STANFORD SPRING 2017 ANALYSIS QUAL

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1. Looking For Immediate Full Tenured Professorship

Stanford Faculty ought to understand that I am seeking tenure not for my mathematical skills but my *Scientific Accomplishments* over the past year, not only for Four-Sphere Theory which is a fundamental physics but extensive set of empirical results in (a) establishing the empirical validity of Aristotle's Virtue-Eudaimonia theory (b) establishing the universality of human moral nature refuting strongly theories of both *Friedrich Nietzsche* and *Immanuel Kant*, (c) establishing ethnicity-independence of human moral nature, (d) work on evolution of romantic love before the emergence of homo sapiens. Much of the data I used for my quantitative human nature and morality came from World Values Surveys. The mathematical skills I am testing just to get to levels necessary for deeper examination of the relationship between Nature and Mathematics. I have worked with *Jeff McNeal* who was undergraduate supervisor for a prize-winning Mathematics thesis at Princeton which was publishable quality in 1995. I have worked with Daniel Stroock with whom I have a publication in 2000 [1]. I am seeking immediate tenure with plans to settle in Mission San Francisco rather than work at the Palo Alto campus and I have extensive plans for development of quantitative positive psychology for applications for billions.

2. Problem I.1

(a) Let $f(x) = 1/x^2$ then

$$\int_\epsilon^1 f(x)dx = 1/\epsilon^2 - 1$$

therefore

$$\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)dx = \infty$$

This is for $f \to 0$ but the integrals do not. We use contrapositive. If $f \notin L^1$ we can attempt to prove $\int_a^1 f dx$ does not converge. This statement is true. If $f \notin L^1$ then

$$\int |f|dx = \infty$$

therefore $\lim_{a\to 0}\int_a^1 f dx = \infty$

(b) Suppose $f : \mathbf{R} \to \mathbf{R}$ and for each $x \in \mathbf{R}$ there is a quadratic polynomial $P_x(y) = a_x(y-x)^2 + b_x(y-x) + c_x$. Assume

$$\lim_{y\to x}\frac{|f(y) - P_x(y)|}{|y-x|^2} < \infty$$

Date: January 1, 2022.

Then is f twice differentiable.

Let $g_x(y) = f(y) - P_x(y)$. We will prove $g_0(y)$ is twice differentiable, and since $P_0(y)$ is twice differentiable, we can conclude $f = g_0 + P_0$ is as well.

We have

$$\lim_{y\to x} \frac{g_x(y)}{|x-y|^2} < \infty$$

The idea is that this is equivalent to

$$\lim_{h\to 0} \frac{|g_0(x+h) - g_0(x)|}{h^2} < \infty$$

This proves g_0 is twice differentiable.

3. Problem I.2

Let $T_t \in L(L^2(\mathbf{R}))$ be defined by

$$T_t f(x) = f(x-t)$$

(a) Show if $\|T_t - T_s\| < 2$ then $t = s$.

(b) Give with proof a locally convex topology on $L(L^2(\mathbf{R}))$ where $t \to T_t$ is continuous.

This problem is deceptive because my instincts are good at knowing what sort of problems require a lot of organisational skills. So I won't rush on this problem. There is no point in rushing either, because there is a danger of Jambalaya here.

Let us then, recall the operator norm definition and stare at it with mute fascination.

$$\|T_t\| = \sup_{\|f\|\leq 1} \|T_t f\|$$

The norms on the right are all L^1. We want to use boxes of size 1×1 which are indicators of intervals of length 1 which have L^1 norm 1.

We forget about the general conclusion, and try to prove that if only these boxes are involved, then we have the conclusion of (a) first.

As for (b) rest assured that I will consult Reed-Simon and every other text to do the problem because this 'locally convex' topology was probably invented by Von Neumann, and whenever Von Neumann is involved in inventing anything, then it's never anything intuitive. He was a technical virtuoso who always invented things that were meant to be totally incomprehensible to good decent reasonable people.

3.1. **Translations Form A Group.**

I want to examine the group structure of translations. The identity has operator norm 1. Next

$$\begin{aligned}\|T_t f - T_s f\| &= \int |f(x-s+(t-s)) - f(x-s)|dx \\ &= \int |f(w+(t-s)) - f(w)|dw\end{aligned}$$

I've used translation invariance of Lebesgue measure on $\mathbf{R}$ and this tells me

$$\|T_t - T_s\| = \|T_{t-s} - I\|$$

This argument then reduces the problem (a) to showing something slightly easier: if $\|T_a - I\| < 2$ then $a = 0$.

Let us consider the following functions on $[0,1]$.

$$g_n(x) = \begin{cases} n & 0 \le x \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

For any $a > 0$, we can find n such that $a > 1/n$ and so

$$\|T_a g_n - g_n\| = 2$$

This gives us an argument to show if $\|T_a - I\| < 1$ then $a = 0$. So let us examine whether the above integral is 1. The mass is concentrated close to $x = 0$. When we move the function by a, the mass is now in $x < 0$ but g_n has no mass for $x < 0$. Explicitly,

$$g_n(x-a) - g_n(x) = \begin{cases} n & -a \le x \le -a + 1/n < 0 \\ 0 & -a + 1/n < x < 0 \\ -n & 0 < x < 1/n \\ 0 & \text{otherwise} \end{cases}$$

Then, we have

$$2 \le \sup_{\|f\| \le 1} \|T_a f - f\|$$

whenever $a > 0$. Therefore $a = 0$ if $\|T_a - I\| < 2$.

