Journal of Sound and Vibration (1980) **69**(3), 345–359

RAYLEIGH–RITZ VIBRATION ANALYSIS OF MINDLIN PLATES

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(*Received* 17 *August* 1979)

The Rayleigh–Ritz method is applied to the prediction of the natural frequencies of flexural vibration of square plates having general boundary conditions. The analysis is based on the use of Mindlin plate theory so that the effects of shear deformation and rotary inertia are included. The spatial variations of the plate deflection and the two rotations over the plate middle surface are assumed to be series of products of appropriate Timoshenko beam functions. Results are presented for a number of types of plate and these demonstrate the manner of convergence of the method as the number of terms in the assumed series increases.

1. INTRODUCTION

The great majority of results available for the flexural vibration of plates are based on the classical (or Lagrange) theory which involves the Kirchhoff assumption that straight lines originally normal to the plate median surface remain straight and normal during the deformation process. It follows that the rotations of plate cross-sections, ψ_x and ψ_y, along the in-plane co-ordinates x and y, respectively, are directly proportional to the first derivatives of the normal displacement of the median surface, w, and thus that w is the sole reference quantity of classical theory. This simplifies the problem considerably but introduces errors since the effects of transverse shear are ignored and at the same time rotary inertia terms are also neglected. The classical thin plate theory overestimates the plate frequencies which it identifies since the plate flexibility and inertia are under-estimated. For simply supported rectangular plates comparison studies [1–3] with three-dimensional elasticity solutions show that classical theory solutions are significantly in error for all but the lower modes of very thin plates and that the errors increase with increase in the ratio of plate thickness to typical wavelength of vibration.

An improved "thick" plate theory which includes the effects of transverse shear and rotary inertia was presented by Mindlin [4] and, in the comparison studies referred to above, has been shown to predict frequencies of flexural vibration which are in close agreement with elasticity solutions for a wide range of plate geometries and modes of vibration. (The thickness-twist and thickness-shear modes of vibration [5] are also accurately predicted by Mindlin theory. These modes occur at much higher frequencies than do corresponding flexural modes and will not be considered further here.) In Mindlin plate theory any line originally normal to the plate median surface will, during deformation, remain straight but not generally normal to the median surface. In the true situation such a line is generally curved but the assumed straight non-normal line can closely approximate this in an average sense at all points. A shear factor is introduced to take account of the fact that the shear strain distribution through the plate thickness is not uniform. For isotropic materials this factor is often taken to be $\pi^2/12$ ($=0{\cdot}822467$) though

0022–460X/80/070345+15 $02.00/0

numerical comparisons with elasticity solutions [1, 6], for particular values of material and geometric parameters, suggest that the higher value of about 0·88 may be more appropriate. The inclusion of shear effects in Mindlin plate theory means that the cross-sectional rotations ψ_x and ψ_y can no longer be expressed solely in terms of the deflection w; thus there exist three basic reference quantities, viz. w, ψ_x and ψ_y, at the median surface rather than simply w alone as in classical plate theory.

Despite its simplifications, exact solutions of the classical theory equations for the free vibration problem and rectangular geometry are known only for cases where two opposite edges are simply supported. For other cases many solutions have been generated by using approximate methods [7] and amongst these the Rayleigh and Rayleigh–Ritz energy methods have proved to be very popular. Applications of these methods have often been associated with the use of trial functions for the spatial variation of w, that is $W(x, y)$, which are expressed as series of products of beam functions in the form

$$W(x, y) = \sum_{n=1}^{r} \sum_{m=1}^{r} A_{mn} X_m(x) Y_n(y). \tag{1}$$

Here X_m and Y_n are the fundamental mode shapes of uniform beams which have boundary conditions at the beam ends that are appropriate to the plate under consideration. In classical plate theory two boundary conditions are specified at each edge and these may pertain to displacement (the geometric boundary conditions) or to force (the natural boundary conditions) or to both. For mathematical admissibility in the Rayleigh–Ritz method only the geometric boundary conditions need to be strictly satisfied at the outset since the natural conditions should arise from the minimization procedure. The use of the beam functions allows all geometric and natural boundary conditions to be precisely satisfied for simply supported or clamped edges but the natural boundary conditions are not precisely met at a free edge. The beam functions mentioned above are, of course, based on the classical Bernouilli–Euler theory, which is the one-dimensional equivalent of the classical plate theory, and the functions have the useful property of orthogonality. Warburton [8] has presented solutions for rectangular plates having all 21 possible combinations of simply supported, clamped and free edges using the (single-term) Rayleigh method whilst Leissa [9] has later given detailed results for the 21 combinations using the multi-term ($r = 6$) Rayleigh–Ritz method.

In contrast to the situation pertaining in the realm of classical plate theory there exists only a very limited number of solutions to the problem of predicting the natural frequencies of thick plates. Exact solutions of the Mindlin plate equations have been presented by Mindlin *et al.* [5] for rectangular plates with all edges simply supported or with one opposite pair of edges simply supported and the other pair free, and it has been mentioned above that elasticity solutions [1–3] exist for the former case. For other boundary conditions Cheung and Chakrabarti [10] have presented approximate results based on a finite layer modelling of the three-dimensional problem. The finite layer solutions compare well with exact elasticity solutions [1] for thick plates with all edges simply supported but for other boundary conditions their validity has been questioned by Nelson [11]. The latter author has presented some approximate results for plates with clamped and free edges based on the use of Bolotin's solution fitting technique. A small number of finite element and finite strip approaches to the thick plate problem have been made and most of these are noted in the work of Dawe [6]. In reference [6] it is shown that the natural frequencies of thick rectangular plates having one pair of opposite edges simply supported can be calculated to a high level of accuracy using the finite strip method.

