Technische Universität München

Math 3

MA9803 | B.Sc. Engineering Science and B.Sc. Aerospace

Modeling and Simulation with Ordinary Differential Equations

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October 29, 2024

Abstract

Some people think that stiff challenges are the best device to induce learning, but I am not one of them. The natural way to learn something is by spending vast amounts of easy, enjoyable time at it. This goes whether you want to speak German, sight-read at the piano, type, or do mathematics. Give me the German storybook for fifth graders that I feel like reading in bed, not Goethe and a dictionary. The latter will bring rapid progress at first, then exhaustion and failure to resolve.

L. N. Trefethen. Trefethen's index cards - Forty years of notes about People, Words and Mathematics. World Scientific, 2011, S. 86

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1 Introduction

Previously: We had equations like $y^2 + 4y + 1 = 0$, where the solution is a number.

$$\int_a^b f(x)dt = \text{number} = F(b) - F(a)$$

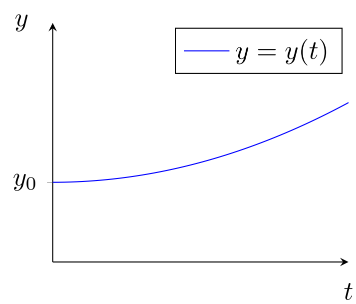
New: $f'(x)$ is given, determine $f(x)$. The solution is a function.

$$\begin{aligned} \text{Velocity}(t) &= \text{Position}'(t) \quad && \text{given} \\ &= \text{Position}(t) && \text{wanted} \end{aligned}.$$

1.1 Differential Equations

Example: Interest rate
$y(t):$ assets at time t.
$\lambda < 0:$ constant interest rate.

$$\text{IVP} \begin{cases} y'(t) &= \lambda \cdot y(t), \\ y(0) &= y_0. \end{cases}$$



Example: Radioactive Decay
$y(t):$ mass of a substance at time t.
$\lambda < 0:$ decay rate.

$$y'(t) = \lambda \cdot y(t).$$

1.2 Problem Solving Steps in Science & Technology

"Reality" $\longrightarrow$ Model = "Mathematisation"

- Laws of nature (Physics, Chemistry).
- Assumtions, hypoteses $\rightarrow$ differential equations.
- Analytical or symbolic solution (exact solution). not practical
- Numerical solution. in practise

$$\textbf{Change of state} \;=\; \textbf{Function (State)}.$$

Ordenary Diff. Eqn. (ODE: one scalar variable (e.g. time) this semester

$$y'(t) = \lambda y(t).$$

Partial Diff. Eqn. (PDE): multiple variables (e.g. time and space) next semester (MA9804)

$$u = u(t, x) \quad x = (x_1, x_2, x_3).$$

$$\frac{\delta u}{\delta t} = k \cdot (\frac{\delta^2 u}{\delta x_1^2} \frac{\delta^2 u}{\delta x_2^2} \frac{\delta^2 u}{\delta x_3^2}).$$

Heat equation. u: temperature

Oftentimes: Solution depends on (design) parameters. $y = y(t;\rho)$, $u = u(t, x, \rho)$

- Parameter identification Inverse Problem
- Design optimisation
- Random / Stochastic parameters Uncertainty quantification

1.3 Application: Solid Mechanics

Newton's Laws of motion: $\frac{dp}{dt} = F$

Change of momentum = force action.

a) Calculate orbits of satellites; today: GPS

b) Vibrations in automobiles; reliability of structures. (earthquakes; spring-mass-system)

1.4 Application: Electric Ciruits

Ohm's Law: $U = R \cdot I$ (voltage = resistance * current)

Kirchhoff's Cirquit Laws: $\rightarrow$ System of ODEs

Law of Induction: $U = n \cdot I'$ (Coil); $I = C \cdot U'$ (Capacitor). Here the $'$ is the time derivative.

1.5 Application: Chemical relations

Chemical reaction: $A + B \xrightarrow[k]{} C.$

Here $c_A(t), c_B(t), c_C(t)$ are the Concentrations of the substance A, B, C in $\frac{\text{Mol}}{\text{l}}$.

Law of mass action: Change of concentration is proportional to the concentration of the resulting substances:

$$\begin{aligned} c_C' &= \ \ k \cdot c_A \cdot c_B \\ c_A' &= -k \cdot c_A \cdot c_B \\ c_B' &= -k \cdot c_A \cdot c_B \end{aligned}$$

1.6 Classification of ODEs

$$y(x).$$

x: independent variable scalar $\qquad$ y: dependent variable

Ordenary DE:

- System of ODEs: $y(x) \in \mathbb{R}^n, n > 1$.
- Single ODE: $\qquad y(x) \in \mathbb{R}, n = 1$.

