```
displacements(activeDof)=U;
end
```

12.5 Free vibrations of Mindlin plates

By using the Hamilton principle [15], we may express the equations of motion of Mindlin plates as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \tag{12.28}$$

where $\mathbf{M}, \mathbf{K}, \mathbf{f}$ are the system mass and stiffness matrices, and the force vector, respectively, and $\ddot{\mathbf{u}}, \mathbf{u}$ are the accelerations and displacements. Assuming a harmonic motion we obtain the natural frequencies and the modes of vibration by solving the generalized eigenproblem [8]

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{X} = \mathbf{0} \tag{12.29}$$

where ω is the natural frequency and $\mathbf{X}$ the mode of vibration.

By using the kinetic energy for the plate

$$T^e = \frac{1}{2} \int_A \rho \left[h\dot{w}^2 + \frac{h^3}{12}\dot{\theta}_x^2 + \frac{h^3}{12}\dot{\theta}_y^2 \right] dA \tag{12.30}$$

we may compute the mass matrix as [15]

$$\mathbf{M}^e = \int_A \rho \mathbf{N}^T \begin{bmatrix} h & 0 & 0 \\ 0 & \frac{h^3}{12} & 0 \\ 0 & 0 & \frac{h^3}{12} \end{bmatrix} \mathbf{N} dA \tag{12.31}$$

being the stiffness matrix already obtained before for static problems.

We consider a square plate (side a), with thickness-to-side ratio $h/a = 0.01$ and $h/a = 0.1$. The non-dimensional natural frequency is given by

$$\bar{\omega} = \omega_{mn} a \sqrt{\frac{\rho}{G}},$$

where ρ is the material density, G the shear modulus ($G = E/(2(1+\nu))$), E the modulus of elasticity and ν the Poisson's coefficient. Indices m and n are the vibration half-waves in axes x and y. In this problem we consider fully simply-supported (SSSS) and fully clamped (CCCC) plates, as well as SCSC and CCCF plates where F means free side.

For CCCC and CCCF we use a shear correction factor $k = 0.8601$, while for SCSC plates we use $k = 0.822$. For SSSS plates we consider $k = 5/6$.

In table 12.2 we show the convergence of the fundamental frequency for CCCC plate with $h/a = 0.01, k = 0.8601, \nu = 0.3$. We obtain quite good agreement with the analytical solution [19].

In table 12.3 we show the convergence of the fundamental frequency for SSSS plate with $h/a = 0.01, k = 0.8333, \nu = 0.3$. Again, we obtain quite good agreement with a analytical solution [19]. Tables 12.4 and 12.5 consider $h/a = 1$ and in all of them results agree very well with analytical solution.

Tables 12.6 and 12.7 list the natural frequencies of a SSSS plate with $h/a = 0.1$ and $h/a = 0.01$, being $k = 0.833, \nu = 0.3$. Our finite element solution agrees with the tridimensional solution and analytical solution given by Mindlin [8].

Tables 12.8 and 12.9 compare natural frequencies with the Rayleygh-Ritz solution [8] and a solution by Liew [20].

Tables 12.10 and 12.11 compare natural frequencies for SCSC plate with $h/a = 0.1$ and $h/a = 0.01$, being $k = 0.822, \nu = 0.3$, respectively. Sides located at $x = 0; L$ are simply-supported.

Table 12.2 Convergence of natural frequency $\bar{\omega}$ for CCCC plate with $h/a = 0.01, k = 0.8601, \nu = 0.3$

10×10 Q4	0.1800	Analytical [19] 0.1754
15×15 Q4	0.1774	
20×20 Q4	0.1765	
25×25 Q4	0.1761	

Table 12.3 Convergence of natural frequency $\bar{\omega}$ for SSSS plate with $h/a = 0.01, k = 0.8333, \nu = 0.3$

10×10 Q4	0.0973	Analytical [19] 0.0963
15×15 Q4	0.0968	
20×20 Q4	0.0965	
25×25 Q4	0.0965	

