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Nhân bản – Phụng sự – Khai phóng

Algorithm Analysis

Data Structures & Algorithms

CONTENTS

- **Introduction to algorithm**
- **Algorithm analysis**
- **Estimating running time**
- **Algorithm growth rates** (Big O, Omega, Theta)
- **Worst-Case, Best-Case, Average-Case**

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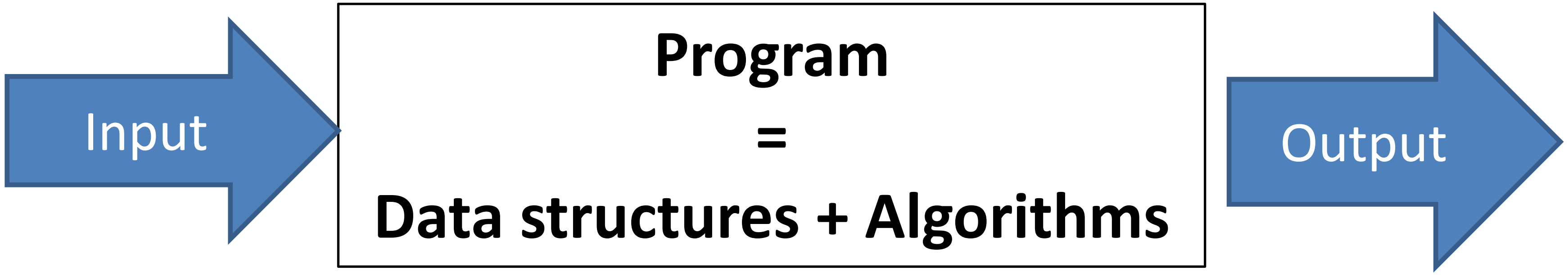
Introduction

- **An algorithm is a sequence of instructions to be followed to solve a problem**
    - There are often many solutions/algorithms to solve a given problem
    - An algorithm can be implemented using different programming languages on different platforms

- **An algorithm must be correct. It should correctly solve the problem**

- **Once we have a correct algorithm for a problem, we have to determine the efficiency of that algorithm**

- **Program = Data structures + Algorithms**



- **Correctness**
  - An algorithm is said to be **correct** if for every input instance, it halts with the correct output.
- **Efficiency**
  - **Computing time** and **memory space** are two important resources.

…Introduction

- **Time**
  - Instructions take time
  - How fast does the algorithm perform?
  - What affects its running time?
- **Space**
  - Data structures take space
  - What kind of data structures can be used?
  - How does choice of data structure affect the running time?

⇨ Focusing on **running time**

  - How to estimate the time required for an algorithm?
  - How to reduce the required time?

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- **Why do we need algorithm analysis?**
  - Showing the algorithm is correct
  - Writing a working program is not good enough
  - The program may be inefficient
  - If the program is run on a **large data set**, then the running time becomes an issue

- **Example: Selection Problem (1/3)**
  - Given a list of N numbers, determine the ***k***th largest, where **k ≤ N**.
  - Algorithm 1

    (1) Read **N** numbers into an array

    (2) Sort the array in decreasing order by some simple algorithm

    (3) Return the element in position ***k***

- **Example: Selection Problem (2/3)**
  - Algorithm 2

    (1) Read the first ***k*** elements into an array and sort them in decreasing order

    (2) Each remaining element is read one by one

    – If smaller than the kth element, then it is ignored

    – Otherwise, it is placed in its correct position in the array, getting one element out of the array

    (3) The element in the ***k***th position is returned as the answer

- **Example: Selection Problem (3/3)**
  - Which algorithm is better when

1. **N** = 100 and ***k*** = 100?

2. **N** = 100 and ***k*** = 1?

…Algorithm Analysis

- **Factors affecting the running time**
  - computer
  - compiler
  - algorithm
  - input to the algorithm
    - The content of the input affects the running time
    - typically, the **input size** (number of items in the input) is the main consideration
      - E.g. sorting problem ⇒ the number of items to be sorted
      - E.g. multiply 2 matrices together ⇒ the total number of elements in the 2 matrices

…Algorithm analysis

- **Analyzing algorithms**
  - Employing mathematical techniques that analyze algorithms independently of specific compilers, computers.

- **To analyze algorithms:**
  1. Starting to count the number of **significant operations** in a particular solution to assess its efficiency
  2. Expressing the efficiency of algorithms using **growth functions: T(n)**

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Estimating running time

- Each operation in algorithm/program has a **cost**

  ⇨ Each operation takes a certain of running time

Ex: count = count + 1;

⇨ takes a certain amount of time, but it is **constant: 1**

A **sequence** of operations:

```
count = count + 1;        //cost: c1
sum = sum + count;        //cost: c2
```

⇨ **Total Cost = c1 + c2**

…Estimating running time

- **Ex: Simple If-Statement**

	Cost	Times
if (n < 0)	c1	1
absval = -n	c2	1
else		
absval = n;	c3	1

⇨ **Total Cost <= c1 + max(c2,c3)**

…Estimating running time

- **Ex: Simple Loop**

```
                                    Cost        Times
i = 1;                               c1           1
sum = 0;                             c2           1
while (i <= n) {                     c3            n+1
        i = i + 1;                   c4            n
        sum = sum + i;               c5            n
}
```

