Università della Svizzera italiana
Year 2015–2016

Calculus

Course Notes

Amedeo Zucchetti
February 27, 2017

Contents

1 Sets, groups and fields

Definition 1.1 (Natural numbers). The set of natural numbers is defined with the following properties

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition 1.2 (Group). A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists!\ 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists!\ (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abelian if the following rule is satisfied too

(v) Commutativity: $a \circ b = b \circ a$

***Example* 1.2.1.** $(\mathbb{Z}_2, \oplus)$ is an abelian group (where $\mathbb{Z}_2 = \{0, 1\}$ and $\oplus$ is exclusive or)

(i) Closure: $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$

(ii) Associativity: we have two elements, so we don't need to prove it

(iii) Identity: $0 \Rightarrow 0 \oplus 0 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$

(iv) Inverse: $(-1) = 1, (-0) = 0 \Rightarrow 1 \oplus 1 = 0$, $0 \oplus 0 = 0$

(v) Commutativity: $1 \oplus 0 = 1 = 0 \oplus 1$

***Example* 1.2.2.** $(\mathbb{N}, +)$ is not a group, since it doesn't have the identity element.

Definition 1.3 (Field). Given a set X, then $(X, +, \cdot)$ is a field if the following are satisfied for all $a, b, c \in X$

(i) $a + b \in X$ and $a \cdot b \in X$

(ii) $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists!\ 0 \in X : a + 0 = 0 + a = a$ and $\exists!\ 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!\ (-a) \in X : a + (-a) = (-a) + a = 0$ and $\forall\ a \neq 0,\ \exists!\ a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$ and $a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

***Example* 1.3.1.** Proposition: $a \cdot b = 0 \Rightarrow$ either a or b are equal to 0.
Proof: we suppose $b \neq 0$, meaning that $0 = 0 \cdot b^{-1} = (a \cdot b) \cdot b^{-1} = a \cdot (b \cdot b^{-1}) = a \cdot 1 = a \Rightarrow a = 0$ (the same can be done supposing $a \neq 0$).

***Example* 1.3.2.** Proposition: $a \cdot 0 = 0$.
Proof: $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow$ subtracting $(-\ a \cdot 0)$ from both sides we get $a \cdot 0 + (-\ a \cdot 0) = a \cdot 0 + a \cdot 0 + (-\ a \cdot 0) \iff 0 = a \cdot 0$

Definition 1.4 (Rational numbers). $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark. $(\mathbb{Q}, +, \cdot)$ is a field.

Definition 1.5 (Ordered Field). Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

***Example* 1.5.1.** Let's take $(\mathbb{Q}, +, \cdot, \leq)$, $a, b \in \mathbb{Q}$. We want to show that if $a \leq b$, then $(-b) \leq (-a)$.

Definition 1.6 (Countable Infinite). A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark. Let A, B sets, then

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \leq |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition 1.1. $\mathbb{Z}$ *is countably infinite*

Proposition 1.2. $\mathbb{Q}$ *is countably infinite.*

Proposition 1.3. $\mathbb{R}$ *is not countable.*

Proposition 1.4. $|\mathbb{R}| = |\mathbb{R}^2|$

Definition 1.7 (Power set). Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition 1.5. $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition 1.6. $\sqrt{2} \notin \mathbb{Q}$

Definition 1.8 (Bounds). Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition 1.9 (Supremum and infimum). Let A be a set

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition 1.10 (Maximum and minimum). Let A be a set

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition 2.1 (Topology). Let X be a set. Then $\tau \subseteq 2^X$ is a topology if

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

***Example* 2.1.1.** Let $X = \{1, 2, 3, 4\}$

1. $\tau = \{\emptyset, X\}$ is topology, since $\emptyset \cup X = X \in \tau$ and $\emptyset \cap X = \emptyset \in \tau$.
2. $\tau = \{\emptyset, \{2\}, \{2, 3\}, X\}$ is topology. The cases with $\emptyset$ and X are trivial. $\{2\} \cup \{2, 3\} = \{2, 3\} \in \tau$ and $\{2\} \cap \{2, 3\} = \{2\} \in \tau$.
3. $A = \{\emptyset, \{2\}, \{3\}, X\}$ is not a topology. In fact, $\{2\} \cup \{3\} = \{2, 3\} \notin A$.

Definition 2.2 (Topological space). Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition 2.3 (Neighborhood in a topological space (X, τ)). A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition 2.4 (Metric). Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

***Example* 2.4.1.** $d(x, y) = |x - y|$

***Example* 2.4.2.** $d(x, y) = ||x - y||_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$

Definition 2.5 (Metric space). Let X be a set, d be a metric, then (X, d) is a metric space.

Definition 2.6 (Ball in a metric space (X, d)). $B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition 2.7 (Open set in a topological space (X, τ)). A set U is open in (X, τ) if $U \in \tau$.

