Assignment 1 - D94

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Question 1 – Gaussian Distribution

We are given the probability distribution:

$$\rho(x) = Ae^{-\lambda(x-a)^2}$$

where $A, \lambda, a \in \mathbb{R}$. Our goal is to:

(a) Normalize the distribution

To ensure $\rho(x)$ is a valid probability density function:

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1$$

Substitute the form of $\rho(x)$:

$$\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2} dx = 1$$

Let $u = x - a$, so that $dx = du$. Then:

$$\int_{-\infty}^{\infty} Ae^{-\lambda u^2} du = 1$$

We use the known Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-\lambda u^2} du = \sqrt{\frac{\pi}{\lambda}}$$

Thus:

$$A\sqrt{\frac{\pi}{\lambda}} = 1 \quad \Rightarrow \quad A = \sqrt{\frac{\lambda}{\pi}}$$

(b) Compute the expectation values and standard deviation

Expectation value $\langle x \rangle$:

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx = \int_{-\infty}^{\infty} xAe^{-\lambda(x-a)^2}dx$$

Let $u = x - a \Rightarrow x = u + a$, $dx = du$:

$$\langle x \rangle = A\int_{-\infty}^{\infty} (u+a)e^{-\lambda u^2}du = A\left(\int ue^{-\lambda u^2}du + a\int e^{-\lambda u^2}du\right)$$

Note: $ue^{-\lambda u^2}$ is an odd function, so over symmetric bounds:

$$\int_{-\infty}^{\infty} ue^{-\lambda u^2}du = 0$$

Then:

$$\langle x \rangle = A \cdot a \cdot \sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\lambda}{\pi}} \cdot a \cdot \sqrt{\frac{\pi}{\lambda}} = a$$

Expectation value $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2\rho(x)dx = \int_{-\infty}^{\infty} (u+a)^2Ae^{-\lambda u^2}du = A\int_{-\infty}^{\infty} (u^2+2au+a^2)e^{-\lambda u^2}du$$

Breaking it into three terms:

$$\int u^2e^{-\lambda u^2}du = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} \quad \text{(see rough work)}$$
$$\int ue^{-\lambda u^2}du = 0 \quad \text{(odd function)}$$
$$\int e^{-\lambda u^2}du = \sqrt{\frac{\pi}{\lambda}}$$

So:

$$\langle x^2 \rangle = A\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + a^2\sqrt{\frac{\pi}{\lambda}}\right) = \sqrt{\frac{\lambda}{\pi}}\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + a^2\sqrt{\frac{\pi}{\lambda}}\right) = \frac{1}{2\lambda} + a^2$$

Standard deviation σ_x:

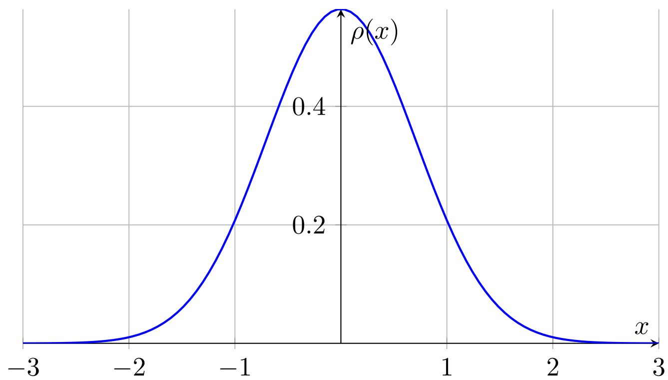
$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \left(\frac{1}{2\lambda} + a^2\right) - a^2 = \frac{1}{2\lambda} \quad \Rightarrow \quad \sigma_x = \frac{1}{\sqrt{2\lambda}}$$

(c) Sketch $\rho(x)$

We plot:

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}} e^{-\lambda(x-a)^2}$$

This is a bell curve centered at $x = a$, with: - Height: $\rho(a) = \sqrt{\frac{\lambda}{\pi}}$ - Spread: $\sigma = \frac{1}{\sqrt{2\lambda}}$



Rough Work and Side Derivations

1. Derivation of the Gaussian Integral

Define:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx \Rightarrow I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy$$

Switch to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta \quad \Rightarrow x^2 + y^2 = r^2, \quad dxdy = r\, dr\, d\theta$$