⇨ **Total Cost = c1 + c2 + (n+1)*c3 + n*c4 + n*c5**

⇨ **The time required for this algorithm is proportional to n: T(n)=n**

– When n tends to infinity

…Estimating running time

- **Ex: Nested Loop**

```
                                        Cost          Times
i=1;                                     c1           1
sum = 0;                                 c2           1
while (i <= n) {                         c3           n+1
        j=1;                             c4           n
        while (j <= n) {                 c5           n*(n+1)
                sum = sum + i;           c6           n*n
                 j = j + 1;              c7           n*n
         }
        i = i + 1;                       c8           n
}
```

⇨ **Total Cost = c1 + c2 + (n+1)*c3 + n*c4 + n*(n+1)*c5+n*n*c6+n*n*c7+n*c8**

⇨ **The time required for this algorithm is proportional to n^2: $T(n)=n^2$**

– When n tends to infinity

…Estimating running time

- General rules for running time estimation
  - Consecutive Statements: Just **add the running times** of those consecutive statements
  - Conditional Statements (If/Else): Never more than the running time of **the test** plus **the larger of running times** of two branches
  - Loops: The running time of a loop is at most the running time of the statements inside of that loop times **the number of iterations**
  - Recursion: Determine and solve the recurrence relation (we don’t focus on this case in this course)

…Estimating running time

- **Consecutive Statements**
  - Just add the running times of those consecutive statements

```
for (i=0;i<n;i++)
        k++
```

T(n) = n

```
for (i=0; i<n; i++)
        for (j=0; j<n; j++)
                k++
```

T(n) = n²

…Estimating running time

- **Conditional Statement**
  - Less more than the running time of **the test** plus **the larger** of the running times of S1 and S2

```
if  (condition)
        S1
else
        S2
```

- **Loops**
  - The running time of a loop is at most the running time of the statements inside of that loop (including tests) times the **number of iterations**

```
for (i=0;i<n;i++)
      k++
```

T(n) = n

…Estimating running time

- **Nested loops**
  - The total runing time of a statement inside a group of nested loops is the running time of the statement **multiplied** by the product of the **sizes of all the loops**

```
for (i=0; i<n; i++)
        for (j=0; j<n; j++)
                k++
```

$\boxed{T(n) = n^2}$

…Estimating running time

- **Function calls**
  - Non recursive calls
    - A function call is considered as a statement
      ⇨ The runing time of a function call is considered as the runing time of a statement
  - Recursive calls
    - Set up the recurrence relation
    - Solve the recurrence
      - May be very complicated

...Estimating running time

- **Example**

$T(n) = n$

```
sum = 0
for (j=0; j<n; j++)
        sum++;
```

- **Example**

$T(n) = n^2$

```
sum = 0
for (j=0; j<n; j++)
    for (k=0; k<n; k++)
        sum++;
```

…Estimating running time

- **Example**

$T(n) = n^3$

```
sum = 0
for (j=0; j<n; j++)
  for (k=0; k<n*n; k++)
    sum++;
```

- **Example**

$T(n) = n^4$

```
sum = 0;
for (j=0; j<n; j++)
  for (k=0; k<j*j; k++)
    if (k%j == 0)
          for (m=0; m<k; m++)
              sum++;
```

…Estimating running time

- **Example**

```
int fact(int n){
        if (n==0)
              return 1;
        else
              return (n * fact(n-1));
  }
```