Implicit DE of order m:

$$F\big(x, y(x), y'(x), ..., y^{(m)}(x)\big) = 0.$$

Explicit DE of order m:

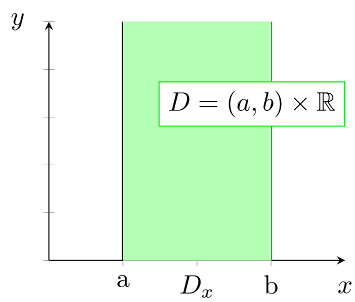
$$y^{(m)}(x) = f\big(x, y(x), y'(x), ..., y^{(m-1)}(x)\big).$$

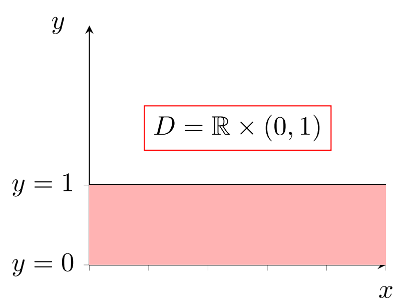
2 First-order ODEs

For now we have $m = 1, n = 1$. Consider $y'(x) = f\big(x, y(x)\big)$. The **explicit form** of the ODE is noted like $y' = f(x, y)$ explicit form of ODE. The function f is defined on $D = D_x \times D_y \in \mathbb{R}^2$.

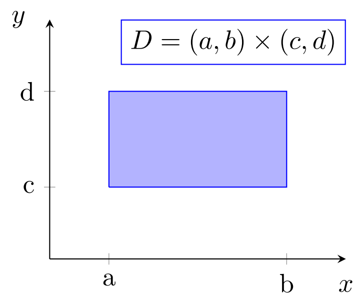
2.1 Examples

"Strip":

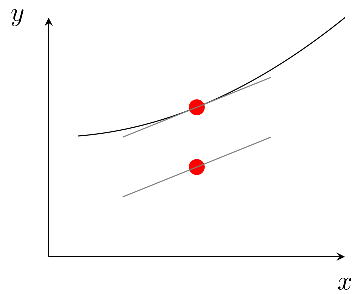




"Rectangle":



2.2 Geometric Interpretation: Direction Field

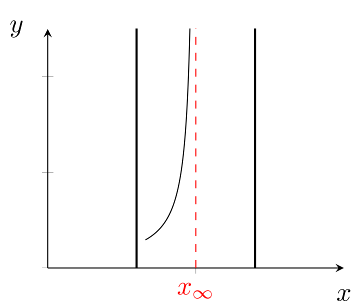


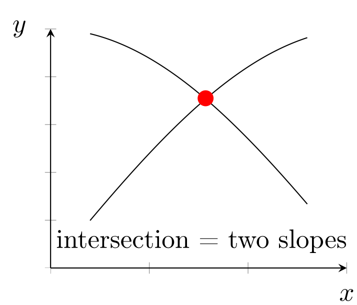
$$y' = f(x, y).$$

$f(x, y)$: Slop of tangent line of $y(x)$ in point (x, y). Using software like *dfield* we can visualize $f(x, y)$ as an vector field.

2.3 Observations

1. Through each point $(x_0, y_0) \in D$ there passes **exactly one** solution curve.
2. Each solution curve is **maximal**, meaning that the curve continues until the boundary of D (this includes a blow up to $+\infty$).
3. Solution curves **don't intersect**!





2.4 Existence and Uniqueness of a Solution

IVP (initial value problem): $y' = f(x, y)$, $y(x_0) = y_0$, domain D

2.4.1 Theorem: Peano

Assume that f is continuous on D and $(x_0, y_0) \in D$. Then the IVP has at leas one solution. This solution is minimal, meaning that we can continue the solution for $x < x_0$ and $x > x_0$ until the boundary of D.

2.4.2 Theorem: Picard-Lindelöf

Let f be continuous on D and let f be continuously differentiable with respect to y, that is, $\frac{\delta f}{\delta y}$ is continuous. Let $(x_0, y_0) \in D$. Then the IVP has a unique solution.

Example:

implicit form of ODE

$$(x^2 - x)y' = (2x-1)y, \quad y(x_0) = y_0.$$

explicit form

$$y' = \frac{2x-1}{x^2-x}y \quad f(x,y) = \frac{2x-1}{x^2-x}y.$$

Three cases:

1. $x_0 \notin \{0; 1\}$: Unique solution (due to Picard-Lindelöf Thm.).
2. $x_0 \in \{0; 1\}$ and $y_0 = 0$: Infinitely many solution
   $y(x) = cx(x-1),\ c \in \mathbb{R}$ Check it!
3. $x_0 \in \{0; 1\}$ and $y_0 \neq 0$: No solution.

2.5 Examples and Recap

$$\text{IVP} \begin{cases} y' &= \lambda y, \\ y(x_0) &= y_0. \end{cases}$$

$\lambda \in \mathbb{R}$ is a constant.

a) Domain of Definition D Here: $f(x,y) = \lambda y$ with x being the independent and y the dependent variable.

$$D = \mathbb{R} \times \mathbb{R}.$$

Meaning: f is well defined for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$

b) Direction field, software dfield, geometric interpretation.

c) Existence and uniquness of the soluton of the DE (differential equation).

d) $y(x) = c \cdot e^{\lambda x}, \quad c \in \mathbb{R}$ is the general solution of the DE. Check:

$$y' = c \cdot \lambda \cdot e^{\lambda x} = \lambda c e^{\lambda x} = \lambda y(x).$$