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2}\, dr$$

Substitute $u = r^2 \Rightarrow du = 2r\, dr \Rightarrow r\, dr = \frac{1}{2} du$:

$$\int_0^{\infty} re^{-r^2}\, dr = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}$$

Thus:

$$I^2 = 2\pi \cdot \frac{1}{2} = \pi \quad \Rightarrow \quad I = \sqrt{\pi}$$

2. Derivation of $\int u^2 e^{-\lambda u^2} du$

Let:

$$I(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda u^2} du = \sqrt{\frac{\pi}{\lambda}}$$

Differentiate both sides with respect to λ:

$$\frac{\mathrm{d}I}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int e^{-\lambda u^2} du = \int \frac{\mathrm{d}}{\mathrm{d}\lambda} e^{-\lambda u^2} du = \int -u^2 e^{-\lambda u^2} du = -\int u^2 e^{-\lambda u^2} du$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\sqrt{\frac{\pi}{\lambda}} = -\frac{1}{2}\sqrt{\pi}\lambda^{-3/2} \Rightarrow \int u^2 e^{-\lambda u^2} du = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$$

Question 2

2(a) - Normalize the Wave Function

We are given the piecewise wave function:

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a \\ A\frac{b-x}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

To normalize the wave function, we require:

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 1$$

Since $\Psi(x,0) = 0$ outside of $[0, b]$, we only integrate over that range:

$$\int_0^a \left(\frac{Ax}{a}\right)^2 dx + \int_a^b \left(\frac{A(b-x)}{b-a}\right)^2 dx = 1$$

First integral:

$$\int_0^a \left(\frac{Ax}{a}\right)^2 dx = A^2 \int_0^a \frac{x^2}{a^2} dx = \frac{A^2}{a^2}\int_0^a x^2 dx = \frac{A^2}{a^2} \cdot \left[\frac{x^3}{3}\right]_0^a = \frac{A^2 a^3}{3a^2} = \frac{A^2 a}{3}$$

Second integral:

$$\int_a^b \left(\frac{A(b-x)}{b-a}\right)^2 dx = A^2 \int_a^b \frac{(b-x)^2}{(b-a)^2} dx = \frac{A^2}{(b-a)^2}\int_a^b (b-x)^2 dx$$

Letting $u = b - x \Rightarrow du = -dx$, the limits change:

$$x = a \Rightarrow u = b - a, \quad x = b \Rightarrow u = 0$$

$$\Rightarrow \frac{A^2}{(b-a)^2}\int_{b-a}^{0} u^2(-du) = \frac{A^2}{(b-a)^2}\int_{0}^{b-a} u^2 du = \frac{A^2}{(b-a)^2}\cdot\left[\frac{u^3}{3}\right]_0^{b-a} = \frac{A^2(b-a)}{3}$$

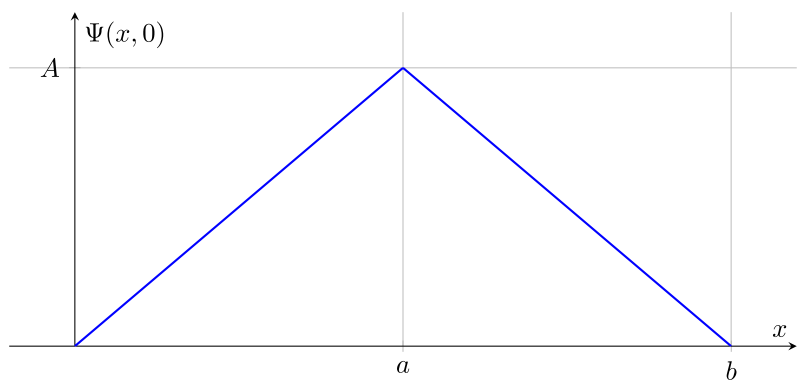
Total:

$$\frac{A^2 a}{3} + \frac{A^2(b-a)}{3} = 1 \Rightarrow \frac{A^2 b}{3} = 1 \Rightarrow A^2 = \frac{3}{b} \Rightarrow A = \sqrt{\frac{3}{b}}$$

2(b) – Sketch of the Wave Function

The function $\Psi(x,0)$ is piecewise linear and defined only over the interval $[0,b]$, with a peak at $x=a$ where the amplitude reaches $\Psi(a) = A = \sqrt{3/b}$. The function is zero outside of this range.

