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BUCKLING OF THICK SKEW PLATES

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SUMMARY

The paper considers the elastic buckling of thick skew plates. Up to now, very little work has been done on this subject and the paper aspires to fill this gap. Based on the principle of stationary total potential energy and the recently developed pb-2 Rayleigh–Ritz method as a solution procedure, buckling solutions for skew plates of various aspect ratios, skew angles and boundary conditions are determined. The tabulation of these results, not available thus far, should be useful to designers and researchers who may use them as benchmark values to test the validity and convergence of their numerical techniques and softwares for thick plate analysis.

1. INTRODUCTION

The elastic buckling of thin skew plates has been extensively researched since the pioneering papers by Salvadori,[1] Guest[2] and Wittrick.[3] A comprehensive list of references (over 20 papers) on this subject was presented in a recent paper by Wang *et al.*[4] Surprisingly, in the case of thick skew plates, there are virtually no papers in the open literature; although there have been several studies on the buckling of thick rectangular and circular plates (see for example, References 5–15).

The aim of this paper is to fill this gap by presenting a study of the elastic buckling of thick skew plates including a tabulation of buckling loads for designers. The Mindlin[16] plate theory will be used and the recently developed pb-2 Rayleigh–Ritz method employed[17,15] for solution. The special feature of the method lies in the definition of the Ritz function which takes the product of a two-dimensional polynomial function (p-2) and the boundary expressions (b). Each of the equations of the boundary is raised to appropriate powers. These pb-2 Ritz functions ensure satisfaction of the free, simply supported and clamped edges. Thus, pb-2 Ritz functions satisfy the geometric boundary conditions at the outset, making the Rayleigh–Ritz method general for plates of arbitrary shape and boundary conditions. The usual problem in the Rayleigh–Ritz method of finding a suitable Ritz function is overcome since the function is automatically defined by the prescribed boundary shape and conditions.

2. ENERGY FUNCTIONAL FOR MINDLIN PLATES

Consider a flat, isotropic, thick, skew plate of uniform thickness, t, length, a, oblique width, b, Young's modulus, E, shear modulus, G and Poisson's ratio, ν. The plate may have any prescribed

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combination of supporting edges. The plate is subject to a normal in-plane load, N_x, as shown in Figure 1. The problem is to determine the elastic buckling load of the plate.

The energy functional of the plate is given by

$$U = \frac{1}{2}\int_V \varepsilon_L^T [B] \varepsilon_L \, dV + \int_V \tau^T \varepsilon_N \, dV \tag{1}$$

in which

V = volume of the skew plate,
ε = column matrix of strains,
τ = column matrix of stresses,
$[B]$ = material property matrix,

and the subscripts L and N denote linear and non-linear components, respectively. Note that the much higher-order non-linear strain product $\varepsilon_N^T [B] \varepsilon_N$ has been neglected.

The displacement fields of the plate in orthogonal co-ordinates can be expressed as

$$u(x, y, z) = z\theta_x(x, y) \tag{2a}$$

$$v(x, y, z) = z\theta_y(x, y) \tag{2b}$$

$$w(x, y, z) = w(x, y) \tag{2c}$$

in which

u, v, w = displacements in the x, y and z directions, respectively,
$\theta_x(x, y)$ = bending slope along y direction;
$\theta_y(x, y)$ = bending slope along x direction.

Note that θ_x and θ_y are independent variables and note that the transverse displacement, w, is assumed to be independent of z (i.e. no thickness deformation is allowed).

In view of equation (2) and using Green's definition for strains,

$$\varepsilon_L = \begin{Bmatrix} \varepsilon_{xxL} \\ \varepsilon_{yyL} \\ \gamma_{xyL} \\ \gamma_{xzL} \\ \gamma_{yzL} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{Bmatrix} = \begin{Bmatrix} z\dfrac{\partial \theta_x}{\partial x} \\ z\dfrac{\partial \theta_y}{\partial y} \\ z\left(\dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x}\right) \\ \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{3}$$

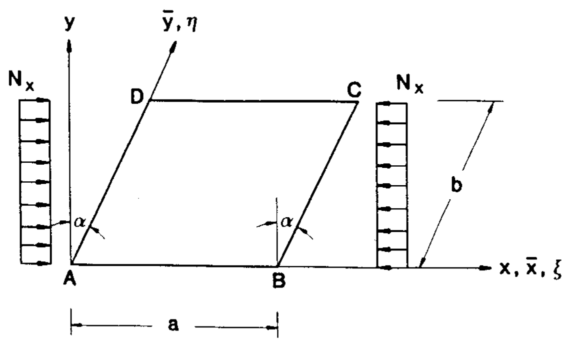

Figure 1. Skew plate under in-plane uniaxial loading

$$\varepsilon_{\mathrm{N}} = \begin{Bmatrix} \varepsilon_{xx\mathrm{N}} \\ \varepsilon_{yy\mathrm{N}} \\ \gamma_{xy\mathrm{N}} \\ \gamma_{xz\mathrm{N}} \\ \gamma_{yz\mathrm{N}} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \end{Bmatrix} \tag{4}$$