The objective of the present paper is to examine the use of the Rayleigh–Ritz method in calculating the natural frequencies of flexural vibration of isotropic plates based on

Mindlin theory, for a range of boundary conditions. To do this trial functions are required for the spatial variation of each of the three quantities w, ψ_x and ψ_y and these are taken to be series of products of beam functions, in a similar way to that described above for classical plate analysis. Here, however, the beam functions used are those of Timoshenko beam theory [12] which is the one-dimensional equivalent of Mindlin plate theory. It is noted that in Mindlin theory three boundary conditions have to be specified at each edge, compared with the two of classical theory. The results presented here are for plates of square geometry only, although the approach described can, of course, be used for rectangular plates. Two ratios of plate thickness to side length are considered, these being 0·01 and 0·10. Various numbers of series terms are used in obtaining the solutions so that the convergence of the method can be examined in detail for a limited number of types of plate.

2. TIMOSHENKO BEAM FUNCTIONS

The calculation of the natural frequencies of vibration of Mindlin plates considered herein is based on the assumption, within a Rayleigh–Ritz procedure, of trial functions for the lateral displacement and the two cross-sectional rotations, each of which are series of products of unidirectional Timoshenko beam functions. These functions are considerably less well known than are the corresponding classical beam functions. The functions are briefly described and recorded here but the reader is referred to the works mentioned below for further details.

A basic assumption of Timoshenko beam theory is that the cross-sectional rotation, ψ, depends not only on the slope of the (total) deflection curve, as in Bernouilli–Euler theory, but also on the value of the shear strain. Thus the deformation of any point in the beam is related to the values of two reference quantities, w (the deflection) and ψ, at the neutral axis (rather than to w alone as in Bernouilli–Euler theory). It follows that the dynamic Timoshenko beam problem is governed by a pair of coupled differential equations. These, for a beam lying in the x-direction in the xz plane, take the forms [12]

$$EI\frac{\partial^2\psi}{\partial x^2}+\kappa^2\bar{A}G\left(\frac{\partial w}{\partial x}-\psi\right)-I\rho\frac{\partial^2\psi}{\partial t^2}=0, \qquad \rho\bar{A}\frac{\partial^2 w}{\partial t^2}-\kappa^2\bar{A}G\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\psi}{\partial x}\right)=0. \quad (2, 3)$$

Here E is the modulus of elasticity, G the modulus of rigidity, I the second moment of area of the cross-section, $\bar{A}$ the area of the cross-section, ρ the material mass density and κ^2 the shear factor.

Traill-Nash and Collar [13] and later Huang [14] have pointed out the existence of two separate spectra of harmonic frequencies which are solutions to these governing equations. In the first spectrum the bending and shearing angles are in phase and the vibrational modes resemble those of classical theory and are referred to as "flexural" modes. In the second spectrum the contributions of bending and shear are of similar order but are in opposition such that the vibrational modes exhibit relatively large rotation of the sections and strong shearing action whilst the lateral deflection is small; these are the "thickness-shear" modes. The lower flexural modes of the first spectrum occur at much lower frequencies than do the modes of the second spectrum, for practical geometries, and are the modes of interest here.

On the assumption of harmonic motion, i.e.,

$$w = W(x)\,e^{ipt}, \qquad \psi = \Psi(x)\,e^{ipt}, \quad (4)$$

the general solution of the governing differential equations for the flexural modes of vibration of a uniform beam of length L whose longitudinal axis runs in the x-direction is [13, 14]

$$W(x) = C_1 \sin b\beta\xi + C_2 \cos b\beta\xi + C_3 \sinh b\alpha\xi + C_4 \cosh b\alpha\xi, \tag{5}$$

$$\Psi(x) = C_1 k_1 \cos b\beta\xi - C_2 k_1 \sin b\beta\xi + C_3 k_2 \cosh b\alpha\xi + C_4 k_2 \sinh b\alpha\xi. \tag{6}$$

The quantities that occur in these equations are defined as follows:

$$\xi = x/L, \qquad b^2 = p^2 \rho \bar{A} L^4 / EI, \tag{7, 8}$$

$$\binom{\alpha}{\beta} = (1/\sqrt{2})\{(\mp)(r^2+s^2) + [(r^2-s^2)^2 + 4/b^2]^{1/2}\}^{1/2}, \tag{9}$$

$$k_1 = b(\beta^2 - s^2)/\beta L, \qquad k_2 = b(\alpha^2 + s^2)/\alpha L, \tag{10, 11}$$

$$r^2 = I/\bar{A}L^2, \qquad s^2 = EI/\kappa^2 \bar{A} G L^2; \tag{12}$$

$C_1 \ldots C_4$ are constants and p is the radian frequency of vibration. (The notation used here is largely that of Huang [14].)

