Precourse SoM+SoED

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1 Statements And Sets

Main Porpose of Mathematics: Formulation of **Statements** and assessing weather certain statements are **true (t,1)** or **false (f,0)**

Definition (informal): A "statement" is an expression that is either true or false.

Examples:

1. "It is raining" is a statement
2. "$x = 5$" is a statement
3. "The Navier-Stokes eq. have a unique solution in three dimensions." is a statement.

1.1 Logical Connectives:

Lofical connectives combine / modify simple statements to createnew ones. Main examples: Negation, And, Or, Implication, Equivalance

1.1.1 Negation

If A is a statement, the $\neg A$ is the negation of A.
It holds:

1. $\neg A$ is true if A is false
2. $\neg A$ is false if A is true

Example: $\neg(x = 5)$ means $x \neq 5$

1.1.2 And

If A and B are statements, then "$A \wedge B$" means "A and B"

1. $A \wedge B$ is true if both A and B are true.
2. $A \wedge B$ is false if at least one of the statements A and B is false.

Example: If A is the statement "$x \leq 3$ " and B the statement "x is a natural number", then $A \wedge B$ is "x is 1, 2, or 3".

1.1.3 Or

If A and B are statements, then "$A \vee B$" means "A or B".
It holds:

1. $A \vee B$ is true if at least one of the statements A and B is true

2. $A \vee B$ is false if both the statements A and B are false

Note: The "or" is not exclusive.

Example: If A is the statement "x is a natural number smaller than 4" and B is the statement "x is a natural number greater than 2", then $A \vee B$ is "x is a natural number".

1.1.4 Implication

"$A \Rightarrow B$" means "If A is true, then B is true."

1. "$A \Rightarrow B$" means "if both A and B are true of if A is false"

2. "$A \Rightarrow B$" is false, if A is true and B is false.

Example:

$$(\text{m, n} \in \mathbb{N} \wedge \text{m is even}) \Rightarrow m * n \text{ is even.}$$

Proof: Assume m, n are natural numbers. Then m is even

$\Rightarrow$ $m = 2 * m'$ for some $m' \in \mathbb{N}$

$\Rightarrow$ $m * n = 2 * m' * n$ with $m' \in \mathbb{N}$

$\Rightarrow$ $m * n$ is even

Note:

$$(A \Rightarrow B) \not\Leftarrow (B \rightarrow B).$$

1.1.5 Equivalance

"$A \Leftrightarrow B$" means "A is true if and only if B is true".
It holds:

1. "$A \Leftrightarrow B$" is true if both A and B are true or if both A and B is false.

2. "$A \Leftrightarrow B$" is false if A is false and B is true or vice versa

Note:

1. $((A \Leftrightarrow B) \Leftrightarrow \{(A \Rightarrow B) \wedge (B \Rightarrow A)\}$

2. $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Example: For m, n natural numbers:

$$m * n \text{ is even} \Leftrightarrow (m \text{ is even} \vee n \text{ is even}).$$

Proof: Show:"$B \Rightarrow B$" and "$A \Rightarrow B$"

1. "$B \Rightarrow A$" is already proven; see above

2. "$A \Rightarrow B$":
We show the equivalent "$\neg B \Rightarrow \neg A$"
Suppose m is odd and n is odd, i.e.,
$m = 2m' + 1$, $n = 2n' + 1$, $m', n' \in \mathbb{N}_0$

   $\Rightarrow$ $m * n = 4m'n' + 2(m' + n') + 1 = 2 * k + 1$;
   with $k := 2m' * n' + (m' + n') \in \mathbb{N}_0$

   $\Rightarrow$ $m * n$ is odd

1.2 Quantifiers

Quantifiers describe quantitative properties:

1. $\forall$: for all

2. $\exists$: exists

3. $\exists_1$, $\exists!$: there exists precisely one

4. $\nexists$: there does not exist (i.e., $\neg\exists$)

Example:

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x.$$

means:
"For all real numbers x, there exists a natural number n such that n is bigger than x"

Note: The order matters

1.3 Sets

Definition (informal): A collection of well-defined distinct objects is called a set. Teh objects contained in a set are called elements.

