A General Relativity Workbook

Solutions

Contents

1. Introduction

Problem 1.1 Geodesic in spacetime.

a. • For ball:

$$t = \frac{10m}{5m/s} = 2s \tag{1.1}$$

$$h = \frac{1}{2}g(\frac{t}{2})^2 = \frac{1}{2}10m/s^2(\frac{2s}{2})^2 = 5m \tag{1.2}$$

• For bullet:

$$t = \frac{10m}{500m/s} = 0.02s \tag{1.3}$$

$$h = \frac{1}{2}g(\frac{t}{2})^2 = \frac{1}{2}10m/s^2(\frac{0.02s}{2})^2 = 5\times 10^{-4}m = 0.5mm \tag{1.4}$$

b.

$$\begin{cases} h & = R[1-\cos(\frac{1}{2}\theta)] = \frac{1}{8}R\theta^2 \\ D & = ct = R\theta \end{cases} \Rightarrow \begin{cases} \theta & = \frac{8h}{ct} \\ R & = \frac{c^2t^2}{8h} \end{cases} \tag{1.5}$$

• For ball:

$$\begin{cases} \theta & = \frac{8*5m}{3*10^8 m/s*2s} = 6.7*10^{-8} \\ R & = \frac{(3*10^8 m/s)^2(2s)^2}{8*5m} = 9*10^{15}m \approx 1\times 10^{16}m \approx 1ly \end{cases} \tag{1.6}$$

- For bullet:

$$\begin{cases} \theta &= \frac{8*5\times10^{-4}m}{3*10^8 m/s*0.02s} = 6.7*10^{-10} \\ R &= \frac{(3*10^8 m/s)^2(0.02s)^2}{8*5\times10^{-4}m} = 9*10^{15}m \approx 1\times10^{16}m \approx 1ly \end{cases} \tag{1.7}$$

$$1ly = 365.25*24*3600s*3*10^8 m/s = 9.47*10^{15}m \approx 1\times10^{16}m \tag{1.8}$$

Problem 1.2 Blue shifted of light in non-inertial frame.

a.

$$t = \frac{d}{c} \Rightarrow v = gt = \frac{gd}{c} \tag{1.9}$$

$$\frac{\lambda}{\lambda_0} = \sqrt{\frac{1-v/c}{1+v/c}} \approx 1 - \frac{v}{c} \Rightarrow \frac{\lambda_0-\lambda}{\lambda_0} = 1 - \frac{\lambda}{\lambda_0} = \frac{v}{c} = \frac{gd}{c^2} \tag{1.10}$$

b.

$$\frac{\lambda_0-\lambda}{\lambda_0} = \frac{gd}{c^2} = \frac{10m/s^2\times25m}{(3\times10^8 m/s)^2} = 2.78\times10^{-15} \tag{1.11}$$

c.

$$\bar{g} = \frac{GM}{R^2} = \frac{6.6743\times10^{-11}m^3kg^{-1}s^{-2}\times3.0\times10^{30}kg}{(12\times10^3 m)^2} = 1.39\times10^{12}m/s^2 \tag{1.12}$$

$$\frac{\lambda_0-\lambda}{\lambda_0} = \frac{\bar{g}d}{c^2} = \frac{GMd}{R^2c^2} = \frac{6.6743\times10^{-11}m^3kg^{-1}s^{-2}\times3.0\times10^{30}kg\times25m}{(12\times10^3 m)^2\times(3\times10^8 m/s)^2} = 3.8\times10^{-4} \tag{1.13}$$

Problem 1.3 Bent of light in gravitational field.

a.

$$\begin{cases} t &= \frac{d}{c} \\ L &= \frac{1}{2}gt^2 \end{cases} \Rightarrow \theta \approx \sin\theta = \frac{L}{d} = \frac{gd}{2c^2} = \frac{10m/s^2\times3m}{2\times(3\times10^8 m/s)^2} = 1.67\times10^{-16} \tag{1.14}$$

b.

$$\theta \approx \sin\theta = \frac{\bar{g}d}{2c^2} = \frac{1.39\times10^{12}m/s^2\times3m}{2\times(3\times10^8 m/s)^2} = 1.67\times10^{-16} = 2.32\times10^{-5} \tag{1.15}$$

where we have used (1.12).

Problem 1.4 Light deflected by the sun.

$$a = \frac{GM}{r^2} \Rightarrow a_y = a\frac{R}{r} = \frac{GMR}{r^3} = \frac{GMR}{(R^2+x^2)^{3/2}} \tag{1.16}$$

$$v_y = \int a_y dt = \int_{-\infty}^{\infty} \frac{GMR}{(R^2+x^2)^{3/2}} \frac{dx}{c} \tag{1.17}$$

$$\begin{aligned}\delta \approx \sin\delta = \frac{v_y}{c} &= \int_{-\infty}^{\infty} \frac{GMR}{(R^2+x^2)^{3/2}} \frac{dx}{c^2} = \frac{2GM}{c^2R} \\ &= \frac{2\times 6.67\times 10^{-11} m^3 kg^{-1} s^{-2} \times 1.98\times 10^{30} kg}{(3\times 10^8 m/s)^2 \times 7\times 10^8 m} = 4.2\times 10^{-6} rad = 0.87 arcsec\end{aligned} \tag{1.18}$$

where we have used

$$1 arcsec = \frac{1}{3600} degree = \frac{1}{3600}\frac{\pi}{180} rad = 4.85\times 10^{-6} rad. \tag{1.19}$$