In Timoshenko beam theory the expressions for bending moment M and shearing force Q are

$$M = -EI\, \partial\psi/\partial x, \qquad Q = \kappa^2 \bar{A} G(\partial w/\partial x - \psi). \tag{13}$$

With these definitions in mind the boundary conditions required on W and Ψ at the three standard types of beam end are thus as follows:

$$\text{simply supported end:} \quad W = 0, \qquad \mathrm{d}\Psi/\mathrm{d}\xi = 0; \tag{14}$$

$$\text{clamped end:} \quad W = 0, \qquad \Psi = 0; \tag{15}$$

$$\text{free end:} \quad \mathrm{d}\Psi/\mathrm{d}\xi = 0, \qquad (1/L)\,\mathrm{d}W/\mathrm{d}\xi - \Psi = 0. \tag{16}$$

Applying the appropriate boundary conditions to the general solution, equations (5) and (6), yields frequency equations for each of the six types of beam having all combinations of standard ends. These frequency equations are highly transcendental (except for the simply supported beam) but can be solved in each case to yield the roots b_i ($i = 1, 2, 3, \ldots$) which are proportional to the square of natural frequency and are the eigenvalues of the problem. The corresponding eigenfunctions are the normal modes W_i and Ψ_i. The frequency equations and the normal modes have been determined by Traill-Nash and Collar [13] and by Huang [14] for the six standard beams and are, in somewhat different form, as follows (for convenience the subscript i is omitted from the quantities W, Ψ, b, etc.; in describing the type of beam the first specified condition is at the end $x = 0$ and the second at the end $x = L$):

(a) simply supported-simply supported beam:

$$W = \sin q\pi\xi, \qquad \Psi = \cos q\pi\xi \qquad (q = 1, 2, 3 \ldots); \tag{17, 18}$$

(b) clamped-clamped beam:

$$W = \delta \sin b\beta\xi + \cos b\beta\xi - (k_1/k_2)\delta \sinh b\alpha\xi - \cosh b\alpha\xi, \tag{19}$$

$$\Psi = \delta k_1 \cos b\beta\xi - k_1 \sin b\beta\xi - k_1\delta \cosh b\alpha\xi - k_2 \sinh b\alpha\xi, \tag{20}$$

where

$$\delta = [\cos b\beta - \cosh b\alpha]/[(k_1/k_2)\sinh b\alpha - \sin b\beta], \tag{21}$$

and where the values of b are obtained as roots of the equation

$$2-2\cosh b\alpha\cos b\beta+\frac{b[b^2s^2(r^2-s^2)^2+(3s^2-r^2)]}{\sqrt{1-b^2r^2s^2}}\sinh b\alpha\sin b\beta=0; \tag{22}$$

(c) clamped-simply supported beam: in this case the beam functions can be obtained from the antisymmetric modes of the clamped-clamped beam of length $2L$;

(d) clamped-free beam:

$$W=-\delta\sin b\beta\xi+\cos b\beta\xi+\delta(k_1/k_2)\sinh b\alpha\xi-\cosh b\alpha\xi, \tag{23}$$

$$\Psi=-k_1\delta\cos b\beta\xi-k_1\sin b\beta\xi+k_1\delta\cosh b\alpha\xi-k_2\sinh b\alpha\xi, \tag{24}$$

where

$$\delta=(k_2\alpha\cosh b\alpha+k_1\beta\cos b\beta)/(k_1\alpha\sinh b\alpha+k_1\beta\sin b\beta), \tag{25}$$

and where the values of b are obtained as roots of the equation

$$2+[b^2(r^2-s^2)^2+2]\cosh b\alpha\cos b\beta-\frac{b(r^2+s^2)}{\sqrt{1-b^2r^2s^2}}\sinh b\alpha\sin b\beta=0; \tag{26}$$

(e) free-free beam:

$$W=-\lambda\delta\sin b\beta\xi+\theta\cos b\beta\xi+\delta\sinh b\alpha\xi+\cosh b\alpha\xi, \tag{27}$$

$$\Psi=-\lambda\delta k_1\cos b\beta\xi-\theta k_1\sin b\beta\xi+\delta k_2\cosh b\alpha\xi+k_2\sinh b\alpha\xi, \tag{28}$$

where

$$\theta=k_2\alpha/k_1\beta, \qquad \lambda=(b\alpha-k_2)/(b\beta-k_1), \tag{29}$$

$$\delta=[\cos b\beta-\cosh b\alpha]/[(\lambda/\theta)\sin b\beta+\sinh b\alpha], \tag{30}$$

and where the values of b are obtained as roots of the equation

$$2-2\cosh b\alpha\cos b\beta+b\frac{[b^2r^2(r^2-s^2)^2+(3r^2-s^2)]}{\sqrt{1-b^2r^2s^2}}\sinh b\alpha\sin b\beta=0. \tag{31}$$

These free-free beam modes, given by equations (27) and (28) and obtained from the general solution (equations (5) and (6)), do not constitute the complete set of trial functions required for the analysis of plates having a pair of opposite edges free; the above modes supply the third and higher trial functions and can be supplemented by first and second trial functions having the forms

$$W=1, \qquad \Psi=0 \qquad \text{and} \qquad W=1-2\xi, \qquad \Psi=1, \tag{32, 33}$$

respectively;

(f) free-simply supported beam: in this case the beam functions can be obtained from the antisymmetric modes of the free-free beam of length $2L$.

