ZULF'S STANFORD ANALYSIS SPRING 2010 QUAL

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1. Problem I.1

Suppose X is a Banach space

(a) Suppose weak and weak * topologies on X are the same then show that X is reflexive.

1.1. The Weak And Weak Star Topologies.

I am grateful to Haim Brezis' wonderful book to gain some sense for the variety of topologies in Banach spaces. The symbolism is extremely helpful to remember the issues.

The *weak* topology is $\sigma(X, X*)$ and the *weak* * topology is $\sigma(X, X^{***})$ on X itself I think. I will not do this problem without assistance, so forget about me being an expert on the various topologies. They are confusing to me. Usually the weak * topology is defined for X^* in terms of X so here just getting the bearing requires some accounting.

The mnemonic $\sigma(X, X^{***})$ is symbolic but there is a substantial issue here which is that we need to understand what weak * topology means for X in the first place. In order for X to have any weak * topology we need to find another Banach space Y such that X is isomorphic to Y^* and then weak * topology is $\sigma(X = Y^*, Y)$.

Thus let us admit that the question is not well-defined unless there exists a Banach space Y such that X is isomorphic to Y^*.

In this case, weak * topology is the coarsest topology such that $f_y : Y^* \to \mathbf{R}$ are continuous for all $y \in Y$ and $f_y(\ell) = \ell(y)$.

Now let's consider the weak topology $\sigma(X, X^*)$. This is the coarsest topology such that $\ell : X \to \mathbf{R}$ are continuous.

We need to prove isomorphism between X and X^{**} as topological vector spaces which is what reflexive means. Now we begin with the observation that $X \subset X^{**}$ and we assume this known for now. So we are interested then in showing $X^{**} \subset X$.

The first idea I have is to consider the two topologies and name them something reasonable. We'll call $\mathcal{A}$ the weak topology on X and $\mathcal{B}$ the weak * topology. We are given that $\mathcal{A} = \mathcal{B}$.

I have an idea. The idea is let $\mathcal{B}_{\mathbf{R}}$ be the Borel sigma-algebra – oops the metric topology I mean. We'll assume known that it is generated by $\mathcal{C}$ consisting of $[a, \infty)$. Then the idea is that we just consider the pull-back of these sets and show that there is an isomorphism between Y and X^*. But since $X = Y^*$ we have $X^* = Y^{**}$, and so $Y \subset X^*$ is automatic.

Suppose $U \in \mathcal{A}$. Then there exists some $\ell \in X^*$ and some $S_1 \subset \mathbf{R}$ such that

$$U = \ell^{-1}(S_1)$$

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Since $\mathcal{A} = \mathcal{B}$ we have $U \in \mathcal{B}$ and there exists a $y \in Y$ such that

$$f_y^{-1}(S_2) = U$$

We're looking to prove reflexivity huh? Suppose $\ell \in X^* - Y$ and maybe a contradiction can be arranged. So this would be non-zero. I don't have a clear path yet.

Let

$$\delta = \inf_{\ell' \in Y} \|\ell - \ell'\|$$

We just assume known that $\delta > 0$. Maybe a geometric Hahn-Banach theorem would let us separate ℓ from Y by a hyperplane because Y is *closed* in X^* in the norm-topology and $\{\ell\}$ is compact since every cover can be reduced to a single open set. This gives us a linear functional g on X^* such that $g(\ell) > 0$ and $g(\ell') < 0$ for all $\ell' \in Y$. Then we can look at the dual in X which is $x_g \in X$.

This is promising here but still not seeing things clearly.

1.2. Pass Two.

Pass Two. I made a major blunder by assuming existence of $x_g \in X$ in the dual because that is precisely what needs to be proved!

Now we turn back from failure and take a step back. We have $\ell \in X^*$ and $d(\ell, \ell') \geq \delta > 0$ for all $\ell' \in Y$. For any $a \in \mathbf{R}$ we consider the set

$$U_a = \ell^{-1}((a, \infty)) \in \mathcal{A}$$

then there exists $\ell' \in Y$ and some open $A \subset \mathbf{R}$ with

$$U_a = \ell'^{-1}(A)$$

For every $x \in U_a$ we have

$$a < \ell(x) < \infty$$

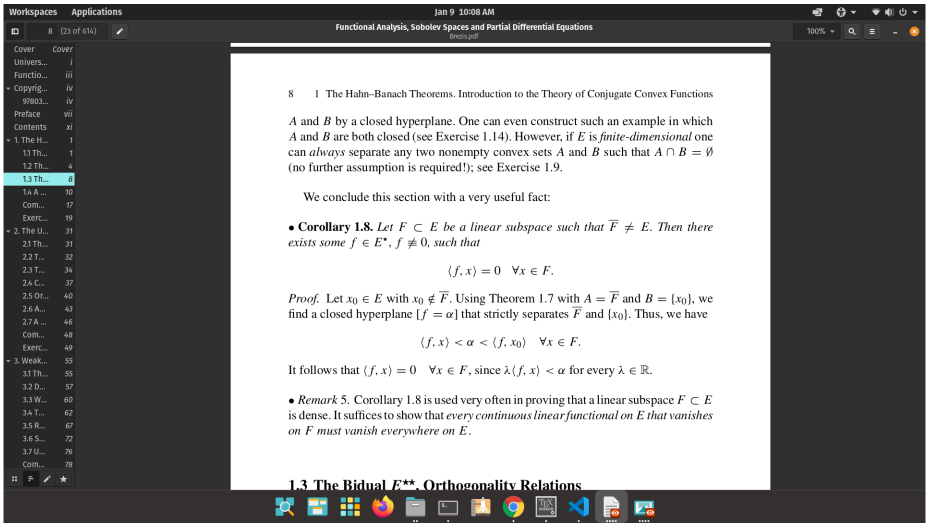
and now use the fact that open sets $A \subset \mathbf{R}$ are countable unions of intervals

$$A = \bigcup_j I_j$$

and label the endpoints $a_j < a_j'$.

At this point it is *very clear to me* that this is a Hahn-Banach theorem problem. So that's the key issue. This is a Hahn-Banach theorem problem and we need to understand how.

Let's look at Brezis' sharp Hahn-Banach theorem formulation. I don't really worry about doing this without assistance now since I need to improve my understanding rather than prove that I am an expert on things in which I am not an expert at all.



8 1 The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

A and B by a closed hyperplane. One can even construct such an example in which A and B are both closed (see Exercise 1.14). However, if E is *finite-dimensional* one can *always* separate any two nonempty convex sets A and B such that $A \cap B = \emptyset$ (no further assumption is required!); see Exercise 1.9.

We conclude this section with a very useful fact:

• **Corollary 1.8.** *Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^\star$, $f \not\equiv 0$, such that*

$$\langle f, x\rangle = 0 \quad \forall x \in F.$$

Proof. Let $x_0 \in E$ with $x_0 \notin \overline{F}$. Using Theorem 1.7 with $A = \overline{F}$ and $B = \{x_0\}$, we find a closed hyperplane $[f = \alpha]$ that strictly separates $\overline{F}$ and $\{x_0\}$. Thus, we have

$$\langle f, x\rangle < \alpha < \langle f, x_0\rangle \quad \forall x \in F.$$

It follows that $\langle f, x\rangle = 0 \quad \forall x \in F$, since $\lambda\langle f, x\rangle < \alpha$ for every $\lambda \in \mathbb{R}$.

• *Remark* 5. Corollary 1.8 is used very often in proving that a linear subspace $F \subset E$ is dense. It suffices to show that *every continuous linear functional on E that vanishes on F must vanish everywhere on E.*

1.3 The Bidual $E^{\star\star}$. Orthogonality Relations

Well, let's see. Let's consider

$$|\ell(x) - \ell'(x)|$$

over $x \in U_a$. These are real numbers and I would like to say

$$|\ell(x) - \ell'(x)| \geq \delta\|x\|$$

for all $xinU_a$. This is not a clear path either yet.

I will have to think about this. This is a difficult problem.

1.3. Closed Subspaces in X are Weakly Closed. Suppose $N \subset X$ is closed. Then for any Cauchy – or Bolzano – sequence x_j has a limit $x \in N$. For any $\ell \in X^*$ then

$$\langle x_j - x_k, \ell\rangle \leq \|x_j - x_k\|\|\ell\|$$

and that shows that every weakly Cauchy sequence will have a weak limit. The strong limit is x, so

$$\lim_{j\to\infty}\langle x_j, \ell\rangle = \langle x, \ell\rangle$$

We can actually just use

$$\langle x - x_j, \ell\rangle \leq \|x - x_j\|\|\ell\|$$

and then we have the result since $\|x - x_j\| \to 0$.

2. Problem I.2

We have to show that for bounded measurable K on $\{(x, y) : 0 \leq y \leq x \leq 1\}$ the operator

$$Tf(x) = \int_0^x K(x, y)f(y)dy$$

is bounded in L^p and has spectrum $\{0\}$.

