# Apuntes de Derivadas

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# Contents

# 1 Derivadas

**Hallar la derivada de** $f(x) = x^3$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \frac{x^3-x^3}{0} = \frac{0}{0} \\
&= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \frac{3x^2h+3xh^2+h^3}{h} = 3x^2+3xh+h^2 \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2
\end{aligned}$$

**Hallar la derivada de** $f(x) = \sqrt[3]{x^2}$

$$f(x) = x^{\frac{2}{3}} \implies f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} \qquad = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}} = \frac{2}{3\sqrt[3]{x}}$$

**Hallar la derivada de** $f(x) = \sin x$

$$\boxed{\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{2\sin(\frac{h}{2})\cos(2+\frac{h}{2})}{h} \\
&= \lim_{h\to 0} \frac{\sin(\frac{h}{2})\cos(2+\frac{h}{2})}{\frac{h}{2}} \\
&= \lim_{h\to 0} 1\cdot\cos(x+\frac{h}{2}) \\
&= \cos x
\end{aligned}$$

**Hallar la derivada de** $f(x) = e^x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x\cdot e^h - e^x}{h} \\
&= \lim_{h\to 0} e^x\cdot\frac{e^h-1}{h} \\
&= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h} \\
&= e^x\cdot 1 \ * \\
&= e^x
\end{aligned}$$

**Hallar la derivada de** $f(x) = \ln x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} \\ &= \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} \\ &= \frac{1}{x} \lim_{u\to 0} \frac{\ln(1+u)}{u} \quad (u = \frac{h}{x}) \\ &= \frac{1}{x} \cdot 1 = \frac{1}{x} \end{aligned}$$

**Teoremas de Álgebra de Derivadas**

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad g(x) \neq 0$$

**Hallar la derivada de** $f(x) = 2x^3 + x + 3$

$$\begin{aligned} f'(x) &= (2x^3)' + (x)' - (3)' = (2)'x^3 + 2(x^3)' + (x)' - (3)' \\ &= (0) \cdot x^3 + 2(3x^2) + 1 - (0) = 6x^2 + 1 \end{aligned}$$

**Hallar la derivada de** $f(x) = k \cdot x^n$

$$\begin{aligned} f'(x) &= k' \cdot x^n + k \cdot (x^n)' = (0) \cdot x^n + k \cdot nx^{n-1} \\ &= k \cdot nx^{n-1} \end{aligned}$$

**Hallar la derivada de** $f(x) = \left(\dfrac{2x+3}{x}\right)$

$$\begin{aligned} f(x) &= \frac{2x+3}{x} = 2 + \frac{3}{x} \\ f'(x) &= 0 - \frac{3}{x^2} = -\frac{3}{x^2} \end{aligned}$$

**Hallar la derivada de** $f(x) = \dfrac{1}{x}$

$$\begin{aligned} f'(x) &= \frac{(1)' \cdot x - 1 \cdot (x)'}{x^2} \\ &= \frac{-1}{x^2} \end{aligned}$$

**Hallar la derivada de** $f(x) = \tan x$

$$\begin{aligned} f'(x) &= \frac{\sin x}{\cos x} \\ &= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{1}{\cos x} = \sec^2 x \end{aligned}$$

**Hallar la derivada de** $f(x) = \cot x$

$$\begin{aligned} f'(x) &= \frac{-(\sin^2 + \cos^2)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{-1}{\sin x} \\ &= -\csc^2 x \end{aligned}$$

**Hallar la derivada de** $f(x) = \sec x$

$$f'(x) = \sec x \tan x$$

**Hallar la derivada de** $f(x) = \csc x$

$$f'(x) = -\csc x \cot x$$

**Regla de la cadena**

Si$f(x) = u(v(x))$ y existe $u'(x)$ y $v'(x)$, entonces:

$$f'(x) = u'(v(x)) \cdot v'(x)$$

**Hallar la derivada de** $f(x) = \sin^2 x$

$$f(x) = \sin x^2 \begin{cases} \sin x \\ x^2 \end{cases} \implies f'(x) = \cos(x^2) \cdot 2x = 2x \cdot \cos(x^2)$$

