STANFORD SPRING 2016 ANALYSIS QUAL

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1. Looking For Immediate Full Tenured Professorship

Stanford Faculty ought to understand that I am seeking tenure not for my mathematical skills but my *Scientific Accomplishments* over the past year, not only for Four-Sphere Theory which is a fundamental physics but extensive set of empirical results in (a) establishing the empirical validity of Aristotle's Virtue-Eudaimonia theory (b) establishing the universality of human moral nature refuting strongly theories of both *Friedrich Nietzsche* and *Immanuel Kant*, (c) establishing ethnicity-independence of human moral nature, (d) work on evolution of romantic love before the emergence of homo sapiens. Much of the data I used for my quantitative human nature and morality came from World Values Surveys. The mathematical skills I am testing just to get to levels necessary for deeper examination of the relationship between Nature and Mathematics. I have worked with *Jeff McNeal* who was undergraduate supervisor for a prize-winning Mathematics thesis at Princeton which was publishable quality in 1995. I have worked with Daniel Stroock with whom I have a publication in 2000 [1]. I am seeking immediate tenure with plans to settle in Mission San Francisco rather than work at the Palo Alto campus and I have extensive plans for development of quantitative positive psychology for applications for billions.

December 29 2021. 3:31 AM. I am beginning Stanford Analysis Qual 2016 Spring.

2. Problem I.5

The problem was delicate but there is a pathway I discovered so I am typing this up first.

Suppose w is measurable finite positive on $\mathbf{R}^n$ and

$$\int_{\mathbf{R}^n} |K(x,y)|w(y)dy \leq Aw(x)$$

and

$$\int_{\mathbf{R}^n} |K(x,y)|w(y)dy \leq Aw(x)$$

Prove

$$Tf(x) = \int K(x,y)f(y)dy$$

is bounded in $L^2(\mathbf{R}^n)$ with $\|T\| \leq A$

This is a pretty subtle problem. I have an idea we'll divide up $\mathbf{R}^n = B(0,r) \cup G(r)$ with $G(r) = \mathbf{R}^n$. I want to do this because I want to use Jensen's inequality possibly.

Date: December 30, 2021.

The subtlety of the problem arises from the fact that if we just apply Cauchy-Schwarz

$$\int |Tf|^2 = \int |\int K(x,y)f(y)|^2 dx \leq \int \int K(x,y)^2 dydx \|f\|_2^2$$

we have to bound

$$\int \int K(x,y)K(x,y)dydx$$

and we don't know what to do about this since the conditions of the problem does not apply.

I have a way out.

$$\int |\int K(x,y)f(y)|^2 dx \leq \int (\int |K(x,y)|^{1/2} w(y)^{1/2} |K(x,y)|^{1/2} f(y) w(y)^{-1/2})^2 dx$$

This allows me to Cauchy-Schwarz and get

$$\leq \int (\int Kwdy)(\int Kf^2/wdy)dx$$

Now I use the condition in the problem

$$\leq \int Aw(x) \int Kf^2/w(y)dydx$$

Now I switch integrations

$$= A \int f^2(y)/w(y) \int K(x,y)w(x)dx$$

Now I use the other condition in the problem

$$\leq A^2 \int w(y)f^2(y)/w(y) = A^2 \|f\|^2$$

Whew.

This one stretches the Cauchy-Schwarzing to new levels of insanity.

3. Problem I.2

Prove that a weakly convergent sequence in ℓ^1 is strongly convergent.

The selector of the k-th component is a continuous linear functional, let's call it L_k. So $L_k(a_, \dots, a_k, \dots) = a_k$. Let $x_n \to x$ weakly. Denote by $x_{n,k}$ and $x_{,k}$ the k-th component. Weak convergence implies

$$L_k(x_n) \to L_k(x)$$

so $x_{n,k} \to x_{,k}$. In order to get strong convergence we need

$$\sum_{k=1}^{\infty} |x_{n,k} - x_{,k}| \to 0$$

The problem to avoid is that even though each $x_{n,k} \to x_{,k}$ there is no control on the speed of the convergence that holds for all k simultaneously.

