Technische Universität München

Math 3

MA9803 | B.Sc. Engineering Science and B.Sc. Aerospace

Modeling and Simulation with Ordinary Differential Equations

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October 25, 2024

Abstract

Some people think that stiff challenges are the best device to induce learning, but I am not one of them. The natural way to learn something is by spending vast amounts of easy, enjoyable time at it. This goes whether you want to speak German, sight-read at the piano, type, or do mathematics. Give me the German storybook for fifth graders that I feel like reading in bed, not Goethe and a dictionary. The latter will bring rapid progress at first, then exhaustion and failure to resolve.

L. N. Trefethen. Trefethen's index cards - Forty years of notes about People, Words and Mathematics. World Scientific, 2011, S. 86

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1 Introduction

Previously: We had equations like $y^2 + 4y + 1 = 0$, where the solution is a number.

$$\int_a^b f(x)dt = \text{number} = F(b) - F(a)$$

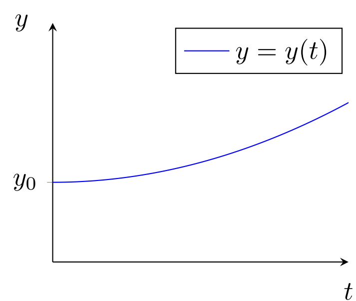
New: $f'(x)$ is given, determine $f(x)$. The solution is a function.

$$\begin{aligned} \text{Velocity}(t) &= \text{Position}'(t) \quad \text{given} \\ &= \text{Position}(t) \quad \text{wanted} \end{aligned}.$$

1.1 Differential equations

Example: Interest rate
$y(t)$: assets at time t.
$\lambda < 0$: constant interest rate.

$$\text{IVP} \begin{cases} y'(t) &= \lambda \cdot y(t), \\ y(0) &= y_0. \end{cases}$$



Example: Radioactive Decay
$y(t)$: mass of a substance at time t.
$\lambda < 0$: decay rate.

$$y'(t) = \lambda \cdot y(t).$$

1.2 Problem Solving Steps in Science & technology

"Reality" $\longrightarrow$ Model = "Mathematisation"

- Laws of nature (Physics, Chemistry).
- Assumtions, hypoteses $\rightarrow$ differential equations.
- Analytical or symbolic solution (exact solution). not practical
- Numerical solution. in practise

Change of state = **Function (State)**.

Ordenary Diff. Eqn. (ODE: one scalar variable (e.g. time) this semester

$$y'(t) = \lambda y(t).$$

Partial Diff. Eqn. (PDE): multiple variables (e.g. time and space) next semester (MA9804)

$$u = u(t, x) \quad x = (x_1, x_2, x_3).$$

$$\frac{\delta u}{\delta t} = k \cdot (\frac{\delta^2 u}{\delta x_1^2} \frac{\delta^2 u}{\delta x_2^2} \frac{\delta^2 u}{\delta x_3^2}).$$

Heat equation. u: temperature

Oftentimes: Solution depends on (design) parameters. $y = y(t; \rho)$, $u = u(t, x, \rho)$

- Parameter identification Inverse Problem
- Design optimisation
- Random / Stochastic parameters Uncertainty quantification

1.3 Application: Solid Mechanics

Newton's Laws of motion: $\frac{dp}{dt} = F$

Change of momentum = force action.

a) Calculate orbits of satellites; today: GPS

b) Vibrations in automobiles; reliability of structures. (earthquakes; spring-mass-system)

1.4 Application: Electric ciruits

Ohm's Law: $U = R \cdot I$ (voltage = resistance * current)

Kirchhoff's Cirquit Laws: $\rightarrow$ System of ODEs

Law of Induction: $U = n \cdot I'$ (Coil); $I = C \cdot U'$ (Capacitor). Here the $'$ is the time derivative.

1.5 Application: Chemical relations

Chemical reaction: $A + B \xrightarrow[k]{} C$.

Here $c_A(t), c_B(t), c_C(t)$ are the Concentrations of the substance A, B, C in $\frac{\text{Mol}}{\text{l}}$.

