Calculus – Lecture Notes

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Contents

1 Sets, groups and fields

Definition Natural numbers $\mathbb{N}$

(i) $1 \in \mathbb{N}$

(ii) $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$ ($n+1$ is the successor of n)

(iii) $\nexists n \in \mathbb{N} : n+1 = 1$ (no number is predecessor of 1)

(iv) $m, n \in \mathbb{N}$ and $m+1 = n+1 \Rightarrow m = n$

(v) $A \subseteq \mathbb{N}$, $n \in A$ and $n+1 \in A \Rightarrow A = \mathbb{N}$

Definition Group
A set X and an operation $\circ$ form a group $(X, \circ)$ if the following rules are satisfied for all $a, b, c \in X$:

(i) Closure: $a \circ b \in X$

(ii) Associativity: $(a \circ b) \circ c = a \circ (b \circ c)$

(iii) Identity: $\exists! 0 \in X : a \circ 0 = 0 \circ a = a$

(iv) Inverse: $\exists! (-a) \in X : a \circ (-a) = (-a) \circ a = 0$

The group $(X, \circ)$ is abellian if the following rule is satisfied too:

(v) Commutativity: $a \circ b = b \circ a$

Example

(1) $(\mathbb{Z}_2, \oplus)$ is an abellian group (where $\mathbb{Z}_2 = \{0, 1\}$ and $\oplus$ is exclusive or):

  (i) Closure: $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$
  (ii) Associativity: we have two elements, so we don't need to prove it
  (iii) Identity: $0 \Rightarrow 0 \oplus 0 = 0$, $1 \oplus 0 = 1$, $0 \oplus 1 = 1$
  (iv) Inverse: $(-1) = 1, (-0) = 0 \Rightarrow 1 \oplus 1 = 0$, $0 \oplus 0 = 0$
  (v) Commutativity: $1 \oplus 0 = 1 = 0 \oplus 1$

(2) $(\mathbb{N}, +)$ is not a group, since it doesn't have the identity element.

Definition Field
Given a set X, then $(X, +, \cdot)$ is a field if it satisfies the following properties for all $a, b, c \in X$:

(i) $a + b \in X$
$a \cdot b \in X$

(ii) $(a + b) + c = a + (b + c)$
$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iii) $\exists! 0 \in X : a + 0 = 0 + a = a$
$\exists! 1 \in X : a \cdot 1 = 1 \cdot a = a$

(iv) $\exists!(-a) \in X : a + (-a) = (-a) + a = 0$
$\forall a \neq 0, \exists! a^{-1} : a \cdot a^{-1} = a^{-1} \cdot a = 1$

(v) $a + b = b + a$
$a \cdot b = b \cdot a$

(vi) $a \cdot (b + c) = a \cdot b + a \cdot c$

Example Application of field axioms

(1) If $a \cdot b = 0 \Rightarrow$ either a or b are equal to 0. We suppose $b \neq 0$:
$0 = 0 \cdot b^{-1} = (a \cdot b) \cdot b^{-1} = a \cdot (b \cdot b^{-1}) = a \cdot 1 = a \Rightarrow a = 0$ (the same can be done supposing $a \neq 0$)

(2) $a \cdot 0 = 0$:
$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \Rightarrow$ we subtract from both sides $(-\ a \cdot 0)$:
$a \cdot 0 + (-\ a \cdot 0) = a \cdot 0 + a \cdot 0 + (-\ a \cdot 0) \iff 0 = a \cdot 0$

Definition $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Remark $(\mathbb{Q}, +, \cdot)$ is a field.

Definition Ordered field
Let $\leq$ be an order relation. Then the field $(X, +, \cdot, \leq)$ is an ordered field if the following properties are satisfied for $a, b, c \in X$:

(i) Either $a \leq b$ or $b \leq a$

(ii) If $a \leq b$ and $b \leq a$, then $a = b$

(iii) If $a \leq b$ and $b \leq c$, then $a \leq c$

(iv) If $a \leq b$, then $a + c \leq b + c$

(v) If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$

Example Application of the order axioms
Let's take $(\mathbb{Q}, +, \cdot, \leq)$, $a, b \in \mathbb{Q}$. We want to show that if $a \leq b$, then $(-b) \leq (-a)$:
$a \leq b \iff a + ((-a) + (-b)) \leq b + ((-a) + (-b)) \iff (a + (-a)) + (-b) \leq (-a) + (b + (-b))$
$\iff (-b) + 0 \leq (-a) + 0 \iff (-b) \leq (-a)$

Definition A set A is countably infinite if it exists a function $f : A \to \mathbb{N}$ bijective.