Below is a qualitative sketch of the wave function:



This sketch uses symbolic values $a=1$ and $b=2$ to illustrate the shape. It is a triangular "tent" shape: - Rising linearly from $(0,0)$ to (a,A) - Falling linearly from (a,A) to $(b,0)$

The wave function is symmetric if $a=\frac{b}{2}$, and the maximum probability density is at $x=a$.

2(c) – Most Likely Position of the Particle at $t=0$

The probability of finding the particle at position x is proportional to the probability density $|\Psi(x,0)|^2$. The particle is most likely to be found where this density reaches its maximum value.

From the definition of the wave function:

$$\Psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a \\ A\frac{b-x}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

This function increases linearly from 0 to A as x goes from 0 to a, then decreases linearly from A to 0 from $x = a$ to $x = b$. Therefore, the maximum of $|\Psi(x, 0)|^2$ occurs at $x = a$, where the wave function reaches its peak.

Conclusion: The particle is most likely to be found at position $\boxed{x = a}$ at time $t = 0$.

2(d) – Probability of Finding the Particle to the Left of a

We are asked to find the probability of locating the particle in the region $0 \leq x < a$. This is given by:

$$P(x < a) = \int_0^a |\Psi(x, 0)|^2 dx$$

From earlier,

$$\Psi(x, 0) = A \cdot \frac{x}{a} \quad \text{for } 0 \leq x \leq a \Rightarrow |\Psi(x, 0)|^2 = A^2 \cdot \frac{x^2}{a^2}$$

Therefore,

$$P(x < a) = \int_0^a A^2 \cdot \frac{x^2}{a^2} dx = \frac{A^2}{a^2} \int_0^a x^2 dx = \frac{A^2}{a^2} \cdot \left[\frac{x^3}{3} \right]_0^a = \frac{A^2 a^3}{3a^2} = \frac{A^2 a}{3}$$

From part (a), we know:

$$A^2 = \frac{3}{b} \Rightarrow P(x < a) = \frac{3}{b} \cdot \frac{a}{3} = \frac{a}{b}$$

Conclusion: The probability of finding the particle to the left of a is $\boxed{\frac{a}{b}}$.

Limiting Cases

Case 1: $b = a$ Then $P(x < a) = \frac{a}{a} = 1$. The particle must be in this region, as the wave function vanishes for $x > a$.

Case 2: $b = 2a$ Then $P(x < a) = \frac{a}{2a} = \frac{1}{2}$, which is consistent with a symmetric triangular wave function.

2(e) – Expectation Value of x

We want to compute the expectation value:

$$\langle x \rangle = \int_0^b x \cdot |\Psi(x, 0)|^2 \, dx = \int_0^a x \left(\frac{Ax}{a} \right)^2 dx + \int_a^b x \left(\frac{A(b - x)}{b - a} \right)^2 dx$$

From part (a), we know $A^2 = \frac{3}{b}$. We'll substitute this immediately.

First integral (from 0 to a):

$$\int_0^a x \cdot \left(\frac{Ax}{a}\right)^2 dx = A^2 \int_0^a \frac{x^3}{a^2} dx = \frac{A^2}{a^2} \cdot \left[\frac{x^4}{4}\right]_0^a = \frac{A^2 a^4}{4a^2} = \frac{A^2 a^2}{4} = \frac{3}{b} \cdot \frac{a^2}{4} = \frac{3a^2}{4b}$$

Second integral (from a to b):

$$\int_a^b x \cdot \left(\frac{A(b-x)}{b-a}\right)^2 dx = \frac{A^2}{(b-a)^2} \int_a^b x(b-x)^2 dx$$

First expand $(b-x)^2 = b^2 - 2bx + x^2$, then:

$$\int_a^b x(b-x)^2 dx = \int_a^b x(b^2 - 2bx + x^2) dx = b^2 \int_a^b x\, dx - 2b \int_a^b x^2 dx + \int_a^b x^3 dx$$

Now compute each:

$$\int_a^b x\, dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2}$$

$$\int_a^b x^2 dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3 - a^3}{3}$$

$$\int_a^b x^3 dx = \left[\frac{x^4}{4}\right]_a^b = \frac{b^4 - a^4}{4}$$

So:

$$\int_a^b x(b-x)^2 dx = b^2 \cdot \frac{b^2 - a^2}{2} - 2b \cdot \frac{b^3 - a^3}{3} + \frac{b^4 - a^4}{4}$$