Parameter c is determined by the initial condition

$$y(x_0) = c e^{\lambda x_0} \overset{!}{=} y_0 \to c = y_0 e^{-\lambda x_0}.$$

Remark on "general solution": What does it mean? Suppose the DE $y' = \lambda y$ has another solution $\widetilde{y}$. Then it holds:

$$\begin{aligned}\left(\frac{\tilde{y}}{e^{\lambda x}}\right)' &= \frac{\tilde{y}' e^{\lambda x} - \tilde{y}\lambda e^{\lambda x}}{(e^{\lambda x})^2} \\ &= \frac{e^{\lambda x}(\tilde{y}' - \lambda \tilde{y})}{e^{2\lambda x}} \\ &= e^{-\lambda x}(\tilde{y}' - \lambda \tilde{y}) = e^{-\lambda x} \cdot 0 = 0\end{aligned}$$

$\tilde{y}' - \lambda \tilde{y} = 0$ since $\tilde{y}$ solves DE $\tilde{y}' = \lambda \tilde{y}$ by assumption.
Hence $\dfrac{\tilde{y}}{e^{\lambda x}} =$ constant and $\tilde{y} =$ constant $\cdot\, e^{\lambda x}$.
Hence all solutions of $y' = \lambda y$ have the form $y = c e^{\lambda x}$, $c \in \mathbb{R}$.

Today's topic: Analytical techniques for the solution of first-order differential equations.

Be careful: Techniques do not work for all DEs; require assumptions!

2.6 Separation of Variables

$$y'(x) = f(x) \cdot g(y).$$

1. Particular solution: The zeros of g are stationary (constant) solutions.
   Example: $g(y^* = 0 \to \tilde{y} \equiv y^*$ solves the DE.
   Check: $0 = (y^*)' = \tilde{y}' = f(x) \cdot g(y^*) \qquad g(y^*) = 0$

2. If f is continuous and g, g' are continuous, we couclude with (2.4) that the DE has a unique solution (Picard-Lindelöf Theorem).

3. Assume $g(y) \neq 0$:

   (a) $\frac{dy}{dx} = f(x)g(x)$

   (b) $\frac{dy}{g(y)} = f(x)dx$ Separation of variables

   (c) $\int \frac{dy}{g(y)} = \int f(x)dx + c$ Parameter

   (d) Calculate the antiderivatives.

   (e) If possible, rearrange (d). and isolate y.

2.7 Example, Separation of Variables

$$y' = -x^2 y \qquad \text{(see previous lecture).}$$

- $f(x) = -x^2$
- $g(y) = y$

1. Stationary solution: $y \equiv 0$
2. $$\begin{aligned} \frac{dy}{dx} &= -\,x^2 y \\ \frac{dy}{y} &= -\,x^2 dx \\ \int \frac{dy}{y} &= -\int x^2 dx + c, \quad c \in \mathbb{R} \\ \ln|y| &= -\,\frac{1}{3}x^3 + c \\ |y| &= \exp\left(-\frac{1}{3}x^3 + c\right) = e^c \cdot e^{-\frac{1}{3}x^3} \qquad e^c > 0 \\ y &= \pm\, e^c \cdot e^{-\frac{1}{3}x^3} \quad \text{plus} \quad y \equiv 0 \end{aligned}$$

Summary: $y(x) = \tilde{c} \cdot e^{-\frac{1}{3}x^3}, \quad \tilde{c} \in \mathbb{R}$

general solution of DE

From now on "antilogarithm":

$$\begin{aligned} \ln|y| &= (...) + c, \quad c \in \mathbb{R} \\ y &= \tilde{c} \cdot e^{(\dots)}, \quad \tilde{c} \in \mathbb{R} \end{aligned}$$

2.8 Application: Water Container

Given:: Cross-section area of cylindrical container A and cross-section area of (circular) outlet B.

Toricelli's Law:

$$v_{out} = \sqrt{2gh}.$$

Wanted: $h = h(t)$ water level in container.

Question: At which time is the container empty? For witch t^* does it hold $h(t^*) = 0$?
consider the change of volume ΔV of water in container.

$$\begin{aligned} \Delta V_{cont} &= A \cdot \Delta h \\ \Delta V_{out} &= \gamma \cdot B \cdot v_{out} \cdot \Delta t \end{aligned}$$

γ: Borda factor $\quad \gamma = 0.62$ for circular outlet.

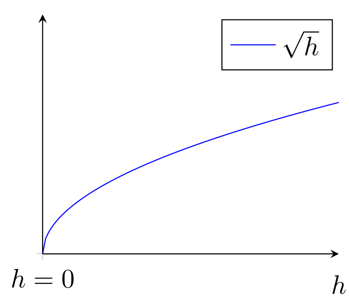
Conservation of Mass:

$$\Delta V_{cont} + \Delta V_{out} = 0 \qquad \Delta V_{cont} = -\Delta V_{out}$$
$$A \cdot \Delta h = -\gamma \cdot B \cdot \sqrt{2gh}\Delta t$$
$$A \cdot \frac{\Delta h}{\Delta t} = -\gamma B\sqrt{2gh}$$
$\Delta t \to 0$
$$A * h' = -\gamma B\sqrt{2gh}$$
$$\text{IVP for } h(t) \begin{cases} h'(t) = -\gamma B\sqrt{2gh(t)} \\ h(0) = h_0. \end{cases}$$

Solve my separation of variables!

1. $\tilde{g}(h) = \sqrt{h}$
   $\tilde{f}(t) = -\gamma \frac{B}{A}\sqrt{2g}$
   $h'(t) = \tilde{f}(t) \cdot \tilde{g}(h)$
   Stationary solution: $h \equiv 0$ is zero of $\tilde{g}(h) = \sqrt{h}$.

