Question 1: Valid Qubit States and Measurement Probabilities

Prompt: Determine which of the following are valid qubit states. For each valid state, provide:

- The probabilities of measuring $|0\rangle$ and $|1\rangle$ in the standard computational basis.
- The probabilities of measuring $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ in the Hadamard basis.

Note: A qubit state $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ is **valid** if and only if it is **normalized**, i.e., $|\alpha|^2 + |\beta|^2 = 1$. Additionally, since the Hadamard basis is orthonormal, we always have:

$$P(+) + P(-) = 1$$

Therefore, we compute $P(+)$ explicitly and use $P(-) = 1 - P(+)$.

(a) $\dfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Normalized: $\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1$
- Computational basis: $P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{2}$
- Hadamard basis:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = 1 \Rightarrow P(+) = 1, \quad P(-) = 0$$

(b) $\dfrac{\sqrt{3}}{2}|1\rangle - \dfrac{1}{2}|0\rangle$

- Normalized: $\frac{1}{4} + \frac{3}{4} = 1$
- Computational basis: $P(0) = \frac{1}{4}, \quad P(1) = \frac{3}{4}$
- Hadamard basis:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}} \Rightarrow P(+) = \left(\frac{-1+\sqrt{3}}{2\sqrt{2}}\right)^2 \approx 0.073, \quad P(-) \approx 0.927$$

(c) $0.7|0\rangle + 0.3|1\rangle$

- Not normalized: $0.7^2 + 0.3^2 = 0.58 \neq 1$

- Not a valid qubit state

(d) $0.8|0\rangle + 0.6|1\rangle$

- Normalized: $0.64 + 0.36 = 1$
- Computational basis: $P(0) = 0.64, \quad P(1) = 0.36$
- Hadamard basis:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}(0.8 + 0.6) = \frac{1.4}{\sqrt{2}} \Rightarrow P(+) = \left(\frac{1.4}{\sqrt{2}}\right)^2 = 0.98, \quad P(-) = 0.02$$

(e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$

- Normalized: $\cos^2\theta + \sin^2\theta = 1$
- Computational basis: $P(0) = \cos^2\theta, \quad P(1) = \sin^2\theta$
- Hadamard basis:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}(\cos\theta + i\sin\theta) \Rightarrow P(+) = \frac{1}{2}, \quad P(-) = \frac{1}{2}$$

(f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$

- Not normalized in general: $\cos^4\theta + \sin^4\theta \leq 1$
- Only valid for special values like $\theta = 0, \frac{\pi}{2}$; otherwise not a valid qubit state

(g) $\left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

- Normalized: $\left|\frac{1+i}{2}\right|^2 + \left|\frac{1-i}{2}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1$
- Computational basis: $P(0) = 0.5, \quad P(1) = 0.5$
- Hadamard basis:

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1+i}{2} + \frac{1-i}{2}\right) = \frac{1}{\sqrt{2}} \Rightarrow P(+) = \frac{1}{2}, \quad P(-) = \frac{1}{2}$$

Question 2: Post-Measurement State and Probability

We are given a two-qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{30}} \left(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle\right)$$

Step 1: Measurement of the first qubit yields 1
Only the components where the first qubit is $|1\rangle$ are retained:

$$|\psi'\rangle \propto -3|10\rangle - 4i|11\rangle = |1\rangle \otimes (-3|0\rangle - 4i|1\rangle)$$

This is the unnormalized post-measurement state.

Step 2: Normalize the second qubit's state
We isolate the second qubit's part:

$$|\phi\rangle = -3|0\rangle - 4i|1\rangle$$

Compute its norm:

$$\|\phi\| = \sqrt{|-3|^2 + |-4i|^2} = \sqrt{9+16} = \sqrt{25} = 5$$

So the normalized post-measurement state is:

$$|\psi'\rangle = |1\rangle \otimes \left(\frac{1}{5}(-3|0\rangle - 4i|1\rangle)\right)$$

Step 3: Probability that the second qubit is measured as $|1\rangle$
Let

$$|\phi\rangle = \frac{1}{5}(-3|0\rangle - 4i|1\rangle)$$

Then:

$$P(1) = \left|\frac{-4i}{5}\right|^2 = \frac{16}{25}$$

Final Answers:

- Post-measurement state:

$$|\psi'\rangle = |1\rangle \otimes \left(\frac{1}{5}(-3|0\rangle - 4i|1\rangle)\right)$$

- Probability that the second qubit is measured as $|1\rangle$: $\boxed{\frac{16}{25}}$

Question 3: Unitarity of Hadamard and Pauli Matrices

A matrix U is **unitary** if it satisfies:

$$U^\dagger U = I$$

where $U^\dagger$ is the conjugate transpose of U, and I is the identity matrix.

