

# Precourse SoM+SoED

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## 1 Statements And Sets

**Main Purpose of Mathematics:** Formulation of **Statements** and assessing whether certain statements are **true (t,1)** or **false (f,0)**

**Definition (informal):** A "statement" is an expression that is either true or false.

### Examples:

1. "It is raining" is a statement
2. " $x = 5$ " is a statement
3. "The Navier-Stokes eq. have a unique solution in three dimensions." is a statement.

## 1.1 Logical Connectives:

**Logical connectives** combine / modify simple statements to create new ones. Main examples: Negation, And, Or, Implication, Equivalence

### 1.1.1 Negation

If  $A$  is a statement, the  $\neg A$  is the negation of  $A$ . It holds:

1.  $\neg A$  is true if  $A$  is false
2.  $\neg A$  is false if  $A$  is true

**Example:**  $\neg(x = 5)$  means  $x \neq 5$

### 1.1.2 And

If  $A$  and  $B$  are statements, then " $A \wedge B$ " means " $A$  and  $B$ "

1.  $A \wedge B$  is true if both  $A$  and  $B$  are true.
2.  $A \wedge B$  is false if at least one of the statements  $A$  and  $B$  is false.

**Example:** If  $A$  is the statement " $x \leq 3$ " and  $B$  the statement " $x$  is a natural number", then  $A \wedge B$  is " $x$  is 1, 2, or 3".

### 1.1.3 Or

If  $A$  and  $B$  are statements, then " $A \vee B$ " means " $A$  or  $B$ ".

It holds:

1.  $A \vee B$  is true if at least one of the statements  $A$  and  $B$  is true

2.  $A \vee B$  is false if both the statements  $A$  and  $B$  are false

**Note:** The "or" is not exclusive.

**Example:** If  $A$  is the statement " $x$  is a natural number smaller than 4" and  $B$  is the statement " $x$  is a natural number greater than 2", then  $A \vee B$  is " $x$  is a natural number".

### 1.1.4 Implication

" $A \Rightarrow B$ " means "If  $A$  is true, then  $B$  is true."

- " $A \Rightarrow B$ " means "if both  $A$  and  $B$  are true of if  $A$  is false"
- " $A \Rightarrow B$ " is false, if  $A$  is true and  $B$  is false.

**Example:**

$$(m, n \in \mathbb{N} \wedge m \text{ is even}) \Rightarrow m * n \text{ is even.}$$

**Proof:** Assume  $m, n$  are natural numbers. Then  $m$  is even

$$\Rightarrow m = 2 * m' \text{ for some } m' \in \mathbb{N}$$

$$\Rightarrow m * n = 2 * m' * n \text{ with } m' \in \mathbb{N}$$

$$\Rightarrow m * n \text{ is even}$$

**Note:**

$$(A \Rightarrow B) \not\Rightarrow (B \rightarrow A).$$

### 1.1.5 Equivalence

" $A \Leftrightarrow B$ " means " $A$  is true if and only if  $B$  is true". It holds:

- " $A \Leftrightarrow B$ " is true if both  $A$  and  $B$  are true or if both  $A$  and  $B$  is false.
- " $A \Leftrightarrow B$ " is false if  $A$  is false and  $B$  is true or vice versa

**Note:**

- $((A \Leftrightarrow B) \Leftrightarrow \{(A \Rightarrow B) \wedge (B \Rightarrow A)\})$
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

**Example:** For  $m, n$  natural numbers:

$$m * n \text{ is even} \Leftrightarrow (m \text{ is even} \vee n \text{ is even}).$$

**Proof:** Show: " $B \Rightarrow A$ " and " $A \Rightarrow B$ "

- " $B \Rightarrow A$ " is already proven; see above
- " $A \Rightarrow B$ ":  
We show the equivalent " $\neg B \Rightarrow \neg A$ "  
Suppose  $m$  is odd and  $n$  is odd, i.e.,  
 $m = 2m' + 1, n = 2n' + 1, m', n' \in \mathbb{N}_0$   
 $\Rightarrow m * n = 4m'n' + 2(m' + n') + 1 = 2 * k + 1$ ;  
with  $k := 2m' * n' + (m' + n') \in \mathbb{N}_0$   
 $\Rightarrow m * n$  is odd

## 1.2 Quantifiers

Quantifiers describe quantitative properties:

- $\forall$ : for all
- $\exists$ : exists
- $\exists_1, \exists!$ : there exists precisely one
- $\nexists$ : there does not exist (i.e.,  $\neg \exists$ )

**Example:**

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x.$$

**means:**

"For all real numbers  $x$ , there exists a natural number  $n$  such that  $n$  is bigger than  $x$ "

**Note:** The order matters

## 1.3 Sets

**Definition (informal):** A collection of well-defined distinct objects is called a set. The objects contained in a set are called elements.