Let $K(x,y) \leq B < \infty$. Then

$$\begin{aligned} |Tf(x)|^p &\leq |\int_0^x K(x,y)f(y)dy| \\ &\leq C\int_0^x |K(x,y)|^p|f(y)|^p dy \\ &\leq CB^p \int_0^x |f|^p dy \\ &\leq CB^p \|f\|^p \end{aligned}$$

and we just integrate over x and get

$$\|Tf\|_p \leq CB^p\|f\|_p$$

Now for the spectrum, let's see.

$$(T-s)f(x) = \int_0^x K(x,y)f(y)dy - sf(x)$$

I want to just look at what happens at the point $x = 0$ and functions f with $|f(0)| > 0$. We have $Tf(0) = 0$ and $|sf(0)| > 0$ for all $s \neq 0$. So $|(T-s)f(0)| > 0$ for these functions.

This is positive but not yet a firm path for the spectrum conclusion. Ok this tells us that for all $s \neq 0$

$$Ker(T-s) \cap C_+ = \emptyset$$

where $C_+ = \{f \in C([0,1]) : |f(0)| > 0\}$. Now I want to look for density of this sort of C_+. I will work on this some more; it's not right yet.

The next idea is for any $N \geq 1$ large we divide up $[0,1]$ into partition intervals $I_k = [k/N, (k+1)/N]$ and then for every $f \in L^p([0,1])$ we consider

$$g_k = f1_{I_k}$$

The point of this breakup is to note that $Tg_k(k/N) = 0$. Then we attempt to use this to ensure that we have no kernel for $(T-s)g_k$ for all the $k = 0,\dots,N-1$ whenever $|s| > 0$ and $|g_k(k/N)| > 0$. And then we try to produce a bounded inverse using this idea. Here we are exploiting the fact that

$$\int_0^x K(x,y)g_k(y)dx = 0$$

for $x \leq k/N$ because $y \leq x$. This is the main substantial issue in this problem.

3. Motivation For Stanford Undergraduates

Competition is training wheel. It's good for people who want to get the feeling of camaderie and so on. But to be truly great, competitiveness is worthless. Yes worthless. It's not worth a dime. You see, true greatness is within you, and it is unique. They are things that you alone were born to do. To know what they are you have to give up competitiveness altogether, and dive deeper within, and you need resilience, ability to suffer, to deal with people mistreating you and not giving you respect, kicking you in the teeth, throwing you out of their places. Because great things do not get good treatment at all. It's always a pattern in history. The greater your actual success the more destructive the response. Resilience and endurance and your inner qualities are what matters. Competitiveness is the training wheel. Enjoy and cherish memories of this happy youth. But do not become dependent on it. Truly great things are too far away from anyone even caring about them, let alone competing with you. They will spit on you then, when you know what they are. Prepare for it. Then success will come only after huge efforts and you need to survive through it. Your best friends will abandon you. That's just part of the show. Don't be surprised, and don't be resentful. This is part of every great man and great woman's life. Be prepared for it. Don't complain about it. Fine, complain about it if it helps. But survive it.

4. I'll Say A Bit More About This To Others Not So Young

It is my visceral experience in life that what matters most to me, what I believe is significant and serious will not be even considered valuable enough to talk about for the entire world. And that is where the great discoveries are always. Robert Musil died in squalor and poverty and so did Friedrich Hölderlin. They were profound artistic geniuses. But this is the cost of *success* and not failure. Failure is rewarded more, mild successes, keeping with the main movements and so on. But true greatness is never nearby. Bill Gates is mad with envy so much that he used enormous powers to stop my breathing to kill me. You can be curious about why; I am not curious. My successes are profound in Four-Sphere Theory and this is threatening to his whole reality and so he almost might not have any option. Now I do think that the United States Government ought to have killed him immediately but they did not. And that is part of the cost of such profound success as I have had with Four-Sphere Theory. You see mild successes are never threatening in the same way. They get prizes. Only truly profound transformative and revolutionary successes bring out all the murderers from the woodworks. With me, it was Bill Gates. With others it will be others. And that's pattern of great genius and scientific revolutionaries in all ages. You have Giordano Bruno. Burned at the stake. There are many others. These are not signs of misfortune. They are signs of unimaginably powerful successes. And the pattern has ever been the same.

5. Problem II.4

Suppose μ_n are sigma-finite measure such that for any $f \in C_c(\mathbf{R}^N)$ we have

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$$

(a) Show that for U open

$$\mu(U) \leq \liminf_{n\to\infty} \mu_n(U).$$

(b) Show that if S has $\mu(\partial S) = 0$ then

$$\lim_{n\to\infty} \mu_n(S) = \mu(S)$$

Assume first the existence of $\varphi_k(x) \in C_c(\mathbf{R}^N)$ with

$$\lim_{k\to\infty} \phi_k(x) = 1_U(x) = \varphi(x)$$

We want to use Fatou's lemma and then track two limits in this problem.

$$\liminf_{n\to\infty} \int \liminf_{k\to\infty} \varphi_k(x) d\mu_n \liminf_{n\to\infty} \liminf_{k\to\infty} \int \varphi_k(x) d\mu_n$$

Here we used Fatou's Lemma. Left side is $\mu(U)$. Take the k limit on the right $\int 1_U d\mu_n \le \liminf_k \int \varphi_k d\mu_n$. Take liminf n and get

$$\mu(U) \le \liminf_{n\to\infty} \mu_n(U)$$

5.1. **Problem II.4b.** We have to show $\mu(S) = \lim_{n\to\infty} \mu_n(S)$ for bounded Borel sets when $\mu(\partial S) = 0$.

Let's try to prove this for bounded open sets first. The idea is to consider S open to be contained in a large open ball B. Then we want to use

$$B = S_0 \cup \partial S_0 \cup S_1$$

where we just set $S_0 = S$ for convenience. Part (a) gives us

$$\mu(S_j) \le \liminf_{n\to\infty} \mu_n(S_j)$$

Now what we do is conside a sequence $\varphi_k \in C_c(B)$ with $\varphi_k(x) = 1$ on S and $\varphi_k(x) = 0$ for $d(x,S) \ge 1/k$. Now for every $k \ge 1$

$$\int_{S_0} \varphi_k d\mu_n + \int_{\partial S_0} \varphi_k d\mu_n + \int_{S_1} \varphi_k d\mu_n$$

will converge to

$$\int \varphi_k d\mu$$

The idea here is to use (a) to get two of the integrals to converge to 0 as $k \to \infty$.

5.2. **Further Issues for II.4(b).** There delicate issue is how to get the convergence to zero of the two integrals. One way is to combine S_0 and ∂S_0 together to obtain a compact set. The remaining integrals will go to zero by Lebesgue's Dominated Convergence theorem.

Let me backtrack here and tell you about how to consruct $\varphi_n(x)$. Begin with the function $g(x)$ on $\mathbf{R}$

$$g(x) = \begin{cases} \exp(1 - \frac{1}{1-|x|}) & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

This function is standard and has the property that it is $C^\infty(\mathbf{R})$ with support obviously in $[-1,1]$. Clearly $(1-|x|)^{-1} \to \infty$ because the $1-|x| > 0$ for all $|x| < 1$. This is what allows smoothness at the endpoints $x = -1, 1$. The support of the function reduces by replacing $|x|$ with $|nx|$ as $n \to \infty$. Define

$$g_n(x) = g(nx)$$

and it's easy to see that g_n has support in $[-1/n, 1/n]$. Now we use this function for $x \in B \subset \mathbf{R}^N$ as follows

$$\varphi_n(x) = g_n(d(\bar{S}_0, x)).$$

The distance is zero for $x \in \bar{S}_0$ including the boundary. Now we have concreteness to the φ_n in the arguments of the last subsection and we can divide the limit in two parts instead of three:

$$\int_B \varphi_k d\mu_n = \int_{\bar{S}_0} \varphi_k d\mu_n + \int_{S_1} \varphi_k d\mu_n$$

On B we have perfectly good $\varphi_n \in C_c(\mathbf{R}^N)$ so for each $k \geq 1$ we have

$$\lim_{n\to\infty} \int \varphi_k d\mu_n = \int \varphi_k d\mu$$

Now take this apart. On the right side we have $\mu(\bar{S_0})$ because Lebesgue Dominated Convergence theorem does apply for $k \to \infty$. On the left side we have

$$\lim_{n\to\infty} \mu_n(\bar{S_0}) + \lim_{n\to\infty} \lim_{k\to\infty} \int_{S_1} g_k(d(\bar{S_0}, x)) d\mu_n(x)$$

And we apply the Dominated convergence theorem again and get zero for the second term. Then we are left with

$$\lim_{n\to\infty} \mu_n(\bar{S_0}) = \mu(\bar{S_0})$$

Now we can take away the boundary on the right side:

$$\lim_{n\to\infty} \mu_n(\bar{S_0}) = \mu(S_0)$$

because $\mu(\partial S_0) = 0$. Now *we have a problem* we have no control over $\mu_n(\partial S_0)$. In fact I am not sure the statement of the problem is actually true without the assumption $\mu_n(\partial S_0) = 0$ for all $n \geq 1$.