2. Domain of definition: $D = \mathbb{R} \times [0, \infty)$ (time t $\times$ water level h)

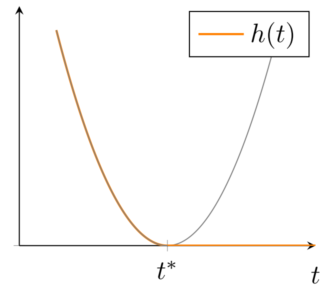


$h \geq 0$ existence (Peano)
$h > 0$ existence and uniqueness.
(Picard-Lindelöt)

3. Separation of variables.

$$\text{let } \mu := \gamma\frac{B}{A}\sqrt{2g}, \quad h > 0$$
$$\frac{dh}{dt} = -\mu\sqrt{h}$$
$$\int \frac{dh}{\sqrt{h}} = -\mu \cdot \int 1 \cdot dt + c$$
$$2\sqrt{h} = -\mu \cdot t + c$$

Initial condition: $h(0) = h_0 > 0 \to c = \sqrt{h_0} = -\mu \cdot 0 + c$
$2\sqrt{h} = -\mu \cdot t + 2\sqrt{h_0}$ Rearrange, isolate h
$h(t) = \frac{1}{4}(2\sqrt{h_0} - \mu t)^2$ as long as $h > 0$!



Complete Solution:

$$h(t) = \begin{cases} \frac{1}{4}(2\sqrt{h_0} - \mu t)^2 & t \leq t^*, \\ 0 & t > t^*. \end{cases}$$

$$\begin{aligned} \text{When is } h(t^*) = &\ 0 \\ 2\sqrt{h_0} = &\ \mu t^* \\ t^* = &\ \frac{2\sqrt{h_0}}{\mu}. \end{aligned}$$

2.9 Linear DE of first Order

Standard form: $y'(x) = +p(x)y(x) = r(x)$ p, r can be nonlinear functions in x

DE is linear in y, y', not allowed are terms like $(y')^2, \sin(y), y \cdot y'$ etc.

$$p = p(x), r = r(x) \text{ continuous} \to \begin{array}{c}\text{existance and uniqueness} \\ \text{solution}\end{array}.$$

Recall: $y' = f(x, y)$

Here:

$$\begin{aligned} y' &= r(x) - p(x) \cdot y(x) =: f(x, y) \\ \frac{\delta f}{\delta y} &= -p(x) \end{aligned}.$$

Concepts:

$$\begin{aligned} y'(x) + p(x)y(x) = 0 &\quad \text{homogeneous linear DE} \\ y*(x) + p(x)y(x) = r(x) &\quad \text{inhoogenous linear DE } r(x) \not\equiv 0 \end{aligned}$$

Remarks:

$$\begin{aligned} Ax = b &\quad \text{linear system of equations} \\ Ax = 0 &\quad \text{homogeneous linear system of equations} \\ b \neq 0 &\quad \text{inhomogeneous linear system of equations.} \end{aligned}$$

2.10 General Solution of the homogeneous DE

$$\begin{aligned} y*(x) + p(x)y(x) = 0 &\qquad y' = \frac{dy}{dx} \\ \frac{dy}{dx} &= -p(x)y(x) \end{aligned}$$

Variables of $p(x)$ and $y(x)$ are separate.

1. Stationary solution: $y \equiv 0$
2. $y \neq 0$, separation of variables.

$$\begin{aligned}\frac{dy}{dx} &= -p(x)y(x)\\ \int \frac{dy}{y} &= -\int_{x_0}^{x} p(z)dz + c\\ ln|y| &= -\int_{x_0}^{x} p(z)dz + c = P(x) - P(x_0) + c\\ &= -P(x) + \tilde{c}\end{aligned}$$

Let $P(\cdot)$ denote the antiderivative of $p(\cdot)$, that is, $P(z) = p(z)$.

"antilogarithm":

$$y(x) = \hat{c} \cdot e^{-P(x)}, \quad \hat{c} \in \mathbb{R}.$$

Reminder: $\int_{x_0}^{x} p(z)dz =: F(x)$

$$\frac{dF}{dx} = p(x) \cdot \frac{dx}{dx} - p(x)\frac{dx_0}{dx} + 0 = p(x).$$

by the Leibniz rule for parametric integrals.

2.11 General Solution for inhomogeneous DE

Let y_1, y_2 denote solutions of the inhomogeneous DE:

$$\begin{aligned}y_1'(x) + p(x)y_1(x) &= r(x)\\ y_2 * (x) + p(x)y_2(x) &= r(x)\end{aligned}$$

subtract:

$$(y_1 - y_2)' + p(x)(y_1 - y_2) = 0$$

$$\begin{aligned}y_D := y_1 + y_2 \rightarrow y_1 &= y_2 + y_D\\ y_D'(x) + p(x)y_D(x) &= 0\end{aligned}$$

Using (2.10):

$$\begin{aligned}y_D(x) &= c * e^{-P(x)}, \quad c \in \mathbb{R}\\ y_1(x)y_2(x) &+ ce^{-P(x)}, \quad c \in \mathbb{R}\end{aligned}$$

general solution of inhomogeneous DE = particular solution of inhomogeneous DE + general solution of homogeneous DE.