Recurrence relation

$C(n) = C(n\text{-}1) + 1, \;\; C(0) = 0$

$T(n) = n$

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Algorithm growth rates

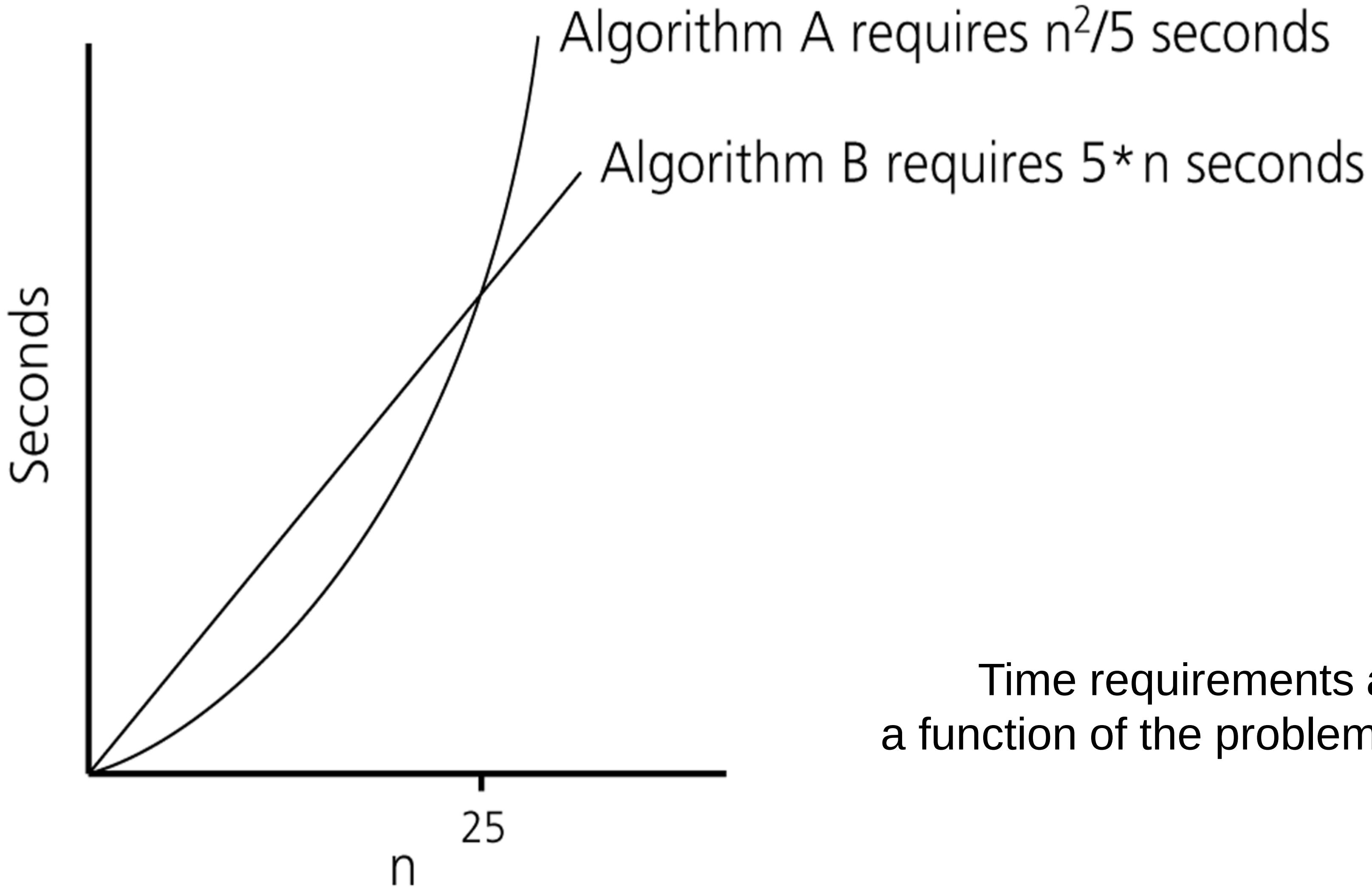
- **Measuring an algorithm’s time requirement as a function of the problem size**
- **Problem size depends on the application**

  Ex: number of elements in a list for a sorting algorithm

  If the problem size is n:

  - Algorithm A requires **$5*n^2$** time units to solve a problem of size **n**
  - Algorithm B requires **$7*n$** time units to solve a problem of size **n**
- **The algorithm’s time requirement grows as a function of the problem size**
  - Algorithm A requires time **proportional** to **n^2: $T(n) = n^2$**
  - Algorithm B requires time **proportional** to **n: $T(n) = n$**
- **An algorithm’s proportional time requirement is known as growth rate**
- **Comparing the efficiency of 2 algorithms by comparing their growth rates**