I need to review this. I have grave doubts that the problem statement is true.

Hold we have

$$\mu(S_0) \leq \liminf_{n\to\infty} \mu_n(S_0) \leq \liminf_{n\to\infty} \mu_n(\bar{S_0}) = \mu(S_0)$$

Interesting perhaps I am wrong here because this squeezing tells us

$$\liminf_{n\to\infty} \mu_n(partial S_0) = 0$$

by subtracting off all the S_0 terms. This is very strange to me.

The general Borel S case is some sigma-algebra operations that I will do later.

5.3. **Zulf Is Extremely Unimpressed With Bourbaki Style Abstract-Orientation.**

I am extremely interested these days in Analysis. I need Analysis. I need Analysis because my Four-Sphere Theory is the Final Theory of Macroscopic Nature with enormous evidence of its fundamental truth, and it is based on very precise Mathematics of geometry and analysis of the spinor fields of a four-sphere and the wave equation on these.

In the early twentieth century abstract mathematics grew and many mathematicians considered it purely a good thing. For me, it was not a good thing. You see, abstraction is elucidating sometimes; much more often it takes all the life out of genuinely important issues of Mathematics. I think there has always been a question about where it is actually valuable and where it is arid and substanceless. Great Mathematicians always naturally had great *taste* and knew where the *Mathematical Substance* was located, and naturally could balance the two to find clearest paths to deep issues. Deep issues of Mathematical interest are not well-served by obfuscation but better served by transparent and clear focus. Here the truly great genius was David Hilbert. Richard Courant wrote the two-volume tome *Methods Of Mathematical Physics* originally in 1937, at least Volume II. And my reverence for these gentlemen is immense because they were opposing strongly the spirit of their age mathematically. I am strongly distrustful over time of abstraction in Mathematics that is so excessive that insight and lucidity is no longer possible because the concrete situations have totally been eliminated. This is where Richard Feynman was right to some extent although he was wrong to accuse *all mathematicians* of falling into this sort of *Vice* of being enamoured of form without substance.

Sometimes, the application of very general theorems is beautiful, and some theorems like Hahn-Banach theorem is beautiful and increases lucidity. But very often, there is nothing really that one can learn from very abstract mathematical situations at all. They are conveniences and should not be considered substantial mathematical issues. The substance of Mathematics is harder to define but it's not abstruse abstraction is such great generality that it takes several days to know what's in the jargon. That is *bad mathematics* .

5.4. **In praise of Raoul Bott and John Milnor.** I am 49 and on Wednesday January 12 2022 I might be incarcerated in mental health circuit that I believe was motivated by *political support for Bill Gates white supremacist agenda as political repression* on totally fabulous and delusional idea that if I make a humourous and edgy poster about 'pulling the trigger' with a red bull's eye on Bill Gates' forehead that even though I have never owned any firearms at all, and he is 2100 miles away with $131 billion in security, that I am a threat to him, rather than he being a threat to my life by illegitimate use of power destroying my *health and career and sabotaging my legitimate income of $620 million.* So I will put in my praise for the beautiful work of Raoul Bott and John Milnor before I am *disappeared* by the United States Government like the Nazi Kristallnacht of 1930s. I read Raoul Bott's *Differential Forms In Algebraic Geometry* in high school and I was mesmerised. Then at Princeton 1991-1995, I studied Milnor's *Morse Theory* and *Characteristic Classes.* I was too young to appreciate how profound these texts are. They go through mathematical substance extremely fluidly without all the jargon-filled explosion that results from people who are less familiar with the territory. I won't go into denigrating anyone else. But they are great mathematicians, and that is *why* their work seems clear and simple. Only time and experience taught me that this level of clarity is work of great genius and not accidental. I do not have any problems giving them homage for I would not even have worked on four-sphere theory without having studied their works at an early age. I used Michael Atiyah's works with I. M. Singer more directly, but Bott and Milnor are always in the background because they first enlightened my awareness to topological issues and their relationship to analysis. In fact,the actual universe has regular spacing of energy precisely because the toplogy of absolute space is not very complex. You don't really have any sense for these things easily without great teachers like Milnor and Bott who had explored these areas and returned with their great knowledge so that we, the ones who were not willing to wade through the unknown, could just follow along with them and only work on a few things. The mathematics they worked on was so beautiful and profound that nature could be addressed anew. Well I will return to Analysis now but they deserve their homage.

6. I Am Quite Resentful Now At The FBI Intrusion Because Bill Gates Is Still Harming Me Nonstop

I could not sleep well at all and have been constantly considered by Bill Gates who has caused massive damage to my health and life with all manner of destructive power. Why doesn't he prove that he can do any of these Problem sets. This worthless charlatan should be shot by the United States Government immediately. I am extremely resentful at how I am abused by the FBI and government officials while a racial criminal murderer keeps harming me with total impunity while I am on American soil. I am extremely resentful and I intend to prepare litigation for punitive damages of $ 1.5 trillion against the United States Government quite soon. I need help for this. I think the United States Government ought to pay for allowing this to happen. He would have been killed in every country in Europe for the destruction he caused in my life. I will make this a serious agenda for myself. I would like to ask Stanford Law School faculty to assist. I know the government situation here. Now that I have signed a consent form they will incarcerate me attempting to coerce me to admit that my experience is hallucination. I assure you that it is not hallucination and I believe this order – obviously from senior leadership of FBI and such is *political repression* of the type that existed in Nazi Germany and not innocent at all. I need to do something about this.

7. Well The Consent Form Was Not Signed Exactly Voluntarily

FBI woke up three people in the house in the morning with loud knocking. Then after showing me the particular poster which I don't take seriously because I have never owned a fire-arm and don't like guns all that much at all, I was asked to talk to the mental health officials. They wanted me to admit that Bill Gates interference could be a hallucination. I said there was a remote possibility that it was but I am sure now after further scrutiny that it is not a hallucination at all. I strongly resent the intrusion in my life. The FBI official told my aunt that I had broken the law and could be put behind bars for it. I consider the signing of the form to be under coercive pressure of the government.

I believe the claim of physical threat to Bill Gates is totally frivolous and I am extremely resentful that they had decided to intervene and put my life in jeopardy. Asian-Americans are a minority of 6% in the nation roughly. I consider what the FBI did to be political repression of the Nazi German type with fabricated threats that are fantastic and unrealistic to take significant destructive measures by the state. I am outraged that Bill Gates had been allowed by United States Government to exercise illegitimate power against me.

Bill Gates is the murderous criminal here and not myself. Why should I be subjected to this *absolutely humiliating treatment* by the United States Federal Government? I did not get the sense that they thought that my concerns were legitimate while they thought that their decision, that I am a physical threat to Bill Gates, which is frivolous and outrageous, they took seriously. I am extremely resentful and angry and I want to take steps for litigation against the United States Government for this. In particular I believe this is a flagrant supression of my natural rights of Life, Liberty and Pursuit of Happiness what their decision had been, to give Bill Gates impunity to devastate my Deep Interior while considering me a violent criminal in a frivolous way for posters that are meaningless and humorous.

8. Problem II.1

(a) We are given f is increasing and differentiable almost everywhere with respect to the Lebesgue measure on $[0, 1]$. We are asked to prove

$$\int_0^1 f'(x)dx \leq f(1) - f(0)$$

I introduce some large $n \geq 1$ and set up the cancellations thus:

$$f(1)-f(0) = \sum_{j=1}^{n} f(\frac{j}{n}) - f(\frac{j-1}{n})$$
$$= \sum_{j=1}^{n} \frac{1}{n} \frac{f(\frac{j}{n}) - f(\frac{j-1}{n})}{\frac{1}{n}}$$

Now we delicately use that f is increasing to claim that for any n we have

$$f'(\frac{j-1}{n}) \leq \frac{f(\frac{j}{n}) - f(\frac{j-1}{n})}{\frac{1}{n}}$$

This gives us

$$\sum_{j=1}^{n} f'(\frac{j-1}{n})\frac{1}{n} \leq \sum_{j=1}^{n} \frac{1}{n} \frac{f(\frac{j}{n}) - f(\frac{j-1}{n})}{\frac{1}{n}}$$

Then we want to use the step function approximation of $f'(x)$ at the partition points and apply monotone convergence theorem letting $n \to \infty$. The right side stays $f(1) - f(0)$.

8.1. **Problem II.1(b).** For $\epsilon > 0$ define $u_\epsilon \in \mathcal{S}'(\mathbf{R})$ by

$$u_\epsilon(\phi) = \int (x+i\epsilon)^{-k}\phi(x)dx$$

Show that u_ϵ converges to some $u \in \mathcal{S}'(\mathbf{R})$ as $\epsilon \to 0$.

