# MTL732- Financial Mathematics

## Lecture 1

### Course Structure

1. Quizzes: 2 in total $$ 2\times 12.5 = 25$
2. Minor : 25
3. Major: 50-> Flexible. Can be changed with projects

### Terminologies in financial market

- **Financial assets**: Anything which can be bought/sold is called a financial asset. Also known as security
- **Financial Markets**: The place where financial trading takes place.
  - Two types of financial assets
    1. Risk-free assets: They guarantee payoff e.g. Bonds
    2. Risky assets: Payoff is random e.g. Stocks
- **Time Horizon**: The epochs where trading takes place. Can be discrete / Continuous(Realistic)
- **Single Period Models**: Simplest models. Trading takes place at $t = 0$ or $t = T$ only

**Examples** $B(0)$= Price of bond at $t = 0$ which is 100
$B(1)$ = Price of Bond at $t =1$ which is 110
$S(0)$ = Price of Stock at $t = 0$ which is 100
$S(1)$ = Price of Stock at $t = 1$ which is unknown
We treat $S(0)$ as random variable $r_B$ = rate of return on bond[Sometimes we call it as returns] $$r_B = \frac{B(1) - B(0)}{B(0)}$$ Similarly we define return on stock as $$r_S = \frac{S(1) - S(0)}{S(0)}$$ Since $S(1)$ is a random variable, $r_S$ is also a random variable In this case we talk of expected return Consider the case $$ S(0) = \begin{cases} 110 \text{ with probability } = 0.8\ 80 \text{ with probability } = 0.2\ \end{cases} $$ Hence, we have $$ r_S = \begin{cases} 0.1 \text{ with probability } = 0.8\ -0.2 \text{ with probability} = 0.2\ \end{cases} $$ Hence, we have $\mathbb{E}\left[r_S\right] = 0.8 \times 0.1 + 0.2 \times (-0.2) = 0.04$

- **Portfolio**: A collection of financial assets i.e. stocks and bonds is called a portfolio. A portfolio in which there are $\boldsymbol{x}$ bonds and $\boldsymbol{y}$ stocks is denoted as $\left(\boldsymbol{x}, \boldsymbol{y}\right)$
  The value of portfolio is given as $V_P(t) = \boldsymbol{x}\boldsymbol{B(t)} +\boldsymbol{y}\boldsymbol{S(t)}$ The rate of return of portfolio is hence defined as $$r_P = \frac{V_P(1) - V_P(0)}{V_P(0)}$$

- If $x, y$ are allowed to take fractional values, it means that the asset has divisibility property.

- If $x, y$ is allowed to any real value, it means the asset has liquidity property

- If $x>0 \Rightarrow$ Long Position

- If $x<0 \Rightarrow$ Short Position

- Short position is said to be closed when the investor returns the money with interest.

- **Short Selling**: When we borrow stocks from a broker and sell them and make investment and return whenever the broker demands, it is known as short selling

- Short position is said to be closed when the investor returns the stock to the broker

**Arbitrage**: A market is said to have arbitrage when an investor can earn some profit with positive probability without any investment in a risk free manner. In other words a market is said to have arbitrage if there exists a portfolio $P$ such that

- $V_P(0) = 0$
- $\mathbb{P}\left(V_P(1)\geq 0\right) = 1$
- $\mathbb{P}\left(V_P(1) >0\right)>0$

# Lecture 2

**No Arbitrage Principle**

There is no portfolio P such that

- $V_P(0) = 0$
- $\mathbb{P}\left(V_P(1)\geq 0\right) = 1$
- $\mathbb{P}\left(V_P(1) >0\right)>0$

**Law of same price**: If two financial instruments have exactly the same payoff, then they have the same price. It follows from no arbitrage principle. Consider two portfolios *P* and *Q* such that $V_P(1) = V_Q(1)$, then $V_P(0) = V_Q(0)$
**Proof**: $V_P(1) = V_Q(1)$ Let $\Omega$ be the set of all possible states of the financial market.
We have $$V_P(1)(\omega) = V_Q(1)(\omega)\text{ } \forall \omega \in \Omega$$
Suppose WLOG if possible $V_P(0) > V_Q(0)$, then the investor can go long in Q and short in P leading to $V_P(0) - V_Q(0)$ profit.
Hence $V_P(0) - V_Q(0)$ is the arbitrage possibility. By no arbitrage pricing, we get $V_P(0) \leq V_Q(0)$ which is a contradiction

## Time Value of Money

$q_1:$ What is the future value of amount invested/borrowed today
$q_2:$ What is the present value of amount which is received/paid at some time in future

## Simple Interest

At $t = 0,$ rate of interest($r$) is $r%$ and amount is $P$
We have $V(0) = P$
$V(1) = P + rP$
$V(2) = P + rP + rP = P + 2rP$
Hence, we have $V(t) = P(1 + rt)$
Hence Growth factor is $1 + rt$ and discount factor $d(0, t) = \displaystyle{\frac{1}{1 + rt}}$

## Periodic Compounding

$P$ is principle, $r$ is the rate of interest(annual)

