Assignment 10

1. (**40**) Consider flipping two biased (a priori, the bais of each coin is uniformly distributed on $[0,1]$) coins. We repeat the following experiment 5 times: (i) Pick one of the two coins at random (ii) Flip the selected coin 10 times. The results of the five experiments are as follows:

$$X = (\texttt{HTTTHHTHTH}, \texttt{HHHHTHHHHH}, \texttt{HTHHHHHTHH}, \texttt{HTHTTTHHTT}, \texttt{THHHTHHHTH}) .$$

   Estimate the biases of the coins in two ways p_1, p_2:

   - (**20**) By optimizing directly the marginal likelihood:

   $$p_1, p_2 = \text{argmax}_{p_1,p_2} \sum_Z P(X|Z, p_1, p_2),$$

     where Z is a (unobservable) sequence of the coin choices.
   - (**20**) By running the iterative EM algorithm.

   Compare the results.

2. (**35**) Consider the model

$$x_t = x_{t-1} + u_{t-1} + \epsilon_{t-1}, \quad z_t = x_t + \eta_t$$

   with $\epsilon_t \sim \mathcal{N}(0, \sigma)$ and $P(\eta_t) \sim \mathcal{N}(0, \rho)$ with constant velocity, $u \equiv 1$. You are given a series of T observations $z_1, ..., z_T$ in `1D_LL.pickle` [pickled dict of z_t]. Your task is to estimate the parameters σ and ρ using the EM (expectation–maximization) algorithm.

   - Derive expression for the log-likelihood $\ln P(x, z, \sigma, \rho)$ where x and z are the hidden state and observed time series.

   Pick some reasonable estimates for σ and ρ (lets denote them σ_0 and ρ_0 correspondingly). Iterate the following two steps ($n = 0, 1, ...$), until reasonable convergence:

   - Compute expected value of $\ln P(x, z, \sigma, \rho)$ – that is, average over x with the distribution, given by $x \sim P(x, z, \sigma_n, \rho_n)$. The result is a function $C_n(\sigma, \rho)$ of a very simple analytical form, only the coefficients are to be computed numerically.
   - Maximize the function $C_n(\sigma, \rho)$ over its arguments and set $(\sigma_{n+1}, \rho_{n+1}) = \text{argmax}_{\sigma,\rho} C_n(\sigma, \rho)$. Iterate if necessary.

   What are the estimated values of σ and ρ? Compare with the results found in the Problem 9.3.

3. (**75***) Consider the local linear trend model:

$$y_t = a_t + \epsilon_t^y, \quad \epsilon_t^y \sim \mathcal{N}(0, \sigma_y)$$

$$a_t = a_{t-1} + b_{t-1} + \epsilon_t^a, \quad \epsilon_t^a \sim \mathcal{N}(0, \sigma_a)$$

$$b_t = b_{t-1} + \epsilon_t^b, \quad \epsilon_t^b \sim \mathcal{N}(0, \sigma_b)$$

with unknown parameters σ_y, σ_b. For the data in `LLT.pickle` [pickled numpy array with observed time series of y_t and σ_a], compute, using the likelihood maximization (in spirit of Problem 9.3), the most probable values of σ_y, σ_b. With these values, compute the smoothed trajectory of the local slope coefficient b_t^* and plot it as a function of time. Compute the prediction for y_{T+1} (that is for the time instant, following the latest available observation).