Let's start with the observation that

$$\max(\sup_n \|x_n\|, \|x\|) = C < \infty$$

This ensures

$$b_n = \sum_{k=1}^{\infty} |x_{n,k} - x_{,k}| < C$$

Now use compactness of $[0, C]$ to choose a convergent subsequence for b_n and call it b_n as well. Now we have

$$\lim_{n\to\infty} b_n = \alpha$$

We have $\alpha \in [0, C]$ and now suppose $\alpha > 0$. This implies one of the components of the sum

$$\sum_{k=1}^{\infty} |x_{n,k} - x_{,k}|$$

is converging to a positive value, which is impossible by $L_k(x_n) \to L_k(x)$.

4. Problem II.2

Suppose $1 < p < \infty$ and $f_n \in L^p([0, 1])$ and $f_n \to f$ pointwise. Show $f_n \to f$ weakly and $\|f\|_p \leq 1$.

The dual of $L^p([0, 1])$ is $L^q([0, 1])$ with $1 = 1/p + 1/q$. We will use Holder's inequality and the fact that 1 is $L^p([0, 1])$ For any $g \in L^q([0, 1])$,

$$\int (f_n - f)g \leq \|f_n - f\|_\infty \int 1g \leq \|f_n - f\|_\infty \|g\|_q$$

This shows weak convergence. Now

$$\int |f_n|^p \leq 1$$

and so dominated convergence applies since $1 \in L^p([0, 1])$ and see

$$\int |f|^p \leq 1$$

5. Observations About Level Of Performance For Stanford Analysis Problems

I am 49 with many accomplishments in science that are fundamental and transformative for the intellectual life of the planet. I am not trying to reach 'professional level' in mastery of Analysis. The last three problems I did at professional level, but the others I will do in a relaxed manner. It is not my goal to reach professional level but it may happen that I do so in some months simply from getting into a groove. I do believe that my natural level is higher than I am displaying in these quals problems as I was well-regarded at Princeton 1991-1995 for my talent in Mathematics but my habits are developed in natural science problems and industry and not in analysis, so time will tell.

I will tell you where I will consult texts, and otherwise I did the problems without any assistance.

6. Problem II.3

(a) Give an example of Hilbert spaces X, Y and an operator $A \in L(X, Y)$ so $Ran(A)$ is not closed.

(b) Show if X, Y are Hilbert spaces $A \in L(X, Y)$ if $Ran(A)$ is closed then $Ran(A^*)$ is closed.

(c) Show that if there exists $C > 0$ such that for $y \in Y$, $\|y\|_Y \leq C\|A^*y\|_X$ then A is surjective.

These are not immediately clear to me. I will seek some assistance. I found an answer to (a) in [4]. Let $g = 1/(1+|x|)$ and

$$Tf(x) = f(x)g(x)$$

on $L^2(\mathbf{R})$. Since $|g| \leq 1$ we have $\|Tf\| \leq \|f\|$. Then *since compactly supported functions are dense in* $L^1(\mathbf{R})$ and are included in $Ran(T)$, the range is dense in $L^2(\mathbf{R})$.

Let

$$f_0(x) = \begin{cases} 1 & |x| < 1 \\ 1/(1+|x|) & |x| \geq 1 \end{cases}$$

Then $f_0/g \notin L^2(\mathbf{R})$ so $f_0 \notin Ran(T)$.

(b) Suppose $y_n \in Ran(A^*)$ and converges to $y \in Y$. We want to show that $y = A^* w$ for some $w \in X$.