3.2. **The Gentleman Doth Protest Too Much Methinks.** I am not comfortable with locally convex topologies at all. Instead I will examine the history of who decided that these technicalities of local convexity are of value.

Metrizable topologies on vector spaces have been studied since their introduction in Maurice Fréchet's 1902 PhD thesis Sur quelques points du calcul fonctionnel (wherein the notion of a metric was first introduced). After the notion of a general topological space was defined by Felix Hausdorff in 1914, although locally convex topologies were implicitly used by some mathematicians, up to 1934 only John von Neumann would seem to have explicitly defined the weak topology on Hilbert spaces and strong operator topology on operators on Hilbert spaces. Finally, in 1935 von Neumann introduced the general definition of a locally convex space (called a convex space by him) [5]

I knew it! It was that Jon von Neumann! He is responsible for all this, to introducing innocent good people to technicalities from Borg technology! He might have intended it to be a secret weapon against Nazis. That would explain the whole thing.

3.3. **Adventures In Looking Up Reed-Simon Seeking Wisdom Of The Von Neumann Contraption.** The substantial issue of part (a) was that if we put the norm topology on $L(L^2(\mathbf{R}))$ then the map $t \to T_t$ does not behave well. In small norm-balls around the identity in $L(L^2(\mathbf{R}))$, none of the translations are in the ball at all.

In other words, the norm topology has too many open sets. It has so many open sets that that we can't get continuity from reasonable seeming maps $\mathbf{R} \to L(L^2(\mathbf{R}))$. The task was therefore to reduce the open set proliferation of the norm topology.

A locally convex topological vector space is one where there is a neighborhood basis consisting only of convex sets.

3.4. What Is A Seminorm? A seminorm is a function $\rho : V \to [0, \infty]$ satisfying

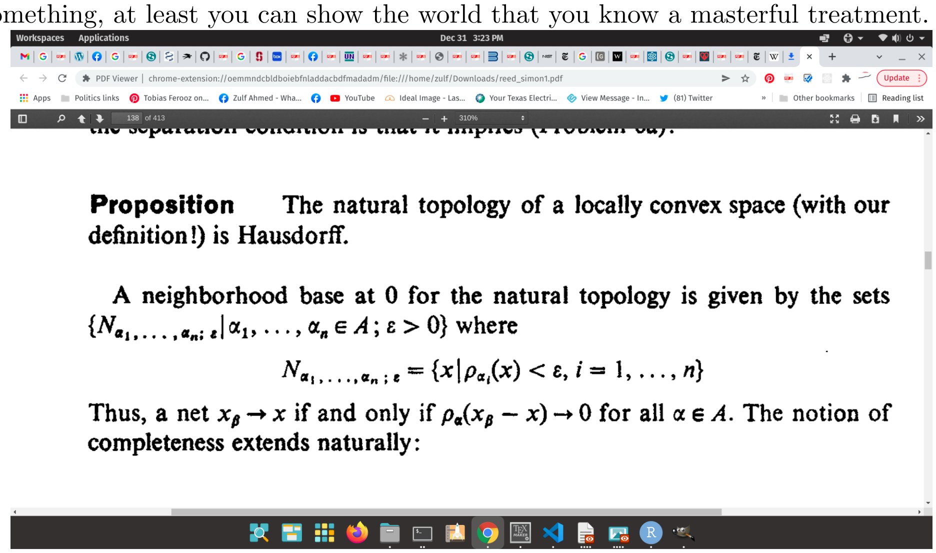
- (a) $\rho(x + y) \le \rho(x) + \rho(y)$
- (b) $\rho(ax) = |a|\rho(x)$

A family of seminorms $(\rho_a)_{a \in A}$ are said to *separate points* if $rho_a(x)$ for all $a \in A$ implies $x = 0$.

A *locally convex space* is one with a family of seminorms $(\rho_a)_{a \in A}$ that separate points. The natural topology is the weakest topology where all the seminorms $(\rho_a)_{a \in A}$ are continuous.

So that's the sort of things we will want to do with $X = L(L^2(\mathbf{R}))$. We want to define a family of seminorms that separate points and then take the weakest topology so all the members of the family are continuous.

3.5. What Reed-Simon p. 125 Said. When you know that you don't know something, at least you can show the world that you know a masterful treatment.



Proposition The natural topology of a locally convex space (with our definition!) is Hausdorff.