Remark Let A, B sets:

- If $|A| = |B| \iff$ exists a bijection between A and B
- If $|A| \leq |B| \iff$ exists an injection from A to B
- If $|A| < |B| \iff$ exists an injection, but not a bijection

Proposition $\mathbb{Z}$ is countably infinite

Proposition $\mathbb{Q}$ is countably infinite

Proposition $\mathbb{R}$ is not countable

Proposition $|\mathbb{R}| = |\mathbb{R}^2|$

Definition Power set
Let A be a set. The power set of A is $2^A = \{A' : A' \subseteq A\}$, the set containing all subsets of A. $|2^A| = 2^{|A|}$

Proposition $|2^{\mathbb{N}}| = |\mathbb{R}|$

Proposition $\sqrt{2} \notin \mathbb{Q}$

Definition Bounds
Let A, X be sets, such that $A \subseteq X$, and $x \in X$, then:

- x is upper bound of A if $a \leq x$, for all $a \in A$
- x is lower bound of A if $x \leq a$, for all $a \in A$

Definition Supremum and infimum
Let A be a set:

- The supremum is the smallest upper bound of A
- The infimum is the greatest lower bound of A

Definition Maximum and minimum
Let A be a set:

- The maximum is the biggest element of A (if $\sup(A) \in A$, then $\max(A) = \sup(A)$)
- The minimum is the smallest element of A (if $\inf(A) \in A$, then $\min(A) = \inf(A)$)

2 Spaces

Definition Topology
Let X be a set. Then $\tau \subseteq 2^X$ is a topology if:

(i) $X \in \tau$

(ii) $\emptyset \in \tau$

(iii) $A_\alpha \in \tau$, then $\bigcup_\alpha A_\alpha \in \tau$ (the union of any element of τ is also contained in τ)

(iv) $A_i \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$ (any finite intersection of elements of τ is also contained in τ)

Example Let $X = \{1, 2, 3, 4\}$

(1) $\tau = \{\emptyset, X\}$ is topology:
$\emptyset \cup X = X \in \tau$
$\emptyset \cap X = \emptyset \in \tau$

(2) $\tau = \{\emptyset, \{2\}, \{2, 3\}, X\}$ is topology:
The cases with $\emptyset$ and X are trivial
$\{2\} \cup \{2, 3\} = \{2, 3\} \in \tau$
$\{2\} \cap \{2, 3\} = \{2\} \in \tau$

(3) $A = \{\emptyset, \{2\}, \{3\}, X\}$ is not a topology:
$\{2\} \cup \{3\} = \{2, 3\} \notin A$

Definition Topological space
Let X be a set, τ a topology, then (X, τ) is a topological space.

Definition Neighborhood in a topological space (X, τ)
A set N is a neighborhood of $x \in X$ if there exists a set $U \in \tau$ such that $x \in U$ and $U \subseteq N$.

Definition Metric
Let X be a set, $x, y, z \in X$. The function $d : X \times X \to \mathbb{R}$ is a metric if:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \iff x = y$

(iii) $d(x, z) \leq d(x, y) + d(x, z)$

Example

(1) $d(x, y) = |x - y|$

(2) $d(x, y) = ||x - y||_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$

Definition Metric space
Let X be a set, d be a metric, then (X, d) is a metric space.

Definition Ball in a metric space (X, d)
$B_r(x) = \{y \in X : d(x, y) < r\}$ is a ball of center x and radius r. $B_r(x)$ is subset of X.

Definition Open set in a topological space (X, τ)
A set U is open in (X, τ) if $U \in \tau$.

Definition Open set in a metric space (X, d)
A set U is open in (X, d) if for all $x \in U$ exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Definition $C \subseteq X$ is closed if $X \setminus C$ is open. A set is closed if its complement is open.

Proposition Let $S = (X, x)$ be a space (x a metric or a topology), then:

(i) X is open in S

(ii) $\emptyset$ is open in S

(iii) For all A_α open in S, then $\bigcup\limits_\alpha A_\alpha$ is open in S (any union of any open set is also open)

(iv) For all A_i open in S, then $\bigcap\limits_{i=1}^n A_i$ is open in S (any finite intersection of any open set is also open)

3 Sequences

Definition Sequence
A sequence (x_n) is a function $x : \mathbb{N} \to X$, where $x(n) = x_n$.
The elements of a sequence can be listed in an ordered set with repetition: $(x_n) = (x_1, x_2, x_3, x_4, \ldots)$

Example

- $a_n = n \Rightarrow (a_n) = (1, 2, 3, 4, 5, \ldots)$
- $b_n = \frac{1}{n} \Rightarrow (b_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$
- $c_n = (-1)^n \Rightarrow (c_n) = (1, -1, 1, -1, \ldots)$

Definition Cauchy sequence
A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N_\varepsilon$.
That is, starting from an index N_ε all values x_n are contained in an interval $[x_{N_\varepsilon} - \varepsilon, x_{N_\varepsilon} + \varepsilon]$.

