A high-order predictor-corrector method for solving nonlinear differential equations of fractional order

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Abstract

An accurate and efficient new class of predictor-corrector schemes are proposed for solving nonlinear differential equations of fractional order. By introducing a new prediction method which is explicit and of the same accuracy order as that of the correction stage, the new schemes achieve a uniform accuracy order regardless of the values of fractional order α. In cases of $0 < \alpha \leq 1$, the new schemes significantly improve the numerical accuracy comparing with other predictor-corrector methods whose accuracy depends on α. Furthermore, by computing the memory term just once for both the prediction and correction stages, the new schemes reduce the computational cost of the so-called memory effect, which make numerical schemes for fractional differential equations expensive in general. Both 2nd-order scheme with linear interpolation and the high-order 3rd-order one with quadratic interpolation are developed and show their advantages over other comparing schemes via various numerical tests.

Keywords: Caputo fractional derivative, fractional differential equations, predictor-corrector methods, explicit schemes, linear and quadratic interpolation

1. Introduction

Recently, fractional differential equations have drawn much attention to researchers, thanks to their various applications in physics and engineering. For example, fractional differential equations can be used as mathematical models for problems in anomalous diffusion transport ([18], [26], [35]), viscoelastic materials ([2], [30]), porous media ([4], [5]), to name but a few. Other applications in chemistry, chaos, finance, signal processing, etc. are discussed in the books, e.g., [25], [30], [31], or [33], and the references therein.

In this work, we consider the following ordinary differential equation (ODE) with fractional order $\alpha \in \mathbb{R}^+$,

$$\begin{cases} D_0^\alpha y(t) = f(t, y(t)), \quad t \in [0, T], \\ y^{(k)}(0) = y_k, \quad k = 0, 1, \dots, m-1, \end{cases} \tag{1.1}$$

where $m - 1 < \alpha \leq m \in \mathbb{Z}^+$, and the fractional derivative is defined in the Caputo sense ([6]), i.e.,

$$D_0^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-1-\alpha} y^{(m)}(\tau) d\tau. \tag{1.2}$$

Here, $\Gamma(\cdot)$ is the conventional *Gamma* function.

We note here that other than the Caputo definition (1.2), there exist different approaches to define fractional derivatives, for example, the Riemann-Liouville approach, the Grünwald-Letnikov definition, or

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recently the modified Riemann-Liouville fractional derivative ([23]). It is well-known that these approaches, under appropriate assumptions, are all related one another. We refer to [12], [31], or [25] for detailed analysis and discussions. In this work, we have chosen the Caputo derivative because it allows us to readily impose the initial problem (1.1) with inhomogeneous conditions.

It is well-known that a continuous function $y(t)$ is the solution of Eq. (1.1) if and only it is the solution of the Volterra integral equation ([12], [31])

$$y(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{1.3}$$

where $g(t) = \sum_{k=0}^{m} y_k \frac{t^k}{k!}$ is the initial condition. Theoretical analysis and numerical approaches for the Volterra integral equations can be found in [3], [9], and [10], etc.

In terms of numerical approaches, the main difficulty one has to tackle is the non-locality property of the solution $y(t)$ due to the kernel $(t-\tau)^{\alpha-1}$ under the integral sign on the right-hand side of Eq. (1.3). In order to illustrate this, let us first discretize the grid to be

$$\Phi_N := \{t_j : 0 = t_0 < t_1 < \ldots < t_j < \ldots < t_n < t_{n+1} < \ldots < t_N = T\}. \tag{1.4}$$

For simplicity, we assume that the grid is uniform, i.e., $h = t_{j+1} - t_j$, $\forall j = 0, \ldots, N-1$. Eq. (1.3) can be rewritten at time t_{n+1} as follows ([10]),

$$y(t_{n+1}) = g(t_{n+1}) + y^*(t_{n+1}) + Y(t_{n+1}), \tag{1.5}$$

where

$$y^*(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{1.6}$$

is called the *lag term* which takes $y(t)$ from the initial time up to $t = t_n$ into consideration in the computation of $y(t_{n+1})$. This is the so-called *memory effect* of the solution (1.3). It is noted that this memory effect is unique for the differential equations with fractional order, and is preferable for engineering applications since it indicates the dependence of $y(t_{n+1})$ on the solution at all prior time points.

The *increment term*

$$Y(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \tag{1.7}$$

determines the advancing of the solution from t_n to t_{n+1}.

For numerical approximations of $y(t)$, a typical approach is to interpolate the right-hand side $f(t, y(t))$, given sufficient continuity, over each interval $I_j = [t_j, t_{j+1}]$, $j = 0, \ldots, N-1$, by some types of interpolating polynomials. For this purpose, Lagrange polynomials are often taken into consideration. For example, the celebrated Adams-Bashforth-Moulton (ABM) by Diethelm et al. ([13], [14]) uses linear interpolation of the right-hand side $f(t, y(t))$, or the recently developed high-order schemes by Li et al. ([27]) in which $f(t)$ is interpolated by a quadratic interpolating polynomial. For all these schemes, a predicted value $\tilde{y}^P_{n+1}$ of $y(t_{n+1})$ has to be sought for the approximation of the increment term $Y(t_{n+1})$. For this purpose, a lower-order interpolation of $f(t)$ is facilitated in order to obtain $\tilde{y}^P_{n+1}$. Then the approximation $\tilde{y}_{n+1} \approx y(t_{n+1})$, which is called a correction, can be carried out. We observe here that since $\tilde{y}^P_{n+1}$ and $\tilde{y}_{n+1}$ are obtained at different accuracy, more precisely, the former is of lower order than the latter, the lag term (1.6) has to be approximated separately in each stage. In other words, the memory takes effects in both the prediction and correction stages. This makes the schemes expensive. It is even more costly for high-order schemes in which more accurate predicted $\tilde{y}^P_{n+1}$ or the approximation of the solution at half-points $t_{j+\frac{1}{2}}$'s are required ([27]).

It is worth mentioning here other numerical approaches to the differential equation with fractional order (1.1). These include the implicit block-by-block schemes by Kumar and Agrawal ([24]) and later improved by Cao and Xu ([11]), the $L1\_2$ scheme ([20]) in which the discretization is carried out on Eq. (1.1) directly instead of on the Volterra integral solution (1.3), or the Grünwald-based methods, e.g., [22], [19]. Fractional numerical methods designed for specifically PDE-modeled applications can be found in [32], [36], [29], [34], etc. Finite element or spectral methods are developed in the works, e.g., [8], [17], and the references therein.

Another issue one faces when developing a prediction-correction scheme for the differential equations with fractional order is the degeneration of the accuracy order in comparison with that of a scheme for classical ODEs, that is, when $\alpha \in \mathbb{Z}^+$. We take the ABM method for instance. It is shown in [14] that this scheme is of p-order where $p = \min(1+\alpha, 2)$. Thus the scheme is less accurate than the two-step Adams-Bashforth method ([21]), which is of order 2, in case of $0 < \alpha < 1$. It is indicated in the below section that this loss of accuracy is due to the usage of a low-order prediction for the interpolation of the increment term.

In this work, we aim to develop a new class of predictor-corrector schemes which overcomes the drawbacks aforementioned. The novelty of our proposed schemes are two-fold. Firstly, the new schemes achieve a uniform accuracy of order regardless of the values of α. This can be achieved by a new prediction method which is explicit and of the same accuracy with that of the correction stage. Secondly, the new designed schemes are more efficient than other corresponding prediction-correction ones in terms of computational costs. By a reduction in the approximation of the lag term $y^*(t_{n+1})$ by applying this for both prediction and correction, we will show that the new scheme with linear interpolation is only about half the cost of that of the ABM. We also discuss schemes of higher order with a quadratic interpolation rule.

We organize our paper as follows. In Section 2, we propose a new efficient scheme with linear interpolation. An error analysis then follows to prove that our new scheme has a uniform order 2 for both cases of $0 < \alpha < 1$ and $\alpha \geq 1$. A detailed check of computational costs show that our new scheme is about of half of the cost of that of the corresponding ABM method. In Section 3, we further improve the scheme to uniformly 3rd order by employing a quadratic interpolation of $f(t, y(t))$ over each interval I_j. Case by case numerical tests are presented in Section 4 to illustrate for our analysis in the previous sections. Finally, we draw our conclusions in the last Section 5.

2. Second-order Predictor-Corrector Scheme with Linear Interpolation

2.1. Description of the Second-order Predictor-Corrector Scheme with Linear Interpolation

Before proceeding, we first define some notations. We denote $y_j = y(t_j)$ the restriction of the exact solution at time t_j, $j = 0, \ldots, N$. Let $\tilde{y}^P_{n+1}, \tilde{y}_{n+1} \approx y_{n+1}$ be the predicted and corrected, respectively, approximations of y_{n+1}. Similarly, we also denote $f_j = f(t_j, y_j)$, $\tilde{f}_j = f(t_j, \tilde{y}_j)$, and $\tilde{f}^P_j = f(t_j, \tilde{y}^P_j)$. The lag and increment terms $y^*(t_{n+1})$, $Y(t_{n+1})$ are approximated by $\tilde{y}^*_{n+1}$ and $\tilde{Y}_{n+1}$, respectively.