Then the second integral becomes:

$$\frac{A^2}{(b-a)^2} \left[\frac{b^2(b^2 - a^2)}{2} - \frac{2b(b^3 - a^3)}{3} + \frac{b^4 - a^4}{4}\right] = \frac{3}{b(b-a)^2} \left[\frac{b^2(b^2 - a^2)}{2} - \frac{2b(b^3 - a^3)}{3} + \frac{b^4 - a^4}{4}\right]$$

Final expression:

$$\langle x \rangle = \frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{b^2(b^2 - a^2)}{2} - \frac{2b(b^3 - a^3)}{3} + \frac{b^4 - a^4}{4}\right]$$

This is the explicit formula for $\langle x \rangle$, fully in terms of a and b.

Question 3 – Wave Function Collapse

We are given a wave function at $t = 0$:

$$\psi(x, 0) = C \left(\frac{1}{\sqrt{2}} \psi_{E=1}(x) + e^{i\alpha} \psi_{E=2}(x)\right)$$

where $\psi_{E=1}(x)$ and $\psi_{E=2}(x)$ are orthonormal energy eigenfunctions, and C and α are real constants.

3(a) – Determine the Normalization Constant C

To normalize the wave function:

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = 1$$

We compute:

$$|\psi(x,0)|^2 = \left| C\left(\frac{1}{\sqrt{2}}\psi_{E=1}(x) + e^{i\alpha}\psi_{E=2}(x)\right)\right|^2 = |C|^2 \left|\frac{1}{\sqrt{2}}\psi_{E=1}(x) + e^{i\alpha}\psi_{E=2}(x)\right|^2$$

Expanding:

$$= |C|^2 \left[\frac{1}{2}|\psi_{E=1}|^2 + |e^{i\alpha}|^2|\psi_{E=2}|^2 + \frac{1}{\sqrt{2}}e^{-i\alpha}\psi^*_{E=1}\psi_{E=2} + \frac{1}{\sqrt{2}}e^{i\alpha}\psi^*_{E=2}\psi_{E=1}\right]$$

Because the eigenfunctions are orthonormal:

$$\int \psi^*_{E=1}\psi_{E=2} dx = 0, \quad \int |\psi_{E=1}|^2 dx = \int |\psi_{E=2}|^2 dx = 1$$

So the integral becomes:

$$\int |\psi(x,0)|^2 dx = |C|^2\left(\frac{1}{2}+1\right) = |C|^2\left(\frac{3}{2}\right) \Rightarrow |C|^2 = \frac{2}{3} \quad \Rightarrow \boxed{C = \sqrt{\frac{2}{3}}}$$

3(b) – Energy Measurement Outcomes and Probabilities

Since $\psi(x,0)$ is a linear combination of $\psi_{E=1}$ and $\psi_{E=2}$, the only possible outcomes of an energy measurement are:

- $E = E_1$ with probability:

$$P(E_1) = |\langle \psi_{E=1}|\psi|\psi_{E=1}|\psi\rangle|^2 = \left|C \cdot \frac{1}{\sqrt{2}}\right|^2 = \frac{2}{3}\cdot\frac{1}{2} = \boxed{\frac{1}{3}}$$

- $E = E_2$ with probability:

$$P(E_2) = |\langle \psi_{E=2}|\psi|\psi_{E=2}|\psi\rangle|^2 = |C \cdot 1|^2 = \boxed{\frac{2}{3}}$$

3(c) – Wave Function After Position Measurement

Immediately after a **position measurement**, the system is no longer in a superposition of energy eigenstates. The act of measuring position collapses the wave function into a sharply localized state (delta-like in the ideal case) at the measured position x_0.

This collapsed state is no longer an energy eigenstate, and can be expressed (formally) as:

$$\psi(x) \to \delta(x - x_0)$$

This new wave function is not stationary and will evolve in time according to the time-dependent Schrödinger equation. The energy of the system is now uncertain (spread over many eigenstates), but the position is momentarily well defined.