2.12 Variation of Constants

Ansatz to obtain a particular solution of the inhomogeneous DE

$$y_{p} = c(x)e^{-P(x)} \quad c = c(x).$$

Insert ansatz in the inhomogeneous DE (product rule!)

$$\begin{aligned} y_P'(x) + p(x)y_P(x)i &= r(x) \\ c'(x)e^{-P(x)} - p(x)e^{-P(x)}c(x) + p(x)c(x)e^{-P(x)} &= r(x) \\ c'(x)e^{-P(x)} &= r(x) \\ c'(x) &= e^{P(x)}r(x) \end{aligned}.$$

$$c(x) = \int_{x_0}^{x} e^{P(z)}r(z)dz + \tilde{c}.$$

Summary: $y_{P}(x) = c(x)e^{-P(x)}$

2.13 Electrical Circuit

$$L \cdot I'(t) + R \cdot I(t) = U \in (\omega t) \tag{1}$$

Current $I = I(t)$, dependent on variable.
ω, L, R, U: Constants, independent of t
t: independent variable

Check linearity of (1):

a) General solution of homogeneous equation:

$$L * I'(t) + R(I) = 0.$$

Sep. of var.: $I(t) = \tilde{c}e^{-R\cdot\frac{t}{L}}, \quad \tilde{c} \in \mathbb{R}$

b) Particular solution if inhomogeneous equation (1).
Ansarz (variation of constants):
$y_p(t) = c(t)e^{-R\cdot\frac{t}{L}}$ Insert in inhomogeneous equation
$Ly_p'(t) + Ry_p(t) = U\sin(\omega t)$
$Lc*(t)e^{-R\frac{t}{L}} + Lc(t)\big(-\frac{R}{L}\big)e^{-R\frac{t}{L}} + Rc(t)e^{-R\frac{t}{L}} = U\sin(\omega t)$

$$c'(t) = \underbrace{\frac{U}{L}\sin(\omega t e^{R\frac{t}{L}})}_{\substack{\text{Integrate with respect to } t. \\ \text{Integration by parts } 2x}}.$$

$$c(t) = Ue^{R\frac{t}{L}} \cdot \frac{R\sin(\omega t) - \omega\cos(\omega t)}{R^2 + \omega^2 L^2}.$$

2.14 Bernoulli DE (nonlinear DE

$$y' + p(x)y = r(x)y^{\alpha}, \quad \alpha \in \mathbb{R} \tag{2}$$

$\alpha \in \{0; 1\}$ 1st order linear DE
$\alpha \notin \{0; 1\}$ Ansatz: $u(x) = y(x)^{1-\alpha}$
$r(x) \neq 0$

$$\begin{aligned} u'(x) &= (1-\alpha)y(x)^{-\alpha} \cdot y'(x) \\ &= (1-\alpha)y(x)^{-\alpha} \cdot \underbrace{(-p(x)y(x) + r(x)y^{\alpha}}_{y' \text{ according to DE (2)}} \\ &= (1-\alpha)(-p(x)u(x) + r(x)) \\ &\Rightarrow u*(x) + (1-\alpha)p(x)u(x) = (1-\alpha)r(x) \end{aligned}.$$

2.15 Application: Population Dynamics

$y(t)$: population density, $y \geq 0$

logistic DE
Verhust, 1838
$$\begin{cases} y' &= ay(1 - \frac{y}{y_\infty}), \\ y(0) &= y. \end{cases}.$$

a, y_∞	:	model parameters; constant in time
$a > 0$	:	population growth rate
$y_\infty > 0$	:	carrying capacity, limit population

$y > y_\infty \Rightarrow y' < 0$

Observation:

$$y' = ay - \frac{a}{y_\infty} \cdot y^2$$
$$y' + p(x)y = r(x)y^{\alpha}$$

nonlinear DE $\rightarrow$ Bernoulli DE, $\alpha = 2$
$p(x) = -a, \quad r(x) = -\frac{a}{y_\infty}$

Ansatz: $u(t) = y(t)^{1-\alpha} = y^{-1}(t) \quad [\alpha = 2]$

$$\begin{aligned} u' = -y^{-2}(t) \cdot y' &= -y^{-2}(ay - \frac{a}{y_\infty}y^2) \\ = -au + \frac{a}{y_\infty} \end{aligned}.$$

homomegous DE: $U' = -au$

1. $u_{hom}0(t) = c \cdot e^{-at}, \quad c \in \mathbb{R}$

2. Variation of constants: $u_p(t) = c(t)e^{-at}$
   ... $c'(t) = e^{at}e^{-at} = \frac{1}{y_\infty}$ constant function!
   Check:
$$u_p' = -au_p + \frac{a}{y_\infty}$$
$$0 = \frac{-a}{y_\infty} + \frac{a}{y_\infty}$$
3. $u(t) = u_{hom}(t) + u_p(t)$ $\longleftarrow$ general solution
   $= ce^{-at} + \frac{1}{y_\infty} = y_\infty^{-1}(1 + cy_\infty e^{-at})$
4. Initial condition:
$$y(0) = y_0$$
$$u(0) = \frac{1}{y(0)} = \frac{1}{y_0}$$
$$\frac{1}{y_0} = u(0) = \frac{1}{y_\infty} + c \to c = \frac{1}{y_0} - \frac{1}{y_\infty} = \frac{y_\infty - y_0}{y_0 y_\infty}$$
5. $y(t) = u(t)^{-1} = \underbrace{\frac{y_\infty}{1 + (\frac{y_\infty - y_0}{y_0})e^{-at}}}_{\text{Solution of the IVP}}$
   Observe: $\lim_{t\to+\infty} y(t) = \frac{y_\infty}{1 + (...) \cdot 0} = y_\infty$
6. Remark: Stationary solutions: $y' = ay(1 - \frac{y}{y_\infty} =: f(y)$
   - zero of f
   - $y_1 \equiv 0$
   - $y_2 \equiv y_\infty$

   Note: The symbol $\equiv$ means a functions is constant equal to the right-hand side.