3. THE RAYLEIGH–RITZ METHOD FOR THE MINDLIN PLATE PROBLEM

In Mindlin plate theory [4] the rectangular plate of thickness h is referred to a right-handed x, y, z system of rectangular co-ordinates, as shown in Figure 1. The surfaces of the plate are the planes $z=\pm h/2$ and the displacement components at any point in the vibrating plate are u, v and w in the x, y and z directions respectively. These components are related to the three reference quantities, w, ψ_x and ψ_y, at the plate middle surface by the

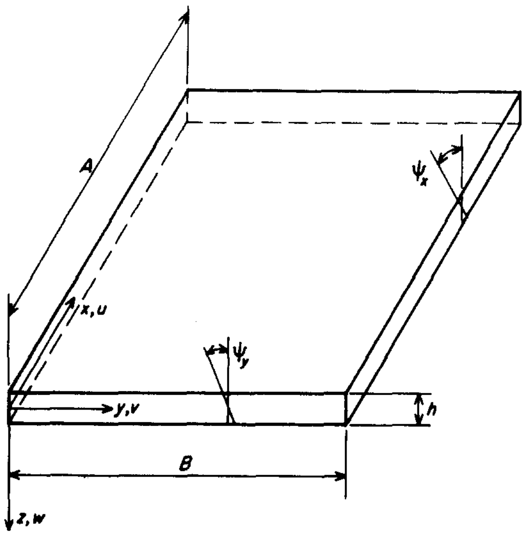

Figure 1. Plate geometry.

equations†

$$u = z\psi_x(x, y, t), \qquad v = z\psi_y(x, y, t), \qquad w = w(x, y, t). \tag{34}$$

The quantities ψ_x and ψ_y are, in an average sense, the cross-sectional rotations in the x and y co-ordinate directions, respectively, and these rotations are expressed in terms of the slopes in the x and y directions of the deformed median surface, $\partial w/\partial x$ and $\partial w/\partial y$, and the shear strains γ_x and γ_y as

$$\psi_x = \gamma_x - \partial w/\partial x, \qquad \psi_y = \gamma_y - \partial w/\partial y. \tag{35}$$

For an isotropic material the strain energy per unit surface area of the vibrating Mindlin plate is

$$\tfrac{1}{4}(D(1+\nu)(\Gamma_x+\Gamma_y)^2 + 2\kappa^2 Gh(\Gamma_{yz}^2 + \Gamma_{xz}^2) + D(1-\nu)[(\Gamma_x-\Gamma_y)^2 + \Gamma_{yx}^2]). \tag{36}$$

Here ν is Poisson's ratio and

$$D = Eh^3/12(1-\nu^2). \tag{37}$$

The plate strain components occurring in equation (36) are defined as

$$\begin{aligned} \Gamma_x &= \partial\psi_x/\partial x, \qquad \Gamma_y = \partial\psi_y/\partial y, \qquad \Gamma_{yx} = \partial\psi_y/\partial x + \partial\psi_x/\partial y, \\ \Gamma_{xz} &= \psi_x + \partial w/\partial x, \qquad \Gamma_{yz} = \psi_y + \partial w/\partial y, \end{aligned} \tag{38}$$

whilst κ^2 is (again) the shear factor, introduced to account for the non-uniformity of the shear strain distribution through the plate thickness.

The kinetic energy of the vibrating plate per unit surface area is

$$(\rho h^3/24)[(\partial\psi_x/\partial t)^2 + (\partial\psi_y/\partial t)^2] + (\rho h/2)(\partial w/\partial t)^2, \tag{39}$$

and this includes the effect of rotary inertia.

† Note that the rotation quantities ψ_x and ψ_y of this section correspond to the *negative* of the rotation ψ of section 2: this agrees with the sign conventions of the source references of the plate [4] and beam [12] theories. The validity of the beam normal modes of section 2 for use in plate analysis is, of course, unaffected by this but it should be borne in mind in relating, for instance, moment and shear force expressions in the beam and plate theories and in consequent definition of boundary conditions.

The expressions for the bending and twisting moments and the transverse shearing forces, per unit length, are of interest in connection with the definition of plate boundary conditions. These moments and shearing forces are given by

$$M_x = D(\partial\psi_x/\partial x + \nu\, \partial\psi_y/\partial y), \qquad M_y = D(\partial\psi_y/\partial y + \nu\, \partial\psi_x/\partial x), \tag{40}$$

$$M_{yx} = \tfrac{1}{2}(1-\nu)D(\partial\psi_y/\partial x + \partial\psi_x/\partial y), \tag{41}$$

$$Q_x = \kappa^2 Gh(\partial w/\partial x + \psi_x), \qquad Q_y = \kappa^2 Gh(\partial w/\partial y + \psi_y). \tag{42}$$

With the strain energy and kinetic energy densities thus defined in terms of the three fundamental quantities w, ψ_x and ψ_y in equations (36)–(39), consideration can now be given to the form of appropriate trial functions for these quantities for use in a Rayleigh–Ritz type analysis. Solutions to the free vibration problem are sought of the form

$$w(x, y, t) = W(x, y)\,e^{ipt}, \qquad \psi_x(x, y, t) = \Psi_x(x, y)\,e^{ipt}, \qquad \psi_y(x, y, t) = \Psi_y(x, y)\,e^{ipt}, \tag{43}$$

