Università della Svizzera italiana

Year 2015–2016

Calculus

Course Notes

Amedeo Zucchetti

February 27, 2017

Contents

1 Sets, groups and fields **3**

2 Spaces **6**

3 Sequences **7**

4 Series **9**

5 Functions and continuity **11**

6 Continuous functions and intervals **13**

7 Uniform continuity **14**

8 Power Series **15**

9 Lipschitz continuity **16**

10 Differentiability and derivatives **17**

11 Integrals **20**

12 Antiderivatives (or indefinite integrals) **21**

1 Sets, groups and fields

Definition 1.1 (Natural numbers). The set of natural numbers is defined with the following properties

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition 1.2 (Group). A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists!\ 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists!\ (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abelian if the following rule is satisfied too

(v) Commutativity: $a \circ b = b \circ a$

Definition 1.3 (Field). Given a set X, then $(X, +, \cdot)$ is a field if the following are satisfied for all $a, b, c \in X$

(i) $a + b \in X$ and $a \cdot b \in X$

(ii) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists!\ 0 \in X : a + 0 = 0 + a = a$ and $\exists!\ 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!\ (-a) \in X : a + (-a) = (-a) + a = 0$ and $\forall\ a \neq 0,\ \exists!\ a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$ and $a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 1.4 (Rational numbers). $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark. $(\mathbb{Q}, +, \cdot)$ is a field.

Definition 1.5 (Ordered Field). Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \le b$ and $b \le a$, then $a = b$

(iii) If $a \le b$ and $b \le c$, then $a \le c$

(iv) If $a \le b$, then $a + c \le b + c$

(v) If $a \le b$ and $0 \le c$, then $a \cdot c \le b \cdot c$

Proof. $a \le b \iff a + ((-a) + (-b)) \le b + ((-a) + (-b)) \iff (a + (-a)) + (-b) \le (-a) + (b + (-b)) \iff (-b) + 0 \le (-a) + 0 \iff (-b) \le (-a)$ □

Definition 1.6 (Countable Infinite). A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark. Let A, B sets, then

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \le |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition 1.1. $\mathbb{Z}$ *is countably infinite*

Proof. We can arrange $\mathbb{Z}$ and $\mathbb{Z}$ in the following way

$$\begin{array}{lllllllllll} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \dots & \} \\ \mathbb{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \dots & \} \end{array}$$

We can take the function $f : \mathbb{Z} \to \mathbb{N}$ such that

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{x}{2} & \text{if } x \text{ even} \\ -\frac{(x-1)}{2} & \text{if } x \text{ odd} \end{cases}, \quad f^{-1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } 0 < x \\ -2x + 1 & \text{if } x < 0 \end{cases}$$

f is bijective, thus $\mathbb{Z}$ is countably infinite. □

Proposition 1.2. $\mathbb{Q}$ *is countably infinite.*

Proof. Idea of the proof. We can arrange $\mathbb{N}$ and $\mathbb{Q}$ as such

$$\begin{array}{llllllllllll} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \dots & \} \\ \mathbb{Q} & = & \{ & \frac{0}{1}, & \frac{1}{1}, & -\frac{1}{1}, & \frac{1}{2}, & -\frac{1}{2}, & \frac{2}{1}, & -\frac{2}{1}, & \frac{1}{3}, & \dots & \} \end{array}$$

Similarly to the proof for $\mathbb{Z}$, we can find a bijection between $\mathbb{N}$ and $\mathbb{Q}$. □

Proposition 1.3. $\mathbb{R}$ *is not countable.*

Proof. Idea of the proof. Let $x \in [0, 1)$. Each x can be written as an infinite succession of digits

1	0.**1**786...
2	0.3**9**09...
3	0.45**0**0...
4	0.097**2**...
...	...

We can construct a new number, taking a digit from each number (each at a different position) and increment it by 1. This way, the new number will be different from any other in the list in the position from where the digit was taken. In our example, the new number would be 0.**2013**...
Since there is one more number than those in the list, then $|\mathbb{N}| < |\mathbb{R}|$, so there is no bijection, and $\mathbb{R}$ is uncountable. $\square$

Proposition 1.4. $|\mathbb{R}| = |\mathbb{R}^2|$

Definition 1.7 (Power set). Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition 1.5. $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition 1.6. $\sqrt{2} \notin \mathbb{Q}$

Proof. By contradiction. We suppose $\sqrt{2} \in \mathbb{Q}$, this means there exists $a, b \in \mathbb{Z}$, $b \neq 0$ and greatest common divisor of a and b is 1, such that $\sqrt{2} = \frac{a}{b}$

$$\sqrt{2} = \frac{a}{b} \iff 2 = \frac{a^2}{b^2} \iff 2b^2 = a^2$$

This means a^2 is even (and a is even), then it exists c such that $a = 2c$

$$2b^2 = a^2 \iff 2b^2 = (2c)^2 = 4c^2 \iff b^2 = 2c^2$$

This means b^2, and b, are even. But if both a and b are even, then the greatest common divisor of a and b is not 1, contradiction. We can conclude that $\sqrt{2} \notin \mathbb{Q}$. $\square$

Definition 1.8 (Bounds). Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition 1.9 (Supremum and infimum). Let A be a set

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition 1.10 (Maximum and minimum). Let A be a set

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition 2.1 (Topology). Let X be a set. Then $\tau \subseteq 2^X$ is a topology if

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\bigcap_{i=1}^n A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Definition 2.2 (Topological space). Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition 2.3 (Neighborhood in a topological space (X, τ)). A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition 2.4 (Metric). Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.5 (Metric space). Let X be a set, d be a metric, then (X, d) is a metric space.

