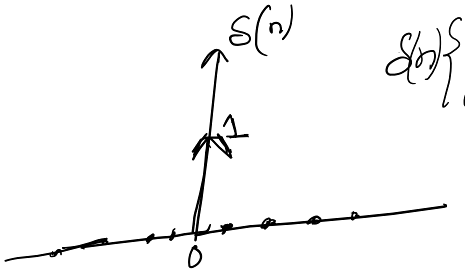
- **Z-transform unit impulse function.**

$x(n) = \delta(n)$



$\delta(n) \begin{cases} 1 & n=0 \\ 0 & \text{other} \end{cases}$

$\therefore\ x(z) = Z.T\,[x(n)]$

$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$

$= \delta(-2)z^{2} + \delta(-1)z^{1} + \delta(0)z^{0} + \delta(1)z^{-1} + \delta(2)z^{-2} \ldots$

$= 0\times z^{2} + 0\times z^{1} + 1\times z^{0} + 0\times z^{-1} + 0\times z^{-2}$

$= 1$

- **Z-transform of Cosine Signal.**

$x(n) = \cos(n\omega)\, u(n)$

$\therefore\ x(z) = Z.T\,[x(n)]$

$= \sum_{n=-\infty}^{\infty} \cos(n\omega) u(n) z^{-n}$

$\left|\ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right.$

$= \sum_{n=0}^{\infty} \cos(n\omega) z^{-n}$

$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right] z^{-n}$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}\right]$

$= \frac{1}{2}\left[\sum_{n=0}^{\infty} \left(e^{j\omega} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(e^{-j\omega} z^{-1}\right)^n\right]$

$= \frac{1}{2}\left[\frac{1}{1-e^{j\omega}/z} + \frac{1}{1-e^{-j\omega}/z}\right]$

$= \frac{1}{2}\left[\frac{z}{z-e^{j\omega}} + \frac{1}{z-e^{-j\omega}}\right]$

$= \frac{1}{2}\left[\frac{z^2 - ze^{-j\omega} + z^2 - ze^{j\omega}}{z^2 - ze^{j\omega} - ze^{-j\omega} + 1}\right] = \frac{1}{2}\left[\frac{2z^2 - z(e^{-j\omega} + e^{j\omega})}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}\right]$

$= \frac{1}{2}\left[\frac{2z^2 - 2z(e^{-j\omega} + e^{j\omega})/2}{z^2 - 2z(e^{j\omega} + e^{-j\omega}) + 1}\right]$

$$= \frac{1}{2}\left[\frac{2z^2 - 2z\cos w}{z^2 - 2z\cos w + 1}\right] = \frac{1}{\cancel{2}}\,\frac{\cancel{2}(z^2 - z\cos w)}{(z^2 - 2z\cos w + 1)}$$

$$x(z) = \frac{z^2 - z\cos w}{z^2 - 2z\cos w + 1}$$ ← Z transform of Cos function.

Initial and Final value Theorem.

- Initial value $x(0)$ in z-transform is given by

$$x(0) = \lim_{z\to\infty} x(z)$$

- Final value $x(\infty)$ in Z-transform is given by

$$x(\infty = \lim_{z\to 1} (z-1)\, X(z)$$

- For unit step signal calculate the initial $(x(0))$ and Final $[x(\infty)]$

we know,

$$u(n) \xleftrightarrow{ZT} \frac{z}{z-1}$$

$$\therefore x(z) = \frac{z}{z-1}$$

- $x(0) = \lim_{z\to\infty} \frac{z}{z-1} = \lim_{z\to\infty} \frac{1}{1-1/z} = \frac{1}{1-1/\infty} = 1$

- $x(\infty) = \lim_{z\to 1} \cancel{(z-1)} \frac{z}{\cancel{(z-1)}} = \lim_{z\to 1} z = 1$

The z-transform of a anticausal system is

$$x(z) = \frac{3-4z}{1-2z+5z^2}$$

the value of $x(0) = ?$

$$x(0) = \lim_{z\to\infty} x(z)$$

$$= \lim_{x\to z}\left[\frac{3-4z}{1-2z+5z^2}\right] = \lim_{z\to\infty} \frac{\cancel{z}}{z^{\cancel{2}}}\left[\frac{3/z - 4}{1/z^2 - 2/z + 5}\right]$$

$$= \frac{1}{\infty}\left[\frac{3/\infty - 4}{1/\infty^2 - 2/\infty + 5}\right]$$

$$= 0\left[\frac{0-4}{0-0+5}\right] = 0$$

$\boxed{\therefore \text{ initial value } x(0) = 0}$

Region of Convergence (ROC)

- ROC is defined as the set of values of z in the z-plane for which the magnitude of $X(z)$ is finite.