where the spatial variations $W(x, y)$, $\Psi_x(x, y)$ and $\Psi_y(x, y)$ are each expressed as a series of products of x-functions and y-functions in the general form of equation (1). The Timoshenko beam normal modes detailed in equations (20)–(33) are to be used as the x-functions and y-functions and clearly the variations of both $W(x, y)$ and $\Psi_x(x, y)$ along the x-direction and of $W(x, y)$ and $\Psi_y(x, y)$ along the y-direction can be directly related to appropriate Timoshenko beam functions. On the other hand the variation of $\Psi_x(x, y)$ along y and the variation of $\Psi_y(x, y)$ along x can not, at first sight, be directly related to any beam function. However, a way out of this difficulty was provided by studying the eigenvectors of finite strip solutions of Mindlin plate vibration problems [6] for rectangular plates with one pair of opposite edges simply supported and various conditions on the other edges. These eigenvectors reveal that for all standard edge conditions the variation of $\Psi_x(x, y)$ along y is of very similar shape to that of the variation of $W(x, y)$ along y and similarly $\Psi_y(x, y)$ and $W(x, y)$ are of very similar shape along x. Consequently, the spatial variations of the three fundamental quantities are taken to be of the following forms

$$W(x, y) = \sum_{n=1}^{r} \sum_{m=1}^{r} a_{mn} W_m(x) W_n(y), \qquad \Psi_x(x, y) = \sum_{n=1}^{r} \sum_{m=1}^{r} b_{mn} \Psi_m(x) W_n(y), \tag{44, 45}$$

$$\Psi_y(x, y) = \sum_{n=1}^{r} \sum_{m=1}^{r} c_{mn} W_m(x) \Psi_n(y). \tag{46}$$

Here the four unidirectional functions $W_m(x)$, $W_n(y)$, $\Psi_m(x)$ and $\Psi_n(y)$ are, of course, those Timoshenko beam normalized modes of section 2 which are appropriate to the boundary conditions of the particular plate under consideration.

The Timoshenko beam modes satisfy precisely all geometric and natural boundary conditions at the ends of each type of *beam*, of course. For the Mindlin *plate* the (three) boundary conditions on an edge $y = \bar{y}$ (a constant) for the "standard" cases are as follows (similar expressions can be written down for an edge $x = \bar{x}$, of course):

simply supported edge:

$$W(x, \bar{y}) = 0, \qquad \Psi_x(x, \bar{y}) = 0, \qquad M_y(x, \bar{y}) = 0; \tag{47}$$

clamped edge:

$$W(x, \bar{y}) = 0, \qquad \Psi_x(x, \bar{y}) = 0, \qquad \Psi_y(x, \bar{y}) = 0; \tag{48}$$

free edge:

$$M_y(x, \bar{y}) = 0, \qquad M_{yx}(x, \bar{y}) = 0, \qquad Q_y(x, \bar{y}) = 0. \tag{49}$$

It can easily be demonstrated that the use of the appropriate Timoshenko beam functions in Mindlin plate analysis allows the precise satisfaction of all geometric and natural boundary conditions where the plate edges are supported (i.e., either simply supported or clamped). Thus for plates with supported edges there need be no concern about the admissibility of the beam functions as trial functions in the Rayleigh–Ritz procedure.

For the case of a free edge, however, the situation is not so clear cut. At such an edge all three boundary conditions (equation (49)) are natural ones and none of these are precisely satisfied by the corresponding beam functions with free ends. This need not affect the admissibility of the free-end Timoshenko beam functions provided that it is *possible* for the plate natural boundary conditions to be met as a result of applying the minimization procedure. In fact, however, it can be shown that the moment acting along the direction normal to a free edge cannot possibly take the required value of zero when the assumed displacement field has the form of equations (44)–(46), except where Poisson's ratio is zero. To demonstrate this consider the situation at a free edge, y = constant. The moment in the direction normal to this edge is $M_y = D(\partial\psi_y/\partial y + \nu\, \partial\psi_x/\partial x)$, from equation (40). Now the quantity $\partial\psi_y/\partial y$ at the free edge of the plate must have the value zero since this quantity has precisely zero value at the free end of each beam function contributing to the complete displacement field of the plate. On the other hand the quantity $\partial\psi_x/\partial x$ at a general point on the free edge will have a non-zero value. It follows that the moment M_y at the free edge cannot possibly have zero value in a Rayleigh–Ritz analysis of the type described here (for $\nu \neq 0$). Effectively the proper free edge condition $M_y = 0$ is replaced by the modified condition $\partial\psi_y/\partial y = 0$ as a consequence of using the trial functions described above. This means that the plate problem considered is slightly more constrained than is intended by the designation "free" and correspondingly the calculated natural frequencies would be expected to converge to answers a little higher than the true "free" ones. This situation has its parallel in classical plate energy solutions where the use of classical beam functions effectively leads to the prescription of zero curvature in the direction normal to a free edge instead of the correct condition of zero moment.

The complete Rayleigh–Ritz procedure requires the generation of solutions for the appropriate normal modes, from equations (20)–(33), and subsequent substitution via equations (44)–(46) into an energy functional containing integrals over the plate middle surface of the strain and kinetic energies, equations (36)–(39). Minimization of the functional determines the coefficients a_{mn}, b_{mn} and c_{mn} which give the best approximation to the governing differential equations of the plate problem, and thence yields the natural frequencies and mode shapes. The procedure is so well known [7, 9] as to require no further comment.