Definition 2.6 (Ball in a metric space (X, d)). $B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition 2.7 (Open set in a topological space (X, τ)). A set U is open in (X, τ) if $U \in \tau$.

Definition 2.8 (Open set in a metric space (X, d)). A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition 2.9 (Closed set). $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition 2.1. *Let $S = (X, x)$ be a space (x a metric or a topology), then*

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup_\alpha A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap_{i=1}^n A_i$ is open in S (any finite intersection of any open set is also open)

3 Sequences

Definition 3.1 (Sequence). A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$. The elements of a sequence can be listed in an ordered set with repetition

$$(x_n) = (x_1, x_2, x_3, x_4, \dots)$$

Definition 3.2 (Cauchy sequence). A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$. That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Definition 3.3 (Convergence in metric space). (X, d) is a metric space. A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Definition 3.4 (Convergence in topological space). (X, τ) is a topological space. A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$. That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Proposition 3.1. *$x_n \to x$ in (X, d) $\iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.*

Proof. Let $U \subseteq X$ open, $x \in U$, it exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

- $\Rightarrow$ Since x_n converges, then $d(x_n, x) < \varepsilon$, for all ε. This means $x \in B_\varepsilon(x) \subseteq U$, thus $x \in U$
- $\Leftarrow$ $x \in B_\varepsilon(x)$ open. This means it exists N such that all $x_n \in B_\varepsilon(x)$, for all $n \geq N$. We can conclude that $d(x_n, x) < \varepsilon$.

$\square$

Theorem 3.2. *If a sequence converges to a limit in a metric space, then the limit is unique.*

Proof. Let's suppose $x_n \to x$ and $x_n \to x'$. It exists N such that for $ngeqN$, $d(x_n, x) < \varepsilon$ and It exists N' such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. We take $n \geq \max\{N, N'\}$. Now we have $0 \leq d(x, x') \leq d(x, x_n) + d(x_n, x') < 2\varepsilon$. Since ε is arbitrarily small, then $d(x, x') \leq 0$. Now, we have $0 \leq d(x, x') \leq 0 \Rightarrow d(x, x') = 0 \Rightarrow x = x'$. $\square$

Remark. This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition 3.3. *$x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.*

Proposition 3.4 (Properties of real sequences). *For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties*

(i) $\lim_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim_{n\to\infty} x_n + \beta \lim_{n\to\infty} y_n$

(ii) $\lim_{n\to\infty} x_n x_y = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$

(iii) $\lim_{n\to\infty} \frac{x_n}{x_y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Proof. We prove each point individually

(i) $\forall\, \varepsilon > 0, \quad \exists\, N : |x_n - x| < \frac{\varepsilon}{2\,|\alpha|} = \varepsilon', \quad \exists\, N' : |y_n - y| < \frac{\varepsilon}{2\,|\beta|} = \varepsilon''$ and we take $n = \max\{N, N'\}$.

$$|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)| = |\alpha(x_n - x) + \beta(y_n - y)| \leq |\alpha(x_n - x)| + |\beta(y_n - y)| =$$

$$= |\alpha|\,|(x_n - x)| + |\beta|\,|(y_n - y)| < |\alpha|\,\varepsilon' + |\beta|\,\varepsilon'' = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(i-ii) Similar to previous demonstration.

□

Definition 3.5 (Bounded sequence). A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Definition 3.6 (Monotonic sequence). A sequence is monotonic if

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Theorem 3.5. *If a sequence monotonic and bounded, then the sequence is convergent.*

Proof. (x_n) increasing and bounded, let $c = \sup(x_n)$. For all $\varepsilon > 0$ exists N such that $c - \varepsilon < x_N$. Since (x_n) increasing, for all $n \geq N$, $x_N \leq x_n \leq c$.

$$c - \varepsilon < x_n \leq c \iff -\varepsilon < x_n - c \leq 0 < \varepsilon \iff |x_n - c| < \varepsilon$$

The last inequality implies convergence. Similarly, the theorem can be proven for decreasing sequences. □

Definition 3.7 (Limit superior and inferior). If (x_n) is a sequence, then

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

Definition 3.8 (Subsequence). $(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Theorem 3.6. *If $x_n \to x$, then $x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.*

Proof. $k \leq n_k$ (it can be proved by induction) and $d(x_k, x) < \varepsilon$. Since $N \leq k \leq n_k$, then $d(x_{n_k}, x) \leq d(x_k, x) < \varepsilon$. This means the subsequence converges to x. □

Definition 3.9 (Dominant term). x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem 3.7. *Every sequence has a monotonic subsequence.*

Proof. Based on dominants terms:

- If we have infinite dominant terms, we take the decreasing subsequence formed by the dominant terms.
- If we have a finite number of dominant terms, then, after the last dominant term, we start taking an increasing subsequence (since, for each term, there will be at some point a greater term).

□

Theorem 3.8 (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

Proof. We take (x_n) bounded. We show it in three steps:

- (x_n) has a monotonic subsequence (x_{n_k})
- Since (x_n) is bounded, then (x_{n_k}) is bounded
- Since (x_{n_k}) is bounded and monotonic, it is convergent

□

Definition 3.10. $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition 4.1 (Series). (x_n) is sequence. $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition 4.2 (Convergence of series). $s_n = \displaystyle\sum_{k=1}^{n} x_k$ a series. $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{n=1}^{\infty} x_k$.