Definition 2.8 (Open set in a metric space (X, d)). A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition 2.9 (Closed set). $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition 2.1. *Let $S = (X, x)$ be a space (x a metric or a topology), then*

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup_\alpha A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap_{i=1}^{n} A_i$ is open in S (any finite intersection of any open set is also open)

3 Sequences

Definition 3.1 (Sequence). A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$. The elements of a sequence can be listed in an ordered set with repetition

$$(x_n) = (x_1, x_2, x_3, x_4, \ldots)$$

***Example* 3.1.1.** $a_n = n \Rightarrow (a_n) = (1, 2, 3, 4, 5, \ldots)$

***Example* 3.1.2.** $b_n = \frac{1}{n} \Rightarrow (b_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

***Example* 3.1.3.** $c_n = (-1)^n \Rightarrow (c_n) = (1, -1, 1, -1, \ldots)$

Definition 3.2 (Cauchy sequence). A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$. That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

***Example* 3.2.1.** $x_n = \frac{1}{n}$ in $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. We have to find, for each ε, an N that satisfies the definition of Cauchy sequence. Let's take $N \leq n \leq m$. Thanks to the triangle inequality, we can first find that:

$$d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |-x_m| = |x_n| + |x_m| = \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}$$

Since $N \leq n \leq m$, then we have $\frac{1}{m} \leq \frac{1}{n} \leq \frac{1}{N}$:

$$\frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

Now, in order to satisfy the definition we must have $\frac{2}{N} \leq \varepsilon$, thus $\frac{2}{\varepsilon} \leq N$. This means, starting from $N = \frac{2}{\varepsilon}$ all $d(x_n, x_m)$ will be smaller than ε. In fact, if we take the previous inequality:

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{2}{N} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$$

Note that it is not important if $d(x, y) < \varepsilon$ or $d(x, y) \leq \varepsilon$.

Definition 3.3 (Convergence in metric space). (X, d) is a metric space. A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

***Example* 3.3.1.** $x_n = \frac{1}{n}$ in $\mathbb{R}$ converges to 0. We take $N \leq n$ and $N = \frac{1}{\varepsilon}$

$$|x_n - 0| = |x_n| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Definition 3.4 (Convergence in topological space). (X, τ) is a topological space. A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$. That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

***Example* 3.4.1.** Let's take $X = \{a, b, c\}$, $(x_n) = (a, b)$ and $\tau = \{\emptyset, \{a\}, X\}$.

- a is not a limit of (x_n), in fact $\{a\}$ contains a, but doesn't contain the tail of (x_n)
- b and c are limits of (x_n), in fact $b \in X$ and $c \in X$, and X contains (x_n) (and its tail too)

Proposition 3.1. *$x_n \to x$ in (X, d) $\iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.*

Theorem 3.2. *If a sequence converges to a limit in a metric space, then the limit is unique.*

Remark. This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition 3.3. *$x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.*

Proposition 3.4 (Properties of real sequences). *For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties*

(i) $\lim\limits_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim\limits_{n\to\infty} x_n + \beta \lim\limits_{n\to\infty} y_n$

(ii) $\lim\limits_{n\to\infty} x_n x_y = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$

(iii) $\lim\limits_{n\to\infty} \dfrac{x_n}{x_y} = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

***Example* 3.4.1.** Knowing that $\frac{1}{n} \to 0$, show that $\frac{1}{n^2} \to 0$.

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0 \cdot 0 = 0$$

***Example* 3.4.2.** Find the limit of the sequence $\frac{2n^2-3n+2}{n^2+n-1}$.

$$\lim_{n\to\infty} \frac{2n^2 - 3n + 2}{n^2 + n - 1} = \frac{\lim(2n^2 - 3n + 2)}{\lim(n^2 + n - 1)} = \frac{\lim 2 - \lim \frac{3}{n} + \lim \frac{2}{n^2}}{\lim 1 + \lim \frac{1}{n} - \lim \frac{1}{n^2}} = \frac{2 - 0 + 0}{1 + 0 - 0} = 2$$

Definition 3.5 (Bounded sequence). A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

***Example* 3.5.1.** $a_n = \frac{1}{n}$ is bounded. We can take $c = 1$, $\left|\frac{1}{n}\right| = \frac{1}{n} \leq 1$, being $a_1 = 1$ the $\sup(a_n)$.

***Example* 3.5.2.** $b_n = (-1)^n$ is bounded. In fact, the values of the sequence are always 1 and -1. If we take $c = 1$, then $|b_n| = 1 \leq 1$.

***Example* 3.5.3.** $x_n = n$ is not bounded, we can prove it by contradiction. We suppose it exists a c such that $|x_n| \leq c$. If we take $x_{c+1} = c + 1$, we have $c + 1 \leq c \iff 0 \leq 1$ contradiction. This means x_n is not bounded.

Definition 3.6 (Monotonic sequence). A sequence is monotonic if

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

***Example* 3.6.1.** $a_n = \frac{1}{n}$ is monotonic decreasing. In fact, $\frac{1}{n+1} \leq \frac{1}{n}$, then $a_{n+1} \leq a_n$.

***Example* 3.6.2.** $b_n = n$ is monotonic increasing. In fact, $b_n = n$ and $b_{n+1} = n+1$. Since $n \leq n+1$, then $b_n \leq b_{n+1}$.

