energy norm is insensitive to errors on small subdomains. In the small neighborhood of boundaries the solution computed from the Kirchhoff model can differ very substantially from the solutions computed from higher models. This substantially different behavior of solutions in the small neighborhood of the boundary is called the *boundary layer effect* or *edge effect* [14.7]. Boundary layer effects are important from the point of view of engineering analysis, since the goal is often to determine moments and shear forces at the boundary, where the solutions corresponding to various plate models can differ very significantly.

Full analysis of boundary layer effects is beyond the scope of this text. For details the reader is referred to [14.8] and [14.9]. In the following the results of an example taken from [14.9] is summarized and some general observations are presented.

Consider the Reissner–Mindlin model of a semi-infinite plate which occupies the half space $y > 0$ loaded by

$$q = Q \cos \frac{x}{L} \tag{14.43}$$

where $L > 0$ and Q are constants. A similar problem was discussed in [14.10] and [14.11]. The solution of this problem, given in [14.9], is summarized in Table 14.1. At the top of the table the solution (β_x, β_y, w) is given in terms of the plate constant D, the modulus of rigidity G, the shear correction factor κ, and the coefficients c_1, c_2, c_3, c_4, which are defined for various boundary conditions below. Although this is a very special problem, it illustrates well the boundary layer effects for cases where the boundary and the loading are smooth.

The boundary layer effect is caused by the term $e^{-\gamma y/d}$, which is multiplied by the coefficient c_4. Observe that c_4 depends on the boundary conditions, and that the boundary layer term is multiplied by different powers of d for different boundary conditions. The lower the power of d which multiplies this term, the stronger the boundary layer effect. In the case of the semi-infinite plate problem the hard simple support exhibits no boundary layer effect, since $c_4 = 0$. For the fixed boundary condition $c_4 \sim d$, and for free and soft simply supported edge $c_4 \sim 1$. Thus the strongest boundary layer effects are associated with the free and soft simply supported edge conditions. Since computation of moments and shear forces involves differentiation with respect to y, which increases the power of d in the denominator by one, the boundary layer effects are more pronounced for moments and shear forces than for β_x, β_y, and w. The Kirchhoff model has no boundary layer. This is because the Kirchhoff model is the limiting case with respect to $d \to 0$ and the boundary layer term vanishes in the limit.

The significance of the boundary layer is twofold. First, if one is interested in the behavior of the solution at the boundary, for example the reactions, then the boundary layer effects cannot be neglected. Second, the boundary layer effect makes the numerical solution of the problem more difficult: the mesh has to be refined at the boundary. Since the solution is smooth along smooth edges, refinement is needed only in the direction normal to the boundary. At corners

Table 14.1. Solution of the Reissner–Mindlin model of the semi-infinite plate problem (from Ref. [14.9] with permission by ASME)

$$\beta_x = \frac{QL^5}{d^3}\left[-\frac{d^3}{DL^2} - c_1 e^{-y/L} - c_2 \frac{y}{L} e^{-y/L} + c_3 \frac{d^2 e^{-y/L}}{\kappa G L^4} - c_4 \frac{\gamma d e^{-\gamma y/d}}{\kappa G L^3}\right] \sin \frac{x}{L}$$

$$\beta_y = \frac{QL^5}{d^3}\left[-c_1 e^{-y/L} + c_2\left(1 - \frac{y}{L}\right)e^{-y/L} + c_3 \frac{d^2 e^{-y/L}}{\kappa G L^4} - c_4 \frac{d^2 e^{-\gamma y/d}}{\kappa G L^4}\right] \cos \frac{x}{L}$$

$$\frac{w}{L} = \frac{QL^5}{d^3}\left[\frac{d^3}{DL^2} + \frac{d^2}{\kappa G L^4} + c_1 e^{-y/L} + c_2\left(\frac{2D}{\kappa G L^2 d} + \frac{y}{L}\right)e^{-y/L} - c_3 \frac{d^2 e^{-y/L}}{\kappa G L^4}\right] \cos \frac{x}{L}$$

$$\gamma = \sqrt{12\kappa + (d/L)^2}$$

fixed
$$\begin{cases} c_1 = -d^3/(DL^2) \\ c_2 = (-\gamma\kappa G d^3/D - \gamma(d/L)^2 + (d/L)^3)/f \\ c_3 = -\gamma\kappa G L^2/f \\ c_4 = -\kappa G d L/f \\ f = \gamma\kappa G L^2 + 2\gamma D/d - 2D/L \end{cases}$$

hard simply supported
$$\begin{cases} c_1 = -d^3/(DL^2) \\ c_2 = -d^3/(2DL^2) \\ c_3 = 0 \\ c_4 = 0 \end{cases}$$

soft simply supported
$$\begin{cases} c_1 = -d^3/(DL^2) \\ c_2 = \{2\gamma\kappa G \nu d^4/(DL) + (d/L - \gamma)^2[\kappa G d^3/D + (1-\nu)(d/L)^2]\}/(2f) \\ c_3 = -\gamma\kappa G L d(1-\nu)/f \\ c_4 = -\kappa G d^3[\kappa G L^2 + D(1-\nu)/d]/(Df) \\ f = -\kappa G L^2(\gamma^2 + (d/L)^2) + (1-\nu)(d/L)[\gamma\kappa G L^2 - (DL/d^2)(\gamma - d/L)^2] \end{cases}$$

free
$$\begin{cases} c_1 = \nu[\kappa G d^3/D - (\gamma^2 + (d/L)^2)]/f \\ c_2 = \kappa G \nu d^3/(Df) \\ c_3 = 0 \\ c_4 = 2\kappa G \nu L^2/f \\ f = -2\kappa G L^2 + (1-\nu)[\kappa G L^2 - (DL^2/d^3)(\gamma - d/L)^2] \end{cases}$$

the mesh has to be refined in the usual way. Following are some general observations concerning the boundary layer effect:

1. The boundary layer effect depends on the loading. For example, in circular plates under constant load there are no boundary layer effects. Hence benchmark computations in which only constant load is considered can be misleading.