Definition 4.3 (Absolute convergence of series). $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series. s_n converges absolutely if

$$\sum_{n=1}^{\infty} |x_k| < \infty$$

Proposition 4.1. *Absolute convergence ⇒ convergence. If* $\displaystyle\sum_{n=1}^{\infty} |x_k| < \infty$*, then* $\displaystyle\sum_{n=1}^{\infty} x_k < \infty$.

Proof. We know that

$$\sum_{n=1}^{\infty} |x_k| < \infty \text{ and } x_n \leq |x_n|$$

then

$$\sum_{n=1}^{\infty} x_k \leq \sum_{n=1}^{\infty} |x_k| < \infty$$

□

Definition 4.4 (Cauchy criterion for series). $s_n = \sum_{k=1}^{n} x_k$, and $\sum_{n=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall \ N \leq m \leq n \Rightarrow |s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m}^{n} x_k \right| < \varepsilon$$

Proposition 4.2 (Comparison test). *For x_n, y_n sequences and $x_n \geq 0$*

(i) If $\sum_{n=1}^{\infty} x_k < \infty$ *and* $|y_n| \leq x_n \Rightarrow \sum_{n=1}^{\infty} y_k < \infty$

(ii) If $\sum_{n=1}^{\infty} x_k = +\infty$ *and* $x_n \leq y_n \Rightarrow \sum_{n=1}^{\infty} y_k = +\infty$

Proof. We prove the two point individually

(i) $\left| \sum_{k=m}^{n} y_k \right| \leq \sum_{k=m}^{n} |y_k| \leq \sum_{k=m}^{n} x_k < \varepsilon \Rightarrow \sum_{n=1}^{\infty} y_k < \infty$

(ii) $+\infty = \sum_{n=1}^{\infty} x_k \leq \sum_{n=1}^{\infty} y_k \Rightarrow \sum_{n=1}^{\infty} y_n = +\infty$

□

Proposition 4.3 (Ratio test). *For x_n sequence, $x_n \neq 0$ and $s_n = \sum_{k=1}^{n} x_k$ series:*

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| > 1$

Proposition 4.4 (Root test). *Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$:*

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

Proof. $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}, \varepsilon > 0, \alpha + \varepsilon < 1$:

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = \lim_{n\to\infty} \sup\{\sqrt[k]{|x_k|} : k > n\} \Rightarrow \exists\, N : \left|\sup\{\sqrt[n]{|x_n|} : n > N\} - \alpha\right| < \varepsilon$$

$$\alpha - \varepsilon < \left|\sup\{\sqrt[n]{|x_n|} : n > N\}\right| < \alpha + \varepsilon \Rightarrow \sqrt[n]{|x_n|} < \alpha + \varepsilon \iff |x_n| < (\alpha+\varepsilon)^n$$

Since the geometric series $\sum_{n=1}^{\infty}(\alpha+\varepsilon)^n < \infty$, then $\sum_{n=1}^{\infty}|x_n| < \sum_{n=1}^{\infty}(\alpha+\varepsilon)^n < \infty$, the series converges absolutely.

□

5 Functions and continuity

Definition 5.1 (Image). Given a function $f : X \to Y$, the image of f is defined as $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition 5.2 (Preimage). Given a function $f : X \to Y$, the preimage of f is defined as $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition 5.3 (Continuity in metric space). $f : (X, d_x) \to (Y, d_y)$ is continuous at $x \in X$ if

$$\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 :\ \forall\ x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark. Continuity can also be defined as follows

$$\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition 5.4 (Continuity in topological space). $f : (X, \tau_x) \to (Y, \tau_y)$ is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Proposition 5.1. *Continuous functions map open sets into open sets.*

$$\textit{If } f : (X, d_x) \to (Y, d_y) \textit{ continuous, then } PreIm_f(A) \textit{ is open, for all } A \subseteq Y \textit{ open}$$

Proof. Let $A \subseteq Y$ open, $x \in PreIm_f(A)$, $f(x) \in A$. Then, it exists $\varepsilon > 0$ such that $B^{d_y}_{\varepsilon}(f(x)) \subseteq A$. Since f is continuous, then it exists δ_ε such that:

$$PreIm_f(B^{d_x}_{\delta_\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x)) \subseteq A \Rightarrow B^{d_x}_{\delta_\varepsilon}(x) \subseteq PreIm_f(A) \Rightarrow A \text{ is open}$$

□

Theorem 5.2. *Continuous functions map limits to limits*

$$f \textit{ continuous, } x_n \to x \iff f(x_n) \to f(x)$$

Proof, topology. (only for "⇒") Let $f : (X, \tau_x) \to (Y, \tau_y)$, $A \in \tau_y$, $f(x) \in A$. Since f continuous, then $PreIm_f(A) \in \tau_x$ and $x \in PreIm_f(A)$. Since x_n converges to x, we have that:

$$\exists\, N :\ \forall\, n \geq N, (x_n) \subseteq PreIm_f(A) \Rightarrow Im_f(x_n) \subseteq A \Rightarrow f(x_n) \to f(x)$$