For clarification, we re-derive the ABM ([13]) as follows. The exact solution (1.3) is written as

$$\begin{aligned} y(t_{n+1}) = g(t_{n+1}) &+ \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \end{aligned} \tag{2.1}$$

On each interval $I_j = [t_j, t_{j+1}]$, we interpolate $f(t)$ by a linear Lagrange polynomial

$$f(t) \approx f_j L_j(t) + f_{j+1} L_{j+1}(t), \quad t \in I_j, \tag{2.2}$$

where

$$L_k(t) = \frac{t - t_l}{t_k - t_l}, \quad (k, l) = (j, j+1), (j+1, j).$$

Substituting the approximation (2.2) into Eq. (2.1) with taking into accounts the approximated values $\tilde{y}_j$ and that the grid is uniform, we obtain that

$$\tilde{y}_{n+1} = g_{n+1} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} [B^{0,j}_{n+1} \tilde{f}_j + B^{1,j}_{n+1} \tilde{f}_{j+1}] + \frac{1}{\Gamma(\alpha)} [B^{0,n}_{n+1} \tilde{f}_n + B^{1,n}_{n+1} \tilde{f}^P_{n+1}], \tag{2.3}$$

where the coefficients are

$$B_{n+1}^{0,j} = \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} L_j(\tau) d\tau, \quad B_{n+1}^{1,j} = \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} L_{j+1}(\tau) d\tau,$$

which can be evaluated explicitly.

Comparing Eq. (2.3) with Eq. (1.5), we notice that the second term on the right-hand side of (2.3) is the approximation of the lag term $y^*(t_{n+1})$, and the last term approximates the increment $Y(t_{n+1})$. Expanding the summation, and collecting $\tilde{f}_j$, one can easily obtain the formula for $a_{j,n+1}$ given in [13].

In order to compute the predicted $\tilde{y}_{n+1}^P$, we invoke a lower-order approximation of $f(t)$, that is, a constant interpolation over each interval $t \in I_j$,

$$f(t) \approx f(t_j)\chi_{I_j}(t),$$

where the indicator function

$$\chi_{I_j}(t) = \begin{cases} 1, & \text{if } t \in I_j, \\ 0, & \text{otherwise,} \end{cases}$$

and substituting into Eq. (2.1), we obtain that

$$\tilde{y}_{n+1}^P = g_{n+1} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} C_{n+1}^j \tilde{f}_j, \tag{2.4}$$

where

$$C_{n+1}^j = \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} d\tau = \frac{h^\alpha}{\alpha}[(n+1-j)^\alpha - (n-j)^\alpha],$$

which is the coefficient $b_{j,n+1}$ in [13] multiplying with $\dfrac{h^\alpha}{\alpha}$.

Remark 2.1.

- For the ABM method, the lag term $y^*(t_{n+1})$ needs re-computing twice for the corrected $\tilde{y}_{n+1}$ and the predicted $\tilde{y}_{n+1}^P$, i.e., the $\sum$ terms in Eqs. (2.3) and (2.4), which is costly. Indeed, one can reduce the cost by reducing the computation of $\tilde{y}_{n+1}^*$ just once for both prediction and correction steps (see the below section).
- It is shown in [13], [14] that the order of the ABM scheme is

  $$\max_{0\le j\le N-1} |y_{n+1} - \tilde{y}_{n+1}| = \mathcal{O}(h^p),$$

  where $p = \min(1+\alpha, 2)$.

  One can check that the order reduction for cases of $0 < \alpha < 1$ comes from the implementation of the low-order prediction $\tilde{y}_{n+1}^P$. In the below section, we propose a new scheme which overcomes this shortage, and obtain a uniform order $\mathcal{O}(h^2)$ for any α.

We now proceed to develop our new scheme. For this purpose, we need the following lemma.

Lemma 2.1. *Assume that $\phi(t) \in \mathbb{P}_1[0,T]$ where $\mathbb{P}_1$ is the space of all polynomials of degree less than or equal to one. Let ϕ_n, $n = 0, \ldots, N$ be the restricted value of $\phi(t)$ on grid Φ_N defined in Eq. (1.4). Then, there exist coefficients b_{n+1}^0 and b_{n+1}^1 such that the following equality*

$$\int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1}\phi(\tau) d\tau = \frac{h^\alpha}{\alpha(\alpha+1)}(b_{n+1}^0\phi_{n-1} + b_{n+1}^1\phi_n), \tag{2.5}$$

is exact. More precisely,

$$b_{n+1}^0 = -1, \quad b_{n+1}^1 = \alpha+2. \tag{2.6}$$

Proof. Since $\phi \in \mathbb{P}_1[0,T]$, there exist $\xi_0, \xi_1 \in \mathbb{R}$ such that

$$\phi(t) = \xi_0 + \xi_1 t. \tag{2.7}$$

Substituting Eq. (2.7) into Eq. (2.5) and evaluating the integral on the left-hand side, we obtain that

$$\frac{h^\alpha}{\alpha}\left[\xi_0 + \xi_1\left(\frac{h}{\alpha+1} + t_n\right)\right] = \frac{h^\alpha}{\alpha(\alpha+1)}\left[b^0_{n+1}(\xi_0 + \xi_1 t_{n-1}) + b^1_{n+1}(\xi_0 + \xi_1 t_n)\right]. \tag{2.8}$$

Equating Eq. (2.8) terms by terms with respect to ξ_k, $k = 0, 1$, one obtains a non-singular linear system with unknowns b^0_{n+1} and b^1_{n+1} whose values are given in Eq. (2.6). This proves the lemma. □

We then propose our new scheme with linear interpolation as follows. We write our prediction and correction stages of the new scheme in terms of the lag and increment term (1.6), (1.7) as below.

$$\tilde{y}_{n+1} = g_{n+1} + \tilde{y}^*_{n+1} + \tilde{Y}_{n+1}, \tag{2.9}$$

where

$$\tilde{y}^*_{n+1} = \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n-1}[B^{0,j}_{n+1}\tilde{f}_j + B^{1,j}_{n+1}\tilde{f}_{j+1}],$$
$$\tilde{Y}_{n+1} = \frac{1}{\Gamma(\alpha)}[B^{0,n}_{n+1}\tilde{f}_n + B^{1,n}_{n+1}\tilde{f}^P_{n+1}];$$

where the prediction is as follows,

$$\tilde{y}^P_{n+1} = g_{n+1} + \tilde{y}^*_{n+1} + \frac{h^\alpha}{\Gamma(\alpha+2)}[b^0_{n+1}\tilde{f}_{n-1} + b^1_{n+1}\tilde{f}_n], \tag{2.10}$$

with b^0_{n+1} and b^1_{n+1} given in Eq. (2.6).

Remark 2.2. We notice that the prediction and correction of y_{n+1} are different from each other only by the approximations of the increment $Y(t_{n+1})$, whereas the lag term $y^*(t_{n+1})$ is computed only one time. This thus reduces the overall cost of the new scheme. (Comparing Eqs. (2.9), (2.10) with Eqs. (2.3), (2.4) of the ABM scheme.)

We now show that the new scheme achieves a uniform accuracy order regardless of α.

2.2. Error Analysis

2.2.1. Truncation and Global Error Analysis for Predictor

From herein, we denote C a generic constant which is independent of all grid parameters and may change case by case.

We need the following lemmas.

Lemma 2.2. *(*Interpolation Errors*) Let $f \in \mathcal{C}^{n+1}[a,b]$ and $p_n \in \mathbb{P}_n[a,b]$ interpolate the function f at the grid Φ_n in (1.4) with $a = t_0$ and $b = t_n$, then there exists $\xi \in (a,b)$ such that, for any $t \in [a,b]$,*

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(t - t_j).$$

Lemma 2.3. *For $\alpha \in \mathbb{R}^+$,*

$$\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_{n+1} - \tau)^{\alpha-1}d\tau \leq \frac{T^\alpha}{\alpha}.$$

Proof.

$$\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau \le \sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau$$
$$= \int_0^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau = \frac{t_{n+1}^\alpha}{\alpha} \le \frac{T^\alpha}{\alpha}.$$

□

Lemma 2.4. *(*Discrete Gronwall's Inequality*) ([15], [11]) Let* $\{a_n\}_{n=0}^N$*,* $\{b_n\}_{n=0}^N$ *be non-negative sequences with* b_n*'s monotonic increasing, and satisfy that*

$$a_n \le b_n + Mh^\gamma \sum_{j=0}^{n-1}(n-j)^{\gamma-1}a_j, \quad 0 \le n \le N,$$

where $M > 0$ *is bounded and independent of* h*, and* $0 < \gamma \le 1$*. Then,*

$$a_n \le b_n E_\gamma(M\Gamma(\gamma)(nh)^\gamma),$$

where $E_\gamma(\cdot)$ *is the Mittag-Leffler function ([12]). It is noted that when* $\gamma = 1$*, the Mittag-Leffler function becomes an exponential one.*

Theorem 2.1. *(Truncation Error) Suppose that* $f(\cdot, u(\cdot)) \in \mathcal{C}^2[0,T]$*. Let the truncation error at time step* t_{n+1} *be*

$$r_{n+1}^P = \Bigg| \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}f(\tau, y(\tau))d\tau - \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\left[B_{n+1}^{0,j}f_j + B_{n+1}^{1,j}f_{j+1}\right] - \frac{h^\alpha}{\Gamma(\alpha+2)}\left[b_{n+1}^0 f_{n-1} + b_{n+1}^1 f_n\right]\Bigg|. \tag{2.11}$$