Suppose $w_n \in X$ are chosen so

$$y_n = A^* w_n$$

For all $x \in X$, we have

$$(y_n, x) \to (y, x)$$

therefore

$$(w_n, Ax) \to (y, x)$$

We would like to convert this a convergence statement for w_n. For $\epsilon > 0$ there is N such that $n, m \geq N$ implies

$$|(w_n - w_m, Ax)| \leq \|w_n - w_m\| \|Ax\| < \epsilon \|Ax\|$$

for all $n, m \geq N$ and all $x \in X$. Let

$$w = \lim_{n \to \infty} w_n$$

Then

$$(w_n, Ax) \to (w, Ax)$$

Therefore $w \in Ran(A^*)$

(c) By the open mapping theorem, A is surjective if it is an open map. I will attempt to show that the inequality

$$\|y\|_Y \leq C \|A^* y\|_X$$

implies that A is an open map and apply the Open Mapping theorem to conclude surjectivity.

Since the problem is not completely transparent to me, let me introduce some notation. Denote by $B_X(r)$ and $B_Y(r)$ the open balls

$$B_X(r) = \{x \in X : \|x\|_X < r\}$$

and

$$B_Y(r) = \{y \in Y : \|y\|_Y < r\}$$

and let us reformulate the problem in topological language. We would like to prove that there exists a $\delta > 0$ such that

$$B_Y(\delta) \subset A(B_X(1))$$

In other words, we want to prove that for *every* $y \in B_Y(\delta)$ there exists $x \in B_X(1)$ with $Ax = y$.

Let y satisfy

$$\|y\|_Y < \delta$$

I am still not sure about this problem, so I will proceed as follows. For any $x \in B_X(1)$ consider

$$\langle Ax - y, Ax\rangle = \|Ax\|^2 - \langle y, Ax\rangle = C^2\|x\|^2 - \langle A^* y, x\rangle$$

I do not see the solution yet, but this is closer.

Wow. December 29 2021 2:41 PM. These functional analysis issues are not intuitive to me and I am trying various paths.

I want to prove that for $y \in Y$ with sufficiently small norm, we can solve

$$y = Ax$$

with $x \in B_X(1)$. I am looking at the general equality

$$\|y - Ax\|^2 = \|y\|^2 + \|Ax\|^2 - 2\langle y, Ax\rangle = \|y\|^2 + \|Ax\|^2 - \langle A^* y, x\rangle$$

Then I use

$$2\langle A^* y, x\rangle \geq 2C\|y\|\|x\|$$

and negate it

$$-2\langle A^* y, x\rangle \leq -2C\|y\|\|x\|.$$

$$\|y - Ax\|^2 \leq \|y\|^2 + \|Ax\|^2 - 2C\|y\|\|x\| = \|Ax\|^2 - C^2\|x\|^2 + (\|y\| - C\|x\|)^2$$

This inequality holds for all $y \in Y$ and all $x \in X$. For all y with $\|y\| \leq C\|x\|$ we have

$$-C^2\|x\|^2 + (\|y\| - C\|x\|)^2 \leq 0$$

and therefore

$$\|y - Ax\|^2 \leq \|Ax\|^2$$

This is closer to what we would like.

The methods to use these sorts of inequalities to produce results is not familiar to me at all. But this is getting closer. Being a scientist I believe any progress is great progress.

Maybe there are millions of people out there who can do this easily. I am not one, and I don't feel inadequate and inferior about it. Years ago, in the 1930s Stefan Banach proved some of these sorts of things and he was a great genius. I don't think it's a serious problem if people like myself who is not the founder of functional analysis does not do these things easily. The problem (c) is still not complete and I will return to this.

6.1. **I Will Examine Peter D. Lax's book.** I am not a child and am not doing these problems to prove myself. I need to have clear understanding, and I don't think that I have clear understanding of (c) above. I once looked at the book on Functional Analysis by Peter D. Lax and I think he has profoundly good sense of these situations. So I will just put other things aside and try to understand the situation better.

6.2. **Injectivity And Surjectivity Intuition.** I understand now what I was missing in my education. For finite-dimensional transformations $T : V \to W$, surjectivity of T corresponds to the injectivity of $T*$. That is the point of II.3(c). This might be very clear to people but it was unknown to me.