□

Proof, metric. (only for "⇒") Let $\varepsilon > 0$, $f : (X, d_x) \to (Y, d_y)$ continuous. Then, it exists $\delta > 0$ such that for all $x' \in X$, $d_x(x, x') < \delta$. This means $d_y(f(x), f(x')) < \varepsilon$. Since x_n converges to x

$$\exists\, N :\ \forall\, n \geq N, d_x(x, x) < \delta \Rightarrow d_y(f(x_n), f(x)) < \varepsilon \Rightarrow f(x_n) \to f(x)$$

□

Proposition 5.3. *$f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.*

Proposition 5.4. *f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.*

Proof. We have the following implications

(1) f continuous at $x \Rightarrow$ for $x_n \to x$, then $f(x_n) \to f(x)$

(2) g continuous at $y \Rightarrow$ for $y_n \to y$, then $g(y_n) \to f(y)$

(3) In particular, for $y_n = f(x_n) \Rightarrow g(f(x_n)) \to g(f(x))$

□

Definition 5.5 (Contraction). $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \leq c < 1$ such that $d(f(x), f(y)) \leq c d(x, y)$, for all $x, y \in X$.

Theorem 5.5 (Banach fixed point). *Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then*

(i) $\exists!\ x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

Definition 5.6 (Convergence of a function). f converges to c at $x_0 \iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim\limits_{x \to x_0} f(x) = c$. Moreover

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim\limits_{x \to x_0^+} f(x) = c$.
- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim\limits_{x \to x_0^-} f(x) = c$.

Proposition 5.6. *f continuous at $a \iff \lim\limits_{x \to a} f(x) = f(a)$*

Proposition 5.7. $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

6 Continuous functions and intervals

Definition 6.1 (Bounded function). $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

Theorem 6.1 (Extreme value). *If $f : \mathbb{R} \to \mathbb{R}[a, b]$ is continuous, then:*

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$, meaning that

$$\exists\ x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}),\ \forall\ x \in [a, b]$$

Proof. Proof by contradiction, we assume f unbounded. We proceed in two steps

(1) Since f undounded, for all $n \in \mathbb{N}$ it exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Then, $(x_n) \subseteq [a, b]$ is bounded and has a subsequence (x_{n_k}) that converges to a $x_0 \in [a, b]$ (Bolzano-Weierstrass). Since f is continuous at x_0, then $f(x_{n_k})$ converges to $f(x_0)$. If f is unbounded, then $f(x_n)$ diverges: contradiction. This means f is bounded.

(2) Let's take $M = \sup\{f(x) : x \in [a, b]\}$ the smallest upper bound of $Im([a, b])$, then $M - \frac{1}{n}$ is not an upper bound. We know it exists x_n such that $M - \frac{1}{n} < f(x_n) \leq M$. This means:

$$\lim_{n\to\infty} M - \frac{1}{n} \leq \lim_{n\to\infty} f(x_n) \leq M \iff M \leq \lim_{n\to\infty} f(x_n) \leq M \iff \lim_{n\to\infty} f(x_n) = M$$

Moreover, $(x_n) \subseteq [a, b]$ is bounded, and it has a subsequence (x_{n_k}) convergent to $x_0 \in [a, b]$. Since f is continuous, then $f(x_{n_k})$ converges to $f(x_0)$. This means $f(x_0) = M$, where x_0 is the maximizer.

□

Theorem 6.2 (Intermediate value). *f continuous on $[a, b]$, $f(a) < c < f(b) \Rightarrow\ \exists\ x \in [a, b] : f(x) = c$.*

Proof. Let's assume $f(a) < c < f(b)$ (the same can be done for the opposite). Let's have $S = \{x \in [a, b] : f(x) < c\}$ not empty, since at least $f(a) \in S$. Let $x_0 = \sup S \in [a, b]$, then $x_0 - \frac{1}{n}$ is not an upper bound, and it exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \leq x_0$. This means s_n converges to x_0. We now have $f(s_n) < c$ and $f(x_0) = \lim f(s_n) \leq c$.
Let's take $t_n = \min\{x_0 + \frac{1}{n}, b\} \in [a, b]$, where $x_0 < t_n \leq t_n + \frac{1}{n}$, meaning that t_n converges to x_0. Now $t_n \notin S$ (since $t_n > \sup S$), $f(t_n) \geq c$ and $f(x_0) = \lim t_n \geq c$. Therefore $c \leq f(x_0) \leq c$, so $f(x_0) = c$. □

Definition 6.2 (Darboux function). A Darboux function is a function that satisfies the intermediate value property.

Proposition 6.3. *Continuous implies Darboux, but not the opposite.*

Proposition 6.4. *Continuous functions map intervals to intervals.*

Definition 6.3 (Connectedness). Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold

- There exists $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exists $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$

N.B.: here $\overline{U}$ doesn't mean complementary set of U, but set closure of U. That is, the smallest closed set containing U.
A set is connected if it is not disconnected.

