Università della Svizzera italiana
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Calculus

Course Notes

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Contents

1 Sets, groups and fields

Definition 1.1 (Natural numbers). The set of natural numbers is defined with the following properties

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition 1.2 (Group). A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists!\ 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists!\ (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abelian if the following rule is satisfied too

(v) Commutativity: $a \circ b = b \circ a$

Definition 1.3 (Field). Given a set X, then $(X, +, \cdot)$ is a field if the following are satisfied for all $a, b, c \in X$

(i) $a + b \in X$ and $a \cdot b \in X$

(ii) $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists!\ 0 \in X : a + 0 = 0 + a = a$ and $\exists!\ 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!\ (-a) \in X : a + (-a) = (-a) + a = 0$ and $\forall\ a \neq 0,\ \exists!\ a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$ and $a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 1.4 (Rational numbers). $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark. $(\mathbb{Q}, +, \cdot)$ is a field.

Definition 1.5 (Ordered Field). Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

Definition 1.6 (Countable Infinite). A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark. Let A, B sets, then

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \leq |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition 1.1. $\mathbb{Z}$ *is countably infinite*

Proposition 1.2. $\mathbb{Q}$ *is countably infinite.*

Proposition 1.3. $\mathbb{R}$ *is not countable.*

Proposition 1.4. $|\mathbb{R}| = |\mathbb{R}^2|$

Definition 1.7 (Power set). Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition 1.5. $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition 1.6. $\sqrt{2} \notin \mathbb{Q}$

Definition 1.8 (Bounds). Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition 1.9 (Supremum and infimum). Let A be a set

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition 1.10 (Maximum and minimum). Let A be a set

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition 2.1 (Topology). Let X be a set. Then $\tau \subseteq 2^X$ is a topology if

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Definition 2.2 (Topological space). Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition 2.3 (Neighborhood in a topological space (X, τ)). A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition 2.4 (Metric). Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.5 (Metric space). Let X be a set, d be a metric, then (X, d) is a metric space.

Definition 2.6 (Ball in a metric space (X, d)). $B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition 2.7 (Open set in a topological space (X, τ)). A set U is open in (X, τ) if $U \in \tau$.

Definition 2.8 (Open set in a metric space (X, d)). A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition 2.9 (Closed set). $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition 2.1. *Let $S = (X, x)$ be a space (x a metric or a topology), then*

(i) X is open in S

(ii) $\emptyset$ *is open in* S

(iii) *For all* A_α *open in* S*, then* $\bigcup_\alpha A_\alpha$ *is open in* S *(any union of any open set is also open)*

(iv) *For all* A_i *open in* S*, then* $\bigcap_{i=1}^{n} A_i$ *is open in* S *(any finite intersection of any open set is also open)*

3 Sequences

Definition 3.1 (Sequence). A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$. The elements of a sequence can be listed in an ordered set with repetition

$$(x_n) = (x_1, x_2, x_3, x_4, \dots)$$

Definition 3.2 (Cauchy sequence). A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$. That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Definition 3.3 (Convergence in metric space). (X, d) is a metric space. A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Definition 3.4 (Convergence in topological space). (X, τ) is a topological space. A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$. That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Proposition 3.1. $x_n \to x$ *in* (X, d) $\iff$ *for all* $U \subseteq X$ *open exists* N_U *such that* $x_n \in U$*, for all* $n \geq N_U$*.*

Theorem 3.2. *If a sequence converges to a limit in a metric space, then the limit is unique.*

Remark. This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition 3.3. $x_n \to x$ *in* (X, d) *metric space, then for all* $y \in X$*,* $d(x_n, y) \to d(x, y)$*.*

Proposition 3.4 (Properties of real sequences). *For all* $(x_n), (y_n)$ *such that* $x_n \to x$*,* $y_n \to y$*, we have the following properties*

(i) $\lim_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim_{n\to\infty} x_n + \beta \lim_{n\to\infty} y_n$

(ii) $\lim_{n\to\infty} x_n x_y = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$

(iii) $\lim_{n\to\infty} \frac{x_n}{x_y} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Definition 3.5 (Bounded sequence). A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Definition 3.6 (Monotonic sequence). A sequence is monotonic if

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Theorem 3.5. *If a sequence monotonic and bounded, then the sequence is convergent.*

Definition 3.7 (Limit superior and inferior). If (x_n) is a sequence, then

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

Definition 3.8 (Subsequence). $(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Theorem 3.6. *If $x_n \to x$, then $x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.*

Definition 3.9 (Dominant term). x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem 3.7. *Every sequence has a monotonic subsequence.*

Theorem 3.8 (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

Definition 3.10. $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition 4.1 (Series). (x_n) is sequence. $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition 4.2 (Convergence of series). $s_n = \displaystyle\sum_{k=1}^{n} x_k$ a series. $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{n=1}^{\infty} x_k$.

