

Topological Recursion, explicit ABCD tensors and implementation

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Abstract

We provide explicit expressions of ABCD tensors for the most classical classes of spectral curves. And we discuss algorithmic implementation of Topological Recursion.

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1 Introduction

Topological Recursion was invented in order to compute recursively the large size asymptotic expansion of random matrices [Eyn04], and then it was realized that Topological Recursion is ubiquitous in mathematical physics, beyond random matrices, it has applications to enumerative geometry, string theory, integrable systems, conformal field theory, combinatorics of maps, statistical physics...etc.

In [EO07] Topological Recursion was formulated as a process that to a spectral curve (a Riemann surface with extra structure) associates a sequence of differential forms, called the *invariants* of the spectral curve. These invariants often coincide with classical geometric invariants. For example:

spectral curve		→ T.R. invariants
Airy	$y = \sqrt{x}$	Witten-Kontsevich intersection numbers
Sine	$y = \sin(2\pi z)$	Weil-Petersson volumes
Mirror of toric CY3-fold semi-circle (GUE)	$P(e^x, e^y) = 0$	Gromov-Witten invariants
A-polynomial of knot classical stress-energy tensor etc...	$y = \sqrt{4 - x^2}$ $P(e^x, e^y) = 0$ $y^2 = T(x)$	higher Catalan numbers Jones polynomial CFT conformal blocs

There exists many lectures, introductory lecture notes and review articles on topological recursion and its applications.

In [KS17], Topological Recursion was reformulated as "**Quantum Airy Structure**", in terms of 4 tensors called A, B, C, D , which was further developed in [ABCO24].

The ABCD tensor formulation amounts to decomposing the invariants on a basis of forms, and consider the recursion for the coefficients. This is how Topological recursion is actually implemented algorithmically.

In this article, we give explicit expressions of the ABCD tensors for many classical spectral curves, in a way used for actual implementation in a python package. We also discuss the algorithmic issues and implementation.

2 Topological Recursion and ABCD tensors

Consider a Riemann surface $\overset{\circ}{\Sigma}$ called the base curve. Most often $\overset{\circ}{\Sigma} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ the Riemann sphere, and this is what we shall consider in all examples below.

2.1 Spectral curve

A **spectral curve** is the following data

$$\left\{ \begin{array}{l} \Sigma = \text{Riemann surface} \\ x : \Sigma \rightarrow \overset{\circ}{\Sigma} = \text{ramified covering map} \\ y = ydx = \text{meromorphic 1-form on } \Sigma \\ B(z_1, z_2) = \text{fundamental 2nd kind differential , } B \in H^0(\Sigma \times \Sigma, K_\Sigma \overset{\text{Sym}}{\boxtimes} K_\Sigma)_{\text{Res}=1} \\ R = \{\text{ramification points}\} \\ a \in R : \sigma_a, \text{ local analytic involution such that } x(\sigma_a(z)) = x(z) \text{ and } \sigma_a(a) = a. \end{array} \right. \quad (2-1)$$

For regular spectral curves, ramification points are the zeros of dx , and we assume that they are simple zeros. Therefore near $z = a$, x is $2 : 1$, and there exists a unique local involution $\sigma_a \neq \text{Id}$, such that $\sigma_a(a) = a$ and $x(\sigma_a(z)) = x(z)$ in a neighborhood of a . We shall consider non-regular spectral curves in section 7 below.

Then we define the **recursion kernel**:

$$K_a(z_1, z) = \frac{dS(z_1, z)}{(y(z) - y(\sigma_a(z)))dx(z)} \quad \text{where} \quad dS(z_1, z) = \frac{1}{2} \int_{\sigma_a(z)}^z B(z_1, .) \quad (2-2)$$

which is meromorphic for z in a neighborhood of the ramification point a , with a simple pole at $z = a$.

2.2 TR invariants

Topological Recursion is a recursive procedure that to a spectral curve, associates its **invariants** $\omega_{g,n}$ defined as follows:

$$\begin{aligned} \omega_{0,1} &= ydx \\ \omega_{0,2} &= B \\ \omega_{0,3}(z_1, z_2, z_3) &= \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) [B(z, z_2)B(\sigma_a(z), z_3) + B(z, z_3)B(\sigma_a(z), z_2)] \\ &= -2 \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, z_2) B(z, z_3) \\ &= - \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} \frac{B(z, z_1)B(z, z_2)B(z, z_3)}{dx(z)dy(z)} \\ \omega_{1,1}(z_1) &= \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, \sigma_a(z)) \end{aligned} \quad (2-3)$$

and for $n \geq 1$ and $2g - 2 + n > 1$

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) &= 2 \sum_{j=2}^n \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) \\ &\quad B(z, z_j) \omega_{g,n}(\sigma_a(z), z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \\ &+ \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) \left[\omega_{g-1,n+1}(z, \sigma_a(z), z_2, \dots, z_n) \right. \\ &\quad \left. + \sum_{g_1+g_2=g} \sum_{I_1 \sqcup I_2 = \{z_2, \dots, z_n\}}^{\text{stable}} \omega_{g_1,1+|I_1|}(z, I_1) \omega_{g_2,1+|I_2|}(\sigma_a(z), I_2) \right] \end{aligned} \quad (2-4)$$

For $n = 0$, the TR invariants are defined for $g \geq 2$ as

$$\mathcal{F}_g = \omega_{g,0} = \frac{1}{2-2g} \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} \omega_{g,1}(z) F_{0,1}(z) \quad (2-5)$$

where $dF_{0,1} = \omega_{0,1}$ (any antiderivative of $\omega_{0,1} = y = ydx$ gives the same result).

Here we won't consider \mathcal{F}_0 and \mathcal{F}_1 .

Homogeneity property:

If we rescale $ydx \rightarrow \lambda ydx$ (i.e. we rescale either x or y or both), then

$$\omega_{g,n} \mapsto \lambda^{2-2g-n} \omega_{g,n} \quad (2-6)$$

Moreover $\omega_{g,n}$ is invariant under translations $x \mapsto x + c$.

2.3 Basis of 1-forms

Consider a vector space V , subspace of the space of meromorphic 1-forms on Σ , such that for all $2g - 2 + n > 0$ we have

$$\omega_{g,n} \in V \otimes \cdots \otimes V.$$

V can be chosen as a subspace of meromorphic 1-forms that have poles at the ramification points, generated by the coefficients of the Taylor expansion of $B(z, z')$ at $z' \in R$, i.e. by derivatives of $B(z', z)$ with respect to z' , at $z' \in R$:

$$V \subset \text{Span}_{a \in R, d \in \mathbb{Z}_+} \left\langle (d/dx(z'))^d (B(z', z)/d\zeta_a(z'))_{z'=a} \right\rangle \quad (2-7)$$

(using the local coordinate $\zeta_a(z') = \sqrt{x(z') - x(a)}$ near $a \in R$). V can be chosen smaller for example when the spectral curve has symmetries.

Consider a basis $\{d\xi_\alpha\}_{\alpha \in J}$ of V . $\omega_{g,n}$ can be decomposed on the basis:

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\alpha_1; \dots; \alpha_n} F_{g,n}[\alpha_1; \dots; \alpha_n] \prod_{i=1}^n d\xi_{\alpha_i}(z_i) \quad (2-8)$$

where we assume that the basis is chosen so that the number of terms is finite. Despite our notation $d\xi_\alpha$ needs not be an exact form.

The goal is to be able to compute the coefficients $F_{g,n}[\alpha_1; \dots; \alpha_n]$.

We also consider the antiderivative (only defined locally):

$$\mathcal{F}_{g,n}(z_1, \dots, z_n) = \sum_{\alpha_1; \dots; \alpha_n} F_{g,n}[\alpha_1; \dots; \alpha_n] \prod_{i=1}^n \xi_{\alpha_i}(z_i). \quad (2-9)$$

Examples of basis:

- any basis of 1-forms $d\xi_{a,d}(z)$, such that $d\xi_{a,d}(z)$ has a pole of degree $2d + 2$ at $z = a$, and $d\xi_{a,d}(z) + d\xi_{a,d}(\sigma_a(z))$ has no pole at $z = a$.
- $d\xi_{a,d}(z) = \text{Res}_{z' \rightarrow a} (x(z') - x(a))^{-(d+1/2)} B(z, z')$
- $d\xi_{a,0}(z) = \text{Res}_{z' \rightarrow a} (x(z') - x(a))^{-1/2} B(z, z')$ and $d\xi_{a,d+1}(z) = d(d\xi_{a,d}(z)/dx(z))$.
- many other choices are possible...

2.4 TR and ABCD Tensors

Topological Recursion (2-4), written for the coefficients amounts to:

$$\begin{aligned} F_{0,3}[\alpha_1; \alpha_2; \alpha_3] &= A[\alpha_1; \alpha_2; \alpha_3] \\ F_{1,1}[\alpha_1] &= D[\alpha_1] \end{aligned} \quad (2-10)$$

and for $2g - 2 + n > 1$:

$$\begin{aligned} F_{g,n}[\alpha_1; \dots; \alpha_n] &= 2 \sum_{i=2}^n \sum_{\alpha \in J} B[\alpha_1; \alpha_i | \alpha] F_{g,n-1}[\alpha; \alpha_2; \dots; \widehat{\alpha}_i; \dots; \alpha_n] \\ &\quad \sum_{\alpha, \alpha' \in J} C[\alpha_1 | \alpha; \alpha'] \left[F_{g-1, n+1}[\alpha; \alpha'; \alpha_2; \dots; \alpha_n] \right. \\ &\quad \left. + \sum_{g_1+g_2=g} \sum_{I_1 \cup I_2 = \{\alpha_2; \dots; \alpha_n\}}^{\text{stable}} F_{g_1, 1+|I_1|}(\alpha; I_1) F_{g_2, 1+|I_2|}(\alpha'; I_2) \right] \end{aligned} \quad (2-11)$$

where we have introduced the following tensors:

- $A \in V^* \otimes V^* \otimes V^*$:

$$\begin{aligned} &\sum_{\alpha_1, \alpha_2, \alpha_3} A[\alpha_1, \alpha_2, \alpha_3] d\xi_{\alpha_1}(z_1) d\xi_{\alpha_2}(z_2) d\xi_{\alpha_3}(z_3) \\ &= \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} \frac{B(z, z_1) B(z, z_2) B(z, z_3)}{dx(z) dy(z)} \\ &= -2 \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, z_1) B(z, z_2) \end{aligned} \quad (2-12)$$

- $B \in V^* \otimes V^* \otimes V$:

$$\begin{aligned} \sum_{\alpha_1} \sum_{\alpha_2} B[\alpha_1, \alpha_2 | \alpha_3] d\xi_{\alpha_1}(z_1) d\xi_{\alpha_2}(z_2) &= \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, z_2) d\xi_{\alpha_3}(\sigma_a(z)) \\ &= - \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, z_2) d\xi_{\alpha_3}(z) \end{aligned} \quad (2-13)$$

- $C \in V^* \otimes V \otimes V$:

$$\sum_{\alpha_1} C[\alpha_1 | \alpha_2, \alpha_3] d\xi_{\alpha_1}(z_1) = \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_{a_1}(z_1, z) d\xi_{\alpha_2}(z) d\xi_{\alpha_3}(\sigma_a(z)) \quad (2-14)$$

- $D \in V^*$:

$$\sum_{\alpha_1} D[\alpha_1] d\xi_{\alpha_1}(z_1) = \sum_{a \in R} \underset{z \rightarrow a}{\text{Res}} K_a(z_1, z) B(z, \sigma_a(z)). \quad (2-15)$$

These tensors are the same as those of Quantum Airy Structures of Kontsevich Soibelman [KS17] and [ABCO24]. Now we shall compute explicitly A, B, C, D for several examples of spectral curves.

3 Airy and KdV

3.1 Spectral curve

The **KdV** spectral curve with times $\{\tilde{t}_k\}_{k \in \mathbb{Z}_+}$ is defined as

$$\begin{cases} \Sigma = \mathbb{C} \\ x = z^2 + \tilde{t}_1 \\ y = -\frac{1}{2} \sum_{k \geq 3} \tilde{t}_k z^{k-2} \\ B = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \\ R = \{0\}, \quad \sigma_0(z) = -z \end{cases} \quad (3-1)$$

Airy is the case

$$\tilde{t}_k = -2\delta_{k,3}. \quad (3-2)$$

The **KP-times** are related to **KdV-times** as follows:

$$t_k = -\delta_{k,1}\tilde{t}_1 + \sum_{j \geq 0} \binom{-k/2}{j} \tilde{t}_{k+2j} \tilde{t}_1^j. \quad (3-3)$$

In particular if $\tilde{t}_1 = 0$, KdV and KP times are the same $t_k = \tilde{t}_k$.

The **Minimal model** of type $(p, 2)$ is such that $\tilde{t}_k = 0$ for $k > p + 2$.

Airy is the minimal model $(1, 2)$.

The minimal model $(3, 2)$ is often called **pure-gravity**, or **Painlevé 1**.

The minimal model $(5, 2)$ is often called **Lee-Yang**.

The case $\tilde{t}_{2k+3} = t_{2k+3} = \frac{(-1)^k (2\pi)^{2k}}{(2k+1)!}$ is called **Weil-Petersson**.