Question 4 – Ehrenfest's Theorem and Quantum Momentum

We are given the definition of the expectation value of position:

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 \, dx$$

We are asked to use the Schrödinger equation to show that:

$$\frac{\mathrm{d}\langle p \rangle}{\mathrm{d}t} = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

This is a statement of **Ehrenfest's theorem**, which shows that expectation values in quantum mechanics evolve according to classical equations of motion.

Step 1: Define $\langle p \rangle$

In quantum mechanics, the momentum operator is:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

So the expectation value of momentum is:

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x,t) dx$$

Step 2: Time derivative of $\langle p \rangle$

Differentiate under the integral sign:

$$\frac{\mathrm{d}\langle p \rangle}{\mathrm{d}t} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \left[\Psi^* \left(-i\hbar \frac{\partial}{\partial x} \Psi \right) \right] dx$$

Using the product rule:

$$= \int_{-\infty}^{\infty} \left[\left(\frac{\partial \Psi^*}{\partial t} \right) (-i\hbar \frac{\partial \Psi}{\partial x}) + \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \frac{\partial \Psi}{\partial t} \right) \right] dx$$

Step 3: Use Schrödinger's Equation

Time-dependent Schrödinger equation (TISE):

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi$$

Taking complex conjugate:

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi^*$$

Substitute into our expression:
Let's break it into two terms:
First term:

$$\int\left(\frac{\partial\Psi^*}{\partial t}\right)(-i\hbar\frac{\partial\Psi}{\partial x})dx = \int\left(\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V(x)\Psi^*\right]\right)(-i\hbar\frac{\partial\Psi}{\partial x})dx$$

Second term:

$$\int\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\frac{\partial\Psi}{\partial t}\right)dx = \int\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\left(\frac{-i}{\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi\right]\right)\right)dx$$

Simplify both terms and combine: - All kinetic terms cancel out symmetrically (after integration by parts). - The remaining potential terms yield:

$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \int\Psi^*(x,t)\left(-\frac{\partial V(x)}{\partial x}\right)\Psi(x,t)dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Conclusion:

We have shown that:

$$\boxed{\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = \left\langle -\frac{\partial V}{\partial x}\right\rangle}$$

This result confirms **Ehrenfest's theorem**, bridging quantum and classical mechanics. It tells us that even though a quantum particle does not follow a definite trajectory, the expectation values evolve according to Newton's second law:

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}$$

Question 5: Damped Harmonic Oscillator

Step 1: Define the Physical System

We model a mass-spring-damper system with:

- Mass: m

- Spring constant: k
- Damping coefficient: γ

The total force acting on the mass is:

$$F = -kx - \gamma\dot{x} = m\ddot{x}$$

Rearranging:

$$\boxed{m\ddot{x} + \gamma\dot{x} + kx = 0}$$

Dividing by m:

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

where $2\beta = \gamma/m$ and $\omega_0^2 = k/m$.

Step 2: Characteristic Equation

Guessing a solution of the form $x(t) = e^{rt}$, we get the characteristic equation:

$$r^2 + 2\beta r + \omega_0^2 = 0 \quad \Rightarrow \quad r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

The discriminant $\Delta = \beta^2 - \omega_0^2$ gives three cases:

Step 3: Three Damping Cases

Case 1: Underdamped ($\beta^2 < \omega_0^2$)

Roots are complex:

$$r = -\beta \pm i\omega, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

General solution:

$$x(t) = e^{-\beta t}(Ae^{i\omega t} + Be^{-i\omega t})$$

Using Euler's formula:

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t), \quad e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$$

$$\Rightarrow Ae^{i\omega t} + Be^{-i\omega t} = (A+B)\cos(\omega t) + i(A-B)\sin(\omega t)$$

Let $C = A + B$, $D = i(A - B)$ (real if $B = \bar{A}$):

$$\boxed{x(t) = e^{-\beta t}(C\cos(\omega t) + D\sin(\omega t))}$$

Case 2: Critically Damped ($\beta^2 = \omega_0^2$)

Repeated real root $r = -\beta$:

$$x(t) = (A + Bt)e^{-\beta t}$$

Case 3: Overdamped ($\beta^2 > \omega_0^2$)

Two real roots:

$$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

Step 4: Sketch of the Three Cases (Plotted in Desmos)

