

ELECTRICITY & MAGNETISM

Lecture notes for Phys 121

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Abstract

These notes are intended as an addition to the lectures given in class. They are NOT designed to replace the actual lectures. Some of the notes will contain less information than in the actual lecture, and some will have extra info. Not all formulas which will be needed for exams are contained in these notes. Also, these notes will NOT contain any up to date organizational or administrative information (changes in schedule, assignments, etc.) but only physics. If you notice any typos - let me know at vitaly@njit.edu. I will keep all notes in a single file - each time you can print out only the added part. A few other things:

Graphics: Some of the graphics is deliberately unfinished, so that we have what to do in class.

Preview topics: can be skipped upon the 1st reading, but will be useful in the future.

Advanced topics: these will not be represented on the exams. Read them only if you are really interested in the material.

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I. INTRODUCTION

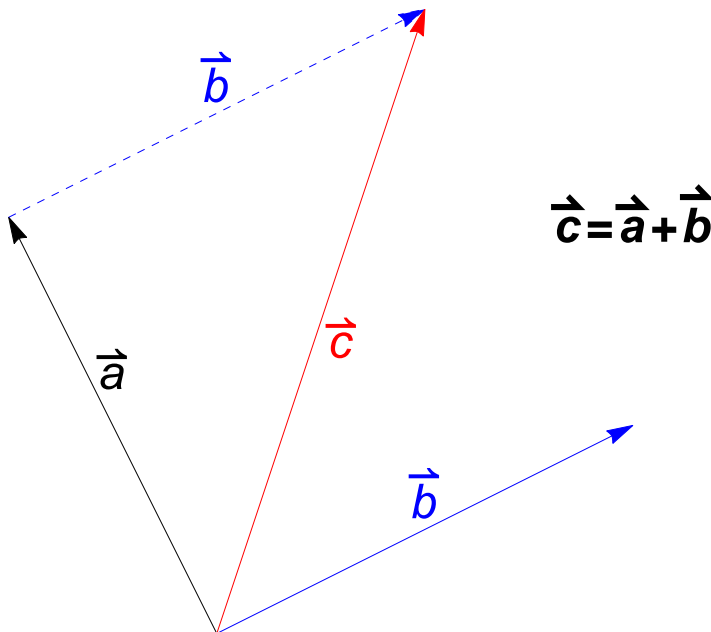
A. Vectors

A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("rule of parallelogram"). (Equivalently, components of a vector are transformed according to certain rules if the system of coordinates is rotated).

This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are mass, kinetic energy and (the forthcoming) charge. Examples of vectors are the displacement, velocity and force.



Tail-to-Head addition rule.

1. Single vector

Consider a vector \vec{a} with components a_x and a_y (let's talk 2D for a while). There is an associated scalar, namely the magnitude (or length) given by the Pythagorean theorem

$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2} \quad (1)$$

Note that for a different system of coordinates with axes x' , y' the components $a_{x'}$ and $a_{y'}$ can be very different, but the length in eq. (1), obviously, will not change, which just means that it is a scalar.

Another operation allowed on a single vector is multiplication by a scalar. Note that the physical dimension ("units") of the resulting vector can be different from the original, as in $\vec{F} = m\vec{a}$.

2. Two vectors: addition

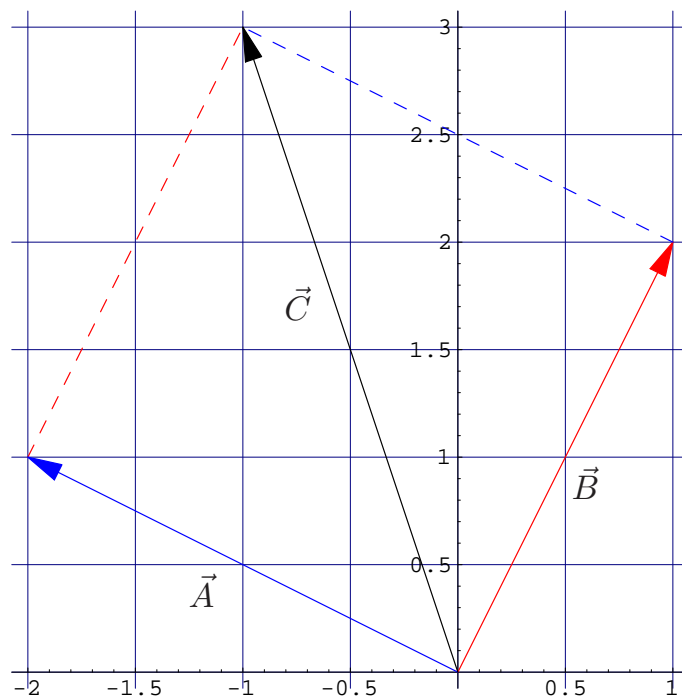


FIG. 1: Adding two vectors: $\vec{C} = \vec{A} + \vec{B}$. Note the use of rule of parallelogram (equivalently, tail-to-head addition rule). Alternatively, vectors can be added by components: $\vec{A} = (-2, 1)$, $\vec{B} = (1, 2)$ and $\vec{C} = (-2 + 1, 1 + 2) = (-1, 3)$.

For two vectors, \vec{a} and \vec{b} one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x, \quad c_y = a_y + b_y \quad (2)$$

The magnitude of \vec{c} then follows from eq. (1). Note that physical dimensions of \vec{a} and \vec{b} must be identical.

Preview. Addition of vectors plays a key role in E&M in that it enters the so-called "superposition principle".

3. Two vectors: scalar (dot) product

If \vec{a} and \vec{b} make an angle ϕ with each other, their scalar (dot) product is defined as

$$\vec{a} \cdot \vec{b} = ab \cos(\phi)$$

or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \quad (3)$$

Example. See Fig. 1.

$$\vec{A} = (-2, 1), \vec{B} = (1, 2) \Rightarrow \vec{A} \cdot \vec{B} = (-2)1 + 1 \cdot 2 = 0$$

(thus angle is 90°).

Example Find angle between 2 vectors \vec{B} and \vec{C} in Fig. 1.

$$\text{General: } \cos \theta = \frac{\vec{a} \cdot \vec{b}}{ab} \quad (4)$$

In Fig. 1:

$$\vec{B} = (1, 2), \vec{C} = (-1, 3) \Rightarrow B = \sqrt{1^2 + 2^2} = \sqrt{5}, C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\cos \theta = \frac{(-1) \cdot 1 + 3 \cdot 2}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}}, \theta = 45^\circ$$

A different system of coordinates can be used to evaluate $\vec{a} \cdot \vec{b}$, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$ in any system of coordinates. The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta \vec{r}$, with $\Delta \vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!

Preview. We will learn that magnetic force on a moving particle is always perpendicular to velocity. Thus, this force makes no work, and the kinetic energy of such a particle is conserved.

Example: Prove the Pythagorean theorem $c^2 = a^2 + b^2$.

4. Two vectors: vector product

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 2. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands). If \vec{a} and \vec{b} make

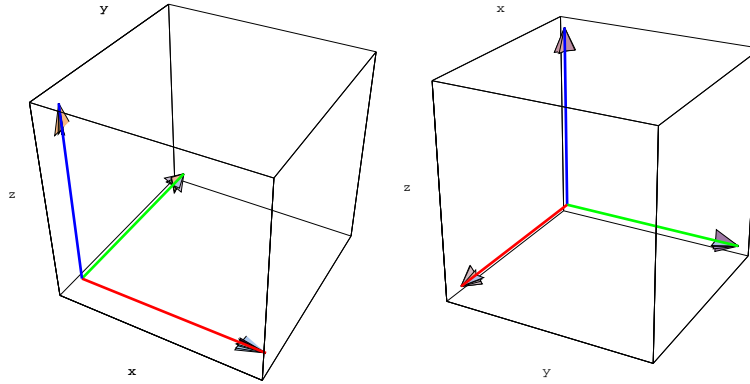


FIG. 2: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x (red) to y (green), then the thumb should point into the z direction (blue). (Note that axes labeling of the figures is outside of the boxes, not necessarily near the corresponding axes; also, for the figure on the right the origin of coordinates is at the *far* end of the box, if it is hard to see in your printout).

an angle $\phi \leq 180^\circ$ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude

$$c = ab \sin(\phi)$$

The direction is defined as perpendicular to both \vec{a} and \vec{b} using the following rule: curl the fingers of the right hand from \vec{a} to \vec{b} in the shortest direction (i.e., the angle must be smaller than 180°). Then the thumb points in the \vec{c} direction. Check with Fig. 3.

Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

Ring Diagram:

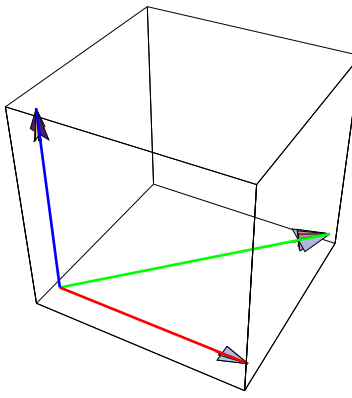
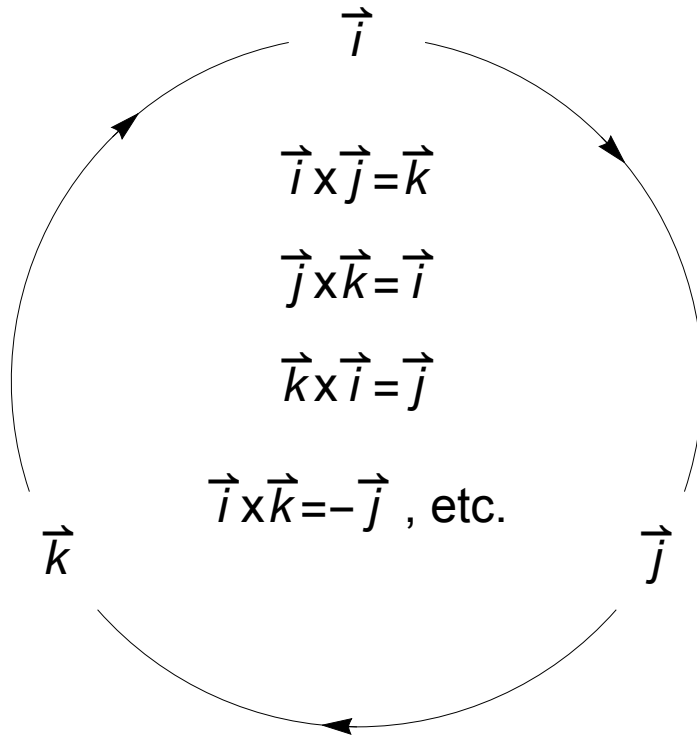


FIG. 3: Example of a cross product \vec{c} (blue) = \vec{a} (red) \times \vec{b} (green). (If you have no colors, \vec{c} is vertical in the example, \vec{a} is along the front edge to lower right, \vec{b} is diagonal).



Suppose now a system of coordinates is introduced with unit vectors \hat{i} , \hat{j} and \hat{k} pointing in the x , y and z directions, respectively. First of all, if \hat{i} , \hat{j} , \hat{k} are written "in a ring", the cross product of any two of them equals in clockwise direction the third one, i.e.

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

etc.

Example. Fig. 1:

$$\vec{A} = -2\hat{i} + \hat{j}, \vec{B} = \hat{i} + 2\hat{j}$$

$$\begin{aligned}\vec{A} \times \vec{B} &= (-2\hat{i} + \hat{j}) \times (\hat{i} + 2\hat{j}) = (-2) \cdot 2\hat{i} \times \hat{j} + \hat{j} \times \hat{i} = \\ &= -4\hat{k} - \hat{k} = -5\hat{k}\end{aligned}$$

(Note: in Fig. 1 \hat{k} goes out of the page; the cross product $\vec{A} \times \vec{B}$ goes into the page, as indicated by "-".)

More generally, the cross product is expressed as a 3-by-3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \quad (5)$$

The two-by-two determinants can be easily expanded. In practice, there will be many zeroes, so calculations are not too hard.

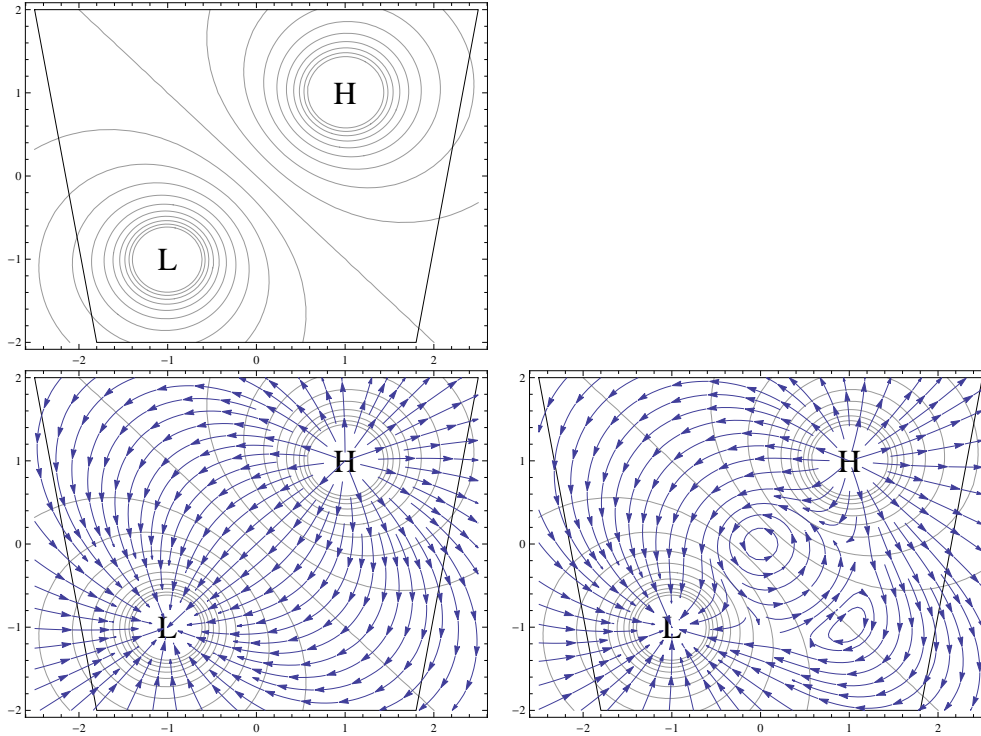
Preview. Vector product is most relevant to magnetism; it determines, e.g. the magnetic force on a particle in a field, $\vec{F} = q\vec{v} \times \vec{B}$ with q being the charge, \vec{v} the velocity, and \vec{B} the intensity of magnetic field at the location of the particle.

Example. See Fig. 1.

$$\vec{A} \times \vec{B} = \hat{k}((-2)2 - 1 \cdot 1) = -5\hat{k}$$

B. Advanced: Fields

So far we were dealing with scalars or vectors attributed to a *single* particle (or a single point, if you prefer). Consider now a much more general situation when a scalar or a vector is attributed to *every* point in space. This brings us to a concept of a *field*, scalar or vector, respectively. Field can also depend on time. A good example of a scalar field is the temperature (or pressure) map which you see in the weather forecast. Similarly, the velocities of the air flow (usually superimposed on the same map) give a vector field.



Examples of scalar and vector fields: weather maps. Top - pressure field (scalar); lines connect points with identical pressure. Lower: wind velocity fields; left - regular flow from high to lower pressure, right - turbulent flow (note regions with non-zero circulation, "tornadoes"). The left maps are similar to those for potential V and electrostatic field \vec{E} of an electric dipole. The type of the map on the right is encountered in time dependent fields, such as those which lead to electromagnetic radiation.

1. Representation of a field; field lines

How to represent a field in a picture? For a scalar field the best way is to draw lines of a constant level, e.g. lines with constant temperature every 10°C (another good example is a topographic map which indicates levels of constant height. Try to sketch maps of a hill top, of a crest and of a "saddle").

For a vector field graphical representation can be harder. The easiest approach would be to select a large number of points in space and to draw vectors from each of them (see, e.g., the example of gravitational field later in these notes). You might not always enjoy the picture, however, since it will look too "discrete", while one feels that field should be continuous. A much better way is to draw the "field lines" - see Fig. 4. They give information about both magnitude and direction of the vector field. Many non-trivial mathematical theorems about the field are easily justified in terms of such pictures. Field lines also provide an enormous boost for physical intuition since rather abstract vector constructions are replaced by simple, easy to understand pictures.

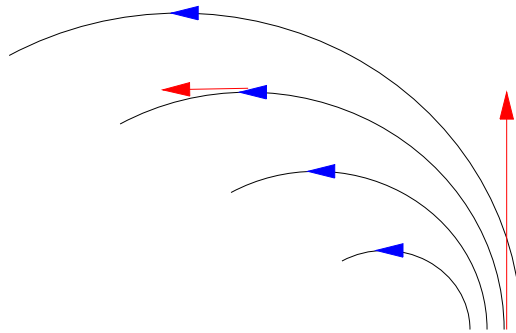


FIG. 4: Example of vector field lines. At each point the direction of vector field is tangent to the line. The magnitude of the vector field at a given point is proportional to the density of lines.