Law of mass action: Change of concentration is proportional to the concentration of the resulting substances:

$$\begin{aligned} c_C' &= \ \ k \cdot c_A \cdot c_B \\ c_A' &= -k \cdot c_A \cdot c_B \\ c_B' &= -k \cdot c_A \cdot c_B \end{aligned}$$

1.6 Classification of ODEs

$$y(x).$$

x: independent variable scalar $\quad$ y: dependent variable

Ordenary DE:

- System of ODEs: $y(x) \in \mathbb{R}^n, n > 1$.
- Single ODE: $y(x) \in \mathbb{R}, n = 1$.

Implicit DE of order m:

$$F\big(x, y(x), y'(x), ..., y^{(m)}(x)\big) = 0.$$

Explicit DE of order m:

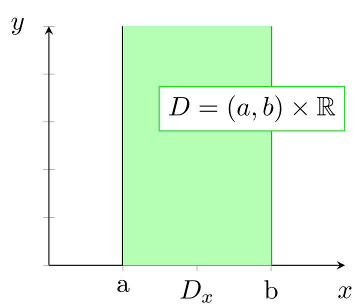
$$y^{(m)}(x) = f\big(x, y(x), y'(x), ..., y^{(m-1)}(x)\big).$$

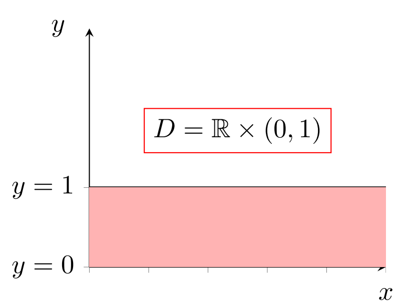
2 First-order ODEs

For now we have $m = 1, n = 1$. Consider $y'(x) = f\big(x, y(x)\big)$. The **explicit form** of the ODE is noted like $y' = f(x, y)$ explicit form of ODE. The function f is defined on $D = D_x \times D_y \in \mathbb{R}^2$.

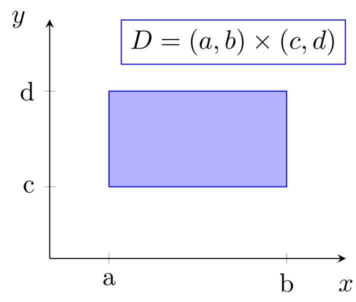
2.1 Examples

"Strip":

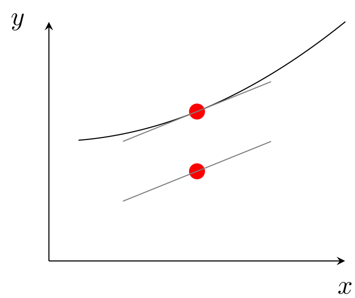




"Rectangle":



2.2 Geometric interpretation: Direction field

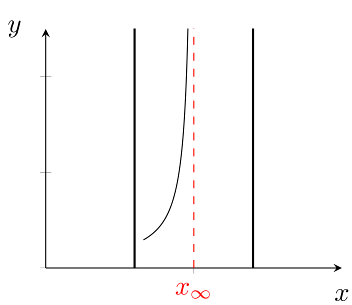


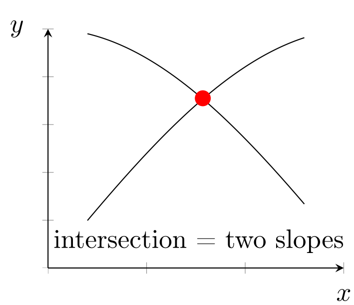
$$y' = f(x, y).$$

$f(x, y)$: Slop of tangent line of $y(x)$ in point (x, y). Using software like *dfield* we can visualize $f(x, y)$ as an vector field.

2.3 Observations

1. Through each point $(x_0, y_0) \in D$ there passes **exactly one** solution curve.
2. Each solution curve is **maximal**, meaning that the curve continues until the boundary of D (this includes a blow up to $+\infty$).
3. Solution curves **don't intersect**!





2.4 Existence and Uniqueness of a Solution

IVP (initial value problem): $y' = f(x, y)$, $y(x_0) = y_0$, domain D

2.4.1 Theorem: Peano

Assume that f is continuous on D and $(x_0, y_0) \in D$. Then the IVP has at leas one solution. This solution is minimal, meaning that we can continue the solution for $x < x_0$ and $x > x_0$ until the boundary of D.