Proposition 6.5. *Continuous functions preserve connectedness.*

$$f : (X, \tau_x) \to (Y, \tau_y), A \subseteq X \text{ connected in } (X, \tau_x) \Rightarrow Im(A) \subseteq Y \text{ is connected in } (Y, \tau_y)$$

Proof. By contradiction. We suppose A connected and $Im(A)$ disconnected. Since $Im(A)$ is disconnected, exist $V_1, V_2 \in \tau_y$ such that:

- $(Im(A) \cap V_1) \cap (Im(A) \cap V_2) = \emptyset$, and
- $(Im(A) \cap V_1) \cup (Im(A) \cap V_2) = Im(A)$, and
- $Im(A) \cap V_1 \neq \emptyset \neq Im(A) \cap V_2$

Let $U_1 = PreIm(V_1)$ and $U_2 = PreIm(V_2)$ it follows (it should be proved) that:

- $(PreIm(A) \cap U_1) \cap (PreIm(A) \cap U_2) = \emptyset$, and
- $(PreIm(A) \cap U_1) \cup (PreIm(A) \cap U_2) = PreIm(A)$, and
- $PreIm(A) \cap U_1 \neq \emptyset \neq PreIm(A) \cap U_2$

This implies that A is disconnected, contradiction. Therefore $Im(A)$ is connected. □

7 Uniform continuity

Definition 7.1 (Uniform continuity). $f : (X, d_x) \to (Y, d_y)$ is uniformly continuous on X if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_\varepsilon > 0 : \;\forall\, x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark. Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall\, \varepsilon > 0 \quad \exists\, \delta_{\varepsilon, x_0} > 0 : \;\forall\, x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem 7.1. *f continuous on A, closed and bounded $\Rightarrow f$ is uniformly continuous on A.*

Theorem 7.2. *f uniformly continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow f(s_n)$ is Cauchy sequence.*

Proof. Let $(s_n) \subseteq S$ a Cauchy sequence, $\varepsilon > 0$ and f uniformly continuous

1. Exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$.
2. Exists N_ε such that for all $n, m \geq N$, then $|s_n - s_m| < \delta$

Combining (1) and (2) we have that for all $n, m \geq N$, $|f(s_n) - f(s_m)| < \varepsilon$. This means $f(s_n)$ is a Cauchy sequence. □

Definition 7.2 (Sequence of functions). $(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Definition 7.3 (Pointwise convergence). f_n converges pointwise to $f \iff \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall\, \varepsilon > 0, x \in S \;\exists\; N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Definition 7.4 (infinite norm). $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)|\}$

Definition 7.5 (Uniform convergence). f_n converges uniformly to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Theorem 7.3. *Uniform limit of a continuous function is continuous.*

$$f_n(x) \text{ continuous and } f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x) \text{ is continuous}$$

Proof. Let $\varepsilon > 0$ Since $f_n \xrightarrow{unif.} f(x)$, it exists N_ε such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N$. Since f_n continuous, it exists $\delta > 0$ such that for all x, x_0 such that $|x_0 - x| < \delta$, then $|f_N(x_0) - f_N(x)| < \frac{\varepsilon}{3}$. By triangle inequality we have

$$|f(x_0) - f(x)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f(x)| \leq$$

$$\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

□

8 Power Series

Definition 8.1 (Power series). Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.

- The series converges for some bounded interval.

Theorem 8.1. *Let* $\beta = \limsup \sqrt[n]{|a_n|}$ *and* $R = \frac{1}{\beta}$ *($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then* $\sum_{n=0}^{\infty} a_n x^n$

- *Converges for* $|x| < R$.
- *Diverges for* $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.

Proof. With root test. Let $\alpha = \limsup \sqrt[n]{|a_n|}$, then $\sum_{n=1}^{\infty} a_n < \infty$ if $\alpha < 1$ or $\sum_{n=1}^{\infty} a_n = \infty$ if $\alpha = 1$. Let $\alpha_x = \limsup \sqrt[n]{|a_n x^n|} = \limsup |x| \sqrt[n]{|a_n|} = |x| \limsup \sqrt[n]{|a_n|} = \beta|x|$. Then

(1) If $0 < R < \infty$, then $\alpha_x = \beta|x| = \frac{|x|}{R}$.

  - If $|x| < R$, then $\alpha_x < 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ converges
  - If $|x| > R$, then $\alpha_x > 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ diverges

(2) If $R = \infty$, then $\alpha_x = 0 < 1$ independently of x. The series always converges.

(3) If $R = 0$, then $\alpha_x = \infty > 1$ independently of x. The series always diverges.

□

9 Lipschitz continuity

Definition 9.1 (Lipschitz continuity). $f : (X, d_x) \to (Y, d_y)$ is Lipschitz continuous if it exists $c \in [0, +\infty)$ such that $d_y(f(x), f(x')) \leq c d_x(x, x')$.

Proposition 9.1. *Lipschitz continuity* $\Rightarrow$ *uniform continuity.*

Proof. We need to show that $\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 :\ \forall\ x, x' \in X : d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$. We choose $\delta = \frac{\varepsilon}{c} \Rightarrow d_y(f(x), f(x')) \leq c d_x(x, x') < c\delta = c\frac{\varepsilon}{c} = \varepsilon$. □

Theorem 9.2 (Weierstrass approximation). *Every continuous function on* $[a, b]$ *can be uniformly approximated by polynomials on* $[a, b]$

$$\exists\ (a_n) \subseteq \mathbb{R} : p_n(x) = \sum_{k=1}^{n} a_k x^k \xrightarrow{unif.} f(x)\ \ on\ \ [a, b]$$

Theorem 9.3 (Bernstein polynomials). $b_{m,n}(x) = \binom{n}{m} x^m (1-x)^{n-m}$

$$span\{b_{0,n}(x), \ldots, b_{n,n}(x)\} = \left\{ \sum_{k=1}^{n} a_k x^k, a_i \in R \right\}$$

Theorem 9.4. $f : [0,1] \to \mathbb{R}$ *continuous, then*

- $B_n(f)(x) = \sum_{m=0}^{n} f(\frac{m}{n}) b_{m,n}(x)$
- $B_n(f)(x) \to f(x)$ *uniformly continuous on* $[0,1]$

10 Differentiability and derivatives

Definition 10.1 (Derivative). The derivative of a function f at point a is defined as one

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\varepsilon \to 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

Definition 10.2 (Differentiability). f is differentiable if the derivative f' exists.

