Calculus – Lecture Notes

Amedeo Zucchetti

May 1, 2016

Based on the course of Prof. Michael Bronstein at USI

Contents

1 Sets, groups and fields

Definition Natural numbers $\mathbb{N}$

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition Group
A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$:

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists! 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists! (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abellian if the following rule is satisfied too:

(v) Commutativity: $a \circ b = b \circ a$

Definition Field
Given a set X, then $(X, +, \cdot)$ is a field if it satisfies the following properties for all $a, b, c \in X$:

(i) $a + b \in X$
$a \cdot b \in X$

(ii) $(a + b) + c = a + (b + c)$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists! 0 \in X : a + 0 = 0 + a = a$
$\exists! 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists! (-a) \in X : a + (-a) = (-a) + a = 0$
$\forall a \neq 0, \exists! a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$
$a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark $(\mathbb{Q}, +, \cdot)$ is a field.

Definition Ordered field
Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$:

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

Example Application of the order axioms
Let's take $(\mathbb{Q}, +, \cdot, \leq)$, $a, b \in \mathbb{Q}$. We want to show that if $a \leq b$, then $(-b) \leq (-a)$:
$a \leq b \iff a + ((-a) + (-b)) \leq b + ((-a) + (-b)) \iff (a + (-a)) + (-b) \leq (-a) + (b + (-b))$
$\iff (-b) + 0 \leq (-a) + 0 \iff (-b) \leq (-a)$

Definition A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark Let A, B sets:

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \leq |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition $\mathbb{Z}$ is countably infinite

Proof We can arrange $\mathbb{Z}$ and $\mathbb{Z}$ in the following way:

$$\begin{array}{lllllllllll} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots & \} \\ \mathbb{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \dots & \} \end{array}$$

We can take the function $f : \mathbb{Z} \to \mathbb{N}$ such that:

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{x}{2} & \text{if } x \text{ even} \\ -\frac{(x-1)}{2} & \text{if } x \text{ odd} \end{cases}, \quad f^{-1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } 0 < x \\ -2x + 1 & \text{if } x < 0 \end{cases}$$

f is bijective, thus $\mathbb{Z}$ is countably infinite.

$\square$

Proposition $\mathbb{Q}$ is countably infinite

Proof Idea of the proof. We can arrange $\mathbb{N}$ and $\mathbb{Q}$ as such:

$$\begin{array}{lclccccccccl} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \dots & \} \\ \mathbb{Q} & = & \{ & \frac{0}{1}, & \frac{1}{1}, & -\frac{1}{1}, & \frac{1}{2}, & -\frac{1}{2}, & \frac{2}{1}, & -\frac{2}{1}, & \frac{1}{3}, & \dots & \} \end{array}$$

Similarly to the proof for $\mathbb{Z}$, we can find a bijection between $\mathbb{N}$ and $\mathbb{Q}$.

$\square$

Proposition $\mathbb{R}$ is not countable

Proof Idea of the proof. Let $x \in [0, 1[$. Each x can be written as an infinite succession of digits:

1	0.**1**786...
2	0.3**9**09...
3	0.45**0**0...
4	0.097**2**...
...	...

We can construct a new number, taking a digit from each number (each at a different position) and increment it by 1. This way, the new number will be different from any other in the list in the position from where the digit was taken. In our example, the new number would be 0.**2013**...
Since there is one more number than those in the list, then $|\mathbb{N}| < |\mathbb{R}|$, so there is no bijection, and $\mathbb{R}$ is uncountable.