3.2 Basis

We define

$$\xi_d(z) = -\frac{(2d-1)!!}{z^{2d+1}} \quad , \quad d\xi_d(z) = \frac{(2d+1)!! dz}{z^{2d+2}}. \quad (3-4)$$

We have

$$B(z, z_1) = \sum_{k=1}^{\infty} k z^{k-1} dz \frac{dz_1}{z_1^{k+1}} \quad (3-5)$$

keeping only the odd part under the involution $\sigma_0(z) = -z$, we have

$$B_{\text{odd}}(z, z_1) = \sum_{k=0}^{\infty} (2k+1) z^{2k} dz \frac{dz_1}{z_1^{2k+2}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k-1)!!} z^{2k} dz d\xi_d(z_1) \quad (3-6)$$

whose antiderivative wrt z is then

$$dS_{\text{odd}}(z_1; z) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!!} z^{2k+1} d\xi_d(z_1). \quad (3-7)$$

3.3 Kernel

The kernel is

$$\begin{aligned} K_0(z_1, z) &= \frac{dS_{\text{odd}}(z_1; z)}{(y(z) - y(-z))dx(z)} \\ &= \frac{-1}{2zdz} \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)!!} z^{2k} d\xi_d(z_1) \right) \left(\sum_{k \geq 0} \tilde{t}_{2k+3} z^{2k} \right)^{-1} \\ &= \frac{-1}{2\tilde{t}_3} \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)!!} z^{2k} d\xi_d(z_1) \right) \left(1 + \sum_{k \geq 1} \frac{\tilde{t}_{2k+3}}{\tilde{t}_3} z^{2k} \right)^{-1} \frac{1}{zdz} \end{aligned} \quad (3-8)$$

Let us define:

$$\begin{aligned} T_0 &= 1 \\ T_k &= \sum_{n=1}^k \frac{(-1)^n}{\tilde{t}_3^n} \sum_{j_1+\dots+j_n=k, j_i>0} \tilde{t}_{3+2j_1} \dots \tilde{t}_{3+2j_n}. \end{aligned} \quad (3-9)$$

We have

$$K_0(z_1, z) = \frac{-1}{2\tilde{t}_3} \left(\sum_{d=0}^{\infty} \frac{d\xi_d(z_1)}{(2d+1)!!} z^{2d} \right) \left(\sum_{k \geq 0} T_k z^{2k} \right) \frac{1}{zdz}. \quad (3-10)$$

3.4 First invariants

We have

$$\begin{aligned} \omega_{0,3}(z_1, z_2, z_3) &= \underset{z \rightarrow 0}{\text{Res}} K_0(z_1, z) (B(z, z_2)B(-z, z_3) + B(z, z_3)B(-z, z_2)) \\ &= -2 \underset{z \rightarrow 0}{\text{Res}} K_0(z_1, z) B(z, z_2) B(z, z_3) \\ &= \frac{1}{\tilde{t}_3} \frac{dz_1}{z_1^2} \underset{z \rightarrow 0}{\text{Res}} \left(1 - z^2/z_1^2 \right)^{-1} \left(1 + \sum_{k \geq 1} T_k z^{2k} \right) \frac{1}{zdz} \frac{dz^2 dz_2 dz_3}{(z-z_2)^2 (z-z_3)^2} \\ &= \frac{1}{\tilde{t}_3} \prod_{i=1}^3 \frac{dz_i}{z_i^2} \end{aligned}$$

$$= \frac{1}{\tilde{t}_3} \prod_{i=1}^3 d\xi_0(z_i). \quad (3-11)$$

We have

$$\begin{aligned}
\omega_{1,1}(z_1) &= \operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) B(z, -z) \\
&= -\operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) \frac{dz^2}{4z^2} \\
&= \frac{1}{2\tilde{t}_3} \frac{dz_1}{z_1^2} \operatorname{Res}_{z \rightarrow 0} (1 - z^2/z_1^2)^{-1} \left(1 + \sum_{k \geq 1} T_k z^{2k} \right) \frac{1}{z} \frac{dz^2}{4z^2} \\
&= \frac{1}{8\tilde{t}_3} \frac{dz_1}{z_1^2} \operatorname{Res}_{z \rightarrow 0} (1 - z^2/z_1^2)^{-1} \left(1 + \sum_{k \geq 1} T_k z^{2k} \right) \frac{dz}{z^3} \\
&= \frac{1}{8\tilde{t}_3} \frac{dz_1}{z_1^2} \left(\frac{1}{z_1^2} + T_1 \right) \\
&= \frac{1}{8\tilde{t}_3} \left(\frac{1}{3} d\xi_1(z_1) + T_1 d\xi_0(z_1) \right)
\end{aligned} \tag{3-12}$$

3.5 ABCD Tensors

- A

$$A[0, 0, 0] = \frac{1}{\tilde{t}_3} \tag{3-13}$$

- D

$$D[1] = \frac{1}{24\tilde{t}_3}, \quad D[0] = \frac{-\tilde{t}_5}{8\tilde{t}_3^2} = \frac{T_1}{8\tilde{t}_3}. \tag{3-14}$$

- B

$$\begin{aligned}
\sum_{d_1} \sum_{d_2} B[d_1, d_2 | d_3] d\xi_{d_1}(z_1) d\xi_{d_2}(z_2) &= \operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) B(z, z_2) d\xi_{d_3}(-z) \\
&= -\operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) B(z, z_2) d\xi_{d_3}(z)
\end{aligned} \tag{3-15}$$

This implies

$$\begin{aligned}
B[d_1, d_2 | d_3] &= \frac{1}{2\tilde{t}_3(2d_1+1)!!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} T_k z^{2k} \right) \frac{z^{2d_1-1}}{dz} \frac{z^{2d_2} dz}{(2d_2-1)!!} d\xi_{d_3}(z) \\
&= \frac{1}{2\tilde{t}_3(2d_1+1)!!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} T_k z^{2k} \right) \frac{z^{2d_1-1}}{dz} \frac{z^{2d_2} dz}{(2d_2-1)!!} \frac{(2d_3+1)!! dz}{z^{2d_3+2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!(2d_2 - 1)!!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} T_k z^{2k} \right) z^{2d_1} z^{2d_2} z^{-(2d_3 + 2)} \frac{dz}{z} \\
&= \frac{(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!(2d_2 - 1)!!} T_{d_3+1-d_1-d_2} \delta_{d_1+d_2 \leq d_3+1}.
\end{aligned} \tag{3-16}$$

• C

$$\begin{aligned}
\sum_{d_1} C[d_1|d_2, d_3] d\xi_{d_1}(z_1) &= \operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) d\xi_{d_2}(z) d\xi_{d_3}(-z) \\
&= -\operatorname{Res}_{z \rightarrow 0} K_0(z_1, z) d\xi_{d_2}(z) d\xi_{d_3}(z)
\end{aligned} \tag{3-17}$$

This implies

$$\begin{aligned}
C[d_1|d_2, d_3] &= \frac{1}{2\tilde{t}_3(2d_1 + 1)!!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} T_k z^{2k} \right) \frac{z^{2d_1}}{z dz} \frac{(2d_2 + 1)!! dz}{z^{2d_2+2}} \frac{(2d_3 + 1)!! dz}{z^{2d_3+2}} \\
&= \frac{(2d_2 + 1)!!(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} T_k z^{2k} \right) z^{2(d_1-d_2-d_3-2)} \frac{dz}{z} \\
&= \frac{(2d_2 + 1)!!(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!} T_{d_2+d_3+2-d_1} \delta_{d_1 \leq d_2+d_3+2}.
\end{aligned} \tag{3-18}$$

3.6 Summary KdV

3.6.1 General KdV case

$$\begin{aligned}
\omega_{g,n}(z_1, \dots, z_n) &= \sum_{d_1, \dots, d_n} F_{g,n}[d_1, \dots, d_n] \prod_{i=1}^n d\xi_{d_i}(z_i) \\
\mathcal{F}_{g,n}(z_1, \dots, z_n) &= \sum_{d_1, \dots, d_n} F_{g,n}[d_1, \dots, d_n] \prod_{i=1}^n \xi_{d_i}(z_i)
\end{aligned} \tag{3-19}$$

$$\xi_d(z) = -\frac{(2d-1)!!}{z^{2d+1}} , \quad d\xi_d(z) = \frac{(2d+1)!! dz}{z^{2d+2}}. \tag{3-20}$$

$$\begin{aligned}
A[0, 0, 0] &= \frac{1}{\tilde{t}_3} \\
D[1] &= \frac{1}{24\tilde{t}_3} , \quad D[0] = \frac{T_1}{8\tilde{t}_3} \\
B[d_1, d_2|d_3] &= \frac{(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!(2d_2 - 1)!!} T_{d_3+1-d_1-d_2} \delta_{d_1+d_2 \leq d_3+1} \\
C[d_1|d_2, d_3] &= \frac{(2d_2 + 1)!!(2d_3 + 1)!!}{2\tilde{t}_3(2d_1 + 1)!!} T_{d_2+d_3+2-d_1} \delta_{d_1 \leq d_2+d_3+2}.
\end{aligned} \tag{3-21}$$

We recall that for the KdV spectral curve, the invariants are the generating functions of Witten-Kontsevich intersection numbers.

Case $\tilde{t}_1 = 0, \tilde{t}_3 = -2$:

$$F_{g,n}[d_1, \dots, d_n] = (-2)^{2-2g-n} \langle \tau_{d_1} \dots \tau_{d_n} e^{\frac{1}{2} \sum_k (2k-1)!! t_{2k+1} \tau_k} \rangle_g . \quad (3-22)$$

The case $\tilde{t}_1 = 0$ and $\tilde{t}_3 \neq -2$ can be obtained by the homogeneity (2-6):

$$\begin{aligned} F_{g,n}[d_1, \dots, d_n] &= \tilde{t}_3^{2-2g-n} \langle \tau_{d_1} \dots \tau_{d_n} e^{-\frac{1}{\tilde{t}_3} \sum_{k \geq 2} (2k-1)!! \tilde{t}_{2k+1} \tau_k} \rangle_g \\ &= (-2)^{2-2g-n} \langle \tau_{d_1} \dots \tau_{d_n} e^{\frac{1}{2} \sum_{k \geq 1} (2k-1)!! (\tilde{t}_{2k+1} + 2\delta_{k,1}) \tau_k} \rangle_g . \end{aligned} \quad (3-23)$$

The case $\tilde{t}_1 \neq 0$ is more subtle, see the litterature, for example [Eyn16].

3.6.2 Summary Airy

This is the case $T_k = \delta_{k,0}$ and $t_3 = -2$:

$$\begin{cases} x = z^2 \\ y = z \\ y^2 - x = 0 \end{cases} \quad (3-24)$$

This is also the minimal model (1, 2).

$$\begin{aligned} A[0,0,0] &= -\frac{1}{2} \\ D[1] &= \frac{-1}{48} \\ B[d_1, d_2 | d_3] &= \frac{-1}{4} \frac{(2d_3 + 1)!!}{(2d_1 + 1)!! (2d_2 - 1)!!} \delta_{d_3+1, d_1+d_2} \\ C[d_1 | d_2, d_3] &= \frac{-1}{4} \frac{(2d_2 + 1)!! (2d_3 + 1)!!}{(2d_1 + 1)!!} \delta_{d_2+d_3+2, d_1} \end{aligned} \quad (3-25)$$

For the Airy spectral curve, the coefficients $F_{g,n}$ in this basis, are the Witten-Kontsevich intersection numbers

$$F_{g,n}[d_1, \dots, d_n] = (-1)^n 2^{2-2g-n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g . \quad (3-26)$$

This spectral curve is related to the Airy function as follows:

$$\begin{aligned} \psi(z^2) &= z^{-\frac{1}{4}} e^{\frac{2}{3} \hbar^{-1} z^{\frac{3}{2}}} e^{\sum_{k=0}^{\infty} \hbar^k \sum_{2g-2+n=k} \frac{1}{n!} \mathcal{F}_{g,n}(z, \dots, z)} \\ &= z^{-\frac{1}{4}} e^{\frac{2}{3} \hbar^{-1} z^{\frac{3}{2}}} e^{\sum_{k=0}^{\infty} \hbar^k \sum_{2g-2+n=k} \frac{1}{n!} \sum_{d_1+\dots+d_n=3g-3+n} \prod_{i=1}^n \xi_{d_i}(z)} \end{aligned} \quad (3-27)$$

satisfies the Airy equation

$$\hbar^2 \psi''(x) = x \psi(x) . \quad (3-28)$$

3.6.3 Summary Painlevé 1

This is the case

$$\begin{cases} x = z^2 - 2u \\ y = z^3 - 3uz \\ y^2 - 2u^3 = x^3 - 3u^2x \end{cases} \quad (3-29)$$

i.e.

$$\tilde{t}_1 = -2u, \quad \tilde{t}_3 = 6u, \quad \tilde{t}_5 = -2. \quad (3-30)$$

$$t_5 = \tilde{t}_5 = -2, \quad t_3 = \tilde{t}_3 - 3/2\tilde{t}_5\tilde{t}_1 = 0, \quad t_1 = \frac{-1}{2}\tilde{t}_3\tilde{t}_1 + \frac{3}{8}\tilde{t}_5\tilde{t}_1^2 = 3u^2. \quad (3-31)$$

Painlevé 1 is also the minimal model (3, 2).

This yields

$$T_k = (3u)^{-k}. \quad (3-32)$$

and thus

$$\begin{aligned} A[0, 0, 0] &= \frac{1}{6u} \\ D[0] &= \frac{1}{144u^2}, \quad D[1] = \frac{1}{144u} \\ B[d_1, d_2 | d_3] &= \frac{1}{12u} \frac{(2d_3 + 1)!!}{(2d_1 + 1)!!(2d_2 - 1)!!} (3u)^{d_1 + d_2 - d_3 - 1} \delta_{d_1 + d_2 \leq d_3 + 1} \\ C[d_1 | d_2, d_3] &= \frac{1}{12u} \frac{(2d_2 + 1)!!(2d_3 + 1)!!}{(2d_1 + 1)!!} (3u)^{d_1 - 2 - d_2 - d_3} \delta_{d_1 \leq d_2 + d_3 + 2}. \end{aligned} \quad (3-33)$$

The coefficients $F_{g,n}$ are related to the Painlevé 1 equation as follows. Define

$$\begin{aligned} U &= u - \sum_{g=1}^{\infty} \hbar^{2g} F_{g,2}[0, 0] \\ &= u - \frac{\hbar^2}{432u^4} - \frac{49\hbar^4}{373248u^9} - \frac{25.49\hbar^6}{2^{11}3^9u^{14}} + O(\hbar^8) \\ &= \frac{1}{\sqrt{3}}t_1^{\frac{1}{2}} - \frac{\hbar^2}{48}t_1^{-2} - \frac{49\hbar^4}{512.3^{\frac{3}{2}}}t_1^{-\frac{9}{2}} - \frac{25.49\hbar^6}{2^{11}3^2}t_1^{-7} + O(\hbar^8). \end{aligned} \quad (3-34)$$

Regarded as a function of $t_1 = 3u^2$, U satisfy the Painlevé 1 equation

$$3U^2 - \frac{1}{2}\hbar^2 U'' = t_1. \quad (3-35)$$

In fact this is general: for every minimal model $(p, 2)$, $U = -\frac{1}{2}\tilde{t}_1 - \sum_{g=1}^{\infty} \hbar^{2g} F_{g,2}[0, 0]$ satisfies a non-linear differential equation of the Painlevé 1 hierarchy (i.e. a Gelfand-Dikii equation) of order $p - 1$ and degree $(p + 1)/2$.