Let's see

$$|(x+i\epsilon)|^{-k} = (x^2+\epsilon^2)^{-k/2}$$

Schwartz space ϕ will certainly have

$$\sup_x \sum_{k=1}^{n} |x^k \phi(x)| < \infty$$

And that allows us to get

$$|u_\epsilon(\phi)| \leq \int (\frac{x^2}{x^2+\epsilon^2})^{-k/2} |x^k\phi(x)|dx$$

Now

$$\frac{x^2}{x^2+\epsilon^2} = 1 - \frac{\epsilon^2}{x^2+\epsilon^2}$$

This will tend to 1 when $\epsilon \to 0$. So the limit u is bounded and a member of $\mathcal{S}'(\mathbf{R})$.

9. Personal Experiences In Importance Of Failures

After I left MIT and Daniel Stroock's tutelage in 2000 roughly, I had several years of difficulty in New York, as I was depressed and could not secure employment in Finance. I did not want to move from New York and Sine Jensen and I lived together. I was seeking something but things did not look promising. Then finally I got something at Alan Brace's group in BNP Paribas, but I said to myself, I don't want an internship. I took a paycut and took a job in Biospect, a South San Francisco biotech. At first I worked on coding but soon got involved in Science. Roughly 2002-2005 I was there. One of the best habits I learned there was *careful record of failures in a lab notebook.* I wasn't in Analytical Chemistry, John Stults' group but I observed them and was quite amazed at the diligence in record-keeping in their work and I adopted it myself.

That's a long time ago, and in the last few years I decided that I will just play Starcraft II not for entertainment anymore. I will play against the AI for a theory of *habituation*. The theory was that beyond our ability to actually notice changes, habituation subtly alters our abilities and understanding, in ways that defy comprehension, due to genetic inheritance that all humans, an extremely rich sophisticated and still mysterious set of things that have yet to be explored adequately. Recording failures and examining them I decided would be good for mathematical issues as well.

10. Problem I.3

(a) If $u \in \mathcal{S}'(\mathbf{R}^n)$ and $x_j u = 0$ for $j = 1, \dots, n$ Show there is c with $u = c\delta_0$

(b) If $f \in \mathcal{S}'(\mathbf{R}^n)$ and $\int \psi_0 dx \neq 0$ and $a \in \mathbf{R}$, show there is a $u \in \mathcal{S}'(\mathbf{R}^n)$ with $u' = f$ and $u(\psi_0) = a$.

For (a) we take a Taylor expansion approach. An arbitrary Schwartz function has Taylor series at $x = 0$ as

$$\varphi(x) = \sum_{|\alpha| \geq 0} c_\alpha \partial^\alpha \varphi(0) x^\alpha$$

The content of the condition is that if there is any x_j in the expansion the term will be killed by u. The argument is as follows. Suppose $g(x) = x_j h(x)$ for some other $h \in \mathcal{S}(\mathbf{R}^n)$. The hypothesis gives $u(g(x)) = 0$. This implies that for the Taylor expansion, we have

$$u(\varphi) = Ac_0\varphi(0)$$

10.1. The integral.

We first consider the integral as

$$u_0(\varphi) = f(\int_0^x \varphi(s) ds)$$

Then we check it on $\psi_0(x)$

$$u_0(\psi_0) = f(\int_0^x \psi_0(s) ds)$$

We could attempt

$$u(\varphi) = u_0(\varphi) \times \frac{a}{u_0(\psi_0)}$$

11. Problem I.5

(a) Show that if P is elliptic, $u \in \mathcal{S}'$ and $Pu \in \mathcal{S}$ then $u \in C^\infty$.

(b) Show that if P is elliptic of order m then $P : H^m \to L^2$ has finite dimensional nullspace.

11.1. Problem I.5(a).

We have $F(Pu) \in \mathcal{S}$. Now

$$F(Pu)(\xi) = P(\xi)Fu(xi)$$

For $\xi \neq 0$ just

$$Fu(xi) = \frac{F(Pu)(\xi)}{P(\xi)}$$

Now I am not extremely familiar with the details so I will just say that $F(Pu)$ is Scwartz so $F(Pu)(\xi)/P(\xi)$ is Schartz. Then we're done because F^{-1} will map to Schartz functions.

11.2. **Problem I.5(b).** Finite dimensional nullspace. These things happen when they are the image of a compact operator. The idea I have is

$$Ker(P) = Ran(P^*)^{\perp}$$

And somehow I have to use ellipticity to get a compact operator whose range is $Ker(P)$.

I need to think about this still.

12. Problem II.2

Suppose $(X, \mathcal{T})$ is a compact Hausdorff topological space and f_j with $j \in \mathbf{N}$ separates points. This means if $x \neq y$ then there exists j such that $f_j(x) \neq f_j(y)$. Show $(X, \mathcal{T})$ is metrizable.

We define

$$a(x, y) = \sum_{j=1}^{\infty} 2^{-j} |f_j(x) - f_j(y)|$$

and

$$d(x, y) = \frac{a(x, y)}{1 + a(x, y)}$$

Let's see. Symmetry is obvious. The separation condition of f_j gives $d(x, y) = 0$ implies $x = y$. Triangle follows from ordinary triangle on $\mathbf{R}$.

There is no novelty here at all as I have seen this construction in Reed-Simon Volume I for seminorms.

13. Problem II.3

(a) Prove if $f \in L^p(\mathbf{T})$ then

$$\int_{\mathbf{T}} |f(x + h) - f(x)|^p dx \to 0$$

as $h \to 0$.

The idea is to use the translation invariance of the Lebesgue measure on the circle here, i.e.

$$\int_{\mathbf{T}} g(x + h) dx = \int_{\mathbf{T}} g(x) dx$$

But we need to handle the $| \cdot |^p$. Here $1 \leq p < \infty$ for this problem.

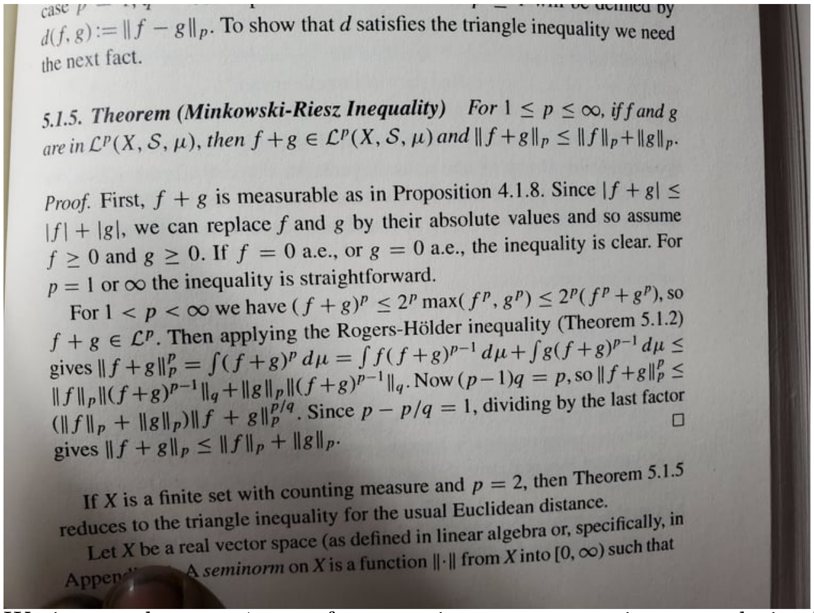
(b) If

$$\sup_{h \neq 0, |h| \leq 1} \int_{\mathbf{T}} |\frac{f(x + h) - f(x)}{h}|^p dx < \infty$$

then the distributional derivative f' belongs to L^p.

Even though this is a clear problem, I will seek some assistance here because the manipulation of $|x - y|^p$ is worth examining carefully.

We examine for this purpose R. M. Dudley's Theorem 5.1.5 on p. 155.



$d(f,g) := \|f-g\|_p$. To show that d satisfies the triangle inequality we need the next fact.

5.1.5. Theorem (Minkowski-Riesz Inequality) *For $1 \le p \le \infty$, if f and g are in $\mathcal{L}^p(X,\mathcal{S},\mu)$, then $f+g \in \mathcal{L}^p(X,\mathcal{S},\mu)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.*

Proof. First, $f+g$ is measurable as in Proposition 4.1.8. Since $|f+g| \le |f|+|g|$, we can replace f and g by their absolute values and so assume $f \ge 0$ and $g \ge 0$. If $f = 0$ a.e., or $g = 0$ a.e., the inequality is clear. For $p = 1$ or ∞ the inequality is straightforward.