2.16 Order Reduction

Autonomous 2nd order DE.
$$F(y, y', y'') = 0.$$

$y = y(x)$ The independent variable x does not occur explicitly in the DE.

Idea: Consider y as independent variable and $v = \frac{dy}{dx} = v(y)$ as dependent variable.
$$F(y, \underbrace{v(y)}_{=y'}, ?) = 0$$
$$y'' = \frac{dy'}{dx} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} \cdot v(y)$$

Summary: $F(y, v(y), v(y) \cdot \frac{dv}{dy}) = 0$

Step 1: Solve 1st order DE to obtain $v = v(y)$.

Step 2:

$$\frac{dy}{dx} = v(y) \quad \text{1st order DE for } y$$
$$\frac{dy}{v(y)} = dx \quad \text{Separtation of the variables!}$$

2.17 Example: Oscillation

A mass m is secured to the wall with a spring (coefficient k) allowing vertical translation $y = y(t)$.

$$m \cdot y'' + k \cdot y = 0$$

Order reduction: $F(y, y', y'') = m \cdot y'' + k \cdot y = 0$ Variable t does not occur explicitly in DE $\rightarrow$ autonomous DE

$m > 0$ mass

$k > 0$ spring constant

Ansatz: $v = v(y) = y' \longrightarrow y'' = v(y) \cdot \frac{dv}{dy}$

DE:

$$my'' + ky = 0 \qquad y\text{: independent variable}$$
$$mv\frac{dv}{dy} + dy = 0 \qquad v\text{: dependent variable}$$

1. Seperation of var.

$$v\frac{dv}{dy} = -\frac{k}{m}y$$
$$\int vdv = -\frac{k}{m}\int ydy + \underbrace{\tilde{c}}_{\geq 0}$$
$$\frac{1}{2}v^2 = -\frac{1}{2}\frac{k}{m}y^2 + \frac{1}{2}c_1^2$$
$$v = v(y) = \pm(c_1^2 - \frac{k}{m}y^2)^{\frac{1}{2}}$$

2. $y' = \pm(c_1^2 - \frac{k}{m}y^2)^{\frac{1}{2}}$
   Solve my separation of var. $y = y(t), y' = \frac{dy}{dt}$

$$\underbrace{\pm\int\frac{dy}{(c_1^2 - \frac{k}{m}y^2)}}_{(*)} = \int dt + c_2.$$

Note: $\left(\frac{1}{|\alpha|\arcsin(\alpha y)}\right)' = \frac{1}{|\alpha|}\cdot\alpha\cdot\frac{1}{\sqrt{1-\alpha^2y^2}} = \pm\frac{1}{\sqrt{1-\alpha^2y^2}}$

$$
\begin{aligned}
(*) \quad &= \frac{\pm 1}{|c_1|}\int \frac{dy}{\left(1-\frac{k}{m}\frac{1}{c_1^2}y^2\right)^{\frac{1}{2}}} \\
&= \frac{1}{|c_1|}\sqrt{\frac{k}{m}}^{-1}|c_1|\arcsin\left(\sqrt{\frac{k}{m}}c_1^{-1}y\right)
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\frac{k}{m}}^{-1}\arcsin\left(\sqrt{\frac{k}{m}}c_1^{-1}y\right) &= t + c_2 \\
\arcsin\left(\sqrt{\frac{k}{m}}c_1^{-1}y\right) &= \sqrt{\frac{k}{m}}t + \tilde{c}_2 \\
\sqrt{\frac{k}{m}}c_1^{-1}y &= \sin\left(\sqrt{\frac{k}{m}}t + \tilde{c}_2\right) \\
y &= \sqrt{\frac{m}{k}}c_1\sin\left(\sqrt{\frac{k}{m}}t + \tilde{c}_2\right) \\
y(t) &= \tilde{c}_1\sin\left(\sqrt{\frac{k}{m}}t + \tilde{c}_2\right)
\end{aligned}
$$

$\rightarrow$ two parameters in general solution

3 Second-order Linear Differential Equations

3.1 Standard Form

$$
\text{IVP}\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \\ \left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}\right\} \text{initial conditions} \end{cases}
$$

Existance and uniqueness of solution for p, q, r continuous ($\rightarrow$ Picard-Lindelöf).

Idea: Rewrite IVP as system of two 1st order DEs

Concepts:
$r(x) \equiv 0$ homogeneous DE
$r(x) \not\equiv 0$ inhomogeneous DE
(see 1st order DEs)

Solution: $y(x) = y_{\text{hom}}(x) + y_p(x)$

general solution of inhomoegnous DE = general solution of hom. DE + (one) particular solution of inhomog. DE.

$$
\begin{aligned}
y''_{\text{hom}} + p(x)y'_{\text{hom}}(x) + q(x)y_{\text{hom}}(x) &= 0 \\
y''_p(x) + p(x)y'_p(x) + q(x)y_p(x) &= r(x)
\end{aligned}
$$

y_{hom} has TWO parameters ($\rightarrow$ two initial conditions)

3.2 Superposition Principle

Let y_1, y_2 denote solutions of the homogeneous DE.