Hadamard matrix:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Its conjugate transpose is:

$$H^\dagger = H^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Multiplying:

$$H^\dagger H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, H is unitary.

Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Check each one:

$$X^\dagger X = X^T X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Y^\dagger Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z^\dagger Z = Z^T Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Conclusion: The Hadamard and all three Pauli matrices (X, Y, Z) satisfy $U^\dagger U = I$, and are therefore unitary.

Question 4: Matrix Representations of $I \otimes H$ and $H \otimes I$

Let:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

We compute the Kronecker (tensor) products.

1. $I \otimes H$:

$$I \otimes H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2. $H \otimes I$:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Conclusion: These matrices represent how the Hadamard gate acts on either the second qubit ($I \otimes H$) or the first qubit ($H \otimes I$) in a two-qubit system.

Question 5: Applying a Pauli-X Gate and Measuring Outcomes

We are given the initial two-qubit state:

$$|\psi\rangle = 0.8|00\rangle + 0.6|11\rangle$$

Step 1: Apply Pauli-X gate to the second qubit
The Pauli-X gate flips the second qubit:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Applying $I \otimes X$, we transform the state:

$$|00\rangle \to |01\rangle, \quad |11\rangle \to |10\rangle$$

$$|\psi'\rangle = 0.8|01\rangle + 0.6|10\rangle$$

Step 2: Measure in the computational basis

The possible measurement outcomes are $|01\rangle$ and $|10\rangle$. The probabilities are given by the squared magnitudes of the amplitudes:

$$P(01) = |0.8|^2 = 0.64, \quad P(10) = |0.6|^2 = 0.36$$

$$P(01) + P(10) = 1 \quad \text{(state is normalized)}$$

Final Answer:

- Probability of observing $|01\rangle$: $\boxed{0.64}$
- Probability of observing $|10\rangle$: $\boxed{0.36}$

Question 6: Show that $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ is not a product state

To show that the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is **not** a tensor product of two single-qubit states, suppose the contrary: that there exist single-qubit states

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\chi\rangle = \gamma|0\rangle + \delta|1\rangle$$

such that:

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$$

Compute the tensor product:

$$|\psi\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

Now compare with:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Matching coefficients:

$$\alpha\gamma = \frac{1}{\sqrt{2}}, \quad \alpha\delta = 0, \quad \beta\gamma = 0, \quad \beta\delta = \frac{1}{\sqrt{2}}$$

From $\alpha\delta = 0$, we conclude: either $\alpha = 0$ or $\delta = 0$ From $\beta\gamma = 0$, we conclude: either $\beta = 0$ or $\gamma = 0$

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Case 1: Suppose $\alpha = 0$. Then $\alpha\gamma = 0$, contradicts $\alpha\gamma = \frac{1}{\sqrt{2}}$
Case 2: Suppose $\delta = 0$. Then $\beta\delta = 0$, contradicts $\beta\delta = \frac{1}{\sqrt{2}}$

Case 3: Suppose $\beta = 0$. Then $\beta\delta = 0$, again contradicts $\beta\delta = \frac{1}{\sqrt{2}}$
Case 4: Suppose $\gamma = 0$. Then $\alpha\gamma = 0$, again contradicts $\alpha\gamma = \frac{1}{\sqrt{2}}$
—
Conclusion: Every possible assignment leads to a contradiction. Therefore,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

cannot be written as a tensor product of two single-qubit states. Hence, it is an **entangled** state.

Question 7: Matrix Form of the Controlled-Hadamard Gate

We are asked to construct the matrix for the two-qubit **controlled-Hadamard** (CH) gate, where:

- The **first qubit** is the *control*,
- The **second qubit** is the *target*,
- If the control qubit is $|0\rangle$, the state remains unchanged,
- If the control qubit is $|1\rangle$, the Hadamard gate H is applied to the second qubit.

