# Precourse SoM+SoED

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## Contents



## 1 Statements And Sets

**Main Porpose of Mathematics:** Formulation of **Statements** and assessing weather certain statements are **true (t,1)** or **false (f,0)**

**Definition (informal):** A "statement" is an expression that is either true or false.

**Examples:**

1. "It is raining" is a statement
2. "$x = 5$" is a statement
3. "The Navier-Stokes eq. have a unique solution in three dimensions." is a statement.

### 1.1 Logical Connectives:

**Lofical connectives** combine / modify simple statements to createnew ones. Main examples: Negation, And, Or, Implication, Equivalance

#### 1.1.1 Negation

If $A$ is a statement, the $\neg A$ is the negation of $A$.
It holds:

1. $\neg A$ is true if $A$ is false
2. $\neg A$ is false if $A$ is true

**Example:** $\neg(x = 5)$ means $x \neq 5$

#### 1.1.2 And

If $A$ and $B$ are statements, then "$A \wedge B$" means "$A$ and $B$"

1. $A \wedge B$ is true if both $A$ and $B$ are true.
2. $A \wedge B$ is false if at least one of the statements $A$ and $B$ is false.

**Example:** If $A$ is the statement "$x \leq 3$ " and $B$ the statement "$x$ is a natural number", then $A \wedge B$ is "$x$ is 1, 2, or 3".

#### 1.1.3 Or

If $A$ and $B$ are statements, then "$A \vee B$" means "$A$ or $B$".
It holds:

1. $A \vee B$ is true if at least one of the statements $A$ and $B$ is true

2. $A \vee B$ is false if both the statements $A$ and $B$ are false

**Note:** The "or" is not exclusive.

**Example:** If $A$ is the statement "$x$ is a natural number smaller than 4" and $B$ is the statement "$x$ is a natural number greater than 2", then $A \vee B$ is "$x$ is a natural number".

#### 1.1.4 Implication

"$A \Rightarrow B$" means "If $A$ is true, then $B$ is true."

1. "$A \Rightarrow B$" means "if both $A$ and $B$ are true of if $A$ is false"

2. "$A \Rightarrow B$" is false, if $A$ is true and $B$ is false.

**Example:**

$$(\text{m, n} \in \mathbb{N} \wedge \text{m is even}) \Rightarrow m * n \text{ is even.}$$

**Proof:** Assume m, n are natural numbers. Then m is even

$\Rightarrow\ m = 2 * m'$ for some $m' \in \mathbb{N}$

$\Rightarrow\ m * n = 2 * m' * n$ with $m' \in \mathbb{N}$

$\Rightarrow\ m * n$ is even

**Note:**

$$(A \Rightarrow B) \not\Leftrightarrow (B \rightarrow B).$$

#### 1.1.5 Equivalance

"$A \Leftrightarrow B$" means "A is true if and only if B is true". It holds:

1. "$A \Leftrightarrow B$" is true if both $A$ and $B$ are true or if both $A$ and $B$ is false.

2. "$A \Leftrightarrow B$" is false if $A$ is false and $B$ is true or vice versa

**Note:**

1. $((A \Leftrightarrow B) \Leftrightarrow \{(A \Rightarrow B) \wedge (B \Rightarrow A)\}$

2. $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$

**Example:** For m, n natural numbers:

$$m * n \text{ is even} \Leftrightarrow (m \text{ is even} \vee n \text{ is even}).$$

**Proof:** Show:"$B \Rightarrow B$" and "$A \Rightarrow B$"

1. "$B \Rightarrow A$" is already proven; see above

2. "$A \Rightarrow B$":
   We show the equvalent "$\neg B \Rightarrow \neg A$"
   Suppose $m$ is odd and $n$ is odd, i.e.,
   $m = 2m' + 1$, $n = 2n' + 1$, $m', n' \in \mathbb{N}_0$

   $\Rightarrow\ m * n = 4m'n' + 2(m' + n') + 1 = 2 * k + 1$; with $k := 2m' * n' + (m' + n') \in \mathbb{N}_0$

   $\Rightarrow\ m * n$ is odd

### 1.2 Quantifiers

Quantifiers describe quantitative properties:

1. $\forall$: for all

2. $\exists$: exists

3. $\exists_1$, $\exists!$: there exists precisely one

4. $\nexists$: there does not exist (i.e., $\neg\exists$)

**Example:**

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x.$$

**means:**
"For all real numbers $x$, there exists a natural number $n$ such that $n$ is bigger than $x$"

**Note:** The order matters

### 1.3 Sets

**Definition (informal):** A collection of well-defined distinct objects is called a set. Teh objects contained in a set are called elements.