Superposition = linear combination:

$$y(x) := \underbrace{a}_{\text{constant}} \cdot y_1(x) + \underbrace{b}_{\text{constant}} \cdot y_2(x).$$

$$\begin{aligned} y'(x) &= a y_1'(x) + b y_2'(x) \\ y''(x) &= a y_1''(x) + b y_2''(x) \end{aligned}$$

$$\left.\begin{aligned} y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) &= 0 \quad | \cdot a \\ y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) &= 0 \quad | \cdot b \end{aligned}\right] +$$

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

- Any superposition of solutions of homogeneous DE is again a solution of homogeneous DE.
- General solution of inhomegeneous DE has the form:

$$y(x) = c_1 \cdot y_1(x) + c_2 \cdot y_2(x) + y_p(x).$$

Question: Under which conditions can we choose c_1 and c_2 such that we can impose any pair (x_0, y_0) and (x_0, y_0) or initial conditions?

Answer: $y(x_0) = y_0$ and $y'(x_0) = y_0'$

Ansatz: $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

Initial conditions:

$$(x) \begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) & = \overbrace{y(x_0)}^{=! y_0} - y_p(x), \\ c_1 y_1'(x_0) + c_2 y_2 * (x_0) & = \underbrace{y'(x_0)}_{=! y_0'} - y_p'(x_0). \end{cases} .$$

Note that $(*)$ is a 2×2 linear system of equns for unknowns c_1, c_2

$$\underbrace{\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\vec{c}} = \underbrace{\begin{pmatrix} y_0 - y_p(x_0) \\ y_0' - y_p'(x_0) \end{pmatrix}}_{\vec{b} \in \mathbb{R}^2}.$$

The vector $\vec{b} \in \mathbb{R}^2$ takes on arbitrary values. To ensure that $A\vec{c} = \vec{b}$ has a section we require: $\det A = |A| \neq 0$ A is an invertable matrix

$$w(y_1, y_2)\Big|_{x=x_0} := \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \begin{matrix} - \\ + \end{matrix}$$
$$= y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$$

w	:	Wronski determinant (function of x!)
w	$=$	$w(x)$
$w_{(y_1,y_2)}(x)$	$=$	a

Recall: $y''(x) + p(x)y'(x) + q(x)y(x) = r(x)$

General solution: $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x), \quad c_1, c_2 \in \mathbb{R}$

y_p	:	some solution of inhom. DE	$r(x) \neq 0$
y_1, y_2	:	solutions of hom. DE	$r(x) \equiv 0$

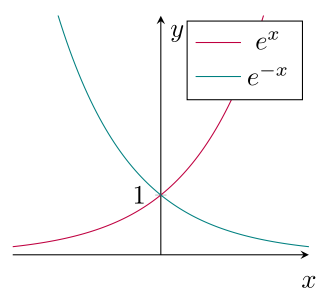
Important condition: $w_{(y_1,y_2)}(x) \neq 0$

$$w_{(y_1,y_2)}(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

3.3 Fundamental System ("basis")

Definition: Let y_1, y_2 be solutions of the homogeneous DE. y_1 and y_2 are called <u>fundamental system</u> if $w_{(y_1,y_2)}(x) \neq 0$ for all x in a suitable interval $[a, b]$.

Ex.: $y''(x) - y(x) = 0$
$y_1(x) = e^x$
$y_2() = e^{-x}$



$$w(y_1, y_2) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1) - 2 \neq 0.$$

$\Rightarrow \{y_1; y_2\}$ is a fundamental system for the DE $y'' - y = 0$

Ex.: $y''(x) - y'(x) = 0, \quad y_1(x) = e^x, \quad y_2(x), \equiv 1$

$$w(y_1), y(y_2) = \begin{vmatrix} e^x & 1 \\ e^x & 0 \end{vmatrix} = 0 - e^x = -e^x \neq 0.$$

$\Rightarrow \{y_1; y_2\}$ is a fundamental system for the DE $y'' - y' = 0$

Remark: $w = w(x)$ satisfies a 1st order DE:

$$w' = (y_1 y_2' - y_1' y_2)' = y_1' y_2' + \underbrace{y_1 y_2''' - y_1'' y_2} = w' - y_1' y_2'.$$

$$\left.\begin{array}{ll} y_1'' + p(x) y_1' + q(x) y_1(x) = 0 & | \cdot (-y_2) \\ y_2'' + p(x) y_2' + q(x) y_2(x) = 0 & | \cdot y_1 \end{array}\right] +$$

$$w'(x) + p(x) w(x) + 0 = 0.$$

1st order linear homogeneous DE $\rightarrow$ separation of variables

Solution: $w(x) = \underbrace{w(x_0)}_{\text{initial condition}} \cdot \underbrace{e^{-\int_{x_0}^{x} p(z) dz}}_{>0}$

Hence we conclude that only two cases are possible:

1. $w(y_1(x), y_2(x)) \equiv 0 \quad$ for all x,
2. $w(y_1(x), y_2(x)) \neq 0 \quad$ for all x.