The material property matrix is given by

$$[B] = \begin{bmatrix} \frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 & 0 & 0 \\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & \kappa G & 0 \\ 0 & 0 & 0 & 0 & \kappa G \end{bmatrix} \tag{5}$$

in which $G = E/[2(1+v)]$ and $\kappa = 5/6$ = the shear correction factor which was first presented by Reissner[18] to compensate for the errors when assuming a constant shear strain distribution.

For uniaxial normal loading the column matrix of stresses, τ, is given by

$$\tau^{\mathrm{T}} = \{-N_x/t \;\; 0 \;\; 0 \;\; 0 \;\; 0\} \tag{6}$$

Substituting equations (2)–(6) into equation (1) yields

$$\Pi = \frac{1}{2}\int_{\mathrm{V}} \left\{ \frac{Ez^2}{1-v^2}\left[\left(\frac{\partial\theta_x}{\partial x} + \frac{\partial\theta_y}{\partial y}\right)^2 - 2(1-v)\left(\frac{\partial\theta_x}{\partial x}\frac{\partial\theta_y}{\partial y} - \frac{1}{4}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right)^2\right)\right] \right.$$
$$\left. + \kappa G\left[\left(\theta_x + \frac{\partial w}{\partial x}\right)^2 + \left(\theta_y + \frac{\partial w}{\partial y}\right)^2\right] - \frac{N_x}{t}\left(\frac{\partial w}{\partial x}\right)^2 \right\} \mathrm{d}V \tag{7}$$

Note that if $\theta_x = -\partial w/\partial x$ and $\theta_y = -\partial w/\partial y$, equation (7) reduces to the well-known energy functional for thin plates.

In order to satisfy the boundary conditions of the skew plate, it is expedient to use oblique co-ordinates $(\bar{x}, \bar{y})$ instead of the above orthogonal co-ordinates (x, y). From simple geometry, the oblique co-ordinates $\bar{x}$ and $\bar{y}$ are given by

$$\bar{x} = x - y\tan\alpha \tag{8a}$$

$$\bar{y} = y\sec\alpha \tag{8b}$$

The rotations θ can be expressed in the oblique co-ordinates as

$$\theta_x(x, y) = \theta_{\bar{x}}(\bar{x}, \bar{y})\cos\alpha \tag{8c}$$

$$\theta_y(x, y) = -\theta_{\bar{x}}(\bar{x}, \bar{y})\sin\alpha + \theta_{\bar{y}}(\bar{x}, \bar{y}) \tag{8d}$$

and the partial derivatives are related by

$$\frac{\partial(\circ)}{\partial x} = \frac{\partial(\circ)}{\partial \bar{x}} \tag{8e}$$

$$\frac{\partial(\circ)}{\partial y} = -\frac{\partial(\circ)}{\partial \bar{x}}\tan\alpha + \frac{\partial(\circ)}{\partial \bar{y}}\sec\alpha \tag{8f}$$

in which α is the skew angle (see Figure 1).

For generality and convenience, the oblique co-ordinates are normalised with respect to the plate dimensions, i.e.

$$\xi = \frac{\bar{x}}{a}, \quad \eta = \frac{\bar{y}}{b} \tag{9}$$

In view of equations (8) and (9), the integration of equation (7) with respect to z yields the following energy functional expression:

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^1\int_0^1 \Bigg\{ D\Bigg[&\sec^2\alpha\left(\frac{1}{a}\frac{\partial\theta_{\bar{x}}}{\partial\xi} - \frac{\sin\alpha}{b}\frac{\partial\theta_{\bar{x}}}{\partial\eta} - \frac{\sin\alpha}{a}\frac{\partial\theta_{\bar{y}}}{\partial\xi} + \frac{1}{b}\frac{\partial\theta_{\bar{y}}}{\partial\eta}\right)^2 \\ &- 2(1-\nu)\left(\frac{1}{ab}\frac{\partial\theta_{\bar{x}}}{\partial\xi}\frac{\partial\theta_{\bar{y}}}{\partial\eta} - \frac{1}{4}\left(\frac{1}{b}\frac{\partial\theta_{\bar{x}}}{\partial\eta} + \frac{1}{a}\frac{\partial\theta_{\bar{y}}}{\partial\xi}\right)^2\right)\Bigg] \\ &+ \kappa Gt\left[\left(\cos\alpha\,\theta_{\bar{x}} + \frac{1}{a}\frac{\partial w}{\partial\xi}\right)^2 + \left(-\sin\alpha\,\theta_{\bar{x}} + \theta_{\bar{y}} - \frac{\tan\alpha}{a}\frac{\partial w}{\partial\xi} + \frac{\sec\alpha}{b}\frac{\partial w}{\partial\eta}\right)^2\right] \\ &- N_x\left(\frac{1}{a}\frac{\partial w}{\partial\xi}\right)^2\Bigg\} ab\cos\alpha\,\mathrm{d}\xi\,\mathrm{d}\eta\end{aligned} \tag{10}$$

in which $D = Et^3/[12(1-\nu^2)]$.

3. BOUNDARY CONDITIONS FOR MINDLIN PLATES

The support conditions for Mindlin plates[19] are;

(1) Free edge (F)—For this type of edge condition,

$$Q_n = 0, \quad M_n = 0 \quad \text{and} \quad M_{nt} = 0 \tag{11}$$

in which Q_n is the shearing force, M_n is the bending moment and M_{nt} is the twisting moment.

(2) Simply supported edge (S and S*)—There are two kinds of simply supported edges in the Mindlin plate theory. The first kind (S) requires

$$w = 0, \quad M_n = 0 \quad \text{and} \quad \theta_t = 0 \tag{12}$$

in which θ_t is the rotation of the mid-plane normal in the tangent plane tz to the plate boundary.

The boundary conditions for the second kind (S*) are such that

$$w = 0, \quad M_n = 0 \quad \text{and} \quad M_{nt} = 0 \tag{13}$$

(3) Clamped edge (C)—For this kind of edge condition,

$$w = 0, \quad \theta_n = 0 \quad \text{and} \quad \theta_t = 0 \tag{14}$$

in which θ_n is the rotation of the mid-plane normal to the clamped edge.

4. pb-2 RAYLEIGH–RITZ METHOD

For Mindlin plates, the transverse deflection and rotational surfaces may be parameterized by

$$w(\xi, \eta) = \sum_{i=1}^{m} c_i \phi_i(\xi, \eta) \tag{15a}$$

$$\theta_{\bar{x}}(\xi, \eta) = \sum_{i=1}^{n} d_i \psi_{xi}(\xi, \eta) \tag{15b}$$

$$\theta_{\bar{y}}(\xi, \eta) = \sum_{i=1}^{l} e_i \psi_{yi}(\xi, \eta) \tag{15b}$$

where

$$\phi_i(\xi, \eta) = f_i(\xi, \eta)\phi_1(\xi, \eta) \tag{16a}$$

$$\psi_{xi}(\xi, \eta) = f_i(\xi, \eta)\psi_{x1}(\xi, \eta) \tag{16b}$$

$$\psi_{yi}(\xi, \eta) = f_i(\xi, \eta)\psi_{y1}(\xi, \eta) \tag{16c}$$

f_i is a two-dimensional polynomial function generated as follows:

$$f_i(\xi, \eta) = \xi^r \eta^s (\cos^2 \pi q) + \xi^s \eta^r (\sin^2 \pi q) \tag{17a}$$

where

$$r = \lceil \sqrt{(i-1)} \rceil \tag{17b}$$

$$q = \frac{i - r^2 - 1}{2} \tag{17c}$$

$$s = q(\cos^2 \pi q) + (q - \tfrac{1}{2})(\sin^2 \pi q) \tag{17d}$$

$\lceil \quad \rceil$ denotes the greatest integer value, for example $\lceil \sqrt{2} \rceil = 1$. $\phi_1, \psi_{x1}, \psi_{y1}$ are the basic functions which must satisfy the geometric boundary conditions (equations (12)–(14)). The basic function for the deflection[17] can be expressed as

$$\phi_1 = (\eta)^{\Omega_1}(\xi - 1)^{\Omega_2}(\eta - 1)^{\Omega_3}(\xi)^{\Omega_4} \tag{18}$$

in which the subscripts of Ω, 1, 2, 3, 4 refer to the supporting edges AB, BC, CD and DA, respectively (see Figure 1) and Ω_i, depending upon the support edge condition, takes on

$\Omega_i = 0$ if the ith edge is free (F) (19a)

