\sigma^2)$. Then $dy = x\,dx/\sigma^2$, and so

$$E(X^2) = 2\sigma^2 \int_0^{\infty} y e^{-y}\,dy = 2\sigma^2 \tag{2.94}$$

Hence, by Eq. (2.31),

$$\sigma_X{}^2 = E(X^2) - [E(X)]^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2 \approx 0.429\sigma^2 \tag{2.95}$$

2.35. Consider a continuous r.v. X with pdf $f_X(x)$. If $f_X(x) = 0$ for $x < 0$, then show that, for any $a > 0$,

$$P(X \ge a) \le \frac{\mu_X}{a} \tag{2.96}$$

where $\mu_X = E(X)$. This is known as the *Markov inequality*.

From Eq. (2.23),

$$P(X \ge a) = \int_a^{\infty} f_X(x)\,dx$$

Since $f_X(x) = 0$ for $x < 0$,

$$\mu_X = E(X) = \int_0^{\infty} x f_X(x)\,dx \ge \int_a^{\infty} x f_X(x)\,dx \ge a\int_a^{\infty} f_X(x)\,dx$$

Hence,

$$\int_a^{\infty} f_X(x)\,dx = P(X \ge a) \le \frac{\mu_X}{a}$$

2.36. For any $a > 0$, show that

$$P(|X - \mu_X| \ge a) \le \frac{\sigma_X{}^2}{a^2} \tag{2.97}$$

where μ_X and $\sigma_X{}^2$ are the mean and variance of X, respectively. This is known as the *Chebyshev inequality*.

From Eq. (2.23),

$$P(|X - \mu_X| \ge a) = \int_{-\infty}^{\mu_X - a} f_X(x)\,dx + \int_{\mu_X + a}^{\infty} f_X(x)\,dx = \int_{|x - \mu_X| \ge a} f_X(x)\,dx$$

By Eq. (2.29),

$$\sigma_X{}^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx \ge \int_{|x - \mu_X| \ge a} (x - \mu_X)^2 f_X(x)\,dx \ge a^2 \int_{|x - \mu_X| \ge a} f_X(x)\,dx$$

Hence, $$\int_{|x-\mu_X| \geq a} f_X(x)\,dx \leq \frac{\sigma_X{}^2}{a^2} \quad \text{or} \quad P(|X - \mu_X| \geq a) \leq \frac{\sigma_X{}^2}{a^2}$$

Note that by setting $a = k\sigma_X$ in Eq. (2.97), we obtain

$$P(|X - \mu_X| \geq k\sigma_X) \leq \frac{1}{k^2} \tag{2.98}$$

Equation (2.98) says that the probability that a r.v. will fall k or more standard deviations from its mean is $\leq 1/k^2$. Notice that nothing at all is said about the distribution function of X. The Chebyshev inequality is therefore quite a generalized statement. However, when applied to a particular case, it may be quite weak.

SPECIAL DISTRIBUTIONS

2.37. A binary source generates digits 1 and 0 randomly with probabilities 0.6 and 0.4, respectively.

(a) What is the probability that two 1s and three 0s will occur in a five-digit sequence?

(b) What is the probability that at least three 1s will occur in a five-digit sequence?

(a) Let X be the r.v. denoting the number of 1s generated in a five-digit sequence. Since there are only two possible outcomes (1 or 0), the probability of generating 1 is constant, and there are five digits, it is clear that X is a binomial r.v. with parameters $(n, p) = (5, 0.6)$. Hence, by Eq. (2.36), the probability that two 1s and three 0s will occur in a five-digit sequence is

$$P(X = 2) = \binom{5}{2}(0.6)^2(0.4)^3 = 0.23$$

(b) The probability that at least three 1s will occur in a five-digit sequence is

$$P(X \geq 3) = 1 - P(X \leq 2)$$

where $$P(X \leq 2) = \sum_{k=0}^{2} \binom{5}{k}(0.6)^k(0.4)^{5-k} = 0.317$$

Hence, $$P(X \geq 3) = 1 - 0.317 = 0.683$$

2.38. A fair coin is flipped 10 times. Find the probability of the occurrence of 5 or 6 heads.

Let the r.v. X denote the number of heads occurring when a fair coin is flipped 10 times. Then X is a binomial r.v. with parameters $(n, p) = (10, \frac{1}{2})$. Thus, by Eq. (2.36),

$$P(5 \leq X \leq 6) = \sum_{k=5}^{6} \binom{10}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{10-k} = 0.451$$

2.39. Let X be a binomial r.v. with parameters (n, p), where $0 < p < 1$. Show that as k goes from 0 to n, the pmf $p_X(k)$ of X first increases monotonically and then decreases monotonically, reaching its largest value when k is the largest integer less than or equal to $(n + 1)p$.