$\square$

Proposition $|\mathbb{R}| = |\mathbb{R}^2|$

Definition Power set
Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition $\sqrt{2} \notin \mathbb{Q}$

Proof By contradiction
We suppose $\sqrt{2} \in \mathbb{Q}$, this means there exists $a, b \in \mathbb{Z}$, $b \neq 0$ and greatest common divisor of a and b is 1, such that $\sqrt{2} = \frac{a}{b}$:

$$\sqrt{2} = \frac{a}{b} \iff 2 = \frac{a^2}{b^2} \iff 2b^2 = a^2$$

This means a^2 is even (and a is even), then it exists c such that $a = 2c$:

$$2b^2 = a^2 \iff 2b^2 = (2c)^2 = 4c^2 \iff b^2 = 2c^2$$

This means b^2, and b, are even. But if both a and b are even, then the greatest common divisor of a and b is not 1, contradiction.
We can conclude that $\sqrt{2} \notin \mathbb{Q}$.

$\square$

Definition Bounds
Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then:

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition Supremum and infimum
Let A be a set:

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition Maximum and minimum
Let A be a set:

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition Topology
Let X be a set. Then $\tau \subseteq 2^X$ is a topology if:

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\bigcup\limits_{\alpha} A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\bigcap\limits_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Definition Topological space
Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition Neighborhood in a topological space (X, τ)
A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition Metric
Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(x, z)$

Definition Metric space
Let X be a set, d be a metric, then (X, d) is a metric space.

Definition Ball in a metric space (X, d)
$B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition Open set in a topological space (X, τ)
A set U is open in (X, τ) if $U \in \tau$.

Definition Open set in a metric space (X, d)
A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition Let $S = (X, x)$ be a space (x a metric or a topology), then:

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup\limits_{\alpha} A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap\limits_{i=1}^{n} A_i$ is open in S (any finite intersection of any open set is also open)

3 Sequences

Definition Sequence
A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$.
The elements of a sequence can be listed in an ordered set with repetition: $(x_n) = (x_1, x_2, x_3, x_4, \ldots)$

Definition Cauchy sequence
A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$.
That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Definition Convergence in a metric space (X, d)
A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Definition Convergence in a topological space (X, τ)
A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$.
That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Proposition $x_n \to x$ in (X, d) $\iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.

Proof Let $U \subseteq X$ open, $x \in U$, it exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

$\Rightarrow$ Since x_n converges, then $d(x_n, x) < \varepsilon$, for all ε. This means $x \in B_\varepsilon(x) \subseteq U$, thus $x \in U$

$\Leftarrow$ $x \in B_\varepsilon(x)$ open. This mean it exists N such that all $x_n \in B_\varepsilon(x)$, for all $n \geq N$. We can conclude that $d(x_n, x) < \varepsilon$.

□

Theorem If a sequence converges to a limit in a metric space, then the limit is unique.

Proof Let's suppose $x_n \to x$ and $x_n \to x'$. It exists N such that for $ngeqN$, $d(x_n, x) < \varepsilon$ and It exists N' such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. We take $n \geq \max\{N, N'\}$. Now we have $0 \leq d(x, x') \leq d(x, x_n) + d(x_n, x') < 2\varepsilon$.
Since ε is arbitrarily small, then $d(x, x') \leq 0$. Now, we have $0 \leq d(x, x') \leq 0 \Rightarrow d(x, x') = 0 \Rightarrow x = x'$. □

Remark This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition $x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.

Proposition Properties of real sequences
For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties:

(i) $\lim\limits_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim\limits_{n\to\infty} x_n + \beta \lim\limits_{n\to\infty} y_n$

(ii) $\lim\limits_{n\to\infty} x_n x_y = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$

(iii) $\lim\limits_{n\to\infty} \dfrac{x_n}{x_y} = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Proof

(i) $\forall\varepsilon > 0, \quad \exists N : |x_n - x| < \dfrac{\varepsilon}{2\,|\alpha|} = \varepsilon', \quad \exists N' : |y_n - y| < \dfrac{\varepsilon}{2\,|\beta|} = \varepsilon'' \to$ we take $n = \max\{N, N'\}$.

$$|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)| = |\alpha(x_n - x) + \beta(y_n - y)| \leq |\alpha(x_n - x)| + |\beta(y_n - y)| =$$

$$= |\alpha|\,|(x_n - x)| + |\beta|\,|(y_n - y)| < |\alpha|\,\varepsilon' + |\beta|\,\varepsilon'' = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(i-ii) Similar to previous demonstration.

