8

Buckling of Plates

8.1 INTRODUCTION

Thin plates of various shapes used in naval and aeronautical structures are often subjected to normal compressive and shearing loads acting in the middle plane of the plate (in-plane loads). Under certain conditions such loads can result in a plate buckling. Buckling or elastic instability of plates is of great practical importance. The buckling load depends on the plate thickness: the thinner the plate, the lower is the buckling load. In many cases, a failure of thin plate elements may be attributed to an elastic instability and not to the lack of their strength. Therefore, plate buckling analysis presents an integral part of the general analysis of a structure.

In this chapter, we consider a systematic but simplified analysis of plate buckling and obtain some useful relations between the critical loads and plate parameters.

8.2 GENERAL POSTULATIONS OF THE THEORY OF STABILITY OF PLATES

This section contains some fundamentals of classical stability analysis of thin elastic plates. It should be noted that the stability analysis of plates is qualitatively similar to the Euler stability analysis of columns [1].

Consider an ideal thin, elastic plate which is assumed initially to be perfectly flat and subjected to external in-plane compressive and shear loads acting strictly in the middle plane of the plate. The resulting deformations of this plate are characterized by the absence of deflections ($u \neq 0$, $v \neq 0$, and $w = 0$) and, consequently, of the bending and twisting moments, as well as the transverse shear forces. Such a plane stress condition of the plate is referred to as an *initial* or *flat configuration of equilibrium,* assuming the equilibrium conditions between applied external loads and the corresponding in-plane stress resultants.

Depending mainly on values of the applied in-plane loads, an initial, flat configuration of a plate equilibrium may be stable or unstable. The initial configuration of elastic equilibrium is *stable*, if when the plate is displaced from this equilibrium state by an infinitesimal disturbance, say a small lateral force, the deflected plate will tend to come back to its initial, flat configuration when the disturbance is removed. The initial configuration of equilibrium is said to be *unstable* if, when the plate is displaced from this equilibrium position by a small lateral load, it will tend to displace still further when the load is removed. The unstable plate will find other (new) equilibrium state(s), which may be in the vicinity of the initial state or may be far away from the initial equilibrium configuration. If the plate remains at the displaced position even after the small lateral load is removed, it is said to be in *neutral equilibrium*; thus, the plate in neutral equilibrium is neither stable nor unstable.

The transition of the plate from the stable state of equilibrium to the unstable one is referred to as *buckling* or *structural instability*. The smallest value of the load producing buckling is called the *critical* or *buckling load*.

The importance of buckling is the initiation of a deflection pattern, which if the loads are further increased above their critical values, rapidly leads to very large lateral deflections. Consequently, it leads to large bending stresses, and eventually to complete failure of the plate.

It is important to note that a plate leading from the stable to unstable configuration of equilibrium always passes through the neutral state of equilibrium, which thus can be considered as a bordering state between the stable and unstable configurations. In the mathematical formulation of elastic stability problems, neutral equilibrium is associated with the existence of bifurcation of the deformations. According to this formulation, the *critical load can be identified with the load corresponding to the bifurcation of the equilibrium states, or in other words, the critical load is the smallest load at which both the flat equilibrium configuration of the plate and slightly deflected configuration are possible*.

The goal of the buckling analysis of plates is to determine the critical buckling loads and the corresponding buckled configuration of equilibrium. We consider below the linear buckling analysis of plates based on the following assumptions:

(a) Prior to loading, a plate is ideally flat and all the applied external loads act strictly in the middle plane of the plate.
(b) States of stress is described by equations of the linear plane elasticity. Any changes in the plate dimensions are neglected prior to buckling.
(c) All the loads applied to the plate are dead loads; that is, they are not changed either in magnitude or in direction when the plate deforms.
(d) The plate bending is described by Kirchhoff's plate bending theory discussed in Chapter 2.

The linear buckling analysis of plates based on these assumptions makes it possible to determine accurately the critical loads, which are of practical importance in the stability analysis of thin plates. However, this analysis gives no way of describing the behavior of plates after buckling, which is also of considerable interest. The postbuckling analysis of plates is usually difficult because it is basically a nonlinear problem. Some postbuckling plate problems will be discussed in Sec. 8.5. Classical buckling problems of plates can be formulated using (1) the equilibrium method, (2) the energy method, and (3) the dynamic method.

The equilibrium method

Consider an initial state of equilibrium of a plate subjected to the external edge loads acting in the middle plane of the plate. Let the corresponding in-plane stress resultants in this initial state be N_x, N_y, and N_{xy}. They may be found from the solution of the plane stress problem for the given plate geometry and in-plane external loading. However, for a complex plate geometry and complex in-plane load configurations, such a problem may involve sufficient difficulties. We confine the bucking analysis in this chapter by considering such load configurations and plate geometry, when determining the above-mentioned in-plane resultants presents no difficulties, and they can be directly expressed via the given external forces. Note that such problems are of a practical importance. Next, assume that for certain values of the external forces, the plate has buckled slightly. We formulate the differential equation of equilibrium for this neighboring state assuming that the latter represents a slightly bent configuration of equilibrium. For the plate, the in-plane external edge loads that result in an elastic instability as in the case of a beam column, are independent of the lateral loads. Thus, the governing differential equation of the linear buckling analysis of plates is obtained from Eq. (3.92) by making p equal zero. We have the following:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right), \tag{8.1}$$

where N_x, N_y, and N_{xy} are the internal forces acting in the middle surface of the plate due to the applied in-plane loading. The right-hand side of Eq. (8.1) can be interpreted as some fictitious, transverse surface load, p_{f}, created by the normal projections of the in-plane internal forces acting in the slightly curved configuration of the plate.

Equation (8.1) is a homogeneous, partial differential equation. The mathematical problem is to solve this equation with appropriate homogeneous boundary conditions. In general, such a problem has only a trivial solution corresponding to the initial, flat configuration of equilibrium (i.e., $w = 0$). However, the coefficients of the governing equation depend on the magnitudes of the stress resultants, which are, in turn, connected with the applied in-plane external forces, and we can find values of these loads for which a nontrivial solution is possible. The smallest value of these loads will correspond to a critical load.

A more general formulation of the equilibrium method transforms the stability problem into an eigenvalue problem. For this purpose, we multiply a reference value of the stress resultants ($\bar{N}_x$, $\bar{N}_y$, and $\bar{N}_{xy}$) by a load parameter λ, i.e.,

$$N_x = -\lambda \bar{N}_x, \quad N_y = -\lambda \bar{N}_y, N_{xy} = -\lambda \bar{N}_{xy}. \tag{8.2}$$

Substituting Eqs (8.2) into Eq. (8.1), we obtain an alternative form of the governing differential equation of plate buckling problems:

$$\nabla^4 w + \frac{\lambda}{D}\left(\bar{N}_x \frac{\partial^2 w}{\partial x^2} + \bar{N}_y \frac{\partial^2 w}{\partial y^2} + 2\bar{N}_{xy}\frac{\partial^2 w}{\partial x \partial y}\right) = 0. \tag{8.3}$$

The solution of Eq. (8.3), $w(x, y)$, obtained by some analytical or numerical methods, introduced in Chapters 3 and 6, involves arbitrary constant coefficients C_i $(i = 1, 2, \ldots, n)$ to be determined from the prescribed boundary conditions. Consequently, Eq. (8.3) is reduced to a system of homogeneous, linear algebraic equations in C_i. For an existence of a nontrivial solution of the system, its determinant must be equal to zero. This results in the so-called characteristic equation in λ. Solving this characteristic equation, we obtain some specific values $\lambda_1, \lambda_2, \ldots, \lambda_n$ (the *characteristic numbers* or *eigenvalues*) and the corresponding nonzero solutions, called *characteristic functions* or *eigenfunctions*. The smallest of the characteristic numbers or eigenvalues not equal to zero will be the *critical value*, λ_{cr}, and the corresponding eigenfunctions will be the *buckling modes*. Then, the critical load is calculated by multiplying λ_{cr} and the corresponding reference value of the load.

The energy method

The energy method is based on the general theorems and principles of the equilibrium of mechanical systems discussed in Sec. 2.6. As mentioned in Sec. 2.6, the potential energy of a system has an extremum at equilibrium. Based on this statement, we can reformulate the concepts of stability and instability presented earlier. The equilibrium will be *stable* if the potential energy in that state has a minimum value in comparison with values corresponding to any possible states close to the state of equilibrium, *unstable* if the potential energy is a maximum, and *neutral* if the potential energy in the equilibrium state is neither a maximum nor a minimum.

Let us apply this potential energy criterion to the buckling analysis of plates. Two states of the plate are considered: an initial state of equilibrium under the given in-plane edge loads, in which the middle surface remains flat; a neighboring state, in which the middle surface is slightly curved due to small virtual displacements applied to the plate. Let Π_0 and Π be the potential energies in the flat and neighboring states of equilibrium, respectively. The equilibrium will be stable if for all possible small deflections $\Pi_0 < \Pi$, unstable if $\Pi_0 > \Pi$, and neutral if $\Pi_0 = \Pi$.

The increment in the total potential energy of the plate loaded by the edge in-plane external loads, after the transition of this plate from an initial configuration of equilibrium to the above-mentioned neighboring configuration of equilibrium is given by the following relationship:

$$\Delta\Pi = \Pi - \Pi_0 = \Delta U_0 + U_b + \Delta\Omega_\Gamma, \tag{8.4}$$

where ΔU_0 is the increment of the strain energy of the plate middle surface in buckling; U_b is the strain energy of bending and twisting of the plate; and $\Delta\Omega_\Gamma$ is the increment in the potential of the in-plane external edge forces applied to the plate. Bifurcation of an initial configuration of equilibrium (corresponding to the neutral equilibrium) occurs when

$$\Delta\Pi = 0. \tag{8.5a}$$

This is *the general energy criterion for the buckling analysis of plates* (*and shells also*). The critical loads may be determined from this criterion at an additional condition of the minimum of the load parameter λ.

It can be shown [1,2] that the governing differential equation (8.1) can be obtained from the condition (8.5a). The latter can also be employed for constructing an approximate solution of the plate buckling problems, in particular for determining the critical loads, by the Ritz method (see Sec. 6.6). In this case, the deflection surface of the plate, $w(x, y)$, in the neighboring state of equilibrium is sought in the form of Eq. (6.42). If all external forces acting on the plate vary in the proportion to a load parameter λ, then substituting Eq. (6.42) into the condition (8.5a) yields

$$\Delta\Pi = \Delta\Pi(\lambda, C_1, C_2, \ldots, C_n) = 0, \tag{8.5b}$$

where C_i $(i = 1, 2, \ldots, C_n)$ are undetermined coefficients. Numerical implementation of the conditions (8.5) requires the usual minimization process (Eq. (6.44)). The latter also leads to a system of linear algebraic, homogeneous equations. So, applying the procedure discussed earlier for determining a nontrivial solution in the equilibrium method, we can find all the characteristic numbers λ_i, and their smallest value corresponding to the critical eigenvalue, λ_{cr}.

