Precourse SoM+SoED

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October 8, 2023

Contents

1 Statements And Sets

Main Purpose of Mathematics: Formulation of **Statements** and assessing weather certain statements are **true (t,1)** or **false (f,0)**

Definition (informal): A ”statement” is an expression that is either true or false.

Examples:

1. ”It is raining” is a statement
2. ”$x = 5$” is a statement
3. ”The Navier-Stokes eq. have a unique solution in three dimensions.” is a statement.

1.1 Logical Connectives

Logical connectives combine / modify simple statements to create new ones. Main examples: Negation, And, Or, Implication, Equivalence

1.1.1 Negation

If A is a statement, the $\neg A$ is the negation of A.
It holds:

1. $\neg A$ is true if A is false
2. $\neg A$ is false if A is true

Example: $\neg(x = 5)$ means $x \neq 5$

1.1.2 And

If A and B are statements, then ”$A \wedge B$” means ”A and B”

1. $A \wedge B$ is true if both A and B are true.
2. $A \wedge B$ is false if at least one of the statements A and B is false.

Example: If A is the statement "$x \leq 3$ " and B the statement "x is a natural number", then $A \wedge B$ is "x is 1, 2, or 3".

1.1.3 Or

If A and B are statements, then "$A \vee B$" means "A or B".
It holds:

1. $A \vee B$ is true if at least one of the statements A and B is true
2. $A \vee B$ is false if both the statements A and B are false

Note: The "or" is not exclusive.

Example: If A is the statement "x is a natural number smaller than 4" and B is the statement "x is a natural number greater than 2", then $A \vee B$ is "x is a natural number".

1.1.4 Implication

"$A \Rightarrow B$" means "If A is true, then B is true."

1. "$A \Rightarrow B$" means "if both A and B are true of if A is false"
2. "$A \Rightarrow B$" is false, if A is true and B is false.

Example:

$$(\text{m, n} \in \mathbb{N} \wedge \text{m is even}) \Rightarrow m * n \text{ is even.}$$

Proof: Assume m, n are natural numbers. Then m is even

$\Rightarrow$ $m = 2 * m'$ for some $m' \in \mathbb{N}$

$\Rightarrow$ $m * n = 2 * m' * n$ with $m' \in \mathbb{N}$

$\Rightarrow$ $m * n$ is even

Note:

$$(A \Rightarrow B) \not\Leftrightarrow (B \to B).$$

1.1.5 Equivalence

"$A \Leftrightarrow B$" means "A is true if and only if B is true".
It holds:

1. "$A \Leftrightarrow B$" is true if both A and B are true or if both A and B is false.
2. "$A \Leftrightarrow B$" is false if A is false and B is true or vice versa

Note:

1. $((A \Leftrightarrow B) \Leftrightarrow \{(A \Rightarrow B) \wedge (B \Rightarrow A)\}$
2. $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

Example: For m, n natural numbers:

$$m * n \text{ is even} \Leftrightarrow (m \text{ is even} \vee n \text{ is even}).$$

Proof: Show:"$B \Rightarrow B$" and "$A \Rightarrow B$"

1. "$B \Rightarrow A$" is already proven; see above
2. "$A \Rightarrow B$":
   We show the equivalent "$\neg B \Rightarrow \neg A$"
   Suppose m is odd and n is odd, i.e.,
   $m = 2m' + 1$, $n = 2n' + 1$, $m', n' \in \mathbb{N}_0$

   $\Rightarrow$ $m * n = 4m'n' + 2(m' + n') + 1 = 2 * k + 1;$
   with $k := 2m' * n' + (m' + n') \in \mathbb{N}_0$

   $\Rightarrow$ $m * n$ is odd

1.2 Quantifiers

Quantifiers describe quantitative properties:

1. $\forall$: for all
2. $\exists$: exists
3. $\exists_1$, $\exists!$: there exists precisely one
4. $\nexists$: there does not exist (i.e., $\neg\exists$)

Example:

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x.$$

means:
"For all real numbers x, there exists a natural number n such that n is bigger than x"

Note: The order matters

1.3 Sets

Definition (informal): A collection of well-defined distinct objects is called a set. The objects contained in a set are called elements.