3.6.4 Summary Weil-Petersson

This is the case where y is the sine function

$$\begin{cases} x = z^2 \\ y = \frac{-1}{4\pi} \sin 2\pi z \end{cases} \quad (3-36)$$

i.e. the KdV times

$$\tilde{t}_{2k+3} = t_{2k+3} = \frac{(-1)^k (2\pi)^{2k}}{(2k+1)!} \quad (3-37)$$

which correspond to the T_k being Bernoulli numbers or equivalently even zeta values:

$$T_0 = 1 \quad , \quad T_k = \frac{2(2^{2k-1} - 1)|B_{2k}|}{(2k)!} (2\pi)^{2k} = 4(2^{2k-1} - 1)\zeta(2k). \quad (3-38)$$

The A, B, C, D tensors are thus

$$\begin{aligned} A[0, 0, 0] &= 1 \\ D[1] = \frac{1}{24}, \quad D[0] &= \frac{T_1}{8} = \frac{1}{2}\zeta(2) = \frac{\pi^2}{12} \\ B[d_1, d_2 | d_3] &= \frac{(2d_3 + 1)!!}{2(2d_1 + 1)!!(2d_2 - 1)!!} T_{d_3 + 1 - d_1 - d_2} \\ C[d_1 | d_2, d_3] &= \frac{(2d_2 + 1)!!(2d_3 + 1)!!}{2(2d_1 + 1)!!} T_{d_2 + d_3 + 2 - d_1}. \end{aligned} \quad (3-39)$$

The coefficients $F_{g,n}$ are related to Weil-Petersson volumes:

$$\begin{aligned} F_{g,n}[d_1, \dots, d_n] &= \left\langle e^{2\pi^2 \kappa_1} \prod_{i=1}^n \tau_{d_i} \right\rangle_g \\ \sum_{d_1 + \dots + d_n \leq 3g-3+n} F_{g,n}[d_1, \dots, d_n] \prod_{i=1}^n \frac{L_i^{2d_i}}{2^{d_i} d_i!} &= \int_{\mathcal{M}_{g,n}(L_1, \dots, L_n)} \text{WeilPetersson} \\ &= \int_{\overline{\mathcal{M}}_{g,n}} e^{2\pi^2 \kappa_1 + \frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i} \\ &= \left\langle e^{2\pi^2 \kappa_1} e^{\frac{1}{2} \sum_{i=1}^n L_i^2 \psi_i} \right\rangle_g. \end{aligned} \quad (3-40)$$

4 GUE - Random matrices

The following spectral curve appears in random matrices, and in particular applications to enumeration of maps, see for example [Eyn16]:

4.1 Spectral curve

$$\begin{cases} \Sigma = \mathbb{C}P^1 \\ x = z + 1/z \\ y = \sum_j u_j z^{-j} \\ B = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \\ R = \{1, -1\}, \sigma_a(z) = 1/z. \end{cases} \quad (4-1)$$

There are 2 branchpoints $a = \pm 1$, and the involution is $\sigma_a(z) = 1/z$.

The spectral curve satisfies an algebraic equation

$$P(x, y) = \left(2y - \sum_j u_j T_j(x)\right)^2 - (x^2 - 4) \left(\sum_j u_j U_j(x)\right)^2 = 0 \quad (4-2)$$

where $T_j(z + z^{-1}) = z^j + z^{-j}$ and $U_j(z + z^{-1}) = \frac{z^j - z^{-j}}{z - z^{-1}}$ are Chebyshev polynomials. $V'(x) = \sum_j u_j T_j(x)$ is the derivative of $V(x)$ called the potential.

The equation is hyperelliptical because of degree 2 in y , but in fact the spectral curve has genus 0 (indeed it is parametrized by rational functions on $\Sigma = \mathbb{C}P^1$).

The Gaussian matrix model (GUE) is the case $u_j = \delta_{j,1}$, i.e. $V'(x) = x$, i.e. $V(x) = \frac{1}{2}x^2$, in which case the spectral curve obeys the equation

$$(2y - x)^2 = (x^2 - 4) \quad (4-3)$$

i.e.

$$y^2 - xy + 1 = 0. \quad (4-4)$$

$\Im y = \frac{1}{2}\sqrt{4 - x^2}$ is the equation of a semi-circle of radius 2, which is the famous Wigner semi-circle spectral curve of the Gaussian unitary matrix model.

The relationship between the Topological Recursion invariants of this spectral curve and a random matrix model is that the expectation value of the trace of the resolvent of a random matrix drawn with probability $e^{-\hbar^{-1}\text{Tr}V(M)}\mathcal{D}M$ (with $\mathcal{D}M$ the Lebesgue measure on H_N) has the following asymptotic expansion in the limit $\hbar \rightarrow 0$, $N \rightarrow \infty$ such that $\hbar N = O(1)$:

$$\frac{\int_{H_N} e^{-\hbar^{-1}\text{Tr}V(M)}\mathcal{D}M \text{Tr}(x_1 - M)^{-1}}{\int_{H_N} e^{-\hbar^{-1}\text{Tr}V(M)}\mathcal{D}M} dx_1 = \sum_{g=0}^{\infty} \hbar^{2g-2+1} \omega_{g,1}(z_1), \quad (4-5)$$

and where $x_1 = x(z_1) = z_1 + 1/z_1$. A similar relation also holds for any $n > 1$, but instead of expectation value, the left hand side is the cumulant expectation value (generalization of covariance). See for example [Eyn16] for further details.

4.1.1 Coordinates

Instead of x or z , it is convenient to use other coordinates:

$$x = z + 1/z \quad , \quad z = e^{2\phi} = \frac{1 + \zeta}{1 - \zeta} \quad , \quad \zeta = \tanh \phi = \sqrt{\frac{x - 2}{x + 2}} = \frac{z - 1}{z + 1} \quad (4-6)$$

$$x = z + 1/z = 2 \cosh 2\phi = 2 \frac{1 + \zeta^2}{1 - \zeta^2} = \frac{4}{1 - \zeta^2} - 2 \quad , \quad dx = \frac{8\zeta d\zeta}{(1 - \zeta^2)^2} \quad (4-7)$$

We define the basis of 1-forms with poles at $z = \pm 1$, i.e. at $\zeta = 0, \infty$, for $a = \pm 1$ and $k \in \mathbb{Z}_+$:

$$\begin{aligned} d\xi_{a,k}(z) &= -a(2k+1)\zeta^{-a(2k+1)} \frac{d\zeta}{\zeta} \\ \xi_{a,k}(z) &= \zeta^{-a(2k+1)} = \left(\frac{z+1}{z-1}\right)^{a(2k+1)} = \left(\frac{z+a}{z-a}\right)^{2k+1}. \end{aligned} \quad (4-8)$$

4.2 Bergman kernel

All the following expressions are equivalent for B :

$$\begin{aligned} B(z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2} = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} = \frac{d(1/\zeta_1)d(1/\zeta_2)}{(1/\zeta_1 - 1/\zeta_2)^2} \\ &= d_1 \otimes d_2 \ln(z_1 - z_2) \\ &= d_1 \otimes d_2 \ln(\zeta_1 - \zeta_2) \\ &= d_1 \otimes d_2 \ln(1/\zeta_1 - 1/\zeta_2) \\ &= d_1 \otimes d_2 \ln(1 - \zeta_1 \zeta_2) \\ &= d_1 \otimes d_2 \ln(1 - \zeta_2 \zeta_1). \end{aligned} \quad (4-9)$$

Expansion near $z_2 \rightarrow a = \pm 1$, i.e. $\zeta_2^a \rightarrow 0$

$$\begin{aligned} B(z_1, z_2) &= d_1 \otimes d_2 \ln(1 - \zeta_2^a/\zeta_1^a) \\ &= -d_1 \otimes d_2 \sum_{k=1}^{\infty} \frac{1}{k} \frac{\zeta_2^{ak}}{\zeta_1^{ak}} \\ B_{\text{odd}}(z_1, z_2) &= -a \sum_{k=0}^{\infty} \zeta_2^{a(2k+1)-1} d\zeta_2 \, d\xi_{a,k}(z_1) \\ &= - \sum_{k=0}^{\infty} \zeta_2^{2ak} d\zeta_2^a \, d\xi_{a,k}(z_1). \end{aligned} \quad (4-10)$$

Integrating over z_2 gives

$$\begin{aligned} dS(z_1, z_2) &= -d_1 \sum_{k=1}^{\infty} \frac{1}{k} \frac{\zeta_2^{ak}}{\zeta_1^{ak}} \\ dS_{\text{odd}}(z_1, z_2) &= -d_1 \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{\zeta_2^{a(2k+1)}}{\zeta_1^{a(2k+1)}} \\ &= - \sum_{k=0}^{\infty} \frac{1}{2k+1} \zeta_2^{a(2k+1)} d\xi_{a,k}(z_1). \end{aligned} \quad (4-11)$$

4.2.1 Kernel

The kernel is

$$K_a(z_1, z) = \frac{dS_{\text{odd}}(z_1, z)}{(y(z) - y(1/z))dx(z)}. \quad (4-12)$$

The denominator is

$$\begin{aligned}
& (y(z) - y(1/z))dx(z) \\
= & - \sum_{j \geq 1} u_j (z^j - z^{-j}) dx(z) \\
= & - \frac{8\zeta d\zeta}{(1 - \zeta^2)^2} \sum_{j \geq 1} u_j a^j ((1 + \zeta^a)^j (1 - \zeta^a)^{-j} - (1 - \zeta^a)^j (1 + \zeta^a)^{-j}) \\
= & -8\zeta d\zeta \sum_{j \geq 1} u_j a^j \left(\frac{(1 + \zeta^a)^{2j}}{(1 - \zeta^{2a})^j} - \frac{(1 - \zeta^a)^{2j}}{(1 - \zeta^{2a})^j} \right) (1 - \zeta^2)^{-2} \\
= & -8\zeta d\zeta \sum_{j \geq 1} u_j a^j ((1 + \zeta^a)^{2j} - (1 - \zeta^a)^{2j}) (1 - \zeta^{2a})^{-2-j} \frac{(1 - \zeta^{2a})^2}{(1 - \zeta^2)^2} \\
= & -8\zeta d\zeta \sum_{j \geq 1} u_j a^j ((1 + \zeta^a)^{2j} - (1 - \zeta^a)^{2j}) (1 - \zeta^{2a})^{-2-j} (a\zeta^{a-1})^2 \\
= & -16\zeta^{2a} \frac{d\zeta}{\zeta} \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{j-1} \binom{2j}{2l+1} \zeta^{a(2l+1)} \right) (1 - \zeta^{2a})^{-2-j} \\
= & -16\zeta^{3a} \frac{d\zeta}{\zeta} \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{j-1} \binom{2j}{2l+1} \zeta^{2al} \right) \left(\sum_{k \geq 0} \binom{-j-2}{k} (-1)^k \zeta^{2ak} \right) \\
= & -16a\zeta^{3a} \frac{d\zeta^a}{\zeta^a} \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{j-1} \binom{2j}{2l+1} \zeta^{2al} \right) \left(\sum_{k \geq 0} \binom{-j-2}{k} (-1)^k \zeta^{2ak} \right) \\
= & -16a\zeta^{3a} \frac{d\zeta^a}{\zeta^a} \sum_{k \geq 0} \zeta^{2ak} \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{j-1} \binom{2j}{2l+1} \binom{-j-2}{k-l} (-1)^{k-l} \right) \\
= & -16a\zeta^{3a} \frac{d\zeta^a}{\zeta^a} \sum_{k \geq 0} \zeta^{2ak} \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{j-1} \binom{2j}{2l+1} \binom{j+1+k-l}{j+1} \right)
\end{aligned} \tag{4-13}$$

Let us write it

$$\begin{aligned}
(y(z) - y(1/z))dx(z) &= -16a\zeta^{2a} d\zeta^a \sum_{k \geq 0} \zeta^{2ak} y_{a,k} \\
y_{a,k} &= \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{\min(j-1, k)} \binom{2j}{2l+1} \binom{j+1+k-l}{j+1} \right)
\end{aligned} \tag{4-14}$$

and let's write its inverse as

$$\begin{aligned}
\frac{1}{(y(z) - y(1/z))dx(z)} &= -\frac{a}{16\zeta^{2a} d\zeta^a} \frac{1}{y_{a,0}} \left(1 + \sum_{k \geq 1} \zeta^{2ak} y_{a,k} / y_{a,0} \right)^{-1} \\
&= -\frac{a}{16\zeta^{2a} d\zeta^a} \left(\sum_{k \geq 0} \zeta^{2ak} Y_{a,k} \right)
\end{aligned} \tag{4-15}$$

where

$$\begin{aligned} Y_{a,0} &= 1/y_{a,0} \\ Y_{a,k} &= \sum_{n=1}^k \frac{(-1)^n}{y_{a,0}^{n+1}} \sum_{j_1+\dots+j_n=k, j_i>0} y_{a,j_1} \dots y_{a,j_n} \quad k \geq 1. \end{aligned} \quad (4-16)$$