Then, there exists a constant C *independent of all grid parameters such that*

$$r_{n+1}^P \le Ch^2. \tag{2.12}$$

Proof. In view of the interpolation (2.2), Eq. (2.11) is written as

$$\begin{aligned} r_{n+1}^P \le{}& \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\left|\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}\left\{f(\tau,y(\tau)) - [f_jL_j(\tau) + f_{j+1}L_{j+1}(\tau)]\right\}d\tau\right| \\ &+ \left|\frac{1}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}f(\tau,y(\tau))d\tau - \frac{h^\alpha}{\Gamma(\alpha+2)}\left[b_{n+1}^0 f_{n-1} + b_{n+1}^1 f_n\right]\right| \\ ={}& I + II. \end{aligned}$$

Since $f(\cdot, u(\cdot)) \in \mathcal{C}^2[0,T]$, by Lemma 2.2,

$$\Bigg| f(t,y(t)) - [f_jL_j(t) + f_{j+1}L_{j+1}(t)] \Bigg| \le \frac{1}{2}|f''(\xi_j)|h^2,$$

for some $\xi_j \in (t_j, t_{j+1})$. Let

$$M := \max_{0\le j\le N}|f''(\xi_j)|,$$

by Lemma 2.3, we have that

$$I \leq \frac{Mh^2}{2\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} d\tau \leq \frac{MT^\alpha}{2\Gamma(\alpha+1)} h^2.$$

By the Taylor's expansion of $f(\cdot, u(\cdot))$ around t_n, we have that

$$f(t) = p_1(t) + \frac{1}{2} f''(\xi_n)(t-t_n)^2, \quad \xi_n \in (t_n, t),$$

where

$$p_1(t) = f_n + f'(t_n)(t-t_n), \text{ thus } p_1(t) \in \mathbb{P}_1[0,T],$$

then by Lemma 2.1, with a note that $p_1(t_n) - f_n = 0$ and $|p_1(t_{n-1}) - f_{n-1}| \leq (M/2)h^2$,

$$\begin{aligned} II &= \left| \frac{1}{\Gamma(\alpha)} \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} \left\{ p_1(\tau) + \frac{1}{2} f''(\xi_n)(\tau-t_n)^2 \right\} d\tau - \frac{h^\alpha}{\Gamma(\alpha+2)} \left[b_{n+1}^0 f_{n-1} + b_{n+1}^1 f_n \right] \right| \\ &\leq \frac{1}{2\Gamma(\alpha)} \left| \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f''(\xi_n)(\tau-t_n)^2 d\tau \right| + \frac{h^\alpha}{\Gamma(\alpha+2)} \left| b_{n+1}^0 \right| \left| p_1(t_{n-1}) - f_{n-1} \right| \\ &\leq \frac{M}{2\Gamma(\alpha+1)} h^{2+\alpha} + \frac{M}{2\Gamma(\alpha+1)} h^{2+\alpha}. \end{aligned}$$

Combining I and II, we conclude that

$$r_{n+1}^P(h) \leq \frac{MT^\alpha}{2\Gamma(\alpha+1)} h^2 + \frac{M}{\Gamma(\alpha+1)} h^{2+\alpha} \leq Ch^2. \tag{2.13}$$

□

Theorem 2.2. *(Global Error) Suppose that $f(\cdot, u(\cdot)) \in \mathcal{C}^2[0,T]$, and furthermore is Lipschitz continuous in the second argument, i.e.,*

$$|f(t,u_1) - f(t,u_2)| \leq L|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R},$$

then the global error

$$E_{n+1}^P = |y_{n+1} - \tilde{y}_{n+1}| \leq Ch^2,$$

given that the starting error $E_1^P \leq Ch^2$.

Lemma 2.5. *For $k = 0, 1$ and $j = 0, 1, \cdots, n-1$*

$$|B_{n+1}^{k,j}| \leq \begin{cases} (n-j)^{\alpha-1} h^\alpha, & 0 < \alpha < 1, \\ T^{\alpha-1} h, & \alpha \geq 1. \end{cases}$$

Proof.

$$|B_{n+1}^{0,j}| \leq \frac{1}{h} \left| \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} (\tau - t_{j+1}) d\tau \right| \leq \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} d\tau.$$

- $\alpha \geq 1$:

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} d\tau \leq (t_{n+1}-t_j)^{\alpha-1} h \leq T^{\alpha-1} h.$$

• $0 < \alpha < 1$:

$$\int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau \le (t_{n+1} - t_{j+1})^{\alpha-1} h = (n-j)^{\alpha-1} h^\alpha.$$

We then deduce the result. The proof for $B_{n+1}^{1,j}$ follows similarly. □

Lemma 2.6. *For $\alpha \ge 1$,*

$$\frac{h^\alpha}{\alpha} \le T^{\alpha-1} h.$$

Proof.

$$\frac{h^\alpha}{\alpha} = \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau \le T^{\alpha-1} h.$$

□

Proof. (of Theorem 2.2)

$$\begin{aligned}
E_{n+1}^P = \Bigg| & \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \\
& - \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left[B_{n+1}^{0,j} \tilde{f}_j + B_{n+1}^{1,j} \tilde{f}_{j+1} \right] - \frac{h^\alpha}{\alpha(\alpha+1)\Gamma(\alpha)} \left[b_{n+1}^0 \tilde{f}_{n-1} + b_{n+1}^1 \tilde{f}_n \right] \Bigg| \\
& \le r_{n+1}^P + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left[|B_{n+1}^{0,j}| |\tilde{f}_j - f_j| + |B_{n+1}^{1,j}| |\tilde{f}_{j+1} - f_{j+1}| \right] \\
& + \frac{h^\alpha}{\alpha(\alpha+1)\Gamma(\alpha)} \left[|b_{n+1}^0| |\tilde{f}_{n-1} - f_{n-1}| + |b_{n+1}^1| |\tilde{f}_n - f_n| \right].
\end{aligned}$$

• $\alpha \ge 1$: By Lemmas 2.4 and 2.6, and that $f(t)$ is Lipschitz, we deduce that

$$\begin{aligned}
E_{n+1}^P & \le r_{n+1}^P + \frac{LT^{\alpha-1}h}{\Gamma(\alpha)} \sum_{j=0}^{n-1} [E_j^P + E_{j+1}^P] + \frac{LT^{\alpha-1}h}{(\alpha+1)\Gamma(\alpha)} E_{n-1}^P + \frac{(\alpha+2)LT^{\alpha-1}h}{(\alpha+1)\Gamma(\alpha)} E_n^P \\
& \le r_{n+1}^P + C\frac{LT^{\alpha-1}h}{\Gamma(\alpha)} \sum_{j=0}^{n} E_j^P \le r_{n+1}^P \exp\left(C\frac{Lh(n+1)T^{\alpha-1}}{\Gamma(\alpha)} \right) \le r_{n+1}^P \exp\left(CT^\alpha \right),
\end{aligned}$$

where we have invoked the Gronwall inequality Lemma 2.4. Thanks to the truncation error (2.12), we deduce that

$$E_{n+1}^P \le Ch^2.$$

• $0 < \alpha < 1$: Similarly, we have that

$$\begin{aligned}
E_{n+1}^P & \le r_{n+1}^P + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left[|B_{n+1}^{0,j}| |\tilde{f}_j - f_j| + |B_{n+1}^{1,j}| |\tilde{f}_{j+1} - f_{j+1}| \right] \\
& \quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[|b_{n+1}^0| |\tilde{f}_{n-1} - f_{n-1}| + |b_{n+1}^1| |\tilde{f}_n - f_n| \right] \\
& \le r_{n+1}^P + \frac{Lh^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{n-1} (n-j)^{\alpha-1} (E_j^P + E_{j+1}^P) + \frac{Lh^\alpha}{\Gamma(\alpha+2)} E_{n-1}^P + \frac{(\alpha+2)Lh^\alpha}{\Gamma(\alpha+2)} E_n^P \\
& \le r_{n+1}^P + C\frac{Lh^\alpha}{\Gamma(\alpha)} \sum_{j=1}^{n} (n+1-j)^{\alpha-1} E_j^P \le r_{n+1}^P E_\alpha \left(CL(n+1)h)^\alpha \right) \le r_{n+1}^P E_\alpha \left(CT^\alpha \right)
\end{aligned}$$

By the truncation error (2.12), the theorem completes. □

2.3. Truncation and Global Error Analysis for Corrector

Carrying a similar procedure, we obtain the following results.