2.4.2 Theorem: Picard-Lindelöf

Let f be continuous on D and let f be continuously differentiable with respect to y, that is, $\frac{\delta f}{\delta y}$ is continuous. Let $(x_0, y_0) \in D$. Then the IVP has a unique solution.

Example:

implicit form of ODE	explicit form
$(x^2 - x)y' = (2x-1)y, \quad y(x_0) = y_0.$	$y' = \dfrac{2x-1}{x^2-x}y \quad f(x,y) = \dfrac{2x-1}{x^2-x}y.$

Three cases:

1. $x_0 \notin \{0; 1\}$: Unique solution (due to Picard-Lindelöf Thm.).
2. $x_0 \in \{0; 1\}$ and $y_0 = 0$: Infinitely many solution
   $y(x) = cx(x-1), \; c \in \mathbb{R}$ Check it!
3. $x_0 \in \{0; 1\}$ and $y_0 \neq 0$: No solution.

2.5 Examples and Recap

$$IVP \begin{cases} y' &= \lambda y, \\ y(x_0) &= y_0. \end{cases}$$

$\lambda \in \mathbb{R}$ is a constant.

a) Domain of Definition D Here: $f(x, y) = \lambda y$ with x being the independent and y the dependent variable.

$$D = \mathbb{R} \times \mathbb{R}.$$

Meaning: f is well defined for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$

b) Direction field, software dfield, geometric interpretation.

c) Existence and uniquness of the soluton of the DE (differential equation).

d) $y(x) = c \cdot e^{\lambda x}, \quad c \in \mathbb{R}$ is the <u>general solution</u> of the DE. Check:

$$y' = c \cdot \lambda \cdot e^{\lambda x} = \lambda c e^{\lambda x} = \lambda y(x).$$

Parameter c is determined by the initial condition

$$y(x_0) = ce^{\lambda x_0} \stackrel{!}{=} y_0 \to c = y_0 e^{-\lambda x_0}.$$

Remark on "general solution": What does it mean? Suppose the DE $y' = \lambda y$ has another solution $\widetilde{y}$. Then it holds:

$$\begin{aligned}\left(\frac{\tilde{y}}{e^{\lambda x}}\right)' &= \frac{\tilde{y}' e^{\lambda x} - \tilde{y}\lambda e^{\lambda x}}{(e^{\lambda x})^2} \\ &= \frac{e^{\lambda x}(\tilde{y}' - \lambda \tilde{y})}{e^{2\lambda x}} \\ &= e^{-\lambda x}(\tilde{y}' - \lambda \tilde{y}) = e^{-\lambda x} \cdot 0 = 0\end{aligned}$$

$\tilde{y}' - \lambda \tilde{y} = 0$ since $\tilde{y}$ solves DE $\tilde{y}' = \lambda \tilde{y}$ by assumption.
Hence $\dfrac{\tilde{y}}{e^{\lambda x}}$ = constant and $\tilde{y}$ = constant $\cdot\, e^{\lambda x}$.
Hence all solutions of $y' = \lambda y$ have the form $y = ce^{\lambda x}$, $c \in \mathbb{R}$.

Today's topic: Analytical techniques for the solution of first-order differential equations.

Be careful: Techniques do not work for all DEs; require assumptions!

2.6 Separation of Variables

$$y'(x) = f(x) \cdot g(y).$$

1. Particular solution: The zeros of g are stationary (constant) solutions.
   Example: $g(y^* = 0 \to \tilde{y} \equiv y^*$ solves the DE.
   Check: $0 = (y^*)' = \tilde{y}' = f(x) \cdot g(y^*) \qquad g(y^*) = 0$

2. If f is continuous and g, g' are continuous, we couclude with (2.4) that the DE has a unique solution (Picard-Lindelöf Theorem).

3. Assume $g(y) \neq 0$:
   (a) $\frac{dy}{dx} = f(x)g(x)$
   (b) $\frac{dy}{g(y)} = f(x)dx$ Separation of variables
   (c) $\int \frac{dy}{g(y)} = \int f(x)dx + c$ Parameter
   (d) Calculate the antiderivatives.
   (e) If possible, rearrange (d). and isolate y.

