The Gelfand Problem in Tubular Domains

Francisco José Vial Prado [*]

Abstract

We construct stable solutions of $\Delta u + \lambda e^u = 0$ with Dirichlet boundary conditions in small tubular domains (i.e. geodesic ε–neighbourhoods of a curve Λ embedded in $\mathbb{R}^n$), adapting the arguments of Pacard-Pacella-Sciunzi. We also show unicity of these solutions, in particular, we show that the stable branch of the bifurcation diagram is similar to the well-known nose-shaped diagram of the standard Gelfand problem in the unit ball. In this work, Λ can be replaced by any compact smooth manifold embedded in $\mathbb{R}^n$.

1 Introduction

Let Ω be a domain of $\mathbb{R}^n$, and for $\lambda > 0$ consider the Gelfand problem

$$(1.1) \qquad (G_\lambda) \begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, let Λ be a smooth closed curve embedded in $\mathbb{R}^n$, and given $\varepsilon > 0$ we define the tubular neighbourhood of radius ε centered about Λ by

$$(1.2) \qquad T_\varepsilon(\Lambda) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Lambda) < \varepsilon\}.$$

In this article, we study the Gelfand problem (G_λ) in this kind of tubular domains. Let us first give some properties of this problem in all generality, assuming that Ω is bounded and connected.

(i) If u solves (G_λ), $\lambda > 0$ if and only if $u > 0$. This is a straightforward consequence of the maximum principle.

(ii) If (G_λ) admits a solution u, then $\lambda \leq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ stands for the first eigenvalue of the Dirichlet Laplacian in Ω. Indeed, let $\phi > 0$ be a principal eigenfunction of the Laplacian in Ω with Dirichlet boundary conditions, and multiply (G_λ) by ϕ. Integrating by parts and using $u < e^u$ gives $(-\lambda_1(\Omega) + \lambda) \int_\Omega \phi\, u < 0$.

(iii) With implicit function methods, one can establish a local solution curve $(\lambda, u) \in [0, \infty) \times C(\bar{\Omega})$, which emanates from the stable solution $\lambda = 0, u = 0$. Because of (ii) this curve is contained in $[0, \lambda_1(\Omega)] \times C(\bar{\Omega})$.

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[*]DCC, Pontificia Universidad Católica de Chile (`fovial@uc.cl`).

We can therefore identify $(\lambda, u) \in [0, \lambda_1] \times C^2(\bar{\Omega})$ with a pair $(\lambda, ||u||_{L^\infty}) \in [0, \lambda_1] \times \mathbb{R}^+$, and draw the bifurcation diagram in the plane. However, determining multiplicity of these solutions require an understanding in the general domain Ω. Nevertheless, theorem 4.6 of [Ban80] implies that the spectrum of (G_λ) (i.e. the values for which (G_λ) is solvable) is an interval $[0, \lambda_*(\Omega))$, the value λ_* depending heavily on the geometry of Ω.

2 Preliminaries

2.1 The stable branch

Let $L(u) := -\Delta u - \lambda e^u$. We say that a solution u of (G_λ) is *stable* if the linearized operator of L at the point u is positive definite. This notion of stability comes from the following argument: For Ω consider the functional $\mathcal{E} : C_0^2(\Omega) \to \mathbb{R}$ defined by

$$(2.1) \qquad \mathcal{E}(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega e^u \, dx.$$

We say that u is a critical point of $\mathcal{E}$ if for every $\phi \in C_0^2(\Omega)$, 0 is a critical point of $E : \mathbb{R} \to \mathbb{R}$ given by $E(t) := \mathcal{E}(u + t\phi)$. Indeed, the equation $E'(0) = 0$ gives

$$(2.2) \qquad \int_\Omega (-\Delta u - e^u) \, \phi \, dx = 0 \ \text{ and } \ E''(0) = \int_\Omega |\nabla u|^2 - \int_\Omega e^u \phi^2.$$

It is hence natural to define stability as follows.