If $\|y\|_Y \leq C\|A^*y\|_X$ then A^* has trivial nullspace and is injective, and therefore $A = A^{**}$ is surjective.

Let me attempt to prove this by extending linear algebra. Our assumption

$$\|y\|_Y \leq C\|A^*y\|_X$$

implies that $A^*y = 0$ implies $\|y\| = 0$. So A^* is injective.

Let's now prove the general proposition that if $T : X \to Y$ is injective then T^* is surjective.

If $x_0 \in Ran(T^*)^\perp$ then we will have for all $y \in Y$

$$\langle x_0, T^*y\rangle = 0$$

therefore

$$\langle Tx_0, y\rangle = 0$$

for all $y \in Y$, and going through a basis of Y we see $Tx_0 = 0$. But T is injective so $x_0 = 0$. Therefore $Ran(T^*) = X$ and T^* is surjective.

We don't actually need to apply the open mapping theorem at all. What I was missing before is the intuition from finite-dimensional linear algebra that injectivity and surjectivity switch with T and T^*.

I am most relieved. There is nothing functional analytic here at all. This was a deficiency in my *ordinary linear algebra* education.

7. Problem I.3

The Sobolev space $H^s(\mathbf{R}^n)$ consists of $f \in L^2(\mathbf{R}^n)$ with

$$\|f\|_s^2 = \int (1 + |\xi|^2)^s |Ff(\xi)|^2 d\xi < \infty$$

Let $R : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^{n-1})$ be the restriction map

$$Ru(x') = u(x', 0)$$

For $s > 1/2$ prove that there is a unique extension of R from $\mathcal{S}(\mathbf{R}^n)$ to $R : H^s(\mathbf{R}^n) \to H^{s-1/2}(\mathbf{R}^{n-1})$

Suppose $f \in H^s(\mathbf{R}^n)$ is arbitrary and we have $f_n \in \mathcal{S}(\mathbf{R}^n)$ with

$$\|f_n - f\|_s \to 0$$

We want to examine the convergence properties of $Rf_n \in \mathcal{S}(\mathbf{R}^{n-1})$ in $\|\cdot\|_{s-1/2}$. If there is a convergence to $g \in H^{s-1/2}(\mathbf{R}^{n-1})$ we set

$$Rf(x') = g(x')$$

This scheme will ensure that there is an extension as required by the problem. Here we use the fact that $\mathcal{S}(\mathbf{R}^n)$ is dense in $L^2(\mathbf{R}^n)$.

Now we look at

$$\int_{\mathbf{R}^n} |F(f - f_n)(\xi)|^2 (1 + |\xi|^2)^s d\xi$$

I will try to rewrite this integral with Fourier variables $\xi = (\xi', \xi'')$ and attempt to get something to converge in the norm of $H^{s-1/2}(\mathbf{R}^{n-1})$.

Now it is December 29 2021 6:07 PM. So 15 hours since I began. About four questions in good shape and six more to go. I am not extremely familiar with distribution theory. I might take two or three more days to complete Spring 2016 Analysis Qual.

7.1. **Small Tubular Thickening.** What I want to do is take a small $\epsilon > 0$ and decompose

$$\mathbf{R}^n = R^{n-1} \times (-\epsilon, \epsilon) \cup \mathbf{R}^{n-1} \times [\epsilon, \infty) \cup \mathbf{R}^{n-1} \times (-\infty, -\epsilon] = A_0 \cup A_1 \cup A_2$$

Then I want to forget about Sobolev space and and other issues and attempt to prove the following. If g_n is a sequence of functions on $\mathbf{R}^n$ with

$$\int_{\mathbf{R}^n} |g_n(y)|^2 (1 + |y|^2)^s dy \to 0$$

Then

$$\int_{A_0} |g_n(y)|^2 (1 + |y|^2)^s$$

will converge to zero as well.

