FRESNEL INTEGRAL COMPUTATION

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This is not research but a complex analysis exercise from Stein-Shakarchi Complex Analysis.

Compute

$$\int_0^\infty \sin(x^2)dx$$

1. Major Steps

The key step is to note that rotation by fourth root of unity $\theta_4 = e^{i\pi/4}$ allows us to consider the following scenario.

On the slanted line we have e^{-t^2}. On the horizontal line we have $\sin(x^2) + i\cos(x^2)$.

This is the crucial step because the slanted line integral is known. It is

$$\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$$

We need to take limit as $R \to \infty$ and integrate around a closed contour. The arc will have

$$|\int_0^{\pi/4} e^{-R^2 g(\theta)}d\theta| \leq Ce^{-R^2}$$

So it will vanish in the limit.

The algebra for exact computation is

$$\frac{\sqrt{\pi}}{2} = |a + ia|$$

This then leads to

$$2a^2 = \pi/4$$

This gives $\sqrt{\pi/8} = \sqrt{2\pi/16} = \sqrt{2\pi}/4$.

2. Beauty of Mathematics Here

The use of $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$ by rotation in the complex plane is just remarkable. This is an eighteenth century affair with complex plane entering calculus. It was Augustin-Louis Cauchy who wanted a rigorous theory of functions to explain how these miraculous change of variables produced correct answers, which developed into the standard vanishing of the arc integral. Here it is a simple estimate. The last part in algebra.

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