2. The boundary layer effect depends on the choice of the plate or shell model, i.e., on the number of terms in (14.25*a*–*c*).
3. Singularities in the neighborhood of corners were discussed in Chapter 10. In the case of the Kirchhoff, Reissner–Mindlin and higher plate models, the solution has a similar singular behavior in the neighborhood of corners. Since as $d \to 0$ the solution converges to the Kirchhoff model (with the exception of the immediate neighborhood of the boundary), plate models exhibit characteristics of the Kirchhoff model at distances greater than a few times the thickness from the boundary. As in the case of plane elasticity discussed in Chapter 10, the singular term is of the form $r^\lambda \psi(\theta)$. For $r \gg d$ the value of λ is the same as in the case of the Kirchhoff solution. For $r < d$ the value of λ is related to the singularity of the particular plate model. Between $r \approx d$ and $r \gg d$ there is a transition of λ between these values [14.12].

   Consider, for example, a uniformly loaded square plate with two opposite sides fixed and the other two sides free. In this case in the neighborhood of the corners the leading term of the asymptotic expansion for the bending moments is of the form $Cr^\lambda \psi(\theta)$ and the values of λ for various plate models are as follows [14.13]:
   (a) Kirchhoff model: $\lambda = 0.0686 + 0.438i$.
   (b) Reissner–Mindlin model: $\lambda = -0.241$.
   (c) Model characterized by $n_x = n_y = 1$, $n_z = 2$: $\lambda = -0.289$.
   (d) Fully three-dimensional model: $\lambda = -0.289$.

For an analysis of the solution of the Reissner–Mindlin model of simply supported rhombic plates we refer to [14.14].

Exercise 14.7. Compute the moments M_x, M_y, M_{xy} and the shear forces Q_x, Q_y, from the solution of the semi-infinite plate given in Table 14.1 for $L = 1$, $d = 0.1, 0.05, 0.01$. Plot the results at $x = 0$ in the interval $0 \le y \le 2d$.

Exercise 14.8. Consider the limit of the displacement function w given in Table 14.1 with respect to $d \to 0$ for all of the boundary conditions listed. Show that

(a) w satisfies the biharmonic equation, i.e., w is the solution of the Kirchhoff model (see Exercise 14.3).
(b) Compute the bending moments and shear forces from the solution of the Kirchhoff model, using (14.12*a*, *b*, *c*) and (14.14*a*, *b*).
(c) Compute the bending moments and shear forces from the solution in Table 14.1 using (14.18*a*, *b*, *c*) and (14.19*a*, *b*). Let $d \to 0$ and compare the results with the results obtained in part (b).

This exercise demonstrates that, when $d \to 0$, the exact solutions corresponding to the Kirchhoff and Reissner-Mindlin models are the same inside the domain but differ on the boundaries. This exercise also demonstrates that the Kirchhoff model cannot distinguish between soft and hard simple supports.

14.4. THE TRANSVERSE VARIATION OF DISPLACEMENTS

In the derivation of plate and shell models the transverse variation of the displacement components was represented by polynomials. In this section it is shown that for homogeneous plates of small thickness the transverse variation of the displacements is indeed best represented by polynomials and, for laminated plates, by piecewise polynomials.

Consider once again the infinite strip shown in Figure 14.3. Assume for the moment that the load is periodic with period L. Consider the limit process $(d/L) \to 0$. Observe that in this process the case d fixed and $L \to \infty$ is equivalent to the case L fixed and $d \to 0$. It will be convenient to regard d as fixed and $L \to \infty$. The case of periodic loading can be generalized to nonperiodic loading by the Fourier integral method.

Let $\beta \overset{\text{def}}{=} 1/L$ and assume that the solution is of the form:

$$u_x = \phi(\beta, y) \sin \beta x \tag{14.44a}$$

$$u_y = \psi(\beta, y) \cos \beta x \tag{14.44b}$$

where $\phi(\beta, y)$ is antisymmetric, and $\psi(\beta, y)$ is symmetric with respect to the middle surface of the strip, i.e., the x-axis. The strain components corresponding to the solution (14.44*a*, *b*) are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \beta\phi \cos \beta x$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y} = \psi' \cos \beta x$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = (\phi' - \beta\psi) \sin \beta x$$

where the primes represent differentiation with respect to y. In the case of isotropic materials:

$$\sigma_x = (E_1 \beta\phi + E_2 \psi') \cos \beta x$$

$$\sigma_y = (E_2 \beta\phi + E_1 \psi') \cos \beta x$$

$$\tau_{xy} = E_6(\phi' - \beta\psi) \sin \beta x.$$

From the equilibrium equations (5.5*a*, *b*), with zero body force components, i.e., $X = Y = 0$:

$$-E_1\beta^2\phi - E_2\beta\psi' + (E_6\phi')' - (E_6\beta\psi)' = 0 \tag{14.45a}$$

$$E_6\beta\phi' - E_6\beta^2\psi + (E_2\beta\phi)' + (E_1\psi')' = 0. \tag{14.45b}$$