4. NUMERICAL APPLICATIONS

In presenting numerical results of the Rayleigh–Ritz procedure described above the aim is to provide a detailed examination of the convergence of the procedure in a limited number of problems, rather than to consider a large range of possible boundary conditions and plate aspect ratios. Consequently, attention is restricted to plates of square planform (of side length A), to two thickness ratios ($h/A = 0{\cdot}01$ and $0{\cdot}10$) and to five combinations of the standard boundary conditions. In the expressions for the trial functions, equations (20)–(33), the number of terms in each direction in each product series (that is, r) is taken to be up to five. Although some of the plates considered have one or two axes of symmetry no account has been taken of this in the present work and the full plate has been analyzed in each case. The number of degrees of freedom in any eigenvalue analysis is thus $3r^2$.

In most applications the value of Poisson's ratio is taken to be 0·3 but some consideration is given to the value zero where free edges are present. For the shear factor, κ^2, the most appropriate value to assume is not absolutely clear. Mindlin suggested and used the value $\pi^2/12$ [4, 5] and this is a commonly used value. However, numerical comparisons of Mindlin theory results with elasticity solutions [1, 6] suggest that a higher value, perhaps 0·88, is more appropriate (for the corresponding Poisson's ratio value of 0·3). Fortunately, a limited range of variation in the assumed value of κ^2 does not affect the magnitudes of the calculated natural frequencies to a great extent. In the applications described here two values of κ^2 are used (0·822 and 0·8601), chosen partly to facilitate comparison with available alternative solutions.

The results presented in this section are defined in terms of a frequency parameter Ω given by

$$\Omega = p\sqrt{2(1+\nu)\rho A^2/E}. \tag{50}$$

The various plates are described by a symbolism defining the boundary conditions at their four edges; a SCSF plate, for instance, means a plate whose edges $x = 0$, $y = 0$, $x = A$ and $y = B$ (with $B = A$ here) are simply supported, clamped, simply supported and free, respectively. Vibrational mode shapes are described in the form i, j where i and j are the numbers of half-waves in the x and y directions, respectively.

In Tables 1–8 are shown the results generated by the Rayleigh–Ritz procedures ($r = 1, 2, 3, 4$ and 5) for the first six frequencies of flexural vibration of three types of square plate—SCSC, SCSS and SCSF—having one pair of opposite edges simply supported. For these plates the exact solutions of the classical plate equations are available [9] as are accurate solutions based on Mindlin theory [6]. The latter solutions are seen to be substantially lower than the classical solutions for the thick plate ($h/A = 0·1$) and slightly lower for the thin plate ($h/A = 0·01$). The Mindlin-theory solutions of reference [6] are obtained by using the finite strip method with the correct assumption of sinusoidal variation of the three reference quantities between the pair of opposite simply supported edges, and with piecewise-polynomial variation between the other opposite pair of edges. This latter variation corresponds to the use of quintic polynomials across the width of each strip, with sufficient strips used to achieve complete convergence. For the supported plates—SCSC and SCSS—all boundary conditions are precisely satisfied and the results in Tables 1–4 demonstrate a very close agreement between the present results and the finite strip results, for all values of r: the greatest difference for even the smallest possible value of r associated with any mode is only about one per cent. The rate of convergence to the

TABLE 1

Frequency parameters Ω for SCSC plate: $h/A = 0·1$, $\kappa^2 = 0·822$, $\nu = 0·3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r = 1$	1·307					
$r = 2$	1·307	2·419	2·892	3·867		
$r = 3$	1·303	2·403	2·892	3·867	4·244	4·941
$r = 4$	1·303	2·403	2·888	3·852	4·244	4·941
$r = 5$	1·302	2·398	2·888	3·852	4·237	4·939
Comparative solution [6]	1·300	2·394	2·885	3·839	4·232	4·936
Classical solution [9]	1·413	2·671	3·383	4·615	4·988	6·299

TABLE 2

Frequency parameters Ω for SCSC plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r=1$	0·1414					
$r=2$	0·1414	0·2683	0·3378	0·4618		
$r=3$	0·1412	0·2670	0·3378	0·4618	0·4984	0·6280
$r=4$	0·1412	0·2670	0·3377	0·4608	0·4984	0·6280
$r=5$	0·1411	0·2668	0·3377	0·4608	0·4979	0·6279
Comparative solution [6]	0·1411	0·2668	0·3376	0·4604	0·4977	0·6279
Classical solution [9]	0·1413	0·2671	0·3383	0·4615	0·4988	0·6299

TABLE 3

Frequency parameters Ω for SCSS plate: $h/A = 0{\cdot}1$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r=1$	1·098					
$r=2$	1·094	2·305	2·546	3·631		
$r=3$	1·093	2·301	2·545	3·623	4·192	4·546
$r=4$	1·093	2·299	2·544	3·619	4·189	4·544
$r=5$	1·092	2·298	2·543	3·616	4·187	4·543
Comparative solution [6]	1·092	2·296	2·542	3·611	4·184	4·541
Classical solution [9]	1·154	2·521	2·862	4·203	4·893	5·525