Examples:

1. The set of all countries on earth
2. The set of all colors

Description of Sets:

1. Explicit definition(write all elements down)

$$A = \{a, b, c, d\}.$$

2. Characterization by property

$$A = \{\text{countries} \mid \text{contains the letter a}\}.$$

$$\mathbb{Q} := \{x \in \mathbb{R} \mid \exists q \in \mathbb{N} : q * x \in \mathbb{Z}\}.$$

Examples of Sets:

$\rightarrow$ $\mathbb{N}$: natural numbers

$\rightarrow$ $\mathbb{N}_0$: natural numbers and zero

$\rightarrow$ $\mathbb{Z}$: integers

$\rightarrow$ $\mathbb{Z}$: rational numbers

$\rightarrow$ $\mathbb{R}$: real numbers

$\rightarrow$ $\mathbb{P}$: set of all prime numbers

$\rightarrow$ $\mathbb{C}$: complex nubers

Def.: Intervals: Let $a, b \in \mathbb{R}$. We define:

$\rightarrow$ $[a, b] := \{s \in \mathbb{R} \mid a \leq s \leq b\}$

$\rightarrow$ $(a, b] :=]a, b]. = \{s \in \mathbb{R} \mid a < s \leq b\}$

$\rightarrow$ $\mathbb{R}_{\geq 0} := [0, \inf)$

1.4 Basic Set-Operations and Relations

$\rightarrow$ $a \in A$: a is an element of A

$\rightarrow$ $a \notin A$: a is not an element of A

$\rightarrow$ $A \subset B$: A is a subset of B, i.e., $a \in A \Rightarrow a \in B$

$\rightarrow$ $A \not\subset B$: A is not a subset of B, $\exists a \in A : a \notin B$

$\rightarrow$ $A \subsetneq B$: A is equal to B, i.e.,

$$(A \subset B) \wedge (\exists b \in B : b \notin A).$$

$\rightarrow$ $A = B$: A is equal to B, i.e., $(A \subset B) \wedge (B \subset A)$

$\rightarrow$ $A \cup B := \{x \mid x \in A \vee x \in B\}$ (union)

$\rightarrow$ $A \cap B := \{x \mid x \in A \wedge x \in B\}$ (intersection)

$\rightarrow$ $A \backslash B = \{x \in A \mid x \notin B\}$ (A without B)

$\rightarrow$ $C_A(B) := A \backslash B$ in the situation $B \subset A$

$\rightarrow$ $\mid A \mid$: the cardinality of A, i.e., number of elements

$\rightarrow$ $A \times B := \{(a, b) \mid a \in A, b \in B\}$ Cartesian product of A and B.

$\rightarrow$ $\emptyset$: empty set

Example:

$$\emptyset < \mathbb{P} < \mathbb{N} < \mathbb{N}_0 < \mathbb{Z} < \mathbb{R} < \mathbb{C}.$$

Note: The operation $\cup$, $\cap$, $\times$ can be iterated.

$$\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup ... \cup A_n, \; n \in \mathbb{N}.$$

$$\bigcap_{i=1}^{n} A_i := A_1 \cap A_2 \cap ... \cap A_n, \; n \in \mathbb{N}.$$

$$\prod_{i=1}^{n} A_1 := A_1 \times A_2 \times ... \times A_n.$$

Example:
$A := \{1, 2, 5, 7\}$, $B := \{n \in \mathbb{N} \mid n \text{ is odd}\}$,
$C := \{2, \sqrt{2}, B\}$, $D := \{1, 5, 7\}$

$D \subsetneq B$, $D \subsetneq A$, $C \not\subset B$, $B \in C$, $B \not\subset C$, $\{B\} \subset C$,
$B \setminus A = \{3\} \cup \{n \in \mathbb{N} \mid n \text{ is odd and } n \geq 9\}$,

$$\begin{aligned} C \times D = \{&(2,1),(2,5),(2,7), \\ &(\sqrt{2},1),(\sqrt{2},5),(\sqrt{2},7), \\ &(B,1),(B,5),(B,7)\} \end{aligned}$$

$$\mid C \times D \mid = 9 = \mid C \mid * \mid D \mid .$$

1.4.1 Disjoint Sets

Two sets A and B are called disjoint if $A \cap B = \emptyset$

1.4.2 Calculus Rules for set operations:

If A, B, C are sets, then it holds:

1. $A \cup B = B \cup A$, $A \cap B = B \cap A$ (commutativity)
2. $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap ...$ (associativity)
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap ...$ (distributivity)
4. $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
   $C \setminus (A \cup ...$ (De Morgan)
5. $\mid A \times B \mid = \mid A \mid * \mid B \mid$
6. $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$
7. $\mid A \setminus B \mid = \mid A \mid - \mid A \cap \mid$