By Eq. (2.36), we have

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)} \tag{2.99}$$

Hence, $p_X(k) \geq p_X(k-1)$ if and only if $(n-k+1)p \geq k(1-p)$ or $k \leq (n+1)p$. Thus, we see that $p_X(k)$ increases monotonically and reaches its maximum when k is the largest integer less than or equal to $(n+1)p$ and then decreases monotonically.

2.40. Show that the Poisson distribution can be used as a convenient approximation to the binomial distribution for large n and small p.

From Eq. (2.36), the pmf of the binomial r.v. with parameters (n, p) is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} p^k (1-p)^{n-k}$$

Multiplying and dividing the right-hand side by n^k, we have

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!}(np)^k\left(1-\frac{np}{n}\right)^{n-k}$$

If we let $n \to \infty$ in such a way that $np = \lambda$ remains constant, then

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \xrightarrow[n\to\infty]{} 1$$

$$\left(1-\frac{np}{n}\right)^{n-k} = \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} \xrightarrow[n\to\infty]{} e^{-\lambda}(1) = e^{-\lambda}$$

where we used the fact that

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Hence, in the limit as $n \to \infty$ with $np = \lambda$ (and as $p = \lambda/n \to 0$),

$$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow[n\to\infty]{} e^{-\lambda}\frac{\lambda^k}{k!} \qquad np = \lambda$$

Thus, in the case of large n and small p,

$$\binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda}\frac{\lambda^k}{k!} \qquad np = \lambda \tag{2.100}$$

which indicates that the binomial distribution can be approximated by the Poisson distribution.

2.41. A noisy transmission channel has a per-digit error probability $p = 0.01$.

(a) Calculate the probability of more than one error in 10 received digits.

(b) Repeat (a), using the Poisson approximation Eq. (2.100).

(a) It is clear that the number of errors in 10 received digits is a binomial r.v. X with parameters $(n, p) = (10, 0.01)$. Then, using Eq. (2.36), we obtain

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \\ &= 0.0042 \end{aligned}$$

(b) Using Eq. (2.100) with $\lambda = np = 10(0.01) = 0.1$, we have

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-0.1}\frac{(0.1)^0}{0!} - e^{-0.1}\frac{(0.1)^1}{1!} \\ &= 0.0047 \end{aligned}$$

2.42. The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson r.v. X with $\lambda = 2$.

(a) Find the probability that more than three calls will arrive during any 10-minute period.

(b) Find the probability that no calls will arrive during any 10-minute period.

(a) From Eq. (2.40), the pmf of X is

$$p_X(k) = P(X = k) = e^{-2}\frac{2^k}{k!} \qquad k = 0, 1, \ldots$$

Thus,

$$P(X > 3) = -P(X \le 3) = 1 - \sum_{k=0}^{3} e^{-2}\frac{2^k}{k!}$$
$$= 1 - e^{-2}(1 + 2 + \tfrac{4}{2} + \tfrac{8}{6}) \approx 0.143$$

(b) $P(X = 0) = p_X(0) = e^{-2} \approx 0.135$

2.43. Consider the experiment of throwing a pair of fair dice.

(a) Find the probability that it will take less than six tosses to throw a 7.

(b) Find the probability that it will take more than six tosses to throw a 7.

(a) From Prob. 1.31(a), we see that the probability of throwing a 7 on any toss is $\frac{1}{6}$. Let X denote the number of tosses required for the first success of throwing a 7. Then, from Prob. 2.15, it is clear that X is a geometric r.v. with parameter $p = \frac{1}{6}$. Thus, using Eq. (2.71) of Prob. 2.15, we obtain

$$P(X < 6) = P(X \le 5) = F_X(5) = 1 - (\tfrac{5}{6})^5 \approx 0.598$$

(b) Similarly, we get

$$P(X > 6) = 1 - P(X \le 6) = 1 - F_X(6)$$
$$= 1 - [1 - (\tfrac{5}{6})^6] = (\tfrac{5}{6})^6 \approx 0.335$$

2.44. Consider the experiment of rolling a fair die. Find the average number of rolls required in order to obtain a 6.

Let X denote the number of trials (rolls) required until the number 6 first appears. Then X is a geometrical r.v. with parameter $p = \frac{1}{6}$. From Eq. (2.85) of Prob. 2.27, the mean of X is given by

$$\mu_X = E(X) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$$

Thus, the average number of rolls required in order to obtain a 6 is 6.