A neighborhood base at 0 for the natural topology is given by the sets $\{N_{\alpha_1, \ldots, \alpha_n; \varepsilon} | \alpha_1, \ldots, \alpha_n \in A; \varepsilon > 0\}$ where

$$N_{\alpha_1, \ldots, \alpha_n; \varepsilon} = \{x | \rho_{\alpha_i}(x) < \varepsilon, i = 1, \ldots, n\}$$

Thus, a net $x_\beta \to x$ if and only if $\rho_\alpha(x_\beta - x) \to 0$ for all $\alpha \in A$. The notion of completeness extends naturally:

So now we have the exact form of the neighborhood basis and we want to get some seminorms for the operators in $L(L^2(\mathbf{R}))$.

3.6. She Got Diamonds On The Soles Of Her Shoes. I'm just a poor boy from a poor family. And she's got diamonds on the soles of her shoes. Oh yeah. She's got diamonds in the soles of her shoes.

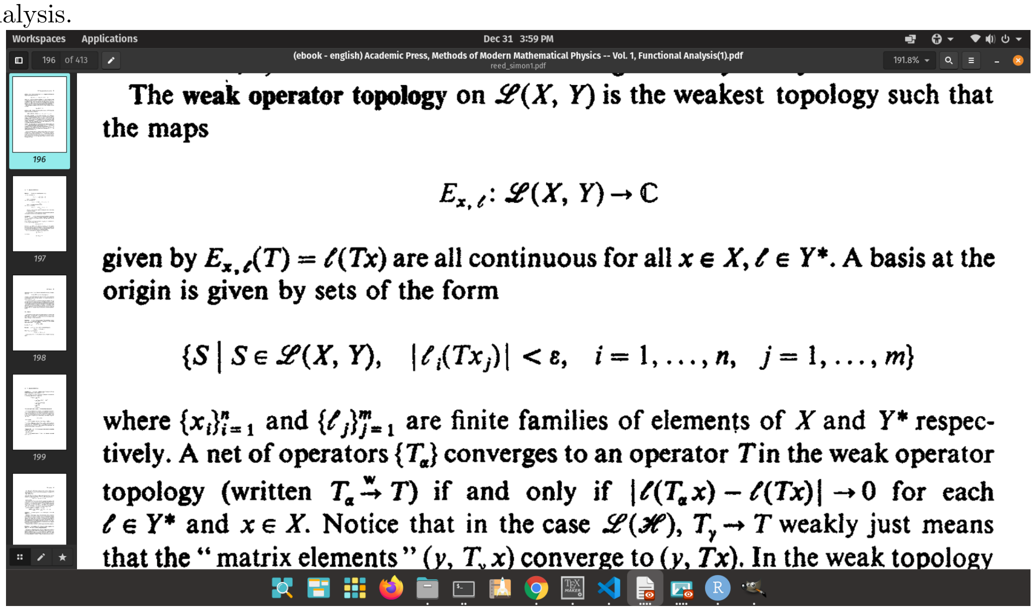
I'm just a poor poor boy from a poor family. Nobody loves me. And she's got diamonds on the soles of her shoes. Oh yeah.

> And I could say ooh ooh ooh
> As if everybody knows
> What I'm talking about
> As if everybody would know
> Exactly what I was talking about
> Talking about diamonds on the soles of her shoes
> She makes the sign of a teaspoon
> He makes the sign of a wave
> The poor boy changes clothes
> And puts on after-shave

To compensate for his ordinary shoes
And she said honey take me dancing
But they ended up by sleeping
In a doorway
By the bodegas and the lights on Upper Broadway
Wearing diamonds on the soles of their shoes
And I could say ooh ooh ooh
As if everybody here would know
What I was talking about
I mean everybody here would know exactly
What I was talking about
Talking about diamonds

4. Zulf Returns To Frightening Families of Seminorms

As promised I did not want to do any work at all for part (b) and so here we have the precise textbook answer p. 183 of Reed-Simon. I highly recommend these particular answers from Reed-Simon whose great virtues will only be apparent after you realise that this whole area of functional analysis is quite anarchic, and Reed-Simon is extremely clear while the world is filled with confusing texts on functional analysis.



The weak operator topology on $\mathcal{L}(X, Y)$ is the weakest topology such that the maps

$$E_{x,\ell}: \mathcal{L}(X, Y) \to \mathbb{C}$$

given by $E_{x,\ell}(T) = \ell(Tx)$ are all continuous for all $x \in X$, $\ell \in Y^*$. A basis at the origin is given by sets of the form

$$\{S \mid S \in \mathcal{L}(X, Y),\quad |\ell_i(Tx_j)| < \varepsilon,\quad i = 1, \dots, n,\quad j = 1, \dots, m\}$$

where $\{x_i\}_{i=1}^n$ and $\{\ell_j\}_{j=1}^m$ are finite families of elements of X and Y^* respectively. A net of operators $\{T_\alpha\}$ converges to an operator T in the weak operator topology (written $T_\alpha \xrightarrow{w} T$) if and only if $|\ell(T_\alpha x) - \ell(Tx)| \to 0$ for each $\ell \in Y^*$ and $x \in X$. Notice that in the case $\mathcal{L}(\mathcal{H})$, $T_\gamma \to T$ weakly just means that the "matrix elements" $(y, T_\gamma x)$ converge to (y, Tx). In the weak topology

5. Problem I.3

(a) What is the spectrum of the operator $T : \ell^2 \to \ell^2$ that is defined by $a \in \ell^\infty$ by

$$Tx_n = a_n x_n?$$

(b) Suppose $K \subset \mathbf{C}$ is compact. Find a Hilbert space H and operator $T \in L(H)$ such that $\sigma(T) = K$.

