Calculus – Lecture Notes

Amedeo Zucchetti

May 1, 2016

Based on the course of Prof. Michael Bronstein at USI

Contents

1 Sets, groups and fields

Definition Natural numbers $\mathbb{N}$

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition Group
A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$:

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists! 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists!(-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abellian if the following rule is satisfied too:

(v) Commutativity: $a \circ b = b \circ a$

Definition Field
Given a set X, then $(X, +, \cdot)$ is a field if it satisfies the following properties for all $a, b, c \in X$:

(i) $a + b \in X$
$a \cdot b \in X$

(ii) $(a+b)+c = a+(b+c)$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists! 0 \in X : a + 0 = 0 + a = a$
$\exists! 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!(-a) \in X : a + (-a) = (-a) + a = 0$
$\forall a \neq 0, \exists! a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$
$a \cdot b = b \cdot a$

(vi) $a \cdot (b+c) = a \cdot b + a \cdot c$

Definition $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark $(\mathbb{Q}, +, \cdot)$ is a field.

Definition Ordered field
Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$:

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

Example Application of the order axioms
Let's take $(\mathbb{Q}, +, \cdot, \leq)$, $a, b \in \mathbb{Q}$. We want to show that if $a \leq b$, then $(-b) \leq (-a)$:
$a \leq b \iff a + ((-a) + (-b)) \leq b + ((-a) + (-b)) \iff (a + (-a)) + (-b) \leq (-a) + (b + (-b))$
$\iff (-b) + 0 \leq (-a) + 0 \iff (-b) \leq (-a)$

Definition A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark Let A, B sets:

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \leq |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition $\mathbb{Z}$ is countably infinite

Proposition $\mathbb{Q}$ is countably infinite

Proposition $\mathbb{R}$ is not countable

Proposition $|\mathbb{R}| = |\mathbb{R}^2|$

Definition Power set
Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition $\sqrt{2} \notin \mathbb{Q}$

Definition Bounds
Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then:

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition Supremum and infimum
Let A be a set:

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition Maximum and minimum
Let A be a set:

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition Topology
Let X be a set. Then $\tau \subseteq 2^X$ is a topology if:

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\displaystyle\bigcup_\alpha A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\displaystyle\bigcap_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Definition Topological space
Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition Neighborhood in a topological space (X, τ)
A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition Metric
Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(x, z)$

Definition Metric space
Let X be a set, d be a metric, then (X, d) is a metric space.

Definition Ball in a metric space (X, d)
$B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition Open set in a topological space (X, τ)
A set U is open in (X, τ) if $U \in \tau$.

Definition Open set in a metric space (X, d)
A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition Let $S = (X, x)$ be a space (x a metric or a topology), then:

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup\limits_{\alpha} A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap\limits_{i=1}^{n} A_i$ is open in S (any finite intersection of any open set is also open)

3 Sequences

Definition Sequence
A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$.
The elements of a sequence can be listed in an ordered set with repetition: $(x_n) = (x_1, x_2, x_3, x_4, \ldots)$

Definition Cauchy sequence
A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$.
That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Definition Convergence in a metric space (X, d)
A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Definition Convergence in a topological space (X, τ)
A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$.
That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Proposition $x_n \to x$ in $(X, d) \iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.

Theorem If a sequence converges to a limit in a metric space, then the limit is unique.

Remark This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition $x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.

Proposition Properties of real sequences
For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties:

(i) $\lim\limits_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim\limits_{n\to\infty} x_n + \beta \lim\limits_{n\to\infty} y_n$

(ii) $\lim\limits_{n\to\infty} x_n x_y = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$

(iii) $\lim\limits_{n\to\infty} \dfrac{x_n}{x_y} = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Definition A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Theorem Convergent real sequences are bounded (not the opposite).

Definition Monotonic sequences

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Theorem If a sequence monotonic and bounded $\Rightarrow$ convergent

Definition Limit superior and inferior of (x_n)

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

Definition Subsequence
$(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Theorem If $x_n \to x \Rightarrow x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.

Definition x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem Every sequence has a monotonic subsequence.

Theorem Bolzano-Weierstrass
Every bounded sequence has a convergent subsequence.

Definition $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition Series

(x_n) sequence. $s_n = \sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition Convergence of series

$s_n = \sum_{k=1}^{n} x_k$ a series. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k$.

Definition Absolute convergence of a series $s_n = \sum_{k=1}^{n} x_k$

s_n converges absolutely if $\sum_{k=1}^{\infty} |x_k| < \infty$.