Example $x_n = \frac{1}{n}$ in $(\mathbb{R}, d)$, where $d(x,y) = |x-y|$. We have to find, for each ε, an N that satisfies the definition of Cauchy sequence.
Let's take $N \leq n \leq m$. Thanks to the triangle inequality, we can first find that:

$$d(x_n, x_m) = |x_n - x_m| \leq |x_n| + |-x_m| = |x_n| + |x_m| = \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m}$$

Since $N \leq n \leq m$, then we have $\frac{1}{m} \leq \frac{1}{n} \leq \frac{1}{N}$:

$$\frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

Now, in order to satisfy the definition we must have $\frac{2}{N} \leq \varepsilon$, thus $\frac{2}{\varepsilon} \leq N$. This means, starting from $N = \frac{2}{\varepsilon}$ all $d(x_n, x_m)$ wil be smaller than ε. In fact, if we take the previous inequality:

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{2}{N} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon$$

Note that it is not important if $d(x,y) < \varepsilon$ or $d(x,y) \leq \varepsilon$.

Definition Convergence in a metric space (X, d)
A sequence (x_n) converges to a limit x if for all $\varepsilon > 0$ exists N_ε such that $d(x_n, x) < \varepsilon$, for all $n \geq N_\varepsilon$.

Example $x_n = \frac{1}{n}$ in $\mathbb{R}$ converges to 0. We take $N \leq n$ and $N = \frac{1}{\varepsilon}$:

$$|x_n - 0| = |x_n| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Definition Convergence in a topological space (X, τ)
A sequence (x_n) converges to a limit x if for all $U \in \tau$ such that $x \in U$, it exists N_U such that $x_n \in U$, for all $n \geq N_U$.
That is, x is a limit of a sequence, if all sets of τ that contain x also contain the tail of the sequence.

Example Let's take $X = \{a, b, c\}$, $(x_n) = (a, b)$ and $\tau = \{\emptyset, \{a\}, X\}$.

- a is not a limit of (x_n), in fact $\{a\}$ contains a, but doesn't contain the tail of (x_n)
- b and c are limits of (x_n), in fact $b \in X$ and $c \in X$, and X contains (x_n) (and its tail too)

Proposition $x_n \to x$ in $(X, d) \iff$ for all $U \subseteq X$ open exists N_U such that $x_n \in U$, for all $n \geq N_U$.

Theorem If a sequence converges to a limit in a metric space, then the limit is unique.

Remark This isn't true in a topological space. In a topological space, a sequence can converge to multiple limits.

Proposition $x_n \to x$ in (X, d) metric space, then for all $y \in X$, $d(x_n, y) \to d(x, y)$.

Proposition Properties of real sequences
For all $(x_n), (y_n)$ such that $x_n \to x$, $y_n \to y$, we have the following properties:

(i) $\lim\limits_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim\limits_{n\to\infty} x_n + \beta \lim\limits_{n\to\infty} y_n$

(ii) $\lim\limits_{n\to\infty} x_n x_y = \lim\limits_{n\to\infty} x_n \lim\limits_{n\to\infty} y_n$

(iii) $\lim\limits_{n\to\infty} \dfrac{x_n}{x_y} = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

Example

(1) Knowing that $\frac{1}{n} \to 0$, show that $\frac{1}{n^2} \to 0$.

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 0 \cdot 0 = 0$$

(2) Find the limit of the sequence $\frac{2n^2-3n+2}{n^2+n-1}$.

$$\lim_{n\to\infty} \frac{2n^2-3n+2}{n^2+n-1} = \frac{\lim(2n^2-3n+2)}{\lim(n^2+n-1)} = \frac{\lim 2 - \lim \frac{3}{n} + \lim \frac{2}{n^2}}{\lim 1 + \lim \frac{1}{n} - \lim \frac{1}{n^2}} = \frac{2-0+0}{1+0-0} = 2$$

Definition A sequence (x_n) is bounded if exists c such that $|s_n| \leq c$.