5.1. **Part a.** We claim $\sigma(T) = \{a_1, \dots\}$. For any k we consider e_n the coordinate basis. These are obviously eigenvectors with eigenvalue a_n. The eigenvectors form a basis so there is nothing else in the spectrum.

5.2. **Part b.** The following is my own solution. Take a countable dense set $\{a_1, \dots, \} \subset K$ and use the same construction as Part (a). Since the resolvent set has to be *open* there won't be any part of it in K at all.

6. Problem I.4

(a) Define the restriction map

$$R : C_0^1(\mathbf{R}) \to C_0^1([0,\infty))$$

Show there is a unique extension to

$$R : H^1(\mathbf{R}) \to H^1([0,\infty))$$

(b) Show that there is a map $E : H^1([0,\infty)) \to H^1(\mathbf{R})$ such that $ER = Id$.

6.1. **Density Argument.** Suppose $f_n \in C_0^1(\mathbf{R})$ with

$$\|f_n - f\|_{H^1(\mathbf{R})} \to 0$$

We can do this for all $f \in H^1(\mathbf{R})$. We want to *define*

$$Rf = f1[0,\infty)$$

and prove that Rf belongs to $H^1([0,\infty))$.

In order to get this sort of thing, we should consider Fourier transforms.

I write

$$\frac{d}{dx}[\frac{1}{-i\xi}e^{-ix\xi}f(x)] = e^{-ix\xi}f(x) + \frac{1}{-i\xi}e^{-ix\xi}f'(x) \tag{1}$$

I do this with the hope of getting some understanding of how to manage

$$F(f1_[0,\infty))(\xi)$$

in terms of $Ff(\xi)$. You see, the problem here is that $H^1(\mathbf{R})$ norm is defined in terms of Fourier transforms. And the substance of this problem is to control Fourier transform of

$$f1_{[0,\infty)}$$

We integrate (1) after norm-squaring both sides from A to ∞.

$$\frac{1}{-i\xi}\int_A^\infty \frac{d}{dx}[e^{ix\xi}f(x)]dx = \int e^{-iA\xi}f(A) + \frac{1}{-i\xi}e^{-iA\xi}f'(A) \tag{2}$$

This is because $f(\infty) = f'(\infty) = 0$. We want to use this to bound

$$\int_0^\infty |f|^2 + |f'|^2 \le C\|f1_[0,\infty)\|_{H^1(\mathbf{R})}$$

In particular we want to use (2) in the evaluation of the Fourier transform *inside* the norm $\|\dot{\|}_{H^1(\mathbf{R})}$ and use some relatively simple inequality techniques to get a bound with C that is independent of n. And that will prove part (a).

6.2. **Part b.** The hint is $Ef(x) = \sum_{j=1}^k a_j f(-jx)$ for some k and a_j for $x < 0$.

7. Problem I.5

Suppose A is bounded and K is compact.

(a) If A has (i) finite dimensional kernel, (ii) infinite dimensional cokernal, (iii) closed range so does $A+K$.

(b) If A has (i) infinite dimensional kernel, (ii) finite dimensional cokernal, (iii) closed range so does $A+K$.

(c) If A has (i) infinite dimensional kernel and (ii) infinite dimensional cokernel, then does (iii) A has closed range imply $A+K$ has closed range?

7.1. Warm Up Exercises.

Suppose x satisfies $(A+K)x = 0$. Let's do things this way. Given $\epsilon > 0$ let $V_\epsilon \oplus W_\epsilon = H$ be an orthogonal division with orthonormal basis $(e_j) \subset H$ arranged so that for $(e_1, \ldots, e_N)$ with have $\|Ke_j\| \geq \epsilon$ and $\|Ke_j\| < \epsilon$ for $j > N+1$. Let's assume this can be done always.

Now suppose $(A+K)x = 0$. Assume $x \notin Ker(A)$. Then $Ax = -Kx$.

We want to be able to choose our ϵ so that $\epsilon \geq \sup_{x \in Ker(A+K) \cap Ker(K)^\perp} \|Kx\|$. Then we want to prove

$$\dim Ker(A+K) \leq \dim Ker(A) + N(\epsilon)$$

For (b) we want to replace A with A^* and apply (a).

8. Problem II.1

(a) Suppose X is a Banach space. Show every norm-closed set is weakly closed.

(b) Suppose $T : X \to Y$ and $S : Y \to Z$ and the composition is bounded and S is bounded and injective. Then T is bounded.

(a) Let K be a norm-closed subset of X and let $x_j \to x$ in K. For $\epsilon > 0$ there exists N such that $n \geq N$ implies

$$\|x_j - x\| < \epsilon$$

For any bounded functional $\ell \in X^*$, we have

$$|\ell(x_j - x)| \leq \|x - x\| \|\ell\| < \epsilon \|\ell\|$$

This implies $x_j \to x$ in the weak topology.