□

Definition A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Theorem Convergent real sequences are bounded (not the opposite).

Proof Since $x_n \to x$, it exists N such that for all $n \geq N$, $|x_n - x| < \varepsilon$. By triangle inequality we have:

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < \varepsilon + |x|$$

We choose $c = \max\{|x_1|, \ldots, |x_{N-1}|, |x| + \varepsilon\}$, then $|x_n| < c$ for each n.

$\square$

Definition Monotonic sequences

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Theorem If a sequence monotonic and bounded $\Rightarrow$ convergent

Proof (x_n) increasing and bounded, let $c = \sup(x_n)$. For all $\varepsilon > 0$ exists N such that $c - \varepsilon < x_N$. Since (x_n) increasing, for all $n \geq N$, $x_N \leq x_n \leq c$.

$$c - \varepsilon < x_n \leq c \iff -\varepsilon < x_n - c \leq 0 < \varepsilon \iff |x_n - c| < \varepsilon$$

The last inequality implies convergence. Similarly, the theorem can be proven for decreasing sequences.

$\square$

Definition Limit superior and inferior of (x_n)

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

Definition Subsequence
$(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Theorem If $x_n \to x \Rightarrow x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.

Proof $k \leq n_k$ (it can be proved by induction) and $d(x_k, x) < \varepsilon$. Since $N \leq k \leq n_k$, then $d(x_{n_k}, x) \leq d(x_k, x) < \varepsilon$. This means the subsequence converges to x.

$\square$

Definition x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem Every sequence has a monotonic subsequence.

Proof Based on dominants terms:

- If we have infinite dominant terms, we take the decreasing subsequence formed by the dominant terms.
- If we have a finite number of dominant terms, then, after the last dominant term, we start taking an increasing subsequence (since, for each term, there will be at some point a greater term).

$\square$

Theorem Bolzano-Weierstrass
Every bounded sequence has a convergent subsequence.

Proof We take (x_n) bounded. We show it in three steps:

- (x_n) has a monotonic subsequence (x_{n_k})
- Since (x_n) is bounded, then (x_{n_k}) is bounded
- Since (x_{n_k}) is bounded and monotonic, it is convergent

$\square$

Definition $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition Series
(x_n) sequence. $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition Convergence of series
$s_n = \displaystyle\sum_{k=1}^{n} x_k$ a series. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k$.

Definition Absolute convergence of a series $s_n = \displaystyle\sum_{k=1}^{n} x_k$

s_n converges absolutely if $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$.

Proposition Absolute convergence $\Rightarrow$ convergence. If $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$, then $\displaystyle\sum_{k=1}^{\infty} x_k < \infty$.

Proof $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$ and $x_n \leq |x_n|$, then $\displaystyle\sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} |x_k| < \infty$.

$\square$

Definition Cauchy criterion for series
$s_n = \displaystyle\sum_{k=1}^{n} x_k$, and$\displaystyle\sum_{k=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall N \leq m \leq n \Rightarrow |s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m}^{n} x_k \right| < \varepsilon$$

Proposition Comparison test, for x_n, y_n sequences and $x_n \geq 0$

(i) If $\sum_{k=1}^{\infty} x_k < \infty$ and $|y_n| \leq x_n \Rightarrow \sum_{k=1}^{\infty} y_k < \infty$

(ii) If $\sum_{k=1}^{\infty} x_k = +\infty$ and $x_n \leq y_n \Rightarrow \sum_{k=1}^{\infty} y_k = +\infty$

Proof

(i) $\left|\sum_{k=m}^{n} y_k\right| \leq \sum_{k=m}^{n} |y_k| \leq \sum_{k=m}^{n} x_k < \varepsilon \Rightarrow \sum_{k=1}^{\infty} y_k < \infty$