Definition 1. *Let Ω be an open set of $\mathbb{R}^n$, and $u \in C_0^2(\Omega)$ a solution of $-\Delta u = f(u)$. We say that u is stable if*

$$Q_u(\phi) := \int_\Omega |\nabla \phi|^2 \, dx - \int_\Omega f'(u) \, \phi^2 \, dx \geq 0$$

for all $\phi \in C_c^1(\Omega)$ (or $\phi \in H_0^1(\Omega)$ if Ω is bounded).

For the sake of completeness, let us give some properties of stable solutions.

Proposition 1. *Local minimisers of the energy $\mathcal{E}$ are stable.*

Proposition 2. *A $C_0^2(\Omega)$ solution of $-\Delta u = f(u)$ is stable if and only if $\lambda_1(-\Delta u - f'(u), \omega) \geq 0$ for every bounded subdomain w of Ω (or simply, $\omega = \Omega$ if Ω is bounded).*

Proposition 3. *A $C_0^2(\Omega)$ solution of $-\Delta u = f(u)$ is stable if and only if it exists $v \in C^2(\Omega)$, $v > 0$, and $-\Delta v - f'(u) \, v \geq 0$.*

Proposition 4. *There is one unique stable solution of (G_λ) for every admissible λ, which is a minimiser of the energy and the solution with the smallest $L^\infty(\Omega)$ norm.*

For more properties and proofs we recomend [Dup11], §1. We define also the lower (or stable) branch of the curve

$$\mathcal{S} := \{(\lambda, ||u_\lambda||_{L^\infty(\Omega)}), \ u_\lambda \text{ is a solution of } (G_\lambda)\}.$$

It turns out (see [Dup11], §3.3), that for every n there exists a maximum value λ_* such that $\mathcal{S}$ is a smooth curve connecting $(0,0)$ with $(\lambda_*, ||u_{\lambda_*}||_{L^\infty(\Omega)})$, which is unbounded in the $|| \cdot ||_{L^\infty}$ direction if $n \geq 10$.

2.2 The case $\Omega = B^n$

The celebrated theorem of Gidas-Ni-Nirenberg (see [GNN79]) establishes that the solutions of

$$(2.3) \qquad \text{(GNN)} \begin{cases} \Delta u + f(u) = 0 & \text{in } B^n, \\ u = 0 & \text{on } \partial B^n, \end{cases}$$

are positive and radially symmetric provided that f is positive and regular, where $B^n = \{x \in \mathbb{R}^n, |x| \leq 1\}$. Therefore, if $\Omega = B^n$, all solutions of (G_λ) are radially symmetric and (G_λ) is equivalent to the problem of finding $u : [0,1] \to \mathbb{R}$ such that

$$(2.4) \qquad \begin{cases} u'' + \frac{n-1}{r} u' + \lambda e^u = 0, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases}$$

Notice that $||u||_{L^\infty} = u(0)$. For $n = 1$, the equation $u'' + \lambda e^u = 0$ may be explicitly solved by means of the Laplace transform (see [Khu04]), giving $u(x) = 2 \log \left(\alpha \operatorname{sech}(\alpha \sqrt{\lambda/2} \cdot x) \right)$, and the boundary conditions require that α is a solution to the transcendental equation $\alpha = \cosh\left(\alpha \sqrt{\lambda/2} \right)$. One verifies that this leads to zero, one, or two solutions for $\lambda > \lambda_c, \lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where $\lambda_c \approx .88$. Observe that $||u||_{L^\infty} = u(0) = 2 \log \alpha$, and from the transcendental equation one computes $\alpha_1 = 1 + \frac{\lambda}{4} + o(1)$, $\alpha_2 = \frac{4}{\lambda} + o(1)$. It thus follows that, for any small and fixed λ, one solution approaches to $||u|| = 0$ and the other one approaches to $||u|| = \infty$, this is, the solution curve $(\lambda, ||u_\lambda||)$ is unbounded, contained in $[0, \lambda_c] \times \mathbb{R}^+$. One can easily check that, in the upper branch of solutions, λ decreases with $||u||$, yielding the nose-shape of the solution curve.

