Mathematical foundation of Computer Science

Mid semester examination 2024
M.Tech. 1st Year
Department of Computer Science & Engineering
University of Calcutta

Full Marks: 30
Time: 1 hour 30 mins

Clearly show all the steps of your calculations.
Marks will be deducted for unusual step jumps.

Part – I
(Answer any 5; each question consists of 2 marks)

Q1. Show that $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent.

Q2. Show that $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are orthogonal.

Q3. Provide a counter example to show that the set of all invertible matrices does not form a vector space under matrix addition operation (the field can be real or complex).

Q4. Let $A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ be two vectors. Show that these two vectors do not span the 3-dimensional vector space. (Hint: find a vector which cannot be written as a linear combination of these two vectors).

Q5. Let $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Show that the columns of A are orthonormal.

Q6. Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ whose eigenvectors are $|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Show that $\langle v_1|A|v_2\rangle = 0$.

Q7. Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$. If A is a linear operator on V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$, find the matrix representation of A.

Part – II

(Answer any 4; each question consists of 5 marks)

Q8. Prove that for any two matrices A and B, $(A.B)^{-1} = B^{-1}.A^{-1}$

Q9. Let $P = |p_1\rangle\langle p_1| + |p_2\rangle\langle p_2|$ where p_1 and p_2 are two matrices such that $\langle p_i|p_j\rangle = \delta_{ij}, i,j \in \{0,1\}$ and $\delta_{ij} = 0$ if $i \neq j$ and 1 otherwise. In other words, p_1 and p_2 forms an orthonormal basis. Show that $P^2 = P$.

Q10. Let $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be two vectors in a 2-dimensional vector space. Construct an orthonormal basis of this vector space from A and B using Gram-Schmidt orthogonality method.

Q11. Show that the eigenvalues of a Hermitian matrix are real.

Q12. For the following set of linear equations, reduce them to the row-echelon form, identify the pivot elements, and solve for x and y.

$$x + 3y = 7$$
$$3x + 4y = 11$$

Q13. Let A = $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Calculate the matrix B = exp(A).

Q14. Suppose for a matrix A whose all entries are real, $A^T = A^{-1}$ (i.e., the transpose is equal to the inverse). If $|a_1\rangle, |a_2\rangle, \dots, |a_n\rangle$ are the columns of A, then show that $\langle a_i|a_j\rangle = 0$ if $i \neq j$.