Proposition 10.1. f *differentiable at* a*, then* f *continuous at* a*.*

Proof. We need to prove that $\lim_{x \to a} f(x) = f(a)$. We know that $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. Then $\lim_{x \to a} (x - a) \frac{f(x) - f(a)}{x - a} + f(a) = \lim_{x \to a} f(x)$. $\Rightarrow \lim_{x \to a} (x - a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} f(a) = 0 \cdot L + f(a) = f(a)$. $\square$

Definition 10.3. $f \in \mathcal{C}^k(\mathbb{R})$, f is differentiable k times, and the derivatives are continuous.

Proposition 10.2. *Properties of derivatives*

- $(f + g)'(x) = f'(x) + g'(x)$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \forall\ g(x) \neq 0$
- $(g \circ f)'(x) = (g' \circ f)(x) f'(x) = g'(x) f(x) f'(x)$
- $f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))}$

Definition 10.4 (Local minimizer). x^* is a local minimizer if exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - \varepsilon, x^* + \varepsilon)$. This means, $f(x^*)$ is local minimum (the smallest image in a given interval).

Theorem 10.3. $f : \mathbb{R} \to \mathbb{R}(a, b)$ *is differentiable and has a local minimum at* $x \Rightarrow f'(x) = 0$*.*

Proof. $\exists\, \varepsilon > 0 : f(x) \le f(y)\ \forall\, y \in (x-\varepsilon, x+\varepsilon)$

$$\lim_{y\to x^-} \frac{f(y)-f(x)}{y-x} \le 0, \quad \lim_{y\to x^+} \frac{f(y)-f(x)}{y-x} \ge 0 \quad \Rightarrow \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = 0$$

Since f is differentiable, $\lim_{y\to x} \dfrac{f(y)-f(x)}{y-x}$ exists, and it has to be 0. □

Theorem 10.4 (Rolle's theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on (a,b) and $f(a) = f(b) \Rightarrow$ it exists $x \in (a,b)$ such that $f'(x) = 0$.*

Proof. There exists x_m and x_M minimizer and maximizer, such that $f(x_m) \le f(x) \le f(x_M)$ for all $x \in [a,b]$.

(1) W.l.o.g., if $x_m = a$, $x_M = b$, then $f(x) = f(a) = f(b)$ constant, meaning that $f'(x) = 0\ \forall\, x$

(2) $x_m, x_M \in (a,b) \Rightarrow$ it exists $x \in (a,b)$ such that $f'(x) = 0$

□

Theorem 10.5 (Mean value theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on $(a,b) \Rightarrow$ it exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*

Proof. Let $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) \Rightarrow g(a) = f(a)$ and $g(b) = f(b)$. By Rolle's theorem we know that it exists $c \in (a,b)$ such that $g'(c) = 0$. Now:

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \iff f'(x) = g'(x) + \frac{f(b)-f(a)}{b-a} \Rightarrow f'(c) = g'(c) + \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

□

Theorem 10.6 (Second order optimality conditions). *Let $f \in \mathcal{C}^2(\mathbb{R})$ and $f'(x) = 0$*

- *If $f''(x) > 0 \Rightarrow x$ is a local minimum*
- *If $f''(x) < 0 \Rightarrow x$ is a local maximum*
- *If $f''(x) = 0 \Rightarrow x$ is an inflection point*

Proof. Let $f'(x) = 0$ and $f''(x) > 0$. It exists $\varepsilon > 0$ such that $\frac{f'(y)}{y-x} > 0$ for all $y : 0 < |x-y| < \varepsilon$. Then $y \in (x-\varepsilon, x+\varepsilon) \setminus \{x\}$. Now we have three cases

- $y \in (x, x+\varepsilon)$, $y > x \Rightarrow f'(y) > 0 \Rightarrow f$ is increasing on $(x, x+\varepsilon)$
- $y \in (x-\varepsilon, x)$, $y < x \Rightarrow f'(y) < 0 \Rightarrow f$ is decreasing on $(x-\varepsilon, x)$
- $f(x) \le f(y)$ for all $y \in (x-\varepsilon, x+\varepsilon)$

□

Definition 10.5 (Convex vector space). Let A be a vector space, $x, y \in A$ and $t \in [0,1]$. Then A is convex if $tx + (1-t)y \in A$.