Theorem 2.3. *(Truncation Error) With the same assumptions as those of Theorem 2.1, we have that*

$$r_{n+1}^C \le Ch^2.$$

Proof. The proof follows that of Theorem 2.1 with a note that

$$\begin{aligned}
r_{n+1}^C &\le \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}\Big|f(\tau,y(\tau))-[f_jL_j(\tau)+f_{j+1}L_{j+1}(\tau)]\Big|d\tau \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}\Big|f(\tau,y(\tau))-\big[f_nL_n(\tau)+f_{n+1}^PL_{n+1}(\tau)\big]\Big|d\tau \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}\Big|f(\tau,y(\tau))-[f_jL_j(\tau)+f_{j+1}L_{j+1}(\tau)]\Big|d\tau \\
&+\frac{1}{\Gamma(\alpha)}|B_{n+1}^{1,n}||f_{n+1}^P-f_{n+1}| \\
&\le \frac{Mh^2}{2\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}d\tau+\frac{Lh^\alpha E_{n+1}^P}{\Gamma(\alpha+1)} \\
&\le \frac{MT^\alpha}{2\Gamma(\alpha+1)}h^2+C\frac{L}{\Gamma(\alpha+1)}h^{2+\alpha}\le Ch^2.
\end{aligned}$$

□

For the global error of the corrector it can be easily checked by the Lemma 2.4 and Theorems 2.2, 2.3 .

Theorem 2.4. *(Global Error) With the same assumptions as those of Theorem 2.2, we have*

$$E_{n+1}^C \le Ch^2.$$

3. Third-order Predictor-Corrector Scheme with with Quadratic Interpolation

3.1. Description of Third-order Predictor-Corrector Scheme with Quadratic Interpolation

In this section, we further improve our scheme by employing a quadratic interpolation of the right-hand side $f(t,y(t))$ over each interval I_j. For this, we observe the following lemma.

Lemma 3.1. *Assume that $\phi(t)\in\mathbb{P}_2[0,T]$ where $\mathbb{P}_2$ is the space of all polynomials of degree less than or equal to two. Let ϕ_n, $n=0,\dots,N$ be the restricted value of $\phi(t)$ on grid Φ_N defined in Eq. (1.4). Then, there exist coefficients a_{n+1}^0, a_{n+1}^1, and a_{n+1}^2 such that the following equality*

$$\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}\phi(\tau)d\tau=\frac{h^\alpha}{\alpha(\alpha+1)(\alpha+2)}(a_{n+1}^0\phi_{n-2}+a_{n+1}^1\phi_{n-1}+a_{n+1}^2\phi_n),$$

is exact. More precisely,

$$a_{n+1}^0=\frac{\alpha+4}{2},\quad a_{n+1}^1=-2(\alpha+3),\quad a_{n+1}^2=\frac{2\alpha^2+9\alpha+12}{2}.$$

The proof follows that of Lemma 2.1, hence omitted. Over $[t_{j-1}, t_{j+1}], j \geq 1$, we interpolate $f(t)$ by a quadratic Lagrange polynomial

$$f(t) \approx \sum_{k=j-1}^{j+1} f_k Q_k^j(t), \tag{3.1}$$

where

$$Q_k^j(t) = \prod_{\substack{m=j-1\\ m\neq k}}^{j+1} \frac{t-t_m}{t_k-t_m},$$

On $[t_0, t_1]$, $f(t)$ can be interpolated by $f(t) \approx f_0 Q_0^0(t) + f_{1/2} Q_{1/2}^0(t) + f_1 Q_1^0(t)$, where

$$Q_0^0(t) = \frac{(t-t_{1/2})(t-t_1)}{(t_0-t_{1/2})(t_0-t_1)}, Q_{1/2}^0(t) = \frac{(t-t_0)(t-t_1)}{(t_{1/2}-t_1)(t_{1/2}-t_1)}, Q_1^0(t) = \frac{(t-t_0)(t-t_{1/2})}{(t_1-t_0)(t_1-t_{1/2})}.$$

Substituting Eq. (3.1) into Eq. (2.1), taking the uniform property of the grid, we obtain that

$$\tilde{y}_{n+1} = g_{n+1} + \tilde{y}_{n+1}^* + \tilde{Y}_{n+1}. \tag{3.2}$$

Here, the lag term is approximated as follows,

$$\tilde{y}_{n+1}^* = \frac{1}{\Gamma(\alpha)}\left[A_{n+1}^{0,0}\tilde{f}_0 + A_{n+1}^{1,0}\tilde{f}_{1/2} + A_{n+1}^{2,0}\tilde{f}_1\right] + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n-1}\left[A_{n+1}^{0,j}\tilde{f}_{j-1} + A_{n+1}^{1,j}\tilde{f}_j + A_{n+1}^{2,j}\tilde{f}_{j+1}\right],$$

where,

$$\begin{aligned}
A_{n+1}^{0,0} &= \int_0^{t_1} (t_{n+1}-\tau)^{\alpha-1} Q_0^0(\tau) d\tau, \\
A_{n+1}^{1,0} &= \int_0^{t_1} (t_{n+1}-\tau)^{\alpha-1} Q_{1/2}^0(\tau) d\tau, \\
A_{n+1}^{2,0} &= \int_0^{t_1} (t_{n+1}-\tau)^{\alpha-1} Q_1^0(\tau) d\tau,
\end{aligned}$$

and, for $1 \leq j \leq n$,

$$A_{n+1}^{i,j} = \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} Q_{i+j-1}^j(\tau) d\tau, \quad i = 0, 1, 2.$$

The increment term is

$$\tilde{Y}_{n+1} = \frac{1}{\Gamma(\alpha)}\left[A_{n+1}^{0,n}\tilde{f}_{n-1} + A_{n+1}^{1,n}\tilde{f}_n + A_{n+1}^{2,n}\tilde{f}_{n+1}^P\right],$$

where the predicted $\tilde{y}_{n+1}^P$ is approximated as follows,

$$\tilde{y}_{n+1}^P = g_{n+1} + \tilde{y}_{n+1}^* + \frac{h^\alpha}{\Gamma(\alpha+3)}\left[a_{n+1}^0\tilde{f}_{n-2} + a_{n+1}^1\tilde{f}_{n-1} + a_{n+1}^2\tilde{f}_n\right], \tag{3.3}$$

where $a_{n+1}^0, a_{n+1}^1, a_{n+1}^2$ can be computed by Lemma 3.1.

3.2. Error Analysis

3.2.1. Truncation and Global Error Analysis for Predictor

By a similar procedure, we can obtain the analysis for the accuracy order of the new scheme using a quadratic interpolation. Let r^P_{n+1} be the truncation error of prediction at t_{n+1} defined by

$$\begin{aligned} r^P_{n+1} = \Bigg| & \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau - \frac{1}{\Gamma(\alpha)} \left[A^{0,0}_{n+1} f_0 + A^{1,0}_{n+1} f_{1/2} + A^{2,0}_{n+1} f_1 \right] \\ & - \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n-1} \left[A^{0,j}_{n+1} f_{j-1} + A^{1,j}_{n+1} f_j + A^{2,j}_{n+1} f_{j+1} \right] - \frac{h^\alpha}{\Gamma(\alpha+3)} \left[a^0_{n+1} f_{n-2} + a^1_{n+1} f_{n-1} + a^2_{n+1} f_n \right] \Bigg|. \end{aligned}$$

Theorem 3.1. *Suppose that $f(\cdot, u(\cdot)) \in \mathcal{C}^3[0,T]$. Then, there exists a constant C independent of all grid parameters such that*

$$r^P_{n+1} \le Ch^3.$$

Proof. From the quadratic interpolation of $f(t)$ in (3.1), we have

$$\begin{aligned} r^P_{n+1} \le & \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} (t_{n+1}-\tau)^{\alpha-1} \left| f(\tau, y(\tau)) - \left[f_0 Q^0_0(\tau) + f_{1/2} Q^0_{1/2}(\tau) + f_1 Q^0_1(\tau) \right] \right| d\tau \\ & + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{\alpha-1} \left| f(\tau, y(\tau)) - \left[f_{j-1} Q^j_{j-1}(\tau) + f_j Q^j_j(\tau) + f_{j+1} Q^j_{j+1}(\tau) \right] \right| d\tau \\ & + \left| \frac{1}{\Gamma(\alpha)} \int_{t_n}^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau - \frac{h^\alpha}{\Gamma(\alpha+3)} \left[a^0_{n+1} f_{n-2} + a^1_{n+1} f_{n-1} + a^2_{n+1} f_n \right] \right| \\ & = I + II + III. \end{aligned}$$

Since $f(\cdot, u(\cdot)) \in \mathcal{C}^3[0,T]$, by Lemma 2.2, we can easily check that

$$I, II \le Ch^3.$$

Taylor's expansion f around t_n gives

$$f(t) = p_2(t) + \frac{1}{3} f'''(\xi_n)(t-t_n)^3, \quad \xi_n \in (t_n, t), \tag{3.4}$$

where

$$p_2(t) = f_n + f'(t_n)(t-t_n) + f''(t_n)(t-t_n)^2/2, \text{ thus } p_2(t) \in \mathbb{P}_2[0,T],$$

then by Lemma 3.1 with (3.4), we have

$$III \le \frac{h^\alpha}{\Gamma(\alpha+3)} (|a^0_{n+1}| p_2(t_{n-2}) - f_{n-2}| + |a^1_{n+1}||p_2(t_{n-1}) - f_{n-1}| + \frac{M}{6\Gamma(\alpha+1)} h^{3+\alpha} \le Ch^{3+\alpha},$$

where

$$M := \max_{0 \le j \le N} |f'''(\xi_j)|.$$

Combining I, II and III, the proof is complete. ∎

By the similar process in the linear interpolation, the following Lemma can be easily checked.

