Introduction to Quantum Computing – Assignment # 1

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Problem Set #1 – Preliminar Topics & Review:
Linear Algebra, Probability & Statistics, ODEs & PDEs, Quantum Mechanics

1 Probability and Statistics

1. Consider the *Gaussian* distribution,

$$\rho(x) = Ae^{-\lambda(x-a)^2} \tag{1}$$

where A, a and λ are real constants.

(a) Assuming $\rho(x)$ represents a proper porbability distribution function, determine the value of A.

(b) Calculate the expected values: $\langle x \rangle$, $\langle x^2 \rangle$, and corresponding standar deviations: σ_x, σ_{x^2}.

(c) Sketch the plot of $\rho(x)$.

2 Wave Function

2.1 Normalization

2. At time $t = 0$ a particle is represented by the *wave function*,

$$\Psi(x,0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & otherwise, \end{cases} \tag{2}$$

where A, a and b are (positive) constants.

(a) Normalize the wave function Ψ; i.e. find A in terms of a and b.

(b) Sketch $\Psi(x,0)$, as a function of x.

(c) Where is the particle most likely to be found at $t = 0$?

(d) What is the probability of finding the particle to the lrft of a? Check that your result makes sense in the limiting cases of $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

3. *Wave function collapse*
A quantum mechanical system is, at $t = 0$, prepared in a state described by the wave function

$$\psi(x, t = 0) = C\left(\frac{1}{\sqrt{2}}\psi_{E=1}(x) + e^{i\alpha}\psi_{E=2}(x)\right) \tag{3}$$

where the wave functions on the right-hand side are *orthonormal energy eigenfunctions*. Both C and α are real constants.

   (a) Determine the constant C.
   (b) An energy measurement is made.
   What are the possible outcomes, and what are the probabilities of those outcomes?
   (c) Subsequently, the position of the particle is measured.
   What do you know about the wave function of the system immediately after this measurement?

2.2 Momentum

4. In classical mechanics, we define the *momentum* of a particle of mass m, as $\vec{p} = m\vec{v}$. This concept can be generalized to quantum mechanics as,

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} \tag{4}$$

where $\langle x \rangle$ denotes the expectation value of x, for a given particle described by the wave function $\Psi(x, t)$ as,

$$\langle x \rangle = \int_{-\infty}^{+\infty} x|\Psi(x, t)|^2 dx \tag{5}$$

Using *Schrödinger equation*, probe that

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle \tag{6}$$

This is an instance of *Ehrenfest's theorem*, which asserts that *expectation values obey the classical laws*. I.e. how can we reconcile this with our expectation in classical mechanics?

3 ODEs

3.1 Damped Harmonic Oscillator

5. Derive the equations of motion for a harmonic oscillator (e.g. masss-spring system) in the presence of a dumping term proportional to the speed of the mass.
Solve the resulting ODE and sketch the solutions to this problem.

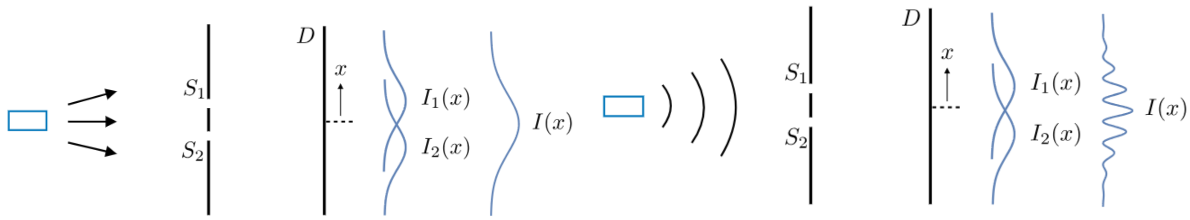

Figure 1: Double-slit experiment: for particles (left) and waves (right).