$\Omega_i = 1$ if the ith edge is simply supported (S and S*) or clamped (C) (19b)

The basic functions for the rotations can be expressed as

$$\psi_{x1} = (\eta)^{\Omega_1}(\xi - 1)^{\Omega_2}(\eta - 1)^{\Omega_3}(\xi)^{\Omega_4} \tag{20}$$

$\Omega_i = 0$ if the ith edge is free (F) or simply supported (S*) or simply supported (S) in the $\bar{y}$-direction (21a)

$\Omega_i = 1$ if the ith edge is simply supported (S) in the $\bar{x}$-direction or clamped (C) (21b)

$$\psi_{y1} = (\eta)^{\Omega_1}(\xi - 1)^{\Omega_2}(\eta - 1)^{\Omega_3}(\xi)^{\Omega_4} \tag{22}$$

$\Omega_i = 0$, if the ith edge is free (F) or simply supported (S*) or simply supported (S) in the $\bar{x}$-direction (23a)

$\Omega_i = 1$, if the ith edge is simply supported (S) in the $\bar{y}$-direction or clamped (C) (23b)

Applying the Rayleigh–Ritz method leads to

$$[K]\begin{Bmatrix}\{c\}\\ \{d\}\\ \{e\}\end{Bmatrix}=\{0\} \tag{24}$$

in which

$$\{c\}=\begin{Bmatrix}c_1\\ c_2\\ \vdots\\ c_m\end{Bmatrix},\quad \{d\}=\begin{Bmatrix}d_1\\ d_2\\ \vdots\\ d_n\end{Bmatrix},\quad \{e\}=\begin{Bmatrix}e_1\\ e_2\\ \vdots\\ e_l\end{Bmatrix} \tag{25}$$

$$[K]=\begin{bmatrix}[K_{cc}] & [K_{cd}] & [K_{ce}]\\ & [K_{dd}] & [K_{de}]\\ \text{symmetric} & & [K_{ee}]\end{bmatrix} \tag{26}$$

$$\begin{aligned}K_{ccij}={}&\frac{b}{a}\cos\alpha(\kappa Gt\sec^2\alpha-N_x)\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\xi}\frac{\partial\phi_j}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &+\frac{a}{b}\sec\alpha\,\kappa Gt\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\eta}\frac{\partial\phi_j}{\partial\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &-\tan\alpha\,\kappa Gt\left(\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\xi}\frac{\partial\phi_j}{\partial\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta+\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\eta}\frac{\partial\phi_j}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\right),\\ &i=1,2,\ldots,m,\ j=1,2,\ldots,m\end{aligned} \tag{27}$$

$$\begin{aligned}K_{cdij}={}&b\kappa Gt\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\xi}\psi_{xj}\,\mathrm{d}\xi\,\mathrm{d}\eta-a\sin\alpha\,\kappa Gt\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\eta}\psi_{xj}\,\mathrm{d}\xi\,\mathrm{d}\eta,\\ &i=1,2,\ldots,m,\ j=1,2,\ldots,n\end{aligned} \tag{28}$$

$$\begin{aligned}K_{ceij}={}&a\kappa Gt\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\eta}\psi_{yj}\,\mathrm{d}\xi\,\mathrm{d}\eta-b\sin\alpha\,\kappa Gt\int_0^1\int_0^1\frac{\partial\phi_i}{\partial\xi}\psi_{yj}\,\mathrm{d}\xi\,\mathrm{d}\eta,\\ &i=1,2,\ldots,m,\ j=1,2,\ldots,l\end{aligned} \tag{29}$$

$$\begin{aligned}K_{ddij}={}&\frac{b}{a}D\sec\alpha\int_0^1\int_0^1\frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{xj}}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &+\frac{a}{b}D\cos\alpha\left(\tan^2\alpha+\frac{(1-\nu)}{2}\right)\int_0^1\int_0^1\frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{xj}}{\partial\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &-D\tan\alpha\left(\int_0^1\int_0^1\frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{xj}}{\partial\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta+\int_0^1\int_0^1\frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{xj}}{\partial\xi}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)\\ &+ab\cos\alpha\,\kappa Gt\int_0^1\int_0^1\psi_{xi}\psi_{xj}\,\mathrm{d}\xi\,\mathrm{d}\eta,\\ &i=1,2,\ldots,n,\ j=1,2,\ldots,n\end{aligned} \tag{30}$$