Properties of ROC:

1) The ROC is a ring in the z-plane centered at the origin.

2) The ROC cannot contain any poles.

3) The ROC must be connected Region.

3) The ROC

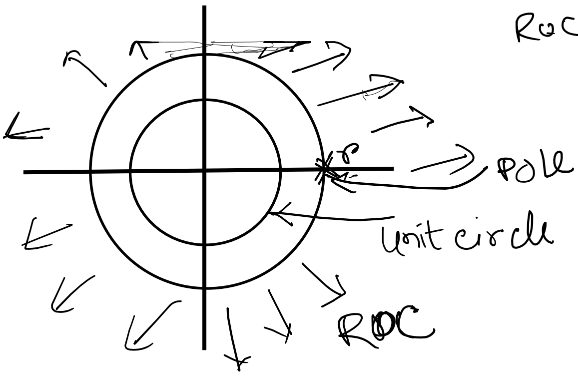
4) If $x(n)$ is right-sided sequence, then the ROC exists outward from the outermost finite pole to $z=\infty$

$\Rightarrow \quad x(n) = f(n) \cdot u(n)$.

| We know,
| $u(n)$ is right-sided sequence.

$$x(n) \longleftrightarrow X(z) = \frac{z}{z-r}$$

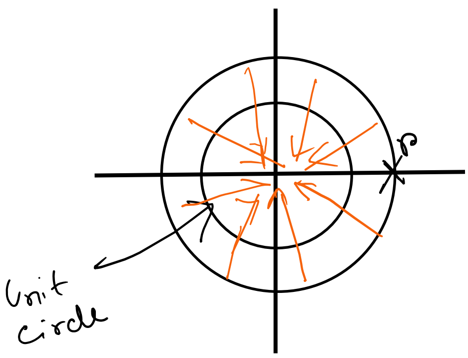
ROC: $|z| > r$



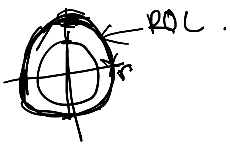
6.) If $x(n)$ is left-sided sequence, the the ROC extends inward from innermost pole to $z=0$

$x(n) = f(n)\, u(-n)$

↑ left-sided sequence.



7) If $x(n)$ is two sided signal i.e $x(n) = f_1 u(n) + f_2 u(-n)$ and if $|z| = r$, then ROC will contain a ring in Z-plane.



- Example of z-transform using differentiation property:—

⇔ Find the inverse Z-transform of $\log\left(\frac{1}{1-a^{-1}z}\right)$, $|z| < |a|$

⇒ Note: if $x(n) \xleftrightarrow{ZT} X(z)$
$nx(n) \xleftrightarrow{ZT} -z\frac{d\,x(z)}{dz}$

now, given,

$$x(z) = \log\left(\frac{1}{1-a^{-1}z}\right) = \log(1-a^{-1}z)^{-1} = -\log(1-a^{-1}z)$$

$$\Rightarrow \frac{dx(z)}{dz} = \frac{d}{dz}\left(-\log(1-a^{-1}z)\right)$$

$$= -\frac{1}{1-a^{-1}z}\left[0 - a^{-1}(1)\right] = \frac{a^{-1}}{1-a^{-1}z}$$

$$\therefore -z\frac{dx(z)}{dz} = -\frac{a^{-1}z}{1-a^{-1}z} = -\frac{z/a}{1-z/a} = \frac{-z}{a-z}$$

$$= \frac{z}{z-a}$$

$a^n u(n) = \frac{z}{z-a}$ Sequence right-sided	$-a^n u(-n-1) = \frac{z}{z-a}$ Sequence left-sided

$\therefore n x(n) = -a^n u(-n-1)$] take Inverse Z-transform at both sides.

$$x(n) = -\frac{a^n}{n}u(-n-1)$$

2) Find the inverse z-transform of $\log(1-az^{-1})$

we know,

if $x(n) \longleftrightarrow X(z)$
$nx(n) \xleftrightarrow{ZT} -z\frac{d\,x(z)}{dz}$

$$\Rightarrow \frac{d\,x(z)}{dz} = \frac{d}{dz}\left(\log(1-az^{-1})\right)$$

$$= \frac{1}{1-az^{-1}}\left[0 - az^{-2}(-1)\right]$$

$$\frac{dx(z)}{dz} = \frac{az^{-2}}{1-az^{-1}}$$

$$-z\frac{dx(z)}{dz} = \frac{-\cancel{z}(az^{-\cancel{2}1})}{1-az^{-1}} = -\left[\frac{az^{-1}}{1-az^{-1}}\right] = -\frac{a/z}{1-a/z} = -\frac{a}{z-a}$$

as no ROC is given, we consider the signal is Right-sided.

we know,

$$a^n u(n) = \frac{z}{z-a}$$

$$a^{(n-1)} u(n-1) = z^{-1} \frac{z}{z-a} = \frac{1}{z-a}$$

$$-a \; a^{(n-1)} u(n-1) = -\frac{a}{z-a}$$

So, Applying Inverse z transform at both sides

$$n x(n) = -a \; a^{(n-1)} u(n-1)$$

$$x(n) = -\frac{a^n}{n} u(n-1)$$

Z transform and Inverse Z-transform.