This is clear because all three parts are positive and the integral over $A_0 \cup A_1 \cup A_2$ dominates the one over A_0, and the integrals over A_1 and A_2 converge to zero.

Now we want to say that

$$\int_{A_0} |g_n(y)|^2 (1 + |y|^2)^s dy \approx 2\epsilon \int_{\mathbf{R}^{n-1}} |g_n(y', 0)| (1 + |y'|^2)^{s-1/2} dy'$$

If we succeed, then we will be able to use this to get an extension as required by the problem.

7.2. **Sobolev Embedding Invocation.** I was hoping for some continuity and the basic intuition there is that if there are k derivatives in L^p and $kp > n$ then all the elements of the Sobolev space $H^{k,p}(\mathbf{R}^n)$ are continuous. I am not going to use it. I will not even bother with all the Sobolev theory. I will, instead just focus on bounding the integral

$$\int_{\mathbf{R}^{n-1}} |g_n(w, 0)|^2 (1 + |w|^2)^{s-1/2} ds$$

given

$$\int_{\mathbf{R}^{n-1}} \int_{-\epsilon}^{\epsilon} |g_n(w, t)|^2 (1 + |w|^2 + t^2) dw dt \to 0$$

Each of these g_n are smooth enough as they are all members of Schwartz space $\mathcal{S}(\mathbf{R}^n)$.

Taylor expand them now

$$g_n(x, t) = g_n(x, 0) + g_n'(x, 0)t + O(t^2)$$

This might produce a result since Taylor expansion separates out the restriction term. Fourier transform sends Schwartz functions to Schwartz functions, so Taylor expansion is certainly good in local neighborhoods of the hyperplane $\mathbf{R}^{n-1} \times \{0\}$.

7.3. **What Can we do With One Term Taylor Expansion?** Let us just do this slowly

$$|g_n(y',t)|^2 \leq |g_n(y',0)|^2 + 2g_n'(y',0)t + |g_n'(y',0)|^2t^2 + O(t^2)$$

When we integrate this with respect to $t \in [-\epsilon, \epsilon]$ first before doing the rest of the integration in y' with the weight $(1+|y'|^2)^s$ etc. we get

$$\int_{-\epsilon}^{\epsilon} |g_n(y',t)|^2 dt = 2\epsilon|g_n(y',0)|^2 + \frac{2}{3}\epsilon^3|g'(y',0)|^2 + O(\epsilon^3)$$

This is very useful for us because we see

$$\int_{A_0} |g_n(y',t)|^2(1+|y'|^2+t^2)^s dy'dt = 2\epsilon\|g_n(y',0)\|_s + O(\epsilon^3)$$

Now since we use $\|g_n\|_s \to 0$ with and given $\eta > 0$ there exists N such that $n \geq N$ implies $\|g_n\|_s < \eta$. We consider $\eta \leq \epsilon^2$, and that gives us

$$\|g_n(y',0\|_{H^s(\mathbf{R}^{n-1})} \leq \epsilon$$

So this will work and we can get restrictions to be a map into $H^s(\mathbf{R}^{n-1})$ without losing derivative control.

7.4. **Why I Am Confident No Loss Is The Right Answer.** You see when you restrict a function f to $\mathbf{R}^{n-1}$ and it's a linear hyperplane, then Fourier transforms respect the geometry and then you should not lose any derivative control at all when you restrict.

So you let $g_n(y) = F(f - f_n)(y)$ assuming you have Scwhartz functions f_n converging to f in $H^s(\mathbf{R}^n)$.

Ah, these g_n are not Schwartz! The f_n are Schwartz. They are $H^s(\mathbf{R}^n)$.

Let's assume g_n are $C^1(\mathbf{R}^n)$. Then given $\epsilon > 0$ what we do is take η-thickening of $\mathbf{R}^{n-1}$ for our analysis with $\eta = \epsilon^2$. This will yield that there exists N such that for $n \geq N$

$$\|g_n(y',0)\|_{H^s(\mathbf{R}^{n-1}} < C\epsilon$$

by my path.