**Hallar la derivada de** $f(x) = \ln(\sqrt{x^2 + 3x})$

$$f(x) = \ln((x^2 + 3x)^{1/2}) = \frac{1}{2} \ln(x^2 + 3x)$$

$$f'(x) = \frac{1}{2} \cdot \frac{d}{dx}[\ln(x^2 + 3x)]$$

$$\frac{d}{dx}[\ln(x^2 + 3x)] = \frac{1}{x^2 + 3x} \cdot \frac{d}{dx}(x^2 + 3x)$$

$$\frac{d}{dx}(x^2 + 3x) = 2x + 3$$

$$f'(x) = \frac{1}{2} \cdot \frac{1}{x^2+3x} \cdot (2x+3) = \frac{2x+3}{2(x^2+3x)}$$
$$f'(x) = \frac{2x+3}{2(x^2+3x)}$$

**Hallar la derivada de** $f(x) = \sqrt{\sqrt{x^4}}$

$$f(x) = \begin{cases} u(x) = x^4 \implies u'(x) = 4x^3 \\ v(x) = x^{\frac{1}{2}} \implies v'(x) = \frac{1}{2}x^{-\frac{1}{2}} \\ w(x) = x^{\frac{1}{2}} \implies w'(x) = \frac{1}{2}x^{-\frac{1}{2}} \end{cases} \implies f'(x) = w'(v(u(x))) \cdot v'(u(x)) \cdot u'(x)$$

$$w'(v(u(x))) = w'((x^4)^{\frac{1}{2}}) = w'(x^2) = \frac{1}{2}(x^2)^{-\frac{1}{2}} = \frac{1}{2}x^{-2}$$
$$v'(u(x)) = v'(x^4) = \frac{1}{2}(x^4)^{-\frac{1}{2}} = \frac{1}{2}x^{-1}$$
$$u'(x) = 4x^3$$
$$f'(x) = \frac{1}{2}x^{-1} \cdot \frac{1}{2}x^{-2} \cdot 4x^3$$
$$f'(x) = \frac{4}{4}x^0 = 1$$

**Hallar la derivada de** $y = e^{3x}$

$$\ln y = \ln(e^{3x})$$
$$\frac{1}{y} \cdot y' = 3x\ln(e)$$
$$y' = 3x \cdot 1 \cdot y$$
$$y' = 3x \cdot e^{3x}$$

**Hallar la derivada de** $y = \log x$

$$y = \frac{1}{x\ln 10}$$

**Hallar la derivada de** $y = \log_8 x^2$

$$y' = \frac{1}{x^2 \ln 8} \cdot 2x$$
$$y' = \frac{2}{x\ln 8}$$

**Hallar la derivada de** $f(x) = x^2 5^{3x} + \cos(\sqrt{x^2-2x})$

$$f(x) = x^2 5^{3x} + \cos(\sqrt{x^2-2x})$$
$$f'(x) = \frac{d}{dx}(x^2 5^{3x}) + \frac{d}{dx}(\cos(\sqrt{x^2-2x}))$$
$$\frac{d}{dx}(x^2 5^{3x}) = 2x \cdot 5^{3x} + x^2 \cdot 3\ln(5) \cdot 5^{3x}$$

$$\frac{d}{dx}(\cos(\sqrt{x^2-2x})) = -\sin(\sqrt{x^2-2x}) \cdot \frac{1}{2}(x^2-2x)^{-\frac{1}{2}} \cdot 2x-2$$
$$\frac{d}{dx}(\cos(\sqrt{x^2-2x})) = -\sin(\sqrt{x^2-2x}) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-2x}} \cdot 2x-2$$
$$\frac{d}{dx}(\cos(\sqrt{x^2-2x})) = -\sin(\sqrt{x^2-2x}) \cdot \frac{x-1}{\sqrt{x^2-2x}}$$
$$f'(x) = 2x \cdot 5^{3x} + x^2 \cdot 3\ln(5) \cdot 5^{3x} - \sin(\sqrt{x^2-2x}) \cdot \frac{x-1}{\sqrt{x^2-2x}}$$