**Examples:**

1. The set of all countries on earth
2. The set of all colors

**Description of Sets:**

1. Explicit definition(write all elements down)

$$A = \{a, b, c, d\}.$$

2. Characterization by property

$$A = \{\text{countries} \mid \text{contains the letter a}\}.$$

$$\mathbb{Q} := \{x \in \mathbb{R} \mid \exists q \in \mathbb{N} : q * x \in \mathbb{Z}\}.$$

**Examples of Sets:**

- $\mathbb{N}$  : natural numbers
- $\mathbb{N}_0$  : natural numbers and zero
- $\mathbb{Z}$  : integers
- $\mathbb{Q}$  : rational numbers
- $\mathbb{R}$  : real numbers
- $\mathbb{P}$  : set of all prime numbers
- $\mathbb{C}$  : complex nubers

**Def.: Intervals:** Let  $a, b \in \mathbb{R}$ . We define:

- $[a, b] := \{s \in \mathbb{R} \mid a \leq s \leq b\}$
- $(a, b] := ]a, b]. = \{s \in \mathbb{R} \mid a < s \leq b\}$
- $\mathbb{R}_{\geq 0} := [0, \infty)$

**1.4 Basic Set-Operations and Relations**

→  $a \in A$  :  $a$  is an element of  $A$

→  $a \notin A$  :  $a$  is not an element of  $A$

→  $A \subset B$  :  $A$  is a subset of  $B$ , i.e.,  $a \in A \Rightarrow a \in B$

→  $A \not\subset B$  :  $A$  is not a subset of  $B$ ,  $\exists a \in A : a \notin B$

→  $A \subseteq B$  :  $A$  is equal to  $B$ , i.e.,

$$(A \subset B) \wedge (\exists b \in B : b \notin A).$$

→  $A = B$  :  $A$  is equal to  $B$ , i.e.,  $(A \subset B) \wedge (B \subset A)$

→  $A \cup B := \{x \mid x \in A \vee x \in B\}$  (union)

→  $A \cap B := \{x \mid x \in A \wedge x \in B\}$  (intersection)

→  $A \setminus B = \{x \in A \mid x \notin B\}$  ( $A$  without  $B$ )

→  $C_A(B) := A \setminus B$  in the situation  $B \subset A$

→  $|A|$  : the cardinality of  $A$ , i.e., number of elements

→  $A \times B := \{(a, b) \mid a \in A, b \in B\}$  Cartesian product of  $A$  and  $B$ .

→  $\emptyset$  : empty set

**Example:**

$$\emptyset < \mathbb{P} < \mathbb{N} < \mathbb{N}_0 < \mathbb{Z} < \mathbb{R} < \mathbb{C}.$$

**Note:** The operation  $\cup, \cap, \times$  can be iterated.

$$\bigcup_{i=1}^n A_i := A_1 \cup A_2 \cup \dots \cup A_n, \quad n \in \mathbb{N}.$$

$$\bigcap_{i=1}^n A_i := A_1 \cap A_2 \cap \dots \cap A_n, \quad n \in \mathbb{N}.$$

$$\prod_{i=1}^n A_i := A_1 \times A_2 \times \dots \times A_n.$$

**Example:**

$A := \{1, 2, 5, 7\}$ ,  $B := \{n \in \mathbb{N} \mid n \text{ is odd}\}$ ,

$C := \{2, \sqrt{2}, B\}$ ,  $D := \{1, 5, 7\}$

$D \subsetneq B$ ,  $D \subsetneq A$ ,  $C \not\subset B$ ,  $B \in C$ ,  $B \not\subset C$ ,  $\{B\} \subset C$ ,

$B \setminus A = \{3\} \cup \{n \in \mathbb{N} \mid n \text{ is odd and } n \geq 9\}$ ,

$$\begin{aligned} C \times D = \{ & (2, 1), (2, 5), (2, 7), \\ & (\sqrt{2}, 1), (\sqrt{2}, 5), (\sqrt{2}, 7), \\ & (B, 1), (B, 5), (B, 7) \} \end{aligned}$$

$$|C \times D| = 9 = |C| * |D|.$$

**1.4.1 Disjoint Sets**

Two sets  $A$  and  $B$  are called disjoint if  $A \cap B = \emptyset$

**1.4.2 Calculus Rules for set operations:**

If  $A, B, C$  are sets, then it holds:

1.  $A \cup B = B \cup A$ ,  $A \cap B = B \cap A$  (commutativity)
  2.  $A \cup (B \cup C) = (A \cup B) \cup C$   
 $A \cap \dots$  (associativity)
  3.  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$   
 $A \cap \dots$  (distributivity)
  4.  $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$   
 $C \setminus (A \cup \dots)$  (De Morgan)
  5.  $|A \times B| = |A| * |B|$
  6.  $|A \cup B| = |A| + |B| - |A \cap B|$
  7.  $|A \setminus B| = |A| - |A \cap B|$
- (5.-7. for  $|A|, |B| < \infty$ )