Definition 10.6 (Convex function). $f : \mathbb{R} \to \mathbb{R}[a, b]$ is convex if for all $x, y \in [a, b]$, $t \in [0, 1]$, then

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

Theorem 10.7. *If f is convex, then global minimum is local minimum.*

Proof. Let x^* a local minimum of f convex. We have to show that $f(x^*) \le f(y)$, for all y. It exists $\varepsilon > 0$ such that $f(x^*) \le f(x)$ for all $x \in (x^* - \varepsilon, x^* + \varepsilon)$. Let y, $t < \frac{\varepsilon}{|y - x^*|}$ and $x = (1-t)x^* + ty$. Then, $|x - x^*| = |(1-t)x^* + ty - x*| = t|y - x^*| < \varepsilon$. Now

$$f(x^*) \le f(x) = f(ty + (1-t)x^*) \le tf(y) + (1-t)f(x^*) \Rightarrow tf(x^*) \le tf(y) \Rightarrow f(x^*) \le f(y)$$

$\square$

Theorem 10.8 (Gradient inequality). $f \in \mathcal{C}^1$ *is convex* $\iff f(x) \ge f(y) + f'(y)(x - y)$

Proof. $f(x + t(y - x)) \le (1-t)f(x) + tf(y)$, for all x, y and for all $t \in [0, 1]$.

$\Rightarrow$ We divide by t, and then take $t \to 0$

$$f(y) \ge f(x) + \frac{f(x + t(y-x)) - f(x)}{t(y-x)}(y - x) \Rightarrow f(y) \ge f(x) + f'(x)(y - x)$$

$\Leftarrow$ Let $x \ne y$, $t \in [0, 1]$ and $z = tx + (1-t)y$ $f(x) \ge f(z) + f'(z)(x - z)$ and $f(y) \ge f(z) + f'(z)(y - z)$, now, multiplying the first inequality by t, the second by $(1-t)$ and adding them

$$tf(x) + (1-t)f(y) \ge tf(z) + tf'(z)(x - z) + (1-t)f(z) + t(1-t)f'(z)(y - z) =$$

$$= f(z) + (-f'(z)z + f'(z)(tx + (1-t)y)) = f(z) = f(tx - (1-t)y) \Rightarrow f \text{ is convex}$$

$\square$

Theorem 10.9 (Newton's method). *Newton's method is a way to approximate a local minimum or maximum of a function. $x^{(0)}$ is the initial guess of a local minimum $\Rightarrow x^{(n+1)} = x^{(n)} - \frac{f'(x^{(n)})}{f''(x^{(n)})}$ is a more precise approximation.*

Proof. $f \in \mathcal{C}^2(\mathbb{R})$. Let $g(\varepsilon) = f(x) + f'(x)\varepsilon + \frac{1}{2}f''(x)\varepsilon^2 \approx f(x + \varepsilon)$. Then, $g'(\varepsilon) = f'(x) + f''(x)\varepsilon = 0$ if and only if $\varepsilon = -\frac{f'(x)}{f''(x)} \Rightarrow f'(x - \frac{f'(x)}{f''(x)}) \approx 0$. $\square$

Theorem 10.10 (Taylor' series). *Taylor series are a way to approximate a function. Let $f \in \mathcal{C}^\infty(\mathbb{R})$, then its Taylor series around point x_0 is $T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.*

Definition 10.7. If $f(x) = T_f(x)$ for all x, then $f(x)$ is analytic.

Theorem 10.11 (Taylor's theorem). *$f \in \mathcal{C}^{n+1}(\mathbb{R})$, then it exists $\xi \in (a, x)$ such that*

$$f(x) = \sum_{k=0}^{n} \left(\frac{f^{(k)}(a)}{k!}(x-a)^k \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Where $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = O((x-a)^{n+1})$ is the error of approximation.

11 Integrals

Definition 11.1 (Partition). Let $f : \mathbb{R} \to \mathbb{R}[a, b]$, $\Delta = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ is a partition of $[a, b]$. Let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. Then

$$L_\Delta(f) = \sum_{k=1}^{n}(x_k - x_{k-1})m_k, \quad U_\Delta(f) = \sum_{k=1}^{n}(x_k - x_{k-1})M_k$$

$L(f) = \sup\{L_\Delta(f)\}$ and $U(f) = \inf\{U_\Delta(f)\}$ are the lower and upper Darboux sums.

Theorem 11.1 (Ross' theorem). *f bounded on $[a, b] \Rightarrow L(f) \leq U(f)$*

Definition 11.2 (Darboux (Riemann) integral). If $L(f) = U(f)$, then f is Darboux integrable and we call the integral $L(f) = U(f) = \int_a^b f(x)dx$.

Proposition 11.2. *f continuous and bounded $\Rightarrow f$ is Riemann integrable.*

Proof. f is continuous and bounded, then f is uniformly continuous. For all $\varepsilon > 0$ exists Δ such that $\varphi_\Delta(x) \leq f(x) \leq \psi_\Delta(x)$ for all $x \in [a, b]$, and $\varphi_\Delta(x) - \psi_\Delta(x) \leq \frac{\varepsilon}{b-a}$. By uniform continuity we have that it exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for all $x, y \in [a, b]$ such that $|x - y| < \delta$. Now, choosing Δ sufficiently dense, we have that $x_k - x_{k-1} < k$ for all k. Then

$$|f(x_k) - f(x_{k-1})| < \frac{\varepsilon}{b-a} \Rightarrow M_k - m_k < \frac{\varepsilon}{b-a}$$

Now:

$$U_\Delta(f) - L_\Delta(f) = \sum_{k=0}^{n}(x_k - x_{k-1})(M_k - m_k) \leq \sum_{k=0}^{n}(x_k - x_{k-1})\frac{\varepsilon}{b-a} = (x_n - x_0)\frac{\varepsilon}{b-a} = (b-a)\frac{\varepsilon}{b-a} = \varepsilon$$

This leads to

$$U(f) \leq U_\Delta(f) \leq L_\Delta(f) + \varepsilon \leq L(f) + \varepsilon \leq U(f) + \varepsilon$$

