Calculus – Lecture Notes

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Contents

1   Sets, groups and fields

Definition   Natural numbers $\mathbb{N}$

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}$ ($n + 1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n + 1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m + 1 = n + 1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n + 1 \in A \Rightarrow A = \mathbb{N}$

Definition   Group
A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$:

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists! 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists! (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abellian if the following rule is satisfied too:

(v) Commutativity: $a \circ b = b \circ a$

Example

(1) $(\mathbb{Z}_2, \oplus)$ is an abellian group (where $\mathbb{Z}_2 = \{0, 1\}$ and $\oplus$ is exclusive or):

   (i) Closure: $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$

   (ii) Associativity: we have two elements, so we don't need to prove it

   (iii) Identity: $0 \Rightarrow 0 \oplus 0 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$

   (iv) Inverse: $(-1) = 1, (-0) = 0 \Rightarrow 1 \oplus 1 = 0$, $0 \oplus 0 = 0$

   (v) Commutativity: $1 \oplus 0 = 1 = 0 \oplus 1$

(2) $(\mathbb{N}, +)$ is not a group, since it doesn't have the identity element.

Definition   Field
Given a set X, then $(X, +, \cdot)$ is a field if it satisfies the following properties for all $a, b, c \in X$:

(i) $a + b \in X$
$a \cdot b \in X$

(ii) $(a + b) + c = a + (b + c)$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists!0 \in X : a + 0 = 0 + a = a$
$\exists!1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!(-a) \in X : a + (-a) = (-a) + a = 0$
$\forall a \neq 0, \exists! a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$
$a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

Example   Application of field axioms

(1) If $a \cdot b = 0 \Rightarrow$ either a or b are equal to 0. We suppose $b \neq 0$:
$0 = 0 \cdot b^{-1} = (a \cdot b) \cdot b^{-1} = a \cdot (b \cdot b^{-1}) = a \cdot 1 = a \Rightarrow a = 0$ (the same can be done supposing $a \neq 0$)

(2) $a \cdot 0 = 0$:
$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow$ we subtract from both sides $(- a \cdot 0)$:
$a \cdot 0 + (- a \cdot 0) = a \cdot 0 + a \cdot 0 + (- a \cdot 0) \iff 0 = a \cdot 0$

Definition $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark $(\mathbb{Q}, +, \cdot)$ is a field.

Definition   Ordered field
Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$:

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

Example   Application of the order axioms
Let's take $(\mathbb{Q}, +, \cdot, \leq)$, $a, b \in \mathbb{Q}$. We want to show that if $a \leq b$, then $(-b) \leq (-a)$:
$a \leq b \iff a + ((-a) + (-b)) \leq b + ((-a) + (-b)) \iff (a + (-a)) + (-b) \leq (-a) + (b + (-b))$
$\iff (-b) + 0 \leq (-a) + 0 \iff (-b) \leq (-a)$

Definition   A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark   Let A, B sets:

- If $|A| = |B| \iff$ exists a bijection between A and B

- If $|A| \leq |B| \iff$ exists an injection from A to B

- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition $\mathbb{Z}$ is countably infinite

Proof   We can arrange $\mathbb{Z}$ and $\mathbb{Z}$ in the following way:

$$\mathbb{N} = \{ \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad \dots \quad \}$$
$$\mathbb{Z} = \{ \quad 0, \quad 1, \quad -1, \quad 2, \quad -2, \quad 3, \quad -3, \quad \dots \quad \}$$

We can take the function $f : \mathbb{Z} \to \mathbb{N}$ such that:

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{x}{2} & \text{if } x \text{ even} \\ -\frac{(x-1)}{2} & \text{if } x \text{ odd} \end{cases}, \quad f^{-1}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } 0 < x \\ -2x+1 & \text{if } x < 0 \end{cases}$$

f is bijective, thus $\mathbb{Z}$ is countably infinite.

$\square$

Proposition $\mathbb{Q}$ is countably infinite

Proof   Idea of the proof. We can arrange $\mathbb{N}$ and $\mathbb{Q}$ as such:

$$\mathbb{N} = \{ \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad \dots \quad \}$$
$$\mathbb{Q} = \{ \quad \frac{0}{1}, \quad \frac{1}{1}, \quad -\frac{1}{1}, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad \frac{2}{1}, \quad -\frac{2}{1}, \quad \frac{1}{3}, \quad \dots \quad \}$$

Similarly to the proof for $\mathbb{Z}$, we can find a bijection between $\mathbb{N}$ and $\mathbb{Q}$.

$\square$

Proposition $\mathbb{R}$ is not countable

Proof   Idea of the proof. Let $x \in [0,1[$. Each x can be written as an infinite succession of digits:

1	0.**1**786...
2	0.3**9**09...
3	0.45**0**0...
4	0.097**2**...
...	...

We can construct a new number, taking a digit from each number (each at a different position) and increment it by 1. This way, the new number will be different from any other in the list in the position from where the digit was taken. In our example, the new number would be 0.**2013**...

Since there is one more number than those in the list, then $|\mathbb{N}| < |\mathbb{R}|$, so there is no bijection, and $\mathbb{R}$ is uncountable.