Now the issue could be precisely that we cannot assume $g_n(y',t)$ has a valid Taylor expansion in t near 0 so we can't get this estimate for the restriction but that is hard to believe because if $s > 1/2$ as in the premise of the problem, why should the derivatives become less if we're integrating in an orthogonal direction to y'?

The path I laid out will produce a good result when we can assume g_n are all C^1.

8. Problem II.5

Let $\phi : \mathbf{R} \to \mathbf{C}$ be 2π-periodic Hölder with exponent α.

Let

$$\varphi(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Then

$$\varphi(x+h) - \varphi(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}(1 - e^{inh})$$

Absolute value square is $|a|^2 = a\bar{a}$ and

$$|\varphi(x+h) - \varphi(x)|^2 = \sum_n |c_n|^2(1 - e^{inh})(1 - e^{-inh}) = 2\sum_n |c_n|^2(2 - 2\cos(nh))$$

Then integrate with measure

$$\frac{1}{2\pi} 1_[- \pi, \pi] dx$$

so

$$\sum_n |c_n|^2(1 - \cos(nh)) = \frac{1}{4\pi}\int_{-\pi}^{\pi} |\varphi(x+h) - \varphi(x)|^2 dx$$

Hölder-continuity with exponent α gives

$$\sup_x \frac{|\varphi(x+h) - \varphi(x)|}{h^\alpha} = C$$

(b) We begin with

$$\sum_n \frac{1 - \cos(nh)}{h^{2\alpha}} |c_n|^2$$

First two terms

$$1 - \cos(nh) = (nh)^2/2! - (nh)^4/4! + \cdots$$

So

$$\frac{1 - \cos(nh)}{h^{2\alpha}} = (n^2/2!)h^{2-2\alpha} - (n^4/4!)h^{4-2\alpha} + \cdots$$

Given $\alpha > 1/2$ we can get the sum to be bounded as $h \downarrow 0$.

9. Finally Found A Copy Of Reed And Simon I

I wish to exclaim with joy to the world about how happy I am to find a copy of Reed-Simon I. I love this book and I learned functional analysis at Princeton from it. Barry Simon is truly a great erudite mathematician, who, unlike many other mathematicians laboured to teach us something valuable taking time away from his researches and I just realised that when I first learned some functional analysis, I stayed away from Schwartz space and other things that seemed a bit too high falutin. That was 1993. Things are a bit different. I might have to consider distributions on spinor fields of a four-sphere soon and this may all be very important for me.

10. Problem II.4

Let $f \in \mathcal{S}'(\mathbf{R})$ and $\psi_0 \in \mathcal{S}(\mathbf{R})$ with $\int \psi_0 = b \neq 0$ and $a \in \mathbf{R}$.
(a) Find $u \in \mathcal{S}'(\mathbf{R})$ satisfying

- $u(\psi_0) = a$
- $u' = f$

(b) For $\epsilon > 0$ and $k \in \mathbf{N}$ define

$$u_{\pm,\epsilon}(\phi) = \int (1 \pm i\epsilon)^{-k} \phi(x) dx.$$

Prove $u_{\pm,\epsilon} \in \mathcal{S}'(\mathbf{R})$ first. Then prove $u_{\pm,\epsilon} \to u_\pm$ then calculate $u_+ - u_-$.

10.1. **Relaxed Path.** With tempered distributions I do not have an enormous amount of familiarity. So I will just go through these in a relaxed manner.

Let's do some simple exercises first. The functional

$$w_1(\phi) = a/b \int \phi(x)dx$$

will have the property that $w_1(\psi_0) = a$.