Dynamic or kinematic method

This method is the most general and universal and is associated with the mathematical problem of the stability of motion. The method is based on examining transverse oscillations of a thin plate subjected to in-plane edge loads. The smallest value of the load that results in unbounded growth of the amplitude of these oscillations in time is considered as a critical value of the applied loads.

The equilibrium and energy methods are now discussed in more detail.

8.3 THE EQUILIBRIUM METHOD

8.3.1 Buckling of rectangular plates

According to the equilibrium method, the critical values of applied in-plane forces may be found from the solution of the governing differential equation (8.1) or its equivalent analog (8.3). As mentioned in Sec. 8.2, this equation is a homogeneous, linear partial differential equation with, generally speaking, variable coefficients. It is impossible to find its analytical solution in the general case. However, for some particular but practically important cases this equation makes it possible to obtain an exact solution. The following examples illustrate the equilibrium method for obtaining the exact solutions associated with determining the critical forces in rectangular plates.

Example 8.1

Determine the critical buckling load for a simply supported plate subjected to a uniformly distributed compressive edge load q_x acting in the x direction, as shown in Fig. 8.1.

Solution

For this particular case $N_x = -q_x$ and $N_y = N_{xy} = 0$. The differential equation (8.1) becomes

$$D\nabla^2\nabla^2 w + N_x \frac{\partial^2 w}{\partial x^2} = 0. \tag{8.6}$$

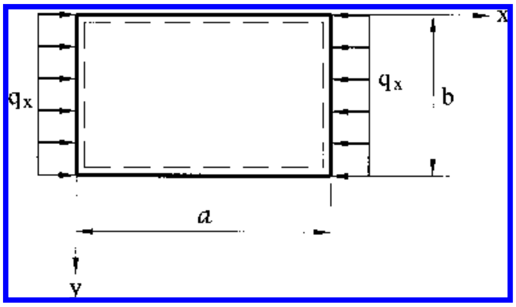

Fig. 8.1

We seek the solution of this equation in the form of Eq. (3.15a) that satisfies the simply supported boundary conditions. Inserting this solution into Eq. (8.1) leads to the following equation:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[D\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - q_x\pi^2\frac{m^2}{a^2}\right]w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=0.$$

One possible solution is $w_{mn}=0$; however, this represents the trivial solution, $w(x,y)=0$, and corresponds to an equilibrium in the unbuckled, flat state of the plate and is of no interest. Another possible solution is obtained by setting the quantity in square brackets to zero, or

$$\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - q_x\pi^2\frac{m^2}{a^2}=0,$$

from which

$$q_x=\frac{\pi^2 D}{b^2}\left(\frac{mb}{a}+\frac{n^2 a}{mb}\right)^2. \tag{a}$$

The constants w_{mn} remain undetermined. Expression (a) gives all values of q_x corresponding to $m=1,2,3,\ldots$; $n=1,2,3,\ldots$ as possible forms of the defected surface (Eq. (3.15a)). From all of these values one must select the smallest, which will be the critical value. Evidently the smallest value of q_x is obtained for $n=1$. For $n=1$ the formula for q_x takes the form

$$N_x=\frac{\pi^2 D}{b^2}\left(\frac{mb}{a}+\frac{a}{mb}\right)^2 \tag{8.7a}$$

or, in an equivalent form,

$$q_x=K\frac{\pi^2 D}{b^2}, \tag{8.7b}$$

where

$$K=\left(\frac{mb}{a}+\frac{a}{mb}\right)^2 \tag{8.8}$$

is the *buckling load parameter*. For a given value of m, the parameter K depends only on the ratio a/b, called the *aspect ratio* of the plate. As follows from Eqs (8.7) and (8.8), the smallest value of q_x, and consequently, the value of the critical force $q_{x,\mathrm{cr}}$, depends on the number of half-sine waves in the longitudinal direction, m. For a given aspect ratio the critical load is obtained by selecting m so that it makes Eq. (8.7b) a minimum. Since only K depends on m, we have the following:

$$\frac{\mathrm{d}K}{\mathrm{d}m} = 2\left(\frac{mb}{a} + \frac{a}{mb}\right)\left(\frac{b}{a} - \frac{a}{m^2 b}\right) = 0.$$

Since the first factor in parentheses of the above is nonzero, we obtain

$$m = \frac{a}{b}. \tag{8.9}$$

This provides the following minimum values of the critical load:

$$\min q_x = q_{x,\mathrm{cr}} = \frac{4\pi^2 D}{b^2}. \tag{8.10}$$

The corresponding value of the buckling load parameter is $K = 4$. The corresponding critical stress is found to be

$$\sigma_{x,\mathrm{cr}} = \frac{N_{x,\mathrm{cr}}}{h} = \frac{q_{x,\mathrm{cr}}}{h} = \frac{4\pi^2 D}{b^2 h} = \frac{\pi^2 E}{3(1-\nu^2)}\left(\frac{h}{b}\right)^2. \tag{8.11}$$

Thus, the critical values of q_x and σ_x correspond to such plate dimensions when its width, b, fits in its length, a, by whole numbers. In this case a bent plate is subdivided into square cells of side dimensions b. In the general case, $q_{x,\mathrm{cr}}$ may be determined from Eqs (8.7) and (8.8).

The variation of the buckling load parameter K as a function of the aspect ratio a/b for $m = 1, 2, 3, 4$ is shown in Fig. 8.2. Referring to this figure, the magnitude of $q_{x,\mathrm{cr}}$ and the number of half-waves m (in the direction of the applied compressive forces) for any value of the aspect ratio can readily be found. For example, if $a/b = 1.5$, we can find that $K = 4.34$ and $m = 2$. The corresponding critical load for this particular case is

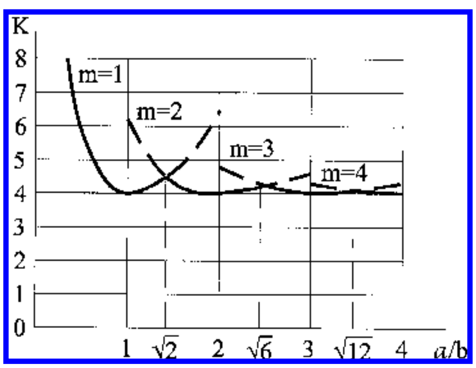

Fig. 8.2

$$q_{x,\mathrm{cr}} = 4.34\frac{\pi^2 D}{b^2}.$$

The plate will buckle under this load into two half-waves in the direction of the applied compressive loads and one-half in the perpendicular direction.

An analysis of the curves in Fig. 8.2 shows that for short and broad plates (for $a/b < 1$) a minimum value of the critical force is obtained for $m = 1$. For $a/b \ll 1$, that is for very short and broad plates, the ratio a/b can be neglected compared with the ratio b/a in Eq. (8.8). As a result, min.$K \cong b^2/a^2$ and the value of the critical force for this particular case is

$$q_{x,\mathrm{cr}} = \frac{\pi^2 D}{a^2}.$$

Thus, in this case, the critical force does not depend on the plate width and depends only upon its length. The above expression represents the Euler critical load for a strip of unit width and of length a whose smallest value of flexural rigidity, EI, is replaced with the flexural rigidity of the plate, D.

Example 8.2

Determine the buckling critical load for a plate subjected to a uniformly distributed compressive edge load acting in the x direction. Assume that the edges $x = 0$ and $x = a$ are simply supported, the edge $y = 0$ is fixed, and the edge $y = b$ is free, as shown in Fig. 8.3.

Solution

It is convenient for this problem to employ Levy's method (see Sec. 3.5) for the solution. The boundary conditions at the edges $x = 0$ and $x = a$ will be automatically satisfied by setting

$$w(x, y) = \sum_{m=1}^{\infty} f_m(y) \sin\frac{m\pi x}{a}. \tag{a}$$

For this case, we also can set $N_x = -q_x$ and $N_y = N_{xy} = 0$. Substituting the above into Eq. (8.1) and imposing the condition that at least one of the terms multiplying $\sin(m\pi x/a)$ must vanish, we determine for $f_m(y)$ the following ordinary differential equation:

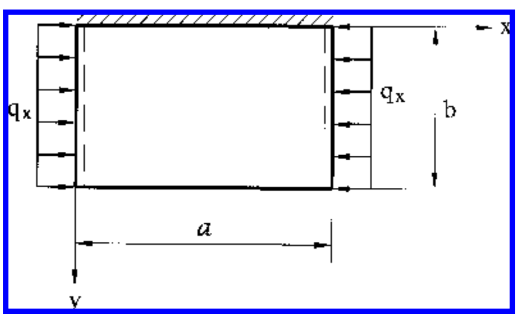

Fig. 8.3

$$\frac{d^4 f_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 f}{dy^2} + \left[\left(\frac{m\pi}{a}\right)^4 - \frac{q_x}{D}\left(\frac{m\pi}{a}\right)^2\right] f_m = 0. \tag{8.12}$$

The solution of Eq. (8.12) is shown below:

$$f_m(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y, \tag{8.13}$$

where

$$\alpha, \beta = \left[\pm\left(\frac{m\pi}{a}\right)^2 + \sqrt{\frac{q_x}{D}\left(\frac{m\pi}{a}\right)^2}\right]^{1/2}. \tag{8.14}$$

The constants C_i $(i = 1, 2, 3, 4)$ are evaluated from the boundary conditions prescribed on the edges $y = 0$ and $y = b$, i.e.,

$$w = 0|_{y=0}, \quad \frac{\partial w}{\partial y} = 0\bigg|_{y=0} \tag{8.15a}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) = 0\bigg|_{y=b}, \; V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] = 0\bigg|_{y=b}. \tag{8.15b}$$

Introducing (a) with (8.13) into the boundary conditions (8.15a), we obtain two algebraic equations for the unknown constants. Solving the above equations, one finds

$$C_1 = -\frac{C_3}{2} + \frac{\beta C_4}{2\alpha}, \quad C_2 = -\frac{C_3}{2} - \frac{\beta C_4}{2\alpha}. \tag{b}$$

Substituting the above into Eq. (8.13), gives

$$f_m(y) = C_3(\cos \beta y - \cosh \alpha y) + C_4\left(\sin \beta y - \frac{\beta}{\alpha} \sinh \alpha y\right). \tag{8.16}$$

Introducing $w(x, y)$ with $f_m(y)$ in the form of Eq. (8.16) into the boundary conditions (8.15b) results in two simultaneous homogeneous algebraic equations. In order to obtain a nontrivial solution, we equate the determinant of these equations to zero, and obtain

$$2gh\left(g^2 + \zeta^2\right) \cos \beta b \cosh \alpha b = \frac{1}{\alpha\beta}\left(\alpha^2 \zeta^2 - \beta^2 g^2\right) \sin \beta b \sinh \alpha b, \tag{8.17}$$

where

$$g = \alpha^2 - \nu\left(\frac{m\pi}{a}\right)^2, \quad \zeta = \beta^2 + \nu\left(\frac{m\pi}{a}\right)^2.$$

For $m = 1$, the minimum eigenvalue of Eq. (8.17), i.e., the critical value of the applied force is

$$q_{x,\mathrm{cr}} = K \frac{\pi^2 D}{b^2}, \tag{8.18}$$

where for $\nu = 0.25$, $K = 1.328$.