$\square$

Proposition $|\mathbb{R}| = |\mathbb{R}^2|$

Definition   Power set
Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition $\sqrt{2} \notin \mathbb{Q}$

Proof   By contradiction
We suppose $\sqrt{2} \in \mathbb{Q}$, this means there exists $a, b \in \mathbb{Z}$, $b \neq 0$ and greatest common divisor of a and b is 1, such that $\sqrt{2} = \frac{a}{b}$:

$$\sqrt{2} = \frac{a}{b} \iff 2 = \frac{a^2}{b^2} \iff 2b^2 = a^2$$

This means a^2 is even (and a is even), then it exists c such that $a = 2c$:

$$2b^2 = a^2 \iff 2b^2 = (2c)^2 = 4c^2 \iff b^2 = 2c^2$$

This means b^2, and b, are even. But if both a and b are even, then the greatest common divisor of a and b is not 1, contradiction.
We can conclude that $\sqrt{2} \notin \mathbb{Q}$.

$\square$

Definition   Bounds
Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then:

- x is upper bound of A if $a \leq x$, for all $a \in A$

- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition   Supremum and infimum
Let A be a set:

- The supremum is the smallest upper bound of A

- The infimum is the greatest lower bound of A

Definition   Maximum and minimum
Let A be a set:

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)

- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2   Spaces

Definition   Topology
Let X be a set. Then $\tau \subseteq 2^X$ is a topology if:

   (i) $X \in \tau$

   (ii) $\emptyset \in \tau$

   (iii) $A_\alpha \in \tau$, then $\bigcup\limits_{\alpha} A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

   (iv) $A_i \in \tau$, then $\bigcap\limits_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Example   Let $X = \{1, 2, 3, 4\}$

   (1) $\tau = \{\emptyset, X\}$ is topology:
        $\emptyset \cup X = X \in \tau$
        $\emptyset \cap X = \emptyset \in \tau$

   (2) $\tau = \{\emptyset, \{2\}, \{2, 3\}, X\}$ is topology:
        The cases with $\emptyset$ and X are trivial
        $\{2\} \cup \{2, 3\} = \{2, 3\} \in \tau$
        $\{2\} \cap \{2, 3\} = \{2\} \in \tau$

   (3) $A = \{\emptyset, \{2\}, \{3\}, X\}$ is not a topology:
        $\{2\} \cup \{3\} = \{2, 3\} \notin A$

Definition   Topological space
Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition   Neighborhood in a topological space (X, τ)
A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition   Metric
Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if:

   (i) $d(x, y) = d(y, x)$

   (ii) $d(x, y) = 0 \iff x = y$

   (iii) $d(x, z) \leq d(x, y) + d(x, z)$

Example

   (1) $d(x, y) = |x - y|$

   (2) $d(x, y) = ||x - y||_2 = \left(\sum\limits_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$

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Definition   Metric space
Let X be a set, d be a metric, then (X, d) is a metric space.

Definition   Ball in a metric space (X, d)
$B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition   Open set in a topological space (X, τ)
A set U is open in (X, τ) if $U \in \tau$.

Definition   Open set in a metric space (X, d)
A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition   Let $S = (X, x)$ be a space (x a metric or a topology), then:

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup_\alpha A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap_{i=1}^{n} A_i$ is open in S (any finite intersection of any open set is also open)

3   Sequences

Definition   Sequence
A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$.
The elements of a sequence can be listed in an ordered set with repetition: $(x_n) = (x_1, x_2, x_3, x_4, \ldots)$

Example

- $a_n = n \Rightarrow (a_n) = (1, 2, 3, 4, 5, \ldots)$

- $b_n = \frac{1}{n} \Rightarrow (b_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

- $c_n = (-1)^n \Rightarrow (c_n) = (1, -1, 1, -1, \ldots)$

Definition   Cauchy sequence
A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$.
That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Example $x_n = \frac{1}{n}$ in $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. We have to find, for each ε, an N that satisfies the definition of Cauchy sequence.

Let's take $N \leq n \leq m$. Thanks to the triangle inequality, we can first find that:

$$d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |-x_m| = |x_n| + |x_m| = \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}$$

Since $N \leq n \leq m$, then we have $\frac{1}{m} \leq \frac{1}{n} \leq \frac{1}{N}$:

$$\frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

Now, in order to satisfy the definition we must have $\frac{2}{N} \leq \varepsilon$, thus $\frac{2}{\varepsilon} \leq N$. This means, starting from $N = \frac{2}{\varepsilon}$ all $d(x_n, x_m)$ wil be smaller than ε. In fact, if we take the previous inequality:

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{2}{N} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$$

Note that it is not important if $d(x, y) < \varepsilon$ or $d(x, y) \leq \varepsilon$.

Definition Convergence in a metric space (X, d)
A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Example $x_n = \frac{1}{n}$ in $\mathbb{R}$ converges to 0. We take $N \leq n$ and $N = \frac{1}{\varepsilon}$:

$$|x_n - 0| = |x_n| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Definition Convergence in a topological space (X, τ)
A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$.

That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Example Let's take $X = \{a, b, c\}$, $(x_n) = (a, b)$ and $\tau = \{\emptyset, \{a\}, X\}$.

- a is not a limit of (x_n), in fact $\{a\}$ contains a, but doesn't contain the tail of (x_n)

- b and c are limits of (x_n), in fact $b \in X$ and $c \in X$, and X contains (x_n) (and its tail too)

Proposition $x_n \to x$ in (X, d) $\iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.