Therefore

$$\begin{aligned} K_a(z_1, z) &= \frac{a}{16\zeta^{2a}d\zeta^a} \left(\sum_{k \geq 0} \zeta^{2ak} Y_{a,k} \right) \left(\sum_{d_1=0}^{\infty} \frac{1}{2d_1+1} \zeta^{a(2d_1+1)} d\xi_{a,d_1}(z_1) \right) \\ &= \frac{a}{16\zeta^a d\zeta^a} \left(\sum_{k=0} \zeta^{2ak} \sum_{j=0}^k Y_{a,k-j} \frac{d\xi_{a,j}(z_1)}{2j+1} \right). \end{aligned} \quad (4-17)$$

4.3 ABCD Tensors

From the definitions we get

• A

$$\begin{aligned} \omega_{0,3}(z_1, z_2, z_3) &= \sum_{a=\pm 1} 2 \operatorname{Res}_a K_a(z_1, z) B(z, z_2) B(1/z, z_3) \\ &= \sum_{a=\pm 1} \operatorname{Res}_a \frac{a}{8y_{a,0}\zeta^a d\zeta^a} (d\xi_{a,0}(z_1) + O(\zeta^{2a})) \\ &\quad (-d\zeta^a d\xi_{a,0}(z_2)) (d\zeta^a d\xi_{a,0}(z_3)) \\ &= - \sum_{a=\pm 1} \frac{a}{8y_{a,0}} d\xi_{a,0}(z_1) d\xi_{a,0}(z_2) d\xi_{a,0}(z_3) \end{aligned} \quad (4-18)$$

i.e.

$$A[a, 0; a, 0; a, 0] = F_{0,3}[a, 0; a, 0; a, 0] = \frac{-aY_{a,0}}{8} = \frac{-a}{8y_{a,0}} \quad (4-19)$$

• D

$$\begin{aligned} \omega_{1,1}(z_1) &= \sum_{a=\pm 1} \operatorname{Res}_a K_a(z_1, z) B(z, 1/z) \\ &= \sum_{a=\pm 1} \operatorname{Res}_0 \frac{a}{16\zeta^a d\zeta^a} (Y_{a,0} + \zeta^{2a} Y_{a,1}) (d\xi_{a,0}(z_1) + \frac{1}{3} \zeta^{2a} d\xi_{a,1}(z_1)) \frac{-(d\zeta^a)^2}{4\zeta^{2a}} \\ &= - \sum_{a=\pm 1} \frac{a}{64} \left(Y_{a,1} d\xi_{a,0}(z_1) + \frac{Y_{a,0}}{3} d\xi_{a,1}(z_1) \right) \end{aligned} \quad (4-20)$$

$$D[a, 0] = \frac{-aY_{a,1}}{64}, \quad D[a, 1] = \frac{-aY_{a,0}}{192} \quad (4-21)$$

• C

$$\operatorname{Res}_{a_1} K_{a_1}(z_1, z) d\xi_{a_2, d_2}(z) d\xi_{a_3, d_3}(1/z)$$

$$\begin{aligned}
&= - \operatorname{Res}_0 \frac{a_1}{16\zeta^{a_1} d\zeta^{a_1}} \left(\sum_k \zeta^{2a_1 k} Y_{a_1, k} \right) \\
&\quad \left(\sum_{d_1} \frac{\zeta^{2a_1 d_1}}{2d_1 + 1} d\xi_{a_1, d_1}(z_1) \right) d\xi_{a_2, d_2}(z) d\xi_{a_3, d_3}(z)
\end{aligned} \tag{4-22}$$

$$\begin{aligned}
&C[a_1, d_1 | a_2, d_2; a_3, d_3] \\
&= - \operatorname{Res}_0 \frac{a_1}{16\zeta^{a_1} d\zeta^{a_1}} \left(\sum_k \zeta^{2a_1 k} Y_{a_1, k} \right) \frac{\zeta^{2a_1 d_1}}{2d_1 + 1} d\xi_{a_2, d_2}(z) d\xi_{a_3, d_3}(z) \\
&= -a_1 a_2 a_3 \frac{(2d_2 + 1)(2d_3 + 1)}{2d_1 + 1} \operatorname{Res}_0 \frac{1}{16\zeta^{a_1} d\zeta^{a_1}} \left(\sum_k \zeta^{2a_1 k} Y_{a_1, k} \right) \\
&\quad \zeta^{2a_1 d_1} \zeta^{-a_2(2d_2+1)} \zeta^{-a_3(2d_3+1)} \frac{(d\zeta^{a_1})^2}{\zeta^{2a_1}} \\
&= -a_1 a_2 a_3 \frac{(2d_2 + 1)(2d_3 + 1)}{2d_1 + 1} \operatorname{Res}_0 \frac{d\zeta^{a_1}}{16\zeta^{a_1}} \left(\sum_k \zeta^{2a_1 k} Y_{a_1, k} \right) \\
&\quad \zeta^{2a_1(d_1-1)} \zeta^{-a_2(2d_2+1)} \zeta^{-a_3(2d_3+1)} \\
&= \frac{-1}{16} a_1 a_2 a_3 \frac{(2d_2 + 1)(2d_3 + 1)}{2d_1 + 1} Y_{a_1, a_1 a_2 d_2 + a_1 a_3 d_3 - d_1 + 1 + a_1(a_2 + a_3)/2}
\end{aligned} \tag{4-23}$$

it is non-zero only if

$$d_1 \leq 1 + a_1(a_2 d_2 + a_3 d_3 + (a_2 + a_3)/2) \tag{4-24}$$

• B

$$\begin{aligned}
&B[a, d_1; a, d_2 | a_3 d_3] \\
&= - \operatorname{Res}_0 \frac{a}{16\zeta^a d\zeta^a} \left(\sum_k \zeta^{2ak} Y_{a, k} \right) \frac{\zeta^{2ad_1}}{2d_1 + 1} (-\zeta^{2ad_2} d\zeta^a) d\xi_{a_3, d_3}(z) \\
&= \frac{a}{16(2d_1 + 1)} \operatorname{Res}_0 \frac{1}{\zeta^a d\zeta^a} \left(\sum_k \zeta^{2ak} Y_{a, k} \right) \zeta^{2ad_1} \zeta^{2ad_2} d\zeta^a d\xi_{a_3, d_3}(z) \\
&= \frac{a}{16(2d_1 + 1)} \operatorname{Res}_0 \frac{1}{\zeta^a} \left(\sum_k \zeta^{2ak} Y_{a, k} \right) \zeta^{2ad_1} \zeta^{2ad_2} \zeta^{-a_3(2d_3+1)} (-a_3(2d_3 + 1)) \frac{d\zeta}{\zeta} \\
&= \frac{1}{16(2d_1 + 1)} \operatorname{Res}_0 \frac{1}{\zeta^a} \left(\sum_k \zeta^{2ak} Y_{a, k} \right) \zeta^{2ad_1} \zeta^{2ad_2} \zeta^{-a_3(2d_3+1)} (-a_3(2d_3 + 1)) \frac{d\zeta^a}{\zeta^a} \\
&= \frac{-a_3(2d_3 + 1)}{16(2d_1 + 1)} \operatorname{Res}_0 \left(\sum_k \zeta^{2ak} Y_{a, k} \right) \zeta^{2a(d_1+d_2-aa_3d_3)} \zeta^{-a(1+aa_3)} \frac{d\zeta^a}{\zeta^a} \\
&= \frac{-a_3(2d_3 + 1)}{16(2d_1 + 1)} Y_{a, aa_3 d_3 - d_1 - d_2 + (1+aa_3)/2}
\end{aligned} \tag{4-25}$$

4.4 Summary GUE

$$\begin{cases} x = z + 1/z \\ y = \sum_j u_j z^{-j} \\ B = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \\ a = \pm 1 \end{cases} \quad (4-26)$$

We define for the ramification points $a = \pm 1$:

$$\begin{aligned} y_{a,k} &= \sum_{j \geq 1} u_j a^j \left(\sum_{l=0}^{\min(j-1, k)} \binom{2j}{2l+1} \binom{j+1+k-l}{j+1} \right) \\ Y_{a,k} &= \sum_{n=0}^k \frac{(-1)^n}{y_{a,0}^{n+1}} \sum_{j_1 + \dots + j_n = k, j_i > 0} y_{a,j_1} \dots y_{a,j_n} \quad k \geq 1 \end{aligned} \quad (4-27)$$

In particular

$$y_{a,0} = 2 \sum_{j \geq 1} j u_j a^j, \quad Y_{a,0} = \frac{1}{y_{a,0}}. \quad (4-28)$$

$$\begin{aligned} A[a, 0; a, 0; a, 0] &= \frac{-a}{8} Y_{a,0} \\ D[a, 1] &= \frac{-a}{192} Y_{a,0} \\ D[a, 0] &= \frac{-a}{64} Y_{a,1} \\ B[a, d_1; a, d_2 | a_3, d_3] &= \frac{-a_3(2d_3 + 1)}{16(2d_1 + 1)} Y_{a, aa_3 d_3 - d_1 - d_2 + (1+aa_3)/2} \\ C[a_1, d_1 | a_2, d_2; a_3, d_3] &= \frac{-a_1 a_2 a_3 (2d_2 + 1)(2d_3 + 1)}{16} \\ &\quad Y_{a_1, a_1 a_2 d_2 + a_1 a_3 d_3 - d_1 + 1 + a_1(a_2 + a_3)/2} \end{aligned} \quad (4-29)$$

For pure GUE ($u_j = \delta_{j,1}$) we have

$$\begin{aligned} y_{a,k} &= (k+2)(k+1)a \\ Y_{a,k} &= a \frac{(-1)^k}{2} \binom{3}{k} \end{aligned} \quad (4-30)$$

$$\begin{aligned} A[a, 0; a, 0; a, 0] &= \frac{-1}{16} \\ D[a, 1] &= \frac{-1}{384} \\ D[a, 0] &= \frac{3}{128} \end{aligned}$$

$$\begin{aligned}
B[a, d_1; a, d_2 | a_3, d_3] &= \frac{-aa_3(2d_3 + 1)}{32(2d_1 + 1)} (-1)^k \binom{3}{k} \\
&\quad \text{with } k = aa_3d_3 - d_1 - d_2 + (1 + aa_3)/2 \\
C[a_1, d_1 | a_2, d_2; a_3, d_3] &= \frac{-a_2a_3}{32} \frac{(2d_2 + 1)(2d_3 + 1)}{2d_1 + 1} (-1)^k \binom{3}{k} \\
&\quad \text{with } k = a_1a_2d_2 + a_1a_3d_3 - d_1 + 1 + a_1(a_2 + a_3)/2.
\end{aligned} \tag{4-31}$$

5 Elliptic curves

Consider $\tau \in \mathbb{C}$ such that $\Im \tau > 0$. Let Σ be the torus of modulus τ :

$$\Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}. \tag{5-1}$$

An elliptic function f is such that $f(z) = f(z + 1) = f(z + \tau)$. We recall that the only elliptic holomorphic functions are constant. Non-constant elliptic meromorphic functions must have at least 2 simple poles or at least one higher degree pole. The sum of residues at all poles must be 0.

5.1 Elliptic spectral curves

We shall consider the following class of spectral curves:

$$\left\{
\begin{array}{l}
\Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \\
z \mapsto x(z) = \text{elliptic meromorphic function} \\
z \mapsto y(z) = \text{elliptic meromorphic function} \\
B = (\wp(z_1 - z_2) + G_2)dz_1 \otimes dz_2 \\
\sigma_a(z) = 2a - z
\end{array}
\right. \tag{5-2}$$

where \wp is the Weierstrass elliptic function (see below), $G_2 \in \mathbb{C}$ is arbitrary (often one chooses $G_2 = G_2(\tau)$ the 2nd Eisenstein series (in which case B is normalized on the \mathcal{A} -cycle $= \mathbb{R}/\mathbb{Z} + \tau\mathbb{Z}$), but any other choice is good too). We also assume that the ramification points are simple, and that the involutions take the form:

$$\sigma_a(z) = 2a - z. \tag{5-3}$$

5.1.1 Reminder Weierstrass elliptic function

Recall that the Weierstrass function is:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(n + \tau m - z)^2} - \frac{1}{(n + \tau m)^2} \\
&= \frac{1}{z^2} + \sum_{k \geq 2} (2k - 1)G_{2k}z^{2k-2}
\end{aligned}$$

$$\text{where } G_{2k} = \sum_{(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(n + \tau m)^{2k}} = 2k\text{-th Eisenstein series. (5-4)}$$

It satisfies the differential equation

$$\wp'^2 = 4(\wp^3 - 15G_4\wp - 35G_6), \quad (5-5)$$

from which we deduce

$$\begin{aligned} \wp'' &= 6(\wp^2 - 5G_4) \\ \wp''' &= 12\wp\wp'. \end{aligned} \quad (5-6)$$

And in general, even derivatives $\wp^{(2k)}$ can be expressed as polynomials of \wp :

$$\wp^{(2k)} = (2k+1)!Q_k(\wp) \quad (5-7)$$

where Q_k is a polynomial of degree $k+1$. It satisfies the recursion

$$Q_0(x) = x, \quad (2k+3)(2k+2)Q_{k+1}(x) = 6(x^2 - 5G_4)Q'_k + 4(x^3 - 15G_4x - 35G_6)Q''_k. \quad (5-8)$$