Ex.: $w(x) \equiv -2$

Ex.: $w(x) = -e^x \neq 0$ for all x

Question: What happens in the case $w(y_1, y_2) \equiv 0$?
$\{y_1; y_2\}$ is not a fundamental system.

$$y_1 y_2' - y_1' y_2 = 0 \text{ for all } x; \quad y_1, y_2 \neq 0 \tag{3}$$

Let's consider the quotient $\frac{y_1}{y_2}$:

$$\frac{y_1}{y_2}' = \frac{y_1' y_2 - y_1 y_2'}{y_2^2} = 0 \quad \text{by (3).}$$

Hence $\frac{y_1}{y_2}$ is constant in x! We conclude:

$$y_1(x) = \tilde{c} \cdot y_2(x), \quad \tilde{c} \in \mathbb{R} \backslash \{0\}.$$

$\Rightarrow$ y_1 and y_2 are not linearly independent and are no basis / fundamental system for the homogeneous DE.

Ex: $y_1(x) = e^x \quad y_2(x) = e^{-x}$: $\frac{y_1}{y_2}$ is not a constant $\rightarrow$ basis.

Ex: $y_1(x) = e^x \quad y_2(x) \equiv_0$: $\frac{y_1}{y_2}$ is not a constant $\rightarrow$ basis.

Next goal: Construct y_1 and y_2

3.4 Method of Lagrange

Setting: y_1 is already known; how to calculate y_2?

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \tag{4}$$

Ansatz: $y_2(x) = u(x)y_1(x)$ $\quad u(x)$ is not a constant function!

$$\begin{aligned} y_2' &= u'y_1 + uy_1' \quad \text{(product rule)} \\ y_2'' &= u''y_1 + u'y_1' + u'y_1' + uy_1'' \end{aligned}$$

$$\begin{aligned} 0 &= y_2'' + py_2' + qy_2 \\ &= u''y_1 + u'y_1' + u'y_1' + uy_1'' + pu'y_1 + puy_1' + quy_1 \\ &= u''y_1 + 2u'y_1' + uy_1'' + pu'y_1 + puy_1'' + quy_1 \\ &= u''y_1 + u'(2y_1' + py_1) + u\underbrace{(y_1''' + py_1'' + qy_1)}_{=0 \text{ by } (4)} \\ &= u''y_1 + u'(2y_1' + py_1) \end{aligned}$$

Substitution: $v = u'$ $\quad v' = u''$ $\quad v = v(x)$

$0 = v'y_1 + v(2y_1' + py_1)$ 1st order homogeneous DE inv $y_1 \neq 0$.

Rearrange:

$$v' + v(2\frac{y_1'}{y_1} + p) = 0 \quad \leftarrow \text{separation of variables.}$$

Obtain one (!) solution $v(x) = exp(...) > 0$

Back substitution:

$$\begin{aligned} u'(x) &= v(x) \\ u(x) &= \int exp(...)dz > 0 \end{aligned}$$

Hence $u(x)$ is not a constant function!

3.5 Example

$$x(x-1)y''(x) = xy'(x) + y(x) = 0 \tag{5}$$

Standard form of (5):

$$y''(x) - (x-1)^{-1}y'(x) + x^{-1}(x-1)^{-1}y(x) = 0.$$

unique solution for $x \neq 0$ and $x \neq 1$. We consider $x > 1$

1. $y_1 = x$ solves (5).

$$0 - (x-1)^{-1} \cdot 1 + x^{-1}(x-1)^{-1}x = 0.$$

2. Method of Lagrange:

$$y_2(x) = u(x) \cdot y_1(x) = u(x) \cdot x.$$

$$y_2' = u' \cdot x + u, \quad y_2'' = u''x + u' \cdot 2.$$

Insert in DE (5):

$$\begin{aligned} x(x-1)(u'' \cdot x + 2u') - x(u' \cdot x + u) + u \cdot x &= 0 \\ u'' \cdot x^2(x-1) + u'(2x^2 - 2x - x^2) &= 0 \\ u'' \cdot x \cdot (x-1) + u' \cdot (x-2) &= 0 \end{aligned}$$

3. Substitution $v = u'$

$$v' \cdot x \cdot (x-1) + v \cdot (x-2) = 0, \quad \frac{dv}{dx} = v'.$$

$$\begin{aligned} x \cdot (x-1) \cdot \frac{dv}{dx} &= -(x-2) \cdot v \\ \int \frac{dv}{v} &= -\int \frac{x-2}{x \cdot (x-1)} dx \end{aligned}$$

0 only one solution, not general solution

$$\ln|v| = \ldots$$

$$\begin{aligned} v(x) &= \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2} = u'(x) \\ u(x) &= \ln|x| + \frac{1}{x} = \ln x + \frac{1}{x} \text{ for } x > 1 \\ y_2(x) &= u(x) \cdot y_1(x) = x \cdot \ln x + 1 \end{aligned}$$

3.6 Inhomogeneous DE and Variation of Constants

Standard form:

$$y'' + p(x)y' + q(x)y = r(x) \tag{6}$$

Fundamental system: $\{y_1; y_2\}$

$$\begin{aligned} y_1'' + py_1' + q &= 0 \\ y_2'' + py_2' + q &= 0 \\ W = y_1y_2' - y_2y_1' &\neq 0 \end{aligned}$$

Goal: Construct a particular solution y_p of the DE (6).