Definition 4.3 (Absolute convergence of series). $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series. s_n converges absolutely if

$$\sum_{n=1}^{\infty} |x_k| < \infty$$

Proposition 4.1. *Absolute convergence $\Rightarrow$ convergence. If* $\sum_{n=1}^{\infty} |x_k| < \infty$, *then* $\sum_{n=1}^{\infty} x_k < \infty$.

Definition 4.4 (Cauchy criterion for series). $s_n = \sum_{k=1}^{n} x_k$, and $\sum_{n=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall\ N \leq m \leq n \Rightarrow |s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m}^{n} x_k \right| < \varepsilon$$

Proposition 4.2 (Comparison test). *For x_n, y_n sequences and $x_n \geq 0$*

(i) If $\sum_{n=1}^{\infty} x_k < \infty$ *and* $|y_n| \leq x_n \Rightarrow \sum_{n=1}^{\infty} y_k < \infty$

(ii) If $\sum_{n=1}^{\infty} x_k = +\infty$ *and* $x_n \leq y_n \Rightarrow \sum_{n=1}^{\infty} y_k = +\infty$

Proposition 4.3 (Ratio test). *For x_n sequence, $x_n \neq 0$ and $s_n = \sum_{k=1}^{n} x_k$ series:*

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

Proposition 4.4 (Root test). *Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|x_n|}$:*

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

5 Functions and continuity

Definition 5.1 (Image). Given a function $f : X \to Y$, the image of f is defined as $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition 5.2 (Preimage). Given a function $f : X \to Y$, the preimage of f is defined as $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition 5.3 (Continuity in metric space). $f : (X, d_x) \to (Y, d_y)$ is continuous at $x \in X$ if

$$\forall\ \varepsilon > 0\ \exists\ \delta_\varepsilon > 0 :\ \forall\ x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark. Continuity can also be defined as follows

$$\forall\, \varepsilon > 0 \; \exists\, \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition 5.4 (Continuity in topological space). $f : (X, \tau_x) \to (Y, \tau_y)$ is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Proposition 5.1. *Continuous functions map open sets into open sets.*

$$\textit{If } f : (X, d_x) \to (Y, d_y) \textit{ continuous, then } PreIm_f(A) \textit{ is open, for all } A \subseteq Y \textit{ open}$$

Theorem 5.2. *Continuous functions map limits to limits*

$$f \textit{ continuous, } x_n \to x \iff f(x_n) \to f(x)$$

Proposition 5.3. *$f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.*

Proposition 5.4. *f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.*

Definition 5.5 (Contraction). $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \le c < 1$ such that $d(f(x), f(y)) \le c d(x, y)$, for all $x, y \in X$.

Theorem 5.5 (Banach fixed point). *Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then*

(i) $\exists!\, x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

Definition 5.6 (Convergence of a function). f converges to c at $x_0 \iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim\limits_{x \to x_0} f(x) = c$. Moreover

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim\limits_{x \to x_0^+} f(x) = c$.
- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim\limits_{x \to x_0^-} f(x) = c$.

Proposition 5.6. *f continuous at $a \iff \lim\limits_{x \to a} f(x) = f(a)$*

Proposition 5.7. $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

6 Continuous functions and intervals

Definition 6.1 (Bounded function). $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

Theorem 6.1 (Extreme value). *If $f : \mathbb{R} \to \mathbb{R}[a, b]$ is continuous, then:*

(i) f is bounded on $[a, b]$

(ii) f has a maximum and a minimum on $[a, b]$, meaning that

$$\exists \ x_{minimizer}, x_{maximizer} \in [a, b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}), \ \forall \ x \in [a, b]$$

Theorem 6.2 (Intermediate value). *f continuous on $[a, b]$, $f(a) < c < f(b) \Rightarrow \ \exists \ x \in [a, b] : f(x) = c$.*

Definition 6.2 (Darboux function). A Darboux function is a function that satisfies the intermediate value property.

Proposition 6.3. *Continuous implies Darboux, but not the opposite.*

Proposition 6.4. *Continuous functions map intervals to intervals.*

Definition 6.3 (Connectedness). Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold

- There exists $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exists $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$

N.B.: here $\overline{U}$ doesn't mean complementary set of U, but set closure of U. That is, the smallest closed set containing U.
A set is connected if it is not disconnected.