2.7 Example, Separation of Variables

$$y' = -x^2 y \qquad \text{(see previous lecture).}$$

- $f(x) = -x^2$
- $g(y) = y$

1. Stationary solution: $y \equiv 0$
2.
$$\begin{aligned}
\frac{dy}{dx} &= -\, x^2 y \\
\frac{dy}{y} &= -\, x^2 dx \\
\int \frac{dy}{y} &= -\int x^2 dx + c, \quad c \in \mathbb{R} \\
\ln|y| &= -\,\frac{1}{3}x^3 + c \\
|y| &= \exp\left(-\frac{1}{3}x^3 + c\right) = e^c \cdot e^{-\frac{1}{3}x^3} \qquad e^c > 0 \\
y &= \pm\, e^c \cdot e^{-\frac{1}{3}x^3} \quad \text{plus} \quad y \equiv 0
\end{aligned}$$

Summary: $y(x) = \tilde{c} \cdot e^{-\frac{1}{3}x^3}, \quad \tilde{c} \in \mathbb{R}$

general solution of DE

From now on "antilogarithm":

$$\begin{aligned}
\ln|y| &= (...) + c, \quad c \in \mathbb{R} \\
y &= \tilde{c} \cdot e^{(\cdots)}, \quad \tilde{c} \in \mathbb{R}
\end{aligned}$$

2.8 Application: Water container

Given:: Cross-section area of cylindrical container A and cross-section area of (circular) outlet B.

Toricelli's Law:

$$v_{out} = \sqrt{2gh}.$$

Wanted: $h = h(t)$ water level in container.

Question: At which time is the container empty? For witch t^* does it hold $h(t^*) = 0$?
consider the change of volume ΔV of water in container.

$$\begin{aligned}
\Delta V_{cont} &= A \cdot \Delta h \\
\Delta V_{out} &= \gamma \cdot B \cdot v_{out} \cdot \Delta t
\end{aligned}$$

γ: Borda factor $\quad \gamma = 0.62$ for circular outlet.

Conservation of Mass:

$$\Delta V_{cont} + \Delta V_{out} = 0 \qquad \Delta V_{cont} = -\Delta V_{out}$$

$$A \cdot \Delta h = -\gamma \cdot B \cdot \sqrt{2gh}\Delta t$$

$$A \cdot \frac{\Delta h}{\Delta t} = -\gamma B \sqrt{2gh}$$

$\Delta t \to 0$

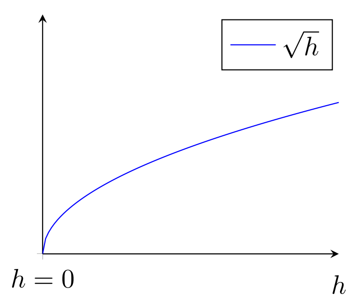
$$A * h' = -\gamma B \sqrt{2gh}$$

$$\text{IVP for } h(t) \begin{cases} h'(t) = -\gamma B \sqrt{2gh(t)} \\ h(0) = h_0. \end{cases}$$

Solve my separation of variables!

1. $\tilde{g}(h) = \sqrt{h}$
   $\tilde{f}(t) = -\gamma \frac{B}{A} \sqrt{2g}$
   $h'(t) = \tilde{f}(t) \cdot \tilde{g}(h)$
   Stationary solution: $h \equiv 0$ is zero of $\tilde{g}(h) = \sqrt{h}$.

2. Domain of definition: $D = \mathbb{R} \times [0, \infty)$ (time t $\times$ water level h)



$h \geq 0$ existence (Peano)
$h > 0$ existence and uniqueness.
(Picard-Lindelöt)

3. Separation of variables.

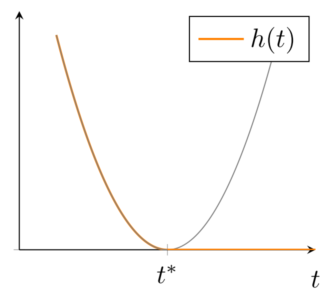
$$\text{let } \mu := \gamma \frac{B}{A} \sqrt{2g}, \quad h > 0$$

$$\frac{dh}{dt} = -\mu \sqrt{h}$$

$$\int \frac{dh}{\sqrt{h}} = -\mu \cdot \int 1 \cdot dt + c$$

$$2\sqrt{h} = -\mu \cdot t + c$$

Initial condition: $h(0) = h_0 > 0 \to c = \sqrt{h_0} = -\mu \cdot 0 + c$
$2\sqrt{h} = -\mu \cdot t + 2\sqrt{h_0}$ Rearrange, isolate h
$h(t) = \frac{1}{4}(2\sqrt{h_0} - \mu t)^2$ as long as $h > 0$!