(ii) $+\infty = \sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} y_k \Rightarrow \sum_{k=1}^{\infty} y_n = +\infty$

□

Proposition Ratio test, for x_n sequence, $x_n \neq 0$ and $s_n = \sum_{k=1}^{n} x_k$ series:

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

Proposition Root test, Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$:

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

Proof $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}, \varepsilon > 0, \alpha + \varepsilon < 1$:

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = \lim_{n\to\infty} \sup\{\sqrt[k]{|x_k|} : k > n\} \Rightarrow \exists N : \left|\sup\{\sqrt[n]{|x_n|} : n > N\} - \alpha\right| < \varepsilon$$

$$\alpha - \varepsilon < \left|\sup\{\sqrt[n]{|x_n|} : n > N\}\right| < \alpha + \varepsilon \Rightarrow \sqrt[n]{|x_n|} < \alpha + \varepsilon \iff |x_n| < (\alpha + \varepsilon)^n$$

Since the geometric series $\sum_{k=1}^{\infty} (\alpha+\varepsilon)^n < \infty$, then $\sum_{k=1}^{\infty} |x_n| < \sum_{k=1}^{\infty} (\alpha+\varepsilon)^n < \infty$, the series converges absolutely.

□

5 Functions and continuity

Definition Given a function $f : X \to Y$, the image of f is defined as: $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition Given a function $f : X \to Y$, the preimage of f is defined as: $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition Continuity of $f : (X, d_x) \to (Y, d_y)$ (in a metric space)
f is continuous at $x \in X$ if:

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark Continuity can also be defined as follows:

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta \varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition Continuity of $f : (X, \tau_x) \to (Y, \tau_y)$ (in a topological space)
f is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Proposition Continuous functions map open sets into open sets. If $f : (X, d_x) \to (Y, d_y)$ continuous, then $PreIm_f(A)$ is open, for all $A \subseteq Y$ open.

Proof Let $A \subseteq Y$ open, $x \in PreIm_f(A)$, $f(x) \in A$. Then, it exists $\varepsilon > 0$ such that $B^{d_y}_{\varepsilon}(f(x)) \subseteq A$. Since f is continuous, then it exists δ_ε such that:

$$PreIm_f(B^{d_x}_{\delta_\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x)) \subseteq A \Rightarrow B^{d_x}_{\delta_\varepsilon}(x) \subseteq PreIm_f(A) \Rightarrow A \text{ is open}$$

$\square$

Theorem Continuous functions map limits to limits:

$$f \text{ continuous}, x_n \to x \iff f(x_n) \to f(x)$$

Proof Topological (only for "$\Rightarrow$")
Let $f : (X, \tau_x) \to (Y, \tau_y)$, $A \in \tau_y$, $f(x) \in A$. Since f continuous, then $PreIm_f(A) \in \tau_x$ and $x \in PreIm_f(A)$. Since x_n converges to x, we have that:

$$\exists N : \forall n \geq N, (x_n) \subseteq PreIm_f(A) \Rightarrow Im_f(x_n) \subseteq A \Rightarrow f(x_n) \to f(x)$$

$\square$

Proof Metrical (only for "$\Rightarrow$")
Let $\varepsilon > 0$, $f : (X, d_x) \to (Y, d_y)$ continuous. Then, it exists $\delta > 0$ such that for all $x' \in X$, $d_x(x, x') < \delta$. This means $d_y(f(x), f(x')) < \varepsilon$. Since x_n converges to x:

$$\exists N : \forall n \geq N, d_x(x, x) < \delta \Rightarrow d_y(f(x_n), f(x)) < \varepsilon \Rightarrow f(x_n) \to f(x)$$

$\square$

Proposition $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.

Proposition f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.