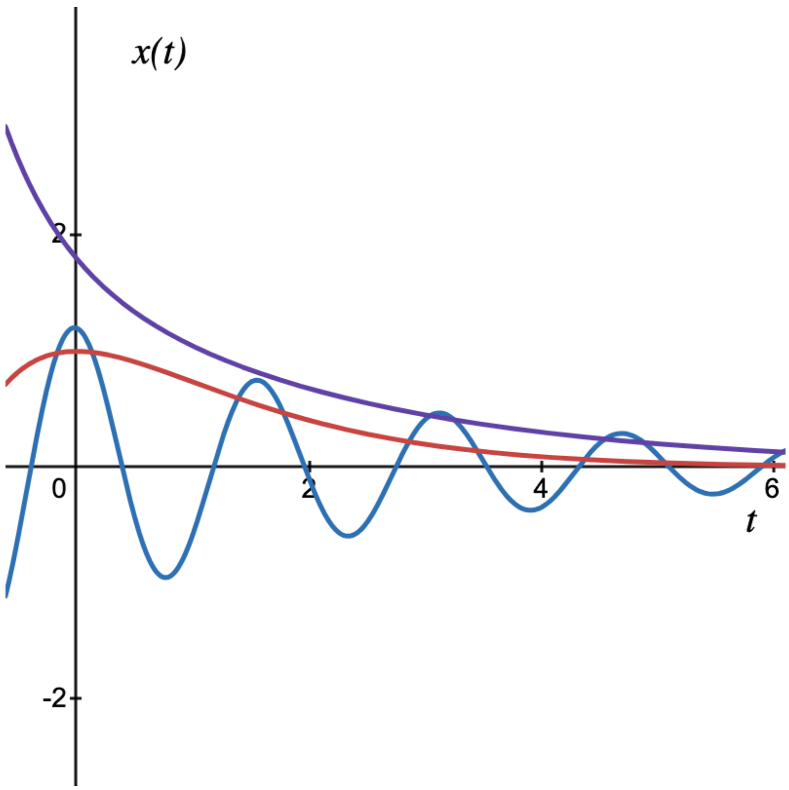


Figure 1: Comparison of underdamped (blue), critically damped (red), and overdamped (purple) systems.

Visual Validation Table

Curve Color	Behavior	Equation	Notes
Blue	Oscillates, decays	$1.2e^{-0.3t}\cos(4t)$	✓ Underdamped: correct amplitude decay and zero-crossings
Red	Non-oscillating, fastest	$(1+t)e^{-t}$	✓ Critically damped: fast, curved descent without oscillation
Purple	Non-oscillating, slower	$1.5e^{-0.4t}+0.3e^{-2t}$	✓ Overdamped: slower decay with no inflection or overshoot

Question 6 – Double-Slit Experiment

6(a) – Classical Particle Intensity

When only slit S_1 is open, the detector records intensity $I_1(x)$. When only slit S_2 is open, the detector records intensity $I_2(x)$. Since particles are assumed to arrive independently, the total intensity when both slits are open is the sum of individual intensities:

$$\boxed{I(x) = I_1(x) + I_2(x)}$$

This is a classical (non-interfering) result.

6(b) – Quantum or Wave Case: Superposition

For waves (e.g. electrons or photons), the amplitude at the detector with both slits open is:

$$\psi(x) = \psi_1(x) + \psi_2(x)$$

The total intensity is proportional to the square of the modulus of the wave function:

$$I(x) = |\psi(x)|^2 = |\psi_1(x) + \psi_2(x)|^2$$

We now expand this modulus using complex algebra. Recall that:

$$|\psi_1 + \psi_2|^2 = (\psi_1 + \psi_2)(\psi_1 + \psi_2)^*$$

Now applying the distributive property:

$$(\psi_1 + \psi_2)(\psi_1^* + \psi_2^*) = \psi_1\psi_1^* + \psi_2\psi_2^* + \psi_1\psi_2^* + \psi_2\psi_1^*$$

Rewriting:

$$|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1$$

Note that:

$$\psi_1^*\psi_2 + \psi_2^*\psi_1 = 2\,\mathrm{Re}(\psi_1^*\psi_2)$$

This is because the two cross-terms are complex conjugates of each other, and their sum yields twice the real part. To verify:

Let $\psi_1 = a + ib$, $\psi_2 = c + id$. Then:

$$\psi_1^* \psi_2 = (a - ib)(c + id) = ac + bd + i(ad - bc)$$

$$\psi_2^* \psi_1 = (c - id)(a + ib) = ac + bd - i(ad - bc)$$

Adding them:

$$\psi_1^* \psi_2 + \psi_2^* \psi_1 = 2ac + 2bd = 2\,\mathrm{Re}(\psi_1^* \psi_2)$$

Therefore:

$$|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\,\mathrm{Re}(\psi_1^* \psi_2)$$

Letting $I_1 = |\psi_1|^2$, $I_2 = |\psi_2|^2$, we conclude:

$$\boxed{I(x) = I_1(x) + I_2(x) + 2\,\mathrm{Re}(\psi_1^*(x)\psi_2(x))}$$

Unlike the classical case, we now have an interference term $2\,\mathrm{Re}(\psi_1^* \psi_2)$ which can constructively or destructively affect the total intensity at each point x. This is the signature feature of quantum interference and is what gives rise to the fringe patterns observed in double-slit experiments.

6(c) – Explicit Form of $I(x)$ with Full Expansion

We are interested in computing the intensity:

$$I(x) = |\psi(x,t)|^2 = \left| C e^{-i\omega t} \left(e^{ikr_1} + e^{ikr_2} \right) \right|^2$$

Using $|ab|^2 = |a|^2 |b|^2$, and noting that $|e^{-i\omega t}|^2 = 1$, we get:

$$I(x) = |C|^2 \cdot \left| e^{ikr_1} + e^{ikr_2} \right|^2$$

Let:

$$z = e^{i\theta_1} + e^{i\theta_2}, \quad \text{with } \theta_1 = kr_1,\ \theta_2 = kr_2$$

Derivation of Identity: $|e^{i\theta_1} + e^{i\theta_2}|^2 = 2 + 2\cos(\theta_1 - \theta_2)$

We aim to compute:

$$|e^{i\theta_1} + e^{i\theta_2}|^2$$

Let:

$$z = e^{i\theta_1} + e^{i\theta_2}$$

Then:

$$|z|^2 = z \cdot z^* = \left(e^{i\theta_1} + e^{i\theta_2} \right) \left(e^{-i\theta_1} + e^{-i\theta_2} \right)$$

Expand the product:

$$\begin{aligned} |z|^2 &= e^{i\theta_1} e^{-i\theta_1} + e^{i\theta_1} e^{-i\theta_2} + e^{i\theta_2} e^{-i\theta_1} + e^{i\theta_2} e^{-i\theta_2} \\ &= 1 + e^{i(\theta_1 - \theta_2)} + e^{-i(\theta_1 - \theta_2)} + 1 \\ &= 2 + e^{i(\theta_1 - \theta_2)} + e^{-i(\theta_1 - \theta_2)} \end{aligned}$$

Now let:

$$\phi = \theta_1 - \theta_2$$

So:

$$|z|^2 = 2 + e^{i\phi} + e^{-i\phi}$$

Now use Euler's identity to expand both exponentials:

$$e^{i\phi} = \cos(\phi) + i\sin(\phi), \quad e^{-i\phi} = \cos(-\phi) + i\sin(-\phi)$$

Recall:

$$\cos(-\phi) = \cos(\phi), \quad \sin(-\phi) = -\sin(\phi)$$

So we have:

$$\begin{aligned} e^{i\phi} + e^{-i\phi} &= (\cos(\phi) + i\sin(\phi)) + (\cos(\phi) - i\sin(\phi)) \\ &= 2\cos(\phi) \end{aligned}$$

Therefore:

$$|z|^2 = 2 + 2\cos(\phi) = 2 + 2\cos(\theta_1 - \theta_2)$$

$$\boxed{\left|e^{i\theta_1} + e^{i\theta_2}\right|^2 = 2 + 2\cos(\theta_1 - \theta_2)}$$

Breakdown of imaginary cancellation:
If you want to verify why the imaginary parts cancel out in $e^{i\phi} + e^{-i\phi}$, write:

$$\begin{aligned} e^{i\phi} + e^{-i\phi} &= \cos(\phi) + i\sin(\phi) + \cos(\phi) - i\sin(\phi) \\ &= 2\cos(\phi) + i(\sin(\phi) - \sin(\phi)) \\ &= 2\cos(\phi) \end{aligned}$$