Other branches of solutions have been computed by numerical means, see for instance [PW02]. The bi-dimensional case $\Omega = B^2$ can also be solved explicitly and it presents a similar behaviour than the previous one. Solutions exist if and only if $0 \leq \lambda \leq 2$, and for $\lambda = 2$ there is only one solution given by $u_*(r) = \log \frac{4}{(1+r^2)^2}$. For other admissible values of λ, solutions are given by

$$u_i(r) = \log \frac{b_i}{(1 + (\lambda b_i/8) r^2)^2},$$

where $b_i = \frac{32}{\lambda^2} \left(1 - \frac{\lambda}{4} + (-1)^i \sqrt{1 - \lambda/2} \right), i = 1, 2$.

There is, however, a remarkable difference between $n = 1$ and $n = 2$: For $\Omega = (-1, 1)$, the unstable solution blows up at every point as $\lambda \to 0$, whereas for $\Omega = B^2$, it blows up only at the origin.

For general dimension $n \geq 3$, the problem is analysed with a suitable change of variables, and the behaviour of the solution yields from a dynamical coupled system that arises from the study of the radial equation. This is the approach considered by Dupaigne in [Dup11]; for other instances of the Gelfand problem we strongly recommend this text. We summarize the results for the radially symmetric case in every dimension in Table 1.

Dimension	Number of solutions	Maximum value of λ
$n = 1, 2$	Two if $\lambda \in (0, \lambda_*)$	
	One if $\lambda = \lambda_*$	
$3 \leq n \leq 9$	One if λ is sufficiently small	$\lambda_*(B^n) > 2(n-2)$
	Finitely many if $\lambda \neq 2(n-2)$	
	More than any given number for λ sufficiently close to $2(n-2)$	
	Infinitely many for $\lambda = 2(n-2)$	
	Two for λ close to $\lambda_*(B^n)$	
	One for $\lambda = \lambda_*(B^n)$	
$n \geq 10$	One unique stable solution	$\lambda_*(B^n) = 2(n-2)$

Table 1: Bifurcation diagram for the Gelfand problem in $\Omega = B^n$ ([Dup11])

2.3 Fermi coordinates

Our framework is a specific case of the one used in [PPS14] for a 1-dimensional manifold, and we will indeed follow their notations. Consider the tubular neighbourhood T_ε as defined in 1.2; The Fermi coordinates parameterise this set as a product space between the curve Λ and B^{n-1} as follows. First identify Λ with the zero-section of $N\Lambda$ (the normal bundle of Λ) and T_ε with

$$\Omega_\varepsilon := \{(y, z) \in N\Lambda; \, y \in \Omega, \, z \in N_y\Lambda, \, |z| \leq \varepsilon\}$$

via the natural mapping $T_\varepsilon \to \Omega_\varepsilon$, $(y, z) \mapsto y + z$.

If $g_z := dz^2$ is the Euclidean metric on normal fibers, and $\mathring{g}$ is the metric induced on Λ, we have that the metric $\bar{g}$ on $N\Lambda$ is induced by the embedding of Λ in $\mathbb{R}^n$, this is $\bar{g} = \mathring{g} + g_z$. Lemma 3.2 of [PPS14] proves that, in these coordinates, the Euclidean Laplacian Δ can be decomposed as

$$(2.5) \qquad \qquad \Delta = \Delta_{\bar{g}} + D,$$

where $\Delta_{\bar{g}} = \Delta_{\mathring{g}} + \Delta_{g_z}$ denotes the Laplace-Beltrami operator on $N\Lambda$ for metric $\bar{g}$, and D is a second-order differential operator of the form

$$(2.6) \qquad \qquad D = \sum_{i=1}^{n-1} z_i D_i^{(2)} + D^{(1)},$$

where $D^{(1)}$ and $D^{(2)}$ are first-order and second-order partial differential operators respectively, whose coefficients are smooth and bounded. Note that in this 1–dimensional manifold, we have a parameterisation $t : \mathbb{R} \to \Lambda$ of the curve, and the Laplace-Beltrami operator of Λ is simply ∂_{tt}. We therefore have

$$(2.7) \qquad \qquad \Delta = \partial_{tt} + \Delta_{g_z} + D.$$

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This decomposition, and in particular the form of the operator D, will allow us to obtain estimates for functions defined in small tubes and prove our main results, which we describe in the next sections.