Proposition 6.5. *Continuous functions preserve connectedness.*

$$f : (X, \tau_x) \to (Y, \tau_y), A \subseteq X \ \textit{connected in} \ (X, \tau_x) \Rightarrow Im(A) \subseteq Y \ \textit{is connected in} \ (Y, \tau_y)$$

7 Uniform continuity

Definition 7.1 (Uniform continuity). $f : (X, d_x) \to (Y, d_y)$ is uniformly continuous on X if

$$\forall\ \varepsilon > 0 \quad \exists\ \delta_\varepsilon > 0 :\ \forall\ x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Remark. Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall\ \varepsilon > 0 \quad \exists\ \delta_{\varepsilon, x_0} > 0 :\ \forall\ x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem 7.1. *f continuous on A, closed and bounded $\Rightarrow$ f is uniformly continuous on A.*

Theorem 7.2. *f uniformly continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow$ $f(s_n)$ is Cauchy sequence.*

Definition 7.2 (Sequence of functions). $(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Definition 7.3 (Pointwise convergence). f_n converges pointwise to $f \iff \lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall\ \varepsilon > 0, x \in S\ \exists\ N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Definition 7.4 (infinite norm). $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)|\}$

Definition 7.5 (Uniform convergence). f_n converges uniformly to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Theorem 7.3. *Uniform limit of a continuous function is continuous.*

$$f_n(x) \textit{ continuous and } f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x) \textit{ is continuous}$$

8 Power Series

Definition 8.1 (Power series). Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.
- The series converges for some bounded interval.

Theorem 8.1. *Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ ($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then* $\displaystyle\sum_{n=0}^{\infty} a_n x^n$

- *Converges for* $|x| < R$.
- *Diverges for* $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.

9 Lipschitz continuity

Definition 9.1 (Lipschitz continuity). $f : (X, d_x) \to (Y, d_y)$ is Lipschitz continuous if it exists $c \in [0, +\infty)$ such that $d_y(f(x), f(x')) \le c d_x(x, x')$.

Proposition 9.1. *Lipschitz continuity* $\Rightarrow$ *uniform continuity.*

Theorem 9.2 (Weierstrass approximation). *Every continuous function on* $[a, b]$ *can be uniformly approximated by polynomials on* $[a, b]$

$$\exists\ (a_n) \subseteq \mathbb{R} : p_n(x) = \sum_{k=1}^{n} a_k x^k \xrightarrow{unif.} f(x)\ \ on\ \ [a, b]$$

Theorem 9.3 (Bernstein polynomials). $b_{m,n}(x) = \binom{n}{m} x^m (1-x)^{n-m}$

$$span\{b_{0,n}(x), \dots, b_{n,n}(x)\} = \left\{\sum_{k=1}^{n} a_k x^k, a_i \in R\right\}$$

Theorem 9.4. $f : [0, 1] \to \mathbb{R}$ *continuous, then*

- $B_n(f)(x) = \sum_{m=0}^{n} f(\frac{m}{n}) b_{m,n}(x)$
- $B_n(f)(x) \to f(x)$ *uniformly continuous on* $[0, 1]$

10 Differentiability and derivatives

Definition 10.1 (Derivative). The derivative of a function f at point a is defined as one

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{\varepsilon \to 0} \frac{f(a + \varepsilon) - f(a)}{\varepsilon}$$

Definition 10.2 (Differentiability). f is differentiable if the derivative f' exists.

Proposition 10.1. f *differentiable at* a, *then* f *continuous at* a.

Definition 10.3. $f \in \mathcal{C}^k(\mathbb{R})$, f is differentiable k times, and the derivatives are continuous.

Proposition 10.2. *Properties of derivatives*

- $(f+g)'(x) = f'(x) + g'(x)$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} \qquad \forall\ g(x) \neq 0$
- $(g \circ f)'(x) = (g' \circ f)(x)f'(x) = g'(x)f(x)f'(x)$
- $f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))}$

Definition 10.4 (Local minimizer). x^* is a local minimizer if exists $\varepsilon > 0$ such that $f(x^*) \leq f(x)$ for all $x \in (x^* - \varepsilon, x^* + \varepsilon)$. This means, $f(x^*)$ is local minimum (the smallest image in a given interval).

Theorem 10.3. *$f : \mathbb{R} \to \mathbb{R}(a,b)$ is differentiable and has a local minimum at $x \Rightarrow f'(x) = 0$.*

Theorem 10.4 (Rolle's theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on (a,b) and $f(a) = f(b) \Rightarrow$ it exists $x \in (a,b)$ such that $f'(x) = 0$.*

Theorem 10.5 (Mean value theorem). *Let $f : \mathbb{R} \to \mathbb{R}[a,b]$ differentiable on $(a,b) \Rightarrow$ it exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*

Theorem 10.6 (Second order optimality conditions). *Let $f \in \mathcal{C}^2(\mathbb{R})$ and $f'(x) = 0$*

- *If $f''(x) > 0 \Rightarrow x$ is a local minimum*
- *If $f''(x) < 0 \Rightarrow x$ is a local maximum*
- *If $f''(x) = 0 \Rightarrow x$ is an inflection point*

Definition 10.5 (Convex vector space). Let A be a vector space, $x, y \in A$ and $t \in [0,1]$. Then A is convex if $tx + (1-t)y \in A$.

