

Assignment 2

1. (10) In this exercise, we will explore three main algorithms available in scientific `python` distributions for computation of SVD: `numpy.linalg.svd`, `scipy.sparse.linalg.svds` and `sklearn.utils.extmath.randomized_svd`. To this end:
 - Construct a random $n \times n$ matrix A (with iid elements sampled from standard normal distribution); consider $n = 2000$.
 - Using these implementations, construct rank-2 approximations to A . You will thus obtain three rank-2 matrices A_{svd} , A_{svds} and A_{rsvd} . Measure the run-time of these three algorithms for the given task.
 - Compute the error norms: $\|A - A_{\text{svd}}\|_F$, $\|A - A_{\text{svds}}\|_F$, $\|A - A_{\text{rsvd}}\|_F$. Explain the result.
2. (5) Let A be $m \times n$ with SVD $A = U\Sigma V^T$. Compute SVDs of the following matrices in terms of U , Σ and V :
 - (i) $(A^T A)^{-1}$, (ii) $(A^T A)^{-1} A^T$, (iii) $A (A^T A)^{-1}$, (iv) $A (A^T A)^{-1} A^T$.
3. (10) Consider the matrix:

$$\begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \quad (1)$$

- List the singular values, left singular vectors and right singular vectors of A . The SVD is not unique, so find the one that has the minimal number of minus signs in U and V .
 - Draw a labeled picture of the unit ball in \mathbb{R}^2 and its image under A , together with the singular vectors with the coordinates of their vertices marked.
 - What are 2-norm and Frobenius norm of A ?
 - Find A^{-1} not directly, but via SVD.
 - Find the eigenvalues λ_1, λ_2 of A .
4. (5) The file `A.hdf` contains the $n \times n$ matrix A . Determine the best approximation of A_{ij} in terms of the following ansatz, where the variables are separated: $A_{ij} \approx h_i \eta_j$. What is the related relative error of such approximation:

$$\delta_{\text{err}} = \frac{\sqrt{\sum_{ij} (A_{ij} - h_i \eta_j)^2}}{\sqrt{\sum_{ij} A_{ij}^2}}? \quad (2)$$

How many terms K would an exact representation of the following form:

$$A_{ij} = \sum_{\alpha=1}^K h_{\alpha i} \eta_{\alpha j} \quad (3)$$

require?

5. (10) In this exercise, you will explore application of SVD to dimensionality reduction to images (and their labels) stored in the file `digits.hdf`.
 - Inspect the dataset: plot examples of images, corresponding to several digits (say 0, 3, 7).
 - Normalize the dataset A .
 - Use SVD to project the dataset A from 64 dimensions to 2 dimensions. Show the colored scatter plot, where colors encode the digits.