Table 12.4 Convergence of natural frequency $\bar{\omega}$ for CCCC plate with $h/a = 0.1, k = 0.8601, \nu = 0.3$

10×10 Q4	1.6259	Analytical [19] 1.5940
15×15 Q4	1.6063	
20×20 Q4	1.5996	

Table 12.5 Convergence of natural frequency $\bar{\omega}$ for SSSS plate with $h/a = 0.1, k = 0.8333, \nu = 0.3$

10×10 Q4	0.9399	Analytical [19] 0.930
15×15 Q4	0.9346	
20×20 Q4	0.9327	

Table 12.6 Natural frequencies of a SSSS plate with $h/a = 0.1, k = 0.833, \nu = 0.3$

Mode no.	m	n	15 × 15 Q4	3D *	Mindlin *
1	1	1	0.9346	0.932	0.930
2	2	1	2.2545	2.226	2.219
3	1	2	2.2545	2.226	2.219
4	2	2	3.4592	3.421	3.406
5	3	1	4.3031	4.171	4.149
6	1	3	4.3031	4.171	4.149
7	3	2	5.3535	5.239	5.206
8	2	3	5.3535	5.239	5.206
9	4	1	6.9413	–	6.520
10	1	4	6.9413	–	6.520
11	3	3	7.0318	6.889	6.834
12	4	2	7.8261	7.511	7.446
13	2	4	7.8261	7.511	7.446

* analytical solution

Table 12.7 Natural frequencies of a SSSS plate with $h/a = 0.01, k = 0.833, \nu = 0.3$

Mode no.	m	n	20 × 20 Q4	Mindlin *
1	1	1	0.0965	0.0963
2	2	1	0.2430	0.2406
3	1	2	0.2430	0.2406
4	2	2	0.3890	0.3847
5	3	1	0.4928	0.4807
6	1	3	0.4928	0.4807
7	3	2	0.6380	0.6246
8	2	3	0.6380	0.6246
9	4	1	0.8550	0.8156
10	1	4	0.8550	0.8156
11	3	3	0.8857	0.8640
12	4	2	0.9991	0.9592
13	2	4	0.9991	0.9592

(* – analytical solution)

Tables 12.12 and 12.13 compare natural frequencies for CCCF plates with $h/a = 0.1$ and $h/a = 0.01$, respectively, being $k = 0.822, \nu = 0.3$. Side located at $x = L$ is free.

The present finite element results are quite accurate.

Figure 12.3 shows the modes of vibration for a CCCC plate with $h/a = 0.1$, using 10×10 Q4 elements.

Figure 12.4 shows the modes of vibration for a SSSS plate with $h/a = 0.1$, using 10×10 Q4 elements.

Figure 12.5 shows the modes of vibration for a SCSC plate with $h/a = 0.01$, using 15×15 Q4 elements.

Table 12.8 Natural frequencies of a CCCC plate with $h/a = 0.1, k = 0.8601, \nu = 0.3$

Mode no.	m	n	20×20 Q4	Rayleygh-Ritz [19]	Liew et al. [20]
1	1	1	1.5955	1.5940	1.5582
2	2	1	3.0662	3.0390	3.0182
3	1	2	3.0662	3.0390	3.0182
4	2	2	4.2924	4.2650	4.1711
5	3	1	5.1232	5.0350	5.1218
6	1	3	5.1730	5.0780	5.1594
7	3	2	6.1587		6.0178
8	2	3	6.1587		6.0178
9	4	1	7.6554		7.5169
10	1	4	7.6554		7.5169
11	3	3	7.7703		7.7288
12	4	2	8.4555		8.3985
13	2	4	8.5378		8.3985

Table 12.9 Natural frequencies of a CCCC plate with $h/a = 0.01, k = 0.8601, \nu = 0.3$