**Examples:**

1. The set of all countries on earth
2. The set of all colors

**Description of Sets:**

1. Explicit definition(write all elements down)

$$A = \{a, b, c, d\}.$$

2. Characterization by property

$$A = \{\text{countries} \mid \text{contains the letter a}\}.$$

$$\mathbb{Q} := \{x \in \mathbb{R} \mid \exists q \in \mathbb{N} : q * x \in \mathbb{Z}\}.$$

**Examples of Sets:**

→ $\mathbb{N}$ : natural numbers

→ $\mathbb{N}_0$ : natural numbers and zero

→ $\mathbb{Z}$ : integers

→ $\mathbb{Z}$ : rational numbers

→ $\mathbb{R}$ : real numbers

→ $\mathbb{P}$ : set of all prime numbers

→ $\mathbb{C}$ : complex nubers

**Def.: Intervals:** Let $a, b \in \mathbb{R}$. We define:

→ $[a, b] := \{s \in \mathbb{R} \mid a \leq s \leq b\}$

→ $(a, b] :=]a, b]. = \{s \in \mathbb{R} \mid a < s \leq b\}$

→ $\mathbb{R}_{\geq 0} := [0, \inf)$

## 1.4 Basic Set-Operations and Relations

→ $a \in A$ : $a$ is an element of $A$

→ $a \notin A$ : $a$ is not an element of $A$

→ $A \subset B$ : $A$ is a subset of $B$, i.e., $a \in A \Rightarrow a \in B$

→ $A \not\subset B$ : $A$ is not a subset of $B$, $\exists a \in A : a \notin B$

→ $A \subsetneq B$ : $A$ is equal to $B$, i.e.,

$$(A \subset B) \wedge (\exists b \in B : b \notin A).$$

→ $A = B$ : $A$ is equal to $B$, i.e., $(A \subset B) \wedge (B \subset A)$

→ $A \cup B := \{x \mid x \in A \vee x \in B\}$ (union)

→ $A \cap B := \{x \mid x \in A \wedge x \in B\}$ (intersection)

→ $A \backslash B = \{x \in A \mid x \notin B\}$ ($A$ without $B$)

→ $C_A(B) := A \backslash B$ in the situation $B \subset A$

→ $\mid A \mid$ : the cardinality of $A$, i.e., number of elements

→ $A \times B := \{(a, b) \mid a \in A, b \in B\}$ Cartesian product of $A$ and $B$.

→ $\emptyset$ : empty set

**Example:**

$$\emptyset < \mathbb{P} < \mathbb{N} < \mathbb{N}_0 < \mathbb{Z} < \mathbb{R} < \mathbb{C}.$$

**Note:** The operation $\cup$, $\cap$, $\times$ can be iterated.

$$\bigcup_{i=1}^{n} A_i := A_1 \cup A_2 \cup ... \cup A_n, \ n \in \mathbb{N}.$$

$$\bigcap_{i=1}^{n} A_i := A_1 \cap A_2 \cap ... \cap A_n, \ n \in \mathbb{N}.$$

$$\prod_{i=1}^{n} A_1 := A_1 \times A_2 \times ... \times A_n.$$

**Example:**
$A := \{1, 2, 5, 7\}$, $B := \{n \in \mathbb{N} \mid n \text{ is odd}\}$,
$C := \{2, \sqrt{2}, B\}$, $D := \{1, 5, 7\}$

$D \subsetneq B$, $D \subsetneq A$, $C \not\subset B$, $B \in C$, $B \not\subset C$, $\{B\} \subset C$,
$B \backslash A = \{3\} \cup \{n \in \mathbb{N} \mid n \text{ is odd and } n \geq 9\}$,

$$\begin{aligned} C \times D = \{ & (2,1), (2,5), (2,7), \\ & (\sqrt{2},1), (\sqrt{2},5), (\sqrt{2},7), \\ & (B,1), (B,5), (B,7)\} \end{aligned}$$

$$\mid C \times D \mid = 9 = \mid C \mid * \mid D \mid .$$

### 1.4.1 Disjoint Sets

Two sets $A$ and $B$ are called disjoint if $A \cap B = \emptyset$

### 1.4.2 Calculus Rules for set operations:

If $A, B, C$ are sets, then it holds:

1. $A \cup B = B \cup A$, $A \cap B = B \cap A$ (commutativity)
2. $A \cup (B \cup C) = (A \cup B) \cup C)$
   $A \cap ...$ (associativity)
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap ...$ (distributivity)
4. $C \backslash (A \cap B) = (C \backslash A) \cup (C \backslash B)$
   $C \backslash (A \cup ...$ (De Morgan)
5. $\mid A \times B \mid = \mid A \mid * \mid B \mid$
6. $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$
7. $\mid A \backslash B \mid = \mid A \mid - \mid A \cap \mid$

(5.-7. for $\mid A \mid, \mid B \mid < \text{inf}$)