It can be computed with the following formula

$$\begin{aligned} Q_k(x) &= \left(x^{k+1} + \sum_{j=0}^{k-1} \alpha_{k,j} x^j \right) \\ \alpha_{k,j} &= \frac{k+1}{j} [z^{2(k+1-j)}] \left(1 + \sum_{l=2}^{k-j} (2l-1)G_{2l} z^{2l} \right)^{-j} \\ \alpha_{k,0} &= G_{2k+2} - (k+1)[z^{2k+2}] \ln \left(1 + \sum_{l=2}^k (2l-1)G_{2l} z^{2l} \right) \\ Q_k(x) &= (G_{2k+2} - (k+1)[z^{2k+2}] \ln(z^2\wp(z) - z^2x)). \end{aligned} \quad (5-9)$$

Given an arbitrary $G_2 \in \mathbb{C}$, it is convenient to define:

$$\hat{\wp}(z) = \wp(z) + G_2. \quad (5-10)$$

5.1.2 Reminder Legendre-Jacobi elliptic functions

The Legendre normalization of an elliptic curve is the algebraic equation

$$y^2 = (1 - x^2)(1 - k^2x^2).$$

Then, define the Legendre elliptic integrals

$$K = K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$\begin{aligned}
K' &= K(\sqrt{1-k^2}) \\
\tau &= \frac{iK'}{K} \\
F(x) &= \int_0^x \frac{dx'}{\sqrt{(1-x'^2)(1-k^2x'^2)}}.
\end{aligned} \tag{5-11}$$

The Jacobi elliptic functions are then defined as

$$\begin{aligned}
\text{sn}(u) &= F^{-1}(u) \\
\text{cn}(u) &= \sqrt{1 - \text{sn}(u)^2} \\
\text{dn}(u) &= \sqrt{1 - k^2 \text{sn}(u)^2}
\end{aligned} \tag{5-12}$$

They satisfy many nice properties, that generalize those of trigonometric functions, K is like $\pi/2$, and:

$$\begin{aligned}
\text{sn}(-u) &= -\text{sn}(u) \\
\text{sn}(u + 4K) &= \text{sn}(u) \\
\text{sn}(u + 2K) &= -\text{sn}(u) \\
\text{sn}(u + 2iK') &= \text{sn}(u) \\
\\
\text{sn}(K) &= 1 = -\text{sn}(3K) \\
\text{sn}(2K) &= 0 \\
\text{sn}(iK') &= \infty \\
\\
\text{cn}(-u) &= \text{cn}(u) \\
\text{cn}(u + 4K) &= \text{cn}(u) \\
\text{cn}(u + 2K) &= -\text{cn}(u) \\
\text{cn}(u + 2iK') &= -\text{cn}(u) \\
\\
\text{cn}(0) &= 1 \\
\text{cn}(K) &= 0 = \text{cn}(3K) \\
\text{cn}(2K) &= -1 \\
\text{cn}(iK') &= \infty \\
\\
\text{dn}(-u) &= \text{dn}(u) \\
\text{dn}(u + 2K) &= \text{dn}(u) \\
\text{dn}(u + 4iK') &= \text{dn}(u) \\
\text{dn}(u + 2iK') &= -\text{dn}(u) \\
\\
\text{dn}(0) &= 1
\end{aligned}$$

$$\begin{aligned}
\operatorname{dn}(K) &= \sqrt{1 - k^2} \\
\operatorname{dn}(iK') &= \infty
\end{aligned}
\tag{5-13}$$

$$\begin{aligned}
\operatorname{sn}'(u) &= \operatorname{cn} u \operatorname{dn} u \\
\operatorname{cn}'(u) &= -\operatorname{sn} u \operatorname{dn} u \\
\operatorname{dn}'(u) &= -k^2 \operatorname{sn} u \operatorname{cn} u.
\end{aligned}
\tag{5-14}$$

We have the relationship to the Weierstrass function

$$\begin{aligned}
\operatorname{sn}(2Kz)^2 &= \frac{\wp(1/2) - \wp(\tau/2)}{\wp(z) - \wp(\tau/2)} \\
\text{where } & k^2 = \frac{\wp(1/2 + \tau/2) - \wp(\tau/2)}{\wp(1/2) - \wp(\tau/2)}.
\end{aligned}
\tag{5-15}$$

5.2 Basis

We choose the basis:

$$a \in R, d \in \mathbb{Z}_+ : d\xi_{a,d}(z) = (\wp^{(2d)}(z-a) + G_2 \delta_{d,0}) dz = \hat{\wp}^{(2d)}(z-a) dz. \tag{5-16}$$

It is odd under involutions $z \mapsto \sigma_a(z) = 2a - z$.

It has the expansion near a :

$$\begin{aligned}
d\xi_{a,d}(z) &\sim_{z \rightarrow a} \frac{(2d+1)!}{(z-a)^{2d+2}} + \sum_{j=0}^{\infty} G_{2d+2j+2} \frac{(2d+2j+1)!}{(2j)!} (z-a)^{2j} \\
&\sim_{z \rightarrow a} \frac{(2d+1)!}{(z-a)^{2d+2}} \left(1 + \sum_{j=0}^{\infty} G_{2d+2j+2} \binom{2d+2j+1}{2j} (z-a)^{2d+2j+2} \right)
\end{aligned}
\tag{5-17}$$

It has the expansion near $b \neq a$:

$$\begin{aligned}
d\xi_{a,d}(z) &\sim_{z \rightarrow b} \sum_{j=0}^{\infty} \frac{1}{(2j)!} \hat{\wp}^{(2d+2j)}(b-a) (z-b)^{2j} \\
&\sim_{z \rightarrow b} \sum_{j=0}^{\infty} \frac{1}{(2j)!} ((2d+2j+1)! Q_{d+j}(\wp(b-a)) + \delta_{d+j,0} G_2) (z-b)^{2j}.
\end{aligned}
\tag{5-18}$$

The Kernel B has expansion

$$B(z_1, z) \sim_{z \rightarrow a} \sum_k \frac{(z-a)^k}{k!} \hat{\wp}^{(k)}(a-z_1) dz dz_1$$

$$\begin{aligned}
B_{\text{odd}}(z_1, z) &\sim_{z \rightarrow a} \sum_d \frac{(z-a)^{2d} dz}{(2d)!} d\xi_{a,d}(z_1) \\
dS_{\text{odd}}(z_1; z) &\sim_{z \rightarrow a} \sum_d \frac{(z-a)^{2d+1}}{(2d+1)!} d\xi_{a,d}(z_1)
\end{aligned} \tag{5-19}$$

5.2.1 Kernel

$$K_a(z_1, z) = \frac{1}{(z-a)dz} \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2k} \right) \sum_{d \geq 0} \frac{d\xi_{a,d}(z_1)}{(2d+1)!} (z-a)^{2d} \tag{5-20}$$

where

$$\frac{1}{(y(z) - y(\sigma_a(z)))x'(z)} = \sum_{k=0}^{\infty} Y_{k,a}(z-a)^{2k-2}. \tag{5-21}$$

In particular

$$Y_{a,0} = \frac{1}{2y'(a)x''(a)} \tag{5-22}$$

• A

$$\begin{aligned}
\omega_{0,3}(z_1, z_2, z_3) &= -2 \sum_{a \in R} \operatorname{Res}_a K_a(z_1, z) B(z, z_2) B(z, z_3) \\
&= -2 \sum_{a \in R} \frac{1}{(z-a)dz} (Y_{a,0} + O((z-a)^2)) (d\xi_{a,0}(z_1) + O((z-a)^2)) \\
&\quad (d\xi_{a,0}(z_2) + O((z-a)^2)) (d\xi_{a,0}(z_3) + O((z-a)^2)) dz^2 \\
&= -2 \sum_{a \in R} Y_{a,0} \prod_{i=1}^3 d\xi_{a,0}(z_i)
\end{aligned} \tag{5-23}$$

$$A[a, 0; a, 0; a, 0] = -2Y_{a,0} \tag{5-24}$$

• D

$$\begin{aligned}
B(z, \sigma_a(z)) &= -(g(2z) + G_2) dz^2 \\
&\sim_{z \rightarrow a} - \left(\frac{1}{4(z-a)^2} + G_2 + O((z-a)^2) \right) dz^2
\end{aligned} \tag{5-25}$$

$$\omega_{1,1}(z_1) = - \sum_{a \in R} \frac{1}{(z-a)dz} (Y_{a,0} + (z-a)^2 Y_{a,1} + O((z-a)^4))$$

$$\begin{aligned}
& \left(d\xi_{a,0}(z_1) + \frac{1}{6}(z-a)^2 d\xi_{a,1}(z_1) + O((z-a)^4) \right) \\
& \left(\frac{1}{4(z-a)^2} + G_2 + O((z-a)^2) \right) dz^2 \\
= & -\frac{1}{4} \sum_{a \in R} \frac{dz}{(z-a)^3} (Y_{a,0} + (z-a)^2 Y_{a,1} + O((z-a)^4)) \\
& \left(d\xi_{a,0}(z_1) + \frac{1}{6}(z-a)^2 d\xi_{a,1}(z_1) + O((z-a)^4) \right) \\
& (1 + 4(z-a)^2 G_2 + O((z-a)^4)) \\
= & -\frac{1}{4} \sum_{a \in R} \frac{1}{6} Y_{a,0} d\xi_{a,1}(z_1) + Y_{a,1} d\xi_{a,0}(z_1) + 4G_2 d\xi_{a,0}(z_1)
\end{aligned} \tag{5-26}$$

$$\begin{aligned}
D[a, 1] &= \frac{-Y_{a,0}}{24} \\
D[a, 0] &= \frac{-1}{4} (Y_{a,1} + 4G_2 Y_{a,0})
\end{aligned} \tag{5-27}$$

• B

$$\begin{aligned}
& B[a, d_1; a, d_2 | a_3, d_3] \\
= & -\operatorname{Res}_a \frac{1}{(z-a)dz} \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2k} \right) \frac{(z-a)^{2d_1}}{(2d_1+1)!} \frac{(z-a)^{2d_2}}{(2d_2)!} dz d\xi_{a_3, d_3}(z) \\
= & -\frac{1}{(2d_1+1)!(2d_2)!} \operatorname{Res}_a \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2k} \right) (z-a)^{2d_1+2d_2-1} d\xi_{a_3, d_3}(z) \\
= & \frac{-1}{(2d_1+1)!(2d_2)!} \operatorname{Res}_a \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2(k+d_1+d_2)-1} \right) \hat{\wp}^{(2d_3)}(z-a_3) dz
\end{aligned} \tag{5-28}$$

If $a_3 \neq a$ this gives 0 except if $0 = k = d_1 = d_2$, and then

$$\begin{aligned}
B[a, 0; a, 0 | a_3, d_3] &= -\frac{Y_{a,0}}{2} \hat{\wp}^{(2d_3)}(a-a_3) \\
&= -\frac{Y_{a,0}}{2} (\wp^{(2d_3)}(a-a_3) + \delta_{d_3,0} G_2)
\end{aligned} \tag{5-29}$$

If $a_3 = a$ this gives

$$\begin{aligned}
& B[a, d_1; a, d_2 | a, d_3] \\
= & \frac{-1}{(2d_1+1)!(2d_2)!} \operatorname{Res}_a \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2(k+d_1+d_2)-1} \right) \hat{\wp}^{(2d_3)}(z-a) dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{(2d_1+1)!(2d_2)!} \operatorname{Res}_a \left(\sum_{k \geq 0} Y_{a,k}(z-a)^{2(k+d_1+d_2)-1} \right) dz \\
&\quad \left(\frac{(2d_3+1)!}{(z-a)^{2d_3+2}} + \sum_{j \geq d_3+1} (2j-1)G_{2j}(z-a)^{2j-2-2d_3} \frac{(2j-2)!}{(2j-2-2d_3)!} \right) \\
&= \frac{-(2d_3+1)!}{(2d_1+1)!(2d_2)!} Y_{a,d_3+1-d_1-d_2} - (2d_3+1)! G_{2d_3+2} Y_{a,0} \delta_{d_1,0} \delta_{d_2,0} \tag{5-30}
\end{aligned}$$

$$B[a, d_1; a, d_2 | a, d_3] = -\frac{(2d_3+1)!}{(2d_1+1)!(2d_2)!} (Y_{a,1+d_3-d_1-d_2} + G_{2d_3+2} Y_{a,0} \delta_{d_1,0} \delta_{d_2,0}). \tag{5-31}$$

• C

$$\begin{aligned}
&C[a_1, d_1 | a_2, d_2; a_3, d_3] \\
&= -\operatorname{Res}_{z \rightarrow a_1} \frac{dz}{z-a_1} \left(\sum_{k \geq 0} Y_{a_1,k}(z-a_1)^{2k} \right) \frac{(z-a_1)^{2d_1}}{(2d_1+1)!} \hat{\phi}^{(2d_2)}(z-a_2) \hat{\phi}^{(2d_3)}(z-a_3) \\
&= \frac{-1}{(2d_1+1)!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} Y_{a_1,k} z^{2(k+d_1)-1} \right) dz \hat{\phi}^{(2d_2)}(z+a_1-a_2) \hat{\phi}^{(2d_3)}(z+a_1-a_3) \tag{5-32}
\end{aligned}$$

If $a_2 \neq a_1$ and $a_3 \neq a_1$:

$$C[a_1, d_1 | a_2, d_2; a_3, d_3] = -Y_{a_1,0} \delta_{d_1,0} \hat{\phi}^{(2d_2)}(a_1-a_2) \hat{\phi}^{(2d_3)}(a_1-a_3) \tag{5-33}$$