Lemma 3.2. *For $k=0,1,2$ and $j=0,1,\cdots,n-1$*

$$|A_{n+1}^{k,j}| \le \begin{cases} (n-j)^{\alpha-1}h^{\alpha}, & 0<\alpha<1, \\ T^{\alpha-1}h, & \alpha \ge 1. \end{cases}$$

Theorem 3.2. *(Global Error) Suppose that $f(\cdot,u(\cdot)) \in \mathcal{C}^3[0,T]$, and is Lipschitz continuous, then the global error*

$$E_{n+1}^{P} = |y_{n+1} - \tilde{y}_{n+1}| \le Ch^3,$$

given that the starting errors $E_1^P \le Ch^3$ and $E_{1/2}^P \le Ch^2$.

Proof.

$$\begin{aligned} E_{n+1}^{P} \le\ & r_{n+1}^{P} + \frac{1}{\Gamma(\alpha)}\left\{|A_{n+1}^{0,0}||\tilde{f}_0 - f_0| + |A_{n+1}^{1,0}||\tilde{f}_{1/2} - f_{1/2}| + |A_{n+1}^{2,0}||\tilde{f}_1 - f_1|\right\} \\ & + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n-1}\left\{|A_{n+1}^{0,j}||\tilde{f}_{j-1} - f_{j-1}| + |A_{n+1}^{1,j}||\tilde{f}_j - f_j| + |A_{n+1}^{2,j}||\tilde{f}_{j+1} - f_{j+1}|\right\} \\ & + \frac{1}{\Gamma(\alpha+3)}\left\{|a_{n+1}^{0}||\tilde{f}_{n-2} - f_{n-2}| + |a_{n+1}^{1}||\tilde{f}_{n-1} - f_{n-1}| + |a_{n+1}^{2}||\tilde{f}_n - f_n|\right\} \end{aligned}$$

- $\alpha \ge 1$: By Lemmas 2.4, 2.6, 3.2, and that $f(t)$ is Lipschitz, we have

$$E_{n+1}^{P} \le r_{n+1}^{P} + \frac{LT^{\alpha-1}h}{\Gamma(\alpha)}E_{1/2}^{P} + C\frac{LT^{\alpha-1}h}{\Gamma(\alpha)}\sum_{j=0}^{n}E_j^p \le \left(r_{n+1}^{P} + \frac{T^{\alpha-1}Lh}{\Gamma(\alpha)}E_{1/2}^{P}\right)\exp\left(CT^{\alpha}\right).$$

- $0<\alpha<1$: Similarly, we have that

$$E_{n+1}^{P} \le r_{n+1}^{P} + \frac{LT^{\alpha-1}h}{\Gamma(\alpha)}E_{1/2}^{P} + C\frac{Lh^{\alpha}}{\Gamma(\alpha)}\sum_{j=1}^{n}(n+1-j)^{\alpha-1}E_j^P \le \left(r_{n+1}^{P} + \frac{LT^{\alpha-1}h}{\Gamma(\alpha)}E_{1/2}^{P}\right)E_{\alpha}\left(CT^{\alpha}\right).$$

□

3.2.2. Truncation and Global Error Analysis for Corrector

Now we have the following truncation error of the corrector with quadratic interpolation in the similar manner.

Theorem 3.3. *(Truncation Error) With the same assumptions as those of Theorem 3.2, we have*

$$r_{n+1}^{C} \le Ch^3.$$

Proof.

$$\begin{aligned} r_{n+1}^{C} \le\ & \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_{n+1}-\tau)^{\alpha-1}\left|f(\tau,y(\tau)) - \left[f_0Q_0^0(\tau) + f_{1/2}Q_{1/2}^0(\tau) + f_1Q_1^0(\tau)\right]\right|d\tau \\ & + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{n-1}\int_{t_j}^{t_{j+1}}(t_{n+1}-\tau)^{\alpha-1}\left|f(\tau,y(\tau)) - \left[f_{j-1}Q_{j-1}^j(\tau) + f_jQ_j^j(\tau) + f_{j+1}Q_{j+1}^j(\tau)\right]\right|d\tau \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_n}^{t_{n+1}}(t_{n+1}-\tau)^{\alpha-1}\left|f(\tau,y(\tau)) - \left[f_{n-1}Q_{n-1}^n(\tau) + f_nQ_n^n(\tau) + f_{n+1}^PQ_{n+1}^n(\tau)\right]\right|d\tau \\ \le\ & C\frac{MT^{\alpha}}{\Gamma(\alpha+1)}h^3 + C\frac{L}{\Gamma(\alpha+1)}h^{3+\alpha} \le Ch^3. \end{aligned}$$

□

From the Lemma 2.4 and Theorems 3.2, 3.3 we obtain the following global error of the corrector in quadratic interpolation.

Theorem 3.4. *(Global Error) With the same assumptions as those of Theorem 3.2, we have*

$$E_{n+1}^C \leq Ch^3.$$

Remark 3.1. As indicated in Theorem 2.2 and 3.2, the global error of the scheme depends on that of the start-up, which is a common difficulty for multi-step methods. For a classical ODE where $\alpha \in \mathbb{Z}^+$, one could instead use a multi-stage method with corresponding accuracy, e.g., Runge-Kutta, for the first few steps then switch back to the multi-step one, when appropriate. For fractional ODEs, in order to preserve the order of accuracy, one could invoke an iteration, say Newton's method, for approximating the solution at the first few time steps. Here, since we would use just explicit schemes, we obtain approximate solutions with finer step size at the first several steps. We propose the start-up for our new scheme with quadratic interpolation in the Appendix.

4. Numerical Results

In this section, we illustrate the accuracy and efficiency of our new schemes, both with linear and quadratic interpolation. For all below tests, we use the following error estimates.

- *Pointwise error at final time $t_N = T$:*

$$E_{pt}(T) = |y_N - \tilde{y}_N|.$$

- *L^2 error over $[0, T]$:*

$$E_{L^2} = \left(h \sum_{j=0}^{N} |y_j - \tilde{y}_j|^2 \right)^{1/2}.$$

We choose the following examples for our numerical experiments. For schemes with linear interpolation, we compare the results obtained from our new scheme Eqs. (2.9) and (2.10) with those from the ABM method Eqs. (2.3) and (2.4). For high-order schemes with quadratic interpolation, we compare the results of our scheme Eqs. (3.2) and (3.3) with the Improved Algorithm II scheme developed by Li et al. ([27]). We denote *MF-PCL* and *MF-PCQ* by our new schemes with linear and quadratic interpolation, respectively. Similarly, the ABM scheme is denoted as *ABM*, and *Alg II* for the improved algorithm II scheme.

Example 1.

$$\begin{cases} D_0^\alpha y(t) = \dfrac{40320}{\Gamma(9-\alpha)} t^{8-\alpha} - 3\dfrac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})} t^{4-\frac{\alpha}{2}} + \dfrac{9}{4}\Gamma(\alpha+1) + \left(\dfrac{3}{2} t^{\frac{\alpha}{2}} - t^4\right)^3 - y(t)^{\frac{3}{2}}, \\ y(0) = 0, \ y'(0) = 0, \end{cases}$$

whose exact solution is

$$y(t) = t^8 - 3t^{4+\frac{\alpha}{2}} + \frac{9}{4} t^\alpha.$$

Example 2.

$$\begin{cases} D_0^\alpha y(t) = \dfrac{\Gamma(4+\alpha)}{6} t^3 + t^{3+\alpha} - y(t), \\ y(0) = 0, \ y'(0) = 0, \end{cases}$$

whose exact solution is

$$y(t) = t^{3+\alpha}.$$

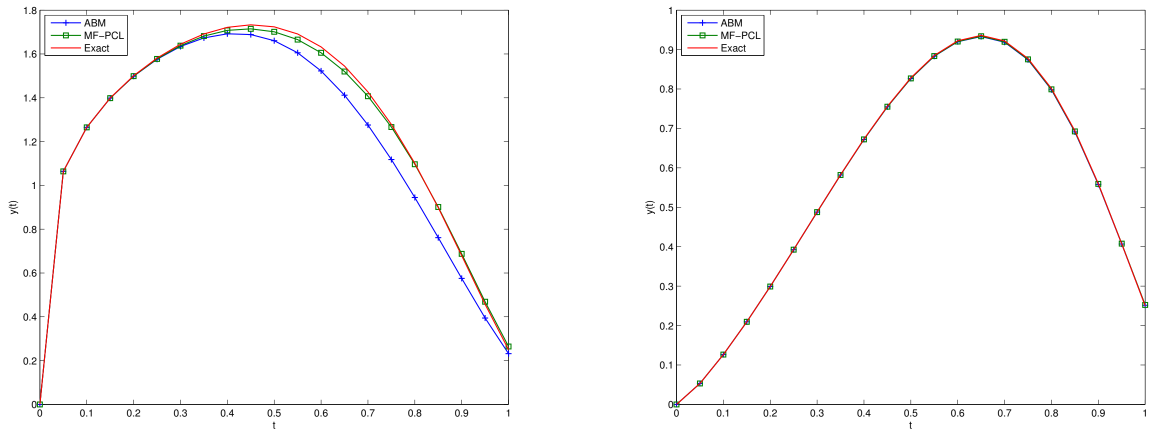

Figure 1: Numerical solutions with grid N=20 in Example 1(Left: $\alpha = 0.25$; Right: $\alpha = 1.25$.).