Proposition Absolute convergence $\Rightarrow$ convergence. If $\sum_{k=1}^{\infty} |x_k| < \infty$, then $\sum_{k=1}^{\infty} x_k < \infty$.

Definition Cauchy criterion for series

$s_n = \sum_{k=1}^{n} x_k$, and$\sum_{k=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall N \leq m \leq n \Rightarrow |s_n - s_m| = \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| = \left|\sum_{k=m}^{n} x_k\right| < \varepsilon$$

Proposition Comparison test, for x_n, y_n sequences and $x_n \geq 0$

(i) If $\sum_{k=1}^{\infty} x_k < \infty$ and $|y_n| \leq x_n \Rightarrow \sum_{k=1}^{\infty} y_k < \infty$

(ii) If $\sum_{k=1}^{\infty} x_k = +\infty$ and $x_n \leq y_n \Rightarrow \sum_{k=1}^{\infty} y_k = +\infty$

Proposition Ratio test, for x_n sequence, $x_n \neq 0$ and $s_n = \sum_{k=1}^{n} x_k$ series:

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

Proposition Root test, Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$:

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

5 Functions and continuity

Definition Given a function $f : X \to Y$, the image of f is defined as: $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition Given a function $f : X \to Y$, the preimage of f is defined as: $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition Continuity of $f : (X, d_x) \to (Y, d_y)$ (in a metric space)
f is continuous at $x \in X$ if:

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 : \forall x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark Continuity can also be defined as follows:

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition Continuity of $f : (X, \tau_x) \to (Y, \tau_y)$ (in a topological space)
f is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Proposition Continuous functions map open sets into open sets. If $f : (X, d_x) \to (Y, d_y)$ continuous, then $PreIm_f(A)$ is open, for all $A \subseteq Y$ open.

Theorem Continuous functions map limits to limits:

$$f \text{ continuous, } x_n \to x \iff f(x_n) \to f(x)$$

Proposition $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.

Proposition f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.

Definition $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Theorem Banach fixed point
Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then:

(i) $\exists! x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

6 Limits of functions

Definition f converges to c at x_0 $\iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim\limits_{x \to x_0} f(x) = c$.

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim\limits_{x \to x_0^+} f(x) = c$.
- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim\limits_{x \to x_0^-} f(x) = c$.

Definition $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

Theorem Extreme value
If $f : [a, b] \to \mathbb{R}$ is continuous, then:

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$
$\exists x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}), \forall x \in [a, b]$

Remark This isn't true if the interval is open:

- $f : \mathbb{R} \to \mathbb{R}(0, 1)$, $f(x) = \frac{1}{x}$ is unbounded, since $f(x)$ goes to infinity for x small
- $f : (-1, 1) \to$, $f(x) = x^2$, doesn't have a max, since $\sup\{Im((-1, 1))\} = 1$ is $f(1)$ or $f(-1)$, but 1 and -1 $\notin (-1, 1)$

Theorem Intermediate value (IVT)
f continuous on $[a, b]$, $f(a) < c < f(b) \Rightarrow \exists x \in [a, b] : f(x) = c$.

Definition A Darboux function is a function that satisfies the intermediate value property.

Proposition Continuous implies Darboux, but not the opposite.

Proposition Continuous functions map intervals to intervals.

Definition Connectedness
Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold:

- There exist $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exist $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$! NOT SURE !

A set is connected if it is not disconnected.

Proposition Continuous functions preserve connectedness.
$f : (X, \tau_x) \to (Y, \tau_y)$, $A \subseteq X$ connected in (X, τ_x), then $Im(A) \subseteq Y$ is connected in (Y, τ_y).

Definition Uniform continuity
$f : (X, d_x) \to (Y, d_y)$ is uniformely continuous on X if:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall \varepsilon > 0 \quad \exists \delta_{\varepsilon, x_0} > 0 : \forall x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem f continuous on A, closed and bounded $\Rightarrow$ f is uniformely continuous on A.

Theorem f uniformely continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow$ $f(s_n)$ is Cauchy sequence.

Definition Sequence of functions
$(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Definition f_n converges pointwise to $f \iff \lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall \varepsilon > 0, x \in S \exists N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Definition $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)| < \varepsilon\}$

Definition f_n converges uniformely to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Theorem Uniform limit of a continuous function is continuous
$f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x)$ is continuous.

7 Power series

Definition Power series
Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases:

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.
- The series converges for some bounded interval.

Theorem Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ ($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then $\sum\limits_{n=0}^{\infty} a_n x^n$:

- Converges for $|x| < R$.
- Diverges for $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.