

Consider the sequence of intervals $A_n = (-\frac{1}{n}, \frac{1}{n})$

Claim 1: $A_{n+1} \subset A_n$

Proof:

Let $x \in A_{n+1}$

$$\begin{aligned}\implies -\frac{1}{n+1} &< x < \frac{1}{n+1} \\ \implies -\frac{1}{n} &< x < \frac{1}{n} \text{ as } \frac{1}{n} > \frac{1}{n+1} \\ \implies x &\in A_n\end{aligned}$$

$$\implies A_{n+1} \subset A_n$$

Claim2:- $\bigcap_{n=1}^{\infty} A_n = \{0\}$

proof :-

Clearly $\forall n \in \mathbb{N} 0 \in A_n$.

$$\begin{aligned}\implies 0 &\in \bigcap_{n=1}^{\infty} A_n \\ \implies \{0\} &\subset \bigcap_{n=1}^{\infty} A_n (1)\end{aligned}$$

Let $x \in \bigcap_{n=1}^{\infty} A_n$

$$\implies \forall n \in \mathbb{N} -\frac{1}{n} < x < \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} -\frac{1}{n} < \lim_{n \rightarrow \infty} x < \lim_{n \rightarrow \infty} \frac{1}{n}$$

$$\text{We know } \lim_{n \rightarrow \infty} -\frac{1}{n} = 0$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Since x is constant(w.r.t n) we have $\lim_{n \rightarrow \infty} x = x$

Also by sandwich theorem we have $\lim_{n \rightarrow \infty} x = 0$

Since limit of a sequence if exists is unique we have $x = 0$

Hence $\bigcap_{n=1}^{\infty} A_n \subset \{0\}$ (2)

From 1 and 2 we have

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$