***Example* 3.6.3.** $c_n = (-1)^n$ is not monotonic. We can take $n = 1$, then $a_1 \leq a_2 \not\leq a_3$.

Theorem 3.5. *If a sequence monotonic and bounded, then the sequence is convergent.*

Definition 3.7 (Limit superior and inferior). If (x_n) is a sequence, then

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

***Example* 3.7.1.**

Definition 3.8 (Subsequence). $(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

***Example* 3.8.1.** $x_n = (-1)^n \cdot n$. We take $k = 2n$, then the subsequence $(x_{n_k}) = (x_{2n})$ of (x_n) takes all the even indexes n of (x_n):

$$\begin{array}{lcccccccccl} (x_n) & = & (& -1, & 2, & -3, & 4, & -5, & 6, & \dots &) \\ (x_{2n}) & = & (& & 2, & & 4, & & 6, & \dots &) \end{array}$$

Theorem 3.6. *If $x_n \to x$, then $x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.*

Definition 3.9 (Dominant term). x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem 3.7. *Every sequence has a monotonic subsequence.*

Theorem 3.8 (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

Definition 3.10. $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition 4.1 (Series). (x_n) is sequence. $s_n = \sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition 4.2 (Convergence of series). $s_n = \sum_{k=1}^{n} x_k$ a series. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{n=1}^{\infty} x_k$.

***Example* 4.2.1.** The following are famous convergent series

- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$
- Geometric: $\sum_{n=1}^{\infty} a^n = \begin{cases} \infty & |a| \geq 1 \\ \frac{1}{1-a} & |a| < 1 \end{cases}$
- Exponential: $\sum_{n=1}^{\infty} \frac{1}{n!} = e$
- Leibniz: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{4}$

Definition 4.3 (Absolute convergence of series). $s_n = \sum_{k=1}^{n} x_k$ is a series. s_n converges absolutely if

$$\sum_{n=1}^{\infty} |x_k| < \infty$$

Proposition 4.1. *Absolute convergence $\Rightarrow$ convergence. If $\sum_{n=1}^{\infty} |x_k| < \infty$, then $\sum_{n=1}^{\infty} x_k < \infty$.*

Definition 4.4 (Cauchy criterion for series). $s_n = \sum_{k=1}^{n} x_k$, and $\sum_{n=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall\ N \leq m \leq n \Rightarrow |s_n - s_m| = \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| = \left|\sum_{k=m}^{n} x_k\right| < \varepsilon$$

Proposition 4.2 (Comparison test). *For x_n, y_n sequences and $x_n \geq 0$*

(i) If $\sum_{n=1}^{\infty} x_k < \infty$ and $|y_n| \leq x_n \Rightarrow \sum_{n=1}^{\infty} y_k < \infty$

(ii) If $\sum_{n=1}^{\infty} x_k = +\infty$ *and* $x_n \leq y_n \Rightarrow \sum_{n=1}^{\infty} y_k = +\infty$

***Example* 4.2.1.** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, in fact

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Proposition 4.3 (Ratio test). *For* x_n *sequence,* $x_n \neq 0$ *and* $s_n = \sum_{k=1}^{n} x_k$ *series:*

(i) s_n *converges absolutely if* $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n *diverges if* $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

***Example* 4.3.1.** $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$

$$\left|\frac{(-\frac{1}{3})^{n+1}}{(-\frac{1}{3})^n}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3} \Rightarrow \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1 \Rightarrow \text{ converges absolutely}$$

***Example* 4.3.2.** $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

- Ratio test:

$$\limsup_{n\to\infty} \left|\frac{n+1}{(n+1)^2+3}\frac{n^2+3}{n}\right| = \limsup_{n\to\infty} \frac{n+1}{(n+1)^2+3}\frac{n^2+3}{n} = 1, \text{ no information}$$

- Comparison test:

$$\frac{n}{n^2+3n^2} \leq \frac{n}{n^2+3} \Rightarrow \frac{n}{n^2+3n^2} = \frac{n}{4n^2} = \frac{1}{4}\frac{n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4}\frac{n}{n^2} \Rightarrow +\infty = \frac{1}{4}\sum_{n=1}^{\infty}\frac{n}{n^2} \leq \sum_{n=1}^{\infty}\frac{n}{n^2+3}$$

The series diverges. Sometimes one test can give more information than others.

Proposition 4.4 (Root test). *Let* $s_n = \sum_{k=1}^{n} x_k$ *a series,* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$*:*

(i) s_n *converges absolutely if* $\alpha < 1$

(ii) s_n *diverges if* $\alpha > 1$

***Example* 4.4.1.** $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n \Rightarrow \limsup_{n\to\infty}\sqrt[n]{-\frac{1}{3}^n} = \limsup_{n\to\infty}\frac{1}{3} = \frac{1}{3} < 1$, the series converges absolutely.