Proof Let $U \subseteq X$ open, $x \in U$, it exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

$\Rightarrow$ Since x_n converges, then $d(x_n, x) < \varepsilon$, for all ε. This means $x \in B_\varepsilon(x) \subseteq U$, thus $x \in U$

$\Leftarrow$ $x \in B_\varepsilon(x)$ open. This mean it exists N such that all $x_n \in B_\varepsilon(x)$, for all $n \geq N$. We can conclude that $d(x_n, x) < \varepsilon$.

$\square$

Theorem   If a sequence converges to a limit in a metric space, then the limit is unique.

Proof   Let's suppose $x_n \to x$ and $x_n \to x'$. It exists N such that for $ngeqN$, $d(x_n, x) < \varepsilon$ and It exists N' such that for $n \geq N'$, $d(x_n, x') < \varepsilon$. We take $n \geq \max\{N, N'\}$. Now we have $0 \leq d(x, x') \leq d(x, x_n) + d(x_n, x') < 2\varepsilon$.
Since ε is arbitrarily small, then $d(x, x') \leq 0$. Now, we have $0 \leq d(x, x') \leq 0 \Rightarrow d(x, x') = 0 \Rightarrow x = x'$. $\square$

Remark   This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition $x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.

Proposition   Properties of real sequences
For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties:

(i) $\lim\limits_{n \to \infty} (\alpha x_n + \beta y_n) = \alpha \lim\limits_{n \to \infty} x_n + \beta \lim\limits_{n \to \infty} y_n$

(ii) $\lim\limits_{n \to \infty} x_n x_y = \lim\limits_{n \to \infty} x_n \lim\limits_{n \to \infty} y_n$

(iii) $\lim\limits_{n \to \infty} \dfrac{x_n}{x_y} = \dfrac{\lim_{n \to \infty} x_n}{\lim_{n \to \infty} y_n}$

Proof

(i) $\forall \varepsilon > 0, \quad \exists N : |x_n - x| < \dfrac{\varepsilon}{2\,|\alpha|} = \varepsilon', \quad \exists N' : |y_n - y| < \dfrac{\varepsilon}{2\,|\beta|} = \varepsilon'' \to$ we take $n = \max\{N, N'\}$.

$|(\alpha x_n + \beta y_n) - (\alpha x + \beta y)| = |\alpha(x_n - x) + \beta(y_n - y)| \leq |\alpha(x_n - x)| + |\beta(y_n - y)| =$

$= |\alpha|\,|(x_n - x)| + |\beta|\,|(y_n - y)| < |\alpha|\,\varepsilon' + |\beta|\,\varepsilon'' = \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$

(i-ii) Similar to previous demonstration.

$\square$

Example

(1) Knowing that $\frac{1}{n} \to 0$, show that $\frac{1}{n^2} \to 0$.

$$\lim_{n \to \infty} \frac{1}{n^2} = \lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} \frac{1}{n} = 0 \cdot 0 = 0$$

(2) Find the limit of the sequence $\frac{2n^2 - 3n + 2}{n^2 + n - 1}$.

$$\lim_{n \to \infty} \frac{2n^2 - 3n + 2}{n^2 + n - 1} = \frac{\lim(2n^2 - 3n + 2)}{\lim(n^2 + n - 1)} = \frac{\lim 2 - \lim \frac{3}{n} + \lim \frac{2}{n^2}}{\lim 1 + \lim \frac{1}{n} - \lim \frac{1}{n^2}} = \frac{2 - 0 + 0}{1 + 0 - 0} = 2$$

Definition   A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Example

- $a_n = \frac{1}{n}$ is bounded. We can take $c = 1$, $\left|\frac{1}{n}\right| = \frac{1}{n} \leq 1$, being $a_1 = 1$ the $\sup(a_n)$.

- $b_n = (-1)^n$ is bounded. In fact, the values of the sequence are always 1 and -1. If we take $c = 1$, then $|b_n| = 1 \leq 1$.

- $x_n = n$ is not bounded, we can prove it by contradiction.
  We suppose it exists a c such that $|x_n| \leq c$. If we take $x_{c+1} = c + 1$, we have $c + 1 \leq c \iff 0 \leq 1$ contradiction. This means x_n is not bounded.

Theorem   Convergent real sequences are bounded (not the opposite).

Proof   Since $x_n \to x$, it exists N such that for all $n \geq N$, $|x_n - x| < \varepsilon$. By triangle inequality we have:

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < \varepsilon + |x|$$

We choose $c = \max\{|x_1|, \ldots, |x_{N-1}|, |x| + \varepsilon\}$, then $|x_n| < c$ for each n.

$\square$

Definition   Monotonic sequences

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n

- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Example

- $a_n = \frac{1}{n}$ is monotonic decreasing. In fact, $\frac{1}{n+1} \leq \frac{1}{n}$, then $a_{n+1} \leq a_n$.

- $b_n = n$ is monotonic increasing. In fact, $b_n = n$ and $b_{n+1} = n + 1$. Since $n \leq n + 1$, then $b_n \leq b_{n+1}$.

- $c_n = (-1)^n$ is not monotonic. We can take $n = 1$, then $a_1 \leq a_2 \nleq a_3$.