(b) Suppose $T : X \to Y$ and $S : Y \to Z$ where X, Y, Z are Banach. Given $A = T \circ S$ is bounded and S is bounded and injective, prove T is bounded.

Suppose $x \in X$ and T is unbounded at x. Since $A = T \circ S$ is bounded, then

$$\|Ax\| \leq C_A \|x\|$$

Since S is injective then by the open mapping theorem S is open surjection on its range $Ran(S) \subset Z$ and the inverse S^{-1} is bounded on $Ran(S)$. Now $Ax \in Ran(S)$. So

$$\|S^{-1}(Ax)\| \leq C_{S^{-1}} \|Ax\|$$

Then

$$\|Tx\| \leq C_{S^{-1}} C_A \|x\|$$

This contradicts assumption that T is not bounded at $x \in X$.

9. Problem II.2

Suppose $(E, \mathcal{E})$ is a measure space and f_n are measurable functions. Show $\inf_n f_n, \sup_n f_n, \liminf_n f, \limsup_n f_n$ are measurable and the set $\{x : f_n \text{converges}\} \in \mathcal{E}$.

9.1. **On This Problem I Will Look At R. M. Dudley's Book For Assistance.** Measurability of various operations I had always left as a technicality without worry for more than thirty years and did not actually ever think to worry about them. So these sorts of problems are not familiar to me. I will just take the opportunity to examine this issue more clearly with assistance from R. M. Dudley's book, chapter 4.2.

Now I do know that since $f_n : X \to \mathbf{R}$ we have $f_n^{-1}(A)$ is measurable for every measurable $A \subset \mathbf{R}$.

There is first a question of which sigma-algebra to use on $\mathbf{R}$. I am not sure whether the *Borel* or *Lebesgue* sets are more appropriate.

I will make the arbitrary decision that it is *Borel measurability* that is implied in the problem for f_n just because I have rarely seen many people explicitly invoke Lebesgue measurability. Borel sets are generated by countable operations of open and closed sets.

In order to prove that the four quantities $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$ and $\limsup_n f_n$ are measurable, we will eliminate $\inf_n f_n$ and $\liminf_n f_n$ by noting that if we prove it for the $\sup_n f_n$ we can apply it to $g_n = -f_n$ and obtain the conclusion for the others.

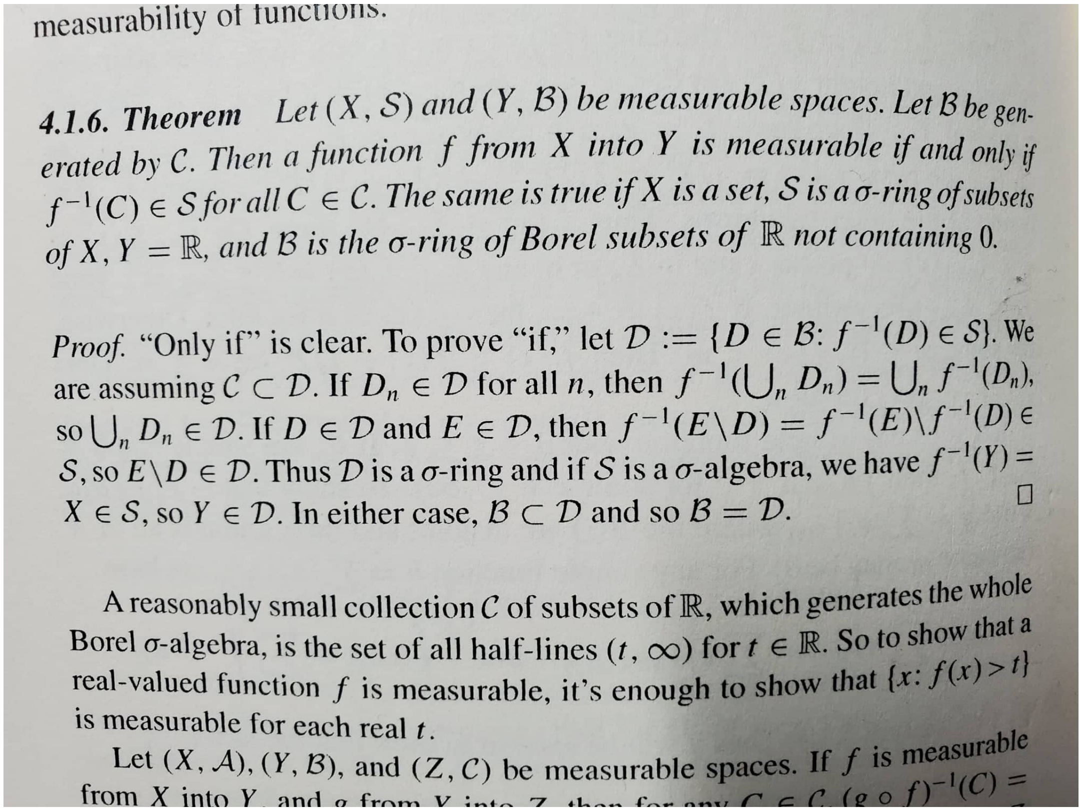
That's good. We are now more specific: we will prove measurability of two functions and a set. Good. I like fewer things having to do with measurability. I don't want to really denigrate measurability, it's great to go around doing whatever one's heart desires with a dismissive wave of the hand and "We'll it's measurable. It's elementary. It's elementary. Why would I, a great genius, actually prove measurability. Go go. Measurability is flexible. I want to do grand projects, not do set intersections with my life. And seriously, you don't have anything better to do with your precious life than worry about where this complicated operation I did is measurable? Shame on you. You know, Riemann would have gone so much further if people like you, the *Weierstrasses of the world* did not keep asking for existence of minimizers to the beautiful functional

$$\int |\nabla f|^2 dx$$

You want to do this all your life?"