…Algorithm growth rates

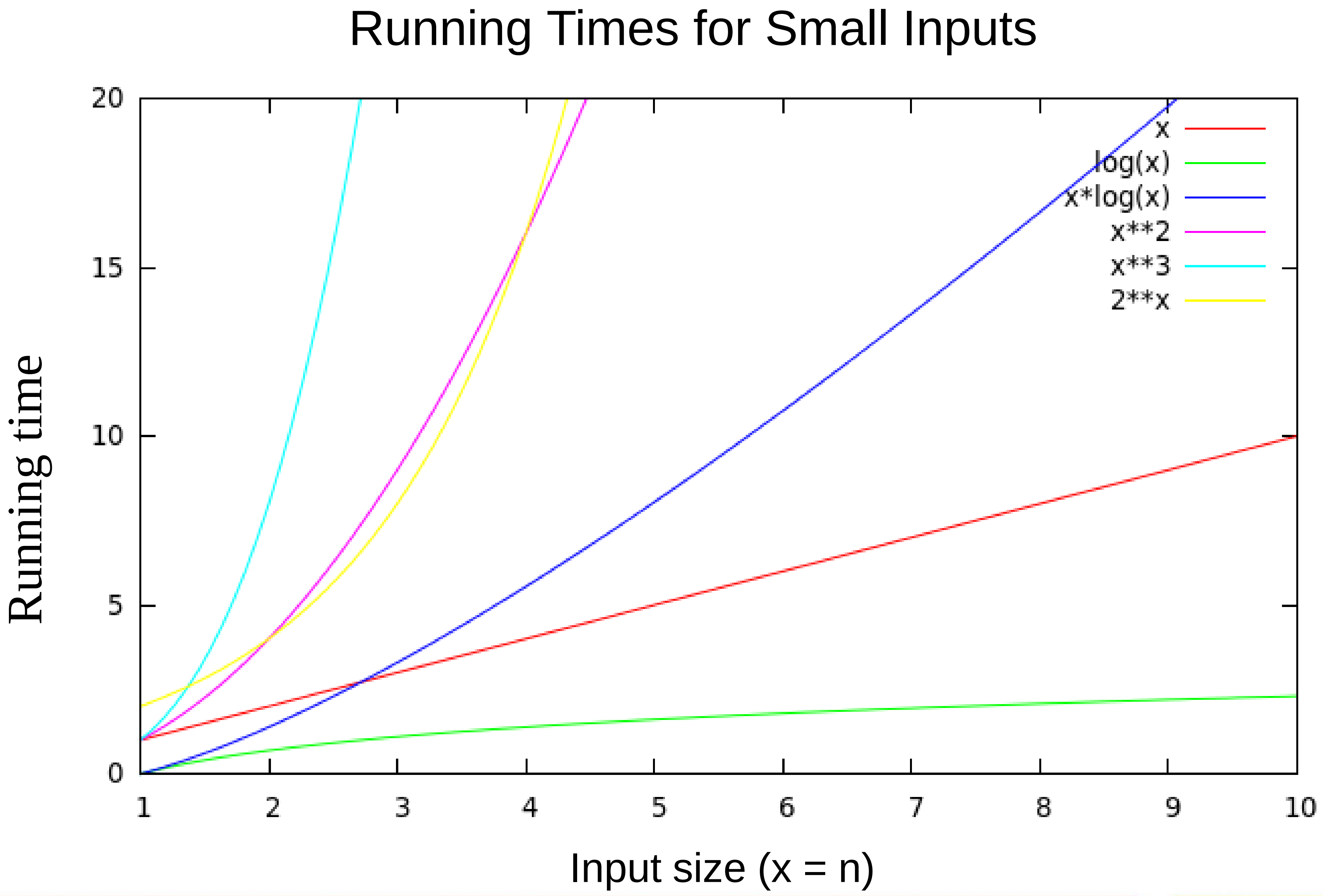


Time requirements as
a function of the problem size **n**

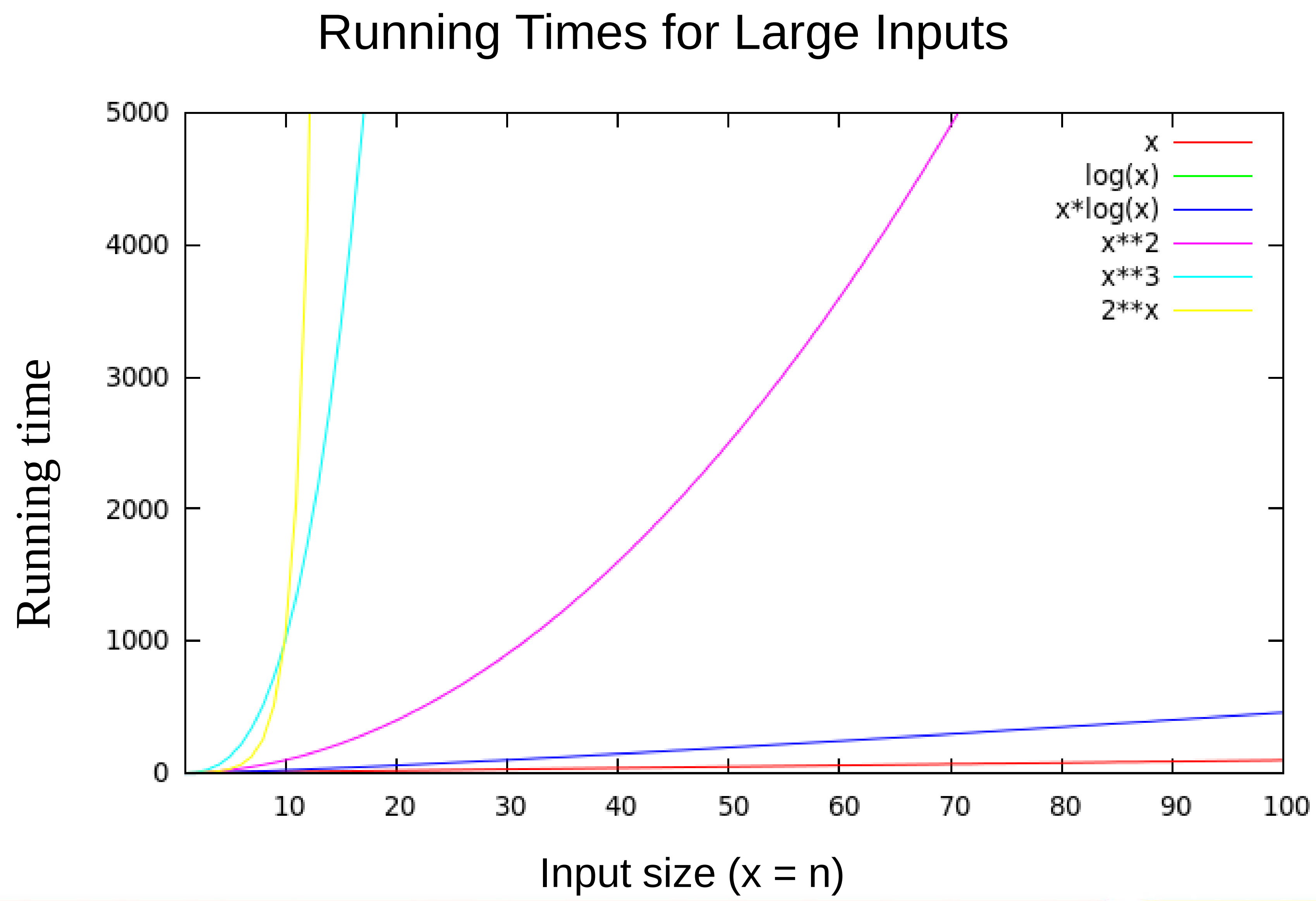
- Common Growth Rates

Function	Growth Rate Name
c	Constant
log N	Logarithmic
$\log^2 N$	Log-squared
N	Linear
N log N	Log-linear
N^2	Quadratic
N^3	Cubic
2^N	Exponential