***Example* 4.4.2.** $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$ converges, in fact

$$\sqrt[n]{2^{(-1)^n-n}} = \begin{cases} 2^{\frac{1}{n}-1} & \text{if } n \text{ even} \\ 2^{-\frac{1}{n}-1} & \text{if } n \text{ odd} \end{cases} \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}-1} = \lim_{n\to\infty} 2^{-\frac{1}{n}-1} = \frac{1}{2} < 1$$

5 Functions and continuity

Definition 5.1 (Image). Given a function $f : X \to Y$, the image of f is defined as $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition 5.2 (Preimage). Given a function $f : X \to Y$, the preimage of f is defined as $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition 5.3 (Continuity in metric space). $f : (X, d_x) \to (Y, d_y)$ is continuous at $x \in X$ if

$$\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 :\ \forall\ x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

***Example* 5.3.1.** Let's take $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\frac{1}{x} & \text{otherwise} \end{cases}$
We want to prove that f is continuous in 0. Let $\varepsilon > 0$, then $|f(x) - f(0)| = |f(x) - 0| = |f(x)| \leq x^2$. If we take $\delta = \sqrt{\varepsilon}$, then

$$|x - 0| < \delta \Rightarrow x^2 < \delta \Rightarrow |f(x) - f(0)| \leq x^2 < \delta^2 = \varepsilon \Rightarrow f \text{ is continuous in } 0$$

Remark. Continuity can also be defined as follows

$$\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition 5.4 (Continuity in topological space). $f : (X, \tau_x) \to (Y, \tau_y)$ is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

***Example* 5.4.1.** Let's take (M, τ_m), (N, τ_n), $M = N = \{1, 2\}$, $\tau_m = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_m = \{\emptyset, \{1, 2\}\}$.

- Let $f : (M, \tau_m) \to (N, \tau_n)$, such that $f(1) = 2$ and $f(2) = 1$:

$$PreIm_f(\emptyset) = \emptyset \in \tau_m, PreIm_f(\{1, 2\}) = \{1, 2\} \in \tau_m \Rightarrow f \text{ is continuous in all } x \in M$$

- Let $g : (N, \tau_n) \to (M, \tau_m)$, such that $f(1) = 2$ and $f(2) = 1$:

$$PreIm_g(\{1\}) = \{2\} \notin \tau_n \Rightarrow g \text{ is not continuous}$$

Proposition 5.1. *Continuous functions map open sets into open sets.*

$$\textit{If } f:(X,d_x)\to(Y,d_y) \textit{ continuous, then } PreIm_f(A) \textit{ is open, for all } A\subseteq Y \textit{ open}$$

Theorem 5.2. *Continuous functions map limits to limits*

$$f \textit{ continuous, } x_n \to x \iff f(x_n)\to f(x)$$

***Example* 5.2.1.** Let's take $f(x)=2x^2+1$ and $\lim_{n\to\infty} x_n = x$. Then:

$$\lim_{n\to\infty} 2x_n^2+1 = 2\left(\lim_{n\to\infty} x_n\right)^2+1 = 2x^2+1$$

This means that for $x_n\to x$, then $f(x_n)\to f(x)$. Therefore, f is continuous.

Proposition 5.3. *$f,g:\mathbb{R}\to\mathbb{R}$ continuous at $x \Rightarrow f+g$, $f\cdot g$ and $\frac{f}{g}$ (for $g(x)\neq 0$) are continuous at x.*

Proposition 5.4. *f continuous at x and g continuous at $f(x)\Rightarrow g\circ f = g(f(x))$ is continuous at x.*

Definition 5.5 (Contraction). $f:(X,d)\to(X,d)$ is a contraction $\iff$ it exists $0\le c<1$ such that $d(f(x),f(y))\le cd(x,y)$, for all $x,y\in X$.

Theorem 5.5 (Banach fixed point). *Let's take (X,d) complete (Cauchy $\iff$ convergence) and $f:(X,d)\to(X,d)$ a contraction, then*

(i) $\exists!\ x^*\in X: f(x^*)=x^*$

(ii) $x_0\in X$, $x_{n+1}=f(x_n)\Rightarrow x_n\to x^*$

Definition 5.6 (Convergence of a function). f converges to c at $x_0 \iff$ for all (x_n) such that $x_n\to x_0$ we have $f(x_n)\to c$. We write $\lim_{x\to x_0} f(x)=c$. Moreover

- f converges from above if, for all (x_n), then $x_0<x_n$. We write $\lim_{x\to x_0^+} f(x)=c$.
- f converges from below if, for all (x_n), then $x_n<x_0$. We write $\lim_{x\to x_0^-} f(x)=c$.

***Example* 5.6.1.** Let $f(x)=\frac{1}{x}\Rightarrow \lim_{x\to 0^+}\frac{1}{x}=+\infty$, $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

***Example* 5.6.2.** Let $f(x)=floor(x)\Rightarrow \lim_{x\to 1^+} floor(x)=1$, $\lim_{x\to 1^-} floor(x)=0$, but

$$\lim_{x\to \frac{1}{2}^+} floor(x)=\frac{1}{2}=\lim_{x\to\frac{1}{2}^-} floor(x)$$

Proposition 5.6. *f continuous at $a \iff \lim_{x\to a} f(x)=f(a)$*

Proposition 5.7. $\lim_{x\to a}(fg)(x)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

6 Continuous functions and intervals

Definition 6.1 (Bounded function). $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

***Example* 6.1.1.** $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin(x)$ is bounded on $\mathbb{R}$, since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$.