Examples:

1. The set of all countries on earth
2. The set of all colors

Description of Sets:

1. Explicit definition(write all elements down)

$$A = \{a, b, c, d\}.$$

2. Characterization by property

$$A = \{\text{countries} \mid \text{contains the letter a}\}.$$
$$\mathbb{Q} := \{x \in \mathbb{R} \mid \exists q \in \mathbb{N} : q * x \in \mathbb{Z}\}.$$

Examples of Sets:

→ $\mathbb{N}$: natural numbers

→ $\mathbb{N}_0$: natural numbers and zero

→ $\mathbb{Z}$: integers

→ $\mathbb{Z}$: rational numbers

→ $\mathbb{R}$: real numbers

→ $\mathbb{P}$: set of all prime numbers

→ $\mathbb{C}$: complex numbers

Def.: Intervals: Let $a, b \in \mathbb{R}$. We define:

→ $[a, b] := \{s \in \mathbb{R} \mid a \leq s \leq b\}$

→ $(a, b] :=]a, b]. = \{s \in \mathbb{R} \mid a < s \leq b\}$

→ $\mathbb{R}_{\geq 0} := [0, \inf)$

1.4 Basic Set-Operations and Relations

→ $a \in A$: a is an element of A

→ $a \notin A$: a is not an element of A

→ $A \subset B$: A is a subset of B, i.e., $a \in A \Rightarrow a \in B$

→ $A \not\subset B$: A is not a subset of B, $\exists a \in A : a \notin B$

→ $A \subsetneq B$: A is equal to B, i.e.,

$$(A \subset B) \wedge (\exists b \in B : b \notin A).$$

→ $A = B$: A is equal to B, i.e., $(A \subset B) \wedge (B \subset A)$

→ $A \cup B := \{x \mid x \in A \vee x \in B\}$ (union)

→ $A \cap B := \{x \mid x \in A \wedge x \in B\}$ (intersection)

→ $A \backslash B = \{x \in A \mid x \notin B\}$ (A without B)

→ $C_A(B) := A \backslash B$ in the situation $B \subset A$

→ $\mid A \mid$: the cardinality of A, i.e., number of elements

→ $A \times B := \{(a, b) \mid a \in A, b \in B\}$ Cartesian product of A and B.

→ $\emptyset$: empty set

Example:

$$\emptyset < \mathbb{P} < \mathbb{N} < \mathbb{N}_0 < \mathbb{Z} < \mathbb{R} < \mathbb{C}.$$

Note: The operation $\cup$, $\cap$, $\times$ can be iterated.

$$\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup ... \cup A_n, \ n \in \mathbb{N}.$$

$$\bigcap_{i=1}^{n} A_i := A_1 \cap A_2 \cap ... \cap A_n, \ n \in \mathbb{N}.$$

$$\prod_{i=1}^{n} A_1 := A_1 \times A_2 \times ... \times A_n.$$

Example:
$A := \{1, 2, 5, 7\}$, $B := \{n \in \mathbb{N} \mid n \text{ is odd}\}$,
$C := \{2, \sqrt{2}, B\}$, $D := \{1, 5, 7\}$

$D \subsetneq B$, $D \subsetneq A$, $C \not\subset B$, $B \in C$, $B \not\subset C$, $\{B\} \subset C$,
$B \backslash A = \{3\} \cup \{n \in \mathbb{N} \mid n \text{ is odd and } n \geq 9\}$,

$$\begin{aligned} C \times D = \{&(2,1), (2,5), (2,7), \\ &(\sqrt{2},1), (\sqrt{2},5), (\sqrt{2},7), \\ &(B,1), (B,5), (B,7)\} \\ \mid C \times D \mid = 9 = &\mid C \mid * \mid D \mid . \end{aligned}$$

1.4.1 Disjoint Sets

Two sets A and B are called disjoint if $A \cap B = \emptyset$

1.4.2 Calculus Rules for set operations:

If A, B, C are sets, then it holds:

1. $A \cup B = B \cup A$, $A \cap B = B \cap A$ (commutativity)
2. $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap ...$ (associativity)
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap ...$ (distributivity)
4. $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$
   $C \backslash (A \cup ...$ (De Morgan)
5. $\mid A \times B \mid = \mid A \mid * \mid B \mid$
6. $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$
7. $\mid A \backslash B \mid = \mid A \mid - \mid A \cap \mid$