Theorem   If a sequence monotonic and bounded $\Rightarrow$ convergent

Proof (x_n) increasing and bounded, let $c = \sup(x_n)$. For all $\varepsilon > 0$ exists N such that $c - \varepsilon < x_N$. Since (x_n) increasing, for all $n \geq N$, $x_N \leq x_n \leq c$.

$$c - \varepsilon < x_n \leq c \iff -\varepsilon < x_n - c \leq 0 < \varepsilon \iff |x_n - c| < \varepsilon$$

The last inequality implies convergence. Similarly, the theorem can be proven for decreasing sequences.

$\square$

Definition   Limit superior and inferior of (x_n)

- $\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup\{x_k : k \geq n\}$

- $\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf\{x_k : k \geq n\}$

Definition   Subsequence
$(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Example $x_n = (-1)^n \cdot n$. We take $k = 2n$, then the subsequence $(x_{n_k}) = (x_{2n})$ of (x_n) takes all the even indexes n of (x_n):

$$
\begin{array}{rcl}
(x_n) & = & (\quad -1, \quad 2, \quad -3, \quad 4, \quad -5, \quad 6, \quad \dots \quad) \\
(x_2 n) & = & (\qquad\quad 2, \qquad\quad 4, \qquad\quad 6, \quad \dots \quad)
\end{array}
$$

Theorem   If $x_n \to x \Rightarrow x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.

Proof $k \leq n_k$ (it can be proved by induction) and $d(x_k, x) < \varepsilon$. Since $N \leq k \leq n_k$, then $d(x_{n_k}, x) \leq d(x_k, x) < \varepsilon$. This means the subsequence converges to x.

$\square$

Definition x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem   Every sequence has a monotonic subsequence.

Proof   Based on dominants terms:

- If we have infinite dominant terms, we take the decreasing subsequence formed by the dominant terms.

- If we have a finite number of dominant terms, then, after the last dominant term, we start taking an increasing subsequence (since, for each term, there will be at some point a greater term).

$\square$

Theorem   Bolzano-Weierstrass
Every bounded sequence has a convergent subsequence.

Proof   We take (x_n) bounded. We show it in three steps:

- (x_n) has a monotonic subsequence (x_{n_k})

- Since (x_n) is bounded, then (x_{n_k}) is bounded

- Since (x_{n_k}) is bounded and monotonic, it is convergent

$\square$

Definition $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

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4 Series

Definition   Series

(x_n) sequence. $s_n = \sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition   Convergence of series

$s_n = \sum_{k=1}^{n} x_k$ a series. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k$.

Example

- Harmonic: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n} = +\infty$

- Geometric: $\displaystyle\sum_{k=1}^{\infty} a^n = \begin{cases} \infty & |a| \geq 1 \\ \frac{1}{1-a} & |a| < 1 \end{cases}$

- Exponential: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n!} = e$

- Leibniz: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{4}$

Definition   Absolute convergence of a series $s_n = \displaystyle\sum_{k=1}^{n} x_k$

s_n converges absolutely if $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$.

Proposition   Absolute convergence $\Rightarrow$ convergence. If $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$, then $\displaystyle\sum_{k=1}^{\infty} x_k < \infty$.

Proof $\displaystyle\sum_{k=1}^{\infty} |x_k| < \infty$ and $x_n \leq |x_n|$, then $\displaystyle\sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} |x_k| < \infty$.

$\square$

Definition   Cauchy criterion for series

$s_n = \displaystyle\sum_{k=1}^{n} x_k$, and $\displaystyle\sum_{k=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall N \leq m \leq n \Rightarrow |s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m}^{n} x_k \right| < \varepsilon$$

Proposition   Comparison test, for x_n, y_n sequences and $x_n \geq 0$

(i) If $\displaystyle\sum_{k=1}^{\infty} x_k < \infty$ and $|y_n| \leq x_n \Rightarrow \displaystyle\sum_{k=1}^{\infty} y_k < \infty$

(ii) If $\displaystyle\sum_{k=1}^{\infty} x_k = +\infty$ and $x_n \leq y_n \Rightarrow \displaystyle\sum_{k=1}^{\infty} y_k = +\infty$

Proof

(i) $\displaystyle\left|\sum_{k=m}^{n} y_k\right| \leq \sum_{k=m}^{n} |y_k| \leq \sum_{k=m}^{n} x_k < \varepsilon \Rightarrow \sum_{k=1}^{\infty} y_k < \infty$

(ii) $\displaystyle +\infty = \sum_{k=1}^{\infty} x_k \leq \sum_{k=1}^{\infty} y_k \Rightarrow \sum_{k=1}^{\infty} y_n = +\infty$

$\square$

Example $\displaystyle\sum_{k=1}^{\infty} \frac{1}{n^2 + 1} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{n^2 + 1} < \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty$, the series converges.

Proposition   Ratio test, for x_n sequence, $x_n \neq 0$ and $s_n = \displaystyle\sum_{k=1}^{n} x_k$ series:

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

Example

- $\displaystyle\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^n$:

$$\left|\frac{\left(-\frac{1}{3}\right)^{n+1}}{\left(-\frac{1}{3}\right)^n}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3} \Rightarrow \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1 \Rightarrow \text{ converges absolutely}$$

- $\displaystyle\sum_{k=1}^{\infty} \frac{n}{n^2 + 3}$

  − Ratio test:

$$\limsup_{n\to\infty} \left|\frac{n+1}{(n+1)^2 + 3} \frac{n^2 + 3}{n}\right| = \limsup_{n\to\infty} \frac{n+1}{(n+1)^2 + 3} \frac{n^2 + 3}{n} = 1, \text{ no information}$$

– Comparison test:

$$\frac{n}{n^2 + 3n^2} \leq \frac{n}{n^2 + 3} \Rightarrow \frac{n}{n^2 + 3n^2} = \frac{n}{4n^2} = \frac{1}{4}\frac{n}{n^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{4}\frac{n}{n^2} \Rightarrow +\infty = \frac{1}{4}\sum_{k=1}^{\infty} \frac{n}{n^2} \leq \sum_{k=1}^{\infty} \frac{n}{n^2 + 3}$$

The series diverges. Sometimes one test can give more information than others.