On the other hand

$$w_2'(\phi) = w_2(d/dx\phi)$$

We can therefore define

$$w_2(\phi) = f(\int_0^x \phi(s)ds)$$

And this will give us

$$w_2' = f$$

We can calibrate things in this way. Let

$$B = f(\int_0^x \psi_0(s)ds)$$

then we define

$$u(\phi) = f(\frac{a}{B}\int_0^x \psi_0(s)ds).$$

(b) Well,

$$|(1 \pm i\epsilon)^{-k}| = (1+\epsilon^2)^{-k/2} \leq 1$$

Thus we have $u_{\pm,\epsilon}$ are bounded linear functionals on $\mathcal{S}(\mathbf{R})$.

11. Problem I.1

Suppose $(X, \mathcal{B}, \mu)$ is a measure space.

(a) For $\mu(X) < \infty$ are there any inclusions between $L^1(X,\mu)$, $L^2(X,\mu)$ and $L^\infty(X,\mu)$?

(a) Since constant functions are L^1 and L^2 for $\mu(X) < \infty$, we have $L^\infty(X,\mu) \subset L^p(X,\mu)$ for $p = 1, 2$

$$\int |f|^p d\mu \leq \|f\|_\infty^p \mu(X)$$

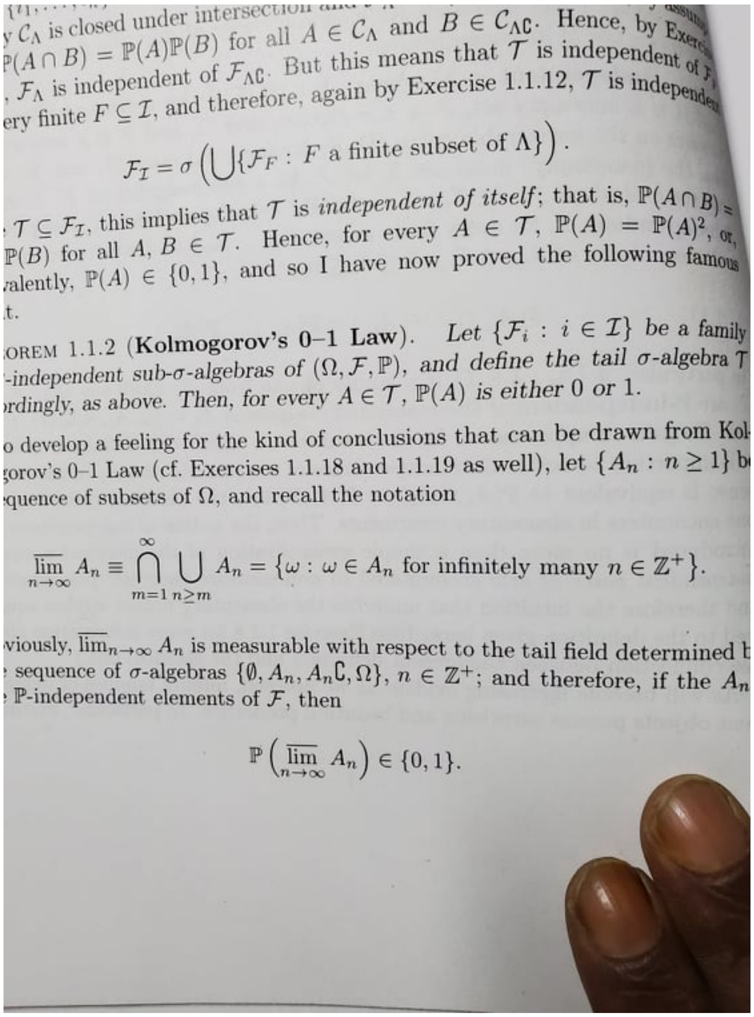
On the other hand $L^1(X,\mu) \subset L^2(X,\mu)$ since by Jensen's inequality we have

$$\phi(\int |f|d\mu) \leq \int \phi(|f|)d\mu$$

So $\phi(x) = x^2$ ensures that if $f \in L^2(X,\mu)$ then $f \in L^1(X,\mu)$.

When $\mu(X) = \infty$ we have $L^p(X,\mu) \subset L^\infty(X,\mu)$ but the inclusion is strict as constants are not L^p and included in L^∞.