$$K_{deij} = D\cos\alpha(\sec^2\alpha - 1 + \nu)\int_0^1\int_0^1 \frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\eta}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$+ D\cos\alpha\left(\tan^2\alpha + \frac{(1-\nu)}{2}\right)\int_0^1\int_0^1 \frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\xi}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$-\frac{b}{a}D\tan\alpha\int_0^1\int_0^1 \frac{\partial\psi_{xi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\xi}\mathrm{d}\xi\,\mathrm{d}\eta - \frac{a}{b}D\tan\alpha\int_0^1\int_0^1 \frac{\partial\psi_{xi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\eta}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$- ab\sin\alpha\cos\alpha\,\kappa G t\int_0^1\int_0^1 \psi_{xi}\psi_{yj}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, l \qquad (31)$$

$$K_{eeij} = \frac{a}{b}D\sec\alpha\int_0^1\int_0^1 \frac{\partial\psi_{yi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\eta}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$+\frac{b}{a}D\cos\alpha\left(\tan^2\alpha + \frac{(1-\nu)}{2}\right)\int_0^1\int_0^1 \frac{\partial\psi_{yi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\xi}\mathrm{d}\xi\,\mathrm{d}\eta$$

$$- D\tan\alpha\left(\int_0^1\int_0^1 \frac{\partial\psi_{yi}}{\partial\xi}\frac{\partial\psi_{yj}}{\partial\eta}\mathrm{d}\xi\,\mathrm{d}\eta + \int_0^1\int_0^1 \frac{\partial\psi_{yi}}{\partial\eta}\frac{\partial\psi_{yj}}{\partial\xi}\mathrm{d}\xi\,\mathrm{d}\eta\right)$$

$$+ ab\cos\alpha\,\kappa G t\int_0^1\int_0^1 \psi_{yi}\psi_{yj}\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$i = 1, 2, \ldots, l; \; j = 1, 2, \ldots, l \qquad (32)$$

The buckling load intensity factor, $k = N_x b^2/(\pi^2 D)$, is obtained by solving the generalized eigenvalue problem defined by equation (24).

5. NUMERICAL RESULTS

Skew plates of Poisson's ratio $\nu = 0{\cdot}3$, various aspect ratios (a/b), skew angles (α), thickness to width ratios (t/b) and different combinations of supporting conditions (see Figure 2) are con-

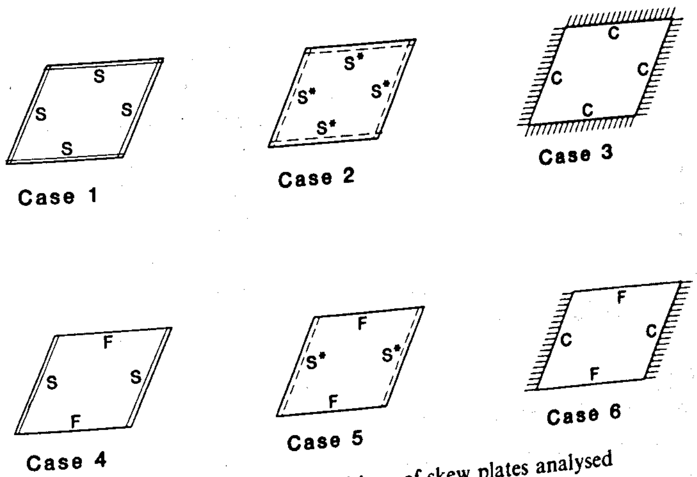

Figure 2. Boundary conditions of skew plates analysed

sidered. For purposes of comparison with results obtained by other researchers, the approximate shear correction factor, $\kappa = 5/6$ is adopted.

5.1. Convergence study

A convergence study was carried out to establish the number of polynomial terms required for accurate solution. Table I presents the solutions for skew Mindlin plates (with $a/b = 1$, $t/b = 0{\cdot}001, 0{\cdot}20$) for various m, n, l values. It can be seen that relatively more terms are required for highly skewed plate than when compared to a rectangular plate ($\alpha = 0$). This is due to the stress concentrations of the corners of the plates. For accurate solutions, the number of terms required is $m = n = l = 120$ and these numbers are used to generate the results.

5.2. Results

Tables II and III present the buckling load intensity factors for six different boundary conditions, i.e. SSSS, S*S*S*S*, CCCC, FSFS, FS*FS* and FCFC taking aspect ratios, $a/b = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5, 2{\cdot}0$ and $2{\cdot}5$; skew angles, $\alpha = 0, 15, 30$ and $45°$ and thickness to width ratios, $t/b = 0{\cdot}001, 0{\cdot}05, 0{\cdot}10, 0{\cdot}15$ and $0{\cdot}20$.