Mode no.	m	n	20×20 Q4	Rayleygh-Ritz [19]	Liew et al. [20]
1	1	1	0.175	0.1754	0.1743
2	2	1	0.3635	0.3576	0.3576
3	1	2	0.3635	0.3576	0.3576
4	2	2	0.5358	0.5274	0.5240
5	3	1	0.6634	0.6402	0.6465
6	1	3	0.6665	0.6432	0.6505
7	3	2	0.8266		0.8015
8	2	3	0.8266		0.8015
9	4	1	1.0875		1.0426
10	1	4	1.0875		1.0426
11	3	3	1.1049		1.0628
12	4	2	1.2392		1.1823
13	2	4	1.2446		1.1823

Table 12.10 Natural frequencies for SCSC plate with $h/a = 0.1, k = 0.822, \nu = 0.3$

Mode no.	m	n	15×15 Q4	Mindlin solution [8]
1	1	1	1.2940	1.302
2	2	1	2.3971	2.398
3	1	2	2.9290	2.888
4	2	2	3.8394	3.852
5	3	1	4.3475	4.237
6	1	3	5.1354	4.936
7	3	2	5.5094	
8	2	3	5.8974	
9	4	1	6.9384	
10	1	4	7.2939	
11	3	3	7.7968	
12	4	2	7.8516	
13	2	4	8.4308	

Table 12.11 Natural frequencies for SCSC plate with $h/a = 0.01, k = 0.822, \nu = 0.3$

Mode no.	m	n	15×15 Q4	Mindlin solution [8]
1	1	1	0.1424	0.1411
2	2	1	0.2710	0.2668
3	1	2	0.3484	0.3377
4	2	2	0.4722	0.4608
5	3	1	0.5191	0.4979
6	1	3	0.6710	0.6279
7	3	2	0.7080	
8	2	3	0.7944	
9	4	1	0.8988	
10	1	4	1.0228	
11	3	3	1.0758	
12	4	2	1.1339	
13	2	4	1.2570	

Table 12.12 Natural frequencies for CCCF plate with $h/a = 0.1, k = 0.8601, \nu = 0.3$

Mode no.	m	n	15×15 Q4	Mindlin solution [8]
1	1	1	1.0923	1.089
2	2	1	1.7566	1.758
3	1	2	2.7337	2.673
4	2	2	3.2591	3.216
5	3	1	3.3541	3.318
6	1	3	4.6395	4.615
7	3	2	4.9746	
8	2	3	5.4620	
9	4	1	5.5245	
10	1	4	6.5865	
11	3	3	6.6347	
12	4	2	7.6904	
13	2	4	8.1626	

Table 12.13 Natural frequencies for CCCF plate with $h/a = 0.01, k = 0.8601, \nu = 0.3$

Mode no.	m	n	15×15 Q4	Mindlin solution [8]
1	1	1	0.1180	0.1171
2	2	1	0.1967	0.1951
3	1	2	0.3193	0.3093
4	2	2	0.3830	0.3740
5	3	1	0.4031	0.3931
6	1	3	0.5839	0.5695
7	3	2	0.6387	
8	2	3	0.7243	
9	4	1	0.8817	
10	1	4	0.9046	
11	3	3	1.0994	
12	4	2	1.1407	
13	2	4	1.1853	

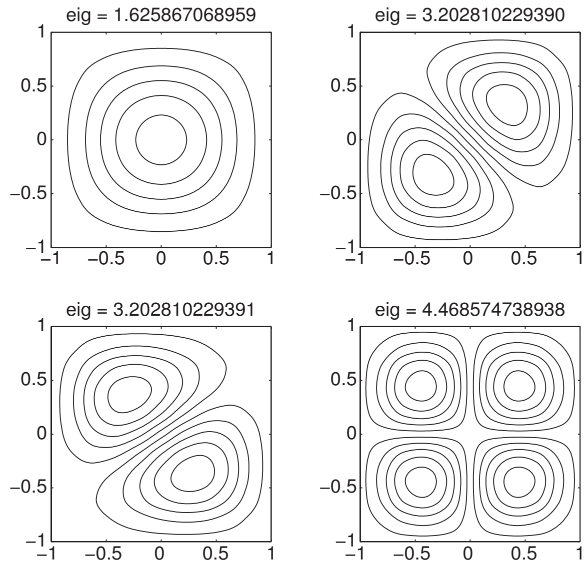

Fig. 12.3 Modes of vibration for a CCCC plate with $h/a = 0.1$, using 10×10 Q4 elements