Theorem 6.1 (Extreme value). *If $f : \mathbb{R} \to \mathbb{R}[a, b]$ is continuous, then:*

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$, meaning that

$$\exists\ x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}),\ \forall\ x \in [a, b]$$

Theorem 6.2 (Intermediate value). *f continuous on $[a, b]$, $f(a) < c < f(b) \Rightarrow\ \exists\ x \in [a, b] : f(x) = c$.*

Definition 6.2 (Darboux function). A Darboux function is a function that satisfies the intermediate value property.

Proposition 6.3. *Continuous implies Darboux, but not the opposite.*

***Example* 6.3.1.** $f(x) = \begin{cases} \sin(\frac{1}{x}) & x > 0 \\ 0 & x = 0 \end{cases}$ is a Darboux function, but it is not continuous.

Proposition 6.4. *Continuous functions map intervals to intervals.*

Definition 6.3 (Connectedness). Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold

- There exists $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exists $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$

N.B.: here $\overline{U}$ doesn't mean complementary set of U, but set closure of U. That is, the smallest closed set containing U.
A set is connected if it is not disconnected.

Proposition 6.5. *Continuous functions preserve connectedness.*

$$f : (X, \tau_x) \to (Y, \tau_y), A \subseteq X \text{ connected in } (X, \tau_x) \Rightarrow Im(A) \subseteq Y \text{ is connected in } (Y, \tau_y)$$

7 Uniform continuity

Definition 7.1 (Uniform continuity). $f : (X, d_x) \to (Y, d_y)$ is uniformly continuous on X if

$$\forall\ \varepsilon > 0 \quad \exists\ \delta_\varepsilon > 0 :\ \forall\ x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

***Example* 7.1.1.** $f(x) = \frac{1}{x^2}$ in $[a, +\infty)$, $a > 1$. To show that f is uniformly continuous, we have to show that for all $\varepsilon > 0$ exists $\delta_\varepsilon > 0$ such that for all x, y such that $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Let $\varepsilon > 0$ and $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \frac{(x+y)(x-y)}{x^2y^2}$. Then, since $a \le x, y \ \ \forall\ x, y$:

$$\frac{(x+y)}{x^2y^2} = \frac{x}{x^2y^2} + \frac{y}{x^2y^2} \le \frac{2}{a^3}$$

We chose $\delta = \frac{\varepsilon a^3}{2}$, then:

$$\forall\ x, y \ge a : |x - y| < \delta \Rightarrow |f(x) - f(y)| = |x - y| \left| \frac{x+y}{x^2y^2} \right| < \delta \frac{2}{a^3} = \frac{\varepsilon a^3}{2} \frac{2}{a^3} = \varepsilon$$

This means f is uniformly continuous.

Remark. Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall\ \varepsilon > 0 \quad \exists\ \delta_{\varepsilon, x_0} > 0 :\ \forall\ x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem 7.1. *f continuous on A, closed and bounded $\Rightarrow$ f is uniformly continuous on A.*

Theorem 7.2. *f uniformly continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow$ $f(s_n)$ is Cauchy sequence.*

***Example* 7.2.1.** $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$. In fact, $s_n = \frac{1}{n}$ is Cauchy, but $f(s_n) = n^2$ is not Cauchy.

Definition 7.2 (Sequence of functions). $(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

***Example* 7.2.1.** $f_n(x) = \frac{x}{n} \to f(x) = 0$

Definition 7.3 (Pointwise convergence). f_n converges pointwise to $f \iff \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall\ \varepsilon > 0, x \in S\ \exists\ N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

***Example* 7.3.1.** $f_n(x) = x^n$, $x \in [0, 1] \Rightarrow f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$. f_n is continuous and f is discontinuous.

Definition 7.4 (infinite norm). $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)|\}$

Definition 7.5 (Uniform convergence). f_n converges uniformly to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

***Example* 7.5.1.** Let $f_n(x) = (1 - |x|)^n$, $x \in (-1, 1)$. Then f converges pointwise (but not uniformly) to $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. In fact

- Pointwise convergence
  For $x = 0$, $f_n(x) = (1-0)^n = 1$, then $\lim_{n\to\infty} f_n = \lim_{n\to\infty} 1 = 1$. For $x \neq 0$, $|x| < 1$. This means $1 - |x| < 1$, then $\lim_{n\to\infty} (1 - |x|)^n = 0$.
- Uniform convergence
  We assume $f_n \xrightarrow{unif.} f$ and we take $\varepsilon = \frac{1}{2}$. Then it exists N such that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $x \in (-1, 1)$. Let's take $x = 1 - 2^{-\frac{1}{n}}$, then $1 - x = 2^{-\frac{1}{n}}$. Thus $(1 - x)^n = (2^{-\frac{1}{n}})^n = \frac{1}{2} \not< \frac{1}{2} = \varepsilon$. Contradiction, f doesn't converge uniformly to f.

Theorem 7.3. *Uniform limit of a continuous function is continuous.*

$$f_n(x) \textit{ continuous and } f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x) \textit{ is continuous}$$

***Example* 7.3.1.** Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n + g_n \xrightarrow{unif.} f + g$. In fact

$$\exists\ N_f :\ \forall\ x \in S\, |f_n(x) - f(x)| < \frac{\varepsilon}{2}\ \forall\ n > N_f$$

$$\exists\ N_g :\ \forall\ x \in S\, |g_n(x) - g(x)| < \frac{\varepsilon}{2}\ \forall\ n > N_g$$

We take $N = \max\{N_f, N_g\}$. Then

$$|f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \forall\ n \geq N$$

This means $f_n + g_n \xrightarrow{unif.} f + g$.