5.3. Confirmation of results

The results for SSSS square plates ($a/b = 1, \alpha = 0°$) are in total agreement with Hinton's closed form solutions[9] of 4·000, 3·944, 3·786 and 3·264 for $t/b = 0{\cdot}001, 0{\cdot}05, 0{\cdot}10$ and $0{\cdot}20$, respectively. These values are only a few percent different from the exact three dimensional solutions given by Srinivas and Rao.[6] For thin skew SSSS plates ($t/b = 0{\cdot}001$ and $a/b = 1$), the buckling loads are in good agreement with those obtained by Mizusawa and Leonard,[20] where $k = 4{\cdot}000, 4{\cdot}394, 5{\cdot}894, 10{\cdot}08$ for $\alpha = 0, 15, 30$ and $45°$, respectively.

5.4. Buckling behaviour of skew plates

As expected, the shear deformation effect becomes more important as the plate thickness to width ratio, t/b, increases. It is seen from Tables II and III that the buckling load intensity factor decreases with increasing thickness to width ratio, t/b. The decrease in the buckling load intensity factor is more pronounced for plates with small aspect ratios than for plates with large aspect ratios and occurs most significantly on the CCCC case and least significantly on FS*FS* case.

The buckling load intensity factor increases with increasing skew angles, α. The increase is more pronounced for plates with small aspect ratios, a/b, than for plates with large aspect ratios and occurs most significantly on the S*S*S*S* case and least on the FCFC case. The results show the importance of the shear deformation effect for very skewed plates and, thus, cannot be ignored. For example, in the CCCC case with aspect ratio $a/b = 0{\cdot}5$ and skew angle, $\alpha = 45°$ (see Table II), the reduction in the buckling load intensity factor from plates with $t/b = 0{\cdot}001$ to $0{\cdot}2$ is almost 85 per cent.

It is worth noting that the buckling load intensity factors for plates with the S and S* boundary conditions are very close in thin plates ($t/b = 0{\cdot}001$) but deviate from one another as plate thickness–width ratio increases.

Table I. Convergence of buckling load intensity factors, $k = N_x b^2/(\pi^2 D)$, for thick skew plates with $a/b = 1$ and $t/b = 0{\cdot}001$ and 0·20

Cases	Terms	Skew angle α							
		$t/b = 0{\cdot}001$				$t/b = 0{\cdot}20$			
	$m = n = l$	0°	15°	30°	45°	0°	15°	30°	45°
SSSS	40	4·0000	4·4001	5·9716	10·5906	3·2637	3·5360	4·4900	6·3996
	90	4·0000	4·3948	5·9131	10·2154	3·2637	3·5331	4·4587	6·1839
	100	4·0000	4·3943	5·9039	10·1534	3·2637	3·5328	4·4542	6·1690
	120	4·0000	4·3938	5·8969	10·1032	3·2637	3·5326	4·4509	6·1572
S*S*S*S*	40	3·9999	4·4030	6·0159	11·1075	2·8817	3·1178	3·9395	5·7027
	90	3·9998	4·3956	5·9254	10·2916	2·8767	3·1112	3·9232	5·5502
	100	3·9998	4·3946	5·9125	10·2234	2·8767	3·1111	3·9228	5·5487
	120	3·9998	4·3940	5·9026	10·1428	2·8766	3·1111	3·9226	5·5479
CCCC	40	10·1062	10·8835	13·7557	22·4861	5·3164	5·4932	6·0587	7·2853
	90	10·0738	10·8345	13·5379	20·1259	5·3156	5·4913	6·0328	6·9750
	100	10·0737	10·8345	13·5377	20·1136	5·3156	5·4913	6·0328	6·9728
	120	10·0738	10·8345	13·5377	20·1115	5·3156	5·4913	6·0328	6·9712

Table II. Buckling load intensity factor, $k = N_x b^2/(\pi^2 D)$, of thick skew plates having SSSS, S*S*S*S* and CCCC boundary conditions subject to uniaxial loads