…Algorithm growth rates



…Algorithm growth rates



…Algorithm growth rates

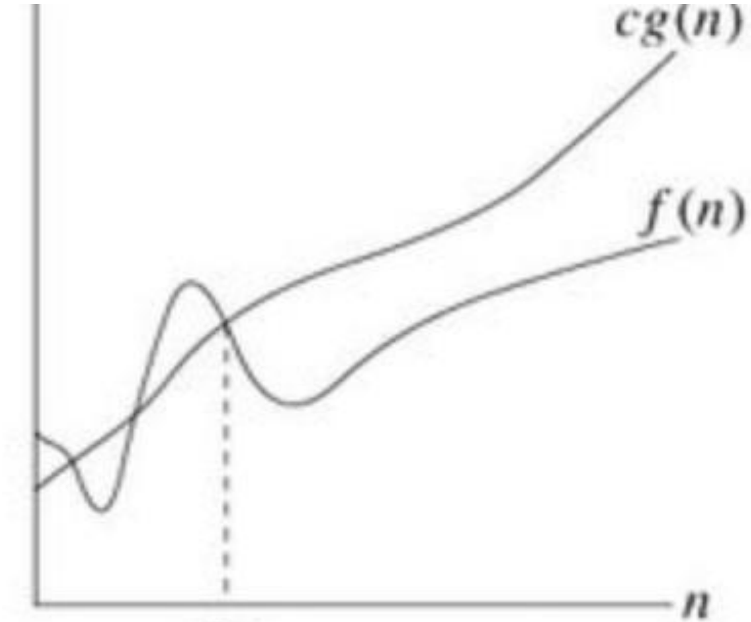
- **Asymptotic notations**
    - Upper bound O(g(n)
    - Lower bound Ω(g(n))
    - Tight bound Θ(g(n))

…Algorithm growth rates

- **Big O**
  - $f(n) = O(g(n))$
  - There are positive constants c and n_0 such that

    $$f(n) \le c\, g(n) \quad \text{when} \quad n \ge n_0$$

  - growth rate of f(n) $\le$ growth rate of g(n)
  - g(n) is an *upper bound* on f(n)



$f(n) = O(g(n))$

…Algorithm growth rates

- **Big O**
  - If **Algorithm A requires time proportional to g(n)**, Algorithm A is said to be **order g(n),** and it is denoted as **O(g(n))**.
  - The function **g(n)** is called the algorithm’s **growth-rate function**.
  - The capital O is used in the notation

    ⇨ called the **Big O notation.**
  - If Algorithm A requires time proportional to **n^2**, it is **O(n^2)**.
  - If Algorithm A requires time proportional to **n**, it is **O(n)**.

- **Big O – Example**

  - Let $f(n) = 2n^2$. Then
    - $f(n) = O(n^4)$
    - $f(n) = O(n^3)$
    - $f(n) = O(n^2)$ (best answer, asymptotically tight)

  - $O(n^2)$: reads "**order n-squared**" or "**Big-O n-squared**"

- **Big O – Some rules**
  - Ignore the lower order terms
  - Ignore the coefficients of the highest-order term
  - If T(n) is an asymptotically positive polynomial of degree **k**, then $T(n) = O(n^k)$

Ex: $\mathbf{7n^2 + 10n + 3 = O(n^2)}$

- **Big O – Some rules**

  - No need to specify the base of logarithm
    - Changing the base from one constant to another changes the value of the logarithm by only a constant factor

  - For logarithmic functions,

    $T(\log_m n) = O(\log n)$, (use: $T(\log_m n) = T((\log_2 n) / (\log_2 n)))$

  - If $T_1(n) = O(f(n)$ and $T_2(n) = O(g(n))$,
    - $T_1(n) + T_2(n) = \max(\ O(f(n)),\ O(g(n))\ )$
    - $T_1(n) * T_2(n) = O(\ f(n) * g(n)\ )$

…Algorithm growth rates

- **Big O – more example**
  - $n^2 / 2 - 3n = O(n^2)$
  - $1 + 4n = O(n)$
  - $7n^2 + 10n + 3 = O(n^2)$
  - $\log_{10} n = \log_2 n / \log_2 10 = O(\log_2 n) = O(\log n)$
  - $10 = O(1)$, $10^{10} = O(1)$
  - $\log n + n = O(n)$