It can be observed that a type of the plate boundary support has an effect on values of critical forces and buckling modes. For example, fixed supports increase a plates stability compared with hinged supports; the presence of a free edge leads to a sharp decrease in critical forces, etc. If the boundary conditions of a plate differ from simply supported ones, determining the critical forces even for simple cases of loadings represents a sufficiently complicated mathematical problem and can be obtained by numerical methods only [2–8].

For rectangular plates subjected to uniform compressive forces acting in the direction of one coordinate axis only (either the x or the y axis) with various boundary conditions, the critical stress, σ_{cr}, can be determined from the expression

$$\sigma_{cr} = K\frac{\pi^2 D}{b^2 h}, \tag{8.19}$$

where the coefficients K are given in Fig. 8.4 versus the aspect ratios a/b [4].

Example 8.3

Determine the critical forces for the rectangular plate with simply supported edges and uniformly compressed in two directions, as shown in Fig. 8.5.

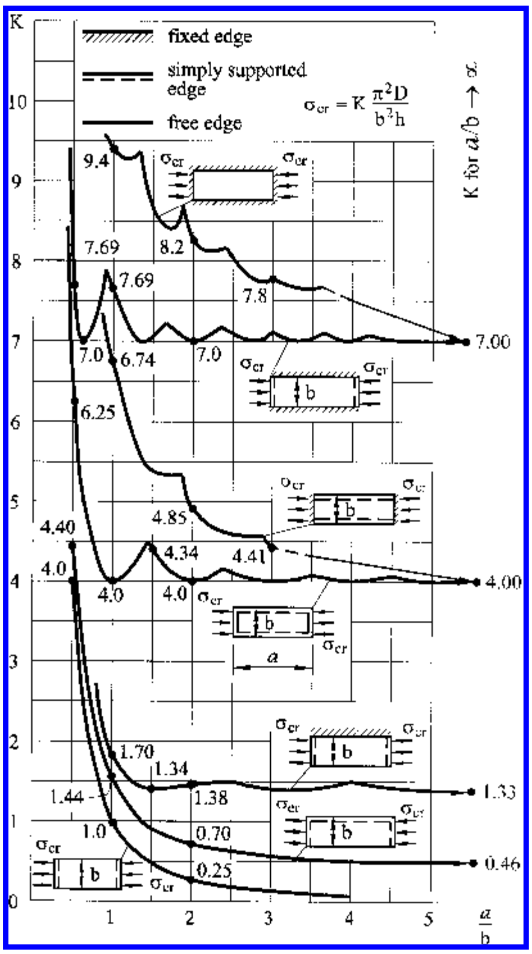

Fig. 8.4

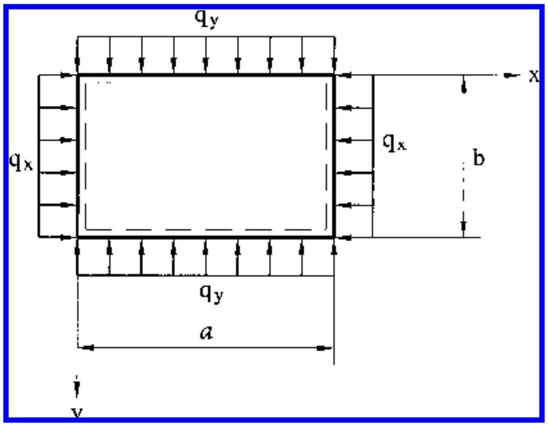

Fig. 8.5

Solution

For this problem, first analyzed by Bryan [9] in 1861, the stress resultants may be easily found: they are $N_x = -q_x$, $N_y = -q_y$, $N_{xy} = 0$. Equation (8.1) for this type of loading becomes

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{1}{D}\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{8.20}$$

Let us take w in the form of the expression (3.15a). The deflection surface equation in this form satisfies the simply supported boundary conditions. Substituting the above into Eq. (8.20) gives

$$w_{mn}\left\{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{1}{\pi^2 D}\left[q_x \frac{m^2}{a^2} + q_y \frac{n^2}{b^2}\right]\right\} = 0. \tag{8.21}$$

The trivial solution of this equation is $w_{mn} = 0$. As mentioned earlier, this solution corresponds to unbuckled, i.e., flat, configuration of equilibrium of the plate and is of no interest for buckling analysis. A nontrivial solution can be obtained by equating the term in braces in Eq. (8.21) to zero, i.e.,

$$q_x\left(\frac{m}{a}\right)^2 + q_y\left(\frac{n}{b}\right)^2 = D\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2. \tag{8.22}$$

Let us consider *several particular cases:*

1. Assume, first, that $q_x = q_y = q = \text{const}$ (a uniform all-round compression). In this case, it follows from Eq. (8.22) that

$$q = \frac{\pi^2 D}{b^2}\left[n^2 + \left(\frac{mb}{a}\right)^2\right].$$

It can be easily shown that the minimum value of q, q_{cr}, corresponds to $m = n = 1$. Thus,

$$q_{\mathrm{cr}} = \frac{\pi^2 D}{b^2}\left[1 + \left(\frac{b}{a}\right)^2\right]. \tag{8.23}$$

In particular, for the square plate ($a = b$), the above expression appears, as follows:

$$q_{\mathrm{cr}} = 2\frac{\pi^2 D}{b^2}.$$

Hence, if the square plate is compressed in two directions by the two equal system of forces, q, the critical value of these forces is two times less than that for the square plate compressed by the same force acting in one direction only (see Example 8.1).

2. Assume now that the compressive forces q_x and q_y applied to the plate of Fig. 8.5 will increase in the proportion to one parameter. For example, $q_x = \lambda$ and $q_y = \alpha\lambda$, where $\alpha > 0$ is some fixed known parameter. Then, from Eq. (8.22), we obtain

$$\lambda = \frac{\pi^2 D}{b^2}\frac{\left[\left(\frac{mb}{a}\right)^2 + n^2\right]}{\left(\frac{mb}{a}\right)^2 + \alpha n^2}. \tag{8.24}$$

For $a > b$, the minimum value of λ may be reached for $n = 1$ only. Thus,

$$\lambda_{\mathrm{cr}} = K\frac{\pi^2 D}{b^2}, \tag{8.25}$$

where

$$K = \frac{\left[(mb/a)^2 + 1\right]}{(mb/a)^2 + \alpha}. \tag{8.26}$$

For each ratio of a/b and each value of α it should be selected a number of the half-sine waves m from the condition of minimum of K as discussed in Example 8.1.

8.3.2 Buckling of circular plates

Circular plates in some measuring instruments are used as sensitive elements reacting to a change in the lateral pressure. In some cases – in temperature changes, in the process of their assembly – these elements are subjected to the action of radial compressive forces from a supporting structure. As a result, buckling of the circular plates can take place.

Let us consider a circular solid plate subjected to uniformly distributed in-plane compressive radial forces q_r, as shown in Fig. 8.6. We confine our buckling analysis to considering only axisymmetric configurations of equilibrium for the plate.

We can use the polar coordinates r and φ to transfer the governing differential equation of plate buckling (Eq. (8.1)), derived for a rectangular plate, to a circular plate. For the particular case of axisymmetric loading and equilibrium configurations, we have

$$N_x = N_y = N_r = -q_r, \quad N_{xy} = 0. \tag{8.27}$$

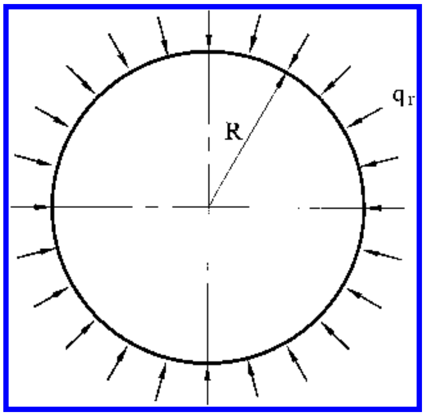

Fig. 8.6

Denoting

$$\mu^2 = \frac{q_r}{D}, \tag{8.28}$$

and using the relations between the polar and Cartesian coordinates, Eqs (4.1)–(4.4), we obtain the following differential equation of the axisymmetrically loaded circular plate subjected to in-plane compressive forces q_r:

$$\frac{d^4 w}{dr^4} + \frac{2}{r}\frac{d^3 w}{dr^3} - \frac{1}{r^2}\frac{d^2 w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} + \mu^2\left[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right] = 0. \tag{8.29}$$

Let us introduce the following new variable:

$$\rho = \mu r, \tag{8.30}$$

which represents a dimensionless polar radius. Using the new variable ρ, we can rewrite Eq. (8.29), as follows:

$$\frac{d^4 w}{d\rho^4} + \frac{2}{\rho}\frac{d^3 w}{d\rho^3} + \left(1 - \frac{1}{\rho^2}\right)\frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\left(1 + \frac{1}{\rho^2}\right)\frac{dw}{d\rho} = 0. \tag{8.31}$$

This is a fourth-order linear, homogeneous differential equation. The general solution of this equation is given by [1] as

$$w(\rho) = C_1 + C_2 \ln \rho + C_3 J_0(\rho) + C_4 Y_0(\rho), \tag{8.32}$$

where $J_0(\rho)$ and $Y_0(\rho)$ are the Bessel functions of the first and second kind of zero orders, respectively. They are tabulated in Ref. [10]. In Eq. (8.32), C_i $(i = 1, \ldots, 4)$ are constants of integration. Since $w(\rho)$ must be finite for all values of ρ, including $\rho = 0$, then the two terms $\ln \rho$ and $Y_0(\rho)$, having singularities at $\rho = 0$, must be dropped for the solid plate because they approach an infinity when $\rho \to \infty$. Thus, for the solid circular plate, Eq. (8.32) must be taken in the form

$$w(\rho) = C_1 + C_3 J_0(\rho). \tag{8.33}$$

Determine the critical values of the radial compressive forces, q_r, applied to the middle plane of solid circular plates for two types of boundary supports.

(1) Circular plate with fixed edge

Let the radius of the plate be R. We denote the corresponding value of μR by β, i.e., $\beta = \mu R$. The boundary conditions are

$$w(\beta) = 0\big|_{\rho=\beta}, \quad \vartheta(\beta) = 0\big|_{\rho=\beta}, \tag{a}$$

where the slope of the plate midsurface, $\vartheta(\rho)$, is given by

$$\vartheta(\rho) = \mu \frac{\mathrm{d}w}{\mathrm{d}\rho} = \mu C_3 \frac{\mathrm{d}}{\mathrm{d}\rho} J_0(\rho). \tag{b}$$

From the Bessel function theory [10], that it follows

$$J_1(\rho) = -\frac{\mathrm{d}}{\mathrm{d}\rho} J_0(\rho). \tag{c}$$

Thus, we can write the following representations for the slope

$$\vartheta(\rho) = -\mu C_3 J_1(\rho), \tag{8.34}$$

and

$$\vartheta(\beta) = -\mu C_3 J_1(\beta) \quad \text{on the boundary,} \tag{8.35}$$

where $J_1(\;)$ is the Bessel function of the first kind of the first order.