For $1 < p < \infty$ we have $(f+g)^p \le 2^p \max(f^p, g^p) \le 2^p(f^p + g^p)$, so $f+g \in \mathcal{L}^p$. Then applying the Rogers-Hölder inequality (Theorem 5.1.2) gives $\|f+g\|_p^p = \int (f+g)^p \, d\mu = \int f(f+g)^{p-1} \, d\mu + \int g(f+g)^{p-1} \, d\mu \le \|f\|_p \|(f+g)^{p-1}\|_q + \|g\|_p \|(f+g)^{p-1}\|_q$. Now $(p-1)q = p$, so $\|f+g\|_p^p \le (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}$. Since $p - p/q = 1$, dividing by the last factor gives $\|f+g\|_p \le \|f\|_p + \|g\|_p$. □

If X is a finite set with counting measure and $p = 2$, then Theorem 5.1.5 reduces to the triangle inequality for the usual Euclidean distance.

Let X be a real vector space (as defined in linear algebra or, specifically, in Appen... A *seminorm* on X is a function $\|\cdot\|$ from X into $[0,\infty)$ such that

We ignore the $p = 1$ case for now since want to gain some clarity here. For $f, g \ge 0$ we see the use here of

$$(f+g)^p \le 2^p \max\{f^p, g^p\}$$

Let's do this for $p = 2$ where things are a bit more explicit first.

$$(f(x+h) - f(x))^2 = f(x+h)^2 + f(x)^2 + 2f(x+h)f(x)$$

Here we are on solid ground, the sort of ground to which I am partial. Here it's clear that we want to use the Cauchy-Schwarz inequality and use the translation invariance of the Lebesgue measure

$$\int f(x+h)f(x)dx \le C(\int f(x+h)^2 dx)^{1/2}(\int f(x)dx)^{1/2} = \|f\|_2^2$$

This then proves the result we seek because

$$\int (f(x+h) - f(x))^2 \le 2\|f\|_2^2 - 2C\|f\|^2 = 2(1-C)\|f\|_2^2$$

so long as C is close to 1.

So our task is to p-ify this argument. I'll return to this later.

13.1. **Distributional Derivative.** Suppose $\varphi \in C^\infty(\mathbf{T})$ and $f \in L^p(\mathbf{T})$.

$$u_f(\varphi) = \int_{\mathbf{T}} f(x) varphi(x) dx$$

Then

$$\partial_x u_f(\varphi) = u_f(\partial_x \varphi)$$

My thoughts are that we ought to use the density of smooth functions in $L^p(\mathbf{T})$ paying attention to get their first derivatives to be close to

$$h^{-1}(f(x+h) - f(x))$$

then integrate by parts with approximants like

$$\int g_n \partial_x \varphi dx = \int \partial_x(g_n\varphi) - \partial_x g_n \varphi dx$$

The fundamental theorem of calculus will eliminate the first term because the value at the endpoints are the same for the primitive. And then we can use various inequalities to get boundededness for the distributional derivative using the assumptions of the problem.

14. Problem I.4

A number ξ is *diophantine* with exponent $k > 0$ if there exists a $C > 0$ so that there are no rational p/q with $p, q \in \mathbf{Z}$ with

$$|\xi - \frac{p}{q}| < Cq^{-k}$$

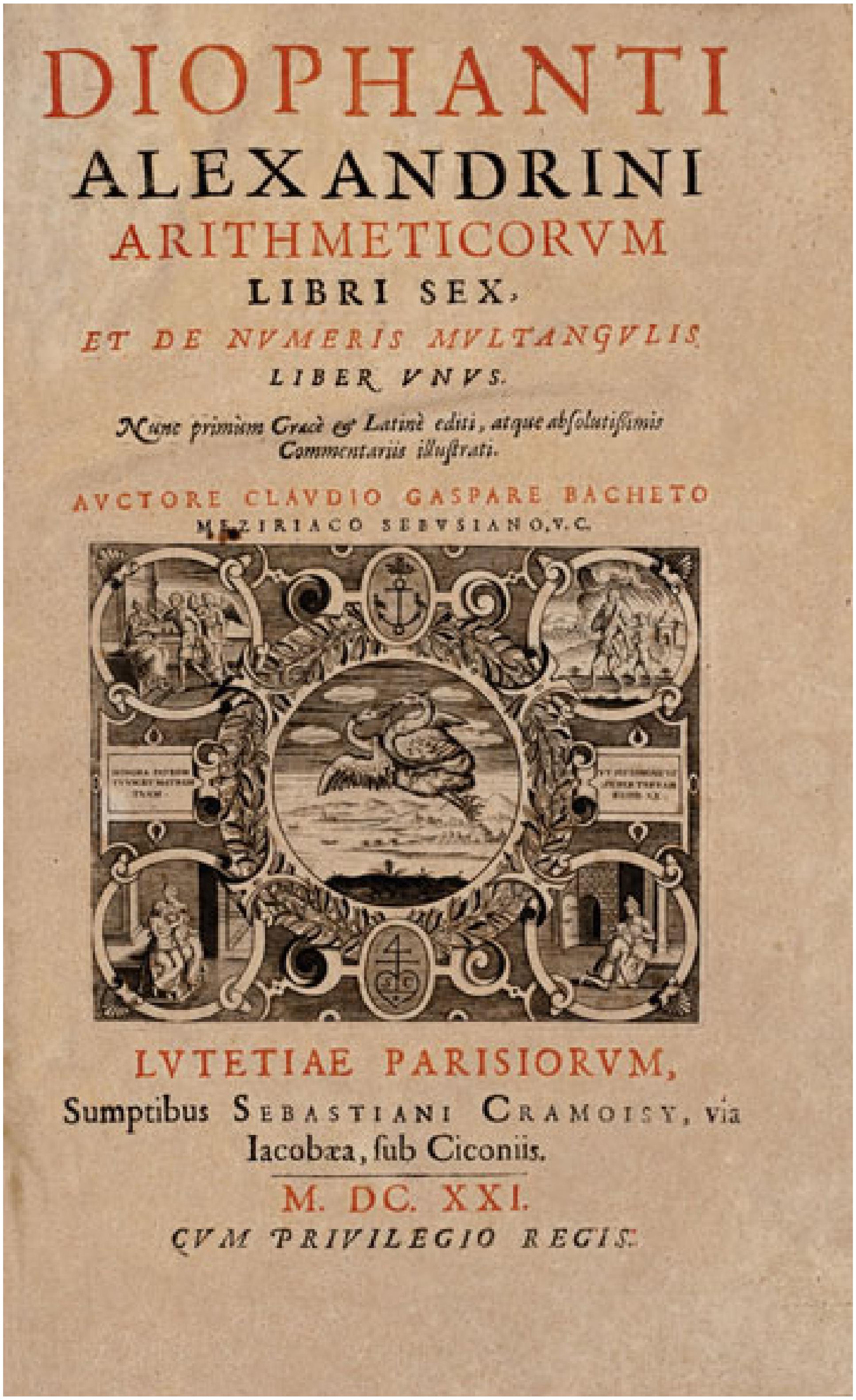
Prove that the set of diophantine equations is first category.

The problem has a significant hint without which this problem is impossible for me.

I find these set theoretic manipulations not very easy, so let me take my time here. We want to prove that the diopantine numbers, and since this is number theory, let us give it an illustrious symbol. Let $\mathcal{D}$ be the diophantine numbers.

That's really nice with a flourish. Let $\mathcal{D} \subset \mathbf{R}$ be the diophantine numbers. Hold on let me get distracted by this because these Baire Category theorem problems are not comfortable and I am more curious about who is Diophantus.

That's beautiful. Diophantus of Alexandria, 200-284, mathematician from Alexandria and wrote *Arithmetica*. Alexandria, a city in Egypt, not Greek at all, is it now, more *Asian* really than Greek. Good, good work Diophantus, you taught those Europeans what's what huh? I approve.



DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

Nunc primùm Græcè & Latinè editi, atque abſolutiſsimis Commentariis illuſtrati.

AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, ſub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

14.1. **Zulf Will Bring Up This Issue.** Diophantus of Alexandria was not Ancient Greek. He was Egyptian. He was more Asian than European. He was the true father of algebraic equations and he was not European. Look I am fine with Europeans taking credit for the works of Europeans, but you cannot take credit for other people's work! I am an Asian-American, and this is disgusting what has happened. Why do you take credit for other people's work so much? I am chalking down Diophantus' *Arithmetica* as *Asian Intellectual Heritage*. Try not to keep taking our things please.

14.2. **I'll Negotiate With European Intellectuals.** If you keep being proud of work of Asians like Diophantus, I'll pass on the word to all of Asia that we will be proud of work of Karl Friedrich Gauss, of Georg Bernhard Riemann, and Pierre-Simon Laplace and consider them part of *Asian Intellectual Tradition* too and you can't complain then.

14.3. **Zulf Will Make This Personal.** I was not supported by any universities or government agencies at all when I worked on Four-Sphere Theory between 2008-2018 although I did get support from Social Security service in the United States. So I could easily have just named the fundamental electromagnetic law, S4 Electromagnetism Ahmed's Law. But I was quite generous and conscientious to the priority and great discovery of the classical wave equation to Jean Le Rond d'Alembert and called the fundamental law Ahmed-d'Alembert Law.