Proposition   Root test, Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$:

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

Proof $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}, \varepsilon > 0, \alpha + \varepsilon < 1$:

$$\limsup_{n\to\infty} \sqrt[n]{|x_n|} = \lim_{n\to\infty} \sup\{\sqrt[k]{|x_k|} : k > n\} \Rightarrow \exists N : \left|\sup\{\sqrt[n]{|x_n|} : n > N\} - \alpha\right| < \varepsilon$$

$$\alpha - \varepsilon < \left|\sup\{\sqrt[n]{|x_n|} : n > N\}\right| < \alpha + \varepsilon \Rightarrow \sqrt[n]{|x_n|} < \alpha + \varepsilon \iff |x_n| < (\alpha + \varepsilon)^n$$

Since the geometric series $\sum_{k=1}^{\infty}(\alpha+\varepsilon)^n < \infty$, then $\sum_{k=1}^{\infty} |x_n| < \sum_{k=1}^{\infty}(\alpha+\varepsilon)^n < \infty$, the series converges absolutely.

$\square$

Example

- $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^n \Rightarrow \limsup_{n\to\infty} \sqrt[n]{-\frac{1}{3}^n} = \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1$, the series converges absolutely

- $\sum_{k=1}^{\infty} 2^{(-1)^n - n} \Rightarrow \sqrt[n]{2^{(-1)^n - n}} = \begin{cases} 2^{\frac{1}{n}-1} & \text{if } n \text{ even} \\ 2^{-\frac{1}{n}-1} & \text{if } n \text{ odd} \end{cases} \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}-1} = \lim_{n\to\infty} 2^{-\frac{1}{n}-1} = \frac{1}{2} < 1$, the series converges.

5   Functions and continuity

Definition   Given a function $f : X \to Y$, the image of f is defined as: $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition   Given a function $f : X \to Y$, the preimage of f is defined as: $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition   Continuity of $f : (X, d_x) \to (Y, d_y)$ (in a metric space)
f is continuous at $x \in X$ if:

$$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : \forall x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Example   Let's take $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin \frac{1}{x} & \text{otherwise} \end{cases}$
We want to prove that f is continuous in 0. Let $\varepsilon > 0$, then $|f(x) - f(0)| = |f(x) - 0| = |f(x)| \le x^2$. If we take $\delta = \sqrt{\varepsilon}$, then:

$$|x - 0| < \delta \Rightarrow x^2 < \delta \Rightarrow |f(x) - f(0)| \le x^2 < \delta^2 = \varepsilon \Rightarrow f \text{ is continuous in } 0$$

Remark   Continuity can also be defined as follows:

$$\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta_\varepsilon}(x)) \subseteq B^{d_y}_\varepsilon(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition   Continuity of $f : (X, \tau_x) \to (Y, \tau_y)$ (in a topological space)
f is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Example   Let's take $(M, \tau_m), (N, \tau_n)$, $M = N = \{1, 2\}$, $\tau_m = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_m = \{\emptyset, \{1, 2\}\}$.

- Let $f : (M, \tau_m) \to (N, \tau_n)$, such that $f(1) = 2$ and $f(2) = 1$:

  $$PreIm_f(\emptyset) = \emptyset \in \tau_m, PreIm_f(\{1, 2\}) = \{1, 2\} \in \tau_m \Rightarrow f \text{ is continuous in all } x \in M$$

- Let $g : (N, \tau_n) \to (M, \tau_m)$, such that $f(1) = 2$ and $f(2) = 1$:

  $$PreIm_g(\{1\}) = \{2\} \notin \tau_n \Rightarrow g \text{ is not continuous}$$

Proposition   Continuous functions map open sets into open sets. If $f : (X, d_x) \to (Y, d_y)$ continuous, then $PreIm_f(A)$ is open, for all $A \subseteq Y$ open.

Proof   Let $A \subseteq Y$ open, $x \in PreIm_f(A)$, $f(x) \in A$. Then, it exists $\varepsilon > 0$ such that $B^{d_y}_\varepsilon(f(x)) \subseteq A$. Since f is continuous, then it exists δ_ε such that:

$$PreIm_f(B^{d_x}_{\delta_\varepsilon}(x)) \subseteq B^{d_y}_\varepsilon(f(x)) \subseteq A \Rightarrow B^{d_x}_{\delta_\varepsilon}(x) \subseteq PreIm_f(A) \Rightarrow A \text{ is open}$$

$\square$

Theorem   Continuous functions map limits to limits:

$$f \text{ continuous}, x_n \to x \iff f(x_n) \to f(x)$$

Proof   Topological (only for "⇒")
Let $f : (X, \tau_x) \to (Y, \tau_y)$, $A \in \tau_y$, $f(x) \in A$. Since f continuous, then $PreIm_f(A) \in \tau_x$ and $x \in PreIm_f(A)$. Since x_n converges to x, we have that:

$$\exists N : \forall n \geq N, (x_n) \subseteq PreIm_f(A) \Rightarrow Im_f(x_n) \subseteq A \Rightarrow f(x_n) \to f(x)$$