**Annual Compounding**:
We have $V(0) = P$
$V(1) = P + rP = P(1 + r)$
$V(2) = P( 1 +r ) + rP(1 + r) =(1 + r)^2 P$
$V(t) = \underbrace{(1 + r)^t}_{\text{Growth factor}}P$
***Semi Annual Compounding***:
*We have $V(0) = P$*
*$V(1) = P(1 + \frac{r}{2})^2$*
*Following in similar way, we get* $$V(t) = \underbrace{\left(1 + \frac{r}{2}\right)^{2t}}_{\text{Growth factor}} P$$
If compounding is done m periodic intervals in each year, we have $V(1) = (1 + \frac{r}{m})^{m}$
Hence, we have
$\boxed{V(t) = \underbrace{\left(1 + \frac{r}{m}\right)^{tm}}_{\text{Growth factor}}P}$
***Continuous Compounding***: *Taking limit $m\rightarrow \infty$, we get* $V(t) = \underbrace{\exp\left(rt\right)}_{\text{Growth factor}}P$ $$\frac{\partial V}{\partial P} = \text{Growth factor}$$
**Which Compounding Method is better?**
Compare growth factor after 1 year.
If for compounding method $C_1$ growth factor after 1 year us greater than the growth factor of compounding method $C_2$, then $C_1$ is preferred over $C_2$
**Effective rate** For a given compounding method where annual rate of interest is $r$, the effective rate is the one which gives the same growth factor over a period of 1 year under annual compounding, i.e. $\left(1 + r/m\right)^m = (1 + r_e)$
Solving, we get
$$\begin{equation*} \boxed{r_e = \left(1 + \frac{r}{m}\right)^m - 1} \end{equation*} $$ For continuous compounding, $r_e = e^{r } - 1$
If $r_e = r'_e$, then the two compounding methods are equivalent
If $r_e > r'_e$, then the first compounding method is preferred.
**Annuity**: It is a finite stream of payments $C$ made at equal intervals If $r$ is the rate of interest(compounded annually), then present value is $$P = \frac{C}{1 + r} + \frac{C}{(1 + r)^2 } + \ldots + \frac{C}{(1 + r)^n} = C\left[\frac{1 - \left(1 + r\right)^{-n}}{1 + r}\right]$$ Taking $n\rightarrow \infty$ $P = \frac{C}{1 + r}\rightarrow$ Perpetuity
Example Consider a person buys a loan of $Rs. 10,000$ and needs to pay that in 5 equal installments due yearly at $15%$ rate of interest. How much money does he need to pay in each installment?
**Sol**: We have 10,000 = $C\frac{1 - 1.15^{-5}}{1.15 } = 22870.85$ Hence the person should pay Rs 22870.86 in each installment

# Lecture 3: Introduction to derivatives

There are some financial securities whose value depends on another financial security. In this case the former is called as derivative and the latter one is called underlying security In short we call them *derivative* and *underlying* Stock is commonly used as underlying derivative Examples of derivatives are

1. Forward Contract
2. Future contract
3. Option contract
4. Swap Contract $\rightarrow$ Commodity

**Forward Contract**

It is derivative where underlying is stock. It has following characteristics

- It is agreement to buy or sell the stock at some future date called delivery date for some fixed price $F$(forward price)
- The investor who decides to buy the stock is said to have long forward contract or take long forward position .
- The investor who decides to sell the stock is said to have entered short forward contract or take a short forward position
- No money is paid at $t = 0$ when the contract is exchanged
- It guarantees that the stock has to bought and sold at delivery Let us denote the delivery time by $T$, $S(T)$ as the value of stock at time $T$ and $F$ as the forward price of the stock

**Pricing Formula for F**

**Theorem**: Let price of a stock at time t= 0 be S(0). Suppose that short selling is allowed. Then the forward price $F(0, T)$ where $T$ is the delivery time is given by $$F(0, T) =\frac{S(0)}{d(0, T)}$$ Proof: Suppose $F(0, T) < \displaystyle \frac{S(0)}{d(0, T)}$

- At t = 0
  - short sell 1 unit of stock to generate 1 unit and sell in the market to generate $S(0)$.
  - Invest $S(0)$ in risk free asset.
  - Enter in long forward contract
  - $V(0) = S(0) - S(0) = 0$
- At $t= T$
  - Withdraw $\displaystyle\frac{S(0)}{d(0, T)}$ from the bank
  - Buy stock at $F(0, T)$.
  - Close the short position by returning the stock
  - $V(T) =\displaystyle \frac{S(0)}{d(0, T)} - F(0, T) > 0$ This is arbitrage. So $F(0, T)\geq \displaystyle \frac{S(0)}{d(0, T)}$ Now suppose $F(0, T) > \displaystyle \frac{S(0)}{d(0, T)}$
- At t = 0
  - Borrow $S(0)$ from the bank and buy 1 unit stock
  - Enter in short forward contract with forward price $F(0, T)$
  - V(0) = S(0) - S(0) = 0
- At $t = T$
  - Close the short forward position by selling the stock at forward price
  - Close the short position in bank by paying $\displaystyle\frac{S(0)}{d(0, T)}$
  - $V(T) = \displaystyle \frac{S(0)}{d(0, T)} - F(0,T) >0$ This is arbitrage Hence, we have $$\boxed{F(0, T) = \frac{S(0)}{d(0, T)}}$$