$$\sum_{i=1}^{N} i \leq N \cdot N = O(N^2)$$

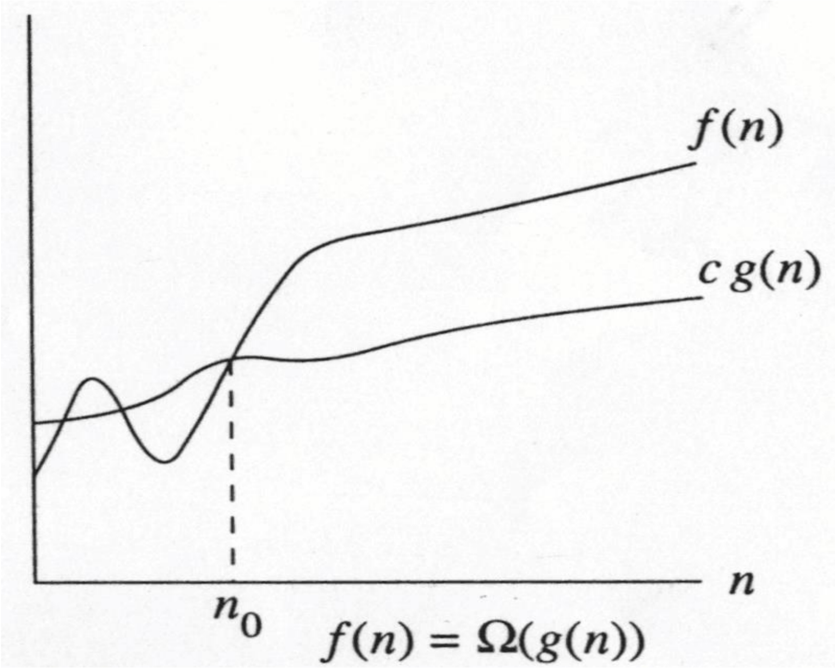
$$\sum_{i=1}^{N} i^2 \leq N \cdot N^2 = O(N^3)$$

…Algorithm growth rates

- **Big Omega**
  - $f(n) = \Omega(g(n))$
  - There are positive constants c and n_0 such that

    $$f(n) \geq c\, g(n) \quad \text{when} \quad n \geq n_0$$

  - growth rate of $f(n) \geq$ growth rate of $g(n)$
  - $g(n)$ is a **lower bound** on $f(n)$
  - $f(n)$ grows no slower than $g(n)$ for "large" n



- Big Omega – Examples

  - Let $f(n) = 2n^2$
    - $f(n) = \Omega(n)$
    - $f(n) = \Omega(n^2)$ (best answer)

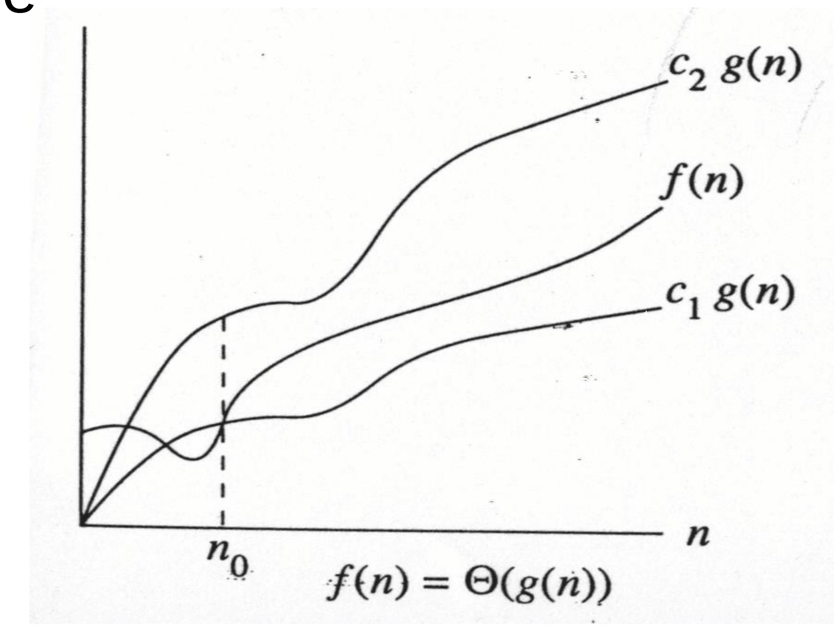
…Algorithm growth rates

- **Big Theta**
- f(n) = Θ(g(n))   iff f(n) = O(g(n)) and f(n) = Ω(g(n))

- growth rate of f(n) = growth rate of g(n)
- Big-Theta means the bound is the **tightest** possible

- Example: Let f(n)=2n^2 , g(n)=n^2
  - since   f(n) = O(g(n)) and f(n) = Ω(g(n)),

    thus   f(n) = Θ(g(n))