□

Proposition 11.3 (Properties of integrals). *$f, g : \mathbb{R} \to \mathbb{R}[a, b]$ integrable, $\lambda \in \mathbb{R}$ and $c \in [a, b]$. Then:*

(1) $\int_a^b (\lambda f)(x)dx = \lambda \int_a^b f(x)dx$

(2) $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(4) *If* $f(x) \leq g(x)\ \forall\ x \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

Proof. Proof of 2. Other proofs can be done similarly.
f, g integrable, then $L(f) = U(f)$ and $L(g) = U(g)$. Moreover, for all $\varepsilon > 0$ exists Δ such that $L_\Delta(f) \geq L(f) - \frac{\varepsilon}{2}$ and $L_\Delta(g) \geq L(g) - \frac{\varepsilon}{2}$. This means that for all ε

$$L(f+g) \geq L_\Delta(f+g) \geq L_\Delta(f) + L_\Delta(g) \geq L(f) + L(g) - \varepsilon \iff L(f+g) \geq L(f) + L(g)$$

Similarly, for all $\varepsilon > 0$ exists Δ such that $U_\Delta(f) \leq U(f) + \frac{\varepsilon}{2}$ and $U_\Delta(g) \leq U(g) + \frac{\varepsilon}{2}$. Then, for all ε

$$U(f+g) \leq U_\Delta(f+g) \leq U_\Delta(f) + U_\Delta(g) \leq U(f) + U(g) + \varepsilon \iff U(f+g) \leq U(f) + U(g)$$

Now we can see that

$$L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g) = L(f) + L(g)$$

From that it follows that $L(f) + L(g) = L(f+g) = U(f+g) = U(f) + U(g)$, thus proving proposition 2. □

Theorem 11.4. *If f is monotonic or continuous, then f is integrable.*

Theorem 11.5. *If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Proof. $-|f(x)| \leq f(x) \leq |f(x)| \Rightarrow -\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx \Rightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$ □

Theorem 11.6 (Mean value theorem for integrals). *$f, g : \mathbb{R} \to \mathbb{R}[a, b]$ continuous, $g(x) \geq 0$ for all $x \in [a, b] \Rightarrow$ it exists $c \in [a, b]$ such that* $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$

Proof. Minimum and maximum of f and g exists, thus $m \leq f(x) \leq M$. Multiplying by $g(x)$ we get $mg(x) \leq f(x)g(x) \leq Mg(x)$. Now we can apply integral, and for some $\mu \in [m, M]$ the following holds

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx \Rightarrow \int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$$

Since f continuous, by intermediate value theorem we can say that it exists $c \in [a, b]$ such that $f(c) = \mu$. □

Corollary 11.6.1. *$f : \mathbb{R} \to \mathbb{R}[a, b]$ continuous, then it exists $c \in [a, b]$ such that* $\int_a^b f(x)dx = f(c)(b - a)$.

Proof. We choose $g(x) = 1 \Rightarrow \int_a^b f(x)1dx = f(c)\int_a^b 1dx = f(c)(b - a)$. □

12 Antiderivatives (or indefinite integrals)

Definition 12.1 (Antiderivative). $F : \mathbb{R} \to \mathbb{R}[a, b]$ differentiable, is the antiderivative of $f : \mathbb{R} \to \mathbb{R}[a, b]$ if $F'(x) = f(x)$. We write $\int f(x)dx$.

Theorem 12.1 (Fundamental theorem of calculus). *$f : \mathbb{R} \to \mathbb{R}[a,b]$ continuous, then f has an unique antiderivative $F(x) = \int_a^x f(t)dt$, with $F(a) = 0$.*

Proof. W.l.o.g. we can assume that $x < y$, for $x, y \in [a,b]$. Then

$$\frac{F(y) - F(x)}{y - x} = \frac{1}{y-x}\left(\int_a^y f(t)dt - \int_a^x f(t)dt\right) = \frac{1}{y-x}\int_x^y f(t)dt$$

We also know that it exists $\xi_y \in [a,b]$ such that $(y-x)f(\xi_y) = \int_x^y f(t)dt$. Since f is continuous, then $\lim\limits_{y\to x} f(\xi_y) = f(\lim\limits_{y\to x} \xi_y) = f(x)$. Now

$$F'(x) = \lim_{y\to x}\frac{F(y)-F(x)}{y-x} = \lim_{y\to x}\frac{1}{y-x}\int_x^y f(t)dt = \lim_{y\to x} f(\xi_y) = f(x)$$

Moreover, we can conclude that $F(a) = \int_a^b f(t)dt = 0$. □

Corollary 12.1.1. *$f : \mathbb{R} \to \mathbb{R}[a,b]$, F antiderivative of f, then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.*

Proof. $F(x) = \int_a^x f(t)dt + c$, $F(a) = 0 + c$ and $F(b) = \int_a^b f(t)dt + c$. Then, $F(b) - F(a) = \int_a^b f(t)dt$. □

Theorem 12.2 (Integration by parts). *$f, g : \mathbb{R} \to \mathbb{R}\int a, b \in \mathcal{C}^1([a,b])$, then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Theorem 12.3 (Integration by substitution). *$f : \mathbb{R} \to \mathbb{R}\int a, b$ continuous $g : \mathbb{R} \to \mathbb{R}\int a, b \in \mathcal{C}^1([a,b])$, then:*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$$