STANFORD ANALYSIS QUAL 2013 II.5

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1. Stanford 2013 Analysis Problem I.5

Suppose $f : \mathbf{R} \to X$ where X is a topological vector space. Tell us whether f is continuous and if it is differentiable. Differentiable means

$$g = \lim_{h \to 0} \frac{1}{h}(f(t+h) - f(t))$$

exists in X.
(a) $X = L^2$ and $f_t(x) = \chi_{[t,t+1]}(x)$
(b) $X = L^2$ and $f_t(x) = \sin(x-t)$ if $x \in [x, x+\pi]$ and 0 otherwise.
(c) $X = \mathcal{S}'$ with weak-* topology and $f = \delta_t$

2. Difference of boxes

(a) Here f_t is continuous. Let

$$g_t = f_{t+h} - f_t$$

Then

$$g_t(x) = \begin{cases} 1 & t \le x \le t+h \\ -1 & t+1 \le x \le t+h+1 \\ 0 & \text{otherwise} \end{cases}$$

What we see is that as $h \to 0$ the quotient g_t/h converges to $\delta_t - \delta_{t+1}$ which does not belong to L^2, so f_t is not differentiable.

3. The Sine

(b) f_t is continuous.

$$g_t = \begin{cases} \sin(x-t-h) - \sin(x-t) & t \le x \le t+\pi-h \\ \sin(x-t-h) & t-h < x < t \\ \sin(x-t) - \sin(x-t-\pi-h) & t+\pi-h < x \\ 0 & \text{otherwise} \end{cases}$$

Then I conveniently look up formula for sine for sums.

$$\sin(x-t-h) = \sin(x-t)\cos(h) + \cos(x-t)\sin(-h)$$

See, this does not look all that promising, because, when divide by h we'll get a delta function here. So this is not differentiable.

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4. The Delta Function

(c) $X = \mathcal{S}'$ and $f_t = \delta_t$. Here the issue is what is the weak-* topology continuity. Ah, it's the weakest topology making duality mapping with $\varphi \in \mathcal{S}$ continuous. So here we can conclude that f_t is continuous and differentiable since the duality pairing is just evaluation at t and Schwartz functions are smooth.