2. Properties of field lines and related definitions

The condition that the magnitude of the vector field at a given point is proportional to the density of lines, generally speaking, would require that some lines should be added or removed at various places in the picture. Remarkably, however, for the fields we are going to consider this happens only at some special points, and otherwise field lines run continuously. Points from which lines start are often called "sources", and points where they vanish are "sinks".

Preview. For electrostatic field \vec{E} sources and sinks for field lines are positive and negative charges, respectively. Only there the lines can start or interrupt. (See the gravitational example below, which is similar to a negative charge; a positive charge will have lines going out). There are *no* magnetic charges in Nature, and thus magnetic field lines never start or end, but either loop (around currents) or come and go to infinity.

Example. Gravitational field at any point \vec{r} outside of a planet is defined as the ratio of a force \vec{F} on a probe to the mass of that probe, m . Show that this equals the gravitational acceleration $\vec{g}(\vec{r})$. Sketch the vector field lines for the field \vec{g} - see Fig. 5.

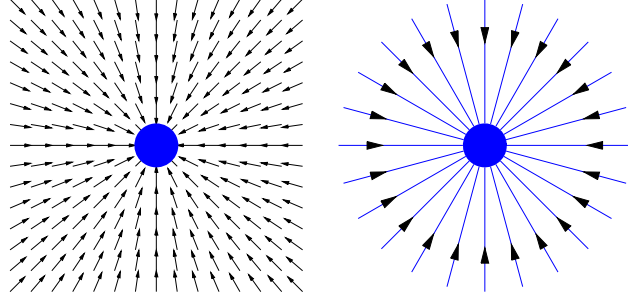


FIG. 5: Gravitational field around a planet. Left - representation by vectors, right - representation by field lines. Since the density of lines determines the magnitude of field, the latter decays inversely proportional to square of the distance from the center. The structure of this field is very similar to the electrostatic field outside a negatively charged sphere.

Gravitational field is detected by a probe, but we need a quantitative feature which is independent of the actual probe m :

$$\vec{F}_g = -G \frac{Mm}{r^3} \vec{r}, \quad \frac{\vec{F}_g}{m} = -G \frac{M}{r^3} \vec{r} = \vec{g}$$

Here \vec{r} is from the center of the planet to the observation point (do not need the probe anymore).

Similarly, can construct a scalar function, the *gravitational potential*.

$$V_g \equiv U_g/m = -G \frac{Mm}{r}/m = -GM/r$$

Note

$$|V_g| = \frac{1}{2} v_{esc}^2, \quad \text{and} \quad |V_g| \ll c^2$$

II. ELECTRIC CHARGE

A. Notations and units

Notations: q , Q or (special) e for the charge of an electron.

Units: C (coulombs). Very large! (Historically, C was introduced as $\mathcal{A} \cdot \text{s}$, with \mathcal{A} being the ampere, for current. Today it is more common to treat C as another fundamental unit, which together with kg (kilogram), m (meter) and s (second) determines the SI system of units. The ampere \mathcal{A} is then derived as C/s).

Charge of an electron

$$e \simeq -1.6 \cdot 10^{-19} \text{ C}$$

In fact, this charge is quite appreciable and can be directly measured in the lab.

B. Superposition of charges

If several charges, positive or negative q_1 , q_2 , ... etc., are placed on a small particle, at large distances that particle will act as a single charge with

$$Q_{tot} = q_1 + q_2 + \dots \quad (6)$$

C. Quantization of charge

The smallest charge is the charge of an electron, i.e. for any observable charge Q one should have

$$Q/e = 0, \pm 1, \pm 2, \dots$$

D. Charge conservation

In a closed system

$$Q_{tot} = \text{const} \quad (7)$$

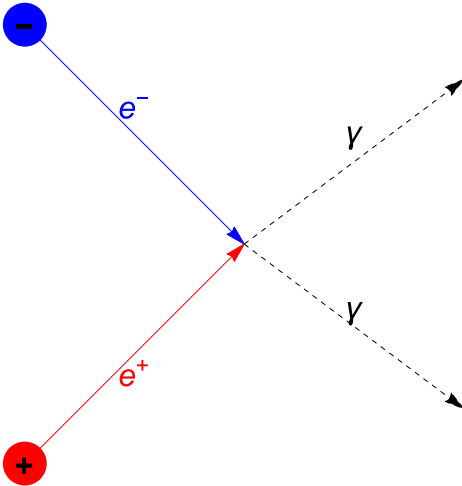
This is a fundamental Law of Nature, which is valid even if the number of elementary particles is not conserved (as in nuclear reactions)!

Examples. Decay of a neutron into a proton and an electron (+ some kind of neutrino which has no charge and is of little interest here):

$$n^0 \rightarrow p^+ + e^- + \nu^0$$

Example Annihilation of the electron e^- and a positron e^+ :

$$e^- + e^+ = 2\gamma^0$$



E. The Coulomb's Law

If two charges q_1, q_2 are separated by a distance r , the force between them is

$$\boxed{F = k \frac{q_1 q_2}{r^2}, \quad k \simeq 9 \cdot 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2} \quad (8)$$

with positive sign referring to repulsion and negative to attraction. The force acts along the line connecting the two charges - see Fig. 6.

(some books write the product of *absolute* values of charges, to emphasize that F is the magnitude of force, which is always positive. However, the form given by eq. (8) is correct, and has more information as long as you know what it means).

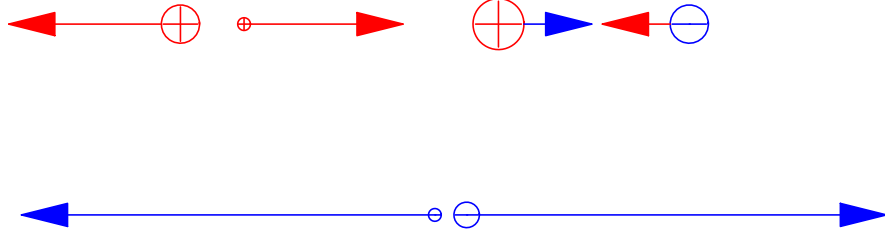


FIG. 6: The Coulomb interaction between charges. Figures are drawn to scale, with radii of charges being proportional to their magnitudes, and forces being proportional to predictions of the Coulomb Law. Positive and negative charges are indicated by red and blue, respectively. Note the following: (a) same charges repel each other, while opposite charges are attracted. (b) Forces acting on each of the two interacting charge are the *same* in magnitude, even if charges are different (otherwise the 3rd Law of Newton would be violated). (c) Forces become extremely large if the two charges are very close to each other, even if both charges are small

If one really wants to be pedantic (e.g., when dealing with a computer which has a poor sense of humor), the Coulomb's law can be formulated in a vector form: If \vec{r}_{12} is the vector which points from charge 1 to charge 2 (with $r = |\vec{r}_{12}|$, as before), then the vector of force \vec{F}_{21} which acts on charge 2 (and is due to interaction with charge 1) is given by

$$\vec{F}_{21} = k \frac{q_1 q_2}{r^3} \vec{r}_{12} \quad (9)$$

Example: check the above equation for a pair of charges from Fig. 6) [in fact, those pictures were generated by a computer using eq. (9)].

The vector version of Coulombs Law is more convenient in large formal calculations with many charges.

F. Superposition of forces

Consider a charge, let's call it q_0 which interacts with many other charges in the system, q_1, q_2, \dots , etc. Then the *total* force which acts on q_0 is the vector superposition of individual forces, i.e.

$$\vec{F}_{0,net} = \vec{F}_{01} + \vec{F}_{02} + \dots = \sum_{i=1}^n k \frac{q_i q_0}{r_{i0}^3} \vec{r}_{i0} \quad (10)$$

This is illustrated in Fig. 7 where the charge of interest, q_0 is the one in lower right.