Example

- $a_n = \frac{1}{n}$ is bounded. We can take $c = 1$, $\left|\frac{1}{n}\right| = \frac{1}{n} \leq 1$, being $a_1 = 1$ the $\sup(a_n)$.
- $b_n = (-1)^n$ is bounded. In fact, the values of the sequence are always 1 and -1. If we take $c = 1$, then $|b_n| = 1 \leq 1$.
- $x_n = n$ is not bounded, we can prove it by contradiction.
  We suppose it exists a c such that $|x_n| \leq c$. If we take $x_{c+1} = c+1$, we have $c+1 \leq c \iff 0 \leq 1$ contradiction. This means x_n is not bounded.

Theorem Convergent real sequences are bounded (not the opposite).

Definition Monotonic sequences

- (x_n) is monotonic increasing if $x_n \leq x_{n+1}$ for all n
- (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ for all n

Example

- $a_n = \frac{1}{n}$ is monotonic decreasing. In fact, $\frac{1}{n+1} \leq \frac{1}{n}$, then $a_{n+1} \leq a_n$.
- $b_n = n$ is monotonic increasing. In fact, $b_n = n$ and $b_{n+1} = n+1$. Since $n \leq n+1$, then $b_n \leq b_{n+1}$.
- $c_n = (-1)^n$ is not monotonic. We can take $n = 1$, then $a_1 \leq a_2 \nleq a_3$.

Theorem If a sequence monotonic and bounded $\Rightarrow$ convergent

Definition Limit superior and inferior of (x_n)

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$
- $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

Definition Subsequence
$(x_{n_k}) \subseteq (x_n)$ is a subsequence of (x_n). Only some terms of a sequence are part of a subsequence.

Example $x_n = (-1)^n \cdot n$. We take $k = 2n$, then the subsequence $(x_{n_k}) = (x_{2n})$ of (x_n) takes all the even indexes n of (x_n):

$$\begin{array}{lcllllllll} (x_n) & = & (& -1, & 2, & -3, & 4, & -5, & 6, & \dots \ ) \\ (x_2 n) & = & (& & 2, & & 4, & & 6, & \dots \ ) \end{array}$$

Theorem If $x_n \to x \Rightarrow x_{n_k} \to x$. If a sequence converges, all subsequences converge to the same limit.

Definition x_n is a dominant term if $x_m < x_n$ for all $n < m$.

Theorem Every sequence has a monotonic subsequence.

Theorem Bolzano-Weierstrass
Every bounded sequence has a convergent subsequence.

Definition $X \subseteq \mathbb{R}^n$ is compact $\iff$ X is closed and bounded (this is not true for $\mathbb{R}^\infty$).

4 Series

Definition Series
(x_n) sequence. $s_n = \displaystyle\sum_{k=1}^{n} x_k$ is a series (also known as the partial sum). A series is the summation of the terms of a sequence.

Definition Convergence of series
$s_n = \displaystyle\sum_{k=1}^{n} x_k$ a series. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k$.

Example

- Harmonic: $\sum_{k=1}^{\infty} \frac{1}{n} = +\infty$
- Geometric: $\sum_{k=1}^{\infty} a^n = \begin{cases} \infty & |a| \geq 1 \\ \frac{1}{1-a} & |a| < 1 \end{cases}$
- Exponential: $\sum_{k=1}^{\infty} \frac{1}{n!} = e$
- Leibniz: $\sum_{k=1}^{\infty} \frac{(-1)^n}{2n-1} = \frac{\pi}{4}$

Definition Absolute convergence of a series $s_n = \sum_{k=1}^{n} x_k$

s_n converges absolutely if $\sum_{k=1}^{\infty} |x_k| < \infty$.

Proposition Absolute convergence $\Rightarrow$ convergence. If $\sum_{k=1}^{\infty} |x_k| < \infty$, then $\sum_{k=1}^{\infty} x_k < \infty$.

Definition Cauchy criterion for series
$s_n = \sum_{k=1}^{n} x_k$, and$\sum_{k=1}^{\infty} x_k < \infty$ is a Cauchy series if for all $\varepsilon > 0$ it exists N such that:

$$\forall N \leq m \leq n \Rightarrow |s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k \right| = \left| \sum_{k=m}^{n} x_k \right| < \varepsilon$$

Proposition Comparison test, for x_n, y_n sequences and $x_n \geq 0$

(i) If $\sum_{k=1}^{\infty} x_k < \infty$ and $|y_n| \leq x_n \Rightarrow \sum_{k=1}^{\infty} y_k < \infty$

(ii) If $\sum_{k=1}^{\infty} x_k = +\infty$ and $x_n \leq y_n \Rightarrow \sum_{k=1}^{\infty} y_k = +\infty$

Example $\sum_{k=1}^{\infty} \frac{1}{n^2+1} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{n^2+1} < \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty$, the series converges.