Proof

(1) f continuous at $x \Rightarrow$ for $x_n \to x$, then $f(x_n) \to f(x)$

(2) g continuous at $y \Rightarrow$ for $y_n \to y$, then $g(y_n) \to f(y)$

(3) In particular, for $y_n = f(x_n) \Rightarrow g(f(x_n)) \to g(f(x))$

$\square$

Definition $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Theorem Banach fixed point
Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then:

(i) $\exists! x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

6 Limits of functions

Definition f converges to c at x_0 $\iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim\limits_{x \to x_0} f(x) = c$.

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim\limits_{x \to x_0^+} f(x) = c$.
- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim\limits_{x \to x_0^-} f(x) = c$.

Definition $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

Theorem Extreme value
If $f : [a, b] \to \mathbb{R}$ is continuous, then:

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$
$\exists x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}), \forall x \in [a, b]$

Proof

(i) Proof by contradiction, we assume f unbounded
This means, for all $n \in \mathbb{N}$ it exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Then, $(x_n) \subseteq [a, b]$ is bounded and has a subsequence (x_{n_k}) that converges to a $x_0 \in [a, b]$ (Bolzano-Weierstrass). Since f is continuous at x_0, then $f(x_{n_k})$ converges to $f(x_0)$. If f is unbounded, then $f(x_n)$ diverges: contradiction. This means f is bounded.

(ii) Let's take $M = \sup\{f(x) : x \in [a, b]\}$ the smallest upper bound of $Im([a, b])$, then $M - \frac{1}{n}$ is not an upper bound. We know it exists x_n such that $M - \frac{1}{n} < f(x_n) \leq M$. This means:

$$\lim_{n\to\infty} M - \frac{1}{n} \leq \lim_{n\to\infty} f(x_n) \leq M \iff M \leq \lim_{n\to\infty} f(x_n) \leq M \iff \lim_{n\to\infty} f(x_n) = M$$

Moreover, $(x_n) \subseteq [a, b]$ is bounded, and it has a subsequence (x_{n_k}) convergent to $x_0 \in [a, b]$. Since f is continuous, then $f(x_{n_k})$ converges to $f(x_0)$. This means $f(x_0) = M$, where x_0 is the maximizer.

□

Remark This isn't true if the interval is open:

- $f : \mathbb{R} \to \mathbb{R}(0, 1)$, $f(x) = \frac{1}{x}$ is unbounded, since $f(x)$ goes to infinity for x small
- $f : (-1, 1) \to$, $f(x) = x^2$, doesn't have a max, since $\sup\{Im((-1, 1))\} = 1$ is $f(1)$ or $f(-1)$, but 1 and -1 $\notin (-1, 1)$

Theorem Intermediate value (IVT)
f continuous on $[a, b]$, $f(a) < c < f(b) \Rightarrow \exists x \in [a, b] : f(x) = c$.

Proof Let's assume $f(a) < c < f(b)$ (the same can be done for the opposite). Let's have $S = \{x \in [a, b] : f(x) < c\}$ not empty, since at least $f(a) \in S$. Let $x_0 = \sup S \in [a, b]$, then $x_0 - \frac{1}{n}$ is not an upper bound, and it exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \leq x_0$. This means s_n converges to x_0. We now have $f(s_n) < c$ and $f(x_0) = \lim f(s_n) \leq c$.
Let's take $t_n = \min\{x_0 + \frac{1}{n}, b\} \in [a, b]$, where $x_0 < t_n \leq t_n + \frac{1}{n}$, meaning that t_n converges to x_0. Now $t_n \notin S$ (since $t_n > \sup S$), $f(t_n) \geq c$ and $f(x_0) = \lim t_n \geq c$. Therefore $c \leq f(x_0) \leq c$, so $f(x_0) = c$.

□

Definition A Darboux function is a function that satisfies the intermediate value property.

Proposition Continuous implies Darboux, but not the opposite.

Proposition Continuous functions map intervals to intervals.

Definition Connectedness
Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold:

- There exist $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exist $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$! NOT SURE !

A set is connected if it is not disconnected.

Proposition Continuous functions preserve connectedness.
$f : (X, \tau_x) \to (Y, \tau_y)$, $A \subseteq X$ connected in (X, τ_x), then $Im(A) \subseteq Y$ is connected in (Y, τ_y).