(5.-7. for $\mid A \mid, \mid B \mid < \inf$)

2 Functions and Inequalities

2.1 Functions

Definition: Let A and B be nonempty sets. A function f from A to B is a rule that assigns to each element of the set A a unique element of the set B, i.e.,

$$\forall x \in A \exists_1 y \in B : y = f(x).$$

Notation:

$$f : A \to B, x \mapsto f(x).$$

Def.: Let $f : A \to B$ be a function between nonempty sets A and B. Then:

- $\to$ A is called the domain.
- $\to$ B is called the codomain.
- $\to$ The element $f(x)$ that a given $x \in A$ is mapped to by f is called the image of x under f.
- $\to$ For $C \subset A$, $f(C) := \{y \in B \mid \exists x \in C : f(x) = y\}$
- $\to$ The set $\{(x, y) \in A \times B \mid y = f(x)\}$ is called the graph of f.

Examples:

1. If $A := \{x \mid x \text{ is a mono-colored car}\}$
   and $B := sety \mid y$ is a color, then

   $$f := A \to B, \quad f(x) := \text{color of } x.$$

2. If we consider for $A = \mathbb{R}$, the rule

   $$A \ni x \mapsto (-\inf, x]$$

   then this does not define a function $f : \mathbb{R} \to \mathbb{R}$. But if we define the so-called power set of $\mathbb{R}$ by

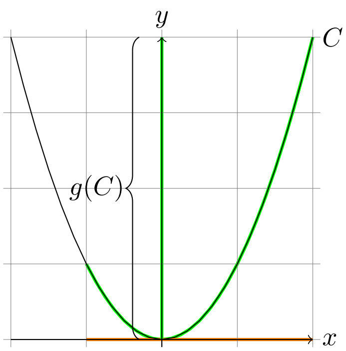
   $$\mathbb{P}(\mathbb{R}) := \{C \mid C \text{ is a subset of } \mathbb{R}\},$$

   then $A \ni x \mapsto (-\inf, x]$ defines a function from $\mathbb{R}$ to $\mathbb{P}(\mathbb{R})$.

3. The rule $g(x) := x^2$ defines a function whose domain and codomain is equal to $\mathbb{R}$, i.e., $g : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

   - $\to$ $g(\mathbb{R}) = [0, \inf)$
   - $\to$ For $C := [-1, 2]$, $D := [1, 4]$:

   $$g(C) = [0, 4].$$



Note: In most applications the appearing functions are functions between Euclidean spaces, i.e., $f : \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$

Example: Minimize air resistance
Function from set of shapes to $\mathbb{R}$

2.2 Calculus Rules for Images and Preimages:

Let $f : A \to B$ be a map between nonempty sets A and B. Then:

- $\to$ $C_1 \subset C_2 \subset A \Rightarrow f(C_1) \subset f(C_2)$
- $\to$ $D_1 \subset D_2 \subset B \Rightarrow f^{-1}(D_1) \subset f^{-1}(D_2)$
- $\to$ $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$, $\forall C_1, C_2 \subset A$
- $\to$ $f^{-1}(C_1 \cap D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$, $\forall D_1, D_2 \subset A$
- $\to$ $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$, $\forall C_1, C_2 \subset A$
- $\to$ $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$, $\forall D_1, D_2 \subset B$
- $\to$ $C \subset f^{-1(f(C))}$, $\forall C \subset A$
  ex.: $f : \mathbb{R} \to \mathbb{R}, x \mapsto 0 \in \mathbb{R}$
  - C={0}
  - $f(C) = \{0\}$
  - $f^{-1}(f(C)) = \mathbb{R}$
- $\to$ $f(f^{-1}(D)) \subset D$, $\forall D \subset B$

2.3 Mapping Properties

2.3.1 Injectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **injective** if:

$$\forall x_1, x_2 \in A, x_1 \neq x_2 : f(x_1) \neq f(x_2).$$

2.3.2 Surjectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **surjective** if:

$$\forall y \in B \exists x \in A : f(x) = y.$$

2.3.3 Bijectivity

Let $f : A \to B$ be a function between non-empty sets. Then f is called **bijective** if f is injective and surjective,

$$\forall y \in B \exists_1 x \in A : f(x) = y.$$

Example: The function $g(x) := x^2$ is neither surjective nor injective as a map from $\mathbb{R}$ to $\mathbb{R}$. However, g is:

- $\to$ surjective as a map $g : \mathbb{R} \to [0, \inf)$
- $\to$ bijective as a map $g : [0, \inf) \to [0, \inf)$

$\Rightarrow$ **choice of domain and co-domain crucial**

2.3.4 Inverse Function

If $f : A \to B$ is a bijective map between nonempty sets, then there exists a unique function $f^{-1} : B \to A$ satisfying

$$f^{-1}(f(x)) = x \quad \forall x \in A.$$
$$f(f^{-1}(y)) = y \quad \forall y \in B.$$

Note: Do not get confused with the notation for preimages here!