Definition 10.6 (Convex function). $f : \mathbb{R} \to \mathbb{R}[a,b]$ is convex if for all $x, y \in [a,b]$, $t \in [0,1]$, then

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Theorem 10.7. *If f is convex, then global minimum is local minimum.*

Theorem 10.8 (Gradient inequality). *$f \in \mathcal{C}^1$ is convex $\iff f(x) \geq f(y) + f'(y)(x-y)$*

Theorem 10.9 (Newton's method). *Newton's method is a way to approximate a local minimum or maximum of a function. $x^{(0)}$ is the initial guess of a local minimum $\Rightarrow x^{(n+1)} = x^{(n)} - \frac{f'(x^{(n)})}{f''(x^{(n)})}$ is a more precise approximation.*

Theorem 10.10 (Taylor' series). *Taylor series are a way to approximate a function. Let $f \in \mathcal{C}^{\infty}(\mathbb{R})$, then its Taylor series around point x_0 is $T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.*

Definition 10.7. If $f(x) = T_f(x)$ for all x, then $f(x)$ is analytic.

Theorem 10.11 (Taylor's theorem). *$f \in \mathcal{C}^{n+1}(\mathbb{R})$, then it exists $\xi \in (a, x)$ such that*

$$f(x) = \sum_{k=0}^{n} \left(\frac{f^{(k)}(a)}{k!}(x-a)^k \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Where $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = O((x-a)^{n+1})$ is the error of approximation.

11 Integrals

Definition 11.1 (Partition). Let $f : \mathbb{R} \to \mathbb{R}[a, b]$, $\Delta = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ is a partition of $[a, b]$. Let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. Then

$$L_\Delta(f) = \sum_{k=1}^{n}(x_k - x_{k-1})m_k, \quad U_\Delta(f) = \sum_{k=1}^{n}(x_k - x_{k-1})M_k$$

$L(f) = \sup\{L_\Delta(f)\}$ and $U(f) = \inf\{U_\Delta(f)\}$ are the lower and upper Darboux sums.

Theorem 11.1 (Ross' theorem). *f bounded on $[a, b] \Rightarrow L(f) \leq U(f)$*

Definition 11.2 (Darboux (Riemann) integral). If $L(f) = U(f)$, then f is Darboux integrable and we call the integral $L(f) = U(f) = \int_a^b f(x)dx$.

Proposition 11.2. *f continuous and bounded $\Rightarrow f$ is Riemann integrable.*

Proposition 11.3 (Properties of integrals). *$f, g : \mathbb{R} \to \mathbb{R}[a, b]$ integrable, $\lambda \in \mathbb{R}$ and $c \in [a, b]$. Then:*

(1) $\int_a^b (\lambda f)(x)dx = \lambda \int_a^b f(x)dx$

(2) $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(4) *If $f(x) \leq g(x)\ \forall\ x \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$*

Theorem 11.4. *If f is monotonic or continuous, then f is integrable.*

Theorem 11.5. *If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.*

Theorem 11.6 (Mean value theorem for integrals). *$f, g : \mathbb{R} \to \mathbb{R}[a, b]$ continuous, $g(x) \geq 0$ for all $x \in [a, b] \Rightarrow$ it exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$*

Corollary 11.6.1. *$f : \mathbb{R} \to \mathbb{R}[a, b]$ continuous, then it exists $c \in [a, b]$ such that $\int_a^b f(x)dx = f(c)(b - a)$.*

12 Antiderivatives (or indefinite integrals)

Definition 12.1 (Antiderivative). $F : \mathbb{R} \to \mathbb{R}[a, b]$ differentiable, is the antiderivative of $f : \mathbb{R} \to \mathbb{R}[a, b]$ if $F'(x) = f(x)$. We write $\int f(x)dx$.

Theorem 12.1 (Fundamental theorem of calculus). *$f : \mathbb{R} \to \mathbb{R}[a, b]$ continuous, then f has an unique antiderivative $F(x) = \int_a^x f(t)dt$, with $F(a) = 0$.*

Corollary 12.1.1. *$f : \mathbb{R} \to \mathbb{R}[a, b]$, F antiderivative of f, then $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$.*

Theorem 12.2 (Integration by parts). *$f, g : \mathbb{R} \to \mathbb{R}\int a, b \in \mathcal{C}^1([a, b])$, then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Theorem 12.3 (Integration by substitution). *$f : \mathbb{R} \to \mathbb{R}\int a, b$ continuous $g : \mathbb{R} \to \mathbb{R}\int a, b \in \mathcal{C}^1([a, b])$, then:*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$$