Example 3.

$$\begin{cases} D_0^\alpha y(t) = \dfrac{\Gamma(5+\alpha)}{24} t^4 + t^{8+2\alpha} - y^2(t), \\ y(0) = 0, \ y'(0) = 0, \end{cases}$$

whose exact solution is

$$y(t) = t^{4+\alpha}.$$

For all these examples, we choose the final time $T = 1$.

4.1. Linear Interpolation

Numerical errors and orders of accuracy for *Example 1* obtained from schemes with linear interpolation are listed in Table 1 for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 1.25$. For both cases $\alpha < 1$ and $\alpha > 1$, we observe that our *MF-PCL* scheme achieves a 2nd order, which well agrees with our error analysis in the above section. We also note the dependence of the accuracy order on α of the ABM scheme. It shows much better results of the *MF-PCL* over the ABM in case $\alpha = 0.25$, in both error magnitudes and orders. For the case where $\alpha > 1$, both schemes behave similarly and achieve 2nd order. Numerical solutions with grid $N = 20$ for *Example 1* are plotted in Fig. 1 for both values of α.

We remark that in case of $\alpha < 1$, as shown in Table 1 (see also Tables 2 and 3), the accuracy of the *MF-PCL* is even better than 2nd-order. We explain this behavior by the competence between the errors in the lag and increment term of the truncation errors (see Eq. (2.13)) when α is small.

Numerical errors, and accuracy orders obtained from the ABM and *MF-PCL* for *Example 2* with different values of α are listed in Table. It is shown that the *MF-PCL* outperforms the ABM in terms of both error magnitudes and orders for $\alpha < 1$.

Similarly, for *Example 3*, we list and plot the errors in Table 3. Again, we observe significantly better results of the *MF-PCL* over those of the ABM scheme.

Table 1: Numerical comparisons of errors and orders with linear interpolation in Example 1.

	ABM				MF-PCL			
$\alpha = 0.25$								
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.5007E-01	-	3.1370E-01	-	1.7439E-01	-	2.1027E-01	-
20	1.8095E-02	3.7886	8.6943E-02	1.8512	1.4567E-02	3.5815	1.4413E-02	3.8668
40	3.6054E-03	2.3274	2.4831E-02	1.8079	2.6392E-03	2.4645	1.6389E-03	3.1366
80	1.4522E-03	1.3119	8.0459E-03	1.6258	5.0674E-04	2.3808	2.4669E-04	2.7319
160	6.5805E-04	1.1420	2.8152E-03	1.5150	9.9526E-05	2.3481	4.1343E-05	2.5770
320	2.9689E-04	1.1483	1.0318E-03	1.4481	2.0182E-05	2.3020	7.4652E-06	2.4694
$\alpha = 0.5$								
	ABM				MF-PCL			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	1.7859E-02	-	4.9366E-02	-	2.6565E-02	-	1.4118E-02	-
20	1.8123E-03	3.3008	1.3769E-02	1.8421	5.2961E-03	2.3265	2.0946E-03	2.7528
40	4.1619E-04	2.1225	4.1517E-03	1.7296	1.0748E-03	2.3008	3.6047E-04	2.5387
80	1.7655E-04	1.2371	1.3188E-03	1.6544	2.3143E-04	2.2155	7.0952E-05	2.3450
160	7.9795E-05	1.1457	4.3342E-04	1.6054	5.3098E-05	2.1238	1.5567E-05	2.1883
320	3.3898E-05	1.2351	1.4570E-04	1.5728	1.2728E-05	2.0606	3.6510E-06	2.0921
$\alpha = 1.25$								
	ABM				MF-PCL			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	5.5326E-03	-	8.1359E-03	-	1.0161E-02	-	6.0442E-03	-
20	1.5932E-03	1.7960	1.8821E-03	2.1120	2.3430E-03	2.1166	1.3773E-03	2.1337
40	4.3283E-04	1.8801	4.4311E-04	2.0866	5.7010E-04	2.0391	3.3411E-04	2.0435
80	1.1434E-04	1.9205	1.0555E-04	2.0697	1.4133E-04	2.0122	8.2552E-05	2.0169
160	2.9741E-05	1.9428	2.5353E-05	2.0577	3.5232E-05	2.0041	2.0531E-05	2.0075
320	7.6631E-06	1.9564	6.1289E-06	2.0485	8.7988E-06	2.0015	5.1201E-06	2.0035

4.2. Quadratic Interpolation

We now turn to check the cases with our new high-order scheme with quadratic interpolation. We denote our new scheme *MF-PCQ*.

In Tables 4 and 5, we show the numerical errors and orders obtained from the *MF-PCQ* for *Example 2* and *Example 3* with various values of α, respectively. We also present the results obtained from the Improved Algorithm II (abbreviated by *Alg II*) by Li et al. ([27]) for comparison purpose. For all these cases, the *MF-PCQ* scheme achieves the designed 3rd order for both cases of $\alpha < 1$ and $\alpha \geq 1$, and are much better than the comparing results of the Alg. II scheme. In case of $\alpha = 1.5$, we notice that although the errors of the *MF-PCQ* are less than those of the *Alg II*, the former remains $3rd$-order whereas the latter's order is around 2.5.

We next compare the performance of our *MF-PCQ* and the *Alg II* in case the right-hand side is just C^0 continuous. For this case, we choose the following test as suggested in [27].

Example 4.

$$D_0^\alpha y(t) = \begin{cases} \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - y(t) + t^2 - t, & \alpha > 1, \\ \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \dfrac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - y(t) + t^2 - t, & \alpha \leq 1, \end{cases} \quad \text{at final time } T = 1.$$

The exact solution is available as follows,

$$y(t) = t^2 - t.$$

Table 2: Numerical comparisons of errors and orders with linear interpolation in Example 2.

	$\alpha = 0.25$							
	ABM				MF-PCL			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	1.1972E-01	-	5.8385E-02	-	2.1664E-02	-	1.3197E-02	-
20	4.4583E-02	1.4250	2.0731E-02	1.4938	3.7111E-03	2.5454	2.2213E-03	2.5707
40	1.6517E-02	1.4326	7.4827E-03	1.4702	6.6351E-04	2.4837	3.9177E-04	2.5033
80	6.1966E-03	1.4144	2.7666E-03	1.4355	1.2453E-04	2.4137	7.2643E-05	2.4311
160	2.3625E-03	1.3911	1.0460E-03	1.4032	2.4580E-05	2.3409	1.4179E-05	2.3571
320	9.1470E-04	1.3690	4.0297E-04	1.3762	5.0945E-06	2.2705	2.9088E-06	2.2852
	$\alpha = 0.5$							
	ABM:				MF-PCL.			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	4.3925E-02	-	2.1407E-02	-	8.3277E-03	-	4.9845E-03	-
20	1.3777E-02	1.6728	6.3232E-03	1.7593	1.5344E-03	2.4402	8.7563E-04	2.5091
40	4.4119E-03	1.6428	1.9563E-03	1.6925	3.2032E-04	2.2601	1.7634E-04	2.3120
80	1.4446E-03	1.6107	6.2784E-04	1.6397	7.2807E-05	2.1374	3.9194E-05	2.1696
160	4.8214E-04	1.5832	2.0706E-04	1.6004	1.7375E-05	2.0671	9.2400E-06	2.0847
320	1.6338E-04	1.5613	6.9655E-05	1.5717	4.2518E-06	2.0309	2.2474E-06	2.0396
	$\alpha = 1.25$							
	ABM:				MF-PCL.			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	1.3331E-02	-	6.4106E-03	-	9.4619E-03	-	4.7489E-03	-
20	3.1725E-03	2.0711	1.4378E-03	2.1566	2.3387E-03	2.0164	1.1042E-03	2.1046
40	7.5985E-04	2.0618	3.3483E-04	2.1024	5.8322E-04	2.0036	2.6690E-04	2.0487
80	1.8296E-04	2.0542	7.9668E-05	2.0714	1.4571E-04	2.0009	6.5638E-05	2.0237
160	4.4265E-05	2.0473	1.9199E-05	2.0530	3.6421E-05	2.0003	1.6277E-05	2.0117
320	1.0755E-05	2.0411	4.6642E-06	2.0413	9.1046E-06	2.0001	4.0528E-06	2.0058

Hence, the initial conditions are

$$y(0) = 0, \quad y'(0) = -1.$$

In Table 6, we list the numerical errors and orders for *Example 4* with $\alpha = 0.3,\ 0.5$ and 1.25. We note that our obtained results from the *Alg II* here are different in terms of error magnitudes comparing with those given in Table 13 and 14 in [27]. Due to the violation of the conditions given in Theorem 3.2, it is shown that both schemes cannot obtained the designed convergence order. Nevertheless, we conclude that our new scheme outperforms the *Alg II*, especially for cases of $\alpha < 1$.

4.3. Computational issues

The proposed scheme does not repeat the memory effect in approximating the predicted value $\tilde{y}^P$ which gives a benefit to reduce computational cost. Comparing with the ABM scheme it reduces a half of computational cost roughly. For the quadratic interpolation we do not analyze the computational cost because *Alg II* scheme uses the intermediate grids. In Tables 7 and 8, we show the computational time (in seconds) for both linear and quadratic interpolation schemes for Example 2, respectively. We observe a much cheaper cost of the proposed schemes comparing with those of the ABM or *Alg II* ones.