3 The stable solution in the tube

In this section, we establish the following:

Theorem 1. *For small ε, there exists a unique stable solution to*

$$(3.1) \qquad \begin{cases} \Delta u + \frac{\lambda}{\varepsilon^2} e^u = 0 & \text{in } T_\varepsilon(\Lambda), \\ u = 0 & \text{on } \partial T_\varepsilon(\Lambda). \end{cases}$$

An outline of the proof follows:

- Let U be the unique radial stable solution of (G_λ) for the $(n-1)$-dimensional ball. We choose u_ε, a rescaled version of U that travels along Λ, this is, for each section of the normal bundle of Λ, consider the $(n-1)$-dimensional ball of radius ε centered about Λ and use a copy of U.

- The function u_ε verifies approximately the equation and it is stable in the sense of section 2.1. To prove this, we find a super-solution to the problem and use a version of the maximum principle.

- We use a fixed-point argument to prove the existence of a genuine solution to 3.1 of the form $u_\varepsilon + v$, where v is small. The fixed-point theorem ensures the uniqueness of the perturbation if v is small enough. Finally, we show that larger perturbations do not lead to stable solutions, proving the result.

Let us first collect some lemmas that will help us prove 1.

3.1 Some elliptic estimates

Throughout this section, $\| \cdot \|$ stands for the L^∞ norm in the tube T_ε, unless stated otherwise. Let $U(r)$ be the unique (radial) stable solution of (G_λ) for $\Omega = B^{n-1}$, i.e.

$$(3.2) \qquad \begin{cases} \Delta U + \lambda\, e^U = 0 & \text{in } B^{n-1}, \\ U = 0 & \text{on } \partial B^{n-1}, \end{cases}$$

and define $u_\varepsilon : T_\varepsilon \to \mathbb{R}$ by

$$u_\varepsilon(y, z) := U\left(\frac{\text{dist}(z, \Lambda)}{\varepsilon} \right).$$

As the non-linearity $\lambda\, e^u$ is not multiplicative, we have that the function u_ε will not verify the Gelfand equation in the tube, nor an approximation, but the following estimate.

Lemma 1. *There is a constant C such that*

$$(3.3) \qquad |\varepsilon^2 \Delta u_\varepsilon + \lambda\, e^{u_\varepsilon}| \le C\varepsilon.$$

Proof. It follows from the definition of u_ε that $\varepsilon^2 \Delta_{g_z} u_\varepsilon + \lambda e^{u_\varepsilon} = 0$, and note that u_ε does not depend on the parameter t. Using the expression of the Euclidean Laplacian in Fermi coordinates, we have

$$\varepsilon^2 \,\Delta u_\varepsilon + \lambda e^{u_\varepsilon} = (\varepsilon^2 \,\Delta_{g_z} u_\varepsilon + \lambda e^{u_\varepsilon}) + \varepsilon^2 (\partial_{tt} + D) u_\varepsilon = \varepsilon^2 D \, u_\varepsilon,$$

therefore, the estimate

$$(3.4)\quad |\varepsilon^2 \Delta u_\varepsilon + \lambda e^{u_\varepsilon}| \;\; \leq \;\; \varepsilon^2 \, ||D u_\varepsilon||$$

$$(3.5)\qquad\qquad\qquad \leq \;\; \sum_{i=1}^{n-1} \left(\varepsilon^2 ||z_i D^{(2)} u_\varepsilon|| + \varepsilon^2 ||D^{(1)} u_\varepsilon|| \right)$$