***Example* 7.3.2.** Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n g_n$ doesn't converge uniformly to fg. In fact, let $h_n(x) = \frac{x}{n}$. By contradiction we can prove h_n doesn't converge uniformly to h. Now, if we take $f_n(x) = \frac{1}{n}$ and $g_n(x) = x$ (uniformly convergent), then $f(x)g(x) = \frac{x}{n} = h(x)$ not uniformly convergent. We found a counter example.

***Example* 7.3.3.** Let $f_n(x)$ continuous on $[a, b]$, $f_n(x) \xrightarrow{unif.} f(x)$, $(x_n) \subseteq [a, b]$ and $x_n \to x$. Then, $f_n(x_n) \to f(x)$. To prove it we have to show that exists N such that for all $n \geq N$, then $|f_n(x_n) - f(x)| < \varepsilon$.

(1) $f_n \xrightarrow{unif.} f$, this means it exists N_1 such that $|f_n(y) - f(y)| < \frac{\varepsilon}{2}$, for all $n \geq N_1$ and $y \in [a, b]$. In particular, $|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$.

(2) Since $f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x)$, then $f(x)$ is continuous. Then $f(x_n) \to f(x)$, this means it exists N_2 such that for all $n \geq N_2$, then $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$.

(3) We chose $N = \max\{N_1.N_2\}$. Then:

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall\, n \geq N$$

We can conclude that $f_n(x_n) \to f(x)$.

8 Power Series

Definition 8.1 (Power series). Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.
- The series converges for some bounded interval.

Theorem 8.1. *Let* $\beta = \limsup \sqrt[n]{|a_n|}$ *and* $R = \frac{1}{\beta}$ *(*$R = \infty$ *if* $\beta = 0$*,* $R = 0$ *if* $\beta = \infty$*). Then* $\displaystyle\sum_{n=0}^{\infty} a_n x^n$

- *Converges for* $|x| < R$.
- *Diverges for* $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.

***Example* 8.1.1.** Let $a_n = 1$. We have the power series $\displaystyle\sum_{n=0}^{\infty} x^n$ and $\beta = \limsup \sqrt[n]{1} = 1$, then $R = 1$. This means the series converges for $x \in (-1, 1)$ and diverges for x such that $|x| > 1$. Moreover, it diverges for $x = 1$, since $\displaystyle\sum_{n=0}^{\infty} 1 = +\infty$, and it is not defined for $x = -1$.

***Example* 8.1.2.** Let $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n} x^n$ a power series. Then $\limsup \left|\frac{a_n}{a_{n+1}}\right| = \limsup \left|\frac{n}{n+1}\right| = 1$, then $R = 1$. For $x = 1$ we have the harmonic series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ wich diverges, for $x = -1$ we have $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n} < \infty$. We can conclude that the power series converges for $x \in [-1, 1)$.

9 Lipschitz continuity

Definition 9.1 (Lipschitz continuity). $f : (X, d_x) \to (Y, d_y)$ is Lipschitz continuous if it exists $c \in [0, +\infty)$ such that $d_y(f(x), f(x')) \leq cd_x(x, x')$.

Proposition 9.1. *Lipschitz continuity $\Rightarrow$ uniform continuity.*

Example **9.1.1.** $\sqrt{x}$ is uniformly continuous but not Lipschitz continuous over $[0, 1]$. We can prove it by contradiction. We assume $\sqrt{x}$ is Lipschitz continuous. This means it exists $c \in [0, +\infty]$ such that for $x' = 0$, then $|\sqrt{x} - \sqrt{0}| \leq c|x - 0| \iff |\sqrt{x}| \leq c|x| \iff c \geq \frac{1}{\sqrt{x}}$. For $x = 0$, then c is not defined. Contradiction.

Theorem 9.2 (Weierstrass approximation). *Every continuous function on $[a, b]$ can be uniformly approximated by polynomials on $[a, b]$*

$$\exists\ (a_n) \subseteq \mathbb{R} : p_n(x) = \sum_{k=1}^{n} a_k x^k \xrightarrow{unif.} f(x)\ \ on\ \ [a, b]$$

Theorem 9.3 (Bernstein polynomials). $b_{m,n}(x) = \binom{n}{m} x^m (1 - x)^{n-m}$

$$span\{b_{0,n}(x), \ldots, b_{n,n}(x)\} = \left\{ \sum_{k=1}^{n} a_k x^k, a_i \in R \right\}$$

Example **9.3.1.**

$$a_0 + a_1 x + a_2 x^2 = b_0(1 - x)^2 + b_1 2x(1 - x) + b_2 x^2 = b_0 + 2(b_1 - b_0) + (b_0 - 2b_1 + b_2)x^2$$

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \Rightarrow \begin{cases} b_0 = a_0 \\ b_1 = a_0 + \frac{1}{2} a_1 \\ b_2 = a_0 + a_1 + a_2 \end{cases}$$

Theorem 9.4. $f : [0, 1] \to \mathbb{R}$ *continuous, then*

- $B_n(f)(x) = \sum_{m=0}^{n} f(\frac{m}{n}) b_{m,n}(x)$
- $B_n(f)(x) \to f(x)$ *uniformly continuous on* $[0, 1]$

10 Differentiability and derivatives

Definition 10.1 (Derivative). The derivative of a function f at point a is defined as one

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\varepsilon \to 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

Definition 10.2 (Differentiability). f is differentiable if the derivative f' exists.

