Assignment 9

1. (**25**) Consider the Linear Dynamic System, with non-stationary variance of the noise:

$$x_t = x_{t-1} + u_{t-1} + \epsilon_{t-1}, \quad z_t = x_t + \eta_t \tag{1}$$

with $\epsilon_t \sim \mathcal{N}(0, \sigma_t)$ and $P(\eta_t) \sim \mathcal{N}(0, \rho_t)$. Using all data available in `nonstationary_LDS.pickle` [pickled dict of u_t, σ_t and ρ_t], compute the smoothed trajectory of the hidden state x_t. Plot it as a function of time together with the error bar. Compute the prediction for the location of the object at $T+1$ (that is for the time instant, following the latest available observation).

2. (**35**) Consider 2D motion with random acceleration. Let z_{1t} and z_{2t} be the horizontal and vertical locations of the object, and $\dot{z}_{1t}$ and $\dot{z}_{2t}$ be the corresponding velocity. We can represent this as a state vector $z_t \in R^4$ as follows:

$$z_t = (z_{1t}, z_{2t}, \dot{z}_{1t}, \dot{z}_{2t})^T.$$

Let us assume that the object would-be-moving at constant velocity, but its motion is perturbed by random Gaussian noise (e.g., due to the wind). Additionally, we can only observe the noised location of the object $y_t \in R^2$, but not its velocity. We can model the system dynamics as follows:

$$z_t = A_t z_{t-1} + \epsilon_t, \quad y_t = C_t z_t + \delta_t,$$

with

$$A_t = \begin{pmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\epsilon_t \sim \mathcal{N}(0, \sigma\hat{1}), \quad \delta_t \sim \mathcal{N}(0, \rho\hat{1}).$$

For the data in `2DMotion.pickle` [pickled dict of y_t, Δ, σ and ρ], compute the filtered trajectory z_t. Draw the filtered trajectory on the plane together with the observations and compute the prediction for the location of the object at $T+1$ (that is for the time instant, following the latest available observation).

3. (**35**) Consider the model in Eq. (1) with stationary but unknown σ and ρ and constant velocity, $u \equiv 1$. You are given a series of T observations $z_1, ..., z_T$ in `1D_LL.pickle` [pickled dict of z_t]. Your task is to estimate the parameters σ and ρ using maximization of the log-likelihood function.

   - The likelihood of given values of σ and ρ is proportional to $P(\sigma, \rho) \propto \int P(x, z, \sigma, \rho) dx$, where the integral can be evaluated explicitly. Using `JAX`, write fast code to compute $\ln P(\sigma, \rho)$ and its gradients up to the 2nd order.
   - Perform numerical maximization to find the optimal values for σ and ρ.

   What are the estimated values of σ and ρ?