$$(3.6)\qquad\qquad\qquad = \;\; \sum_{i=1}^{n-1} \left(\varepsilon ||x_i D^{(2)} U||_{L^\infty(B)} + \varepsilon ||D^{(1)} U||_{L^\infty(B)} \right)$$

holds and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The stability of U implies that there exist $\mu_1 > 0$ and $\phi_1 > 0$ such that

$$(3.7)\qquad \begin{cases} -(\Delta + \lambda e^U)\phi_1 = \mu_1 \phi_1 & \text{in } B^{n-1}, \\ \qquad\qquad\quad \phi = 0 & \text{on } \partial B^{n-1}, \end{cases}$$

and hence the linearized operator about U is invertible. This proves the existence of a function W, which is a super-solution to the linearized equation in B^{n-1}, verifying

$$(3.8)\qquad \begin{cases} -(\Delta + \lambda e^U)W = 1 & \text{in } B^{n-1}, \\ \qquad\qquad W = 0 & \text{on } \partial B^{n-1}. \end{cases}$$

Lemma 2. *There is a constant B such that $|W| \leq B, |\nabla W| \leq B, |\Delta W| \leq B$.*

Proof. By the GNN theorem, W is radially symmetric, positive and $\partial_r W < 0$, which gives $|W| \leq W(0)$. Write $W(z) = a(r)$, then a solves

$$(3.9)\qquad \begin{cases} a''(r) + \frac{n-1}{r^2} a'(r) + \lambda e^U a + 1 = 0, & 0 < r < 1, \\ a(1) = 0, a'(0) = 0. \end{cases}$$

As $a(r)$ is bounded and positive, we have

$$(3.10)\qquad\qquad -C \leq a''(r) + \frac{n-1}{r^2} a'(r) \leq -1$$

for a positive constant C. Multiplying 3.10 by $e^{-\frac{n-1}{r}}$ we have $-Ce^{-\frac{n-1}{r}} \leq \left(e^{-\frac{n-1}{r}} a'(r)\right)' \leq e^{-\frac{n-1}{r}}$, i.e.

$$|a'(r)| \leq C_1 \cdot e^{\frac{n-1}{r}} \int_0^r e^{-\frac{n-1}{t}} \, dt \leq C_1$$

for a constant $C_1 = \max(C, 1)$. This yields $|\nabla W(z)| = |a(r)| \leq C_1$. Finally, write $|\Delta W| \leq 1 + e^U W \leq C_2$ for a positive constant C_2 and define $B := \max(C_1, C_2, W(0))$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Similarly, define $w_\varepsilon : T_\varepsilon \to \mathbb{R}$ by $w_\varepsilon(y, z) := W\big(\frac{\mathrm{dist}(z, \Lambda)}{\varepsilon}\big)$. The corresponding estimate for w_ε will be given by the following.

Lemma 3. *For sufficiently small ε, the function w_ε verifies*

$$(3.11) \qquad \varepsilon^2 \Delta w_\varepsilon + \lambda e^{u_\varepsilon} w_\varepsilon \leq -1/2.$$

Proof. Again note that w_ε does not depend on the parameter t and that $-(\varepsilon^2 \Delta_{g_z} w_\varepsilon + e^{u_\varepsilon} w_\varepsilon) = 1$. We then have that $(\varepsilon^2 \Delta + \lambda e^{u_\varepsilon}) w_\varepsilon = (\varepsilon^2 \Delta_{g_z} + \lambda e^{u_\varepsilon}) w_\varepsilon + \varepsilon^2 (\partial_{tt} + D) w_\varepsilon = -1 + \varepsilon^2 D w_\varepsilon$. As $|\varepsilon^2 D w_\varepsilon| \leq \varepsilon C'$ for a constant C' (as in the proof of lemma 1) , the result follows for small ε. $\qquad\square$