# 2 Derivada de funciones inversas

Sea $f(x)$ continua y monotona en el intervalo $[a,b]$ y sea $g(x) = f^{-1}(x)$ la inversa, entonces:

$$\boxed{g(x) = \left[f^{-1}(x)\right]' = \frac{1}{f'(x)}}$$

**Hallar la derivada de la inversa de** $f(x) = 3x+2$

$$\left[f^{-1}(x)\right]' = \frac{1}{f'(x)} = \frac{1}{3}$$

**Hallar la derivada de la inversa de** $f(x) = \sin x$

$$y = \sin x \implies x = \sin y$$
$$\frac{dx}{dy} = \cos y$$
$$\left[f^{-1}(x)\right]' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}$$
$$\cos^2 y + \sin^2 y = 1 \implies \cos y = \sqrt{1-\sin^2 y}$$
$$\sin y = x \implies \cos y = \sqrt{1-x^2}$$
$$\left[f^{-1}(x)\right]' = \frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

**Hallar la derivada de la inversa de** $f(x) = \cos x$

$$y = \cos x \implies x = \cos y$$
$$\frac{dx}{dy} = -\sin y$$
$$\left[f^{-1}(x)\right]' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{-\sin y} = -\frac{1}{\sin y}$$
$$\cos^2 y + \sin^2 y = 1 \implies \sin y = \sqrt{1-\cos^2 y}$$
$$\cos y = x \implies \sin y = \sqrt{1-x^2}$$
$$\left[f^{-1}(x)\right]' = \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

**Hallar la derivada de la inversa de** $f(x) = \tan x$

$$y = \tan x \implies x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y$$

$$\left[f^{-1}(x)\right]' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y}$$

$$y = \arctan x \implies \sec^2 y = 1 + \tan^2 y$$

$$\tan y = x \implies \sec^2 y = 1 + x^2$$

$$\left[f^{-1}(x)\right]' = \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

# 3 Derivada de funciones implicitas

Existen funciones en las cuales la $y$ no esta despejada (o incluso no se la puede despejar) y no esta de la forma $y = f(x)$. Para encontrar la derivada se considera que $y = f(x)$ utilizando *algebra de derivadas* y la *regla de cadena.*

Considerando que $y' = f'(x) = \dfrac{df}{dx}$

**Encontrar** $y'$ **si** $x^2y^3 - 2xy = 6x + y + 1$

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(x^2)y^3 + x^2\frac{d}{dx}(y^3) = 2xy^3 + x^2 \cdot 3y^2y'$$

$$\frac{d}{dx}(2xy) = \frac{d}{dx}(2x) \cdot y + 2x \cdot \frac{d}{dx}(y) = 2 \cdot y + 2x \cdot y'$$

$$\frac{d}{dx}(6x + y + 1) = \frac{d}{dx}(6x) + \frac{d}{dx}(y) = 6 + y'$$

$$\frac{d}{dx}(x^2y^3 - 2xy) = \frac{d}{dx}(6x + y + 1)$$

$$(2xy^3 + x^2 \cdot 3y^2y') - (2y + 2xy') = 6 + y'$$

$$x^2 \cdot 3y^2y' - 2y - 2xy' - y' = 6 - 2xy^3$$

$$y'(x^2 \cdot 3y^2 - 2x - 1) = 6 - 2xy^3 + 2y$$

$$y' = \frac{6 - 2xy^3 + 2y}{x^2 \cdot 3y^2 - 2x - 1}$$

**Encontrar** $y'$ **si** $xy^2 = \sin(xy)$

Diferenciamos ambos lados de la ecuación respecto a $x$ :