Substituting the expressions (8.34) and (8.35) into the boundary conditions (a) yields the following system of two linear homogeneous equations:

$$\begin{aligned} C_1 + C_3 J_0(\beta) &= 0, \\ -\mu C_3 J_1(\beta) &= 0. \end{aligned}$$

For a nontrivial solution of these equations:

$$J_1(\beta) = 0.$$

From the tables of roots of the Bessel functions [10] it follows that the smallest root of the function $J_1(\beta)$ is $\beta_{\min} = 3.8317$.

Noting that $\beta^2 = (\mu R)^2 = q_r / DR^2$, we obtain the critical value of the compressive forces as

$$q_{r,\mathrm{cr}} = (3.8317)^2 \frac{D}{R^2} = 14.68 \frac{D}{R^2}. \tag{8.36}$$

(2) Circular plate with simply supported edge

The boundary conditions for this type of support are

$$w(\beta) = 0|_{\rho=\beta}, M_r(\beta) = 0|_{\rho=\beta}. \tag{d}$$

The radial bending moment, M_r, for an axisymmetrically loaded circular plate is given by Eq. (4.14). When passing from variable r to the variable ρ, the expression for M_r becomes

$$M_r = -D\mu^2 \left(\frac{\mathrm{d}^2 w}{\mathrm{d}\rho^2} + \frac{\nu}{\rho} \frac{\mathrm{d}w}{\mathrm{d}\rho} \right). \tag{8.37}$$

Using Eq. (8.33) for the deflections and Eq. (8.37) for the radial bending moments, we can represent the second boundary condition (d) in the form

$$-D\mu^2\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}J_0(\rho)+\frac{\nu}{\rho}\frac{d}{d\rho}J_0(\rho)\right]=0\Bigg|_{\rho=\beta}. \tag{8.38}$$

Using the relationships between the Bessel functions of the first kind [10], we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}J_0(\rho)=-J_0(\rho)+\frac{1}{\rho}J_1(\rho). \tag{e}$$

Substituting the expression for the deflections (8.33) into the boundary conditions (d) and taking into account Eqs (8.38) and (e), we arrive at the following system of linear homogeneous equations:

$$\begin{aligned} C_1+C_3J_0(\beta)&=0\\ -D\mu^2C_3[\beta J_0(\beta)-(1-\nu)J_1(\beta)]&=0 \end{aligned} \tag{f}$$

A nontrivial solution of this system of equations leads to the following:

$$\beta J_0(\beta)-(1-\nu)J_1(\beta)=0. \tag{g}$$

Letting $\nu=0.3$ and using the tables of the Bessel function [10], we can determine the smallest nonzero root of Eq. (g). We have

$$\beta_{\min}=2.0485,$$

and the critical value of an intensity of the radial compressive forces is

$$q_{r,\mathrm{cr}}=4.196\frac{D}{R^2}. \tag{8.39}$$

Comparing the values of the critical compressive forces for the clamped and simply supported circular solid plates, we can conclude that the replacement of the supported edges with clamped ones increases the critical force by a factor of 3.5.

8.4 THE ENERGY METHOD

Practical application of the equilibrium method runs into serious mathematical obstacles when determining the buckling loads of the plates with complex geometry and mixed boundary conditions. Under these circumstances a possibility for obtaining a rigorous solution of the differential equation (8.1) becomes very doubtful and, practically, impossible. Therefore, the use of the energy method can be very advantageous.

We will apply the energy criterion (8.5) to the buckling analysis of plates. The increment in the total potential energy of the plate upon buckling is given by Eq. (8.4). Let us derive the expression for the increment in the strain energy of the middle surface of the plate. In deriving this expression, we assume that the in-plane stress resultants are entirely due to external edge loading in the plane of the plate, in which case they are unchanged during bending (created by buckling). The increment of the strain energy of the plate middle surface due to buckling can be obtained from the general expression of the potential energy of an elastic body given by Eq. (2.51). Based on the assumptions 4 and 5 of the classical plate theory, the above expression is simplified and has the following form for the above-mentioned increment of the middle surface:

$$\Delta U = \frac{1}{2}\iiint_V \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right)\mathrm{d}V. \tag{a}$$

This increment of the strain energy can also be expressed in terms of the in-plane stress resultants. The latter are related to the stress components as follows:

$$\sigma_x = \frac{N_x}{h}, \quad \sigma_y = \frac{N_y}{h}, \quad \text{and} \quad \tau_{xy} = \frac{N_{xy}}{h}. \tag{b}$$

Substituting the above into Eq. (a) and integrating over the plate thickness, h, results in the following expression for the increment in the strain energy of the plate middle surface:

$$\Delta U_0 = \iint_A \left(N_x \varepsilon_x + N_y \varepsilon_y + N_{xy}\gamma_{xy}\right)\mathrm{d}x\mathrm{d}y. \tag{8.40}$$

Note that there is no 1/2 coefficient in Eq. (8.40), since the in-plane stress resultants are already acting when additional middle surface strains (due to the buckling) occur. For the sake of simplicity, we derive the expression for ΔU_0 for a rectangular plate of dimensions $a \times b$. Inserting Eqs (7.82) into Eq. (8.40) results in the following expression for the increment of the strain energy of the plate middle surface:

$$\begin{aligned}\Delta U_0 = &\int_0^b\int_0^a \left[N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{2}\int_0^b\int_0^b \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{8.41}$$

We transform the first integral of the right-hand side of Eq. (8.41). Integrating this expression term-by-term, we obtain

$$\begin{aligned}\int_0^b\int_0^a &\left[N_x \frac{\partial u}{\partial x} + N_y \frac{\partial v}{\partial y} + N_{xY}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\mathrm{d}x\mathrm{d}y = \int_0^b \left[\left|N_x u\right|_0^a + \left|N_{xy} v\right|_0^a\right]\mathrm{d}y \\ &+ \int_0^b \left[\left|N_y v\right|_0^b + \left|N_{xy} u\right|_0^b\right]\mathrm{d}x - \int_0^b\int_0^a u\left[\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right]\mathrm{d}x\mathrm{d}y \\ &- \int_0^b\int_0^a v\left[\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right]\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{8.42}$$

Using Eqs (3.90a) and (3.90b), we can conclude that the two last integrals on the right-hand side in the above expression vanish. The first two integrals on the right-hand side of Eq. (8.42) represent the work, W_e, done by the in-plane external forces applied to the middle surface of the plate. Thus, the expression (8.42) can be represented in the form

$$\Delta U_0 = W_e + \frac{1}{2}\int_0^b \int_0^a \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \mathrm{d}x\mathrm{d}y. \tag{8.43}$$

It can be shown that the expression for ΔU_0 in the form of Eq. (8.43) is valid (replacing the limits of integration) for a plate of any geometry, not necessarily, rectangular.

The increment in the potential of the external, in-plane forces applied to the plate is equal to the negative value of the work done by these forces, i.e.,

$$\Delta\Omega_\Gamma = -W_e. \tag{8.44}$$

The strain energy of the bending and twisting of a plate, U_b, is given by Eq. (2.53). Therefore, the increment in the total potential energy of the plate upon buckling, $\Delta\Pi$, can be obtained by substituting Eqs (2.53) and (8.43) with taking into account Eq. (8.44) on the right-hand side of Eq. (8.4).

$$\begin{aligned}\Delta\Pi = &\frac{1}{2}\iint_A D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}\mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{2}\iint_A \left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\mathrm{d}x\mathrm{d}y\end{aligned} \tag{8.45}$$

As mentioned in Sec. 8.2, the energy criterion (8.5) may be employed be obtain the exact and approximate solutions of the plate buckling problems. In the latter case, this criterion may be combined with, for instance, the Ritz method together with the eigenvalue technique introduced in Sec. 8.2.

Note that in practice $\Delta\Pi = 0$ only if the form chosen for the deflection is exact. Any approximation form, when Eq. (8.45) is applied, will give a value of the critical force higher than the true value. Let us demonstrate the advantages of the energy criterion in illustrative examples.

Example 8.4

Using the Ritz method, determine the critical buckling load for the plate with three simply supported edges $x = 0, a$ and $y = 0$ and one free edge $y = b$, as shown in Fig. 8.7. The plate is loaded by linearly distributed compressive in-plane forces $q_x = q\left(1 + \eta\frac{y}{b}\right)$ along the simply supported edges $x = 0, a$, where $\eta > 0$ is some fixed parameter.

Solution

The deflection surface of the plate can be approximated, as follows:

$$w = \sin\frac{n\pi x}{a}\sum_{i=1}^{N} C_i y^i. \tag{a}$$

This approximate solution satisfies exactly the prescribed geometric boundary conditions, i.e.,

$$w = 0|_{x=0,a}, \quad w = 0|_{y=0}. \tag{b}$$

Retaining only one term ($i = 1$) in the series (a), we can find the following:

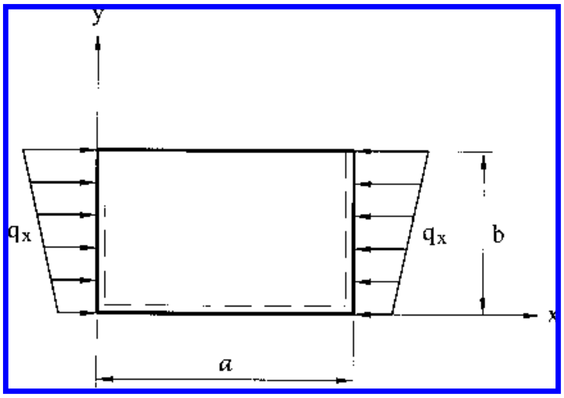

Fig. 8.7

$$\frac{\partial w}{\partial x} = C_1 y\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\right), \frac{\partial^2 w}{\partial x^2} = -C_1 y\left(\frac{n\pi}{a}\right)^2 \sin\frac{n\pi x}{a},$$
$$\frac{\partial^2 w}{\partial x \partial y} = C_1\left(\frac{n\pi}{a}\cos\frac{n\pi x}{a}\right), \ \frac{\partial^2 w}{\partial y^2} = 0.$$

For this type of loading, the solution of the plane stress problem is $N_x = -q_x$, $N_y = N_{xy} = 0$. Substituting the above into Eq. (8.45), and performing the operations of integration, we obtain

$$\Delta\Pi = C_1^2 \frac{ab}{4}\left\{D\left[\frac{b^2}{3}\left(\frac{n\pi}{a}\right)^4 + 2(1-\nu)\left(\frac{n\pi}{a}\right)^2\right] - q\left(\frac{n\pi}{a}\right)^2 b^2\left(\frac{1}{3}+\frac{\eta}{4}\right)\right\}.$$

The eigenvalues of the load q_x can be found from equation $\partial\Pi/\partial C_1 = 0$. We have

$$q_n = \frac{n^2\pi^2 D}{b^2}\left[\frac{b^2/a^2 + 6(1-\nu)/n^2\pi^2}{1+3\eta/4}\right].$$

For $n = 1$, the smallest eigenvalue is approximately equal to the critical value, i.e.,

$$q_{\mathrm{cr}} = \frac{b^2/a^2 + 6(1-\nu)/\pi^2}{1-3\eta/4}\cdot\frac{\pi^2 D}{b^2}. \tag{8.46}$$

Example 8.5

Determine the critical value of uniformly distributed in-plane shear forces q_{xy} for the simply supported rectangular plate shown in Fig. 8.8.