Proof By contradiction. We suppose A connected and $Im(A)$ disconnected.
Since $Im(A)$ is disconnected, exist $V_1, V_2 \in \tau_y$ such that:

- $(Im(A) \cap V_1) \cap (Im(A) \cap V_2) = \emptyset$, and
- $(Im(A) \cap V_1) \cup (Im(A) \cap V_2) = Im(A)$, and
- $Im(A) \cap V_1 \neq \emptyset \neq Im(A) \cap V_2$

Let $U_1 = PreIm(V_1)$ and $U_2 = PreIm(V_2)$ it follows (it should be proved) that:

- $(PreIm(A) \cap U_1) \cap (PreIm(A) \cap U_2) = \emptyset$, and
- $(PreIm(A) \cap U_1) \cup (PreIm(A) \cap U_2) = PreIm(A)$, and
- $PreIm(A) \cap U_1 \neq \emptyset \neq PreIm(A) \cap U_2$

This implies that A is disconnected, contradiction. Therefore $Im(A)$ is connected.

□

Definition Uniform continuity
$f : (X, d_x) \to (Y, d_y)$ is uniformely continuous on X if:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall \varepsilon > 0 \quad \exists \delta_{\varepsilon, x_0} > 0 : \forall x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem f continuous on A, closed and bounded $\Rightarrow f$ is uniformely continuous on A.

Theorem f uniformely continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow f(s_n)$ is Cauchy sequence.

Proof Let $(s_n) \subseteq S$ a Cauchy sequence, $\varepsilon > 0$ and f uniformely continuous:

1. Exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$.
2. Exists N_ε such that for all $n, m \geq N$, then $|s_n - s_m| < \delta$

Combining (1) and (2) we have that for all $n, m \geq N$, $|f(s_n) - f(s_m)| < \varepsilon$. This means $f(s_n)$ is a Cauchy sequence.

□

Definition Sequence of functions
$(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Definition f_n converges pointwise to $f \iff \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall \varepsilon > 0, x \in S \exists N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Definition $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)| < \varepsilon\}$

Definition f_n converges uniformely to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Theorem Uniform limit of a continuous function is continuous
$f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x)$ is continuous.

Proof Let $\varepsilon > 0$
Since $f_n \xrightarrow{unif.} f(x)$, it exists N_ε such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N$.
Since f_n continuous, it exists $\delta > 0$ such that for all x, x_0 such that $|x_0 - x| < \delta$, then $|f_N(x_0) - f_N(x)| < \frac{\varepsilon}{3}$.
By triangle inequality we have:

$$|f(x_0) - f(x)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f(x)| \leq$$

$$\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

7 Power series

Definition Power series
Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases:

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.
- The series converges for some bounded interval.

Theorem Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ ($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then $\sum_{n=0}^{\infty} a_n x^n$:

- Converges for $|x| < R$.
- Diverges for $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.

Proof With root test. Let $\alpha = \limsup \sqrt[n]{|a_n|}$, then $\sum_{k=1}^{\infty} a_n < \infty$ if $\alpha < 1$ or $\sum_{k=1}^{\infty} a_n = \infty$ if $\alpha = 1$. Let $\alpha_x = \limsup \sqrt[n]{|a_n x^n|} = \limsup |x| \sqrt[n]{|a_n|} = |x| \limsup \sqrt[n]{|a_n|} = \beta |x|$. Then:

1. If $0 < R < \infty$, then $\alpha_x = \beta |x| = \frac{|x|}{R}$.
   - If $|x| < R$, then $\alpha_x < 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ converges
   - If $|x| > R$, then $\alpha_x > 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ diverges
2. If $R = \infty$, then $\alpha_x = 0 < 1$ independently of x. The series always converges.
3. If $R = 0$, then $\alpha_x = \infty > 1$ independently of x. The series always diverges.

$\square$