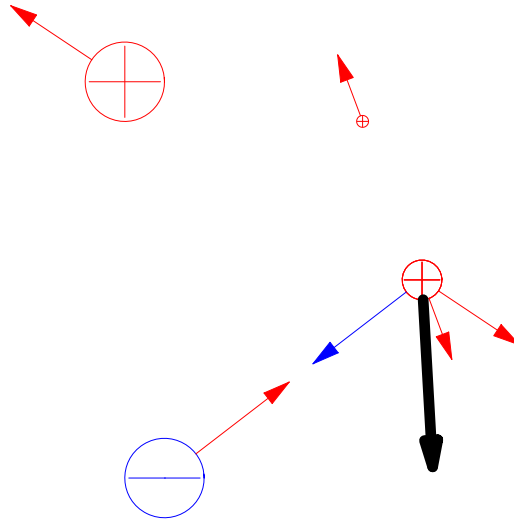
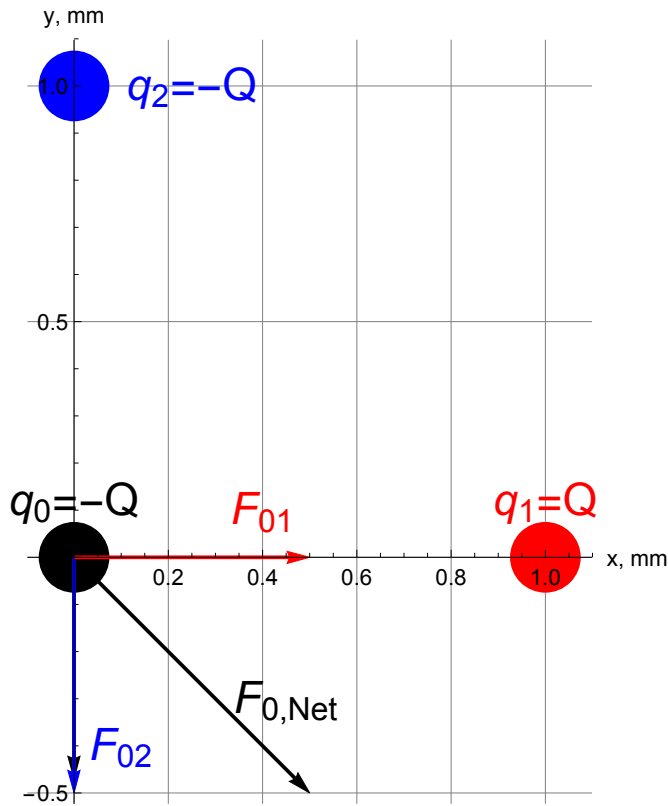


FIG. 7: The principle of superposition. The total force (black arrow in the picture) acting on a given charge equals the *vector* sum of all three individual forces which act on this charge due to its pairwise interaction with every other charge present in the system.

Example: $Q = 2 \mu C$, $a = 1 \text{ mm}$. Find the force on the charge at the origin.



$$F_0 = \sqrt{F_{01}^2 + F_{02}^2} = \sqrt{2} \cdot F_{01}$$

$$F_{01} = k \frac{Q^2}{a^2} \approx 9 \cdot 10^9 (2 \cdot 10^{-6})^2 / (10^{-3})^2 = 3.6 \cdot 10^4 \text{ N}, \quad F_0 = \sqrt{2} \times 3.6 \cdot 10^4 = \dots$$

Continuous q_i : sum over "all other charges" q_i is replaced by a corresponding integral (volume, surface or linear integral depending on the actual charge distribution).

$$\sum \rightarrow \int dV, \text{ or } \int dA, \text{ or } \int dl$$

$$q_i \rightarrow \rho dV, \text{ or } \sigma dA, \text{ or } \lambda dl$$

Here ρ , σ and λ are the volume charge density, surface charge density and linear charge density, respectively, with units

$$[\rho] = \text{C/m}^3, \quad [\sigma] = \text{C/m}^2, \quad [\lambda] = \text{C/m}$$

G. Reaction of a charge to electrostatic and other forces

Recall that the 2nd Law of Newton

$$\vec{F} = m\vec{a} \text{ , or } \vec{F} = d\vec{p}/dt \quad (11)$$

is valid for *any* force, whatever its origin. So, if m is the mass of the charge q_0 and $\vec{F}_{0,net}$ is the total electrostatic force acting on that charge, as in eq. (10), then the 2nd Law allows one to find the acceleration \vec{a} , as for any other particle. If other, non-electrostatic forces also act on the charge, they should be just added to give the *total* force, and the 2nd Law will allow to find acceleration.

Advanced: although we are talking about electrostatics, particles are permitted to move, albeit not too fast. If they do move fast, with speeds comparable to the speed of light, the 2nd Law in the above version need correction, and Coulomb's also needs to be modified to account for retardation. (Equivalently, magnetic fields due to particle motion must be included). In addition, rapidly accelerating charges will emit electromagnetic waves, which are not part of the story (yet).

Example: Estimate the speed of an electron in a hydrogen atom with radius about $0.53 \cdot 10^{-10} \text{ m}$.

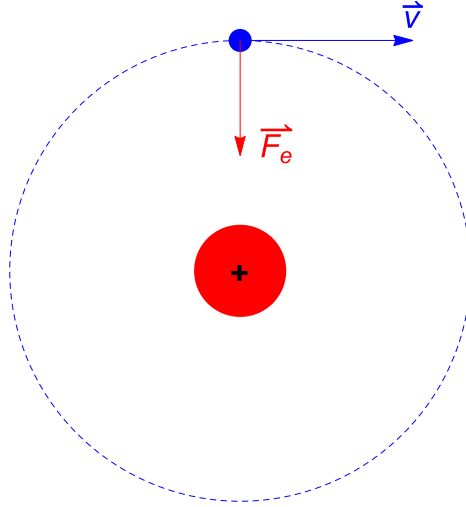


FIG. 8:

Solution: the centripetal acceleration $a = v^2/r$ is due to coulomb interaction between the electron and the proton. Thus,

$$F = k \frac{e^2}{r^2} \simeq 9 \cdot 10^9 \frac{(1.6 \cdot 10^{-19})^2}{(0.53 \cdot 10^{-10})^2} \approx 8.2 \cdot 10^{-8} \text{ N}$$

From 2nd Law find the acceleration of the electron:

$$a_e = F/m_e \approx 8.2 \cdot 10^{-8} / (9.1 \cdot 10^{-31}) = \dots$$

with m_e being the mass of electron.

To find speed v use $F = m_e a_c$

$$m \frac{v^2}{r} = k \frac{e^2}{r^2}$$

(the heavy proton practically does not move). Or,

$$v = \sqrt{ke^2/(m_e \cdot r)} = \sqrt{\frac{9 \cdot 10^9 (1.6 \cdot 10^{-19})^2}{9.1 \cdot 10^{-31} \times 0.53 \cdot 10^{-10}}} \dots$$

(Check that it does not exceed speed of light!).

Acceleration of the proton:

$$a_p = F/m_p = a \frac{m_e}{m_p}$$

with $m_p \sim 1.67 \cdot 10^{-27} \text{ kg}$. Note: F - same (3rd Law !).

What other forces can act on a charge? The answer depends whether we consider an elementary charge or just a charged "macroscopic" particle (which can be tiny on a human scale, like a fine dust particle).

If the charge is elementary, there is *only one* other long range force which can act on it. This is the force of gravity, $F_g = m\vec{g}$ with \vec{g} being the gravitational acceleration. (Nuclear "forces" which can act on protons are of very short range, about 10^{-14} m, not of human scale at all. They are also not "forces" in the strict meaning of word, since they do not lead to anything like the 2nd Law).

The gravitational interaction between 2 elementary charges is negligibly small (estimate!), but if a charge interacts with a huge body, like a planet, the electrostatic and gravitational forces can be comparable, as in the Millican experiment.

Discussion. Relation between the Coulomb's Law and the Newton's Law of gravitation

$$F_G = -G \frac{m_1 m_2}{r^2}$$

with $G \simeq 6.7 \cdot 10^{-11} \text{ N m}^2/\text{kg}^2$.

Compare to Coulomb's law:

r^{-2} - same!

$m_{1,2}$ - analogous to $q_{1,2}$

BUT:

"-" in the formula AND $m_{1,2} > 0$

Compare forces between two electrons:



$$F_G = -G \frac{m_e^2}{r^2}, \quad F_e = k \frac{e^2}{r^2}$$

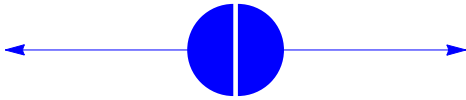
$$\frac{F_G}{F_e} \sim \frac{G m_e^2}{k e^2}$$

$$\frac{F_G}{F_e} \sim \frac{10^{-10-60}}{10^{10-38}} \sim 10^{-42}$$

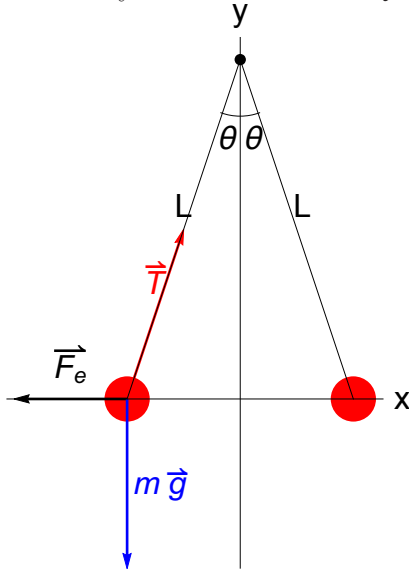
For a non-elementary charge one can introduce other forces, similarly to what is commonly done in regular mechanics. For example, for two suspended light charged pit balls one can discuss the tension force \vec{T} as the third force which equilibrates the gravitational \vec{F}_g and the

electric F_e forces (i.e., $\vec{F}_e + \vec{F}_g + \vec{T} = 0$ if the system is in equilibrium - see example below. In principle, tension is not a fundamental force but is also of electromagnetic origin, but this is only in principle. In reality, one cannot predict the value of T from considering interactions of elementary charges in the thread, and T must be deduced from measurements.