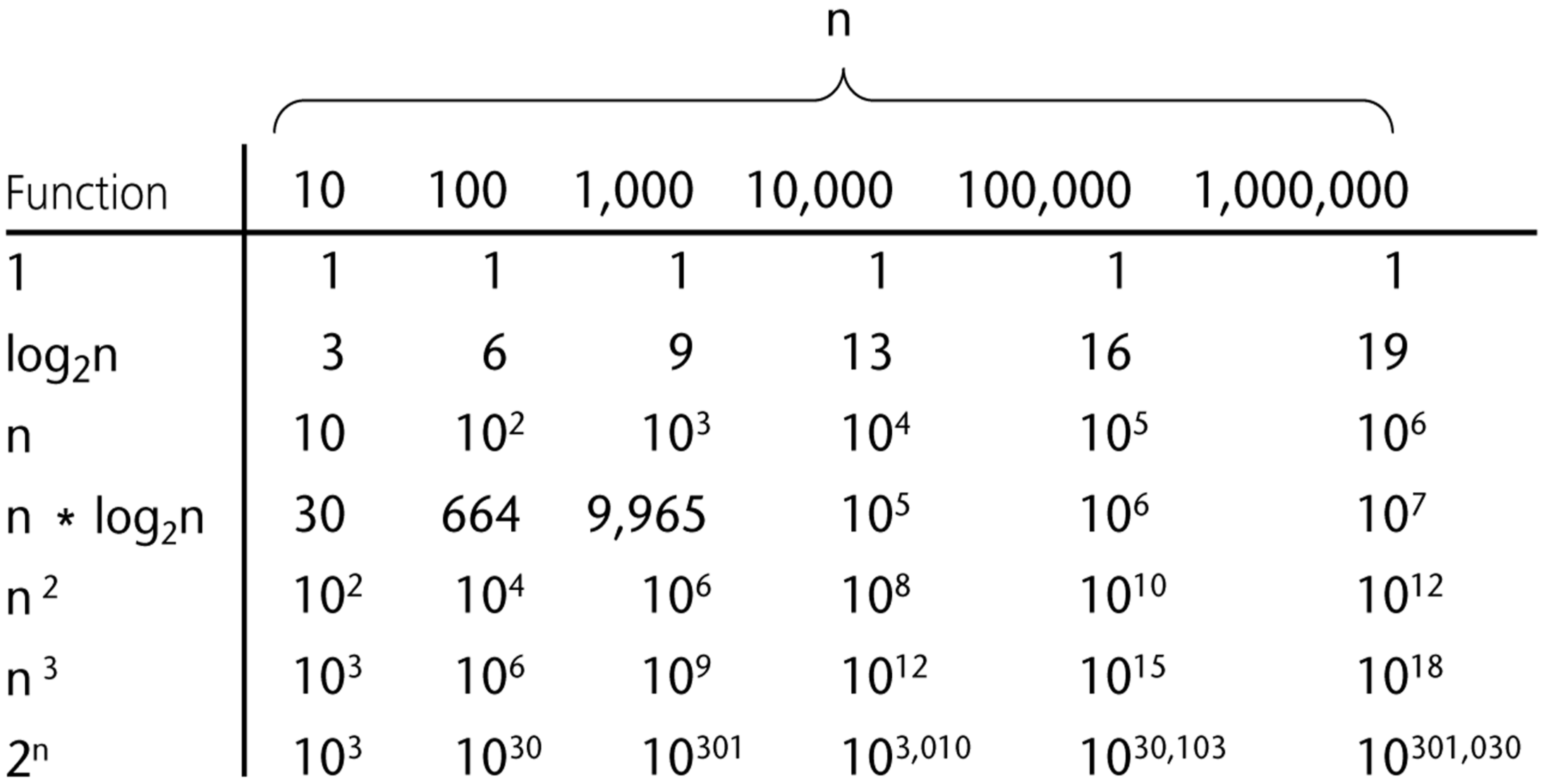
- **Big Theta – Some rules**

  - If $T(n)$ is a asymptotically positive polynomial of degree k, then $T(n) = \Theta(n^k)$

  - For logarithmic functions,

    $T(\log_m n) = \Theta(\log n)$, (use: $T(\log_m n) = T((\log_2 n) / (\log_2 n)))$

A Comparison of Growth-Rate Functions



	n					
Function	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

…Algorithm growth rates

- **Asymptotic notations**
  - When n goes to infinity
    - Upper bound **O(g(n))**
      - **the most popular**
    - Lower bound **Ω(g(n))**
    - Tight bound **Θ(g(n))**

…Algorithm growth rates

Growth-Rate Functions

O(1) Time requirement is **constant**, and it is independent of the problem’s size.

O($\log_2 n$) Time requirement for a **logarithmic** algorithm increases increases slowly as the problem size increases.

O(n) Time requirement for a **linear** algorithm increases directly with the size of the problem.

O($n*\log_2 n$) Time requirement for a **$n*\log_2 n$** algorithm increases more rapidly than a linear algorithm.

O(n^2) Time requirement for a **quadratic** algorithm increases rapidly with the size of the problem.

O(n^3) Time requirement for a **cubic** algorithm increases more rapidly with the size of the problem than the time requirement for a quadratic algorithm.

O(2^n) As the size of the problem increases, the time requirement for an **exponential** algorithm increases too rapidly to be practical.

…Algorithm growth rates

- **Reminder of Properties of Growth-Rate Functions**
  - We can ignore low-order terms in an algorithm’s growth-rate function.
    - If an algorithm is $O(n^3+4n^2+3n)$, it is also $O(n^3)$.
    - We only use the higher-order term as algorithm’s growth-rate function.
  - We can ignore a multiplicative constant in the higher-order term of an algorithm’s growth-rate function.
    - If an algorithm is $O(5n^3)$, it is also $O(n^3)$.
  - $O(f(n)) + O(g(n)) = O(f(n)+g(n))$
    - If an algorithm is $O(n^3) + O(4n)$, it is also $O(n^3 +4n)$ ⇨ it is $O(n^3)$
  - $O(f(n)) * O(g(n)) = O(f(n) * g(n))$