***Example* 10.2.1.** $f(x) = x^2 \Rightarrow \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(x+\varepsilon)^2 - x^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon(2x+\varepsilon)}{\varepsilon} = \lim_{\varepsilon \to 0} 2x+\varepsilon = 2x$

***Example* 10.2.2.** $f(x) = \sqrt{x} \Rightarrow f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt{x}\sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} =$
$\frac{1}{2\sqrt{a}}$

***Example* 10.2.3.** $f(x) = |x| \Rightarrow f'(x) = \lim_{\varepsilon \to 0} \frac{|x+\varepsilon| - |x|}{\varepsilon} = \begin{cases} \lim_{\varepsilon \to 0} \frac{x+\varepsilon-x}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = 1 & x > 0 \\ \lim_{\varepsilon \to 0} \frac{-x-\varepsilon+x}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{-\varepsilon}{\varepsilon} = -1 & x < 0 \end{cases}$
Not defined for $x = 0$.

Proposition 10.1. *f differentiable at a, then f continuous at a.*

Definition 10.3. $f \in \mathcal{C}^k(\mathbb{R})$, f is differentiable k times, and the derivatives are continuous.

Proposition 10.2. *Properties of derivatives*

- $(f+g)'(x) = f'(x) + g'(x)$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \forall\ g(x) \neq 0$
- $(g \circ f)'(x) = (g' \circ f)(x)f'(x) = g'(x)f(x)f'(x)$
- $f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))}$

***Example* 10.2.1.** Let $f(x) = e^x$, $f'(x) = e^x$ and $f^{-1}(y) = \ln(y)$. The derivative of $\ln'(y)$ is

$$\ln'(y) = \frac{1}{e^{\ln(y)}} = \frac{1}{y}$$

Definition 10.4 (Local minimizer). x^* is a local minimizer if exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - \varepsilon, x^* + \varepsilon)$. This means, $f(x^*)$ is local minimum (the smallest image in a given interval).

Theorem 10.3. *$f : \mathbb{R} \to \mathbb{R}(a,b)$ is differentiable and has a local minimum at $x \Rightarrow f'(x) = 0$.*

Theorem 10.4 (Rolle's theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on (a,b) and $f(a) = f(b) \Rightarrow$ it exists $x \in (a,b)$ such that $f'(x) = 0$.*

Theorem 10.5 (Mean value theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on $(a,b) \Rightarrow$ it exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*

Theorem 10.6 (Second order optimality conditions). *Let $f \in \mathcal{C}^2(\mathbb{R})$ and $f'(x) = 0$*

- *If $f''(x) > 0 \Rightarrow x$ is a local minimum*
- *If $f''(x) < 0 \Rightarrow x$ is a local maximum*
- *If $f''(x) = 0 \Rightarrow x$ is an inflection point*

Definition 10.5 (Convex vector space). Let A be a vector space, $x, y \in A$ and $t \in [0, 1]$. Then A is convex if $tx + (1-t)y \in A$.

Definition 10.6 (Convex function). $f : \mathbb{R} \to \mathbb{R}[a, b]$ is convex if for all $x, y \in [a, b]$, $t \in [0, 1]$, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Theorem 10.7. *If f is convex, then global minimum is local minimum.*

Theorem 10.8 (Gradient inequality). *$f \in \mathcal{C}^1$ is convex $\iff f(x) \geq f(y) + f'(y)(x-y)$*

Theorem 10.9 (Newton's method). *Newton's method is a way to approximate a local minimum or maximum of a function. $x^{(0)}$ is the initial guess of a local minimum $\Rightarrow x^{(n+1)} = x^{(n)} - \frac{f'(x^{(n)})}{f''(x^{(n)})}$ is a more precise approximation.*

***Example* 10.9.1.** $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - x$, $f'(x) = x^3 - x - 1$ and $f''(x) = 3x^2 - 1$. Solve $f'(x) = 0$, with initial guess 1:

(1) $x^{(0)} = 1$

(2) $x^{(1)} = 1 - \frac{f'(1)}{f''(1)} = 1.5$

(3) $x^{(2)} \approx 1.3478$

(4) $x^{(3)} \approx 1.3252$

(5) $x^{(4)} \approx 1.32472$

(6) $x^{(5)} \approx 1.32472$

As we can see, the last two approximation are already close.

Theorem 10.10 (Taylor' series). *Taylor series are a way to approximate a function. Let $f \in \mathcal{C}^\infty(\mathbb{R})$, then its Taylor series around point x_0 is $T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.*

***Example* 10.10.1.** $f(x) = e^x$, the Taylor series around 0 is $T_f(x) = \sum_{k=0}^{\infty} \frac{e^0}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \Rightarrow T'_f(x) = e^x$.