Consider now the invertibility problem for the linearized operator $L_\varepsilon := -(\varepsilon^2 \Delta + e^{u_\varepsilon})$ in $T_\varepsilon(\Lambda)$: Given a smooth f in the tube, find ϕ such that

$$(3.12) \qquad \begin{cases} L_\varepsilon \phi = f & \text{in } T_\varepsilon(\Lambda), \\ \phi = 0 & \text{on } \partial T_\varepsilon(\Lambda). \end{cases}$$

Lemma 4. *Let ϕ solve 3.12, then $\phi \leq 2||f||_{L^\infty(T_\varepsilon)} \cdot w_\varepsilon$. Moreover, L_ε is invertible given that ε is small enough.*

Proof. Let us first give the a-priori estimate by means of the maximum principle. From equations 3.11 and 3.12 we have

$$(3.13) \qquad L_\varepsilon w_\varepsilon \geq \frac{1}{2} \quad \text{and} \quad L_\varepsilon\left(\frac{\phi}{2||f||}\right) \leq \frac{1}{2}.$$

By addition, we have $L_\varepsilon\big(w_\varepsilon - \frac{1}{2||f||}\phi\big) \geq 0$. Recall the strong maximum principle: If u is a smooth function verifying $-\Delta u \geq 0$ in a connected domain Ω of $\mathbb{R}^n$, then if u attains a minimum in Ω, u is constant. The same conclusion holds if $-\Delta$ is replaced by $-\Delta + a(x)$ with $a \in L^p(\Omega)$ and $p \leq n/2$ (see [Hof80] for a proof of this theorem with applications to a Helium-like system). Note that the conclusion holds regardless of the sign of $a(x)$. Apply the strong maximum principle to the operator L_ε to conclude that either $w_\varepsilon - \frac{1}{2||f||}\phi$ is constant or it attains a minimum in $\partial T_\varepsilon(\Lambda)$. Because both w_ε and ϕ vanish on the boundary, in both cases $w_\varepsilon - \frac{1}{2||f||}\phi \geq 0$, proving the first assertion. To prove that L_ε is invertible, write the problem in Fermi coordinates $(y, z) \in \Lambda \times B^{n-1}$:

$$(3.14) \qquad \begin{cases} -(\varepsilon^2 \Delta_{\bar{g}} + \lambda e^{u_\varepsilon})\phi - \varepsilon^2 D = f & \text{in } T_\varepsilon(\Lambda), \\ \phi = 0 & \text{on } \partial T_\varepsilon(\Lambda). \end{cases}$$

Recall that D is of the form $\sum_{i=1}^{n-1} z_i D^{(2)} + D^{(1)}$ where $D^{(i)}$ is an i–differential operator with smoothly bounded coefficients. In order to work in a domain with no dependence in ε, use the scaling $z \mapsto z/\varepsilon$ and define $\Phi(y, z) := \phi(y, \varepsilon z), F(y, z) := f(y, \varepsilon z)$. After simple manipulation, the problem reads

$$(3.15) \qquad \begin{cases} (L - \varepsilon D)\Phi = F & \text{in } T_1(\Lambda), \\ \Phi = 0 & \text{on } \partial T_1(\Lambda). \end{cases}$$

where $L = -(\Delta_{\bar{g}} + \lambda e^U)$ is invertible by hypothesis. Indeed, its eigenvalues are numbers of the form $\mu_i + \varepsilon^2 \nu_j$ where $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots$ are the

eigenvalues of $-(\Delta_{g_z} + \lambda e^U)$ in B^{n-1} and $0 = \nu_1 < \nu_2 \leq \nu_3 \leq \dots$ are the eigenvalues of $-\Delta_{\hat{g}}$ on Λ. Therefore, $L - \varepsilon D$ is a small perturbation of L, and we conclude that it is invertible for small ε after a simple fixed-point argument: Write $\Phi + L^{-1}F + \Psi$, then Ψ solves

$$(3.16) \qquad \begin{cases} \Psi = \varepsilon L^{-1}D(L^{-1}F + \Psi) & \text{in } T_1(\Lambda), \\ \Psi = 0 & \text{on } \partial T_1(\Lambda). \end{cases}$$