$\square$

Proof   Metrical (only for "⇒")
Let $\varepsilon > 0$, $f : (X, d_x) \to (Y, d_y)$ continuous. Then, it exists $\delta > 0$ such that for all $x' \in X$, $d_x(x, x') < \delta$. This means $d_y(f(x), f(x')) < \varepsilon$. Since x_n converges to x:

$$\exists N : \forall n \geq N, d_x(x, x) < \delta \Rightarrow d_y(f(x_n), f(x)) < \varepsilon \Rightarrow f(x_n) \to f(x)$$

$\square$

Example   Let's take $f(x) = 2x^2 + 1$ and $\lim\limits_{n \to \infty} x_n = x$. Then:

$$\lim_{n \to \infty} 2x_n^2 + 1 = 2 \left(\lim_{n \to \infty} x_n \right)^2 + 1 = 2x^2 + 1$$

This means that for $x_n \to x$, then $f(x_n) \to f(x)$. Therefore, f is continuous.

Proposition $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.

Proposition f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.

Proof

(1) f continuous at $x \Rightarrow$ for $x_n \to x$, then $f(x_n) \to f(x)$

(2) g continuous at $y \Rightarrow$ for $y_n \to y$, then $g(y_n) \to f(y)$

(3) In particular, for $y_n = f(x_n) \Rightarrow g(f(x_n)) \to g(f(x))$

$\square$

Definition $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Theorem   Banach fixed point
Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then:

(i) $\exists! x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

6 Limits of functions

Definition f converges to c at x_0 $\iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim\limits_{x \to x_0} f(x) = c$.

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim\limits_{x \to x_0^+} f(x) = c$.

- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim\limits_{x \to x_0^-} f(x) = c$.

Example

- Let $f(x) = \dfrac{1}{x} \Rightarrow \lim\limits_{x \to 0^+} \dfrac{1}{x} = +\infty,\ \lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

- Let $f(x) = floor(x) \Rightarrow \lim\limits_{x \to 1^+} floor(x) = 1,\ \lim\limits_{x \to 1^-} floor(x) = 0,$ but $\lim\limits_{x \to \frac{1}{2}^+} floor(x) = \dfrac{1}{2} = \lim\limits_{x \to \frac{1}{2}^-} floor(x)$

Definition $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \le c$ for all $x \in X$.

Example $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin(x)$ is bounded on $\mathbb{R}$, since $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$.

Theorem   Extreme value
If $f : [a, b] \to \mathbb{R}$ is continuous, then:

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$
$\exists x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \le f(x) \le f(x_{maximizer}), \forall x \in [a, b]$

Proof

(i) Proof by contradiction, we assume f unbounded
This means, for all $n \in \mathbb{N}$ it exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Then, $(x_n) \subseteq [a, b]$ is bounded and has a subsequence (x_{n_k}) that converges to a $x_0 \in [a, b]$ (Bolzano-Weierstrass). Since f is continuous at x_0, then $f(x_{n_k})$ converges to $f(x_0)$. If f is unbounded, then $f(x_n)$ diverges: contradiction. This means f is bounded.

(ii) Let's take $M = \sup\{f(x) : x \in [a, b]\}$ the smallest upper bound of $Im([a, b])$, then $M - \frac{1}{n}$ is not an upper bound. We know it exists x_n such that $M - \frac{1}{n} < f(x_n) \le M$. This means:

$$\lim_{n \to \infty} M - \frac{1}{n} \le \lim_{n \to \infty} f(x_n) \le M \iff M \le \lim_{n \to \infty} f(x_n) \le M \iff \lim_{n \to \infty} f(x_n) = M$$

Moreover, $(x_n) \subseteq [a, b]$ is bounded, and it has a subsequence (x_{n_k}) convergent to $x_0 \in [a, b]$. Since f is continuous, then $f(x_{n_k})$ converges to $f(x_0)$. This means $f(x_0) = M$, where x_0 is the maximizer.

$\square$

Remark   This isn't true if the interval is open:

- $f : \mathbb{R} \to \mathbb{R}(0,1)$, $f(x) = \frac{1}{x}$ is unbounded, since $f(x)$ goes to infinity for x small

- $f : (-1,1) \to$, $f(x) = x^2$, doesn't have a max, since $\sup\{Im((-1,1))\} = 1$ is $f(1)$ or $f(-1)$, but 1 and -1 $\notin (-1,1)$

Theorem   Intermediate value (IVT)
f continuous on $[a,b]$, $f(a) < c < f(b) \Rightarrow \exists x \in [a,b] : f(x) = c$.

Proof   Let's assume $f(a) < c < f(b)$ (the same can be done for the opposite). Let's have $S = \{x \in [a,b] : f(x) < c\}$ not empty, since at least $f(a) \in S$. Let $x_0 = \sup S \in [a,b]$, then $x_0 - \frac{1}{n}$ is not an upper bound, and it exists $s_n \in S$ such that $x_0 - \frac{1}{n} < s_n \leq x_0$. This means s_n converges to x_0. We now have $f(s_n) < c$ and $f(x_0) = \lim f(s_n) \leq c$.
Let's take $t_n = \min\{x_0 + \frac{1}{n}, b\} \in [a,b]$, where $x_0 < t_n \leq t_n + \frac{1}{n}$, meaning that t_n converges to x_0. Now $t_n \notin S$ (since $t_n > \sup S$), $f(t_n) \geq c$ and $f(x_0) = \lim t_n \geq c$. Therefore $c \leq f(x_0) \leq c$, so $f(x_0) = c$.
$\square$

Definition   A Darboux function is a function that satisfies the intermediate value property.