…Algorithm growth rates

- **Example 1**

```
                                    Cost          Times
    i = 1;                           c1             1
    sum = 0;                         c2             1
    while (i <= n) {                 c3             n+1
        i = i + 1;                   c4             n
        sum = sum + i;               c5             n
    }
```

$$
\begin{aligned}
T(n) &= c1 + c2 + (n+1)*c3 + n*c4 + n*c5 \\
&= (c3+c4+c5)*n + (c1+c2+c3) \\
&= a*n + b
\end{aligned}
$$

⇨ the growth-rate function for this algorithm is **O(n)**

- **Example 2**

```
for (i=1; i<=n; i++)
          for (j=1; j<=i; j++)
                    for (k=1; k<=j; k++)
                              x=x+1;
```

$$T(n) = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$$

⇨ the growth-rate function for this algorithm is **$O(n^3)$**

…Algorithm growth rates

- **Example 3**

```
                                            Cost          Times
    i=1;                                    c1              1
    sum = 0;                                c2              1
    while (i <= n) {                        c3              n+1
        j=1;                                c4              n
        while (j <= n) {  c5                n*(n+1)
            sum = sum + i;                  c6              n*n
            j = j + 1;                      c7              n*n
      }
      i = i +1;                             c8              n
    }
```

$$
\begin{aligned}
T(n) \quad &= \; c1 + c2 + (n+1)*c3 + n*c4 + n*(n+1)*c5+n*n*c6+n*n*c7+n*c8 \\
&= (c5+c6+c7)*n^2 + (c3+c4+c5+c8)*n + (c1+c2+c3) \\
&= a*n^2 + b*n + c
\end{aligned}
$$

⇨ the growth-rate function for this algorithm is **O(n^2)**

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- **Worst-Case, Best-Case, Average-Case**

Worst-Case, Best-Case, Average-Case

- **An algorithm can require different times to solve different problems of the same size.**
  - Eg. Searching an item in a list of n elements using sequential search. → Cost: 1,2,...,n
- **Worst-Case**
  - The maximum amount of time that an algorithm require to solve a problem of size n.
  - This gives an upper bound for the time complexity of an algorithm.
  - Normally, we try to find worst-case behavior of an algorithm.
- **Best-Case**
  - The minimum amount of time that an algorithm require to solve a problem of size n.
  - The best case behavior of an algorithm is NOT so useful.
- **Average-Case**
  - The average amount of time that an algorithm require to solve a problem of size n.
  - Sometimes, it is difficult to find the average-case behavior of an algorithm.
  - We have to look at all possible data organizations of a given size n, and their distribution probabilities of these organizations.
  - Worst-case analysis is more common than average-case analysis.

…Worst-Case, Best-Case, Average-Case

- **Sequential Search – Analysis**

```
int sequentialSearch(const int a[],  int item,  int n){
     for (int i = 0; i < n && a[i]!= item; i++);
     if (i == n)       return   –1;
     return  i;
}
```

- Unsuccessful Search: O(n)
- Successful Search:

Best-Case: item is in the first location of the array ⇨ O(1)

Worst-Case: item is in the last location of the array ⇨ O(n)

Average-Case: The number of key comparisons 1, 2, ..., n

$$\frac{\sum_{i=1}^{n} i}{n} = \frac{(n^2+n)/2}{n} \quad ⇨ \text{O(n)}$$

…Worst-Case, Best-Case, Average-Case

- **Binary Search – Analysis**

```
int binarySearch(int a[],  int size,  int x) {
   int low =0;
   int high = size –1;
   int mid;                      // mid will be the index of target when it’s found.
   while (low <= high) {
         mid = (low + high)/2;
         if (a[mid] < x)                  low = mid + 1;
         else if (a[mid] > x)             high  = mid – 1;
                 else                     return mid;
   }
   return –1;
}
```

We can do binary search if the array is sorted

Binary Search – Analysis

- Unsuccessful Search
  - The size of the list for each iteration: $n/2$, $n/2^2$, $n/2^3$, …, $n/2^k$
  - Loop stops when $n/2^k = 1$, where k is the number of iterations
  - Then, the number of iterations k in the loop is $\log_2 n$ ⇨ $O(\log_2 n)$

- Successful Search

  Best-Case: The number of iterations is 1 ⇨ $O(1)$

  Worst-Case: The number of iterations is $\log_2 n$ ⇨ $O(\log_2 n)$

  Average-Case: The avg. number of iterations $< \log_2 n$ ⇨ $O(\log_2 n)$

- How much better is *$O(\log_2 n)$*?

.

n	***$O(\log_2 n)$***
16	4
64	6
256	8
1024	10
16,384	14
131,072	17
262,144	18
524,288	19
1,048,576	20
1,073,741,824	30

Conclusion

- **Running time depends on**
  - Input size
    - A function of n (input size)
    - Running time is significant, when n goes to infinity
  - Input contents
    - Best-case, Average-case, Worst-case
- **Steps for estimating running time** (compelxity)
  1. Counting the number of significant operations/statements
     - We often consider **the worst-case**
  2. Applying the growth rate functions (**O**, Ω, Θ)
     - When **n goes to infinity**
     - **O is the most popularly used**
  3. Classifying the algorithm running time/complexity
     - The algorithm is **impractical**, if its running time is more than O(n^3)

SUMMARY

- Introduction to algorithm
- Algorithm analysis
- Estimating running time
- Algorithm growth rates
- Worst-Case, Best-Case, Average-Case

Nhân bản – Phụng sự – Khai phóng

Enjoy the Course…!