Table 3: Numerical comparisons of errors and orders with linear interpolation in Example 3.

	$\alpha = 0.25$							
	ABM				MF-PCL			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.8509E-01	-	9.8618E-02	-	1.1048E-01	-	3.8031E-02	-
20	1.4092E-01	1.0165	3.9967E-02	1.3031	2.4911E-02	2.1489	6.9865E-03	2.4446
40	5.8686E-02	1.2638	1.4877E-02	1.4257	4.0952E-03	2.6048	1.0716E-03	2.7048
80	2.1731E-02	1.4333	5.3180E-03	1.4841	6.1016E-04	2.7467	1.6841E-04	2.6696
160	7.7465E-03	1.4881	1.9111E-03	1.4765	9.5796E-05	2.6712	2.9250E-05	2.5255
320	2.7873E-03	1.4746	7.0315E-04	1.4425	1.6288E-05	2.5561	5.5603E-06	2.3952
	$\alpha = 0.5$							
	ABM				MF-PCL			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	1.1045E-01	-	3.9248E-02	-	3.0503E-02	-	1.1768E-02	-
20	3.7868E-02	1.5444	1.1542E-02	1.7658	4.7293E-03	2.6892	1.8017E-03	2.7074
40	1.2285E-02	1.6240	3.4716E-03	1.7332	7.8440E-04	2.5920	3.2865E-04	2.4547
80	3.9689E-03	1.6301	1.0849E-03	1.6780	1.5036E-04	2.3832	6.9724E-05	2.2368
160	1.3012E-03	1.6089	3.5082E-04	1.6288	3.2419E-05	2.2135	1.6118E-05	2.1130
320	4.3414E-04	1.5836	1.1641E-04	1.5915	7.5169E-06	2.1086	3.8890E-06	2.0512

5. Conclusion

In this work, we have proposed a new class of accurate, efficient, and explicit schemes using linear and quadratic interpolation. By implementing the same discretization method for the lag term in both the prediction and correction stages, we have greatly reduced the overall computational costs. This is a main difficulty issue for the development of numerical schemes for fractional differential equations due to the so-called memory effect. Our new schemes also improve the accuracy to achieve uniform orders for both cases of $0 < alp < 1$ and $\alpha > 1$, especially for the former case where, to our knowledge, other schemes' accuracy depends on this α. This property is obtained by a new prediction method, in which we did not employ a low-order interpolation of the right-hand side $f(t)$ for the prediction step. Instead, we have introduced a new explicit method, which is of the same order with that of the correction stage.

A variety of numerical tests show that our schemes are better than other comparing ones, especially for cases with $\alpha < 1$ in which our new schemes significantly improve both error magnitudes and orders.

The low cost and high accuracy of our new schemes promise various applications, especially in dynamical problems or time derivative discretization for PDE-based problems.

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Appendix A. Start-up of the Scheme

To find a desired accuracy for $\tilde{y}_1, \tilde{y}_2$ we approximate the predicted values at $t_{1/4}, t_{1/2}$ by using the constant, linear and quadratic interpolation as seen in Fig. A.2. Let us define the following forms to be

Table 4: Numerical comparisons of errors and orders with quadratic interpolation in Example 2.

	$\alpha = 0.2$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	1.1282E-01	-	5.4717E-02	-	3.2490E-03	-	2.9278E-03	-
20	4.6245E-02	1.2867	2.1439E-02	1.3518	2.1308E-04	3.9305	2.3158E-04	3.6602
40	1.8376E-02	1.3314	8.3255E-03	1.3646	1.8910E-05	3.4942	2.0182E-05	3.5204
80	7.2652E-03	1.3388	3.2520E-03	1.3562	1.7233E-06	3.4559	1.8304E-06	3.4628
160	2.8853E-03	1.3323	1.2831E-03	1.3417	1.6275E-07	3.4045	1.7131E-07	3.4175
320	1.1547E-03	1.3212	5.1165E-04	1.3264	1.5889E-08	3.3565	1.6512E-08	3.3750
	$\alpha = 0.5$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.1238E-02	-	1.0234E-02	-	5.8073E-04	-	5.6262E-04	-
20	6.8948E-03	1.6231	3.1404E-03	1.7043	5.0855E-05	3.5134	4.6946E-05	3.5831
40	2.2550E-03	1.6124	9.9630E-04	1.6563	5.2410E-06	3.2785	4.5587E-06	3.3643
80	7.4828E-04	1.5915	3.2516E-04	1.6154	5.9340E-07	3.1428	4.9557E-07	3.2015
160	2.5192E-04	1.5706	1.0847E-04	1.5838	7.0647E-08	3.0703	5.7693E-08	3.1026
320	8.5865E-05	1.5528	3.6782E-05	1.5603	8.6282E-09	3.0335	6.9668E-09	3.0498
	$\alpha = 1.5$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.1723E-04	-	9.2811E-05	-	1.1582E-03	-	6.3691E-04	-
20	4.0476E-05	2.4241	1.6089E-05	2.5283	1.5456E-04	2.9056	8.3466E-05	2.9318
40	7.3413E-06	2.4630	2.8107E-06	2.5170	1.9940E-05	2.9544	1.0684E-05	2.9657
80	1.3142E-06	2.4819	4.9353E-07	2.5097	2.5313E-06	2.9777	1.3514E-06	2.9830
160	2.3375E-07	2.4911	8.6925E-08	2.5053	3.1884E-07	2.9890	1.6992E-07	2.9915
320	4.1450E-08	2.4955	1.5336E-08	2.5028	4.0007E-08	2.9945	2.1303E-08	2.9958

used frequently.

$$\hat{A}^{c,d,e}_{n,a,b} = \int_{t_a}^{t_b} (t_n - \tau)^{\alpha-1} \left(\frac{\tau - t_d}{t_c - t_d}\right)\left(\frac{\tau - t_e}{t_c - t_e}\right) d\tau$$
$$\hat{B}^{c,d}_{n,a,b} = \int_{t_a}^{t_b} (t_n - \tau)^{\alpha-1} \left(\frac{\tau - t_d}{t_c - t_d}\right) d\tau.$$

The detail algorithm is described as follows

Algorithm 1. (Start-up Procedure)

1. Approximate $\tilde{y}_{1/4}$:
   - Predict $\tilde{y}^P_{1/4}$: constant interpolation

$$\tilde{y}^P_{1/4} = g_{1/4} + \frac{1}{\Gamma(\alpha+1)}\left(\frac{h}{4}\right)^{\alpha} \tilde{f}_0$$

   - Correct $\tilde{y}_{1/4}$: linear interpolation

$$\tilde{y}_{1/4} = g_{1/4} + \frac{1}{\Gamma(\alpha)}(\hat{B}^{0,1/4}_{1/4,0,1/4}\tilde{f}_0 + \hat{B}^{1/4,0}_{1/4,0,1/4}\tilde{f}^P_{1/4})$$

Table 5: Numerical comparisons of errors and orders with quadratic interpolation in Example 3.

	$\alpha = 0.2$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.3644E-01	-	8.2597E-02	-	3.8752E-02	-	1.3076E-02	-
20	1.2705E-01	0.8961	3.6805E-02	1.1662	4.7954E-03	3.0146	1.2616E-03	3.3735
40	5.9339E-02	1.0983	1.5313E-02	1.2651	3.9686E-04	3.5950	9.2554E-05	3.7688
80	2.4495E-02	1.2765	6.0135E-03	1.3485	2.7356E-05	3.8587	6.6757E-06	3.7933
160	9.4508E-03	1.3740	2.3084E-03	1.3813	1.9914E-06	3.7800	5.4093E-07	3.6254
320	3.5809E-03	1.4001	8.8846E-04	1.3775	1.5859E-07	3.6504	4.7933E-08	3.4963
	$\alpha = 0.5$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	5.1972E-02	-	1.8362E-02	-	4.9248E-03	-	1.9788E-03	-
20	1.8226E-02	1.5117	5.5825E-03	1.7177	3.4502E-04	3.8353	1.4494E-04	3.7711
40	6.0807E-03	1.5837	1.7393E-03	1.6824	2.6946E-05	3.6785	1.3194E-05	3.4575
80	2.0095E-03	1.5974	5.5867E-04	1.6385	2.4626E-06	3.4518	1.4049E-06	3.2313
160	6.6928E-04	1.5862	1.8421E-04	1.6006	2.5616E-07	3.2651	1.6274E-07	3.1099
320	2.2562E-04	1.5687	6.1962E-05	1.5719	2.9021E-08	3.1419	1.9640E-08	3.0507
	$\alpha = 1.5$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	5.2475E-04	-	1.7546E-04	-	3.0570E-03	-	1.4485E-03	-
20	1.0690E-04	2.2954	2.9639E-05	2.5656	4.0006E-04	2.9338	1.7979E-04	3.0101
40	2.0298E-05	2.3969	5.0620E-06	2.5497	5.1235E-05	2.9650	2.2389E-05	3.0055
80	3.7188E-06	2.4484	8.7646E-07	2.5299	6.4831E-06	2.9824	2.7925E-06	3.0031
160	6.6922E-07	2.4743	1.5318E-07	2.5165	8.1536E-07	2.9912	3.4866E-07	3.0017
320	1.1936E-07	2.4872	2.6916E-08	2.5087	1.0223E-07	2.9956	4.3556E-08	3.0009

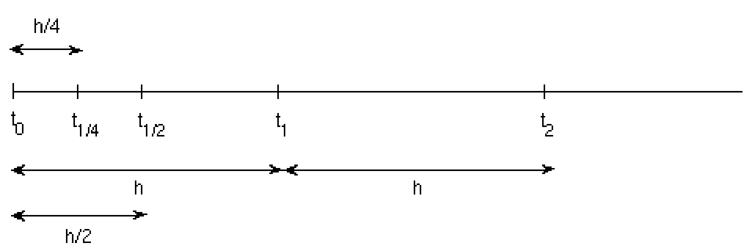

Figure A.2: Grid for the start-up of the new quadratic scheme.