If $a_2 = a_1$ and $a_3 \neq a_1$:

$$\begin{aligned}
&C[a_1, d_1 | a_1, d_2; a_3, d_3] \\
&= \frac{-1}{(2d_1+1)!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} Y_{a_1,k} z^{2(k+d_1)-1} \right) dz \hat{\phi}^{(2d_2)}(z) \hat{\phi}^{(2d_3)}(z+a_1-a_3) \\
&= \frac{-(2d_2+1)!}{(2d_1+1)!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} Y_{a_1,k} z^{2(k+d_1)-1} \right) dz \\
&\quad \left(\frac{1}{z^{2d_2+2}} + G_{2d_2+2} + O(z^2) \right) \hat{\phi}^{(2d_3)}(z+a_1-a_3) \\
&= \frac{-(2d_2+1)!}{(2d_1+1)!} \left(\delta_{d_1,0} G_{2d_2+2} Y_{a_1,0} \hat{\phi}^{(2d_3)}(a_1-a_3) \right. \\
&\quad \left. + \sum_{k=0}^{d_2+1-d_1} \frac{1}{(2k)!} Y_{a_1,d_2+1-d_1-k} \hat{\phi}^{(2d_3+2k)}(a_1-a_3) \right) \tag{5-34}
\end{aligned}$$

If $a_2 = a_3 = a_1$:

$$C[a_1, d_1 | a_1, d_2; a_1, d_3]$$

$$\begin{aligned}
&= \frac{-1}{(2d_1 + 1)!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} Y_{a_1, k} z^{2(k+d_1)-1} \right) dz \hat{\wp}^{(2d_2)}(z) \hat{\wp}^{(2d_3)}(z) \\
&= \frac{-1}{(2d_1 + 1)!} \operatorname{Res}_{z \rightarrow 0} \left(\sum_{k \geq 0} Y_{a_1, k} z^{2(k+d_1)-1} \right) dz \\
&\quad \left(\frac{(2d_2 + 1)!}{z^{2d_2+2}} + \sum_{j=0}^{d_3+1} G_{2d_2+2+2j} z^{2j} \frac{(2d_2 + 2j + 1)!}{(2j)!} + O(z^{2d_3+4}) \right) \\
&\quad \left(\frac{(2d_3 + 1)!}{z^{2d_3+2}} + \sum_{j=0}^{d_2+1} G_{2d_3+2+2j} z^{2j} \frac{(2d_3 + 2j + 1)!}{(2j)!} + O(z^{2d_2+4}) \right) \\
&= -\frac{(2d_2 + 1)!(2d_3 + 1)!}{(2d_1 + 1)!} \left(Y_{a_1, d_2+d_3+2-d_1} \right. \\
&\quad + \sum_{j=0}^{d_3+1-d_1} G_{2d_2+2+2j} \binom{2d_2 + 2j + 1}{2j} Y_{a_1, d_3+1-d_1-j} \\
&\quad + \sum_{j=0}^{d_2+1-d_1} G_{2d_3+2+2j} \binom{2d_3 + 2j + 1}{2j} Y_{a_1, d_2+1-d_1-j} \\
&\quad \left. + \delta_{d_1, 0} Y_{a_1, 0} G_{2d_2+2} G_{2d_3+2} \right) \\
&\tag{5-35}
\end{aligned}$$

5.3 Weierstrass elliptic curve

This is the case

$$\begin{cases} y^2 = 4(x^3 - 15G_4x - 35G_6) \\ x(z) = \wp(z) \\ y(z) = \wp'(z) \\ R = \left\{ \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}, \frac{\tau}{2} \right\}, \sigma_a(z) = -z. \end{cases} \tag{5-36}$$

We have

$$\wp(a) = x_a \quad \text{such that } x_a^3 = 15G_4x + 35G_6. \tag{5-37}$$

We have

$$\begin{aligned}
\sum_{k=0}^{\infty} Y_{a,k} (z - a)^{2k-2} &= \frac{1}{2\wp'(z)^2} \\
\sum_{k=0}^{\infty} Y_{a,k} (z - a)^{2k} &= \frac{1}{2} \left(\sum_{k=0}^{\infty} Q_{k+1}(x_a)(z - a)^{2k} \right)^{-2}
\end{aligned} \tag{5-38}$$

$$Y_{a,0} = \frac{1}{2\wp''(a)^2} = \frac{1}{2Q_1(x_a)^2} = \frac{1}{72(x_a^2 - 5G_4)^2}$$

$$\begin{aligned}
Y_{a,1} &= -\frac{Q_2(x_a)}{Q_1(x_a)^3} \\
Y_{a,2} &= -\frac{Q_3(x_a)}{Q_1(x_a)^3} + \frac{3}{2} \frac{Q_2(x_a)^2}{Q_1(x_a)^4} \\
Y_{a,k} &= \sum_{l=1}^k (-1)^l \frac{l+1}{2} Q_1(x_a)^{-l-2} \sum_{k_1+\dots+k_l=k, k_i \geq 1} \prod_{i=1}^l Q_{k_i+1}(x_a).
\end{aligned} \tag{5-39}$$

5.4 Legendre elliptic curve

This is the case

$$\begin{cases} y^2 = (1-x^2)(1-k^2x^2) \\ x = \operatorname{sn}(2Kz) \\ y = \operatorname{cn}(2Kz) \operatorname{dn}(2Kz) \\ R = \{\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\tau\} \end{cases} \tag{5-40}$$

We have for $a = 1/2$ or $a = 1/2 + \tau/2$:

$$\begin{aligned}
\sum_{k=0}^{\infty} Y_{a,k} (z-a)^{2k-2} &= \frac{1}{4K \operatorname{cn}(2Kz)^2 \operatorname{dn}(2Kz)^2} \\
&= \frac{1}{4K} \frac{(\wp(z) - \wp(\tau/2))^2}{(\wp(z) - \wp(1/2))(\wp(z) - \wp(1/2 + \tau/2))} \\
Y_{\pm\frac{1}{4},0} &= \frac{1}{128K^3(1-k^2)^2} \\
Y_{\frac{1}{2}\tau+\pm\frac{1}{4},0} &= \frac{k^2}{128K^3(1-k^2)^2}
\end{aligned} \tag{5-41}$$

We have

$$Y_{a,k} = \sum_{j=0}^k (\delta_{k-j,1} + (\wp(a) - \wp(\tau/2))U_{a,k-j})(\delta_{j,0} + (\wp(a + \tau/2) - \wp(\tau/2))\tilde{U}_{a,j}) \tag{5-42}$$

where

$$\begin{aligned}
U_{a,k} &= \sum_{l=0}^k (-1)^l (\wp''(a))^{-l-1} \sum_{k_1+\dots+k_l=k, k_i \geq 1} \prod_{i=1}^l Q_{k_i+1}(\wp(a)) \\
\tilde{U}_{a,k} &= \sum_{l=0}^k (-1)^l (\wp(a) - \wp(a + \tau/2))^{-l-1} \sum_{k_1+\dots+k_l=k, k_i \geq 1} \prod_{i=1}^l Q_{k_i}(\wp(a)).
\end{aligned} \tag{5-43}$$

5.5 Painlevé 6, Schlessinger, Liouville CFT

Let Z_i , $i = 1 \dots 4$, four distinct complex points, and α_i , $i = 1, \dots, 4$ four complex values called "charges". We define $\psi(x) = \prod_{i=1}^4 (x - Z_i)$.

Our goal is to consider an elliptic spectral curve of degree 2 ($\deg x = 2$) and such that ydx has simple poles over the preimages $(\zeta_{z_i}, -\zeta_{z_i}) = x^{-1}(Z_i)$ with residues $\pm\alpha_i$ and no other pole.

This implies that the spectral curve's equation must take the form

$$y^2 = \frac{1}{\psi(x)} \left(H + \sum_{i=1}^4 \frac{\alpha_i^2 \psi'(Z_i)}{x - Z_i} \right) = \frac{P_4(x)}{\prod_{i=1}^4 (x - Z_i)^2}. \quad (5-44)$$

where the arbitrary constant $H \in \mathbb{C}$ is called the auxiliary parameter.

The numerator is a polynomial of degree 4

$$P_4(x) = H\psi(x) + \sum_{i=1}^4 \alpha_i^2 \psi'(Z_i) \frac{\psi(x)}{x - Z_i}. \quad (5-45)$$

The underlying curve has genus 1, it is a Torus of some modulus τ . The modulus τ is a function of the zeros of P_4 .

The spectral curve can be parametrized as follows:

$$\begin{cases} \Sigma &= \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \\ x &= \frac{\beta}{\wp(z) - \wp(\zeta_\infty)} + \alpha \\ y &= -\frac{C}{\beta^2} \frac{\wp'(z)}{\prod_{i=1}^4 (\wp(z) - \wp(\zeta_{Z_i}))} \\ B(z_1, z_2) &= (\wp(z_1 - z_2) + G_2) dz_1 \otimes dz_2 \end{cases} \quad (5-46)$$

$$\text{where } \alpha = \frac{\wp(\zeta_1) - \wp(\zeta_\infty)}{\wp(\zeta_1) - \wp(\zeta_0)}, \quad \beta = \alpha(\wp(\zeta_\infty) - \wp(\zeta_0))$$

$$C = \frac{1}{\wp'(\zeta_\infty)} \sqrt{H \prod_{i=1}^4 (\wp(\zeta_\infty) - \wp(\zeta_{Z_i}))} \quad (5-47)$$

where $x(\pm\zeta_Z) = Z$ for each $Z = 0, 1, \infty, Z_1, Z_2, Z_3, Z_4$.

It has 4 ramification points

$$R = \{0, 1/2, \tau/2, 1/2 + \tau/2\} = \frac{1}{2}(\mathbb{Z} + \tau\mathbb{Z}). \quad (5-48)$$

For each of them the involution is

$$z \mapsto \sigma_a(z) = -z. \quad (5-49)$$

We have

$$ydx = \frac{C}{\beta} \frac{\wp'(z)^2}{\prod_{i=1}^4 (\wp(z) - \wp(\zeta_{Z_i}))} dz. \quad (5-50)$$

Therefore, for each $a \in R$, we have

$$\begin{aligned} \frac{1}{dz(z-a)^2} \sum_{k=0}^{\infty} Y_{a,k}(z-a)^{2k} &= \frac{1}{2ydx} \\ &= \frac{\beta}{2Cdz} \frac{\prod_{i=1}^4 (\wp(z) - \wp(\zeta_{Z_i}))}{\wp'(z)^2} \\ \sum_{k=0}^{\infty} Y_{a,k}(z-a)^{2k} &= \frac{\beta}{2C} \frac{(z-a)^2}{\wp'(z)^2} \prod_{i=1}^4 (\wp(z) - \wp(\zeta_{Z_i})). \end{aligned} \quad (5-51)$$

We thus obtain

$$Y_{a,k} = \frac{\beta}{C} \sum_{k_0+k_1+k_2+k_3+k_4=k} Y_{a,k_0}^{\text{Weierstrass}} \prod_{i=1}^4 (Q_{k_i}(\wp(a)) - \delta_{k_i,0} \wp(\zeta_{Z_i})) \quad (5-52)$$

where $Y_{a,k}^{\text{Weierstrass}}$ is given in equation (5-39).

6 Newton's polygon

Consider an algebraic plane curve over a subfield $\mathbb{F} \subset \mathbb{C}$, of equation

$$0 = P(x, y) = \sum_{(i,j) \in \mathcal{N}} P_{i,j} x^i y^j, \quad P_{i,j} \in \mathbb{F}. \quad (6-1)$$

Let

$$\Delta(x) = \text{Resultant}(P(x, .), P_y(x, .)). \quad (6-2)$$

The ramification points $a = (x_a, y_a)$ are the common zeros of $P(x, y)$ and $P_y(x, y)$, and not zeros of $P_x(x, y)$. x_a is thus a zero of $\Delta(x)$. $\Delta(x)$ is proportional to $\prod_{a \in R} (x - x_a)$ and thus has double zeros at each a , whereas $P_y(x, y)$ has simple zeros at a . This implies that the ratio $\Delta(x)/P_y^2$ is finite at ramification points. Its only poles can be at x or y infinite, and thus it must be a polynomial of x, y :

$$\frac{\Delta(x)}{P_y(x, y)} = U(x, y) \in \mathbb{F}[x, y]. \quad (6-3)$$

Consider the Field extension:

$$\mathbb{F}_\Delta = \mathbb{F}[\text{zeros of } \Delta]. \quad (6-4)$$

We have for all branchpoints

$$x_a \in \mathbb{F}_\Delta, \quad y_a \in \mathbb{F}[x_a] \subset \mathbb{F}_\Delta. \quad (6-5)$$

Let us expand near a

$$\begin{cases} x = x_a - \frac{P_{yy}(a)}{2P_x(a)} \zeta_a^2 \\ y = y_a + \zeta_a + \sum_{k \geq 2} y_{a,k} \zeta_a^k \end{cases} \quad y_{a,k} \in \mathbb{F}[x_a] \quad (6-6)$$

The coefficients $y_{a,k}$ are determined by solving $P(x, y) = 0$ to each order in ζ .
 $c_a = (-P_{yy}(a)/2P_x(a))$. For $k \geq 2$ we have the recursion

$$\begin{aligned} P_{yy}(a)y_{a,k} &= -[\zeta_a^{k+1}] \sum_{3 \leq 2p+q \leq k+1} \frac{1}{p!q!} P_{x^p,y^q}(a) c_a^p \zeta_a^{2p+q} \left(1 + \sum_{j \geq 2} y_{a,j} \zeta_a^{j-1} \right)^q \\ &= -[\zeta_a^{k+1}] \sum_{3 \leq 2p+q \leq k+1} \frac{1}{p!q!} P_{x^p,y^q}(a) c_a^p \zeta_a^{2p+q} \sum_{l=0}^q \binom{q}{l} \left(\sum_{j \geq 1} y_{a,j+1} \zeta_a^j \right)^l \\ &= - \sum_{3 \leq 2p+q \leq k+1} \frac{c_a^p}{p!q!} P_{x^p,y^q}(a) \sum_{l=0}^q \binom{q}{l} \sum_{k_1+\dots+k_l=k+1-2p-q} \prod_{j=1}^l y_{a,j+1}. \end{aligned} \quad (6-7)$$