Complete solution:

$$h(t) = \begin{cases} \frac{1}{4}(2\sqrt{h_0} - \mu t)^2 & t \leq t^*, \\ 0 & t > t^*. \end{cases}$$

$$\begin{aligned} \text{When is } h(t^*) =& \ 0 \\ 2\sqrt{h_0} =& \ \mu t^* \\ t^* =& \ \frac{2\sqrt{h_0}}{\mu}. \end{aligned}$$

2.9 Linear DE of first order

Standard form: $y'(x) = +p(x)y(x) = r(x)$ p, r can be nonlinear functions in x
DE is linear in y, y', not allowed are terms like $(y')^2, \sin(y), y \cdot y'$ etc.

$$p = p(x), r = r(x) \text{ continuous} \rightarrow \begin{array}{c}\text{existance and uniqueness} \\ \text{solution}\end{array}.$$

Recall: $y' = f(x, y)$

Here:

$$\begin{aligned} y' =& \ r(x) - p(x) \cdot y(x) =: f(x, y) \\ \frac{\delta f}{\delta y} =& -p(x) \end{aligned}.$$

Concepts:

$$\begin{aligned} y'(x) + p(x)y(x) = 0 & \quad \text{homogeneous linear DE} \\ y * (x) + p(x)y(x) = r(x) & \quad \text{inhoogenous linear DE } r(x) \not\equiv 0 \end{aligned}$$

<u>Remarks</u>:

$$\begin{aligned} Ax = b & \quad \text{linear system of equations} \\ Ax = 0 & \quad \text{homogeneous linear system of equations} \\ b \neq 0 & \quad \text{inhomogeneous linear system of equations.} \end{aligned}$$

2.10 General solution of the homogeneous DE

$$\begin{aligned} y * (x) + p(x)y(x) = 0 & \qquad y' = \frac{dy}{dx} \\ \frac{dy}{dx} =& -p(x)y(x) \end{aligned}$$

Variables of $p(x)$ and $y(x)$ are separate.

1. Stationary solution: $y \equiv 0$
2. $y \neq 0$, separation of variables.

$$\begin{aligned}
\frac{dy}{dx} &= -p(x)y(x) \\
\int \frac{dy}{y} &= -\int_{x_0}^{x} p(z)dz + c \\
ln|y| &= -\int_{x_0}^{x} p(z)dz + c = P(x) - P(x_0) + c \\
&= -P(x) + \tilde{c}
\end{aligned}$$

Let $P(\cdot)$ denote the antiderivative of $p(\cdot)$, that is, $P(z) = p(z)$.

"antilogarithm":

$$y(x) = \hat{c} \cdot e^{-P(x)}, \quad \hat{c} \in \mathbb{R}.$$

Reminder: $\int_{x_0}^{x} p(z)dz =: F(x)$

$$\frac{dF}{dx} = p(x) \cdot \frac{dx}{dx} - p(x)\frac{dx_0}{dx} + 0 = p(x).$$

by the Leibniz rule for parametric integrals.

2.11 General solution for inhomogeneous DE

Let y_1, y_2 denote solutions of the inhomogeneous DE:

$$\begin{aligned}
y_1'(x) + p(x)y_1(x) &= r(x) \\
y_2 * (x) + p(x)y_2(x) &= r(x)
\end{aligned}$$

subtract:

$$(y_1 - y_2)' + p(x)(y_1 - y_2) = 0$$

$$\begin{aligned}
y_D := y_1 + y_2 \to y_1 = y_2 + y_D \\
y_D'(x) + p(x)y_D(x) = 0
\end{aligned}$$

Using (2.10):

$$\begin{aligned}
y_D(x) = c * e^{-P(x)}, \quad c \in \mathbb{R} \\
y_1(x)y_2(x) + ce^{-P(x)}, \quad c \in \mathbb{R}
\end{aligned}$$

general solution of inhomogeneous DE = particular solution of inhomogeneous DE + general solution of homogeneous DE.