Proposition Ratio test, for x_n sequence, $x_n \neq 0$ and $s_n = \sum_{k=1}^{n} x_k$ series:

(i) s_n converges absolutely if $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1$

(ii) s_n diverges if $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| > 1$

Example

- $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^n$:

$$\left|\frac{(-\frac{1}{3})^{n+1}}{(-\frac{1}{3})^n}\right| = \left|-\frac{1}{3}\right| = \frac{1}{3} \Rightarrow \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1 \Rightarrow \text{ converges absolutely}$$

- $\sum_{k=1}^{\infty} \frac{n}{n^2+3}$
  - Ratio test:

$$\limsup_{n\to\infty} \left|\frac{n+1}{(n+1)^2+3}\frac{n^2+3}{n}\right| = \limsup_{n\to\infty} \frac{n+1}{(n+1)^2+3}\frac{n^2+3}{n} = 1, \text{ no information}$$

  - Comparison test:

$$\frac{n}{n^2+3n^2} \leq \frac{n}{n^2+3} \Rightarrow \frac{n}{n^2+3n^2} = \frac{n}{4n^2} = \frac{1}{4}\frac{n}{n^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{4}\frac{n}{n^2} \Rightarrow +\infty = \frac{1}{4}\sum_{k=1}^{\infty} \frac{n}{n^2} \leq \sum_{k=1}^{\infty} \frac{n}{n^2+3}$$

  The series diverges. Sometimes one test can give more information than others.

Proposition Root test, Let $s_n = \sum_{k=1}^{n} x_k$ a series, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$:

(i) s_n converges absolutely if $\alpha < 1$

(ii) s_n diverges if $\alpha > 1$

Example

- $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^n \Rightarrow \limsup_{n\to\infty} \sqrt[n]{-\frac{1}{3}^n} = \limsup_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1$, the series converges absolutely

- $\sum_{k=1}^{\infty} 2^{(-1)^n-n} \Rightarrow \sqrt[n]{2^{(-1)^n-n}} = \begin{cases} 2^{\frac{1}{n}-1} & \text{if } n \text{ even} \\ 2^{-\frac{1}{n}-1} & \text{if } n \text{ odd} \end{cases} \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}-1} = \lim_{n\to\infty} 2^{-\frac{1}{n}-1} = \frac{1}{2} < 1$, the series converges.

5 Functions and continuity

Definition Given a function $f : X \to Y$, the image of f is defined as: $Im_f(X) = \{f(x) : x \in X\}$. It contains all the images of all elements of X.

Definition Given a function $f : X \to Y$, the preimage of f is defined as: $PreIm_f(Y) = \{x : f(x) \in Y\}$. It contains all the elements of X that have an image in Y.

Definition Continuity of $f : (X, d_x) \to (Y, d_y)$ (in a metric space)
f is continuous at $x \in X$ if:

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 : \forall x' \in X, d_x(x, x') < \delta_\varepsilon \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Example Let's take $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin \frac{1}{x} & \text{otherwise} \end{cases}$
We want to prove that f is continuous in 0. Let $\varepsilon > 0$, then $|f(x) - f(0)| = |f(x) - 0| = |f(x)| \leq x^2$. If we take $\delta = \sqrt{\varepsilon}$, then:

$$|x - 0| < \delta \Rightarrow x^2 < \delta \Rightarrow |f(x) - f(0)| \leq x^2 < \delta^2 = \varepsilon \Rightarrow f \text{ is continuous in } 0$$

Remark Continuity can also be defined as follows:

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0 : Im_f(B^{d_x}_{\delta\varepsilon}(x)) \subseteq B^{d_y}_{\varepsilon}(f(x))$$

This means that the image of each ball around each x is contained in another ball around $f(x)$.

Definition Continuity of $f : (X, \tau_x) \to (Y, \tau_y)$ (in a topological space)
f is continuous at $x \in X$ if for all $U \in \tau_y$ such that $f(x) \in U$, then $PreIm_f(U) \in \tau_x$.

Example Let's take (M, τ_m), (N, τ_n), $M = N = \{1, 2\}$, $\tau_m = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\tau_m = \{\emptyset, \{1, 2\}\}$.