I fail to understand why it is that European intellectuals routinely claim Egyptian intellectual production to be *Ancient Greek* and then usurp that which is not theirs at all. This is quite obnoxious. Avicenna produced the first merge of Aristotelian Virtues and Islamic Theology and Thomas Aquinas just did some cosmetic changes to the main new innovations and suddenly he is considered the great and original *Christian intellect* who was responsible for the merge and Avicenna was denied the honours he deserved for deep genius in the merge. It's not tolerable to me that people like Avicenna and Diophantus are either suddenly asked to change their history or denied originality and then the whole of Western peoples are taught rubbish history and lies.

14.4. **I Have Worked On Full Spectrum Of Virtues.** Virtues predate Aristotle's *Nicomachean Ethics* by many centuries. Egypt and India and China have strong record of Virtue theories before Aristotle. But I have concretely worked on Virtues, and have found universal applicability for Life Satisfaction for all human beings, so this is *extraordinarily valuable.* I have acknowledged work of others from America and Europe that led to my conclusions. I am displeased by the cavalier manner in which the great genius of non-Europeans is not respected in Europe and America. This will lead to grave strife in the future, especially with Asia. Asia is the source of Civilisation on Earth, and this is contested fruitlessly by Bill Gates and others in America with accolades and privilege while I, American of Asian origin am denied tenure at Harvard and Stanford frivolously despite groundbreaking advances in multiple areas. This does not bode well for the future.

14.5. **Return to Issues At Hand.** Let us try to understand these diophantine numbers $\mathcal{D}$. A $\xi \in \mathbf{R}$ is Diophantine of exponent $k > 0$ if there exists $C > 0$ such that there are no $p, q \in \mathbf{Z}$ such that

$$|\xi - \frac{p}{q}| < Cq^{-k}$$

These are interesting as $k > 0$ has to be fixed otherwise there these numbers would not exist at all.

Forgive me, but I will just explore this a bit more because this is unfamiliar turf. We have a triple (ξ, k, C) in the specification of the Diophantine number. For every $q \in \mathbf{Z}$ we will have

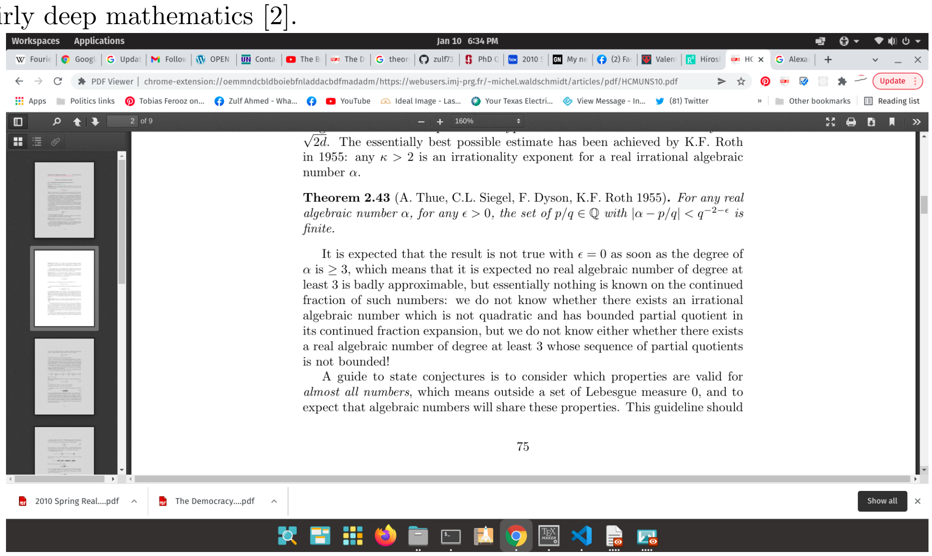
$$(\xi - Cq^{-k}, \xi + Cq^{-k})$$

contain none of the fractions p/q with $p \in \mathbf{Z}$. We immediately have a weak deduction that

$$\mathcal{D} \cap \mathbf{Q} = \emptyset.$$

Yes, yes, this is trivial, but please let the man think without bothering him. These things are quantitatively far from rational.

I am not sure yet, so I will get distracted and ponder about the history of these things. I knew it. This is one of those problems where you need to know some fairly deep mathematics [2].



$\sqrt{2d}$. The essentially best possible estimate has been achieved by K.F. Roth in 1955: any $\kappa > 2$ is an irrationality exponent for a real irrational algebraic number α.

Theorem 2.43 (A. Thue, C.L. Siegel, F. Dyson, K.F. Roth 1955). *For any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbb{Q}$ with $|\alpha - p/q| < q^{-2-\epsilon}$ is finite.*

It is expected that the result is not true with $\epsilon = 0$ as soon as the degree of α is ≥ 3, which means that it is expected no real algebraic number of degree at least 3 is badly approximable, but essentially nothing is known on the continued fraction of such numbers: we do not know whether there exists an irrational algebraic number which is not quadratic and has bounded partial quotient in its continued fraction expansion, but we do not know either whether there exists a real algebraic number of degree at least 3 whose sequence of partial quotients is not bounded!

A guide to state conjectures is to consider which properties are valid for *almost all numbers*, which means outside a set of Lebesgue measure 0, and to expect that algebraic numbers will share these properties. This guideline should

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I will return to this problem last because it's possibly quite hard. All arithmetical questions are frighteningly hard to me.

Let's do this. Let us assume ξ is an irrational number in $[0, 1]$. It is sufficient to prove that diophantine numbers are first category in $[0, 1]$, since then countable union of translates will produce all other elements of $\mathcal{D}$. This is clear since the Diophantine condition does not change by adjusting ξ to $\xi' = \xi - p'/q$. This reduction makes the problem slightly more comfortable. We want to prove $\mathcal{D} \cap [0, 1]$ is nowhere dense in $[0, 1]$.

We know that $\mathbf{Q}$ is dense in $[0, 1]$. Let's consider the dependence. We ought to make $C = C(\xi, k)$. Now let us consider the picture that emerges of the complement. If we enumerate $q = 1, 2, 3, 4, 5, \ldots$ then to test whether $\xi \in \mathcal{D} \subset [0, 1]$ we could take the $C = C(\xi, k)$ and consider for any q

$$U_q = \bigcup_{1 \leq p \leq q-1} (p/q - Cq^{-k}, p/q + Cq^{-k})$$

This gives us clarity. These U_q are a finite number of open intervals in $[0,1]$. And ξ is forbidden to be in these. If $k=1$ they would cover all of $[0,1]$ but for $k>1$ they might not and that allows $\mathcal{D}$ to be nonempty.

Maybe this is not so bad. In order to prove just nowhere dense, we just note that the definition tells us that each ξ is isolated from others. I think I begin to see this problem. If $xi, \xi' \in \mathcal{D}$ we might want to get some sort of lower bound on $|\xi - \xi'|$. This problem is far from done.

15. Problem II.5

A Banach space is *uniformly convex* if for every $\epsilon \in (0,1)$ there exists $\eta < 1$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| < \epsilon$ then $\|(x+y)/2\| < \eta$.

(a) Show every Hilbert space is uniformly convex.

(b) Let X be uniformly convex. Suppose $C \subset X$ is closed and convex. Let $z \in X$. Then $f(x) = \|x - z\|$ achieves its minimum over C, i.e. there exists $x_0 \in C$ such that $f(x_0) = \inf\{f(x) : x \in C\}$.

(c) Show (b) fails without uniform convexity.

15.1. Problem II.5(a).

On a Hilbert space,

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$$

and

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$$

We can use the first to get the condition

$$\|x-y\| < \epsilon$$

along with $\|x\| = \|y\| = 1$ to

$$2 - 2\langle x, y\rangle < \epsilon^2$$

That gives

$$1 - \frac{\epsilon^2}{2} < \langle x, y$$

15.2. Sudden Realisation of Deceptive Nature of Problem II.5(a).

On a Hilbert space for unit vectors, we always have

$$|\langle x, y\rangle| \le 1$$

for unit vectors because these are just cosines. If $\|x\| = \|y\|$ therefore

$$\|x/2 + y/2\| = 1/4 + 1/4 + \langle x, y\rangle \le 1$$

So regardless of what $\epsilon > 0$ is given we set $\eta(\epsilon) = 1$ and that solves the problem. This is unfair in its deceptive nature because earnest candidates can drive themselves insane looking for an η that is not close to 1 and nothing close to zero is possible here.

15.3. **Achieving Closest Point In Uniformly Convex Banach Spaces.** First of all, I want to thank the Stanford Committee for this problem. I am not impressed by (a) because that was *purely devious* but this one, (b) is very important especially for someone who wants my geometric intuition to work in *Banach spaces*. Banach spaces are nice and all, but I *truly hate it* that Banach Spaces are not really ever ready to rock and roll. And by that I mean that I want to draw squiggles on a board, stroke my beard, and wave my hands and say things without thought and I want them to be true. With Banach Spaces you feel like you live in a village without proper drinking water.