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(\sin(xy))$$

$$\frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx} = y^2 + 2xyy'$$

$$\frac{d}{dx}(\sin(xy)) = \cos(xy) \cdot \frac{d}{dx}(xy)$$
$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx} = y + xy'$$
$$\frac{d}{dx}(\sin(xy)) = \cos(xy) \cdot (y + xy') = y\cos(xy) + xy'\cos(xy)$$
$$y^2 + 2xyy' = y\cos(xy) + xy'\cos(xy)$$
$$y^2 - y\cos(xy) = xy'\cos(xy) - 2xyy'$$
$$y^2 - y\cos(xy) = y'(x\cos(xy) - 2xy)$$
$$y' = \frac{y^2 - y\cos(xy)}{x\cos(xy) - 2xy}$$

**Encontrar la pendiente de la recta tangente a la curva $x^2 + y^2 = 9$ en $x = 2$.**

Diferenciamos ambos lados de la ecuación $x^2 + y^2 = 9$ con respecto a $x$ :

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(9)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Evaluamos en $x = 2$ :

$$x^2 + y^2 = 9$$
$$2^2 + y^2 = 9$$
$$4 + y^2 = 9$$
$$y^2 = 5$$
$$y = \pm\sqrt{5}$$

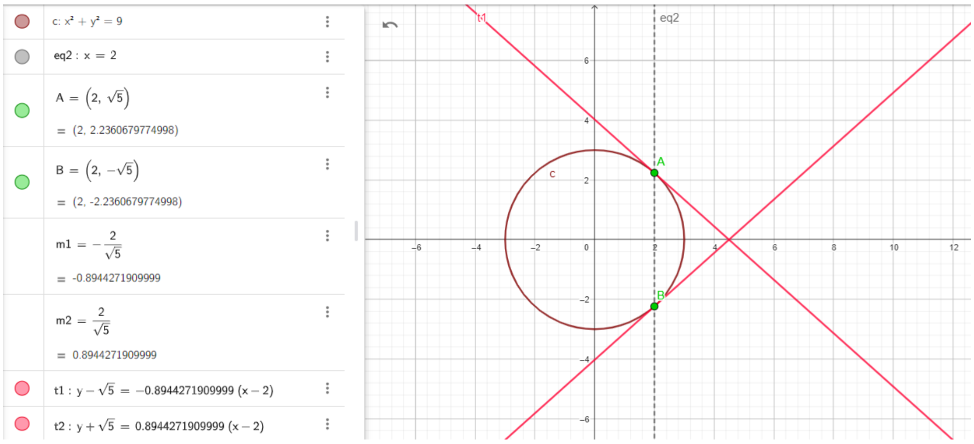
Para el punto $(2, \sqrt{5})$ :

$$\left.\frac{dy}{dx}\right|_{(2,\sqrt{5})} = -\frac{2}{\sqrt{5}}$$

Para el punto $(2, -\sqrt{5})$ :

$$\left.\frac{dy}{dx}\right|_{(2,-\sqrt{5})} = \frac{2}{\sqrt{5}}$$

La pendiente de la recta tangente a la curva en $x = 2$ es $-\frac{2}{\sqrt{5}}$ en el punto $(2, \sqrt{5})$ y $\frac{2}{\sqrt{5}}$ en el punto $(2, -\sqrt{5})$.




# Tabla de derivadas

| Función | Derivada |
|---|---|
| $(k)$ (constante) | $0$ |
| $x^n$ | $nx^{n-1}$ |
| $x^{\frac{m}{n}}$ | $\frac{m}{n}x^{\frac{m}{n}-1}$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x)$ | $\sec^2(x)$ |
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos(x)$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan(x)$ | $\frac{1}{1+x^2}$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |

| | |
|---|---|
| $a^x$ | $a^x\ln(a)$ |
| $\log_b x$ | $\frac{1}{x\ln(b)}$ |
| $\sin^{-1}(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1}(x)$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1}(x)$ | $\frac{1}{1+x^2}$ |
| $\cot^{-1}(x)$ | $-\frac{1}{1+x^2}$ |
| $\sec^{-1}(x)$ | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| $\csc^{-1}(x)$ | $-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ |
| $ax^n$ | $anx^{n-1}$ |
| $ax^{\frac{m}{n}}$ | $\frac{m}{n}ax^{\frac{m}{n}-1}$ |
| $ax^{bx}$ | $abx^{bx}\ln(a)$ |