Solution

We assume that the deflection surface is adequately approximated as follows:

$$w = C_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + C_2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}, \tag{a}$$

where C_1 and C_2 are two unknown coefficients. It is obvious that w in the form of (a) satisfies exactly the prescribed geometrical boundary conditions ($w = 0$ on the boundary).

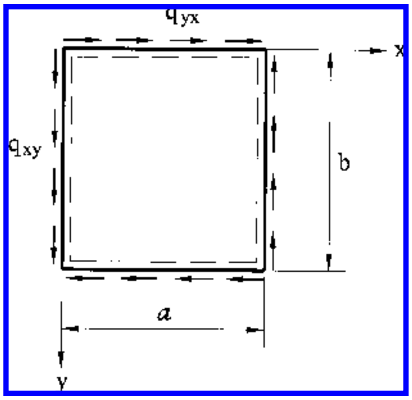

Fig. 8.8

Substituting for w from (a) into the potential energy expression $\Delta\Pi$ given by Eq. (8.45) and setting $N_{xy} = -q_{xy}$, $N_x = N_y = 0$, we obtain

$$\Delta\Pi = -\frac{\pi^4}{8} Dab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 \left(C_1^2 + 16C_2^2\right) - \frac{32}{9} q_{xy} C_1 C_2. \tag{b}$$

Applying the conditions for a stationary value of $\Delta\Pi$ – namely, $\partial(\Delta\Pi)/\delta C_1$ and $\partial(\Delta\Pi)/\partial C_2 = 0$ – results in the following two homogeneous linear equations:

$$\frac{\pi^4}{4} Dab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 C_1 - \frac{32}{9} q_{xy} C_2 = 0,$$

$$-\frac{32}{9} SC_1 + 4\pi^4 Dab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 C_2 = 0.$$

A nontrivial solution can be obtained by equating the determinant of these equations to zero. Consequently, we have

$$\pi^8 D^2 a^2 b^2 \left(a^{-2} + b^{-2}\right)^4 - \left(\frac{32 q_{xy}}{9}\right)^2 = 0,$$

which gives the following approximation for the critical shear forces:

$$q_{xy,\mathrm{cr}} = \pm \frac{9}{32} \pi^4 Dab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2. \tag{8.47}$$

This result, which is larger by 15% than an exact solution for a square plate ($a = b$) [1], can be improved by retaining more terms in the expression (a) [1–3].

8.5 BUCKLING ANALYSIS OF ORTHOTROPIC AND STIFFENED PLATES

8.5.1 Orthotropic plates

Consider a rectangular orthotropic plate whose elastic properties are characterized by four independent constants: the moduli of elasticity E_x and E_y in the two mutually perpendicular principal directions x and y; the shear modulus G; and Poisson's ratio ν_x. The second Poisson's ratio ν_y is related to ν_x by the expression (7.24). The constitutive equations, stress resultants–curvature equations, and the governing differential equation for the orthotropic plate in the framework of small-deflection plate bending theory have been derived in Sec. 7.2. Repeating the derivation of Eq. (3.92), we can represent the governing differential equation of the orthotropic plate buckling, as follows:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = 0. \tag{8.48}$$

Let us analyze the stability of a rectangular, simply supported orthotropic plate subjected to in-plane compressive forces q_x, as shown in Fig. 8.1. The boundary conditions for simply supported edges of the orthotropic plate are

$$w = 0|_{\substack{x=0,a,\\ y=0,b}} \frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2} = 0\bigg|_{x=0,a}, \frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} = 0\bigg|_{y=0,b}. \tag{a}$$

We take the deflection surface of the plate in the form

$$w = w_{11} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{b}$$

The deflection surface in the form of (b) satisfies exactly the boundary conditions (a). Substituting (b) into Eq. (8.48) and letting $N_x = -q_x$, $N_y = N_{xy} = 0$, we obtain from the solution of this equation the following expression for the compressive forces q_x:

$$q_x = \frac{\pi^2 \sqrt{D_x D_y}}{b^2} \left[\sqrt{\frac{D_x}{D_y}} \left(\frac{mb}{a} \right)^2 + \frac{2H}{\sqrt{D_x D_y}} n^2 + \sqrt{\frac{D_y}{D_x}} \left(\frac{a}{mb} \right)^2 \right]. \tag{8.49}$$

It is evident that a minimum value of q_x is reached for $n = 1$. The critical value of the compressive force can be obtained by varying a number of half-waves m. For a plate that is lengthened along the x axis ($a \gg b$), we have the following:

$$q_{x,\mathrm{cr}} = \frac{2\pi^2 \sqrt{D_x D_y}}{b^2} \left(1 + \frac{H}{\sqrt{D_x D_y}} \right). \tag{8.50}$$

For a plate with a finite ratio of sides, it should be taken [4], that

$$m = 1 \text{ for } 0 < \frac{a}{b} < \sqrt[4]{4 \frac{D_x}{D_y}},$$

$$m = 2 \text{ for } \sqrt[4]{4 \frac{D_x}{D_y}} < \frac{a}{b} < \sqrt[4]{36 \frac{D_x}{D_y}},$$

$$m = 3 \text{ for } \sqrt[4]{36 \frac{D_x}{D_y}} < \frac{a}{b} < \sqrt[4]{144 \frac{D_x}{D_y}}, \text{ and so on.}$$

Making $D_x = D_y = H = D$, we obtain the expression (8.10) for $q_{x,\text{cr}}$ in a rectangular, simply supported isotropic plate.

8.5.2 Stiffened plates

In the stability analysis of stiffened plates, two modes of buckling are usually considered. One possible mode is the local buckling of the plate between the stiffeners, provided that the plate is reinforced with strong stiffeners. In the second case, an overall buckling of the plate–stiffener combination occurs. The latter is called primary buckling in the pertinent literature.

A more economical design can be obtained if we permit simultaneous local and primary buckling at about the same stress level. Consequently, in the elastic stability analysis of stiffened plates, the structural interaction of plate and stiffeners should be taken into account.

Two approaches to the stability analysis of stiffened plates are possible. If a plate is reinforced with many equally spaced parallel stiffeners of the same size (or with a grid-stiffening arrangement), such an assembly can effectively be approximated by the orthotropic (structurally orthotropic) plate theory. This approach makes it possible to consider Eq. (8.48) for the buckling analysis of structurally orthotropic plates, and the rigidities on the left-hand side of this equation are determined according to the procedure introduced in Sec. 7.2.3. Evidently, this approach is applicable only in the case when the stiffeners are located close to one another: the value $1/n$ (n is the number of stiffeners or ribs taken over all the width of the plate) should be small compared with unity.

Another approach involves the buckling analysis of a plate which is reinforced with few stiffeners. For such a type of stiffened plates, the convenient orthotropic plate idealization cannot be used to obtain reliable values for the critical loads. Since the stiffeners are rigidly fastened to the plate, we should treat the plate and stiffeners as a structural unit; consequently, at mutual points, the stiffener deflects and twists in the same way as the plate. Since the numerical procedure of the first approach is quite similar to the one discussed in Sec. 8.5.1 and involves no new ideas and principles, the second approach will be introduced below only. The critical load may be determined by the equilibrium or energy methods. The following examples illustrate the application of both methods for the buckling analysis of the stiffened plate.

Example 8.6

This problem was analyzed by Timoshe [1]. Determine the critical load for a simply supported rectangular plate which is reinforced by a single longitudinal rib located along its centerline, as shown in Fig. 8.9. The plate is subjected to the in-plane compressive forces q_x uniformly distributed along the edges $x = 0, a$.

Solution

In our buckling analysis, we take into account only the bending stiffness of the rib in the plane perpendicular to the middle plane of the plate. We apply the differential equation (8.6) to one of the halves of the plate. Its integral is represented in the form

$$w(x, y) = F(y) \sin \frac{m\pi x}{a}. \tag{8.51}$$

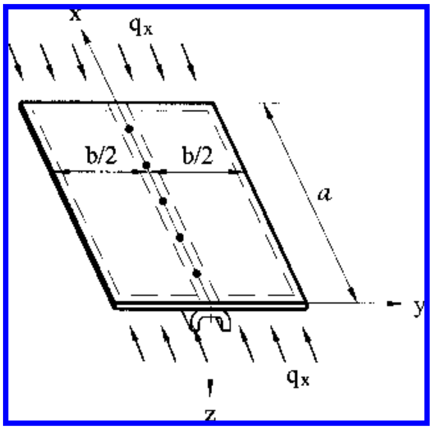

Fig. 8.9

Substituting the above into Eq. (8.6) and letting $N_x = -q_x$, $N_y = 0$, $N_{xy} = 0$, gives

$$\frac{d^4F(y)}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2F(y)}{dy^2} + \left(\frac{m\pi}{a}\right)^2 \left[\left(\frac{m\pi}{a}\right)^2 - \frac{q_x}{D}\right] F(y) = 0. \tag{8.52}$$

Its solution is of the form

$$F(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 \cos \beta y + C_4 \sin \beta y, \tag{8.53}$$

where

$$\alpha = \sqrt{\mu\left(\mu + \sqrt{\frac{q_x}{D}}\right)}, \quad \beta = \sqrt{\mu\left(\sqrt{\frac{q_x}{D}} - \mu\right)} \text{ and } \mu = \frac{m\pi}{a}. \tag{8.54}$$

The boundary conditions on the edge $y = b/2$ are

$$w = 0, \frac{\partial^2 w}{\partial y^2} = 0 \Big|_{y=b/2}. \tag{8.55}$$

Assume that the plate is buckled together with the rib; then the bent surface of the plate must be symmetric about the line $y = 0$. This results in the following condition:

$$\frac{\partial w}{\partial y} = 0 \Big|_{y=0}. \tag{8.56}$$

The difference in the reaction forces from the two strips of the plate given by the expression

$$R_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial y \partial x^2}\right] \tag{8.57}$$

will be transmitted to the rib. Assume that the plate and rib are made of one and the same material. Due to the assumption adopted above for deformations of the rib, it can be easily shown that only the first term in the brackets of Eq. (8.57) should be taken into account. If we assume that the rib together with the plate is subjected to the compressive forces q_x, then the equation for the elastic curve of the rib may be represented in the form

$$\left(EI_i\frac{\partial^4 w}{\partial x^4}+q_x\frac{\partial^2 w}{\partial x^2}+2D\frac{\partial^3 w}{\partial y^3}\right)_{y=0}=0,$$

or replacing $q_x = N_x = \sigma_x A_i$ for the rib, we obtain

$$\left(EI_i\frac{\partial^4 w}{\partial x^4}+\sigma_x A_i\frac{\partial^2 w}{\partial x^2}+2D\frac{\partial^3 w}{\partial y^3}\right)_{y=0}=0, \tag{8.58}$$

where I_i and A_i are the second moment of inertia and area of the rib cross section, respectively.