- Let $f : (M, \tau_m) \to (N, \tau_n)$, such that $f(1) = 2$ and $f(2) = 1$:

$$PreIm_f(\emptyset) = \emptyset \in \tau_m, PreIm_f(\{1, 2\}) = \{1, 2\} \in \tau_m \Rightarrow f \text{ is continuous in all } x \in M$$

- Let $g : (N, \tau_n) \to (M, \tau_m)$, such that $f(1) = 2$ and $f(2) = 1$:

$$PreIm_g(\{1\}) = \{2\} \notin \tau_n \Rightarrow g \text{ is not continuous}$$

Proposition Continuous functions map open sets into open sets. If $f : (X, d_x) \to (Y, d_y)$ continuous, then $PreIm_f(A)$ is open, for all $A \subseteq Y$ open.

Theorem Continuous functions map limits to limits:

$$f \text{ continuous}, x_n \to x \iff f(x_n) \to f(x)$$

Example Let's take $f(x) = 2x^2 + 1$ and $\lim_{n \to \infty} x_n = x$. Then:

$$\lim_{n \to \infty} 2x_n^2 + 1 = 2\left(\lim_{n \to \infty} x_n\right)^2 + 1 = 2x^2 + 1$$

This means that for $x_n \to x$, then $f(x_n) \to f(x)$. Therefore, f is continuous.

Proposition $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $x \Rightarrow f + g$, $f \cdot g$ and $\frac{f}{g}$ (for $g(x) \neq 0$) are continuous at x.

Proposition f continuous at x and g continuous at $f(x) \Rightarrow g \circ f = g(f(x))$ is continuous at x.

Definition $f : (X, d) \to (X, d)$ is a contraction $\iff$ it exists $0 \leq c < 1$ such that $d(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Theorem Banach fixed point
Let's take (X, d) complete (Cauchy $\iff$ convergence) and $f : (X, d) \to (X, d)$ a contraction, then:

(i) $\exists! x^* \in X : f(x^*) = x^*$

(ii) $x_0 \in X$, $x_{n+1} = f(x_n) \Rightarrow x_n \to x^*$

6 Limits of functions

Definition f converges to c at x_0 $\iff$ for all (x_n) such that $x_n \to x_0$ we have $f(x_n) \to c$. We write $\lim_{x \to x_0} f(x) = c$.

- f converges from above if, for all (x_n), then $x_0 < x_n$. We write $\lim_{x \to x_0^+} f(x) = c$.
- f converges from below if, for all (x_n), then $x_n < x_0$. We write $\lim_{x \to x_0^-} f(x) = c$.

Example

- Let $f(x) = \dfrac{1}{x} \Rightarrow \lim_{x \to 0^+} \dfrac{1}{x} = +\infty$, $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$
- Let $f(x) = floor(x) \Rightarrow \lim_{x \to 1^+} floor(x) = 1$, $\lim_{x \to 1^-} floor(x) = 0$, but $\lim_{x \to \frac{1}{2}^+} floor(x) = \dfrac{1}{2} = \lim_{x \to \frac{1}{2}^-} floor(x)$

Definition $f : \mathbb{R} \to \mathbb{R}$ is bounded on $X \subseteq \mathbb{R}$ if $Im(X) = \{f(x) : x \in X\}$ is bounded. That is, it exists c such that $|f(x)| \leq c$ for all $x \in X$.

Example $f : \mathbb{R} \to [-1, 1]$, $f(x) = \sin(x)$ is bounded on $\mathbb{R}$, since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$.

Theorem Extreme value
If $f : [a,b] \to \mathbb{R}$ is continuous, then:

(i) f is bounded on $[a,b]$

(ii) f has a maximum and a minimum on $[a,b]$
$\exists x_{minimizer}, x_{maximizer} \in [a,b] : f(x_{mininizer}) \leq f(x) \leq f(x_{maximizer}), \forall x \in [a,b]$

Remark This isn't true if the interval is open:

- $f : \mathbb{R} \to \mathbb{R}(0,1)$, $f(x) = \frac{1}{x}$ is unbounded, since $f(x)$ goes to infinity for x small
- $f : (-1,1) \to$, $f(x) = x^2$, doesn't have a max, since $\sup\{Im((-1,1))\} = 1$ is $f(1)$ or $f(-1)$, but 1 and -1 $\notin (-1,1)$

Theorem Intermediate value (IVT)
f continuous on $[a,b]$, $f(a) < c < f(b) \Rightarrow \exists x \in [a,b] : f(x) = c$.

Definition A Darboux function is a function that satisfies the intermediate value property.

Proposition Continuous implies Darboux, but not the opposite.