Advanced: There is a fundamental difficulty in E&M, What is the size of an electron? If it is finite, there are enormous forces trying to break it apart (see Coulomb's Law). Which forces prevent it from breaking? (we do not know, and at the moment it seems impossible to introduce such forces consistently, so that they satisfy relativity, conservation of energy and momentum, etc.). The other option is that electron is an infinitesimal point, but then one encounters INFINITY(!) when the center of the electron is approached. The latter is very hard to deal with, both mathematically and conceptionally, but seems to remain the only option which is currently available.



Example: In a Lab demo two light balls with $m = 1$ milli-gram each are suspended on two massless threads with $L = 1$ m. When charged with equal negative charges Q the balls separated by $r = 2$ cm. Find Q and the number of extra electrons on each ball.



$$\vec{T} + m\vec{g} + \vec{F}_e = 0$$

Let $\sin \theta = r/2L \approx \tan \theta$:

$$T \sin \theta - F_e = 0$$

$$T \cos \theta - mg = 0$$

Thus,

$$F_e = mg \tan \theta = k \frac{Q^2}{r^2}$$

$$Q \approx - \left(\frac{mgr^3}{2kL} \right)^{1/2} \sim (0.5 \cdot 8 \cdot 10^{-6+1-6-10})^{1/2}$$

Advanced. Insufficiency of classical mechanics to get the size of an atom

Have $[k] = N \cdot m^2/C^2$, $[e] = C$, $[m] = kg$. Let us try to construct length:

$$[m] = [kg \cdot m^3/s^2 C^2]^\alpha [C]^\beta [kg]^\gamma$$

No solution! What to do? Need a new fundamental constant (Bohr). It is $\hbar \sim 10^{-34} J \cdot s$ (Plank's constant).

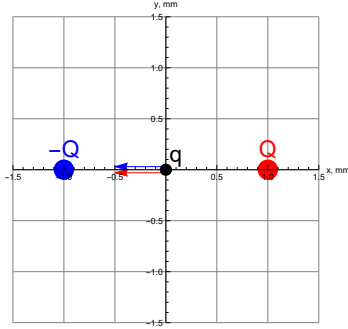
Extra credit (optional): estimate the size of an atom by adding \hbar to previous dimensions.

Example. A dust particle with $m_1 = 4 \mu\text{g}$ and $q_1 = 7\mu\text{C}$ is 3 cm away from another particle with $m_2 = 8\mu\text{g}$ and $q_2 = 5\mu\text{C}$. Find acceleration for each.

$$F = k \frac{q_1 q_2}{r^2} = 9 * 10^9 \frac{7 * 10^{-6} * 5 * 10^{-6}}{(3 * 10^{-2})^2} = 350 \text{ N}$$

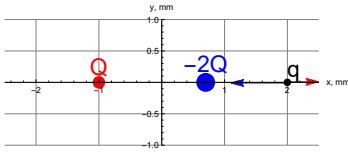
$$a_1 = \frac{F}{m_1} = \frac{350}{4 * 10^{-9}} = 8.75 * 10^{10} \frac{\text{m}}{\text{s}^2}, \quad a_2 = \frac{F}{m_2} = 0.5 a_1$$

In all examples below $Q = 5.0 \mu\text{C}$, $q = 2.0 \mu\text{C}$, distances (if given) are in mm; red is positive, blue is negative. You need to find the resultant force on the black (positive) charge q .



In the above r_1 (from red) = r_2 (from blue) = 1.0 mm. Forces on q add up:

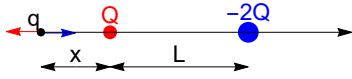
$$F = k \frac{Qq}{r_1^2} + k \frac{Qq}{r_2^2} = 2k \frac{Qq}{r_1^2} \simeq 2 * 9 * 10^9 \frac{5.0 * 10^{-6} * 2.0 * 10^{-6}}{(1.0 * 10^{-3})^2} = 1.8 * 10^5 \text{ N}$$



In the above r_1 (from red) = 3.0 mm, r_2 (from blue) \simeq 1.2 mm. Forces subtract and resultant is towards left:

$$F = k \frac{2Qq}{r_2^2} - k \frac{Qq}{r_1^2} = kQq \left(\frac{2}{r_2^2} - \frac{1}{r_1^2} \right) \simeq 9 * 10^9 * 5.0 * 10^{-6} * 2.0 * 10^{-6} \left(\frac{2}{(1.2 * 10^{-3})^2} - \frac{1}{(3.0 * 10^{-3})^2} \right) = 1.15 * 10^5 \text{ N}$$

Let $L = 3.0 \text{ cm}$. Find x so that $F = 0$ (figure no to scale!).

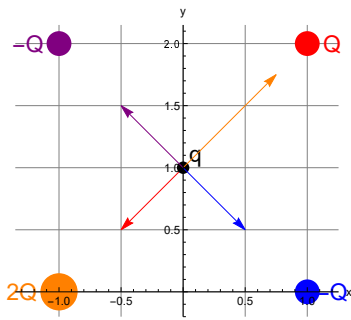
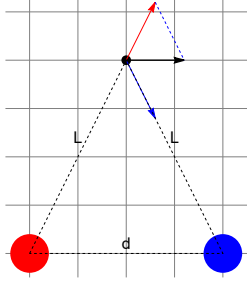


In the above r_1 (from red) $= x$ and r_2 (from blue) $= L + x$. Forces subtract:

$$k \frac{Qq}{x^2} = k \frac{2Qq}{(L+x)^2}, \text{ or } \frac{1}{x^2} = \frac{2}{(L+x)^2} \text{ and } x = \frac{L+x}{\sqrt{2}}$$

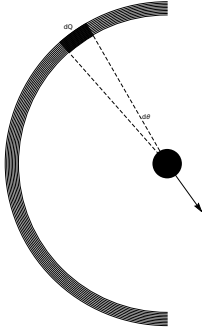
$$\sqrt{2}x = L+x, x(\sqrt{2}-1) = L \text{ and } x = \frac{L}{\sqrt{2}-1} \simeq \frac{3.0 \text{ cm}}{\sqrt{2}-1} = 7.2 \text{ cm (to the left of the smaller charge)}$$

Directions only



$$F = k \frac{2Qq}{r_1^2} - k \frac{Qq}{r_1^2} = k \frac{Qq}{r_1^2} \simeq 4.5 * 10^4 \text{ N}$$

Example. Integration. A charge $Q = 2 \text{ nC}$ is uniformly distributed along a plastic half ring with $R = 3 \text{ cm}$. Find the force which acts on a charge $q = 0.5 \text{ nC}$ at the center.



Solution. From symmetry only $F_x \neq 0$.

$$dF_x = kq * \left(Q * \frac{Rd\theta}{\pi R} \right) * \frac{1}{R^2} \cos \theta$$

$$F_x = \int_{-\pi/2}^{\pi/2} d\theta * kqQ \frac{1}{\pi R^2} \cos \theta = \frac{1}{\pi R^2} kqQ * \sin \theta \Big|_{-\pi/2}^{\pi/2} = \frac{2}{\pi R^2} kqQ$$

III. ELECTRIC FIELD

A. Field due to a point charge

1. Definition and units

Consider the Coulomb's law, eq. (9), but now we treat the charges unequally. The 1st charge is the primary charge, just q , the second charge is a *probe*, a small charge with a value q_0 . The law can now be written as

$$\vec{F}_0 = k \frac{q \cdot q_0}{r^3} \vec{r}$$

with F_0 being the force which acts on the probe and \vec{r} pointing from the primary charge towards the location of the probe.

Now consider the following ratio

$$\vec{F}_0/q_0 = k \frac{q}{r^3} \vec{r}$$

The most remarkable fact about this expression is that it *does not depend on the probe*! Thus, the ratio is a characteristic of the charge q only, but not of q_0 . It deserves a name - *the electric field at point \vec{r}* and a standard notation $\vec{E}(\vec{r})$. The units however, are derived from the known ones: $[\vec{E}] = \text{N/C}$ (and later we learn that this is the same as V/m, *volts per meter*). Explicitly, one has for a field due to a point charge q

$$\vec{E} = k \frac{q}{r^3} \vec{r} \tag{12}$$

or, without vectors

$$\boxed{E = k \frac{q}{r^2}} \tag{13}$$

with positive sign indicating that field goes *away* from the charge and negative sign indicating a field going *towards* the charge, if it happens to be negative. r is just the distance from charge q to the observation point, and we do not need the probe at this point anymore(!)