Proposition   Continuous implies Darboux, but not the opposite.

Example $f(x) = \begin{cases} \sin(\frac{1}{x}) & x > 0 \\ 0 & x = 0 \end{cases}$ is a Darboux function, but it is not continuous.

Proposition   Continuous functions map intervals to intervals.

Definition   Connectedness
Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold:

- There exist $U, V \in \tau$ such that:

  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$

- There exist $U, V \subseteq A$ such that:

  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$! NOT SURE !

A set is connected if it is not disconnected.

Proposition   Continuous functions preserve connectedness.
$f : (X, \tau_x) \to (Y, \tau_y)$, $A \subseteq X$ connected in (X, τ_x), then $Im(A) \subseteq Y$ is connected in (Y, τ_y).

Proof By contradiction. We suppose A connected and $Im(A)$ disconnected.
Since $Im(A)$ is disconnected, exist $V_1, V_2 \in \tau_y$ such that:

- $(Im(A) \cap V_1) \cap (Im(A) \cap V_2) = \emptyset$, and

- $(Im(A) \cap V_1) \cup (Im(A) \cap V_2) = Im(A)$, and

- $Im(A) \cap V_1 \neq \emptyset \neq Im(A) \cap V_2$

Let $U_1 = PreIm(V_1)$ and $U_2 = PreIm(V_2)$ it follows (it should be proved) that:

- $(PreIm(A) \cap U_1) \cap (PreIm(A) \cap U_2) = \emptyset$, and

- $(PreIm(A) \cap U_1) \cup (PreIm(A) \cap U_2) = PreIm(A)$, and

- $PreIm(A) \cap U_1 \neq \emptyset \neq PreIm(A) \cap U_2$

This implies that A is disconnected, contradiction. Therefore $Im(A)$ is connected.

$\square$

Definition   Uniform continuity
$f : (X, d_x) \to (Y, d_y)$ is uniformly continuous on X if:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Example $f(x) = \frac{1}{x^2}$ in $[a, +\infty)$, $a > 1$. To show that f is uniformely continuous, we have to show that for all $\varepsilon > 0$ exists $\delta_\varepsilon > 0$ such that for all x, y such that $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.
Let $\varepsilon > 0$ and $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \frac{(x+y)(x-y)}{x^2 y^2}$. Then, since $a \leq x, y \ \forall x, y$:

$$\frac{(x + y)}{x^2 y^2} = \frac{x}{x^2 y^2} + \frac{y}{x^2 y^2} \leq \frac{2}{a^3}$$

We chose $\delta = \frac{\varepsilon a^3}{2}$, then:

$$\forall x, y \geq a : |x - y| < \delta \Rightarrow |f(x) - f(y)| = |x - y| \left| \frac{x + y}{x^2 y^2} \right| < \delta \frac{2}{a^3} = \frac{\varepsilon a^3}{2} \frac{2}{a^3} = \varepsilon$$

This mean f is uniformely continuous.

Remark   Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall \varepsilon > 0 \quad \exists \delta_{\varepsilon, x_0} > 0 : \forall x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem f continuous on A, closed and bounded $\Rightarrow f$ is uniformely continuous on A.

Theorem f uniformely continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow f(s_n)$ is Cauchy sequence.

Proof   Let $(s_n) \subseteq S$ a Cauchy sequence, $\varepsilon > 0$ and f uniformly continuous:

1. Exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $|x - y| < \delta$.

2. Exists N_ε such that for all $n, m \geq N$, then $|s_n - s_m| < \delta$

Combining (1) and (2) we have that for all $n, m \geq N$, $|f(s_n) - f(s_m)| < \varepsilon$. This means $f(s_n)$ is a Cauchy sequence.

$\square$

Example $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$. In fact, $s_n = \frac{1}{n}$ is Cauchy, but $f(s_n) = n^2$ is not Cauchy.

Definition   Sequence of functions
$(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Example $f_n(x) = \frac{x}{n} \to f(x) = 0$

Definition f_n converges pointwise to $f \iff \lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall \varepsilon > 0, x \in S \exists N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Example $f_n(x) = x^n$, $x \in [0, 1] \Rightarrow f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$. f_n is continuous and f is discontinuous.

Definition $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)| < \varepsilon\}$

Definition f_n converges uniformly to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Example   Let $f_n(x) = (1 - |x|)^n$, $x \in (-1, 1)$. Then f converges pointwise (but not uniformly) to $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. In fact:

- Pointwise convergence
  For $x = 0$, $f_n(x) = (1 - 0)^n = 1$, then $\lim_{n \to \infty} f_n = \lim_{n \to \infty} 1 = 1$.
  For $x \neq 0$, $|x| < 1$. This means $1 - |x| < 1$, then $\lim_{n \to \infty} (1 - |x|)^n = 0$.