Thankfully, salvation comes from the wonderful book of R. M. Dudley who has gifted the world with a precise and polished work where some of the dreariest issues are given clarity.

I want my dear reader to consider his Theorem 4.1.6 from p. 118.



measurability of functions.

4.1.6. Theorem *Let $(X, \mathcal{S})$ and $(Y, \mathcal{B})$ be measurable spaces. Let $\mathcal{B}$ be generated by $\mathcal{C}$. Then a function f from X into Y is measurable if and only if $f^{-1}(C) \in \mathcal{S}$ for all $C \in \mathcal{C}$. The same is true if X is a set, $\mathcal{S}$ is a σ-ring of subsets of X, $Y = \mathbb{R}$, and $\mathcal{B}$ is the σ-ring of Borel subsets of $\mathbb{R}$ not containing 0.*

Proof. "Only if" is clear. To prove "if," let $\mathcal{D} := \{D \in \mathcal{B}: f^{-1}(D) \in \mathcal{S}\}$. We are assuming $\mathcal{C} \subset \mathcal{D}$. If $D_n \in \mathcal{D}$ for all n, then $f^{-1}(\bigcup_n D_n) = \bigcup_n f^{-1}(D_n)$, so $\bigcup_n D_n \in \mathcal{D}$. If $D \in \mathcal{D}$ and $E \in \mathcal{D}$, then $f^{-1}(E \setminus D) = f^{-1}(E) \setminus f^{-1}(D) \in \mathcal{S}$, so $E \setminus D \in \mathcal{D}$. Thus $\mathcal{D}$ is a σ-ring and if $\mathcal{S}$ is a σ-algebra, we have $f^{-1}(Y) = X \in \mathcal{S}$, so $Y \in \mathcal{D}$. In either case, $\mathcal{B} \subset \mathcal{D}$ and so $\mathcal{B} = \mathcal{D}$. □

A reasonably small collection $\mathcal{C}$ of subsets of $\mathbb{R}$, which generates the whole Borel σ-algebra, is the set of all half-lines (t, ∞) for $t \in \mathbb{R}$. So to show that a real-valued function f is measurable, it's enough to show that $\{x: f(x) > t\}$ is measurable for each real t.

Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$, and $(Z, \mathcal{C})$ be measurable spaces. If f is measurable from X into Y, and g from Y into Z, then for any $C \in \mathcal{C}$, $(g \circ f)^{-1}(C) =$

The valuable substance of this theorem is given in the paragraph after the proof, that *it is sufficient* to prove that sets of the form

$$\{x \in X : f(x) > t\}$$

are measurable in order to conclude that f is measurable.

Being not enamoured of these set theoretic considerations, I shall invoke this criterion for our problem.

9.2. **Two Measurable Functions And a Measurable Set.** Let

$$A(t) = \{x \in E : \sup_n f(x) > t\}$$

and

$$B(t) = \{x \in E : \limsup_n f_n(x) > t\}$$

Our task is to prove that these are measurable subsets in E. We define analogously $A_n(t)$ and $B_n(t)$ for convenience.

Now

$$A(t) = \bigcup_{n=1}^{\infty} A_n(t)$$

and since each $A_n(t) \in \mathcal{E}$ we have $A(t) \in \mathcal{E}$.

In order to show $B(t) \in \mathcal{E}$ we have to know the trick of representing $\limsup_n f_n(x) > t$ by set theoretic operations. If you are a master of set theory you can decipher the method by your own genius. Others less gifted, such as yours truly, believe you just need to learn it from Daniel Stroock or someone else and just regurgitate the construction without any understanding of why.

$$B(t) = \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} B_n(t)$$

There are many people who teach that it is wrong to invoke authority instead of thinking about things by your own efforts, being self-reliant. I myself am a firm advocate of the Ralph Waldo Emersonian view of *self-reliance*. Here you see a blatant contradiction in my noble and lofty principles of self-reliance that I routinely advocate for younger people. You see me invoking authority at a drop of a hat. Posterity will have to gauge whether I had been false in my advocacy, and whether these sorts of blatant exceptions constitute a significant failure of virtues for Zulfikar Moinuddin Ahmed. But I will say in my defense that on *technical matters of measurability and sigma algebras there is no shame at all in invoking authority and setting aside one's pride in self-reliance.* One's Character is not fundamentally compromised in this situation.