The vector $\vec{E}(\vec{r})$ is defined for *any* point in space around q . Instead of showing the vectors, however, it is much more convenient to depict the field lines (see the Introduction). Such lines have the property that their tangent coincides with the direction of a vector at a given point. Since \vec{E} always points away from the positive charge (towards a negative charge), for a single charge the field lines will be just straight lines, as in Fig. 9. Note that positive and negative charges serve, respectively, as "sources" and "sinks" for the field lines.

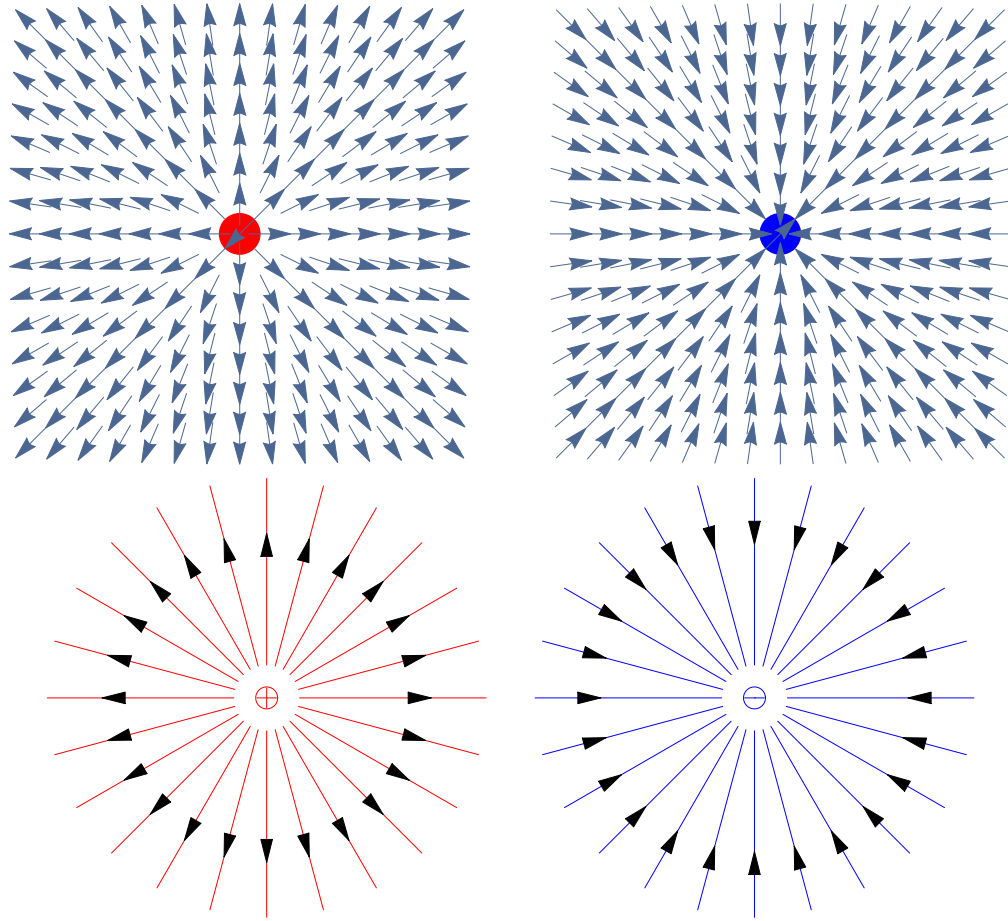


FIG. 9: Vector fields (upper row) and electric field lines (lower row) due to single point charges. Note that the field becomes infinitely strong when a charge is approached.

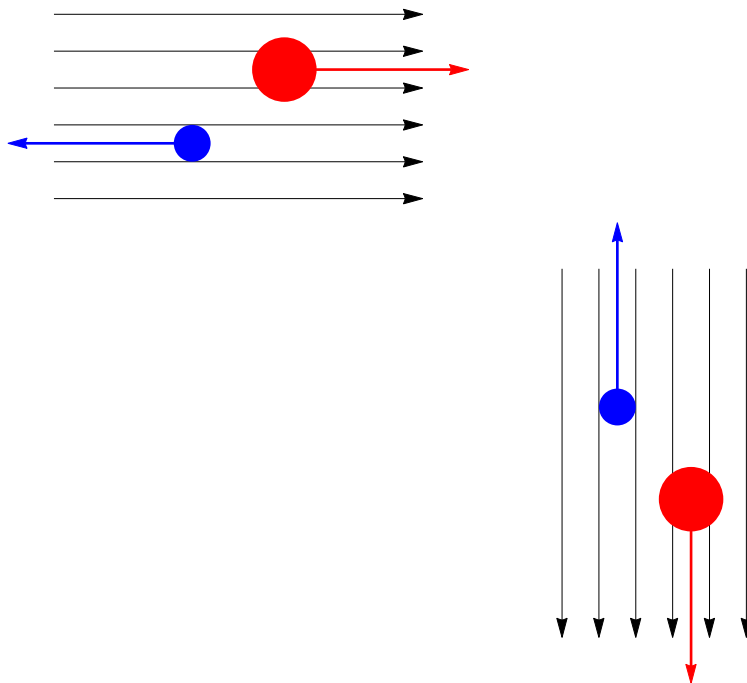
B. Field due to several charges

1. Definition and force on a charge in a field

Similarly to the field of a single charge, in a general case one can introduce field $\vec{E}(\vec{r})$ as a ratio of the force which acts on a small probe placed at \vec{r} to the magnitude of the probe. (After that, the probe does not matter).

In practice, this definition is often reversed. Field \vec{E} is assumed to be known at a given point, and one is asked to find the force on a charge q which is placed there (the charge may or may not be called "probe" in this case). From the definition one has

$$\boxed{\vec{F} = q\vec{E}} \quad (14)$$

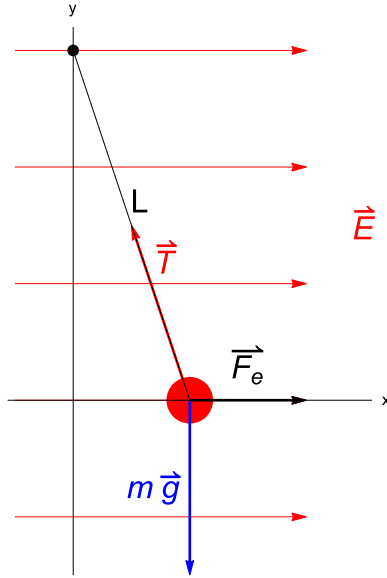


Note that if the charge is negative (blue), the force is *opposite* to the field. If the blue object has mass m and is to be balanced against force of gravity:

$$qE = mg$$

Example. In an oil drop experiment a small droplet with mass $m = 1.5 \mu\text{g}$ (micro-gram) has 100 extra electrons. Find the direction and magnitude of the electric field which would balance the droplet against gravity. (Ans. $E = 7.7 \text{ V/m}$, *down*. Solution in class).

Example. A massless string with a light charged pith ball at the end is placed in a uniform horizontal electric field E . Find the angle which the string makes with the vertical. The mass of the pith ball is m and the charge is q . (Solution in class).



$$\vec{T} + m\vec{g} + \vec{F}_e = 0$$

$$-T \sin \alpha + F_e = 0$$

$$T \cos \alpha - mg = 0$$

Thus,

$$mg \tan \alpha = F_e = qE$$

2. Superposition of fields

Since the force obeys the *superposition principle*, the latter is also valid for the fields. The total field \vec{E} at a given point is determined by a vector sum of contributions of individual charges

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots \quad (15)$$

The fields \vec{E}_1 , \vec{E}_2 , etc. are determined by eq. (12) with \vec{r} replaced by a vector pointing from a corresponding charge to the observation point.

Example Field due to a dipole. We will consider the observation point equally distanced from both charges, as in fig. 10. The distance between charges is d and the distance from each charge to the observation point is L . Both charges are identical in magnitude and equal $\pm q$, respectively.

