Project Description - Practical Programming exam

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1 Introduction

The ordinary cubic spline simetimes makes unpleasant wiggles, for example, around an outlier, or around a step-like feature of the tabulated function. Here is yet another attempt to reduce the wiggles by building a sub-spline.

2 Creating the data set

Suppose we have a data set $\{x_i, y_i\}_{i=1,...,n}$ that represents a tabulated function. For this sub-spline, we also need a estimate of the derivative in each of these points.

We obtain the estimate for the derivative of each inner point x_i by building a quadratic interpolating polynomial through itself and its neighbouring points x_{i-1}, x_i, x_{i+1}. We then evaluate the first derivative p_i of this polynomial in x_i and use this as the estimate.

For the first and last point x_1, x_n, we estimate p_1, p_n using the same quadratic polynomials used to estimate p_2, p_{n-1}, the first and last inner point.

This in total gives us a data set $\{x_i, y_i, p_i\}_{i=1,...,n}$

3 Building the sub-spline

The cubic sub-spline has the form of a third degree polynomial:

$$S_i(x) = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (1)$$

where the three coefficients b_i, c_i, d_i for each interval are determined by three conditions:

$$S_i(x_{i+1}) = y_{i+1} \quad (2)$$

$$S_i'(x_i) = p_i \quad (3)$$

$$S_i'(x_{i+1}) = p_{i+1} \quad (4)$$

Where S_i' denotes the derivative of the cubic sub-spline:

$$S_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2 \tag{5}$$

This gives us three equations with three unknowns. Equation (3) gives us the coefficient b_i:

$$S_i'(x_i) = p_i \Rightarrow b_i = p_i \tag{6}$$

Equation (2) gives the following equation:

$$S_i(x_{i+1}) = y_{i+1} \Rightarrow y_i + p_i\Delta x_i + c_i\Delta x_i^2 + d_i\Delta x_i^3 = y_{i+1} \tag{7}$$

Where $\Delta x_i = x_{i+1} - x_i$. Isolating for c_i, we get the following:

$$y_i + p_i\Delta x_i + c_i\Delta x_i^2 + d_i\Delta x_i^3 = y_{i+1} \Rightarrow \tag{8}$$

$$c_i\Delta x_i^2 = y_{i+1} - y_i - p_i\Delta x_i - d_i\Delta x_i^3 \Rightarrow \tag{9}$$

$$c_i = \frac{\Delta y_i}{\Delta x_i^2} - \frac{p_i}{\Delta x_i} - d_i\Delta x_i \tag{10}$$

where $\Delta y_i = y_{i+1} - y_i$. Equation (3) gives us the final equation:

$$S_i'(x_{i+1}) = p_{i+1} \Rightarrow p_i + 2c_i\Delta x_i + 3d_i\Delta x_i^2 = p_{i+1} \tag{11}$$

Isolating for d_i, we get the following:

$$p_i + 2c_i\Delta x_i + 3d_i\Delta x_i^2 = p_{i+1} \Rightarrow \tag{12}$$

$$3d_i\Delta x_i^2 = p_{i+1} - p_i - 2c_i\Delta x_i \Rightarrow \tag{13}$$

$$d_i = \frac{\Delta p_i}{3\Delta x_i^2} - \frac{2c_i}{3\Delta x_i} \tag{14}$$

Where $\Delta p_i = p_{i+1} - p_i$. Inserting the expression for c_i in equation (10) into the expression for d_i in equation (14) we get the final expression for d_i:

$$d_i = \frac{\Delta p_i}{3\Delta x_i^2} - \frac{2}{3\Delta x_i}\left(\frac{\Delta y_i}{\Delta x_i^2} - \frac{p_i}{\Delta x_i} - d_i\Delta x_i\right) \Rightarrow \tag{15}$$

$$d_i = \frac{\Delta p_i}{3\Delta x_i^2} - \frac{2\Delta y_i}{3\Delta x_i^3} + \frac{2p_i}{3\Delta x_i^2} + \frac{2}{3}d_i \Rightarrow \tag{16}$$

$$\frac{1}{3}d_i = \frac{\Delta p_i}{3\Delta x_i^2} - \frac{2\Delta y_i}{3\Delta x_i^3} + \frac{2p_i}{3\Delta x_i^2} \Rightarrow \tag{17}$$

$$d_i = \frac{\Delta p_i}{\Delta x_i^2} - \frac{2\Delta y_i}{\Delta x_i^3} + \frac{2p_i}{\Delta x_i^2} \tag{18}$$

This can then be reinserted into equation (10) to obtain the final expression for c_i:

$$c_i = \frac{\Delta y_i}{\Delta x_i^2} - \frac{p_i}{\Delta x_i} - \Delta x_i \left(\frac{\Delta p_i}{\Delta x_i^2} - \frac{2\Delta y_i}{\Delta x_i^3} + \frac{2p_i}{\Delta x_i^2} \right) \Rightarrow \tag{19}$$

$$c_i = \frac{\Delta y_i}{\Delta x_i^2} - \frac{p_i}{\Delta x_i} - \frac{\Delta p_i}{\Delta x_i} + \frac{2\Delta y_i}{\Delta x_i^2} - \frac{2p_i}{\Delta x_i} \Rightarrow \tag{20}$$

$$c_i = \frac{3\Delta y_i}{\Delta x_i^2} - \frac{3p_i}{\Delta x_i} - \frac{\Delta p_i}{\Delta x_i} \tag{21}$$

In total, we have the following expressions for the coefficients:

$$b_i = p_i \tag{22}$$

$$c_i = 3\frac{\Delta y_i}{\Delta x_i^2} - 3\frac{p_i}{\Delta x_i} - \frac{\Delta p_i}{\Delta x_i} \tag{23}$$

$$d_i = \frac{\Delta p_i}{\Delta x_i^2} - 2\frac{\Delta y_i}{\Delta x_i^3} + 2\frac{p_i}{\Delta x_i^2} \tag{24}$$