2.12 Variation of Constants

Ansatz to obtain a particular solution of the inhomogeneous DE

$$y_{p} = c(x)e^{-P(x)} \quad c = c(x).$$

Insert ansatz in the inhomogeneous DE (product rule!)

$$\begin{aligned} y'_P(x) + p(x)y_P(x)i &= r(x) \\ c'(x)e^{-P(x)} - p(x)e^{-P(x)}c(x) + p(x)c(x)e^{-P(x)} &= r(x) \\ c'(x)e^{-P(x)} &= r(x) \\ c'(x) &= e^{P(x)}r(x) \end{aligned}.$$

$$c(x) = \int_{x_0}^{x} e^{P(z)}r(z)dz + \tilde{c}.$$

Summary: $y_{P}(x) = c(x)e^{-P(x)}$

2.13 Electrical circuit

$$L \cdot I'(t) + R \cdot I(t) = U \in (\omega t) \tag{1}$$

Current $I = I(t)$, dependent on variable.
ω, L, R, U: Constants, independent of t
t: independent variable

Check linearity of (1):

a) General solution of homogeneous equation:

$$L * I'(t) + R(I) = 0.$$

Sep. of var.: $I(t) = \tilde{c}e^{-R\cdot\frac{t}{L}}, \quad \tilde{c} \in \mathbb{R}$

b) Particular solution if inhomogeneous equation (1).
Ansarz (variation of constants):
$y_p(t) = c(t)e^{-R\cdot\frac{t}{L}}$ Insert in inhomogeneous equation
$Ly'_p(t) + Ry_p(t) = U\sin(\omega t)$
$Lc*(t)e^{-R\frac{t}{L}} + Lc(t)\big(-\frac{R}{L}\big)e^{-R\frac{t}{L}} + Rc(t)e^{-R\frac{t}{L}} = U\sin(\omega t)$

$$c'(t) = \underbrace{\frac{U}{L}\sin(\omega te^{R\frac{t}{L}}}_{\substack{\text{Integrate with respect to } t.\\ \text{Integration by parts } 2x}}.$$

$$c(t) = Ue^{R\frac{t}{L}} \cdot \frac{R\sin(\omega t - \omega\cos(\omega t)}{R^2 + \omega^2 L^2}.$$

2.14 Bernoulli DE (nonlinear DE

$$y' + p(x)y = r(x)y^{\alpha}, \quad \alpha \in \mathbb{R} \tag{2}$$

$\alpha \in \{0; 1\}$ 1st order linear DE
$\alpha \notin \{0; 1\}$ Ansatz: $u(x) = y(x)^{1-\alpha}$
$r(x) \neq 0$

$$\begin{aligned}
u'(x) &= (1-\alpha)y(x)^{-\alpha} \cdot y'(x) \\
&= (1-\alpha)y(x)^{-\alpha} \cdot \underbrace{(-p(x)y(x) + r(x)y^{\alpha}}_{y' \text{ according to DE (2)}} . \\
&= (1-\alpha)(-p(x)u(x) + r(x)) \\
&\Rightarrow u*(x) + (1-\alpha)p(x)u(x) = (1-\alpha)r(x)
\end{aligned}$$

2.15 Application: Population Dynamics

$y(t)$: population density, $y \geq 0$

logistic DE
Verhust, 1838
$$\begin{cases} y' &= ay(1 - \frac{y}{y_\infty}), \\ y(0) &= y. \end{cases} .$$

a, y_∞	:	model parameters; constant in time	
$a > 0$	:	population growth rate	$y > y_\infty \Rightarrow y' < 0$
$y_\infty > 0$	:	carrying capacity, limit population	

Observation:

$$y' = ay - \frac{a}{y_\infty} \cdot y^2$$
$$y' + p(x)y = r(x)y^{\alpha}$$

nonlinear DE $\rightarrow$ Bernoulli DE, $\alpha = 2$
$p(x) = -a, \quad r(x) = -\frac{a}{y_\infty}$