Let the two charges have respective coordinates $\vec{r}_1 = (-d/2, 0)$ and $\vec{r}_2 = (d/2, 0)$; the observation point is then located at $\vec{r}_0 = (0, h)$, with $h = \sqrt{L^2 - d^2/4}$. Let \vec{E}_1 be the field

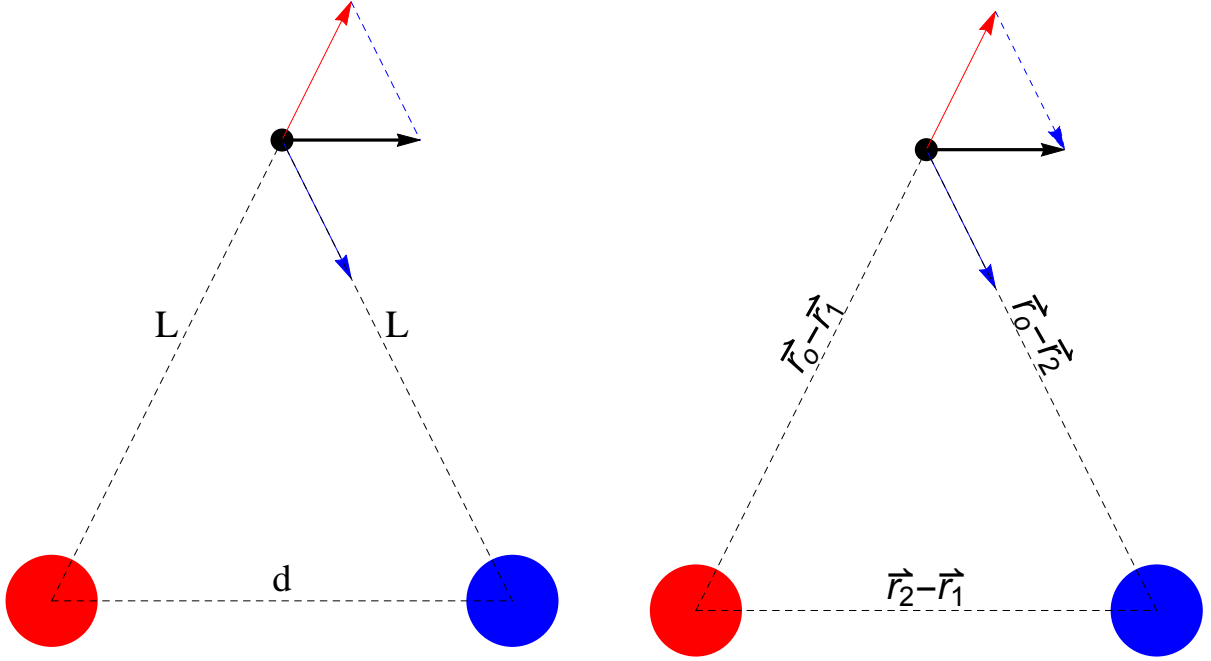


FIG. 10: Evaluation of a field due to a dipole. Left - from similar triangles. Right - from vectors. As a reminder, a tiny probe is shown at the observation point, equally distanced from both charges. In reality, there is nothing present at that point, just field.

from positive (red) charge and \vec{E}_2 field from the negative (blue) charge. The black horizontal field is their resultant $\vec{E}_{dip} = \vec{E}_1 + \vec{E}_2$. From similar triangles

$$E_{dip}/E_1 = d/L \Rightarrow E_{dip} = E_1 \frac{d}{L}$$

$$\text{From } E_1 = kq/L^2: \boxed{E_{dip} = \frac{kqd}{L^3}} \quad (16)$$

Advanced. Alternatively, we can use vectors and the superposition principle:

$$\vec{E}_{dip} = \vec{E}_1 + \vec{E}_2 = kq \frac{\vec{r}_0 - \vec{r}_1}{L^3} - kq \frac{\vec{r}_0 - \vec{r}_2}{L^3} = \frac{kq}{L^3} \{\vec{r}_0 - \vec{r}_1 - \vec{r}_0 + \vec{r}_2\} = \frac{kq}{L^3} \{\vec{r}_2 - \vec{r}_1\}$$

which is a vector pointing to the right (from positive to negative, parallel to the dipole) with the same magnitude.

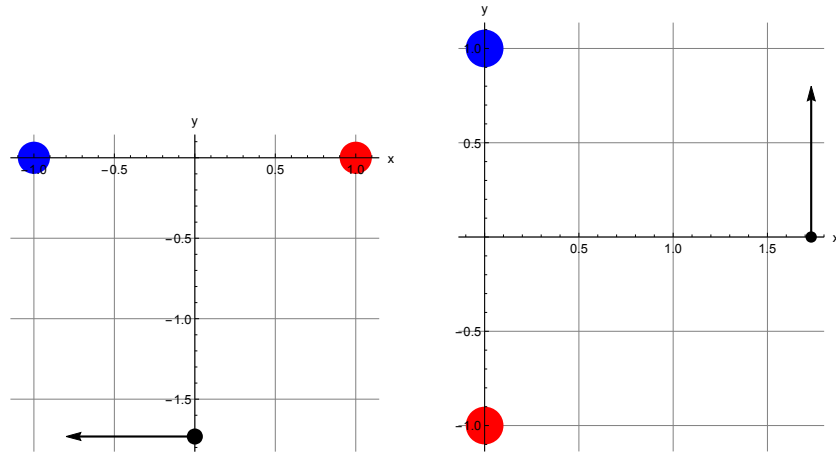


FIG. 11: Picture of the dipole can be rotated.

Example. (a) Find the field from a dipole if the observation point and the charges form an equilateral triangle with side $a = 3.0$ mm with the positive charge $q = 1.0$ nC on the right, as in Fig 10. (b) The same, if the observation point is under the dipole with the same distances from charges.

Solution.(a) Direction from left to right - see Fig. 10. Magnitude: $d = L = a$ and

$$E_{dip} = kqd/L^3 = kq/a^2 = 9 * 10^9 * 1.0 * 10^{-9} \frac{1}{(3.0 * 10^{-3})^2} = 1.0 * 10^6 \text{ N/C}$$

(b) Magnitude - same, direction - same (always parallel to the axis of the dipole, from positive to negative charges.)

Another example. Same arrangement, but both charges are positive - Fig. 12.

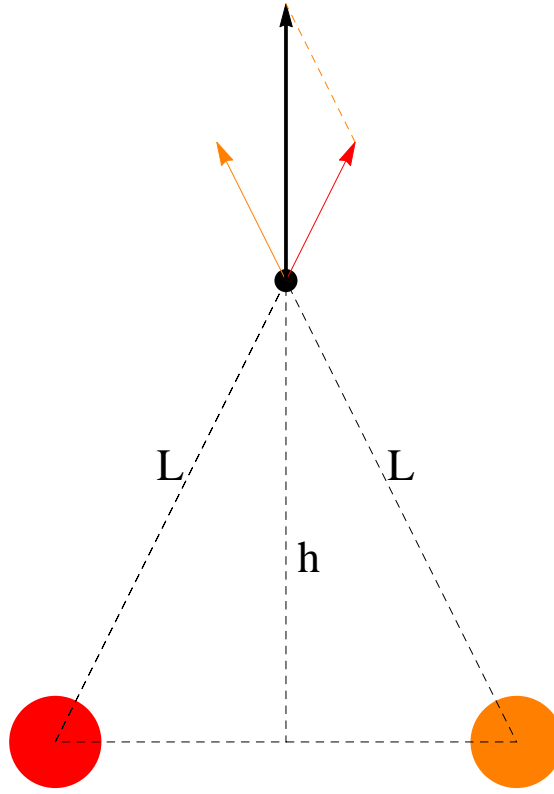


FIG. 12: Example of evaluation of a field due to two identical positive charges.

Now

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = kq \frac{\vec{r}_0 - \vec{r}_1}{L^3} + kq \frac{\vec{r}_0 - \vec{r}_2}{L^3} = \frac{kq}{L^3} \{\vec{r}_0 - \vec{r}_1 + \vec{r}_0 - \vec{r}_2\}$$

or

$$\vec{E} = \frac{kq}{L^3} \{2\vec{r}_0 - \vec{r}_2 - \vec{r}_1\} = \frac{kq}{L^3} (0, 2h) = \frac{2kqh}{L^3} (0, 1)$$

which is a vector pointing up.

In principle, the superposition principle allows one to reconstruct field due to any known charge distribution. If charges are distributed continuously, one just needs to break the distributed charge into small individual domains, and treat each of the as a point charge. This leads to an integral instead of a sum in eq. (15), but otherwise it is the same idea. We will later see how it works on examples.

C. Electrostatic Field Lines (EFL)

In a general case the structure of field lines is more complex than for a single charge; in particular they are not straight lines anymore. Nevertheless, some general properties can be established:

- tangent to the EFL determines the direction of the electric field \vec{E}
- density of EFL determines the magnitude of E
- EFL originate on positive charges
- EFL terminate on negative charges
- EFL can come and go to infinity
- EFL CANNOT start or end in empty space
- EFL CANNOT loop
- as a rule, EFL CANNOT cross

Looping is not allowed since it would contradict conservation of energy. At the point of crossing of two lines it would be impossible to determine the direction of the field. (A special case is the point of zero field; such points however, are extremely rare since all *three* components of \vec{E} must go to zero at the same time).