Definition 10.7. If $f(x) = T_f(x)$ for all x, then $f(x)$ is analytic.

***Example* 10.7.1.** $f(x) = \sin(x)$, $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$ has period of four. Then

$$T_f(x) = \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x)$$

Theorem 10.11 (Taylor's theorem). *$f \in \mathcal{C}^{n+1}(\mathbb{R})$, then it exists $\xi \in (a,x)$ such that*

$$f(x) = \sum_{k=0}^{n} \left(\frac{f^{(k)}(a)}{k!}(x-a)^k \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Where $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = O((x-a)^{n+1})$ is the error of approximation.

11 Integrals

Definition 11.1 (Partition). Let $f : \mathbb{R} \to \mathbb{R}[a,b]$, $\Delta = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}$ is a partition of $[a,b]$. Let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. Then

$$L_\Delta(f) = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k, \quad U_\Delta(f) = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k$$

$L(f) = \sup\{L_\Delta(f)\}$ and $U(f) = \inf\{U_\Delta(f)\}$ are the lower and upper Darboux sums.

***Example* 11.1.1.** $f(x) = cx$ on $[0,1] \Rightarrow L(f) = U(f) = \frac{c}{2}$

***Example* 11.1.2.** $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \Rightarrow L(f) = 0 \neq 1 = U(f)$

Theorem 11.1 (Ross' theorem). *f bounded on $[a,b] \Rightarrow L(f) \leq U(f)$*

Definition 11.2 (Darboux (Riemann) integral). If $L(f) = U(f)$, then f is Darboux integrable and we call the integral $L(f) = U(f) = \int_a^b f(x)dx$.

Proposition 11.2. *f continuous and bounded $\Rightarrow f$ is Riemann integrable.*

Proposition 11.3 (Properties of integrals). *$f, g : \mathbb{R} \to \mathbb{R}[a,b]$ integrable, $\lambda \in \mathbb{R}$ and $c \in [a,b]$. Then:*

(1) $\int_a^b (\lambda f)(x)dx = \lambda \int_a^b f(x)dx$

(2) $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(4) *If* $f(x) \leq g(x)\ \forall\ x \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

Theorem 11.4. *If f is monotonic or continuous, then f is integrable.*

Theorem 11.5. *If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Theorem 11.6 (Mean value theorem for integrals). *$f,g : \mathbb{R} \to \mathbb{R}[a,b]$ continuous, $g(x) \geq 0$ for all $x \in [a,b] \Rightarrow$ it exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$*

Corollary 11.6.1. *$f : \mathbb{R} \to \mathbb{R}[a,b]$ continuous, then it exists $c \in [a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.*

12 Antiderivatives (or indefinite integrals)

Definition 12.1 (Antiderivative). $F : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable, is the antiderivative of $f : \mathbb{R} \to \mathbb{R}[a,b]$ if $F'(x) = f(x)$. We write $\int f(x)dx$.

Theorem 12.1 (Fundamental theorem of calculus). *$f : \mathbb{R} \to \mathbb{R}[a,b]$ continuous, then f has an unique antiderivative $F(x) = \int_a^x f(t)dt$, with $F(a) = 0$.*

Corollary 12.1.1. *$f : \mathbb{R} \to \mathbb{R}[a,b]$, F antiderivative of f, then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.*

***Example* 12.1.1.** $g(x) = \frac{x^{n+1}}{n+1}$, $g'(x) = x^n$. Then $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}$.

Theorem 12.2 (Integration by parts). *$f,g : \mathbb{R} \to \mathbb{R}\int a,b \in \mathcal{C}^1([a,b])$, then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Theorem 12.3 (Integration by substitution). *$f : \mathbb{R} \to \mathbb{R}\int a,b$ continuous $g : \mathbb{R} \to \mathbb{R}\int a,b \in \mathcal{C}^1([a,b])$, then:*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$$

***Example* 12.3.1.** Substitution from left to right
Let $f(x) = \cos(x^2+1)x$, $a = 0$ and $b = 2$. We can chose $g(x) = t = x^2 + 1$, then deriving on both sides we obtain $g'(x) = dt = 2xdx$, hence $\frac{1}{2}g'(x) = \frac{1}{2}dt = xdx$. Now

$$\int_a^b f(x)dx = \int_0^2 \cos(x^2+1)xdx = \int_{0^2+1}^{2^2+1} \cos(t)\frac{1}{2}dt = \frac{1}{2}\int_1^5 \cos(t)dt = \frac{1}{2}\left(\cos(5) - \cos(1)\right)$$

***Example* 12.3.2.** Substitution from right to left
We want to solve $\int_0^1 \sqrt{1-x^2}dx$. We can chose $x = \sin(t)$, this means (deriving on both sides) that $dx =$

$\cos(t)dt$. Now

$$\int_0^1 \sqrt{1-x^2}dx = \int_{\arcsin 0}^{\arcsin(1)} \sqrt{1-\sin^2(t)}\cos(t)dt = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2(t)}\cos(t)dt = \int_0^{\frac{\pi}{2}} \cos^2(t)dt = \frac{\pi}{4}$$