Z- Transform :

In signal processing, the z-transform converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

It can be considered as a dicrete-time equivalent of the Laplace transform.

The Z-transform may be of two type. i.e unilateral (or one sided) and bilateral (two sided)

A bilateral or two-sided Z-transform is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where, z is any complex number (generally $z = re^{j\omega}$)

A unilateral or one-sided z-tranform is defined as,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

where, z is any complex number (generally $z = re^{j\omega}$)

The z-transform is a very useful tool in the analysis of a linear shift invariant (LSI).

The primary limitation of the z-transform is that using z transform, the frequency domain response cannot be obtained and cannot be plotted.

Z-transform properties :

- **Linearity property :**

The linearity property states that if we have two sequences $x_1(n)$ and $x_2(n)$ and their individual Z-transform are $X_1(Z)$ and $X_2(z)$ then we can write,

$$a x_1(n) + b x_2(n) \longleftrightarrow a X_1(z) + b X_2(z)$$

- **Time Shifting property :**

If we have a sequence $x(n)$ and its corresponding z-transform is $X(Z)$. Now, if we time shifted the sequence such as $x(n-K)$, then its Z-transform is given by,

$$x(n-K) \longleftrightarrow z^{-K} X(z)$$

- **Time Reversal property :**

If we have a sequence $x(n)$ and its corresponding Z-transform is $X(Z)$. The time reversal property states that,

If $x(n) \longleftrightarrow X(Z)$

then, $x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$

- **Scalling in Z - domain:**

When we multiply the signal sequence $x(n)$ in the time domain with an exponential factor a^n, the equivalent Z-transform of the new signal is scaled by a factor of a.

If $x(n) \longleftrightarrow X(z)$

then, $a^n x(n) \longleftrightarrow X(z/a)$

or, $a^n x(n) \longleftrightarrow X(a^{-1}z)$

- **Convolution property:**

If we have two sequences $x(n)$ and $y(n)$ and their individual z-transform are $X(z)$ and $Y(z)$ then according to convolution property we have,

$$x(n) * y(n) \longleftrightarrow X(z) . Y(z)$$

- **Differentiation in z-domain:**

If $x(n) \longleftrightarrow X(z)$ then the differentiation in z-domain property states that,

$$x(n) \longleftrightarrow -z \frac{dX(z)}{dz}$$

- **Conjugation property:**

The conjugation property of z-transform states that if,

$$x(n) \longleftrightarrow X(z)$$

then,

$$x^*(n) \longleftrightarrow X^*(z^*)$$

Proof of properties of z-transform:

- **Proof of linearity property:**

Let us consider $x[n] = a x_1(n) + b x_2(n)$

By definition we have,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \{a x_1(n) + b x_2(n)\} z^{-n}$$

$$= \sum_{n=0}^{\infty} a x_1(n) z^{-n} + \sum_{n=0}^{\infty} b x_2(n) z^{-n}$$

$$= a \sum_{n=0}^{\infty} x_1(n) z^{-n} + b \sum_{n=0}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

$$\therefore a x_1(n) + b x_2(n) \longleftrightarrow a X_1(z) + b X_2(z)$$

• **Proof of Time shifting property:**

Let us consider a time-shifted sequence $x(n) = x(n-k)$

By definition we have,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n-k) z^{-n}$$

$$= \sum_{m=0}^{\infty} x(m) z^{-m-k} \quad \left[\text{where } m = n-k\right]$$

$$= \sum_{m=0}^{\infty} x(m) z^{-m} z^{-k}$$

$$= z^{-k} \sum_{m=0}^{\infty} x(m) z^{-m}$$

$$= z^{-k} X(z)$$

$$\therefore x(n-k) \longleftrightarrow z^{-k} X(z)$$

• **Proof of scaling in Z-domain:**

Let us consider $x(n) \rightarrow a^n x(n)$

By definition we have,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) \, (a^{-1} z)^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} = X(z/a)$$

$$\therefore \; a^n x(n) \longleftrightarrow X\left(\frac{z}{a}\right)$$

- Proof of convolution property :

Let us consider $x(n) * h(n) = \sum_{K=0}^{\infty} x(K) \, h(n-K)$

By definition we have,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{K=0}^{\infty} x(K) \, h(n-K) \right\} z^{-n}$$

$$= \sum_{K=0}^{\infty} x(K) \sum_{n=0}^{\infty} h(n-K) \, z^{-n}$$

$$= \sum_{K=0}^{\infty} x(K) \sum_{L=0}^{\infty} h(L) z^{-L-K}$$

$$= \sum_{K=0}^{\infty} x(K) \sum_{L=0}^{\infty} h(L) z^{-L} z^{-K}$$

$$= \sum_{K=0}^{\infty} x(K) z^{-K} \sum_{L=0}^{\infty} h(L) z^{-L}.$$

$$= X(z) \, H(z)$$

$\therefore \; x(n) * h(n) \longleftrightarrow X(z).H(z)$

• Proof of Time reversal property:

Let us consider the statement $x(n) = x(-n)$

By definition we have,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(-n) z^{-n}$$

Let us take $m = -n$ then,

$$\sum_{m=0}^{-\infty} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{0} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{0} x(m) (z^{-1})^{-m}$$

$$= X(z^{-1})$$

$$= X\left(\frac{1}{z}\right)$$

$\therefore \; x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$