# Lecture 4

**Theorem**: Consider a forward contract where delivery date $T$ is $n$-periodic in future. Let asset holding cost be $c(i)$ during period $i$ to $i + 1$, then

$$\boxed{F(0,T) = \frac{S(0)}{d(0,T)} + \sum_{i = 1}^{n - 1}\frac{c(i)}{d(i, n)}}$$ **Proof:** Let $F(0, T) > \displaystyle\frac{S(0)}{d(0,T)} + \sum_{i = 1}^{n - 1}\frac{c(i)}{d(i, n)}$

- At $t = 0$
  - Borrow S(0) from bank
  - Buy 1 unit stock

- Enter into short forward contract
- $V(0) = 0$
- At time period $i + 1$ borrow money $c(i)$ from the bank and pay it as asset holding cost
- At $t = T$
  - Close the short forward contract
  - Close the short position with the bank
  - $V(T) = F(0, T) - \displaystyle\frac{S(0)}{d(0,T)} + \sum_{i = 0}^{n - 1}\frac{c(i)}{d(i, n)} > 0$

This is arbitrage Now let $F(0, T) < \displaystyle\frac{S(0)}{d(0,T)} + \sum_{i = 1}^{n - 1}\frac{c(i)}{d(i, n)}$

- At $t = 0$
  - Short sell stock and generate 1 unit stock and sell it to generate $S(0)$
  - Invest $S(0)$ in risk free asset
  - Enter into long forward contract
- At period $t = i + 1$ receive stock maintenance cost $c(i)$ and invest it in bank
- At $t = T$
  - Collect cash from the bank
  - Buy 1 unit stock at $F(0,T)$
  - Close the short stock position by returning the stock
  - $V(T) = \displaystyle\frac{S(0)}{d(0,T)} + \sum_{i = 0}^{n - 1}\frac{c(i)}{d(i, n)} - F(0,T) > 0$

This is arbitrage Hence, we have $F(0, T) = \displaystyle\frac{S(0)}{d(0,T)} + \sum_{i = 0}^{n - 1}\frac{c(i)}{d(i, n)}$---

**Theorem** Let the asset be stored at zero cost and short selling is allowed and price of stock at $t = 0$ be $S(0)$ and dividend *div* is received at time $0<t<T$, then $$\boxed{F(0,T) = \frac{S(0)}{d(0,T)} - \frac{div}{d(t, T)}}$$ **Proof**:

Suppose that $F(0, T) > \displaystyle\frac{S(0)}{d(0,T)} - \displaystyle\frac{div}{d(t, T)}$

- At $t = 0$
  - Borrow $S(0)$ from the bank
  - Buy 1 unit stock.
  - Enter into short forward position
  - $V(0) = 0$
- At $t = t$
  - Receive $div$ amount as dividend
  - Invest $div$ in risk free asset
- At $t = T$
  - Close the short position by selling the stock
  - Collect cash from the bank earned on $div$ amount
  - Close the short position with bank by paying loan with interest -$V(T) = \displaystyle F(0,T) + \frac{div}{d(t, T)} - \frac{S(0)}{d(0, T)} > 0$

This is arbitrage
Now, let $F(0, T) > \displaystyle\frac{S(0)}{d(0,T)} - \displaystyle\frac{div}{d(t, T)}$

- At $t = 0$
  - Short sell to generate 1 unit of stock

  - Sell the stock and generate S(0) amount of cash
  - Enter into long forward contract
  - Invest S(0) in risk-free asset
  - $V(0) = 0$
- At $t = t$
  - Borrow $div$ amount from the bank
  - Pay the amount to the owner of the stock
- At $t = T$
  - Collect cash from the bank
  - Buy 1 unit stock from the forward contract
  - Close the short position by returning the stock
  - Close the short position with the bank by paying amount $div$ with interest
  - $V(T) = \displaystyle \frac{S(0)}{d(0, T)} - \frac{div}{d(t, T)} - F(0, T)$

This is arbitrage
Hence, we have $F(0,T) = \displaystyle \frac{S(0)}{d(0,T)} - \frac{div}{d(t, T)}$

**Theorem** The forward price of the stock paying divident continuously at rate $r_{div}$ is $$\boxed{F(0, T) = \frac{S(0)}{d(0,T)}e^{-r_{div}T}}$$ **Proof**: Suppose $\displaystyle F(0, T) < \frac{S(0)}{d(0,T)}e^{-r_{div}T}$

- At $t = 0$
  - Short sell to generate $\displaystyle e^{-r_{div}}$ units of stock
  - Sell the stock to generate $\displaystyle \frac{S(0)}{d(0,T)}e^{-r_{div}T}$
  - Enter in long forward contract for 1 unit of stock
  -