1. Field lines due to a dipole

Generally, plotting field lines for several charges is not easy. Two things help. First, directly near charges fields are so strong that other charges do not matter. It is a good start. Second, in many problems there is some special symmetry which helps to understand the structure of field.

Field due to a dipole - Fig. 13: Note that there are no points with zero field.

Field due to two identical charges - Fig. 14: There is one point where the field is zero.

For nonsymmetric arrangements, plotting of a field is a work for a (good) computer. For example, in Fig. 15 there are field of two non-equal charges:

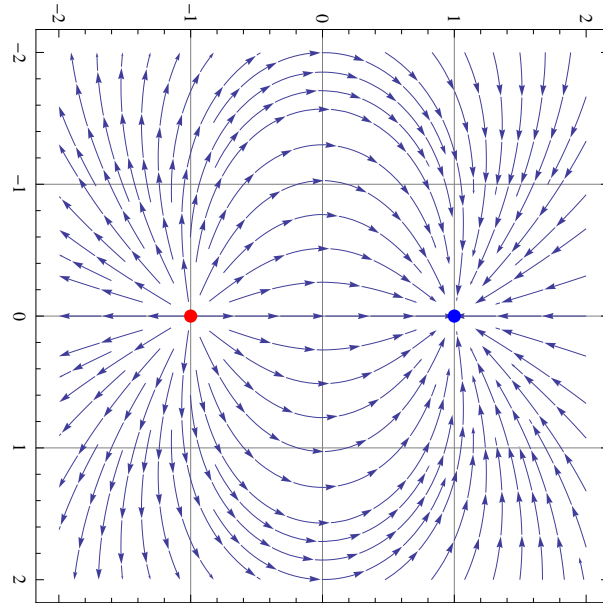


FIG. 13: Electric field lines due to a dipole.

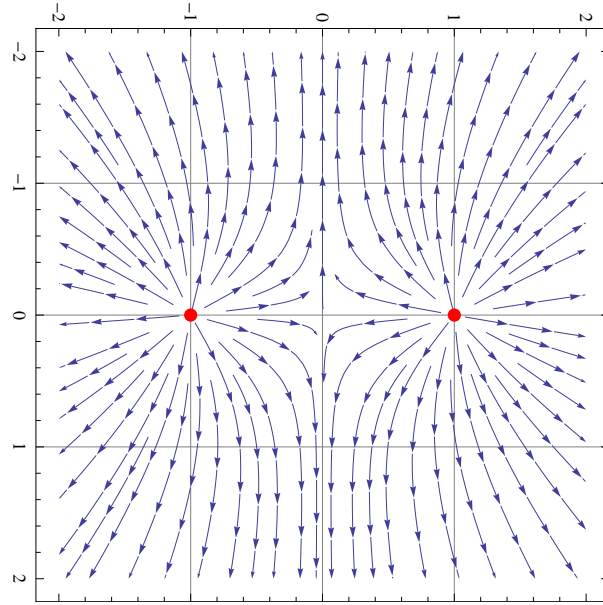


FIG. 14: Electric field lines due to 2 positive charges.

D. Continuous charge distribution

General:

$$1D : q \rightarrow \lambda dx, [\lambda] = C/m$$

$$2D : q \rightarrow \sigma dA, [\sigma] = C/m^2$$

$$3D : q \rightarrow \rho dV, [\rho] = C/m^3$$

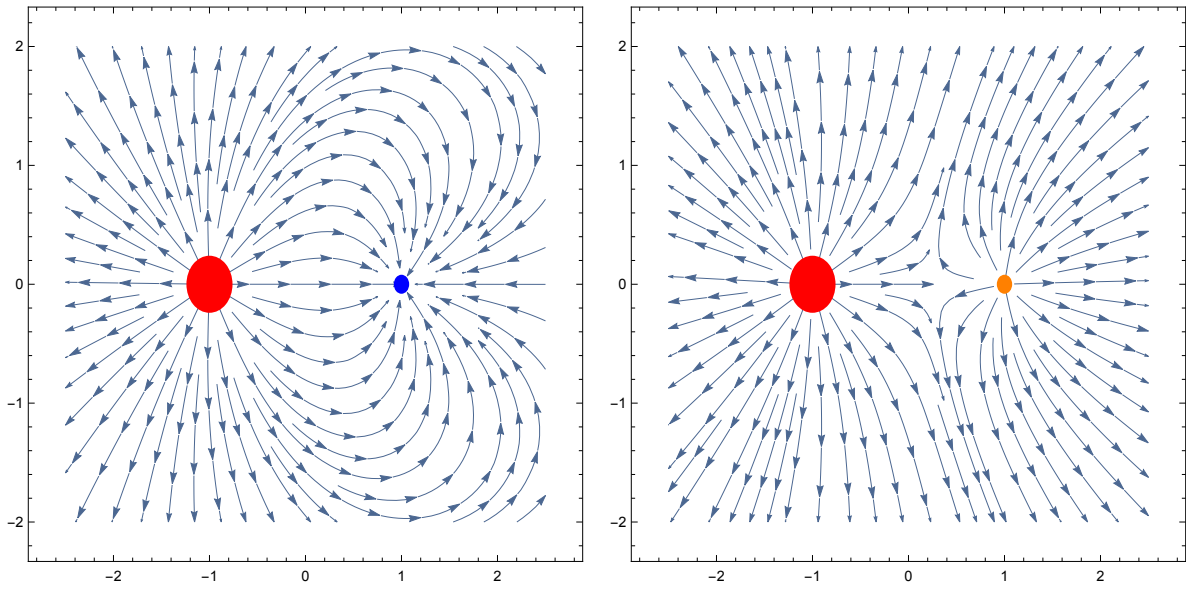
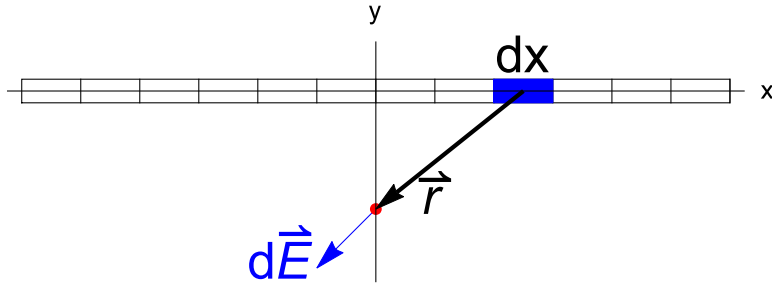


FIG. 15: Electric field lines due to two non-equal charges with the positive charge on the left being 3 times larger. The smaller charge is negative (left figure) and positive (right)

Example. Field from a uniformly charged line at a point (red) equally distanced from the ends



Contribution of the selected (blue) fragment

$$\vec{r} = (-x, y), \quad d\vec{E} = k \frac{\lambda dx}{r^3} \vec{r} = k \frac{\lambda dx}{(\sqrt{x^2 + y^2})^3} (-x, y)$$

Then,

$$E_x = \int dE_x = 0 \text{ (from symmetry)}, \quad E_y = \int dE_y$$

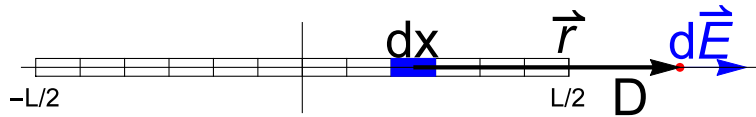
If the line is infinite

$$E_y = k\lambda y \int_{-\infty}^{\infty} \frac{dx}{(\sqrt{x^2 + y^2})^3}$$

Introducing dimensionless integration variable $w = x/y$

$$E_y = \frac{k\lambda}{y} \int_{-\infty}^{\infty} \frac{dw}{(\sqrt{w^2 + 1})^3} = \frac{2k\lambda}{y}$$

Example. Field from a uniformly charged line at a point (red) with distance D from the end, along the rod



Introduce $X = L/2 + D$ -distance of the red point from the center. Contribution of the selected (blue) fragment

$$dE = k\lambda dx / (X - x)^2$$

$$E = \int_{-L/2}^{L/2} dx k\lambda / (X - x)^2 = k\lambda \frac{1}{X - x} \Big|_{-L/2}^{L/2}$$

$$= k\lambda \left(\frac{1}{D} - \frac{1}{D + L} \right) = \frac{k\lambda L}{D(D + L)}$$