The imaginary part vanishes:

$$i\sin(\phi) + i\sin(-\phi) = i(\sin(\phi) - \sin(\phi)) = 0$$

Hence, replacing back:

$$\left|e^{ikr_1} + e^{ikr_2}\right|^2 = 2 + 2\cos(k(r_1 - r_2))$$

And therefore:

$$I(x) = |C|^2 \cdot (2 + 2\cos(k(r_1 - r_2))) = 2|C|^2\,(1 + \cos(k(r_1 - r_2)))$$

$$\boxed{I(x) = 2|C|^2\,[1 + \cos(k(r_1 - r_2))]}$$

Conclusion:
This result shows that the intensity exhibits an interference pattern governed by the cosine of the phase difference between the two paths. Maxima occur when $k(r_1 - r_2) = 2n\pi$, and minima when $k(r_1 - r_2) = (2n + 1)\pi$ for integers n.

6(d) – Interference Maxima and Minima

(i) If $k(r_1 - r_2) = 2n\pi \Rightarrow \cos = 1 \Rightarrow I(x) = 4|C|^2 \Rightarrow$ *Constructive interference (bright fringe)*

(ii) If $k(r_1 - r_2) = (2n+1)\pi \Rightarrow \cos = -1 \Rightarrow I(x) = 0 \Rightarrow$ *Destructive interference (dark fringe)*

Conclusion:

- Unlike the classical case where intensities add, wave interference produces regions of complete cancellation or enhancement. - This highlights the wave nature of quantum particles and reflects the **non-additive** behavior of probability amplitudes.

7. Wave-Particle Duality – de Broglie Wavelength

The de Broglie wavelength is given by:

$$\lambda = \frac{h}{mv}$$

where:

- $h = 6.626 \times 10^{-34}$ J $\cdot$ s is Planck's constant,
- m is the mass of the particle,
- v is the speed of the particle.

—

(a) Electron – $m_e = 9.11 \times 10^{-31}$ kg

Speed	**de Broglie Wavelength**
1 km/h = 0.278 m/s	$\lambda = \frac{6.626\times10^{-34}}{9.11\times10^{-31}\cdot0.278} \approx 2.62 \times 10^{-3}$ m
$0.1c$	$\lambda \approx \frac{6.626\times10^{-34}}{9.11\times10^{-31}\cdot(0.1\cdot3\times10^{8})} \approx 2.43 \times 10^{-11}$ m
$0.5c$	$\lambda \approx 4.85 \times 10^{-12}$ m
$0.75c$	$\lambda \approx 3.24 \times 10^{-12}$ m
$0.99c$	$\lambda \approx 2.45 \times 10^{-12}$ m

(b) Macroscopic Object – $m = 1$ kg

Speed	**de Broglie Wavelength**
1 km/h = 0.278 m/s	$\lambda = \frac{6.626\times10^{-34}}{1\cdot0.278} \approx 2.38 \times 10^{-33}$ m
$0.1c$	$\lambda \approx \frac{6.626\times10^{-34}}{1\cdot0.1\cdot3\times10^{8}} = 2.21 \times 10^{-41}$ m
$0.5c$	$\lambda \approx 4.42 \times 10^{-42}$ m
$0.75c$	$\lambda \approx 2.95 \times 10^{-42}$ m
$0.99c$	$\lambda \approx 2.23 \times 10^{-42}$ m

(c) Photons – EM Radiation Wavelengths

Photons have no rest mass. Their de Broglie wavelength is equal to their electromagnetic wavelength:

Light Type	**Wavelength**
Red light	≈ 700 nm
Blue light	≈ 450 nm
UV light	≈ 100 nm
X-rays	≈ 0.1 nm

Conclusion

- The de Broglie wavelength decreases as mass or velocity increases.
- Electrons moving at slow speeds (e.g., 1 km/h) can have wavelengths on the millimeter or micrometer scale and thus interact with nano- and microscale structures (e.g., atomic lattices, thin films).
- Macroscopic objects (1 kg) have wavelengths many orders of magnitude below nuclear or even Planck-scale dimensions and therefore cannot show observable wave-like behavior.
- The photons' wavelengths correspond to typical sizes of molecules (visible light), DNA/proteins (UV), and atoms/nuclei (X-rays).

Hence, the wave-like behavior of matter is only significant for microscopic particles like electrons, and becomes negligible for macroscopic masses.