

Consider the sequence of intervals $A_n = (-\frac{1}{n}, \frac{1}{n})$

Claim 1: $A_{n+1} \subset A_n$

Proof:

Let $x \in A_{n+1}$

$$\implies -\frac{1}{n+1} < x < \frac{1}{n+1}$$

$$\implies -\frac{1}{n} < x < \frac{1}{n} \text{ as } \frac{1}{n} > \frac{1}{n+1}$$

$$\implies x \in A_n$$

$$\implies A_{n+1} \subset A_n$$

Claim2:- $\bigcap_{n=1}^{\infty} A_n = \{0\}$

proof :-

Clearly $\forall n \in \mathbb{N} \ 0 \in A_n$.

$$\implies 0 \in \bigcap_{n=1}^{\infty} A_n$$

$$\implies \{0\} \subset \bigcap_{n=1}^{\infty} A_n \quad (1)$$

Let $x \in \bigcap_{n=1}^{\infty} A_n$

$$\implies \forall n \in \mathbb{N} \ -\frac{1}{n} < x < \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} -\frac{1}{n} < \lim_{n \rightarrow \infty} x < \lim_{n \rightarrow \infty} \frac{1}{n}$$

We know $\lim_{n \rightarrow \infty} -\frac{1}{n} = 0$

and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Since x is constant (w.r.t n) we have $\lim_{n \rightarrow \infty} x = x$

Also by sandwich theorem we have $\lim_{n \rightarrow \infty} x = 0$

Since limit of a sequence if exists is unique we have $x = 0$

Hence $\bigcap_{n=1}^{\infty} A_n \subset \{0\} \quad (2)$

From 1 and 2 we have

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$