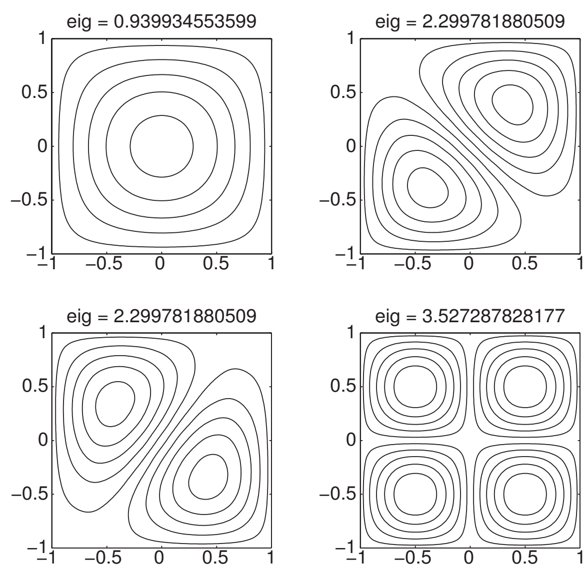

Fig. 12.4 Modes of vibration for a SSSS plate with $h/a = 0.1$, using 10×10 Q4 elements

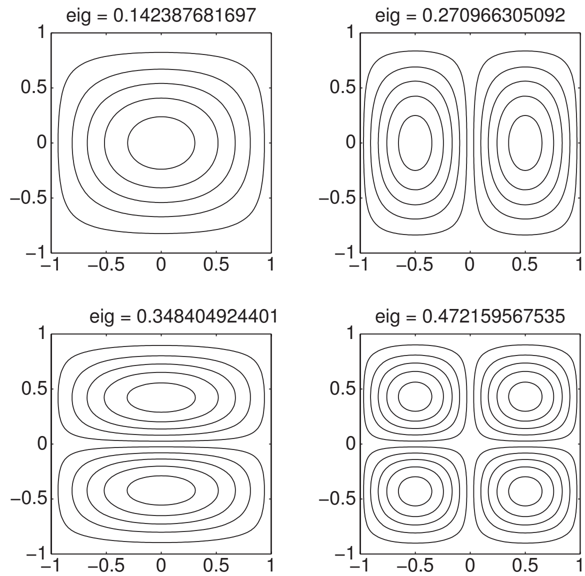

Fig. 12.5 Modes of vibration for a SCSC plate with $h/a = 0.01$, using 15×15 Q4 elements

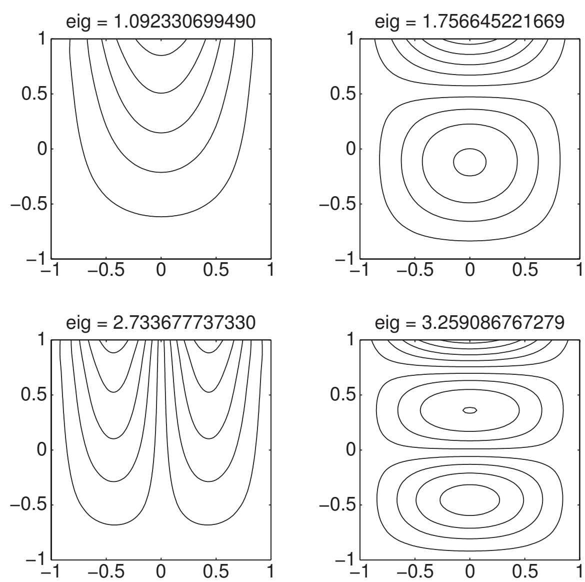

Fig. 12.6 Modes of vibration for a CCCF plate with $h/a = 0.1$, using 15×15 Q4 elements

Figure 12.6 shows the modes of vibration for a CCCF plate with $h/a = 0.1$, using 15×15 Q4 elements.