Example $f(x) = \begin{cases} \sin(\frac{1}{x}) & x > 0 \\ 0 & x = 0 \end{cases}$ is a Darboux function, but it is not continuous.

Proposition Continuous functions map intervals to intervals.

Definition Connectedness
Let (X, τ) a topological space, the $A \subseteq X$ is disconnected if the two equivalent definitions hold:

- There exist $U, V \in \tau$ such that:
  - $(A \cap U) \cap (A \cap V) = \emptyset$, and
  - $(A \cap U) \cup (A \cap V) = A$, and
  - $A \cap U \neq \emptyset \neq A \cap V$
- There exist $U, V \subseteq A$ such that:
  - $A = U \cup V$, and
  - $\overline{U} \cap V = \emptyset = U \cap \overline{V}$! NOT SURE !

A set is connected if it is not disconnected.

Proposition Continuous functions preserve connectedness.
$f : (X, \tau_x) \to (Y, \tau_y)$, $A \subseteq X$ connected in (X, τ_x), then $Im(A) \subseteq Y$ is connected in (Y, τ_y).

Definition Uniform continuity
$f : (X, d_x) \to (Y, d_y)$ is uniformely continuous on X if:

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 : \forall x, x' \in X : d_x(x, x') < \delta \Rightarrow d_y(f(x), f(x')) < \varepsilon$$

Example $f(x) = \frac{1}{x^2}$ in $[a, +\infty)$, $a > 1$. To show that f is uniformely continuous, we have to show that for all $\varepsilon > 0$ exists $\delta_\varepsilon > 0$ such that for all x, y such that $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.
Let $\varepsilon > 0$ and $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \frac{(x+y)(x-y)}{x^2y^2}$. Then, since $a \leq x, y \ \forall x, y$:

$$\frac{(x+y)}{x^2y^2} = \frac{x}{x^2y^2} + \frac{y}{x^2y^2} \leq \frac{2}{a^3}$$

We chose $\delta = \frac{\varepsilon a^3}{2}$, then:

$$\forall x, y \geq a : |x - y| < \delta \Rightarrow |f(x) - f(y)| = |x - y| \left| \frac{x + y}{x^2y^2} \right| < \delta \frac{2}{a^3} = \frac{\varepsilon a^3}{2} \frac{2}{a^3} = \varepsilon$$

This mean f is uniformely continuous.

Remark Uniform continuity is different from normal continuity. In normal continuity the δ depends on both ε and x, while in uniform continuity δ depends solely on ε. In fact, f is "normally" continuous on $x_0 \in X$ if:

$$\forall \varepsilon > 0 \quad \exists \delta_{\varepsilon, x_0} > 0 : \forall x \in X : d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \varepsilon$$

Theorem f continuous on A, closed and bounded $\Rightarrow f$ is uniformely continuous on A.

Theorem f uniformely continuous on S, $(s_n) \subseteq S$ is Cauchy sequence $\Rightarrow f(s_n)$ is Cauchy sequence.

Example $f(x) = \frac{1}{x^2}$ is not uniformely continuous on $(0, 1)$. In fact, $s_n = \frac{1}{n}$ is Cauchy, but $f(s_n) = n^2$ is not Cauchy.

Definition Sequence of functions
$(f_n) \subseteq \{f : S \to \mathbb{R}\}$ is a sequence of functions. A sequence of function can converge to a function: $f_n \to f$.

Example $f_n(x) = \frac{x}{n} \to f(x) = 0$

Definition f_n converges pointwise to $f \iff \lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in S$.

$$\forall \varepsilon > 0, x \in S \exists N_\varepsilon : |f_n(x) - f(x)| < \varepsilon$$

Example $f_n(x) = x^n$, $x \in [0, 1] \Rightarrow f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$. f_n is continuous and f is discontinuous.

Definition $d_\infty(f_n, f) = \sup\{|f_n(x) - f(x)| < \varepsilon\}$

Definition f_n converges uniformely to f if exists N_ε such that $d_\infty(f_n, f) < \varepsilon$ for all $n \geq N_\varepsilon$.

Example Let $f_n(x) = (1 - |x|)^n$, $x \in (-1, 1)$. Then f converges pointwise (but not uniformely) to $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. In fact:

- Pointwise convergence
  For $x = 0$, $f_n(x) = (1-0)^n = 1$, then $\lim\limits_{n\to\infty} f_n = \lim\limits_{n\to\infty} 1 = 1$.
  For $x \neq 0$, $|x| < 1$. This means $1 - |x| < 1$, then $\lim\limits_{n\to\infty} (1 - |x|)^n = 0$.