Hadamard Matrix

Recall the Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Basis and Structure of the CH Gate

The CH gate acts on 2 qubits, so we work in the basis:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

This is equivalent to the standard ordering of computational basis states:

$$|0\rangle \otimes |0\rangle \to \text{index } 0, \quad |0\rangle \otimes |1\rangle \to \text{index } 1, \quad |1\rangle \otimes |0\rangle \to \text{index } 2, \quad |1\rangle \otimes |1\rangle \to \text{index } 3$$

Let's analyze how the gate must act:

Step 1: If control qubit is 0

This corresponds to the basis states $|00\rangle$ and $|01\rangle$. Since the control is 0, we must do nothing — i.e., leave the state unchanged.

Thus, the CH gate must satisfy:

$$CH\,|00\rangle = |00\rangle, \quad CH\,|01\rangle = |01\rangle$$

This means that the first two columns of the matrix must be:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These correspond to projecting $|00\rangle$ and $|01\rangle$ back onto themselves.

Step 2: If control qubit is 1

Now we consider the basis states $|10\rangle$ and $|11\rangle$. Since the control qubit is $|1\rangle$, we apply the Hadamard gate to the second qubit.

We write:

$$|10\rangle = |1\rangle \otimes |0\rangle \Rightarrow CH\,|10\rangle = |1\rangle \otimes H|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

$$|11\rangle = |1\rangle \otimes |1\rangle \Rightarrow CH\,|11\rangle = |1\rangle \otimes H|1\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

Thus, the third and fourth columns must be:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Final Matrix Form

Putting all four columns together, the matrix for the controlled-Hadamard gate is:

$$CH = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Summary and Interpretation

This gate performs the identity on the subspace where the control qubit is $|0\rangle$, and applies the Hadamard gate to the second qubit when the control is $|1\rangle$. The structure reflects this, as the matrix is block-diagonal:

$$CH = \begin{bmatrix} I & 0 \\ 0 & H \end{bmatrix}$$

with I and H each acting on the target qubit, conditional on the control state.

Question 8: Constructing CNOT from Hadamard and Controlled-Z Gates

We are asked to demonstrate that the controlled-NOT (CNOT) gate can be constructed from the Hadamard and controlled-Z (CZ) gates, and to confirm this by matrix multiplication.

Key Identity

The identity we want to prove is:

$$CNOT = (I \otimes H) \cdot CZ \cdot (I \otimes H)$$

That is: - Apply a Hadamard gate to the **target** qubit, - Apply a controlled-Z gate, - Then apply another Hadamard to the target qubit.

This circuit performs the same transformation as CNOT.

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Matrix Definitions

Hadamard:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Identity:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Tensor product $I \otimes H$:

$$I \otimes H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Controlled-Z gate:

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

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Matrix Multiplication

Now compute:

$$CNOT = (I \otimes H) \cdot CZ \cdot (I \otimes H)$$

Let $A = I \otimes H$, then:

$$CNOT = A \cdot CZ \cdot A$$

First compute $CZ \cdot A$:

$$CZ \cdot A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Now multiply with A from the left:

$$CNOT = A \cdot (CZ \cdot A) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is exactly the matrix for the CNOT gate.

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Conclusion

We have verified that:

$$CNOT = (I \otimes H) \cdot CZ \cdot (I \otimes H)$$

This demonstrates that the CNOT gate can be constructed using Hadamard and controlled-Z gates, and that the construction is correct via explicit matrix multiplication.

Question 9: Orthonormality of the Bell States

We are given the four Bell states:

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}$$

We want to verify that these states form an **orthonormal basis** for $\mathbb{C}^4$. That means:

1. Each state must be normalized: $\langle\psi|\psi\rangle = 1$
2. The states must be pairwise orthogonal: $\langle\psi_i|\psi_j\rangle = 0$ for $i \neq j$
3. There are 4 linearly independent states spanning the 4-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

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1. Normalization

We compute the norm of $|\Phi^+\rangle$:

$$\langle\Phi^+|\Phi^+\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle) = \frac{1}{2}(1 + 0 + 0 + 1) = 1$$

By the same argument, all Bell states are normalized:

$$\langle\psi|\psi\rangle = \frac{1}{2}(1 + 1) = 1$$

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2. Orthogonality

We now show that each pair of Bell states is orthogonal.

Example: $\langle\Phi^+|\Phi^-\rangle$

$$\langle\Phi^+|\Phi^-\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle - |11\rangle) = \frac{1}{2}(1 - 1) = 0$$

Example: $\langle\Phi^+|\Psi^+\rangle$

$$= \frac{1}{2}(\langle 00| + \langle 11|)(|01\rangle + |10\rangle) = \frac{1}{2}(0 + 0 + 0 + 0) = 0$$

All other combinations similarly yield 0 inner product.