TABLE 4

Frequency parameters Ω for SCSS plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r=1$	0·1156					
$r=2$	0·1154	0·2524	0·2859	0·4209		
$r=3$	0·1153	0·2521	0·2858	0·4199	0·4889	0·5533
Comparative solution [6]	0·1153	0·2519	0·2858	0·4195	0·4883	0·5513
Classical solution [9]	0·1154	0·2521	0·2862	0·4203	0·4893	0·5525

true solution is good and is monotonic, from the high side, as expected. It is noted that in Table 4 no results are given for $r = 4$ and $r = 5$. This is because some evidence of numerical instability for the thin SCSS plate was apparent, manifested by eigenvalues having small complex parts and rendering these particular results unreliable. (Our concern is not really with plates of this degree of thinness, of course.) Turning now to the SCSF plate one expects some reduction in the accuracy of the results from the Rayleigh–Ritz approach because of the presence of the free edge and the corresponding difficulty associated with satisfying the natural boundary conditions, as discussed in section 3. Two values of Poisson's ratio are considered, 0·3 in Tables 5 and 6 and 0·0 in Tables 7 and 8, since it is recalled that a non-zero value precludes the complete satisfaction of the free-edge boundary conditions, even for large r, whilst the zero value allows such satisfaction in the

TABLE 5

Frequency parameters Ω for SCSF plate: $h/A = 0{\cdot}1$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	1, 2	2, 1	2, 2	1, 3	3, 1
$r = 1$	0·6185					
$r = 2$	0·6066	1·517	1·909	2·823		
$r = 3$	0·6052	1·500	1·904	2·755	3·083	3·861
$r = 4$	0·6041	1·499	1·902	2·753	3·074	3·859
$r = 5$	0·6033	1·495	1·900	2·744	3·073	3·855
Comparative solution [6]	0·5975	1·483	1·884	2·720	3·057	3·827
Classical solution [9]	0·6191	1·613	2·035	3·075	3·533	4·421

TABLE 6

Frequency parameter Ω for SCSF plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	1, 2	2, 1	2, 2	1, 3	3, 1
r = 1	0·06340					
r = 2	0·06234	0·1628	0·2052	0·3153		
r = 3	0·06223	0·1613	0·2047	0·3079	0·3535	0·4442
r = 4	0·06222	0·1613	0·2047	0·3079	0·3528	0·4442
r = 5	0·06217	0·1612	0·2045	0·3075	0·3528	0·4438
Comparative solution [6]	0·06187	0·1611	0·2033	0·3070	0·3526	0·4413
Classical solution [9]	0·06191	0·1613	0·2035	0·3075	0·3533	0·4421

TABLE 7

Frequency parameters Ω for SCSF plate: $h/A = 0{\cdot}1$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}0$

	Mode					
	1, 1	1, 2	2, 1	2, 2	1, 3	3, 1
r = 1	0·5672					
r = 2	0·5374	1·357	1·643	2·555		
r = 3	0·5374	1·329	1·643	2·437	2·701	3·348
r = 4	0·5362	1·327	1·640	2·436	2·687	3·346
r = 5	0·5362	1·322	1·640	2·423	2·686	3·345
Comparative solution [6]	0·5350	1·311	1·639	2·410	2·667	3·344

TABLE 8

Frequency parameters Ω for SCSF plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}822$, $\nu = 0{\cdot}0$

	Mode					
	1, 1	1, 2	2, 1	2, 2	1, 3	3, 1
$r=1$	0·05794					
$r=2$	0·05493	0·1439	0·1739	0·2803		
$r=3$	0·05493	0·1411	0·1739	0·2668	0·3017	0·3742
$r=4$	0·05485	0·1409	0·1737	0·2665	0·3005	0·3739
$r=5$	0·05485	0·1407	0·1737	0·2655	0·3004	0·3739
Comparative solution [6]	0·05483	0·1405	0·1736	0·2650	0·3000	0·3739

limit. Taken as a whole it is seen that the Rayleigh–Ritz method results for the SCSF plate are less accurate than for the supported plates, as expected. However, with $r = 5$ all frequencies are within one per cent of the comparative finite-strip solutions, whether $\nu = 0{\cdot}3$ or $0{\cdot}0$. Thus, in this particular problem at least, it appears that the partial contradiction in the specification of free-edge boundary conditions associated with the use of the Timoshenko beam functions, when ν is non-zero, has very little effect on the values of the calculated frequencies. What effect there is due to this seems to be greatest for modes (1, 1), (2, 1) and (3, 1): i.e., for modes having the simplest shape across the plate between the clamped and free edges.

For plates not having one pair of opposite edges simply supported there are very few comparative results available. However, Nelson has presented approximate results for the square, thick ($h/A = 0{\cdot}1$) CCCC and CCCF plates, based on Bolotin's solution fitting technique, and so results for these two types of plate, with thick and thin geometry, are presented in Tables 9–12. For the thick geometry, Nelson's results are given in the form of the ratio Ω/Ω_0 where Ω and Ω_0 are values of the frequency parameter as calculated by using Mindlin theory and classical theory, respectively, and comparison is made on this basis in Tables 9 and 11. (Note that the classical-theory solutions, Ω_0, are themselves approximate and that it is not known if the Ω_0 values used in reference [11] are precisely the same as those quoted here in Tables 9 and 11.) For the thin geometry, only the

TABLE 9

Frequency parameters Ω for CCCC plate: $h/A = 0{\cdot}1$, $\kappa^2 = 0{\cdot}8601$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r=1$	1·605					
$r=2$	1·605	3·065	3·065	4·311		
$r=3$	1·597	3·053	3·053	4·311	5·045	5·083
$r=4$	1·597	3·049	3·049	4·285	5·045	5·083
$r=5$	1·594	3·046	3·046	4·285	5·035	5·078
Classical solution, Ω_0 [9]	1·756	3·581	3·581	5·280	6·421	6·451
Ω/Ω_0, present work	0·9078	0·8506	0·8506	0·8116	0·7842	0·7872
Ω/Ω_0, reference [11]	0·9007	0·8526	0·8526	0·8107	0·7970	0·7970

