ZULF'S STANFORD ANALYSIS 2012 QUAL PROBLEM 4

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1. Problem 4

Suppose f, g are holomorphic in open set
$Omega \subset \mathbf{C}$ and $|f|^2 + |g|^2$ is constant. Show f and g are both constant.

2. The Derivative Approach

If f is holomorphic, we have

$$\frac{\partial f}{\partial \bar{z}} = 0$$

What we want to do is just take partial derivatives in z and $\bar{z}$ of $|f|^2 + |g|^2$ and set it equal to zero and then attempt to deduce that

$$\frac{\partial f}{\partial z} = 0$$

and

$$\frac{\partial g}{\partial z} = 0$$

In order to see if this approach leads to something without a messy calculation, let's just try to prove a simpler statement. We want to prove if $|f|^2$ is constant then f is constant when f is holomorphic.

$$\begin{aligned}\frac{\partial |f|^2}{\partial \bar{z}} &= \frac{\partial f \bar{f}}{\partial z} \\ &= \frac{\partial f}{\partial z}\bar{f} + f\frac{\partial \bar{f}}{\partial z} \\ &= f\frac{\partial \bar{f}}{\partial z}.\end{aligned}$$

This is from a conjugate magic. Then we assume $f \neq 0$ since otherwise we have constant $f = 0$, and then divide

$$0 = \frac{\partial f}{\partial z} f$$

by f and we have in addition to

$$\frac{\partial f}{\partial \bar{z}} = 0$$

the derivative in the orthogonal $\bar{z}$ direction also zero and can conclude f is constant.

Now let's do the same for $|f|^2 + |g|^2$. Taking $\bar{z}$ derivatives here,

$$\frac{\partial(|f|^2 + |g|^2)}{\partial \bar{z}} = f\frac{\partial \bar{f}}{\partial \bar{z}} + g\frac{\partial \bar{g}}{\partial \bar{z}}$$

Date: December 15, 2021.

Now let's see. If either f or g is identically zero, our previous argument gives the result, so let's assume $f, g \neq 0$.

We use

$$\bar{f}\frac{\partial f}{\partial z} + \bar{g}\frac{\partial g}{\partial z} = 0$$

in the region Ω. We can get $barg/\bar{f}$ is holomorphic from this.