- **Kernel B :** The kernel B is the following $1 \otimes 1$ form (see [Eyn22])

$$B_0((x, y); (x', y')) = \frac{-\frac{P(x, y')P(x', y)}{(x-x')^2(y-y')^2} + Q(x, y; x', y') + S(x, y; x', y')}{P_y(x, y)P_y(x', y')} dx dx' \quad (6-8)$$

where $Q \in \mathbb{F}[x, y, x', y']$ is a polynomial

$$\begin{aligned} Q(x, y; x', y') &= \sum_{(i,j) \in \mathcal{N}} \sum_{(i',j') \in \mathcal{N}} P_{i,j} P_{i',j'} \sum_{\substack{(u,v) \in \mathbb{Z}^2 \cap \text{triangle } (i,j), (i',j'), (i,j') \\ (\delta_{(u,v) \notin \overset{\circ}{\mathcal{N}}} + \delta_{(u,v) \in \overset{\circ}{\mathcal{N}} \text{ and } (i+i'-u, j+j'-v) \in \overset{\circ}{\mathcal{N}}})}} |u-i| |v-j'| \\ &\quad x^{u-1} y^{v-1} x^{i+i'-u-1} y'^{j+j'-v-1} \\ &\quad + \delta_{(u,v) \notin \overset{\circ}{\mathcal{N}}} x'^{u-1} y'^{v-1} x^{i+i'-u-1} y^{j+j'-v-1} \\ &\quad + \frac{1}{2} \delta_{(u,v) \in [(i,j), (i',j')]} x^{u-1} y^{v-1} x^{i+i'-u-1} y'^{j+j'-v-1} \end{aligned} \quad (6-9)$$

and $S \in \mathbb{F}[x, y, x', y']$ is an arbitrary polynomial with $(x, y) \iff (x', y')$ symmetry, and with coefficients only inside the interior $\overset{\circ}{\mathcal{N}}$:

$$S(x, y; x', y') = \sum_{(i,j) \in \overset{\circ}{\mathcal{N}}, (i',j') \in \overset{\circ}{\mathcal{N}}} S_{(i,j), (i',j')} x^{-i-1} y^{j-1} x'^{-i'-1} y'^{j'-1}. \quad (6-10)$$

- **Basis :** We define

$$\begin{aligned} d\xi_{a,0} &= \left(-\frac{P(x, y_a)P(x_a, y)}{(x-x_a)^2(y-y_a)^2} + Q(x_a, y_a; x, y) \right. \\ &\quad \left. + S(x_a, y_a; x, y) \right) \frac{2c_a}{P_{yy}(a)} \frac{dx}{P_y(x, y)} \\ d\xi_{a,k+1} &= d(d\xi_{a,k}/dx) + \sum_{b \neq a} C_{b,a,k} d\xi_{b,0} \end{aligned} \quad (6-11)$$

where the coefficients $C_{b,a,k}$ are chosen such that $d\xi_{a,k}$ is meromorphic on Σ , it has a pole at $z = a$ and no other poles (in particular no pole at ramification points $b \neq a$), and behaves like

$$d\xi_{a,k} \sim_a (-2)^{-k} (2k+1)!! \frac{d\zeta_a}{\zeta_a^{2k+2}} + \text{holomorphic}$$

(6-12)

$$\begin{aligned}
d\xi_{a,k} &= \frac{R_{a,k}(x,y)}{(x-x_a)^{k+1}} \frac{dx}{P_y(x,y)} \\
R_{a,0} &= - \left(\sum_{j=1}^{\deg_x P} \frac{1}{j!} (x-x_a)^{j-1} P_{x^j}(a) \right) \left(\sum_{j=2}^{\deg_y P} \frac{1}{j!} (y-y_a)^{j-2} P_{y^j}(a) \right) \\
&\quad + (x-x_a)(Q(a;x,y) + S(a;x,y)) \in \mathbb{F}_\Delta[x,y] \\
R_{a,k+1} &= -(k+1)R_{a,k} + (x-x_a)\partial_x R_{a,k} \\
&\quad - \frac{U}{\Delta'(x_a)} (P_x P_y \partial_y R_{a,k} + R_{a,k} P_y P_{x,y} - R_{a,k} P_x P_{y,y}) \\
&\quad - \sum_{b \neq a} \frac{(x-x_a)}{\Delta'(x_b)(x-x_b)} \left(U P_x P_y \partial_y R_{a,k} + U R_{a,k} P_y P_{x,y} \right. \\
&\quad \left. - U R_{a,k} P_x P_{y,y} - 2U(b) R_{a,k}(b) \frac{(x-x_a)^k}{(x_b-x_a)^k} R_{b,0} \right) \\
&\in \mathbb{F}_\Delta[x,y]. \tag{6-13}
\end{aligned}$$

We have

$$B_{\text{odd}}(x,y;x',y') = - \sum_k \frac{(-2)^k}{(2k-1)!!} \zeta_a^{2k} d\zeta_a \otimes d\xi_{a,k}(x',y'). \tag{6-14}$$

6.0.1 Example: Cubic elliptic curve

Let t be a formal variable and $\mathbb{F} = \mathbb{Q}[t]$. Consider the algebraic plane curve

$$P(x,y) = x^3 + y^3 + txy + 1. \tag{6-15}$$

Its discriminant is

$$\Delta(x) = 27(x^3 + 1)^2 + 4t^3 x^3. \tag{6-16}$$

We have

$$\begin{aligned}
P_y(x,y) &= 3y^2 + tx \\
U(x,y) &= \frac{\Delta(x)}{P_y(x,y)^2} = 3y^2 + 4tx. \tag{6-17}
\end{aligned}$$

The 6 ramification points $a = (x_a, y_a)$ are the 6 zeros of $\Delta(x)$. They can be written (6 = 3 possible cubic roots and 2 possible square roots):

$$\begin{aligned}
y_a &= \frac{-t}{3} \left(1 \pm \sqrt{1 + 27/t^3} \right)^{\frac{1}{3}} \\
x_a &= \frac{-3}{t} y_a^2 = \frac{-t}{3} \left(1 \pm \sqrt{1 + 27/t^3} \right)^{\frac{2}{3}}. \tag{6-18}
\end{aligned}$$

For the kernel B , equation (6-9) gives

$$\begin{aligned} Q(x, y'x', y') &= xy + 2xy' + 2x'y + x'y' \\ S(x, y'x', y') &= S. \end{aligned} \quad (6-19)$$

This gives

$$\begin{aligned} d\xi_{a,0} &= \frac{R_{a,0}}{(x - x_a)} \frac{dx}{P_y} \\ R_{a,0} &= -(y + 2y_a)(x^2 + xx_a + x_a^2 + ty_a) \\ &\quad + (x - x_a)(xy + 2xy_a + 2yx_a + x_a y_a + S) \\ &= (x - x_a)(S - 3x_a y_a) - (y + 2y_a)(3x_a^2 + ty_a) \\ &= (x - x_a)(S - 3x_a y_a) - (y - y_a)(3x_a^2 + ty_a) - 3y_a(3x_a^2 + ty_a) \end{aligned} \quad (6-20)$$

7 Higher order ramification points

Let us now consider spectral curves whose ramification points can be of higher order, let

$$r = \max\{\text{order of ramification points}\}. \quad (7-1)$$

Regular spectral curves have $r = 2$. Here we consider arbitrary $r \geq 2$, and use the construction of [BE13].

7.1 TR and tensors

For $z \in \Sigma$ generic, define $\tau'_n(z)$ the set of all possible n -uples in $x^{-1}(z) \setminus \{z\}$. $\tau'_n(z) = \emptyset$ if $n + 1 > \#x^{-1}(z) = \deg x$. For $n \geq 3$, let $z_2 \in \Sigma$ and $(z_3, \dots, z_n) \in \tau'_{n-2}(z_2)$, define the kernel

$$K_a^{(n)}(z_1; z_2, \widetilde{z_3, \dots, z_n}) = \frac{\int_a^{z_2} B(z_1, .) dx(z_2)}{\prod_{j=3}^n (y(z_2) - y(z_j)) dx(z_2)}. \quad (7-2)$$

For $n \geq 1$ and $n + m + 2h \geq 3$, we define the tensor $A_{n|m}^{(h)} \in V^{\otimes n} \otimes V^{*\otimes m}$ by:

$$\begin{aligned} &\sum_{\alpha_1, \dots, \alpha_n} A_{n|m}^{(h)}[\alpha_1, \dots, \alpha_n | \alpha_{n+1}, \dots, \alpha_{n+m}] d\xi_{\alpha_1}(z_1) d\xi_{\alpha_2}(\tilde{z}_2) \dots d\xi_{\alpha_n}(\tilde{z}_n) \\ &= \sum_{a \in R} \operatorname{Res}_{z_2 \rightarrow a} \sum_{(z_3, \dots, z_{n+m+2h}) \in \tau'_{n+m+2h-2}(z_2)} K_a^{(n+m+2h)}(z_1; z_2, \dots, z_{n+m+2h}) \\ &\quad \prod_{j=2}^n B(z_j, \tilde{z}_j) \prod_{j=n+1}^{n+m} d\xi_{\alpha_j}(z_j) \prod_{j=1}^h \frac{B(z_{n+m+2j-1}, z_{n+m+2j})}{2^{\delta(n+m+2j-1>2)}}. \end{aligned} \quad (7-3)$$

If $n < 1$ or $n + m + 2h < 3$ we define $A_{n|m}^{(h)} = 0$.

In fact, it was shown in [BE13] that for a given ramification point a of order r_a , its contribution to $A_{n|m}^{(h)}$ vanishes if $n + m + 2h > r_a + 1$. Therefore if $r = \max(\{r_a\}_{a \in R})$, we need all tensors such that $n + m + 2h \leq r + 1$.

For simple ramification points $r = 2$, the 4 possible tensors with $n \geq 1$ and $3 \leq n + m + 2h \leq r + 1 = 3$, i.e. $n + m + 2h = 3$, are the tensors A, B, C, D :

$$\begin{cases} A = A_{3|0}^{(0)} \\ B = A_{2|1}^{(0)} \\ C = A_{1|2}^{(0)} \\ D = A_{1|0}^{(1)} \end{cases} \quad (7-4)$$

The Topological Recursion of [BE13] amounts to, for $2g - 2 + n > 0$ and $n \geq 1$:

$$F_{g,n}[\alpha_1, \dots, \alpha_n] = \sum_{h=0}^{\lfloor r/2 \rfloor} \sum_{p=0}^{r-2h} \sum_{\tilde{\alpha}_1, \dots, \tilde{\alpha}_p} \sum_{\substack{\mu \vdash \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_p\} \\ \ell(\mu)}} \sum_{I_1 \sqcup \dots \sqcup I_{\ell(\mu)} = \{\alpha_2, \dots, \alpha_n\}} \sum_{g_1 + \dots + g_{\ell(\mu)} = g + \ell(\mu) - p} \prod_{i=1}^{\ell(\mu)} \delta(2g_i - 2 + |\mu_i| + |I_i| \geq 0) \delta(U = \bigcup_{i, g_i=0, |\mu_i|=1, |I_i|=1} I_i) \delta(\overline{U} = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_p\} \setminus U) \delta(h = \#\{i, g_i = 0, |\mu_i| = 2, |I_i| = 0\}) A_{1+|U|, p-|U|}^{(h)} [\alpha_1, U|\overline{U}] \prod_{i, 2g_i-2+|\mu_i|+|I_i|>0} F_{g_i, |\mu_i|+|I_i|}[\mu_i, I_i]. \quad (7-5)$$

7.2 (r, s) curves and r-spin

Let $r \geq 2$, and $s > 1 - r$, and $s \not\equiv 0 \pmod{r}$, and as in [Bor+25], consider the (r, s) spectral curve:

$$\begin{cases} x(z) = \frac{1}{r} z^r \\ y(z) = z^s \\ B(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} dz_1 \otimes dz_2 \end{cases} \quad (7-6)$$

There is one ramification point $R = \{0\}$, of order r . Let

$$\rho = e^{2i\pi/r}.$$

Define the basis:

$$d\xi_\alpha(z) = z^{-\alpha-1} dz. \quad (7-7)$$

We have

$$B(z_1, z_2) \sim_{z_2 \rightarrow 0} \sum_{\alpha \geq 1} \alpha z_2^{\alpha-1} dz_2 d\xi_\alpha(z_1)$$

$$dS(z_1, z_2) = \int_0^{z_2} B(z_1, .) \sim_{z_2 \rightarrow 0} \sum_{\alpha \geq 1} z_2^\alpha d\xi_\alpha(z_1). \quad (7-8)$$

Kernel:

$$\begin{aligned} K^{(n)}(z_1; z_2, \dots, z_n) &= \frac{\sum_{\alpha_1 \geq 1} z_2^{\alpha_1} d\xi_{\alpha_1}(z_1)}{(z_2^{r-1} dz_2)^{n-2}} \frac{1}{\prod_{j=3}^n (z_2^s - z_j^s)} \\ &= \sum_{\alpha_1 \geq 1} d\xi_{\alpha_1}(z_1) \frac{z_2^{\alpha_1 - (n-2)(s+r-1)}}{(dz_2)^{n-2} \prod_{j=3}^n (1 - (z_j/z_2)^s)}. \end{aligned} \quad (7-9)$$