2. Approximate $\tilde{y}_{1/2}$:

$$\tilde{y}^*_{1/2} = \frac{1}{\Gamma(\alpha)}(\hat{B}^{0,1/4}_{1/2,0,1/4}\tilde{f}_0 + \hat{B}^{1/4,0}_{1/2,0,1/4}\tilde{f}_{1/4})$$

- Predict $\tilde{y}^{P_1}_{1/2}$: constant interpolation

$$\tilde{y}^{P_1}_{1/2} = g_{1/2} + \tilde{y}^*_{1/2} + \frac{1}{\Gamma(\alpha+1)}\left(\frac{h}{4}\right)^{\alpha}\tilde{f}_{1/4}$$

Table 6: Numerical comparisons of errors and orders with quadratic interpolation in Example 4.

	$\alpha = 0.3$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	2.0352E-02	-	1.4073E-02	-	8.4358E-04	-	4.7666E-03	-
20	7.4511E-03	1.4497	5.0469E-03	1.4795	4.3404E-06	7.6026	1.2145E-03	1.9726
40	2.7176E-03	1.4551	1.8077E-03	1.4813	4.7103E-07	3.2039	3.2340E-04	1.9090
80	1.0005E-03	1.4416	6.5511E-04	1.4643	6.4031E-08	2.8790	8.8761E-05	1.8653
160	3.7297E-04	1.4236	2.4103E-04	1.4425	7.0477E-08	0.1384	2.5085E-05	1.8231
320	1.4079E-04	1.4055	9.0027E-05	1.4208	3.3827E-08	1.0590	7.2972E-06	1.7814
	$\alpha = 0.5$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	6.9797E-03	-	5.3828E-03	-	1.0279E-04	-	3.2469E-03	-
20	2.2242E-03	1.6499	1.6212E-03	1.7313	5.7796E-06	4.1526	8.2940E-04	1.9689
40	7.2329E-04	1.6207	4.9337E-04	1.7163	3.0519E-06	0.9213	2.2983E-04	1.8515
80	2.3990E-04	1.5922	1.5344E-04	1.6850	1.8825E-06	0.6970	6.8351E-05	1.7495
160	8.0882E-05	1.5685	4.9004E-05	1.6467	8.1159E-07	1.2139	2.1553E-05	1.6651
320	2.7619E-05	1.5501	1.6081E-05	1.6076	3.1590E-07	1.3613	7.1038E-06	1.6012
	$\alpha = 1.25$							
	Alg. II				MF-PCQ			
N	E_{pt}	Order	E_{L^2}	Order	E_{pt}	Order	E_{L^2}	Order
10	5.8803E-05	-	7.9315E-05	-	3.0199E-04	-	2.8069E-04	-
20	4.4030E-06	3.7393	3.0361E-05	1.3854	8.6228E-05	1.8083	8.7244E-05	1.6858
40	5.6368E-06	0.3564	1.1538E-05	1.3958	2.4787E-05	1.7986	2.6771E-05	1.7044
80	2.5578E-06	1.1400	4.0096E-06	1.5249	7.2105E-06	1.7814	8.1092E-06	1.7230
160	9.4285E-07	1.4398	1.3174E-06	1.6058	2.1165E-06	1.7684	2.4361E-06	1.7350
320	3.1833E-07	1.5665	4.1836E-07	1.6548	6.2477E-07	1.7603	7.2827E-07	1.7420

- Predict $\tilde{y}_{1/2}^{P_2}$: linear interpolation

$$\tilde{y}_{1/2}^{P_2} = g_{1/2} + \tilde{y}_{1/2}^{*} + \frac{1}{\Gamma(\alpha)}(\hat{B}_{1/2,1/4,1/2}^{1/4,1/2}\tilde{f}_{1/4} + \hat{B}_{\{1/2,1/4,1/2\}}^{1/2,1/4}\tilde{f}_{1/2}^{P_1})$$

- Correct $\tilde{y}_{1/2}$: quadratic interpolation

$$\tilde{y}_{1/2} = g_{1/2} + \frac{1}{\Gamma(\alpha)}(\hat{A}_{1/2,0,1/2}^{0,1/4,1/2}\tilde{f}_0 + \hat{A}_{1/2,0,1/2}^{1/4,0,1/2}\tilde{f}_{1/4} + \hat{A}_{1/2,0,1/2}^{1/2,0,1/4}\tilde{f}_{1/2}^{P_2})$$

3. Approximate $\tilde{y}_1$:

$$\tilde{y}_1^{*} = \frac{1}{\Gamma(\alpha)}(\hat{B}_{1,0,1/2}^{0,1/2}\tilde{f}_0 + \hat{B}_{1,0,1/2}^{1/2,0}\tilde{f}_{1/2})$$

- Predict $\tilde{y}_1^{P_1}$: constant interpolation

$$\tilde{y}_1^{P_1} = g_1 + \tilde{y}_1^{*} + \frac{1}{\Gamma(\alpha+1)}\left(\frac{h}{2}\right)^{\alpha}\tilde{f}_{1/2}$$

- Predict $\tilde{y}_1^{P_2}$: linear interpolation

$$\tilde{y}_1^{P_2} = g_1 + \tilde{y}_1^{*} + \frac{1}{\Gamma(\alpha)}(\hat{B}_{1,1/2,1}^{1/2,1}\tilde{f}_{1/2} + \hat{B}_{1,1/2,1}^{1,1/2}\tilde{f}_1^{P_1})$$

Table 7: Computational time in Example 2: Linear interpolation schemes.

	$\alpha = 0.25$					
N	100	200	400	600	800	1000
ABM	0.002	0.010	0.040	0.090	0.160	0.262
MF-PCL	0.001	0.007	0.025	0.050	0.096	0.150
	$\alpha = 0.5$					
N	100	200	400	600	800	1000
ABM	0.003	0.010	0.040	0.090	0.160	0.250
MF-PCL	0.001	0.006	0.024	0.057	0.096	0.150

Table 8: Computational time in Example 2: Quadratic interpolation schemes.

	$\alpha = 0.2$					
N	100	200	400	600	800	1000
Alg. II	0.010	0.040	0.159	0.357	0.631	0.990
MF-PCQ	0.003	0.013	0.054	0.125	0.214	0.335
	$\alpha = 0.5$					
N	100	200	400	600	800	1000
Alg. II	0.210	0.491	1.102	2.127	2.643	3.224
MF-PCQ	0.123	0.231	0.622	0.981	1.490	1.471

- Correct $\tilde{y}_1$: quadratic interpolation

$$\tilde{y}_1 = g_1 + \frac{1}{\Gamma(\alpha)}(\hat{A}^{0,1/2,1}_{1,0,1}\tilde{f}_0 + \hat{A}^{1/2,01}_{1,0,1}\tilde{f}_{1/2} + \hat{A}^{1,0,1}_{1,0,1}\tilde{f}^{P_2}_1)$$

4. Approximate $\tilde{y}_2$:

$$\tilde{y}^*_2 = \frac{1}{\Gamma(\alpha)}(\hat{A}^{0,1/2,1}_{2,0,1}\tilde{f}_0 + \hat{A}^{1/2,0,1}_{2,0,1}\tilde{f}_{1/2} + A^{1,0,1/2}_{2,0,1}\tilde{f}_1)$$

- Predict $\tilde{y}^{P_1}_2$: constant interpolation

$$\tilde{y}^{P_1}_2 = g_2 + \tilde{y}^*_2 + \frac{h^\alpha}{\Gamma(\alpha+1)}\tilde{f}_1$$

- Predict $\tilde{y}^{P_2}_2$: linear interpolation

$$\tilde{y}^{P_2}_2 = g_2 + \tilde{y}^*_2 + \frac{1}{\Gamma(\alpha)}(\hat{B}^{1,2}_{2,1,2}\tilde{f}_1 + \hat{B}^{2,1}_{2,1,2}\tilde{f}^{P_1}_2)$$

- Correct $\tilde{y}_2$: quadratic interpolation

$$\tilde{y}_2 = g_2 + \tilde{y}^*_2 + \frac{1}{\Gamma(\alpha)}(\hat{A}^{0,1,2}_{2,1,2}\tilde{f}_0 + \hat{A}^{1,0,2}_{2,1,2}\tilde{f}_1 + \hat{A}^{2,0,1}_{2,1,2}\tilde{f}^{P_2}_2)$$

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