2.45. Assume that the length of a phone call in minutes is an exponential r.v. X with parameter $\lambda = \frac{1}{10}$. If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait (a) less than 5 minutes, and (b) between 5 and 10 minutes.

(a) From Eq. (2.48), the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{10}e^{-x/10} & x > 0 \\ 0 & x < 0 \end{cases}$$

Then

$$P(X < 5) = \int_0^5 \tfrac{1}{10}e^{-x/10}\,dx = -e^{-x/10}\Big|_0^5 = 1 - e^{-0.5} \approx 0.393$$

(b) Similarly,

$$P(5 < X < 10) = \int_5^{10} \tfrac{1}{10}e^{-x/10}\,dx = e^{-0.5} - e^{-1} \approx 0.239$$

2.46. All manufactured devices and machines fail to work sooner or later. Suppose that the failure rate is constant and the time to failure (in hours) is an exponential r.v. X with parameter λ.

(*a*) Measurements show that the probability that the time to failure for computer memory chips in a given class exceeds 10^4 hours is e^{-1} (≈ 0.368). Calculate the value of the parameter λ.

(*b*) Using the value of the parameter λ determined in part (*a*), calculate the time x_0 such that the probability that the time to failure is less than x_0 is 0.05.

(*a*) From Eq. (*2.49*), the cdf of X is given by

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Now

$$\begin{aligned} P(X > 10^4) &= 1 - P(X \le 10^4) = 1 - F_X(10^4) \\ &= 1 - (1 - e^{-\lambda 10^4}) = e^{-\lambda 10^4} = e^{-1} \end{aligned}$$

from which we obtain $\lambda = 10^{-4}$.

(*b*) We want

$$F_X(x_0) = P(X \le x_0) = 0.05$$

Hence,

$$1 - e^{-\lambda x_0} = 1 - e^{-10^{-4} x_0} = 0.05$$

or

$$e^{-10^{-4} x_0} = 0.95$$

from which we obtain

$$x_0 = -10^4 \ln(0.95) = 513 \text{ hours}$$

2.47. A production line manufactures 1000-ohm (Ω) resistors that have 10 percent tolerance. Let X denote the resistance of a resistor. Assuming that X is a normal r.v. with mean 1000 and variance 2500, find the probability that a resistor picked at random will be rejected.

Let A be the event that a resistor is rejected. Then $A = \{X < 900\} \cup \{X > 1100\}$. Since $\{X < 900\} \cap \{X > 1100\} = \varnothing$, we have

$$P(A) = P(X < 900) + P(X > 1100) = F_X(900) + [1 - F_X(1100)]$$

Since X is a normal r.v. with $\mu = 1000$ and $\sigma^2 = 2500$ ($\sigma = 50$), by Eq. (*2.55*) and Table A (Appendix A),

$$F_X(900) = \Phi\left(\frac{900 - 1000}{50}\right) = \Phi(-2) = 1 - \Phi(2)$$

$$F_X(1100) = \Phi\left(\frac{1100 - 1000}{50}\right) = \Phi(2)$$

Thus,

$$P(A) = 2[1 - \Phi(2)] \approx 0.045$$

2.48. The radial miss distance [in meters (m)] of the landing point of a parachuting sky diver from the center of the target area is known to be a Rayleigh r.v. X with parameter $\sigma^2 = 100$.

(*a*) Find the probability that the sky diver will land within a radius of 10 m from the center of the target area.

(*b*) Find the radius r such that the probability that $X > r$ is e^{-1} (≈ 0.368).

(*a*) Using Eq. (*2.75*) of Prob. 2.23, we obtain

$$P(X \le 10) = F_X(10) = 1 - e^{-100/200} = 1 - e^{-0.5} \approx 0.393$$

(*b*) Now

$$\begin{aligned} P(X > r) &= 1 - P(X \le r) = 1 - F_X(r) \\ &= 1 - (1 - e^{-r^2/200}) = e^{-r^2/200} = e^{-1} \end{aligned}$$

from which we obtain $r^2 = 200$ and $r = \sqrt{200} = 14.142$ m.