- Uniform convergence
  We assume $f_n \xrightarrow{unif.} f$ and we take $\varepsilon = \frac{1}{2}$. Then it exists N such that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $x \in (-1, 1)$.
  Let's take $x = 1 - 2^{-\frac{1}{n}}$, then $1 - x = 2^{-\frac{1}{n}}$. Thus $(1 - x)^n = (2^{-\frac{1}{n}})^n = \frac{1}{2} \not< \frac{1}{2} = \varepsilon$. Contradiction, f doesn't converge uniformly to f.

Theorem   Uniform limit of a continuous function is continuous
$f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x)$ is continuous.

Proof   Let $\varepsilon > 0$

Since $f_n \xrightarrow{unif.} f(x)$, it exists N_ε such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N$.

Since f_n continuous, it exists $\delta > 0$ such that for all x, x_0 such that $|x_0 - x| < \delta$, then $|f_N(x_0) - f_N(x)| < \frac{\varepsilon}{3}$.

By triangle inequality we have:

$$|f(x_0) - f(x)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f(x)| \leq$$

$$\leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\square$

Example

- Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n + g_n \xrightarrow{unif.} f + g$. In fact:

$$\exists N_f : \forall x \in S \, |f_n(x) - f(x)| < \frac{\varepsilon}{2} \forall n > N_f$$

$$\exists N_g : \forall x \in S \, |g_n(x) - g(x)| < \frac{\varepsilon}{2} \forall n > N_g$$

We take $N = \max\{N_f, N_g\}$. Then:

$$|f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n \geq N$$

This means $f_n + g_n \xrightarrow{unif.} f + g$.

- Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n g_n$ doesn't converge uniformly to fg. In fact, let $h_n(x) = \frac{x}{n}$. By contradiction we can prove h_n doesn't converge uniformly to h. Now, if we take $f_n(x) = \frac{1}{n}$ and $g_n(x) = x$ (uniformely convergent), then $f(x)g(x) = \frac{x}{n} = h(x)$ not uniformly convergent. We found a counter example.

- Let $f_n(x)$ continuous on $[a, b]$, $f_n(x) \xrightarrow{unif.} f(x)$, $(x_n) \subseteq [a, b]$ and $x_n \to x$. Then, $f_n(x_n) \to f(x)$. To prove it we have to show that exists N such that for all $n \geq N$, then $|f_n(x_n) - f(x)| < \varepsilon$.

  1. $f_n \xrightarrow{unif.} f$, this means it exists N_1 such that $|f_n(y) - f(y)| < \frac{\varepsilon}{2}$, for all $n \geq N_1$ and $y \in [a, b]$. In particular, $|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$.

  2. Since $f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x)$, then $f(x)$ is continuous. Then $f(x_n) \to f(x)$, this means it exists N_2 such that for all $n \geq N_2$, then $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$.

  3. We chose $N = \max\{N_1.N_2\}$. Then:

  $$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n \geq N$$

  We can conclude that $f_n(x_n) \to f(x)$.

7   Power series

Definition   Power series
Let $(a_n)_{n \geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases:

- The series converges for all $x \in \mathbb{R}$.

- The series converges for $x = 0$ only.

- The series converges for some bounded interval.

Theorem   Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ ($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then $\sum_{n=0}^{\infty} a_n x^n$:

- Converges for $|x| < R$.

- Diverges for $|x| > R$.

The same can be done with $\beta = \limsup \left| \frac{a_n}{a_{n+1}} \right|$.

Proof   With root test. Let $\alpha = \limsup \sqrt[n]{|a_n|}$, then $\sum_{k=1}^{\infty} a_n < \infty$ if $\alpha < 1$ or $\sum_{k=1}^{\infty} a_n = \infty$ if $\alpha = 1$. Let $\alpha_x = \limsup \sqrt[n]{|a_n x^n|} = \limsup |x| \sqrt[n]{|a_n|} = |x| \limsup \sqrt[n]{|a_n|} = \beta |x|$. Then:

1. If $0 < R < \infty$, then $\alpha_x = \beta |x| = \frac{|x|}{R}$.

   - If $|x| < R$, then $\alpha_x < 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ converges

   - If $|x| > R$, then $\alpha_x > 1$, by root test $\sum_{n=0}^{\infty} a_n x^n$ diverges

2. If $R = \infty$, then $\alpha_x = 0 < 1$ independently of x. The series always converges.

3. If $R = 0$, then $\alpha_x = \infty > 1$ independently of x. The series always diverges.

$\square$

Example

- Let $a_n = 1$. We have the power series $\sum_{n=0}^{\infty} x^n$ and $\beta = \limsup \sqrt[n]{1} = 1$, then $R = 1$. This means the series converges for $x \in (-1, 1)$ and diverges for x such that $|x| > 1$. Moreover, it diverges for $x = 1$, since $\sum_{n=0}^{\infty} 1 = +\infty$, and it is not defined for $x = -1$.

- Let $\sum\limits_{n=0}^{\infty} \dfrac{1}{n} x^n$ a power series. Then $\limsup \left| \dfrac{a_n}{a_{n+1}} \right| = \limsup \left| \dfrac{n}{n+1} \right| = 1$, then $R = 1$. For $x = 1$ we have

  the harmonic series $\sum\limits_{n=0}^{\infty} \dfrac{1}{n}$ wich diverges, for $x = -1$ we have $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{n} < \infty$. We can conclude that

  the power series converges for $x \in [-1, 1)$.