We have X is uniformly convex and $C \subset X$ is closed. We fix $z_0 \in X$

$$A = \inf_{x \in C} \|z_0 - x\|$$

We also let

$$f(x) = \|z_0 - x\|$$

By definition of infimum, we can find some $x_j \in C$ so

$$\lim_{j \to \infty} f(x_j) = A$$

Given $\epsilon > 0$ there is an N such that for $j, k \geq N$ we have

$$|f(x_j) - f(x_k)| < \epsilon$$

This is standard, as every convergent sequence is Cauchy – or, if you go by my history, *Bolzano*. I am truly incensed that Maurice Fréchet was so chauvinistic that he did not call it *Bolzano* who had priority for limits and continuity. Let history record that Zulf was strongly in favour of the world changing the name of the condition from Cauchy to Bolzano.

Next we note that $A > 0$, and we want to translate the inequality to be centered at zero. We have

$$|\|x_j\| - \|x_k\|| < \epsilon$$

This is equivalent to

$$\|x_k\| - \epsilon < \|x_j\| < \|x_k\| + \epsilon$$

Let $y_j = x_j/\|x_j\|$ and let $y_k = x_k/\|x_k\|$. Then what is $\|y_j - y_k\|$? If you are wondering what I am doing, I am trying to inch closer to the *uniform convexity condition* of the problem, because then I will get an inequality

$$\|\frac{1}{2}(y_j + y_k)\| < \eta < 1$$

Now I have a slightly better understanding. So what we are trying to do is something like this. Let $z_0 = 0$ be the origin, and we C closed not containing z_0. Then we get some $x_j \in C$ whose distances to the origin, i.e. $\|x_j\|$ reach the infimum A. Then we rescale the situation so that x_j are unit length, maybe A becomes A'. The key point will be that the averages of some of the x_j will get closer to the origin, and suddenly we have in the *convex* set C a better candidate towards infimum.

16. The Fruits Of Geometric Education

You see, I studied the book of Simon Donaldson and Peter Kronheimer *The Geometry of Four-Manifolds* as an undergraduate, and I am familiar with geometry. Years later, I bought another copy and admiringly looked over the work of great mathematicians, and I said to myself, "Ah great men, these gentlemen, for they have studied all sorts of four-manifolds with $H^2(X)$ nonzero, all the things that are not the universe and I am most filled will gratitude at this, for I have discovered that our universe is exactly a round S^4 without nontrivial topology." See, this is profound, for the world would be horrible if the universe had nontrivial topology in the middle dimension. Brrrrrr. I get the heebie jeebies just thinking about it.

Now I am less dismissive about this sort of trivial topology. It might be trivial *toplogically* but there is the whole of Nature in it. I am nicer about analysis now. But I can assure my dear reader than had I not discovered the exact geometry of Nature, my attitudes toward analysis, that it is the sort of things that industrious people do who don't really have a grand *vision* for where the deep theorems are, which all have to do with extremely sophisticated invariants, and all their work is *local theory local theory* with flapping of the hands dismissively. It is not for nothing that the moment you can put some global product structure for a bundle that does not twist around and deny all nonvanishing sections and such, we call these *trivial, very very trivial* bundles.

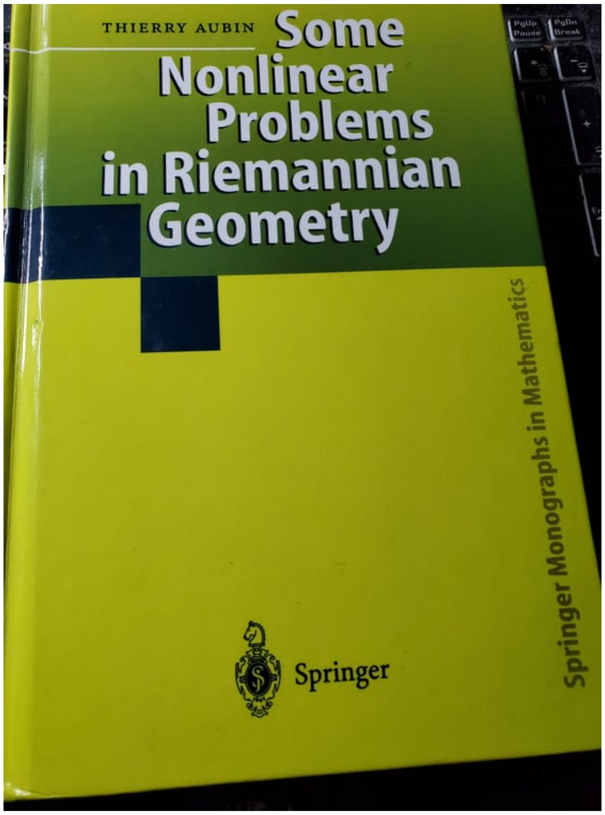
But I am a new man, and turning over a new leaf. Now that the universe is a four-sphere, I am repenting. I am doing inequalities. I am doing all manner of Lebesgue's outrageously dangerous sets that cannot be seen except with all sorts of crazy goggles with orange glowing lenses and so on.

17. Zulf Is Just Extremely Frustrated Now About Wednesday And Mental Health Circuit

I am an American who has been accosted and humiliated by FBI and other government officials right in front of my brother and aunt with absolutely ludicrous charges about how I pose a threat to an intellectually inferior greedy charlatan who has been given enormous powers to harm non-white Americans with US War Power, US Industrial Power used against my *physical body* in massive abuse of national laws against criminal harm and international laws, even Geneva Conventions after Nuremberg, and not Bill Gates, not the habitual racial murderer Bill Gates, but I was disturbed by the governments with threats that *I would be criminalised* and not Bill Gates. The dubious proposition is *clearly indicative of a white supremacist demolition of our Republic, where my natural rights Life, Liberty, and pursuit of Happiness are trampled by frivolous decisions, and I am dragged off with outrageous accusations that I pose a threat to a man with \$131 billion of wealth who lives 2100 miles away, while there my immortal genius, Four-Sphere Theory which will outlast all the nations of the world by a margin of ten million years as the absolute final laws of all physics above* $\delta = 10^{-15}$ *cm against which Quantum Field Theory and General Relativity and Expansionary Cosmology has no possibility of competing is then given off to some white man in America by the dictates of this vile white supremacist savage barbaric criminal Bill Gates. This is totally intolerable to me.*

18. The Sort Of Analysis More Familiar To Me

See, for me some types of Analysis are very familiar. This sort of book by Thierry Aubin, Some Nonlinear Problems In Riemannian Geometry*.*



I have been interested in having a finer understanding of the Sobolev Embedding Theorem for a long time and this might be a good time to appreciate this.

I want to state the major issues for Sobolev Embedding.

$$H^q_k(\mathbf{R}^n) \subset H^p_\ell(\mathbf{R}^n)$$

for

$$\frac{1}{p} = \frac{1}{q} - (k-\ell)/n$$

and $k > \ell \geq 0$ *and* $1 \leq q < p$. *The map* $j : H^q_k \to H^p_\ell$ *is continuous with respect to the norms on each as Banach spaces.*

The first thing about Sobolev Embedding theorem is to try to understand what sort of numbers we are dealing with exactly.

I am most interested obviously in $n = 4$ *for four-spheres. This is a profound truth about* Nature *in this case.*

Let's say you wanted three derivatives. Then you're dealing with a $3/4$ *difference between* $1/p$ *and* $1/q$. *If we want* $1/p > 0$ *we want*

$$1/q - 3/4 > 0$$

which implies $q < 4/3$. *If we want* $q = 1$ *then* $p = 4$. *So that's what Sobolev Embedding will give you. It says in this case*

$$H^1_3 \subset H^4_0$$

and having three derivatives in L^1 *implies being* L^4.

That's quite spectacular in Analysis, and I bet that Frigyes Riesz had no idea at all about such a beautiful play by Sergei Sobolev when he invented L^p spaces. It's a shocking result if you are working on analysis in a direct straightforward way. I mean it was techology from 1938 but it is amazing and more than a liitle disturbing for people used to doing things in L^p for the first time. The theorem does not stay within Hilbert spaces either and spills into Banach spaces.

Probably these things are the first time you get a disturbing realisation that Banach Spaces are actually necessary *which, let me tell you, is not obvious to anyone who has not actually gotten a really good answer from various classes. You might dislike me for this, but I do ask whether all this analytic mumbo jumbo is really necessary or whether it is pretentious sophistry.*

19. Zulf's Guidelines On The Sort Of Things To Prove In Mathematics

The world is a cold heartless place. I am not being negative. I've been hobo for six months in New York. I know that the world is a cold heartless place, so pay attention, because Zulf knows the world quite a bit better than a lot of people.