Using the fact that the norm of the inverse of L is controlled by $1/\mu_1$, it is straightforward to show that the right-hand operator (to which we associate the $||\cdot||_\infty$ norm in the tube) is a contraction mapping for small ε that maps the space of bounded functions in the tube into itself. Thus Ψ exists, allowing to conclude. $\qquad\square$

3.2 Proof of theorem 1

Proof: Write a perturbation of the solution $u = u_\varepsilon + v$. Problem 3.1 reduces to finding v such that

$$(3.17) \qquad \begin{cases} \Delta(u_\varepsilon + v) + \frac{\lambda}{\varepsilon^2} e^{u_\varepsilon + v} = 0 & \text{in } T_\varepsilon(\Lambda), \\ v = 0 & \text{on } \partial T_\varepsilon(\Lambda). \end{cases}$$

Rewrite the differential equation as $(\varepsilon^2 \Delta + \lambda e^{u_\varepsilon})v + (\varepsilon^2 \Delta u_\varepsilon + \lambda e^{u_\varepsilon}) + \lambda e^{u_\varepsilon}(e^v - 1 - v) = 0$, and, recalling that L_ε is invertible for small ε, define the following operator

$$(3.18) \qquad H_\varepsilon := v \mapsto L_\varepsilon^{-1}((\varepsilon^2 \Delta u_\varepsilon + \lambda e^{u_\varepsilon}) + \lambda e^{u_\varepsilon}(e^v - 1 - v)).$$

Thus, any solution of 3.17 is a fixed point of H_ε. Let us introduce the space of functions

$$\mathcal{A}_\varepsilon := \{v \in L^\infty(T_\varepsilon(\Lambda)), \exists C \in \mathbb{R}, ||v|| \leq C\varepsilon\},$$

to which we associate the L^∞ norm in the tube. The following lemmas show that H_ε is a contraction mapping in $\mathcal{A}_\varepsilon$. First, the fact that $H_\varepsilon(\mathcal{A}_\varepsilon) \subset \mathcal{A}_\varepsilon$ for sufficiently small ε is a direct consequence of Lemma 4 for a function in $\mathcal{A}_\varepsilon$.

Lemma 5. *Let v verify 3.17 and define $B = \max(1, 2||\lambda e^U W||_{L^\infty(B^{n-1})})$. If $|v| \leq 1/B$, then $|v| \leq C\varepsilon w_\varepsilon$, where C is a constant close to the constant from Lemma 1.*

Proof. Lemma 4, applied for v and $f = (\varepsilon^2 \Delta u_\varepsilon + \lambda e^{u_\varepsilon}) + \lambda e^{u_\varepsilon}(e^v - 1 - v)$ gives

$$(3.19) \qquad v \quad \leq \quad 2||(\varepsilon^2 \Delta u_\varepsilon + \lambda e^{u_\varepsilon}) + \lambda e^{u_\varepsilon}(e^v - 1 - v)||w_\varepsilon$$
$$(3.20) \qquad \quad \leq \quad C\varepsilon w_\varepsilon + B(e^v - 1 - v),$$

for a constant C from Lemma 1 not depending on ε. Using the fact that $e^t - 1 - t < t^2$ for $t \in [-1, 1]$ (the first non-zero root of $g(t) := t^2 - (e^t - 1 - t)$ is greater than $\ln(3)$), we conclude that, as $|v| \leq 1$, $|v| \leq C\varepsilon w_\varepsilon + Bv^2$. Thus,

$$(3.21) \qquad \left(|v| - C\varepsilon w_\varepsilon + O(\varepsilon^2)\right) \cdot \left(|v| - \frac{1}{B} - C\varepsilon w_\varepsilon + O(\varepsilon^2)\right) \geq 0,$$

from where the result follows. $\qquad\square$

Lemma 6. *H_ε is a contraction mapping of $\mathcal{A}_\varepsilon$.*

Proof. Take $f, g \in \mathcal{A}_\varepsilon$, then,

$$||H_\varepsilon(f - g)|| = ||L_\varepsilon^{-1}\big(\lambda e^{u_\varepsilon}(e^f - e^g - (f - g))\big)||.$$