		Skew angle α											
		Case 1				Case 2				Case 3			
a/b	t/b	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
0·5	0·001	6·2499	6·9782	9·9166	19·2473	6·2495	6·9784	9·9221	19·2749	19·3377	21·5540	30·2876	54·5553
	0·050	6·0372	6·7238	9·4599	17·8531	5·8049	6·4524	9·0251	16·9151	17·2222	19·0005	25·6575	40·2963
	0·100	5·4777	6·0608	8·3206	14·6783	5·1177	5·6594	7·7627	13·7337	13·0260	14·0534	17·4575	22·8724
	0·150	4·7448	5·2033	6·9116	11·1187	4·3680	4·7971	6·4033	10·4511	9·2881	9·7854	11·2613	13·3414
	0·200	3·9963	4·3401	5·5594	8·0084	3·6626	3·9906	5·1647	7·6765	6·6166	6·8359	7·4767	8·3812
1.0	0·001	4·0000	4·3938	5·8969	10·1032	3·9998	4·3940	5·9026	10·1428	10·0738	10·8345	13·5377	20·1115
	0·050	3·9444	4·3280	5·7784	9·7063	3·7835	4·1314	5·4182	8·7328	9·5588	10·2312	12·5711	17·9652
	0·100	3·7865	4·1422	5·4617	8·7991	3·4950	3·8027	4·9324	7·7236	8·2917	8·7741	10·3760	13·6909
	0·150	3·5496	3·8650	4·9980	7·5451	3·1859	3·4564	4·4275	6·6376	6·7595	7·0589	8·0098	9·7994
	0·200	3·2637	3·5326	4·4509	6·1572	2·8766	3·1111	3·9226	5·5479	5·3156	5·4913	6·0328	6·9712
1·5	0·001	4·3403	4·6783	5·9226	9·1658	4·3401	4·6784	5·9270	9·1904	8·3504	8·9333	11·0296	16·2603
	0·050	4·2570	4·5836	5·7774	8·8230	4·1306	4·4356	5·5350	8·2567	7·9431	8·4688	10·3299	14·7817
	0·100	4·0250	4·3217	5·3885	7·9821	3·7885	4·0568	5·0123	7·2878	6·9608	7·3624	8·7331	11·7215
	0·150	3·6900	3·9455	4·8418	6·8627	3·4039	3·6340	4·4350	6·2293	5·8012	6·0778	6·9825	8·7749
	0·200	3·3048	3·5162	4·2327	5·6835	3·0144	3·2069	3·8580	5·2051	4·7156	4·8953	5·4620	6·4829
2·0	0·001	4·0000	4·3417	5·6206	8·9046	3·9999	4·3419	5·6254	8·9238	7·8670	8·3866	10·2834	15·1971
	0·050	3·9444	4·2776	5·5160	8·5781	3·8322	4·1445	5·2964	8·1948	7·4870	7·9578	9·6521	13·8673
	0·100	3·7865	4·0970	5·2327	7·7548	3·5638	3·8445	4·8718	7·2197	6·5736	6·9385	8·2097	11·1128
	0·150	3·5496	3·8273	4·8146	6·6667	3·2619	3·5108	4·4035	6·1458	5·5013	5·7596	6·6266	8·4362
	0·200	3·2637	3·5036	4·2328	5·5372	2·9513	3·1682	3·9224	5·1203	4·5026	4·6775	5·2444	6·3231
2·5	0·001	4·1344	4·4365	5·5556	8·5024	4·1343	4·4370	5·5596	8·5257	7·5731	8·1151	9·9700	14·9399
	0·050	4·0645	4·3579	5·4378	8·2399	3·9590	4·2372	5·2508	7·8731	7·2304	7·7213	9·3633	13·6417
	0·100	3·8683	4·1384	5·1202	7·5635	3·6628	3·9107	4·8039	7·0415	6·3908	6·7442	7·9835	10·9619
	0·150	3·5802	3·8177	4·6639	6·5864	3·3214	3·5369	4·2996	6·1100	5·3629	5·6138	6·4702	8·3682
	0·200	3·2421	3·4436	4·1403	5·4859	2·9697	3·1533	3·7879	5·0929	4·4033	4·5761	5·1475	6·3208

Table III. Buckling load intensity factor, $k = N_x b^2/(\pi^2 D)$, of thick skew plates having FSFS, FS*FS* and FCFC boundary conditions subject to uniaxial loads