- Uniform convergence
  We assume $f_n \xrightarrow{unif.} f$ and we take $\varepsilon = \frac{1}{2}$. Then it exists N such that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $x \in (-1, 1)$.
  Let's take $x = 1 - 2^{-\frac{1}{n}}$, then $1 - x = 2^{-\frac{1}{n}}$. Thus $(1-x)^n = (2^{-\frac{1}{n}})^n = \frac{1}{2} \not< \frac{1}{2} = \varepsilon$. Contradiction, f doesn't converge uniformely to f.

Theorem Uniform limit of a continuous function is continuous
$f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x) \Rightarrow f(x)$ is continuous.

Example

- Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n + g_n \xrightarrow{unif.} f + g$. In fact:

$$\exists N_f : \forall x \in S \, |f_n(x) - f(x)| < \frac{\varepsilon}{2} \forall n > N_f$$

$$\exists N_g : \forall x \in S \, |g_n(x) - g(x)| < \frac{\varepsilon}{2} \forall n > N_g$$

  We take $N = \max\{N_f, N_g\}$. Then:

$$|f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall n \geq N$$

  This means $f_n + g_n \xrightarrow{unif.} f + g$.

- Let $f_n \xrightarrow{unif.} f$ and $g_n \xrightarrow{unif.} g$ on $S \subseteq \mathbb{R}$. Then $f_n g_n$ doesn't converge uniformely to fg. In fact, let $h_n(x) = \frac{x}{n}$. By contradiction we can prove h_n doesn't converge uniformely to h. Now, if we take $f_n(x) = \frac{1}{n}$ and $g_n(x) = x$ (uniformely convergent), then $f(x)g(x) = \frac{x}{n} = h(x)$ not uniformely convergent. We found a counter example.

- Let $f_n(x)$ continuous on $[a, b]$, $f_n(x) \xrightarrow{unif.} f(x)$, $(x_n) \subseteq [a, b]$ and $x_n \to x$. Then, $f_n(x_n) \to f(x)$. To prove it we have to show that exists N such that for all $n \geq N$, then $|f_n(x_n) - f(x)| < \varepsilon$.

  1. $f_n \xrightarrow{unif.} f$, this means it exists N_1 such that $|f_n(y) - f(y)| < \frac{\varepsilon}{2}$, for all $n \geq N_1$ and $y \in [a, b]$. In particular, $|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}$.
  2. Since $f_n(x)$ continuous and $f_n(x) \xrightarrow{unif.} f(x)$, then $f(x)$ is continuous. Then $f(x_n) \to f(x)$, this means it exists N_2 such that for all $n \geq N_2$, then $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$.

3. We chose $N = \max\{N_1.N_2\}$. Then:

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \forall n \geq N$$

We can conclude that $f_n(x_n) \to f(x)$.

7 Power series

Definition Power series
Let $(a_n)_{n\geq 0} \subseteq \mathbb{R}$ a sequence. Then $\sum_{n=0}^{\infty} a_n x^n$ is a power series. We have three cases:

- The series converges for all $x \in \mathbb{R}$.
- The series converges for $x = 0$ only.
- The series converges for some bounded interval.

Theorem Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ ($R = \infty$ if $\beta = 0$, $R = 0$ if $\beta = \infty$). Then $\sum\limits_{n=0}^{\infty} a_n x^n$:

- Converges for $|x| < R$.
- Diverges for $|x| > R$.

The same can be done with $\beta = \limsup \left|\frac{a_n}{a_{n+1}}\right|$.

Example

- Let $a_n = 1$. We have the power series $\sum\limits_{n=0}^{\infty} x^n$ and $\beta = \limsup \sqrt[n]{1} = 1$, then $R = 1$. This means the series converges for $x \in (-1, 1)$ and diverges for x such that $|x| > 1$. Moreover, it diverges for $x = 1$, since $\sum\limits_{n=0}^{\infty} 1 = +\infty$, and it is not defined for $x = -1$.
- Let $\sum\limits_{n=0}^{\infty} \frac{1}{n} x^n$ a power series. Then $\limsup \left|\frac{a_n}{a_{n+1}}\right| = \limsup \left|\frac{n}{n+1}\right| = 1$, then $R = 1$. For $x = 1$ we have the harmonic series $\sum\limits_{n=0}^{\infty} \frac{1}{n}$ wich diverges, for $x = -1$ we have $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n} < \infty$. We can conclude that the power series converges for $x \in [-1, 1)$.