Problem 1.5 Tidal effect.
Denote the radius of the Earth as $r_0 = 6.371\times 10^6 m$.

$$\begin{cases} a_A &= \frac{GM}{(r_0+22m)^2} \\ a_B &= \frac{GM}{(r_0+44m)^2} \\ a_C &= \frac{GM}{r_0^2} \end{cases} \tag{1.20}$$

$$\begin{aligned} a_B - a_A &= \frac{GM}{(r_0+44m)^2} - \frac{GM}{(r_0+22m)^2} = \frac{GM}{r_0^2}\left(\frac{1}{(1+44m/r_0)^2} - \frac{1}{(1+22m/r_0)^2}\right) \\ &\approx \frac{GM}{r_0^2}[1 - 2\times 44m/r_0 - (1-2\times 22m/r_0)] = -\frac{44m}{r_0} \\ &= -10 m/s^2 \frac{44m}{6.371\times 10^6 m} = -6.9\times 10^{-5} m/s^2 \end{aligned} \tag{1.21}$$

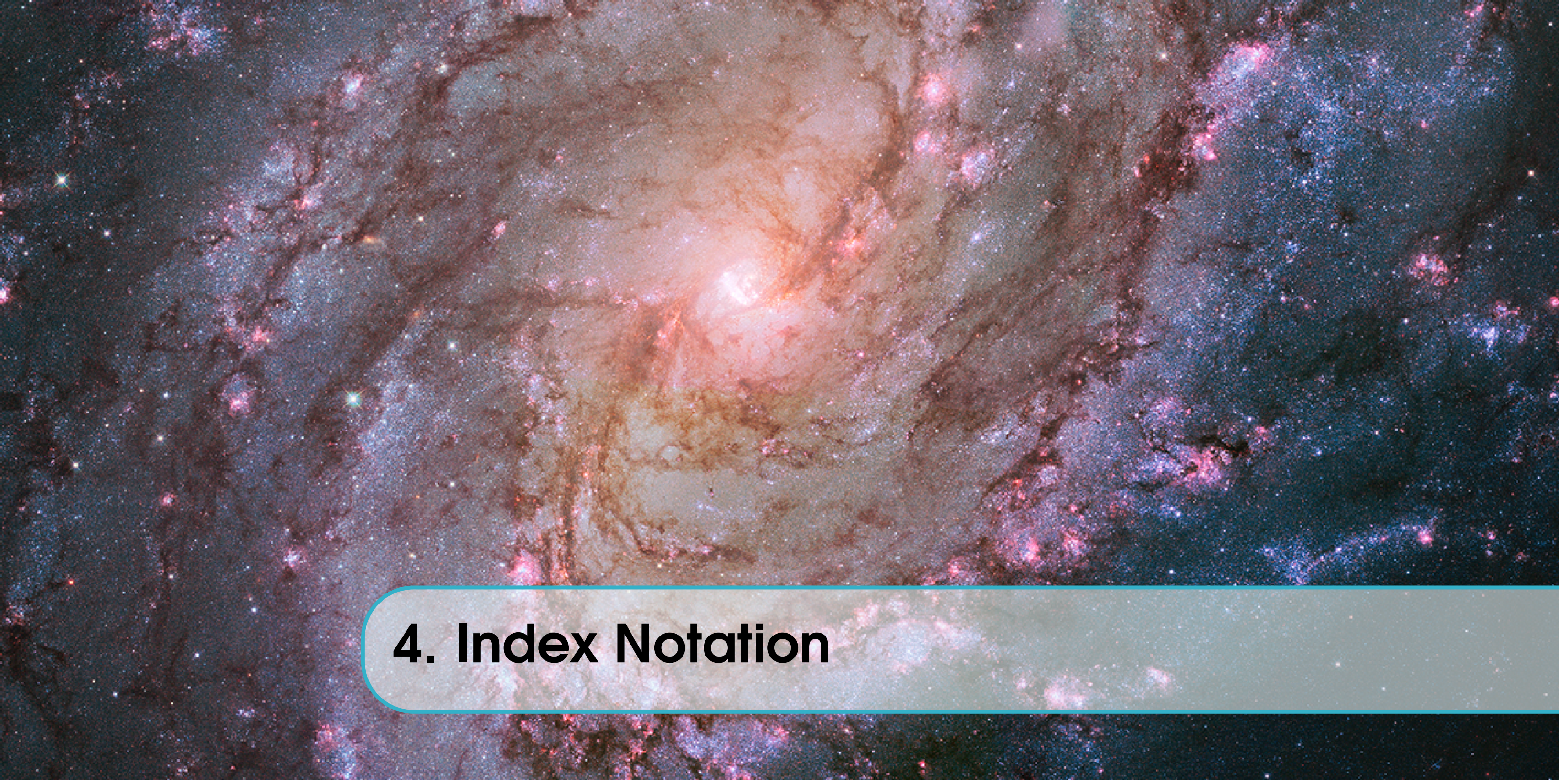
$$\begin{aligned} a_C - a_A &= \frac{GM}{r_0^2} - \frac{GM}{(r_0+22m)^2} = \frac{GM}{r_0^2}\left(1 - \frac{1}{(1+22m/r_0)^2}\right) \\ &\approx \frac{GM}{r_0^2}[1-(1-2\times 22m/r_0)] = \frac{44m}{r_0} = 6.9\times 10^{-5} m/s^2 \end{aligned} \tag{1.22}$$

2. Review of Special Relativity

Problem 2.1 Light been blue shifted in non-inertial frame.

a.
b.
c.

3. Four-Vectors



4. Index Notation