		Skew angle α											
		Case 4				Case 5				Case 6			
a/b	t/b	0°	15°	30°	45°	0°	15°	30°	45°	0°	15°	30°	45°
	0·001	3·8926	4·4093	6·4561	11·7162	3·8926	4·4093	6·4561	11·7148	15·8221	17·6435	23·4293	34·0985
	0·050	3·7760	4·2479	6·0578	10·2566	3·7679	4·2333	5·9907	9·7176	14·1855	15·6654	19·8951	26·8958
0·5	0·100	3·4851	3·8837	5·3584	8·4765	3·4707	3·8585	5·2440	7·6624	10·8894	11·8222	14·0421	17·4404
	0·150	3·0956	3·4160	4·5631	6·7940	3·0777	3·3858	4·4302	5·9658	7·8618	8·3989	9·4702	11·1531
	0·200	2·6797	2·9303	3·8034	5·3637	2·6613	2·8999	3·6748	4·6475	5·6621	5·9663	6·5164	7·4336
	0·001	0·9523	1·0674	1·5128	2·7443	0·9523	1·0674	1·5128	2·7465	3·9193	4·2824	5·6159	8·0948
	0·050	0·9433	1·0523	1·4676	2·5762	0·9421	1·0501	1·4603	2·5450	3·8007	4·1387	5·3660	7·4670
1·0	0·100	0·9222	1·0229	1·4006	2·3767	0·9199	1·0189	1·3887	2·3316	3·5077	3·7937	4·8043	6·3311
	0·150	0·8908	0·9823	1·3198	2·1642	0·8877	0·9773	1·3065	2·1193	3·1152	3·3391	4·1036	5·1556
	0·200	0·8512	0·9330	1·2300	1·9505	0·8477	0·9278	1·2174	1·8766	2·6961	2·8620	3·4071	4·1438
	0·001	0·4168	0·4633	0·6384	1·1080	0·4168	0·4633	0·6385	1·1083	1·7287	1·8528	2·2826	3·2347
	0·050	0·4147	0·4591	0·6244	1·0555	0·4144	0·4586	0·6224	1·0479	1·7023	1·8208	2·2289	3·1148
1·5	0·100	0·4102	0·4521	0·6060	0·9974	0·4096	0·4509	0·6023	0·9852	1·6382	1·7461	2·1111	2·8684
	0·150	0·4036	0·4427	0·5848	0·9365	0·4028	0·4412	0·5804	0·9234	1·5450	1·6388	1·9486	2·5564
	0·200	0·3949	0·4312	0·5613	0·8742	0·3940	0·4296	0·5569	0·8624	1·4327	1·5110	1·7631	2·2301
	0·001	0·2322	0·2566	0·3464	0·5749	0·2322	0·2566	0·3465	0·5753	0·9663	1·0205	1·1971	1·5375
	0·050	0·2315	0·2549	0·3402	0·5522	0·2314	0·2547	0·3394	0·5488	0·9568	1·0090	1·1788	1·5024
2·0	0·100	0·2300	0·2522	0·3325	0·5280	0·2298	0·2518	0·3308	0·5216	0·9350	0·9840	1·1415	1·4336
	0·150	0·2278	0·2488	0·3240	0·5030	0·2275	0·2482	0·3218	0·4954	0·9028	0·9474	1·0887	1·3422
	0·200	0·2250	0·2447	0·3147	0·4776	0·2247	0·2441	0·3124	0·4697	0·8623	0·9018	1·0248	1·2386
	0·001	0·1477	0·1626	0·2165	0·3478	0·1477	0·1626	0·2165	0·3481	0·6151	0·6428	0·7293	0·8827
	0·050	0·1474	0·1617	0·2131	0·3359	0·1474	0·1616	0·2127	0·3342	0·6106	0·6374	0·7211	0·8689
2·5	0·100	0·1468	0·1604	0·2091	0·3234	0·1467	0·1602	0·2081	0·3198	0·6010	0·6266	0·7058	0·8429
	0·150	0·1459	0·1589	0·2048	0·3108	0·1458	0·1586	0·2035	0·3060	0·5870	0·6108	0·6840	0·8079
	0·200	0·1447	0·1570	0·2001	0·2978	0·1446	0·1566	0·1987	0·2926	0·5691	0·5909	0·6570	0·7666

6. CONCLUDING REMARKS

The buckling solutions for Mindlin skew plates having any combination of supporting edge conditions may be readily determined using the pb-2 Rayleigh–Ritz method. The method eliminates the inconvenience of having to seek appropriate displacement functions that will satisfy the geometric boundary conditions. This automatic satisfaction of geometric boundary conditions is achieved by multiplying the Ritz function with the boundary expressions raised to appropriate powers. It is clear that the method can be used to analyse any general plate shape, although this work concentrates on skew plates.

The study shows that the effect of shear deformation on the buckling capacity is more significant for skew plates with relatively large values of thickness to width ratios and skew angles. For such plates, the classical thin plate theory overpredicts considerably the buckling loads and, thus, lead to very unsafe designs.

The tabulated buckling solutions for thick skew plates of various skew angles, aspect ratios and boundary conditions, not available previously, should be useful information to designers and to researchers. They may be used to serve as benchmark values in testing the validity and convergence of their numerical procedures and softwares for plate analysis.

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