For the set $\{x : \lim_n f_n \text{exists}\}$ we have to write this in terms of $\limsup_n f_n(x) = \liminf f_n(x)$ and so we have use $B(t)$ and the analogue $B'(t)$ corresponding to $\liminf$ and then take it's complement and do something like

$$B(t) \cap (E - B'(t))$$

and we get measurability.

10. On The Issues Of Character

I am quite involved in issues of *Human Nature* for some years and *Positive Psychology*. I am a gifted scientist and do not have any problems wading into these issues, as my scientific interests span physics and psychology. I have taken a strong stance against the situationalist challenge to notions of Character. The key issue here that the situationalists do not consider in their empirical work is that Aristotle in *Nicomachean Ethics* is quite explicit in pronouncing that Character Virtues are a matter of *habituation* and at least the Aristotelian theory – which is not the first by any means since in 1300 BC there were Ancient Egyptian texts regarding virtues and there were similar theories also pre-Aristotle in the East – makes very clear that the notion of Virtues *requires habituation* and without this prerequisites issues of Virtue are moot. In the modern times, there is no sensitivity to this aspect and so the challenge of situationalists are not valid. You cannot challenge Virtue based on random measurements of behaviour without first ensuring that the person has actually spent any years of their lives cultivating Virtue. Aristotle's and many others' before him theory of Virtue is correct and cannot be displaced by other theories measuring some other things, such as without habituation what sort of natural behaviour occurs in the population.

11. Problem II.3

Let $\mathcal{T}$ be the weakest topology on $C^\infty(\mathbf{T})$ such that

$$f_{k,\psi}(\varphi) = \|\varphi - \psi\|_{C^k(\mathbf{T})}$$

are continuous for all $k \geq 0$ and for all $\phi \in C^\infty(\mathbf{T})$.

The norm $\|\cdot\|_{C^k(\mathbf{T})}$ is define by

$$\|\varphi\|_{C^k(\mathbf{T})} = \sum_{j\le k} \sup |\partial^j \varphi|$$

(a) Prove that $\mathcal{T}$ is metrizable and write down a metric on $C^\infty(\mathbf{T})$ such that $\mathcal{T}$ is the metric topology.

(b) Show there is no *norm* on $C^\infty(\mathbf{T})$ is the norm-topology for it.

11.1. **Initial Thoughts.** I can just see Jon Von Neumann laughing at me from beyond 1935 when I see this problem. This problem totally smacks of a *family of seminorms* sort of thing, where the metric is determined by $\{\rho_a\}_{a\in A}$ a family of seminorms that are not norms.

Now I have to say, that in this *particular case*, there might be some value to the Von Neumann contraption, since I do love $C^\infty(\mathbf{T})$.

Fine, I will be honest. I will be honest and say that I see $C^\infty(S^4(R))$ right here. My heart is filled with joy and happiness when I contemplate $C^\infty(S^4(R))$ and other smooth things on four-sphere. I am in ecstatic rapture when I contemplate

$$C^\infty(\Gamma\Sigma S^4(R))$$

Let me remind you of what they are. These are smooth spinor fields on a scaled four-sphere. They constitute all things of the universe, my immortal discovery. I am filled with joy at them. The whole universe fills up with love whenever I examine these spinor fields.

Now $C^\infty(\mathbf{T})$ is not as wonderful. It's a toy, a meek and little toy. It's a little kitten who has not grown into a great majestic lion or tiger yet.

Regardless, they are important to me. And since they are important to me, I will repress the horror and shame of Von Neumann laughing at me from behind 1935, swallow my pride and get to using his locally convex neighborhoods with all these *semi-norms* floating around like pirhannas in the water. What I do for love.

11.2. **Opportunity For Four-Sphere Theory.** I suddenly realised that the space of smooth spinor fields on the four-sphere can be topologised by the same method as that which is used in this problem, as they are represented as *matrix functions* of 16 dimensions on the four-sphere, and therefore the technique used in this problem might be carried through to describe explicitly the metrizable topology on the space of smooth spinor fields on a four-sphere. This is actually an extremely serious issue for physics in the way that I have approached it.

I might take off some time away from other problems and use this opportunity to write up in detail a valid description of the metrizable topology for the four-sphere smooth functions and more importantly spinor fields.

By my Four-Sphere Theory, *all matter in nature* arise as formations from spinor fields on a four-sphere of fixed eternally constant radius $R = 3075.69$ Mpc. I have laboured to establish Four-Sphere Theory between 2008-2018 under great adversity and I consider this one of my immortal contributions to my beloved people the human race.

This is the first time in my entire life that I have managed to get any significant motivation to examine Jon Von Neumann's technical work on topologising infinite dimensional linear spaces at all. Knowing my own personality, I feel that I should set aside all other problems and just work this out in detail.

You see, physicists are not as forgiving as mathematicians when one invokes analogy for good reasons that they would like *numbers* so they are unsatisfied with mathematical analogies and want details that directly apply to the scientific model. So this is a slightly more demanding task than just the $C^\infty(\mathbf{T})$ case. I feel compelled to work this out in careful detail leaving nothing to chance. Four-Sphere Theory is my gift to humanity. If I do not care for it, who will?