If you're going to be proving theorems and writing books, prove things that others value in their work. Prove theorems that they will apply to all sorts of things they need to actually do. Word things in ways that make their applicability quite easy. Do not prove theorems about extremely abstract jargon-filled situations that are so general that no one can understand how they have anything to do with anything except by the sage lectures of yourself. If you do that, forget about it. No one will even read your books. They will sell for $300 in Amazon; maybe some University libraries with too much endowment will purchase a copy for the sake of completeness. And then you won't even be a footnote *in history because you're more obscure than all the nonexistent spies from China who are ruining the United States Economy according to the conspiracy theorists.*

Prove theorems that people can understand and use, and publish books around $15 not $300. Otherwise the loveless cold world will punish you.

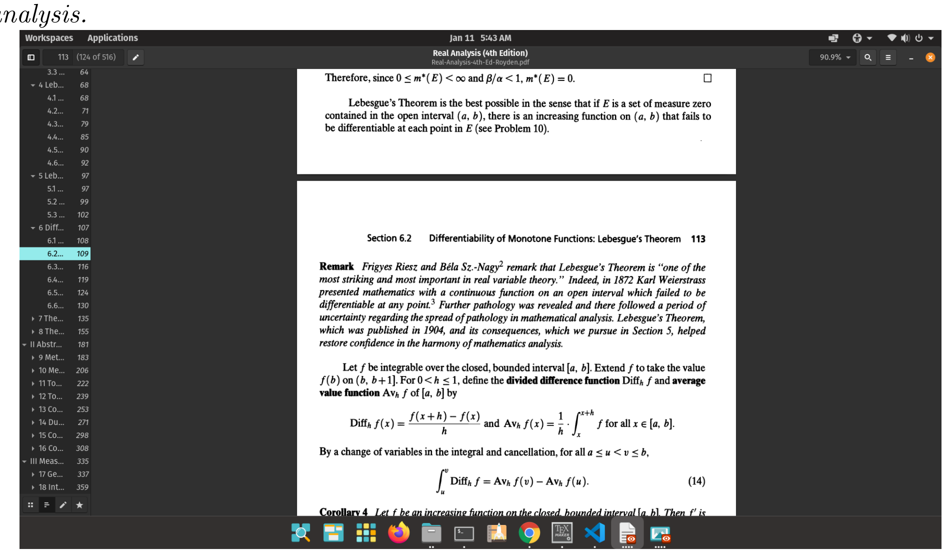
20. A Short Introduction To Life In The Real World For Mathematicians

First of all no one in industry or even minutely empirical sciences can afford to care about mathematical rigour. Let me explain how it works. You see if you're a scientist or engineer or quant or whatever out in the real world, they don't pay you to know how to prove all sorts of fluffy theorems. They want results from all sorts of empirical processes. They want all sorts of funky gizmo working yesterday, and all sorts of medicines that will eliminate Covid, and no matter how little they pay you, they're convinced that they're sacrificing their firstborn every day to keep you employed and their blood is being sucked dry the moment they see the payroll of the company so they will push all sorts of things on your plate and tell you that you're a superstar etc. Then the junior guys will have no clue at all and never actually show up on time or deliver a goddam thing. Then after than, there will be some dork in the other department who had access to your group's code and just as you had planned a 5 day simulation, the checked in buggy code right then. Do you think they give a flying fuck which for which epsilon there is a delta? No. I assure you. They don't think that is all that important. They expect you, the mathematician *to prove* *theorems and give them assurance that the theorem applies. You don't know the factors in their problems. They are worried about things that they get paid $350k to learn, address, and deliver. They want some assistance with models that are likely going to give good results. They have hacked their code to such an extent to fit the numbers that they are too afraid to even touch it, and will pounce and kill anyone who touches it like whatever Crouching Tiger Hidden Dragon even if they are on wheelchair.*

Only the elite guys who are high up and don't have normal work care about mathematical rigour so they can do the I'm Jean-Luc Picard and my dear do you see how cultured and sophisticated I am? That's a luxury. And they should not care about rigour over their professional concerns which are not to apply your high falutin model that has no substantive input of things they know, but to solve the damn problem by hook or by crook. They can't afford to give a fuck.

21. Topic Of Interest: Pathologies In Analysis

Between Karl Weierstrass 1872 nowhere differentiable function and Henri Lebesgue's 1904 theorem that tell us that the strictly increasing measurable function on $[0,1]$ *is differentiable almost everywhere, there was a crisis of pathologies in mathematical analysis.*



Therefore, since $0 \le m^*(E) < \infty$ and $\beta/\alpha < 1$, $m^*(E) = 0$. □

Lebesgue's Theorem is the best possible in the sense that if E is a set of measure zero contained in the open interval (a, b), there is an increasing function on (a, b) that fails to be differentiable at each point in E (see Problem 10).

Section 6.2 Differentiability of Monotone Functions: Lebesgue's Theorem 113

Remark *Frigyes Riesz and Béla Sz.-Nagy[2] remark that Lebesgue's Theorem is "one of the most striking and most important in real variable theory." Indeed, in 1872 Karl Weierstrass presented mathematics with a continuous function on an open interval which failed to be differentiable at any point.[3] Further pathology was revealed and there followed a period of uncertainty regarding the spread of pathology in mathematical analysis. Lebesgue's Theorem, which was published in 1904, and its consequences, which we pursue in Section 5, helped restore confidence in the harmony of mathematics analysis.*

Let f be integrable over the closed, bounded interval $[a, b]$. Extend f to take the value $f(b)$ on $(b, b+1]$. For $0 < h \le 1$, define the **divided difference function** $\mathrm{Diff}_h f$ and **average value function** $\mathrm{Av}_h f$ of $[a, b]$ by

$$\mathrm{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h} \text{ and } \mathrm{Av}_h f(x) = \frac{1}{h} \cdot \int_x^{x+h} f \text{ for all } x \in [a, b].$$

By a change of variables in the integral and cancellation, for all $a \le u < v \le b$,

$$\int_u^v \mathrm{Diff}_h f = \mathrm{Av}_h f(v) - \mathrm{Av}_h f(u). \quad (14)$$

Corollary 4 *Let f be an increasing function on the closed, bounded interval $[a, b]$. Then f' is*

This is H. L. Royden's Real Analysis *which is the textbook I used – let me see – 1992 during sophomore year at Princeton. I did well on the course. Nick Katz, the number theorist with works in number theory and expertise on Deligne's Weil Conjectures and other things taught the course. I am very much interested in this issue. You see, this issue is not a curiousity for me. It is very important to* Four-Sphere Theory *as foundational Science for the next trillion years on Earth. Four-Sphere Theory is based on an mathematical analysis model of the entire universe, and so it is not sufficient that mathematicians have convinced themselves that foundational issues of analysis are resolved. They have not even been addressed for fundamental scientific theory for Four-Sphere Theory.*

For example, is Lebesgue's foundations for analysis on $\mathbf{R}$ *sufficiently strong for physics? What are the possibilities that arise in the physical model and are they all adequately handled?*

We saw earlier evidence that $L^p(\mathbf{R}^n)$ *are necessary for Sobolev embedding theorem. Is the same true for* $L^p(S^4)$ *(this ought to be known)? Is there any reason to re-examine the Lebesgue measurability theory at all on physical grounds. Note that Lebesgue measurable subsets form a* σ *algebra with with lengths being the measure of intervals, with translation-invariance, and countable additivity. On the sphere we have the uniqueness of the Haar measure. I am quite happy with the theory as it exists in mathematics obviously. But I do worry that when the responsibility is for Final Laws of Nature above* $\delta = 10^{-15}$ *cm are being established that we are not making frivolous choices. The stress test has to come from external physical Nature rather than mathematical concerns. We are not yet used to thinking of movement in the actual world as action of a compact Lie group at all. But that is what the truth is.*

22. The Fear Of End Of Mathematics

Joseph-Louis Lagrange in a letter to Jean d'Alembert on September 21 1781 feared that mathematics had reached its limits, and this has been a paranoia for a long time. These two are in my mind for some time now, as the discoverer of the wave equation in 1740s and as the first trigonometric series solution to the vibrating string problem, both of which prefigure the four-sphere theory concretely, now the Dirac on spinor fields replacing the ∂_x *of the vibrating string for the only, the sole, fundamental law governing all of Nature. The paranoia ends here because I am today certain that a vast part of the universe exists beyond human ordinary perception, and only reliance on mathematical solid ground will yield any objective knowledge about objectively real parts of the universe that cannot be examined with the simplistic tools of three dimensional extant science. Thus a new journey is opened for the human race by my efforts, one that will test both our capabilities and our mathematical understanding to their utmost limits, and our race, the human race shall advance tremendously as a result. These are the consequences of my work, and they are consequences that are positive for the future of our people.*

References

[1] https://en.wikipedia.org/wiki/Monotone_convergence_theorem#Convergence_of_a_monotone_sequence_of_real_numbers
[2] https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/HCMUNS10.pdf