The convexity of the exponential function implies that for any reals α, β with $\alpha > \beta$ we have $e^\alpha - e^\beta \le e^\alpha(\alpha - \beta)$, and, taking $\beta = 0$, $e^\alpha - 1 \le \alpha e^\alpha$, therefore,

$$
\begin{aligned}
||H_\varepsilon(f - g)|| &\le C\lambda ||e^{u_\varepsilon}|| \cdot ||e^{\max(f,g)} - 1|| \cdot ||f - g|| \\
&\le C'\varepsilon ||f - g||,
\end{aligned}
$$

for a constant C' not depending on ε. $\qquad\square$

This proves that there exists a unique genuine solution of 3.1 of the form $u_\varepsilon + v$ with v small. The stability of this solution comes from the fact that the spectrum of the operators $-(\Delta + \lambda e^U)$ in $T_1(\Omega)$ and $-(\Delta + \lambda e^{u_\varepsilon}$ in $T_\varepsilon(\Omega)$ are equal: We have

$$-(\Delta + \lambda e^{u_\varepsilon + v}) = -(\Delta + \lambda e^{u_\varepsilon}) + O(\varepsilon) \text{ in } T_\varepsilon(\Omega),$$

and therefore the eigenvalues of the left-hand operator are close to the sequence μ_i, which does not depend on ε. In particular, they form a sequence of positive values for small ε, which implies precisely that the associated quadratic form is positive definite. This allows to conclude that, for sufficiently small ε, there exists a unique small perturbation that leads to a genuine stable solution of 3.1. This uniqueness holds only in a neighbourhood of u_ε; it remains the question of whether there are stable solutions far from u_ε. We answer negatively with the following argument: Suppose that there are two distinct stable solutions of 3.1. Their difference $v := u_2 - u_1$ verifies

$$-\varepsilon^2 \Delta v = \lambda(e^{u_2} - e^{u_1}).$$

Multiplying the above equation by the positive part of v and integrating in the tube gives

$$(3.22) \qquad \varepsilon^2 \int_{T_\varepsilon} |\nabla v_+|^2 \, dx = \lambda \int_{T_\varepsilon} (e^{u_2} - e^{u_1}) \cdot v_+ \, dx.$$

As u_2 is stable, we have $\varepsilon^2 \int_{T_\varepsilon} |\nabla v_+|^2 \ge \lambda \int_{T_\varepsilon} e^{u_2} \cdot v_+^2 \, dx$. Plugging this inequality into 3.22 yields

$$0 \le \lambda \int_{T_\varepsilon} (e^{u_2} - e^{u_1} - e^{u_2} v_+) v_+ \, dx.$$

Note that the integrand is a negative number by strict convexity of the exponential function, therefore, $v_+ = 0$. Changing u_1 and u_2 gives $v_- = 0$, completing the proof of Theorem 1. $\qquad\blacksquare$

4   Concluding remarks

In this article, we adapted the arguments of [PPS14, Dup11] to the Gelfand problem $\Delta u + \lambda e^u = 0$ with Dirichlet boundary conditions in a small geodesic tube around a smooth manifold embedded in $\mathbb{R}^n$. We believe that the same line of thinking in [PPS14] can be carried out without further complications here to complete the bifurcation diagram. This is, to show that the unstable solutions of the Gelfand problem in the unit ball can also be used to construct unstable solutions in tubular domains, for ε possibly outside a set of resonant values in which the linearized operator $-\Delta - \lambda e^{u_\varepsilon}$ is not invertible, where u_ε is the rescaling of an unstable solution. Moreover, an analogous spectral analysis of the linearized operator should yield the Morse index of unstable solutions. This way, we expect the same resonance phenomena, i.e. , that the invertibility of linearized operators holds only for values of ε in a set accumulating around 0 (but not for every small $\varepsilon > 0$).

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