Let us introduce the following parameters:

$$\xi=\frac{a}{b},\quad \gamma=\frac{EI_i}{Db},\quad \delta=\frac{A_i}{bh}. \tag{8.59}$$

Introducing Eq. (8.53) into the conditions (8.55), (8.56), and (8.58), we obtain a system of linear algebraic homogeneous equations for $C_1,\ldots,C_4$. Equating the determinant of this system to zero, yields the following equation:

$$\left(\frac{1}{b\alpha}\tanh\frac{b\alpha}{2}-\frac{1}{b\beta}\tan\frac{b\beta}{2}\right)\left(\frac{\gamma m^2}{\alpha^2}-K\delta\right)\frac{m^2\pi^2}{\alpha^2}-4\frac{m}{\alpha}\sqrt{K}=0, \tag{8.60}$$

where

$$K=\frac{\sigma_{x,\mathrm{cr}}}{\sigma_{x,\mathrm{E}}} \tag{8.61}$$

and

$$\sigma_{x,\mathrm{cr}}=\frac{q_{x,\mathrm{cr}}}{h},\quad \sigma_{x,\mathrm{E}}=4\frac{\pi^2 D}{b^2 h} \tag{8.62}$$

where $\sigma_{x,E}$ is the critical stress for a rectangular, simply supported, unstiffened plate with the dimensions of Fig. 8.9 (see Eq. (8.11)). Equation (8.60) may be solved by the method of trial and errors. For $m = 1$ and $\xi > 2$, its solution, using the parameters introduced in Eqs (8.59), gives the following expression for $\sigma_{x,\mathrm{cr}}$:

$$\sigma_{x,\mathrm{cr}}=\frac{\pi^2 D}{b^2 h}\frac{\left(1+\xi^2\right)^2+2\gamma}{\xi^2(1+2\delta)}. \tag{8.63}$$

Now we analyze the stability problems for stiffened plates by the energy method.

Example 8.7

Determine the critical value of the in-plane compressive forces q_x acting on the plate reinforced by two equally spaced stiffeners, as shown in Fig. 8.10. The plate is simply supported on all edges. Let A_i and B_i ($B_i = EI_i$) be the area of the cross section and the bending stiffness of a stiffener, and c_i be spacing of the stiffeners.

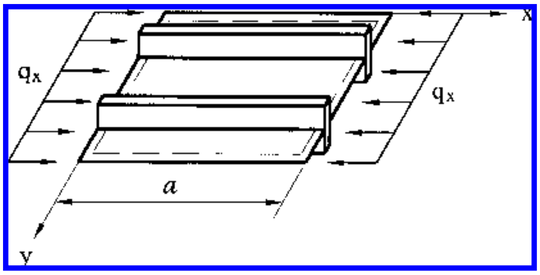

Fig. 8.10

Solution

The expression for the increment in the total potential energy in buckling for the plate reinforced by the discrete stiffeners, $\Delta\Pi_s$, may be represented as follows:

$$\Delta\Pi_s = \Delta\Pi + U_i - W_i, \tag{8.64}$$

where $\Delta\Pi$ is the increment in the potential energy of the unstiffened plate given by Eq. (8.45), U_i is the strain energy of the stiffener in bending, and W_i is the work done by the compressive forces $q_{xi} = q_x A_i/h$ acting on the stiffener, as shown below:

$$U_i = \frac{B_i}{2}\int_0^a \left(\frac{\partial^2 w}{\partial x^2}\right)_{y=c_i} \mathrm{d}x \tag{8.65}$$

and

$$W_i = \frac{q_{xi}}{2}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2_{y=c_i} \mathrm{d}x. \tag{8.66}$$

Let the deflection surface of the plate be approximated by Eq. (3.15a). Substituting for w from the above equation into Eqs (8.45), (8.65), and (8.66), and finally, into Eq. (8.64) for $\Delta\Pi_s$, we obtain the following:

$$\begin{aligned}\Delta\Pi_s = {}& \frac{\pi^4 D}{2}\frac{ab}{4}\sum_m\sum_n w_{mn}{}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{q_x}{2}\frac{ab}{4}\sum_m\sum_n \frac{m^2\pi^2}{a^2}w_{mn}^2 \\ & + \frac{\pi^4 B_i}{4a^3}\sum_m m^4\left(w_{m1}\sin\frac{\pi c_i}{b} + w_{m2}\sin\frac{2\pi c_i}{b} + \dots\right)^2 \\ & - \frac{q_{xi}A_i}{h}\frac{\pi^2}{4a}\sum_m m^2\left(w_{m1}\sin\frac{\pi c_i}{b} + w_{m2}\sin\frac{2\pi c_i}{b} + \dots\right)^2,\end{aligned} \tag{8.67}$$

where D is the flexural rigidity of the unstiffened plate. The critical value of the applied compressive forces, $q_{x,\mathrm{cr}}$, may be found from the energy criterion (8.5). Differentiating $\Delta\Pi_s$ in the form of (8.67) with respect to w_{mn}, we can determine $q_{x,\mathrm{cr}}$ from the nontrivial solution of the system of linear algebraic homogeneous equations, equating the determinant of this system to zero. Dropping the intermediate algebra, we present the final results. The critical force is given by

$$q_{x,\mathrm{cr}} = \frac{\pi^4 D}{b^2} \frac{\left(1+\xi^2\right)+3\gamma}{\xi^2(1-3\delta)}, \tag{8.68}$$

where ξ, δ, and γ are given by formulas (8.59). The solution (8.68) is done in the first approximation, i.e., for $m = n = 1$. If a rectangular, simply supported plate is reinforced by i equally spaced stiffeners then the critical value of the compressive forces is the following:

$$q_{x,\mathrm{cr}} = \frac{\pi^2 D}{b^2} \frac{\left(1+\xi^2\right)^2 + 2\sum_i \gamma_i \sin^2(\pi c_i/b)}{\xi^2\left[1+2\sum_i \delta_i \sin^2(\pi c_i/b)\right]}. \tag{8.69}$$

8.6 POSTBUCKLING BEHAVIOR OF PLATES

8.6.1 Large deflections of plates in compression

Using the linearized buckling analysis introduced in Sec. 8.2, we have considered only the initial elastic buckling of flat plates and the critical forces and stresses have been found for some typical loadings and boundary conditions. The critical force is the force at which bifurcation of equilibrium states occurs and is of fundamental significance to the designer. So, the critical loads found in the previous sections represent merely the loads at which buckling begins. It should be noted that the postbuckling behavior of plates is markedly different from that of thin rods. While a small increase in the critical load for rods will produce a complete collapse, the load-carrying capacity of the plate is not exhausted and elastic plate can carry stresses higher than σ_{cr}. It can be explained, first of all, by the effect of large deflections in the postbuckling stage and, then, by the fact that the longitudinal edges of the plate are usually constrained to remain straight. Thus, the postbuckling mechanism of elastic plates is characterized not only by bending but also by the direct (or in-plane) stresses. It is important that the latter become comparable in magnitude with the former stresses.

The use of an additional strength due to the postbuckling effects is of great practical importance in the design of ship and aerospace structures. By considering the postbuckling behavior of plates, considerable weight savings can be achieved. In these structures, the edges of the plates are usually supported by stringers in such a way that they remain straight during buckling. After buckling, the central part of the plate bulges out, and an increasingly larger portion of the load is carried by the material close to the supported edges (stringers) of the plate.

The nonlinear, large-deflection plate bending theory, discussed in Sec. 7.4, can be used for the analysis of the postbuckling behavior of plates. Because of a nonlinearity of the governing differential equations of this analysis, the resulting mathematical difficulties are considerable and exact solutions can very seldom be obtained. The most generally used techniques for the treatment of postbuckling of plates are based on numerical methods [11,12].

For some simple cases an analytical solution can be obtained under some assumptions regarding the plate behavior, boundary conditions, etc. Consider a rectangular, simply supported plate, as shown in Fig. 8.1. The plate is subjected to the in-plane compressive forces q_x. An approximate expression for the buckled middle surface of the plate is taken in the following form ($m = n = 1$):

$$w = w_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{8.70}$$

The governing differential equations for the investigation of the postbuckling behavior of thin plates of a constant thickness are von Karman's large-deflection equations (7.87), derived in Sec. 7.4. Assuming that the lateral load p is zero, we can rewrite these equations in terms of the stress function ϕ ($\phi = \Phi/h$), as follows:

$$\frac{1}{E}\nabla^4\phi = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{8.71a}$$

$$\frac{D}{h}\nabla^4 w = \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}. \tag{8.71b}$$

Substituting for w from Eq. (8.70) into Eq. (8.71a), we obtain

$$\frac{1}{E}\nabla^4\phi = \frac{1}{2} w_{11}^2 \frac{\pi^4}{a^2 b^2}\left(\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{b}\right). \tag{8.72}$$

Assume that the edge supports do not prevent the in-plane motions of the plate in the y direction or, in other words, the in-plane shear stresses are zero. A particular integral of Eq. (8.72) is taken, as follows:

$$\phi_p = A\cos\frac{2\pi x}{a} + B\cos\frac{2\pi y}{b}.$$

Determine the unknown coefficients A and B by calculating $\nabla^4\phi_p$ and comparing the left- and right-hand sides of Eq. (8.72). We obtain the following:

$$\phi_p = E\frac{w_{11}^2}{32}\left[\left(\frac{a}{b}\right)^2 \cos\frac{2\pi x}{a} + \left(\frac{b}{a}\right)^2 \cos\frac{2\pi y}{b}\right]. \tag{8.73}$$

Assuming that the edge supports do not prevent the in-plane motions of the plate in the y direction, we take the solution of the homogeneous Eq. (8.72) in the form

$$\phi_h = -\frac{q_x y^2}{2h}. \tag{8.74}$$

Finally, the general solution of Eq. (8.72) is

$$\phi = \phi_p + \phi_h = E\frac{w_{11}^2}{32}\left[\left(\frac{a}{b}\right)^2 \cos\frac{2\pi x}{a} + \left(\frac{b}{a}\right)^2 \cos\frac{2\pi y}{b}\right] - \frac{q_x y^2}{2h}. \tag{8.75}$$

In-plane stresses in the plate middle surface are

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = -E\frac{\pi^2}{8}\left(\frac{w_{11}}{a}\right)^2 \cos\frac{2\pi y}{b} - \frac{q_x}{h}; \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = -E\frac{\pi^2}{8}\left(\frac{w_{11}}{b}\right)^2 \cos\frac{2\pi x}{a};$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0. \tag{8.76}$$