(5.-7. for $\mid A \mid, \mid B \mid < \inf$)

2 Functions and Inequalities

2.1 Functions

Definition: Let A and B be nonempty sets. A function f from A to B is a rule that assigns to each element of the set A a unique element of the set B, i.e.,

$$\forall x \in A \exists_1 y \in B : y = f(x).$$

Notation:

$$f : A \to B, x \mapsto f(x).$$

Def.: Let $f : A \to B$ be a function between nenempty sets A and B. Then:

- $\to$ A is called the domain.
- $\to$ B is called the codomain.
- $\to$ The element $f(x)$ that a given $x \in A$ is mapped to by f is called the image of x under f.
- $\to$ For $C \subset A$, $f(C) := \{y \in B \mid \exists x \in C : f(x) = y\}$
- $\to$ The set $\{(x, y) \in A \times B \mid y = f(x)\}$ is called the graph of f.

Examples:

1. If $A := \{x \mid x \text{ is a mono-colored car}\}$
   and $B := sety \mid y$ is a color, then

   $$f := A \to B, \quad f(x) := \text{color of } x.$$

2. If we consider for $A = \mathbb{R}$, the rule

   $$A \ni x \mapsto (-\inf, x]$$

   then this does not define a function $f : \mathbb{R} \to \mathbb{R}$. But if we define the so-called power set of $\mathbb{R}$ by

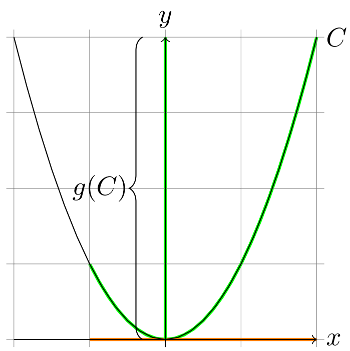
   $$\mathbb{P}(\mathbb{R}) := \{C \mid C \text{ is a subset of } \mathbb{R}\},$$

   then $A \ni x \mapsto (-\inf, x]$ defines a function from $\mathbb{R}$ to $\mathbb{P}(\mathbb{R})$.

3. The rule $g(x) := x^2$ defines a function whose domain and codomain is equal to $\mathbb{R}$, i.e., $g : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

   - $\to$ $g(\mathbb{R}) = [0, \inf)$
   - $\to$ For $C := [-1, 2]$, $D := [1, 4]$:

   $$g(C) = [0, 4].$$



Note: In most applications the appearing functions are functions between Euclidean spaces, i.e., $f : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$

Example: Minimize air resistance
Function from set of shapes to $\mathbb{R}$

2.2 Calculus Rules for Images and Preimages:

Let $f : A \to B$ be a map between nonempty sets A and B. Then:

→ $C_1 \subset C_2 \subset A \Rightarrow f(C_1) \subset f(C_2)$

→ $D_1 \subset D_2 \subset B \Rightarrow f^{-1}(D_1) \subset f^{-1}(D_2)$

→ $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$, $\forall C_1, C_2 \subset A$

→ $f^{-1}(C_1 \cap D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$, $\forall D_1, D_2 \subset A$

→ $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$, $\forall C_1, C_2 \subset A$

→ $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$, $\forall D_1, D_2 \subset B$

→ $C \subset f^{-1(f(C))}$, $\forall C \subset A$
ex.: $f : \mathbb{R} \to \mathbb{R}, x \mapsto 0 \in \mathbb{R}$

- C={0}
- $f(C) = \{0\}$
- $f^{-1}(f(C)) = \mathbb{R}$

→ $f(f^{-1}(D)) \subset D$, $\forall D \subset B$

2.3 Mapping Properties

2.3.1 Injectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **injective** if:

$$\forall x_1, x_2 \in A, x_1 \neq x_2 : f(x_1) \neq f(x_2).$$

2.3.2 Surjectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **surjective** if:

$$\forall y \in B \exists x \in A : f(x) = y.$$

2.3.3 Bijectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **bijective** if f is injective and surjective,

$$\forall y \in B \exists_1 x \in A : f(x) = y.$$

Example: The function $g(x) := x^2$ is nither surjective nor injective as a map from $\mathbb{R}$ to $\mathbb{R}$. However, g is:

→ surjective as a map $g : \mathbb{R} \to [0, \inf)$

→ bijective as a map $g : [0, \inf) \to [0, \inf)$

⇒ **choice of domain and co-domain crucial**