$\Rightarrow$ If f^{-1} exists, then $f^{-1}(B)$
$= $ preimage of B under f
$= $ image of B under f^{-1}

Example: Consider $g : [0, \inf) \to [0, \inf), x \mapsto x^2$

- g bijective, $g^{-1} : [0, \inf) \to [0, \inf)$ given by $g^{-1}(y) :=\sqrt{y}$
- $\forall \in [0, \inf) : g^{-1}(g(x)) =\sqrt{x^2} = x$
  $\forall y \in [0, \inf) : g(g^{-1}(y)) = (\sqrt{y})^2 = y$

2.4 Inequalities

In addition to identities, inequalities pay an essential role in math, we have:

1. $x < y, x > y$: strict inequalities, "x is strictly smaller / greater than y".
2. $x \leq y, x \geq y$: non-strict inequalities

   $\Rightarrow$ "$\leq$" defines a total order on $\mathbb{R}$

2.4.1 Order Properties

1. reflexivity: $x \leq x \quad \forall x \in \mathbb{R}$
2. transitivity: $\forall x, y, z \in \mathbb{R} : (x \leq y \wedge y \leq z) \Rightarrow x \leq z$
3. antisymmetry: $\forall x, y \in \mathbb{R} : (x \leq y \wedge z \leq x) \Rightarrow x = y$
4. totality: $\forall x, y \in \mathbb{R} : x \leq y \vee y \leq x$

Note: The property of antisymmetry is often used to prove identities

$\to$ Show "=" by proving "$\leq$" and "$\geq$".

In practice, one often encounters inequalities of the form $f(x) \leq 0$ involving functions $f : \mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$, and is interested in the solution set:

$$\mathbb{L} := \{x \in \mathbb{R}^n \mid f(x) \leq 0\}.$$

To solve such inequalities, one uses the following properties:

- $a < b, c \in \mathbb{R} \Rightarrow a + c < b + c$
- $(a < c \wedge c < d) \Rightarrow a + c < b + d$
- $(a < b \wedge c > 0) \Rightarrow ac < bc$
- $(0 < a < b \wedge 0 < c < d) \Rightarrow ac < bd$
- $(a < b \wedge c < 0) \Rightarrow ca > cb$
- $(a, b > 0 \wedge a < b) \Rightarrow \frac{1}{b} < \frac{1}{a}$

2.4.2 Monotoniety

If $D \subset \mathbb{R}$ is a nonempty set and $g : D \to \mathbb{R}$ a function, then g is called non-decreasing on D if

$$(x, y \in D \wedge x \leq y) \Rightarrow g(x) \leq g(y).$$

g is called non-increasing if $-g$ is non-decreasing

Example:

1. $g : [0, \inf) \to \mathbb{R}, x \mapsto x^2$, is non-decreasing
2. $g : [0, \inf) \to \mathbb{R}, x \mapsto \sqrt{x}$, is non-decreasing
3. $g : (-\inf, 0] \to \mathbb{R}, x \mapsto x^2$, non-increasing
   ($\Rightarrow$ domain matters)

Example: For all $a, b \geq 0$, we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Proof: For $a, b \geq 0$, we have

$$\begin{aligned}
\sqrt{ab} \leq \frac{a+b}{2} &\Leftrightarrow (\sqrt{ab})^2 \leq (\frac{a+b}{2})^2 \\
&\Leftrightarrow ab \leq \frac{1}{4}(a^2 + 2ab + b^2) \\
&\Leftrightarrow 2ab \leq a^2 + 2ab + b^2 \\
&\Leftrightarrow 0 \leq a^2 - 2ab + b^2 = (a-b)^2
\end{aligned}$$