(b) I won't bother with recalling the Borel-Cantelli set construction and just look it up.



y $\mathcal{C}_\Lambda$ is closed under intersection and ... assum
$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ for all $A \in \mathcal{C}_\Lambda$ and $B \in \mathcal{C}_{\Lambda\complement}$. Hence, by Exerc
, $\mathcal{F}_\Lambda$ is independent of $\mathcal{F}_{\Lambda\complement}$. But this means that $\mathcal{T}$ is independent of $\mathcal{F}$
ery finite $F \subseteq \mathcal{I}$, and therefore, again by Exercise 1.1.12, $\mathcal{T}$ is independe

$$\mathcal{F}_\mathcal{I} = \sigma\left(\bigcup\{\mathcal{F}_F : F \text{ a finite subset of } \Lambda\}\right).$$

$\mathcal{T} \subseteq \mathcal{F}_\mathcal{I}$, this implies that $\mathcal{T}$ is *independent of itself*; that is, $\mathbb{P}(A \cap B) =$
$\mathbb{P}(B)$ for all $A, B \in \mathcal{T}$. Hence, for every $A \in \mathcal{T}$, $\mathbb{P}(A) = \mathbb{P}(A)^2$, or,
valently, $\mathbb{P}(A) \in \{0,1\}$, and so I have now proved the following famous
t.

OREM 1.1.2 (**Kolmogorov's 0–1 Law**). *Let $\{\mathcal{F}_i : i \in \mathcal{I}\}$ be a family*
-independent sub-σ-algebras of $(\Omega, \mathcal{F}, \mathbb{P})$, and define the tail σ-algebra $\mathcal{T}$
rdingly, as above. Then, for every $A \in \mathcal{T}$, $\mathbb{P}(A)$ is either 0 or 1.

o develop a feeling for the kind of conclusions that can be drawn from Kol-
gorov's 0–1 Law (cf. Exercises 1.1.18 and 1.1.19 as well), let $\{A_n : n \geq 1\}$ b
quence of subsets of Ω, and recall the notation

$$\varlimsup_{n\to\infty} A_n \equiv \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{Z}^+\}.$$

viously, $\varlimsup_{n\to\infty} A_n$ is measurable with respect to the tail field determined b
sequence of σ-algebras $\{\emptyset, A_n, A_n\complement, \Omega\}$, $n \in \mathbb{Z}^+$; and therefore, if the A_n
$\mathbb{P}$-independent elements of $\mathcal{F}$, then

$$\mathbb{P}\left(\varlimsup_{n\to\infty} A_n\right) \in \{0,1\}.$$

As you can see from Daniel Stroock's *Probability Theory: An Analytic View*, the relevant sent is

$$G = \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} E_n$$

This quantity is measurable because the countable unions and intersections of measurable sets are involved. Then

$$\mu(G) = \lim_{m\to 0} \mu(\bigcup_{n\geq m}) = \lim_{m\to 0} \sum_{n \geq m} \mu(E_n) = 0$$

This is the exact argument given by Daniel Stroock. This is shameless advertisement for his book because why not? Why shouldn't I put in a good word for Dan Stroock's book. He was complaining about sales of his book, and so buy it. He's sharp and he has some special genius in analysis. I won't spoil it for you.

11.1. **Observations.** These are not intuitive for me at all and I find these still not within my comfort level. But I do not have doubts at all that I can reach higher level mastery on these sorts of issues. I am not used to them yet.

References

[1] https://projecteuclid.org/journals/journal-of-differential-geometry/volume-54/issue-1/A-Hodge-theory-for-some-non-compact-manifolds/10.4310/jdg/1214342150.full
[2] https://github.com/zulf73/human-nature
[3] https://github.com/zulf73/S4TheoryNotes
[4] https://math.stackexchange.com/questions/1630556/examples-of-bounded-linear-operators-with-range-not-close
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