Let us define $\theta_j = 1$ if $j \leq n$ and $\theta_j = -1$ if $j > n$. The operator $A_{n|m}^{(h)}$ is then equal to

$$\begin{aligned} &A_{n|m}^{(h)}[\alpha_1, \dots, \alpha_n | \alpha_{n+1}, \dots, \alpha_{n+m}] \\ &= \text{Res}_{z_2 \rightarrow 0} \frac{z_2^{\alpha_1 - (n+m+2h-2)(s+r-1)}}{(dz_2)^{n+m+2h-2}} \sum_{\sigma \in \mathfrak{S}_{r+1}, \sigma_1=r+1, \sigma_2=r} \frac{1}{\prod_{j=3}^{n+m+2h} (1 - \rho^{s\sigma_j})} \\ &\quad \prod_{j=n+1}^{n+m} \rho^{-\alpha_j \sigma_j} z_2^{-\alpha_j - 1} dz \prod_{j=2}^n \alpha_j \rho^{\alpha_j \sigma_j} z_2^{\alpha_j - 1} dz_2 \\ &\quad \prod_{j=1}^h 2^{-\delta(n+m+2j-1>2)} \frac{\rho^{\sigma_{n+m+2j-1}} \rho^{\sigma_{n+m+2j}}}{(\rho^{\sigma_{n+m+2j-1}} - \rho^{\sigma_{n+m+2j}})^2} \frac{dz_2^2}{z_2^2} \\ &= \frac{2^{\delta_{n+m,1} \text{ and } h>0}}{2^h} \prod_{j=2}^n \alpha_j \text{Res}_{z \rightarrow 0} dz z^{\sum_{j=1}^{n+m} \theta_j \alpha_j - (n+m+2h-1) - (n+m+2h-2)(s+r-1)} \\ &\quad \sum_{\sigma \in \mathfrak{S}_{r+1}, \sigma_1=r+1, \sigma_2=r} \frac{\prod_{j=2}^{n+m} \rho^{\alpha_j \sigma_j \theta_j}}{\prod_{j=3}^{n+m+2h} (1 - \rho^{s\sigma_j})} \prod_{j=1}^h \frac{\rho^{\sigma_{n+m+2j-1}} \rho^{\sigma_{n+m+2j}}}{(\rho^{\sigma_{n+m+2j-1}} - \rho^{\sigma_{n+m+2j}})^2} \\ &= \prod_{j=2}^n \alpha_j \delta \left(\sum_{j=1}^{n+m} \theta_j \alpha_j = (n+m+2h-2)(s+r) \right) \\ &\quad C_{n+m}^{(h)}(\theta_2 \alpha_2, \dots, \theta_{n+m} \alpha_{n+m}) \end{aligned} \quad (7-10)$$

where

$$\begin{aligned} C_n^{(h)}(\alpha_2, \dots, \alpha_n) &= \frac{2^{\delta_{n+m,1} \text{ and } h>0}}{2^h} \sum_{\sigma \in \mathfrak{S}_{r+1}, \sigma_1=r+1, \sigma_2=r} \frac{\prod_{j=2}^n \rho^{\alpha_j \sigma_j}}{\prod_{j=3}^{n+2h} (1 - \rho^{s\sigma_j})} \\ &\quad \prod_{j=1}^h \frac{\rho^{\sigma_{n+2j-1}} \rho^{\sigma_{n+2j}}}{(\rho^{\sigma_{n+2j-1}} - \rho^{\sigma_{n+2j}})^2}. \end{aligned} \quad (7-11)$$

- Example $r = 2, s = \text{odd}$, we have $\rho = -1$. The only operators have $n + m + 2h = 3$: We have

$$C_3^{(0)}(\alpha_2, \alpha_3) = \frac{(-1)^{\alpha_3}}{1 - (-1)^s} = \frac{(-1)^{\alpha_3}}{2}. \quad (7-12)$$

$$C_1^{(1)}(\emptyset) = \frac{1}{1 - (-1)^s} \frac{-1}{(1 - (-1))^2} = \frac{-1}{8}. \quad (7-13)$$

Writting $\alpha_j = 2d_j + 1$ this gives

$$\begin{aligned} A_{3|0}^{(0)}[2d_1 + 1, 2d_2 + 1, 2d_3 + 1] &= \frac{-1}{2}(2d_2 + 1)(2d_3 + 1)\delta(d_1 + d_2 + d_3 = \frac{1}{2}(s - 1)) \\ A_{2|1}^{(0)}[2d_1 + 1, 2d_2 + 1|2d_3 + 1] &= \frac{-1}{2}(2d_2 + 1)\delta(d_1 + d_2 = d_3 + \frac{1}{2}(s + 1)) \\ A_{1|2}^{(0)}[2d_1 + 1|2d_2 + 1, 2d_3 + 1] &= \frac{-1}{2}\delta(d_1 = d_3 + d_2 + \frac{1}{2}(s + 2)) \\ A_{1|0}^{(1)}[2d_1 + 1] &= \frac{-1}{8}\delta(d_1 = 1). \end{aligned} \quad (7-14)$$

- Example $r = 3$ and $s \neq 0 \pmod{3}$, we have $\rho = e^{2i\pi/3}$. Remark that $(1 - \rho^s)(1 - \rho^{-s}) = 3$ and $(\rho - \rho^{-1})^2 = -3$. The only possibilities are such that $n + m + 2h = 3, 4$, which correspond to 11 possible tensors.

We have

$$\begin{aligned} C_4^{(0)}(\alpha_2, \alpha_3, \alpha_4) &= \frac{\rho^{\alpha_3 - \alpha_4} + \rho^{\alpha_4 - \alpha_3}}{(1 - \rho^s)(1 - \rho^{-s})} = \frac{1}{3}(\rho^{\alpha_3 - \alpha_4} + \rho^{\alpha_4 - \alpha_3}) \\ C_2^{(1)}(\alpha_2) &= 2 \frac{1}{3} \frac{\rho^{1+2}}{2(\rho - \rho^{-1})^2} = \frac{-1}{9} \\ C_3^{(0)}(\alpha_2, \alpha_3) &= \frac{\rho^{\alpha_3}}{1 - \rho^s} + \frac{\rho^{-\alpha_3}}{1 - \rho^{-s}} = \frac{\rho^{\alpha_3} - \rho^{s - \alpha_3}}{1 - \rho^s} \\ C_1^{(1)}(\emptyset) &= \sum_{\sigma_3=1}^2 \frac{\rho^{\sigma_3}}{(1 - \rho^{s\sigma_3})(1 - \rho^{\sigma_3})^2} = \frac{-1}{3} \end{aligned} \quad (7-15)$$

and

$$\begin{aligned} &A_{n|m}^{(h)}[\alpha_1, \dots, \alpha_n | \alpha_{n+1}, \dots, \alpha_{n+m}] \\ &= \prod_{j=2}^n \alpha_j \delta \left(\sum_{j=1}^{n+m} \theta_j \alpha_j = (n + m + 2h - 2)(s + r) \right) \\ &\quad C_{n+m}^{(h)}(\theta_2 \alpha_2, \dots, \theta_{n+m} \alpha_{n+m}) \end{aligned} \quad (7-16)$$

8 Algorithmic implementation

8.1 Issues with standard tensor libraries

The implementation of TR is mostly about tensor products and contractions. There exists many tensor libraries, often implemented using GPU, in python and other languages, however, we found several issues to use them:

- Tensor contractions are over indices that can appear in different permuted orderings. In other words we need tensors whose indices are not labeled by

$[0,1,2,3\dots,\text{rank}]$ but by some labels which are not always in the same order. Moreover in the recursive procedure, the same tensor $F_{g,n}$ is re-used many times, with different labellings. The idea is to store tensors only once, don't permute their columns, but only permute the labelling, which is less time consuming.

- The rank n tensors $F_{g,n}$ have finite dimension. But the tensors A, B, C, D and $A_{n|m}^{(h)}$ have typically infinite dimension. For example when applying the tensor B , we need to perform the operation

$$\sum_{\beta} B[\alpha_1, \alpha_j | \beta] F_{g,n-1}[\alpha_2, \dots, \alpha_{j-1}, \beta, \alpha_{j+1}, \dots, \alpha_n] \quad (8-1)$$

For $B[\alpha_1, \alpha_j | \beta]$ in principle there can be an infinite number of values of $\alpha_1, \alpha_j, \beta$. However since the tensor $F_{g,n-1}$ is of finite dimension, there are finitely many possible values of $\alpha_2, \dots, \alpha_{j-1}, \beta, \alpha_{j+1}, \dots, \alpha_n$, in particular of β . Then, remark that for a given β , there are only finitely many values of α_1, α_j for which $B[\alpha_1, \alpha_j | \beta] \neq 0$, so that eventually, the sum is a finite sum, and the result is a finite dim tensor.

Therefore we need a tensor library that contains the information that for given out-indices, only finitely many possible in-indices are attainable. And moreover since the tensors $A_{n|m}^{(h)}$ have infinitely many possible indices, the value can't be stored as a tensor, but as functions of indices.

- A less critical issue, is the fact that the tensors $F_{g,n}$ are symmetric. Storing all the values is very redundant ($n!$ times too much), and could be optimized.

8.2 Implementation

For these reasons, I chose to define a new Tensor implementation (in python) with the following classes:

- **Indices** is a class for list of indices $I = [\alpha_1, \alpha_2, \dots, \alpha_{\dim}]$, where the indices α_i are not necessarily integers, they are typically pairs $\alpha_i = (a_i, d_i) = (\text{point}, \text{integer})$, in fact they can be any objects.

What we need is that Indices-objects are sortable (and thus have comparison methods $\leq, \geq, =, \neq$), and iterable, and one can access their i^{th} elements by a `__getitem__` method. The class must be like that:

Class	Indices
Attributes	list [index ₀ ,...,index _{n-1}]
Methods	$\leq, \geq, =, \neq$ dim = length sort __getitem__ __iter__ __hash__

Example: $I = [\alpha_0, \dots, \alpha_{n-1}]$, $I[1] = \alpha_1$, $I_1 < I_2, \dots$,

for i in I: do something

- **TRTensor** is a class for Tensors. This is the class to store the $F_{g,n}$ s. A Tensor T is a dictionary whose keys are elements of the class Indices, and whose values are numerical values. It must have a `__getitem__` method such that $T[I] = \text{value}$. It must have also linear operations: addition and scalar multiplication. It must also have a tensor product \otimes and a contraction (dot) method. For contractions and tensor products, it is useful to contract on labels rather than column numbers. Therefore each Tensor has a labelling of columns, and the dot and tensor method allow optional relabelling before contracting.

Class	TRTensor	
Attributes	dict of {Indices : value} labels = dict name:rank	
Methods	<code>__getitem__</code> +,- *scalar dot . <code>tensor</code> \otimes	$A[I]$ $A+B$ $A*x$ $A.\text{dot}(B)$: contract over common labels $A.\text{tensor}(B)$

Example: Let $I_1 = \text{Indices}([1,7,2])$, $I_2 = \text{Indices}([0,3,1])$

$T = \text{Tensor}(\{I_1:7, I_2:3\}, \text{labels} = \{\text{"d1":0, "d2":1, "d3":2}\})$

- **TROperator** is a class for the tensors $A_{n|m}^{(h)}$ (and we recall that $A = A_{3|0}^{(0)}$, $B = A_{2|1}^{(0)}$, $C = A_{1|2}^{(0)}$, $D = A_{1|0}^{(1)}$). It can't be a dictionary, because it may have an infinite number of pairs {key:value}. Moreover, the keys are list of Indices (in the class **Indices**), and they are in fact of two types: in-Indices (dim= n) and out-indices (dim = m). For a given out-Indices I of dim m, we must have a function that returns the list of all possible in-Indices, so a function `Indices` \mapsto [list of Indices]. And then for each pair (in-Indices,out-Indices) we must have a function that returns the value. This function can be used to overload the `__getitem__` method. Then, we must have a contraction (dot) method that contracts a **TROperator** with a **TRTensor**, again on labels rather than column numbers. Therefore each **TROperator** has a labelling of columns, and the dot method allows optional relabelling before contracting.

Class	TROperator
Attributes	n,m,h labels_in , labels_out function <code>_outindicesfromin</code> (out-Indices) \mapsto [list of possible in-Indices] function <code>_functionindices</code> (out-Indices,in-Indices) \mapsto value
Methods	<code>-getitem__</code> <code>dot . A.dot(B)</code> \mapsto TRTensor contract over common labels

Example:

$$f_{\text{outindicesfromin}}(d_2) = \{(d_0, d_1) \mid d_0 + d_1 = d_2 + 1\}$$

$$f_{\text{functionindices}}(d_0, d_1, d_2) = -\frac{(2d_2+1)!!}{4(2d_0+1)!!(2d_1-1)!!} \delta_{d_0+d_1, d_2+1}$$

B = TROperator(2,1, f_outindicesfromin,f_functionindices, labels_in = {"d1":0,"d4":1}, labels_out = {"beta":2})

B.dot(Tensor(labels={"d2":0,"d3":1,"beta":2,"d5":3})) the contraction will be over the only common label "beta". The result is a tensor of rank 5: Tensor(labels={"d1","d2","d3","d4","d5"})

- **TRQAS** is the class for Quantum Airy Structures. It contains 2 dictionaries: the dictionary to store A, B, C, D and more generally $A_{n|m}^{(h)}$: {(n,m,h):TROperator}, and a dictionary to store (cache) the $F_{g,n}$: {(g,n):TRTensor}. It also contains a function X_i , that for an index α returns the differential form $d\xi_\alpha$ (as a Lambda function of z), in order to be able to compute the $W_{g,n}$. It may also contain more methods useful for Topological Recursion...

Class	TRQAS (<i>QAS = Quantum Airy Structure</i>)
Attributes	dict { (g,n) : TRTensor } dict { (n,m,h) : TROperator }
Methods	<code>_compute(g,n)</code> function $X_i(d) \mapsto (\text{function}(z) \mapsto \text{value})$

- I also implemented more classes. The purpose is to transform a spectral curve data (typically Σ, x, y, B, R) into Quantum Airy Structure operators. For example a class **KdVTimes** and **SpCurveKdV**, a class **SpCurveGUE** a class **Torus** that implements relations (5-7), (5-9) and **SpCurveTorus**, and **SpCurveNewtonPolygon** and few other examples.

All this is in the gitlab

<https://gitlab.com/toprec/toprec> in the dev branch.

9 Conclusion

All this has been implemented in a python code. However we ask motivated readers to help and contribute to improving and developing this implementation.

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