4 Modern Physics

4.1 Double-Slit Experiment

6. Let's consider a *double-slit experiment* as discussed in class.

   (a) First we will consider **particles** that are shot as shown in Fig. 1. There is a source emitting these particles at a uniform rate in random directions towards a screen with two small slits S_1 and S_2. The particles that pass through one of the slits arrive one at a time at a detector D on the other side of the screen.
   By averaging over time, the detector measures the rate that particles arrive per unit area, as a function of the vertical direction x. We call this the *intensity*.
   If the intensity measured with only S_1 open is $I_1(x)$, and the intensity measured with only S_2 open is $I_2(x)$; what would be the intensity $I(x)$ measured with both S_1 and S_2 open?

   (b) Now let's suppose instead that we have electrons or waves described by a wave function $\psi(x,t)$. There is a source emitting these uniformly towards the screen with the two small slits as described before.
   Assuming that the amplitude at the detector with only S_1 open is $\psi_1(x)$, and the amplitude at the detector with only S_2 open is $\psi_2(x)$; then we will call the amplitude at the detector with both S_1 and S_2 open $\psi(x)$.
   Assuming that the wave amplitude obeys a linear partial differential equation. Then the *principle of superposition* means that

   $$\psi(x) = \psi_1(x) + \psi_2(x) \tag{7}$$

   By averaging over a long period of time, the detector measures the rate that energy is deposited per unit area, as a function of the vertical direction x. We again call this the *intensity*. The energy carried by a wave is proportional to the modulus squared of the amplitude. Ignoring the constant of proportionality,

   $$I_1(x) = |\psi_1(x)|^2, \quad I_2(x) = |\psi_2(x)|^2, \quad I(x) = |\psi(x)|^2 \tag{8}$$

   Compute the total intensity $I(x)$ and compare with the one obtained for the classical particles from part a).

   (c) Assume now that you precisely know the waves $\psi_1(x)$ and $\psi_2(x)$,

   $$\psi_1(x,t) = Ce^{i(kr_1-\omega t)}, \quad \psi_2(x,t) = Ce^{i(kr_2-\omega t)} \tag{9}$$

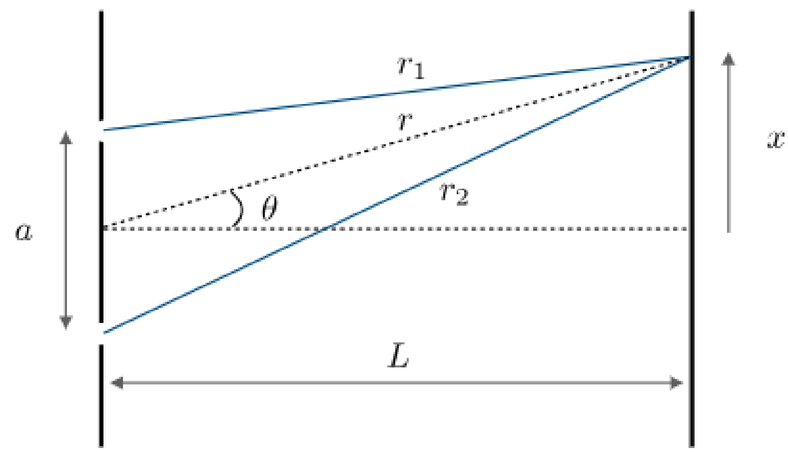

Figure 2: Variables used to compute $I(x)$.

where C is a normalization constant (that won't be relevant for the analysis); $k = \omega/v$, where ω is the angular frequency of the wave and v is the wave velocity; r_1, r_2 are the distances from the slits S_1, S_2 to a point on the detector at height x (see Fig. 2).
Provide an expression for the total intensity $I(x)$ for this specific case.

(d) Consider the following two cases:
   i. $k(r_1 - r_2) = 2n\pi$, and
   ii. $k(r_1 - r_2) = 2(n+1)\pi$;

   where $n \in \mathbb{Z}$.
   What can you conclude from this?
   What differences present with the particles-case study before?

4.2 Wave-Particle Duality

7. Calculate the *de Broglie* wave-length for the following cases:
   (a) an electron moving at 1 km/h, 10%, 50%, 75% and 99% of the speed of light.
   (b) a mass of 1kg moving at 1 km/h, 10%, 50%, 75% and 99% of the speed of light.
   (c) blue and red light, UV and X-rays.

   Which typical size of objects, do you estimate the former waves would be able to interact with?