As seen, the in-plane shear stresses are zero. Let us apply the Galerkin method to solve Eq. (8.71b). Using the general procedure of the method discussed in Sec. 6.5, we obtain

$$\int_0^a \int_0^b E \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \mathrm{d}x\mathrm{d}y = 0, \tag{8.77}$$

where

$$E \equiv \frac{D}{h}\nabla^4 w - \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}. \tag{8.78}$$

Substituting for w and ϕ from Eqs (8.70) and (8.75) into (8.77) and performing the corresponding operations of differentiation and integration, yields the following:

$$D\frac{\pi^4 ab}{4} w_{11}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 - q_x w_{11}\frac{\pi^2}{a^2}\frac{ab}{4} + E\frac{\pi^4 w_{11}^3}{64} h\left(\frac{1}{a^4} + \frac{1}{b^4}\right)ab = 0. \tag{8.79}$$

Assuming that $w_{11} \neq 0$, we obtain

$$q_x = N_x = \sigma_x h = D\frac{\pi^2}{b^2}\left(\frac{b}{a} + \frac{a}{b}\right)^2 + E\frac{\pi^2 h}{16b^2} w_{11}^2\left(\frac{b^2}{a^2} + \frac{a^2}{b^2}\right). \tag{8.80}$$

Since the first term on the right-hand side of the above equation represents the critical force of the linear buckling analysis, $q_{x,\mathrm{cr}}$, introduced in Sec. 8.3, it is evident that the plate can sustain a compression load greater than the linear buckling. It should be noted that the load-carrying capacity is more pronounced when the unloaded edges are constrained to remain straight ($N_y \neq 0$).

The first detailed analysis of the postbuckling behavior of plates loaded by compressive loads was conducted by von Karman *et al.* [13], who suggested a simplified approach to obtain an estimate for the ultimate load carried by the buckled plate. Based on experimental observations, this simplified method assumes that the ultimate buckling load of the plate is carried exclusively by two strips of equal width (the so-called effective width), located along the unloaded edges, and the stringers, jointly with the effective width portion of the plate, act as columns. A detailed numerical analysis of the postbuckling behavior of thin plates with various boundary conditions using the simplified method based on the effective width concept in compression for routine design purposes was developed by Marguerre [14], Cox [2], Schade [15], etc. The interested reader is referred to these references and to Refs [1,3].

8.6.2 Load-carrying capacity of plates in compression

We have seen that a plate after buckling may carry in some cases a compressive force that is many times higher than the critical load at which buckling begins. So, in cases when issues of weight economy are of fundamental importance, as for example in the aircraft industry, it is expedient to determine not only the critical load but also the ultimate load which the plate can carry without failure, which corresponds to the plate load-carrying capacity in compression.

We consider below a rectangular plate that is pin-connected along its edges with rigid stiffeners. The plate is subjected to compressive forces in one direction, as shown in Fig. 8.11a.

Prior to buckling of the plate, the compressive stresses are uniformly distributed over its width, b, but after buckling the distribution of stresses along the loaded edges becomes progressively nonlinear: they increase more intensively in the vicinity of the plate edges and the stresses differ little from their critical values in the central part of the plate. A typical compressive stress distribution along the plate cross section is shown in Fig. 8.11b. The actual distribution of the compressive stresses depends on the boundary conditions and on the length-to-width ratio, a/b, provided that this ratio is less than 3.

When failure of the plate is impending, almost the total compressive load is carried by two strips, located along the unloaded edges.

Following the simplified approach proposed by von Karman *et al*. [13], we determine the ultimate load carried by the compressed plate. This approach is based on the following assumptions:

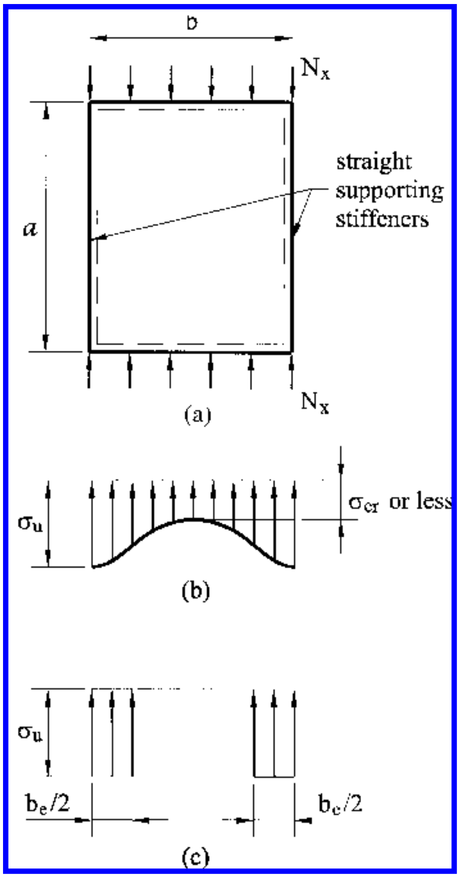

Fig. 8.11

(a) The initial (unloaded) plate is perfectly flat.
(b) The ultimate buckling load is carried exclusively by two strips of equal width, b_e, located along the unloaded edges.
(c) The maximum stress in the edge fiber of the plate, located in the middle surface, σ_u, is uniformly distributed over the two plate strips, b_e, as shown in Fig. 8.11c.
(d) The supporting stiffeners remain straight during the buckling and, jointly with the effective width portion of the plate, b_e, they act as columns.

Since the normal compressive stress is not uniform along the plate cross section, let us introduce the mean compressive stress, σ_m. From the above assumptions, it follows that the effective width, b_e, is

$$b_e = \frac{1}{\sigma_u}\int_0^b \sigma_m \mathrm{d}y. \tag{8.81}$$

Now we can determine the effective width of the bunched plate by using the reference value of the critical stress, σ_{cr}, in the following form:

$$\sigma_{cr} = \frac{N_{1cr}}{h} = K\frac{D}{h}\left(\frac{\pi}{b}\right)^2. \tag{8.82}$$

Due to the assumption (b), an expression similar to Eq. (8.82) can be written for the equivalent plate of width b_e:

$$\sigma_u = K\frac{D}{h}\left(\frac{\pi}{b_e}\right)^2. \tag{8.83}$$

Comparing Eqs (8.82) and (8.83) yields the following:

$$b_e = b\sqrt{\frac{\sigma_{cr}}{\sigma_u}}. \tag{8.84}$$

A more accurate postbuckling analysis estimates b_e as follows [16]:

$$b_e = b \cdot \sqrt[3]{\frac{\sigma_{cr}}{\sigma_u}}. \tag{8.85a}$$

Koiter [17] received the expressions for the effective width of flat plate in the form

$$b_e = b\left[1.2\left(\frac{\sigma_{cr}}{\sigma_u}\right)^{0.4} - 0.65\left(\frac{\sigma_{cr}}{\sigma_u}\right)^{0.8} + 0.45\left(\frac{\sigma_{cr}}{\sigma_u}\right)^{1.2}\right]. \tag{8.85b}$$

The ultimate load carried by the compressed plate is

$$P_u = \sigma_u b_e h. \tag{8.86}$$

To obtain estimates for the maximum edge stresses, σ_u, two cases are considered:

1. If the supporting stiffeners are relatively strong, the yield criterion can be conveniently used, in connection with the effective width concept, to determine σ_u the plate is able to carry. In this case, σ_u is simply equal to the yield stress σ_y, provided that the stiffeners and the adjacent plate

strips reach the yield stresses simultaneously without buckling. The ultimate load carried by the plate in this case is

$$P_{\mathrm{u}} = \sigma_{\mathrm{y}} b_{\mathrm{e}} h. \tag{8.87}$$

2. If the supporting stiffeners are relatively weak, they may fail by buckling before yield stress is developed. In this case, a trial and error procedure should be used to obtain an estimate for the maximum edge stresses, σ_{u}. First, we assign an effective width b_{e} in the first approximation as $b_{\mathrm{e}} = (0.3 - 0.6)b$, and determine $\sigma_u^{(1)}$, considering that the longitudinal supporting stiffener and the effective width portion of the plate act as a column. Using the $\sigma_u^{(1)}$ value obtained, a new estimate for $b_{\mathrm{e}}^{(2)}$ can be calculated, etc. The procedure is repeated until the interrelationship between these two variables is satisfied.

8.7 BUCKLING OF SANDWICH PLATES

The stability analysis of sandwich plate structure has to be taken with regard to various types of its buckling modes as a whole and its separate elements. Let a sandwich plate be loaded by the external in-plane edge loads (compressive and/or shear) symmetrically with respect to its middle plane. Two possible modes of buckling should be distinguished: a general, resulting from bending of the middle surface of the sandwich plate; a local, manifested as bending (wrinkling) of the upper and lower sheets and occurring without bending of the plate as a whole. In this book we consider only the buckling analysis of the general stability of sandwich plates. The local stability problems for sandwich plates were studied in Ref. [18].

The governing differential equations of the small-deflection bending theory of orthotropic and isotropic sandwich plates were derived in Sec. 7.6. The above equations will be valid for the buckling analysis if, according to the general procedure introduced in Sec. 8.2 for plate stability problems, instead of the transverse surface load p, a fictitious load p_{f} (see Sec. 8.2) is inserted in Eqs (7.131) for orthotropic sandwich plates or in Eq. (7.135) for isotropic ones:

$$p_{\mathrm{f}} = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}, \tag{8.88}$$

where N_x, N_y, and N_{xy} are the internal forces acting in the middle plane of the plate corresponding to the applied in-plane loading. Thus, the stability problem of the sandwich orthotropic plate is described by a system of governing equations consisting of Eqs (7.131a) and (7.131b), and (7.131c), where the latter equation is of the form

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -p_{\mathrm{f}}, \tag{8.89}$$

p_{f} is given by Eq. (8.88). In a special case, substituting for p_{f} from Eq. (8.88) into Eq. (7.135), we obtain the following governing differential stability equation for isotropic sandwich shells:

$$D_s \nabla^2 \nabla^2 w = \left(1 - \frac{D_s}{D_Q} \nabla^2\right)\left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y}\right) \tag{8.90}$$

Example 8.8

A rectangular simply supported isotropic sandwich plate of Fig. 7.8 is subjected to an in-plane edge compressive load of intensity q_x, as shown in Fig. 8.1. Determine the critical value of this load if $a = 20\,\text{in.}$, $b = 10\,\text{in.}$, $t = 0.02\,\text{in.}$, $c = 0.2\,\text{in.}$, $E_f = 10 \times 10^6\,\text{psi}$, $G_c = 12{,}000\,\text{psi}$, $\nu = 0.3$.