TABLE 10

Frequency parameters Ω for CCCC plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}8601$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	2, 1	1, 2	2, 2	3, 1	1, 3
$r=1$	0·1761					
$r=2$	0·1761	0·3591	0·3591	0·5297		
$r=3$	0·1755	0·3579	0·3579	0·5297	0·6409	0·6436
$r=4$	0·1755	0·3577	0·3577	0·5274	0·6409	0·6436
$r=5$	0·1754	0·3576	0·3576	0·5274	0·6402	0·6432
Classical solution [9]	0·1756	0·3581	0·3581	0·5280	0·6421	0·6451

TABLE 11

Frequency parameters Ω for CCCF plate: $h/A = 0{\cdot}1$, $\kappa^2 = 0{\cdot}8601$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	1, 2	2, 1	1, 3	2, 2	2, 3
$r=1$	1·100					
$r=2$	1·092	1·794	2·680		3·390	
$r=3$	1·090	1·766	2·677	3·232	3·335	4·685
$r=4$	1·090	1·765	2·674	3·223	3·326	4·616
$r=5$	1·089	1·758	2·673	3·216	3·318	4·615
Classical solution, Ω_0 [9]	1·171	1·953	3·094	3·744	3·938	5·699
Ω/Ω_0, present work	0·9295	0·8999	0·8638	0·8590	0·8426	0·8097
Ω/Ω_0, reference [11]	0·9030	0·8902	0·8614	0·8357	0·8618	—

TABLE 12

Frequency parameters Ω for CCCF plate: $h/A = 0{\cdot}01$, $\kappa^2 = 0{\cdot}8601$, $\nu = 0{\cdot}3$

	Mode					
	1, 1	1, 2	2, 1	1, 3	2, 2	2, 3
$r=1$	0·1181					
$r=2$	0·1173	0·1980	0·3100	0·4012		
$r=3$	0·1172	0·1954	0·3095	0·3753	0·3940	0·5781
$r=4$	0·1172	0·1953	0·3095	0·3744	0·3935	0·5696
$r=5$	0·1171	0·1951	0·3093	0·3740	0·3931	0·5695
Classical solution [9]	0·1171	0·1953	0·3094	0·3744	0·3938	0·5699

(approximate) classical solution is available for comparison (Tables 10 and 12). The Rayleigh–Ritz method results of Tables 9–12 clearly demonstrate well-behaved convergence characteristics for the CCCC and CCCF plates and for thick and thin geometry, although it is likely that for the CCCF plate, with the presence of a free edge, the ultimate convergence will be to levels which are very slightly higher than the true levels. For the thick geometry comparison with Nelson's results is generally very reasonable, given the approximate nature of these comparative results. For the thin geometry the present results converge to levels very slightly below those of the classical theory, as would be expected. On a point of detail, for the CCCC plate it should be noted that the present results forecast equal frequencies for modes (2, 1) and (1, 2), but separate, distinct frequencies for modes (3, 1) and (1, 3). In fact the designations (3, 1) and (1, 3) are perhaps not very appropriate descriptions of these latter mode shapes since the first of these modes is antisymmetric about both plate diagonals (and hence has nodal lines coinciding with the diagonals) whilst the second is symmetric about both diagonals (with a single, "ring-shaped" nodal line). This is in line with the findings of many investigators using classical theory [9].

5. CONCLUSIONS

The Rayleigh–Ritz procedure has been applied to the prediction of the natural frequencies of flexural vibration of isotropic plates when the effects of transverse shear and rotary inertia are taken into account. Timoshenko-beam functions have been used as trial functions in the spatial representation of the deflection w and of the two rotations ψ_x and ψ_y. The procedure has been demonstrated to be very effective in a range of applications.

For supported Mindlin plates all boundary conditions are precisely satisfied and the results show high accuracy, even with the use of very few trial functions. For plates with free edges the natural boundary conditions are not precisely met and indeed for non-zero Poisson's ratio there is some contradiction at the free boundary, associated with the use of the beam functions. Nevertheless, the effect of this contradiction on the calculated frequencies appears to be very small (one per cent or less) and the rate of convergence is still good. The situation parallels that found in obtaining the frequencies of classical plates when using classical beam modes as trial functions in the Rayleigh–Ritz procedure.

The present work lays the foundation for an extended study of the vibration of rectangular Mindlin plates by use of the Rayleigh–Ritz procedure, incorporating anisotropic as well as isotropic materials. Further, the problems of buckling of "thick" plates and of vibration in the presence of a membrane stress system are amenable to solution by the present method.

The study reported here has proceeded concurrently with an investigation of the use of the Timoshenko-beam functions, listed above, as longitudinal functions in the finite strip analysis of thick rectangular plates. This allows the extension of the work described in reference [6] to embrace a much wider range of plate boundary conditions.

ACKNOWLEDGMENT

The authors are pleased to acknowledge the use of the Birmingham University ICL 1906A computer in obtaining the numerical results reported herein. One of the authors (O.L.R.) wishes to express his gratitude to the Egyptian Government for their financial sponsorship during the period of this investigation.

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