- $x(n) \xleftrightarrow{ZT} X(z)$
- $$\boxed{x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

$x(t) \rightarrow$ freq $\rightarrow$ F.T or L.T

$x(n) \rightarrow$ freq $\rightarrow$ DFT, Z.T

- $$F.T[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

$$D.F.T[x(n)] = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Example of Z-transform

1) Find the signal $x(n)$ for which the Z transform is

$$x(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$$

- Expanding the $X(z)$

$$x(z) = \underset{x=-4}{4z^4} - \underset{x=-3}{z^3} + \underset{x=-2}{\underline{0.z^2}} - \underset{x=-1}{3z^1} + \underset{x=0}{\underline{0.z^0}} + \underset{x=1}{4z^{-1}} + \underset{x=2}{3z^{-2}}$$

basic ZT eq $\quad x(n) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= \sum_{-4}^{2} x(n) z^{-n}$$

$$x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}$$

2) If $x(z) = 2z^2 + 3$ find $x(n)$.

$$X(z) = \underset{-2}{2z^2} + \underset{-1}{0z^1} + \underset{0}{3z^0}$$

$$X(z) = \sum_{n=-2}^{0} x(n) z^{-n}$$

$$x(n) = \{2, 0, \underset{\uparrow}{3}\}$$

3) If $x(z) = 3 + 2z^{-1} + 3z^{-3}$ find $x(n)$

$$x(z) = \underset{0}{3z^0} + \underset{1}{2z^{-1}} + \underset{2}{0.z^{-2}} + \underset{3}{3.z^{-3}}$$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$x(n) = \{\underset{\uparrow}{3}, 2, 0, 3\}$$

- Z-transform using convolution property

$\Rightarrow$ Find the convolution of $x_1(n)$ & $x_2(n)$ using property of Z-transform.

$$x_1(n) = \{1, 3, 3, 1\}$$

$$x_2(n) = \{1, 1, 1, 2, 2\}$$

$$\therefore x_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$= 1 \times z^{-0} + 3 \times z^{-1} + 3 \times z^{-2} + 1 \times z^{-3}$$

$$= 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

$$x_2(z) = 1\cdot z^{0} + 1 z^{-1} + 1\cdot z^{-2} + 1\cdot z^{-3} + 2\cdot z^{-4} + 2\cdot z^{-5}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + 2\cdot z^{-4} + 2\cdot z^{-5}$$

$$X(z) = x_1(z) \times x_2(z)$$

$$= \left[1 + 3z^{-1} + 3z^{-2} + z^{-3}\right]\left[1 + z^{-1} + z^{-2} + z^{-3} + 2\cdot z^{-4} + 2\cdot z^{-5}\right]$$

$$= 1 + 3z^{-1} + 3z^{-2} + z^{-3} + z^{-1} + 3z^{-2} + 3z^{-3} + z^{-4} + z^{-2} + 3z^{-3}$$
$$+ 3z^{-4} + z^{5} + z^{-3} + 3z^{-4} + 3z^{5} + z^{-6} + 2z^{-4} +$$
$$+ 6z^{-5} + 2z^{-7} + 2z^{-5} + 6z^{-6} + 6z^{-7} + 2z^{-8}$$

$$= 1 + 4z^{-1} + 7z^{-2} + 8z^{-3} + 9z^{-4} + 12z^{-5} + 13z^{-13} + 8z^{-7}$$
$$+ 2z^{-8}$$

from co-eff

$$x(n) = \{\underset{\uparrow}{1}, 4, 7, 8, 9, 12, 13, 8, 2\}$$

Cross verify

$$x_1 = \{\underset{\uparrow}{1}, 3, 3, 1\} \quad x_2(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 2, 2\}$$

	1	1	1	1	2	2
1	1	1	1	1	2	2
3	3	3	3	3	6	6
3	3	3	3	3	6	6
1	1	1	1	1	2	2

adding diagonaly

$$= \{\underset{\uparrow}{1}, 4, 7, 8, 9, 12, 13, 8, 2\}$$