Solution

The simply supported boundary conditions in terms of the deflections are given by Eqs (3.14). For the problem under consideration, $N_x = -q_x$ and $p_f = -q_x \partial^2 w / \partial x^2$. Therefore, Eq. (8.90) simplifies to the following form:

$$D_s \nabla^2 \nabla^2 w + q_x \left(1 - \frac{D_s}{D_Q} \nabla^2\right) \frac{\partial^2 w}{\partial x^2} = 0. \tag{a}$$

Equation (a) is a constant-coefficient equation. A solution of the form

$$w = C_1 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \quad m, n = 1, 2, 3, \ldots, \tag{b}$$

where C_1 is a constant, is seen to satisfy the boundary conditions (3.14). Introduction of that expression for w into Eq. (a) and rearrangement gives

$$q_x = \frac{\pi^2 D_s}{b^2} \frac{\left[(mb/a) + (n^2 a/mb)\right]^2}{1 + \psi\left[(mb/a)^2 + n^2\right]}, \tag{c}$$

where m and n are positive integers and

$$\psi = \frac{\pi^2 D_s}{b^2 D_Q}. \tag{d}$$

Equation (c) may now be written in the following alternative form:

$$q_x = K_s \frac{\pi^2 D_s}{b^2}, \tag{e}$$

where the nondimensional buckling coefficient K_s for sandwich plates is defined as

$$K_s = \frac{[mb/a + n^2 a/mb)]^2}{1 + \psi\left[n^2 + (mb/a)^2\right]}. \tag{f}$$

For given values of the plate aspect ratio a/b and the shear stiffness parameter ψ, the value of the wave–length parameters m and n may be chosen by trial to give the smallest value of q_x, i.e., $q_{x,\mathrm{cr}}$.

It is seen that for $G_c \to \infty$, $\psi = 0$ and the buckling parameter $K_s \to K$ given by Eq. (8.8) (for $n = 1$). Thus, for this particular case when the transverse shear deformation is neglected, Eq. (c) reduces to the corresponding expression (8.7b) derived for buckling of homogeneous plates.

Using the numerical data of the problem, let us compute the rigidities. We have

$$D_s = \frac{E_s t(c+t)^2}{2(1-\nu^2)} = 5318 \text{ lb-in}, \quad D_Q = G_c \frac{(c+t)^2}{c} = 2904 \text{ lb/in},$$

$$\psi = \frac{\pi^2 D_s}{b^2 D_Q} = 1.831.$$

Introduction of these data into Eq. (f) reveals that the smallest value of the coefficient K_s corresponds to $n = 1$ and $m = 3$. For this value Eq. (g) gives $q_{x,\mathrm{cr}} = 973.2$ lb/in.

If the core were omitted and the two face sheets bonded together to form a thin homogeneous plate 0.04 in. thick, the corresponding critical load would be only 23.4 lb.

PROBLEMS

8.1 Figure 8.2 shows the variation of the buckling load parameter K as a function of aspect ratio a/b for $m = 1, 2, 3, 4$. Based on this figure, determine the condition at which a transition from m to $m+1$ half-sine waves across the span of the plate, compressed in the x direction by a uniform edge loads.

8.2 Calculate the critical value of the uniform compressive edge loads q_x applied over two simply supported edges of length b if two opposite edges of the length a are fixed. Use $a = 2$ m, $b = 1$ m, $h = 0.1$ m, $E = 210$ GPa, and $\nu = 0.3$.

8.3 Calculate the critical value of the uniformly applied compressive edge loads for a rectangular plate of sides a and b with two simply supported opposite edges and two other opposite edges free. Assume that the load is applied over the free edges. Take $a = 10$ m, $b = 5$ m, $h = 0.15$ m, $E = 220$ GPa, and $\nu = 0.3$.

8.4 Let a rectangular, simply supported plate of sides a and b be loaded by uniformly distributed compressive q_x and tensile q_y forces. The q_x forces are applied parallel to the side a and q_y forces act in the direction parallel to the side b. Find the nontrivial solution of Eq. (8.20) for this type of loading and calculate the critical value of the parameter λ if $q_y = \lambda q_x$ and $a = b$. Compare this result with the case when the above plate is compressed in two directions (see Fig. 8.5).

8.5 Consider a circular plate uniformly compressed by in-plane radial forces q_r. Compare the critical values of q_r for two types of plate supports: (a) simply supported edge and (b) fixed edge. Use the radius of the plate $a = 40$ in., $h = 2.5$ in., $E = 25{,}000$ ksi, and $\nu = 0.3$.

8.6 Consider a rectangular plate of sides $2a$ and $2b$. The origin of the Cartesian coordinate system is taken at the plate center. The plate rests at its center upon columns and is compressed by uniformly distributed forces q_x applied parallel to the side $2a$. Assuming that the column support can be approximated by a point support, determine the critical value of q_x. Use the Ritz method and the approximate expression for the deflection in the form

$$w = C\left[\cos\frac{\pi x}{2a} + \cos\frac{\pi y}{2b}\right].$$

8.7 Let a rectangular plate of sides a and b $(a > b)$ with fixed edges be subjected to compressive linearly varying loads $q_x = q_0(1 - \eta y/b)(\eta < 1)$. The load is applied to short plate edges. The origin of the Cartesian coordinate system is attached at the left upper corner of the plate. Determine the critical value of the applied compressive load by the Ritz method. Assume that $a = 3$ m, $b = 1.5$ m, $h = 0.15$ m, $\eta = 0.5$, $E = 200$ GPa, and $\nu = 0.28$, and the approximate expression for deflections is

$$w = C\left(x^2 - a^2\right)^2\left(y^2 - b^2\right)^2.$$

8.8 Derive Eq. (8.48).

8.9 Consider an unstiffened, simply supported rectangular plate of sides a and b and a plate of the same geometry and boundary conditions but reinforced by a single longitudinal rib located along its centerline, as shown in Fig. 8.9. Compare the critical values of the compressive forces q_x uniformly distributed along the edges $x = 0, a$ for the above two plates. Assume that $a = b = 2\,\mathrm{m}$, $h = 0.15\,\mathrm{m}$, $E = 220\,\mathrm{GPa}$, and $\nu = 0.3$, and the rib has a cross section in the form of a channel section C75 $\times$ 6. In the buckling analysis, take into account the buckling stiffness of the rib in the plane that is perpendicular to the plate middle plane.

8.10 A simply supported square plate is reinforced by three equally spaced stiffeners of rectangular cross section of depth h_1 and of width t. The plate is subjected to in-plane uniform compressive forces q_x that act parallel to the stiffeners. Determine the critical value of the applied forces. Assume that the stiffeners and plate are made of the same material and stiffeners are symmetrical about the plate middle plane. Use $a = 5\,\mathrm{m}$, $h = 0.2\,\mathrm{m}$, $h_1 = 0.4\,\mathrm{m}$, $t = 0.1\,\mathrm{m}$, $E = 210\,\mathrm{GPa}$, and $\nu = 0.3$.

8.11 A long, simply supported rectangular plate ($a/b = 3$), subjected to uniform compressive forces q_x acting parallel to the long side a was reinforced by stiffeners symmetrically placed about the plate middle surface. Three variants of locations of the stiffeners had been discussed:

(a) First, the equally spaced transverse stiffeners (i.e., located in the perpendicular direction with respect to q_x) were suggested to locate at a distance equal to b;

(b) Secondly, the equally spaced transverse stiffeners were suggested to locate at a distance $b/2$;

(c) Thirdly, only one longitudinal stiffener located along the plate centerline (i.e., in the direction of the applied load) was suggested to apply.

Assuming buckling of the reinforced plate as a whole, determine the critical values of the applied forces q_x for all the above cases. If the geometrical and mechanical properties of the plate and stiffeners are known, draw conclusions about the efficient arrangement of stiffeners for the problem under consideration from the point of view of the buckling analysis.

8.12 Estimate the ultimate edge load of a rectangular plate, stiffened along both longitudinal edges and uniformly compressed in the x direction. Assume that the plate is simply supported along its edges and that the stiffeners and adjacent plate strips simultaneously approach the compressive yield strength. Let $a = 900\,\mathrm{mm}$, $b = 450\,\mathrm{mm}$, $h = 1\,\mathrm{mm}$, $E = 73\,\mathrm{GPa}$, $\nu = 0.3$, and $\sigma_y = 415\,\mathrm{MPa}$.

REFERENCES

1. Timoshenko, S.P. and Gere, J.M., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.
2. Cox, H.L., *The Buckling of Plates and Shells*, The Macmillan Company, New York, 1963.
3. Bulson, P.S., *The Stability of Plates*, American Elsevier Publishing Company, New York, 1969.
4. Volmir, A.S., *Stability of Elastic Systems*, Gos. Izd-vo Fiz.-Mat. Lit-ry, Moscow, 1963 (in Russian).
5. Carson, W.G. and Newton, R.E., Plate buckling analysis using a fully compatible finite elements, *AIAAJ*, vol. 7, No. 3, pp. 527–529, (1969).
6. Kapur, K.K. and Hartz, B.J., Stability of plates using the finite element method, J *Engng Mech Div, Proc ASCE*, vol. 92, pp. 177–195, 1966.

7. Costa, J.A. and Brebbia, C.A. Elastic buckling of plates using the boundary element methods. In *Boundary Elements, VII* (eds C.A. Brebbia and G. Maier), Springer-Verlag, Berlin, 4-29–4-43 (1985).
8. Manolis, G.D., Beskos, D.E. and Pineros, M.F. Beam and plate stability by boundary elements, *Comput Struct*, vol. 22, pp. 917–923, (1986).
9. Bryan, G.N., On the stability of a plane plate under thrusts in its own plane with application to the buckling of the sides of a ship, *Proc Lond Math Soc*, vol. 22, pp. 54–67, (1891).
10. Watson, G.N., *Theory of Bessel Functions*, 2nd edn, Cambridge University Press, Cambridge, 1944.
11. Murray, D.W. and Wilson, E.L., Finite-element large-deflection analysis of plates, *Proc. ASCE, J Engng Mech, Div EMI*, p. 143 (1969).
12. Yang, T. J., Elastic postbuckling prediction of plates using discrete elements, *AIAAJ*, vol. 9, No. 9, pp. 1665–1666 (1971).
13. von Karman, T., Sechler, E.E., and Donnel, L.H., The strength of thin plates in compression, *Trans ASME*, vo. 54, pp. 53–57 (1932).
14. Marguerre, K., Die mittragende breite der gedrückten platte, *Luftfahrtforschung*, vol. 14, no. 3, pp. 121–128 (1937).
15. Schade, H.A., Larguer effective de toles reinforcees travaillant sous l'action de forces de flexion, *Bulletin Technique du Bureau Veritas*, pp. 215–227 (1953).
16. Marguerre, K., The apparent width of the plate in compression, *Tech Memo* No. 833, NACA, Washington, D.C, 1937.
17. Koiter, W.T., The effective width of flat plates for various longitudinal edge conditions at loads far beyond the buckling load, Rep. No. 5287, *National Luchtvaart Laboratorium* (The Netherlands).
18. Yusuff, S., Theory of wrinkling in sandwich construction, *J Roy Aeron Soc*, vol. 59, No. 529 (1955).