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3. Basis for $\mathbb{C}^4$

There are 4 Bell states, and we have shown they are orthonormal. Therefore, they form an orthonormal basis of the 4-dimensional Hilbert space $\mathbb{C}^4$.

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Conclusion

The Bell states:

$$\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \right\}$$

form an orthonormal basis for $\mathbb{C}^4$.

Question 10: The No-Signalling Principle

Suppose Alice and Bob share the entangled Bell state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

where the first qubit belongs to Alice and the second to Bob. They are spatially separated (e.g., light-years apart), and Alice can choose to measure her qubit at any time.

The Question

Can Bob infer whether or not Alice has performed her measurement, by measuring only his own qubit? If yes, this would violate the **no-signalling principle**, which states that no information can be transmitted faster than light — even through entanglement.

Case 1: Alice does not measure her qubit

If Alice does nothing, Bob's reduced state is obtained by tracing out Alice's qubit from the joint state:

$$\rho_B = \mathrm{Tr}_A \left(|\psi\rangle\langle\psi| \right)$$

Explicitly, we compute:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \Rightarrow |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$$

Tracing out Alice's qubit yields:

$$\rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I$$

Bob's measurement outcomes are therefore:

$$P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{2}$$

Case 2: Alice does measure her qubit in the computational basis

If Alice measures and obtains $|0\rangle$, the entangled state collapses to:

$$|00\rangle \Rightarrow \text{Bob's qubit is } |0\rangle$$

If she obtains $|1\rangle$, it collapses to:

$$|11\rangle \Rightarrow \text{Bob's qubit is } |1\rangle$$

So from Bob's perspective, his qubit is either $|0\rangle$ or $|1\rangle$, each with probability $\frac{1}{2}$. This gives the same mixed state:

$$\rho_B = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I$$

Conclusion: No Signalling Occurs

In both cases — whether Alice measures or not — Bob's reduced density matrix is:

$$\rho_B = \frac{1}{2}I$$

This means Bob's measurement statistics are **completely unchanged** by Alice's choice to measure or not. Therefore:

- Bob cannot infer whether Alice measured.
- No information is transmitted between them.
- This supports the **no-signalling principle**.

Even though quantum entanglement allows for instant correlations, it does not allow for communication faster than light.

Question 11: The No-Cloning and No-Deletion Principles

No-Cloning Theorem

We suppose there exists a unitary operator U that can clone any arbitrary quantum state $|\psi\rangle$ using an ancilla (initialized to $|0\rangle$):

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

Let's assume this holds for at least two distinct (and non-orthogonal) states $|\psi\rangle$ and $|\phi\rangle$. Then:

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$$

Take the inner product of both sides:

$$\langle\psi|\phi\rangle = \langle\psi|\phi\rangle \Rightarrow \text{(initial states)}$$

$$\Rightarrow \langle\psi|\phi\rangle \cdot \langle 0|0\rangle = \langle\psi|\phi\rangle$$

Now take the inner product of the cloned states:

$$\langle\psi|\phi\rangle \cdot \langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2$$

So if cloning were possible, we must have:

$$\langle\psi|\phi\rangle = (\langle\psi|\phi\rangle)^2 \Rightarrow x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \text{ or } x = 1$$

That is, $\langle\psi|\phi\rangle \in \{0, 1\}$. This means that $|\psi\rangle$ and $|\phi\rangle$ must be either orthogonal or identical — which contradicts the assumption that we should be able to clone **any arbitrary state**.

Conclusion

No such universal unitary cloning operator U can exist. **Cloning an arbitrary unknown quantum state is impossible.**
This is the statement of the **No-Cloning Theorem**.

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No-Deletion Principle

The **No-Deletion Theorem** is a logical consequence of the no-cloning theorem due to the time-reversibility of unitary operations in quantum mechanics.

Suppose we had a unitary operator $\tilde{U}$ such that:

$$\tilde{U}(|\psi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |0\rangle$$

Then running $\tilde{U}^{-1}$ would be a universal cloning machine:

$$\tilde{U}^{-1}(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$

But we have already shown that such cloning is impossible via unitaries, therefore such a deletion operator $\tilde{U}$ cannot exist either.

Conclusion

Because quantum operations are reversible (except for measurement), the impossibility of cloning implies the impossibility of deleting. Hence, **quantum states cannot be deleted** by any unitary transformation.