12. Problem II.4

Suppose $f \in C^\infty(\mathbf{R}^n)$ is complex-valued with $Im(f) \geq 0$ and $K \subset \mathbf{R}^n$ compact. Suppose for $x \in K$ with $Im(f) = 0$ the differential of f does not vanish at x. Show that

$$|\int e^{i\omega f(x)} u(x) dx| \leq C\omega^{-N}$$

for some $C > 0$, and all $N \geq 0$ and all $\omega > 1$.

First we do the simple step of separating real and imaginary parts

$$|\int e^{i\omega f(x)} u(x) dx| \leq \int e^{-\omega Im(f(x))} |u(x)| dx$$

since $Re(f)$ gives us a modulus 1 value. This is the most important step.

Next let us do the following. Let $\epsilon > 0$ and let

$$A(\epsilon) = \{x \in \mathbf{R}^n : Im(f(x)) > \epsilon\}$$

and let

$$B(\epsilon) = \mathbf{R}^n - A(\epsilon).$$

Now

$$\int e^{-\omega f(x)} |u(x)| dx = \int_{A(\epsilon)} e^{-\omega f(x)} |u(x)| dx + \int_{B(\epsilon)} e^{-\omega f(x)} |u(x)| dx = I_A + I_B$$

We will handle the two integrals I_A and I_B separately. We have

$$I_A \leq e^{-\omega\epsilon} \int_{A(\epsilon)} |u(x)| dx$$

Now let

$$C_0 = \int |u(x)| dx < \infty$$

since support of $u(x)$ is in $K \subset \mathbf{R}^n$ and u is smooth. This gives us

$$I_A \leq C_0 e^{-\omega\epsilon}$$

Then elementary analysis will produce for $N \geq 0$ and $\omega > 1$ a constant $C_{A,\epsilon}$ such that

$$I_A \leq C_A \omega^{-N}$$

I won't go into the details of the elementary analysis.

Let us turn to the I_B case now. We have $Im(f) \leq \epsilon$. Let $g = Im(f)$ and we have differential of g does not vanish at points x where $g(x) = 0$. Let x be any point where $g(x) = 0$. Then

$$g(x+h) = g(x) + (\nabla g(x), h)) + O(|h|^2) = (\nabla g(x), h) + O(|h|^2)$$

by a two-term Taylor series. Here

$$\exp(-\omega g(x)) = \exp(-\omega(\nabla g(x), h) + O(|h|^2)) \leq C_1 e^{-\omega(\nabla g_(x), h)}$$

Ah, we can use the inverse function theorem to restrict the measure of the points where $g(x) = 0$ since $\nabla g(x) \neq 0$ at these points which implies they are all isolated zeros. For

$$E_0 = \{x \in K : g(x) = 0\}$$

has $\mu(E_0) = 0$.

Now we calibrate $\epsilon(N) > 0$ so that

$$\int_{B(\epsilon(N))} e^{-\omega g(x)} |u(x)| dx \leq C_2 \omega^{-N}$$

This will give us the bound.

13. I Do Not Understand This Racial Obsession Of Bill Gates

When was the last time in history that a rich brown man cut up the eyes of a white man, plotted to murder him, and obstructed $620 million of his income, intervened into a Harvard tenure consider with United States Industrial Power to damage his exoskeleton and continuously repeated 'browns are superior, browns are superior'? What sort of people do white people raise around here. It's absolutely atrocious.

I do consider my inner substance and Character far superior to Bill Gates. I pursued my convictions regarding Nature's fundamental laws against the grain of the entire world, without support from Dan Stroock and others who were quite rightly not interested for their own reasons, and I have struggled and suffered in isolation for them and have triumphed with Four-Sphere Theory after more than a decade of work. Bill Gates is incomparably inferior in his Character, is made of lesser material. I am not racial at all. I do not care about his ethnicity, but I am not impressed by the vile disgusting lowly menial Character of this man.

14. Problem II.5

Suppose X_j are independent identically distributed random variables with common Schwartz density $p(x)$.

$$Z_n = n^{-1} \sum_{j \leq n} X_j$$

Show probability distribution function of Z_n is $n[p * \cdots * p](nx)$ and

$$E(g(Z_n)) \to g(s)$$

for some s.

14.1. Assuming Convolution Relationship.

Assuming the convolution relationship that the probability density of Z_n is $np^{*n}(nx)$ let's do the following.

$$\int g(x) np^{*n}(nx) dx = \int g(y/n) p^{*n}(y) dy$$

Now put in the Fourier transform

$$\int F^{-1} F(g_n p^{*n})(y) dy = \int F^{-1} (\int e^{-i\xi x} g_n(x) p^{*n}(x) dx)(y) dy$$

We want to use the convolution theorem but we don't quite have the right situation.

Let's pretend g_n is convolved inside the Fourier transform. With this wrong assumption, we'd get $\hat{g_n}(\xi) \hat{p}(\xi)^n$ inside the F^{-1}. We're hoping to see something when we now take $n \to \infty$ inside. The Fourier transform of Schwarz is Schwartz, so we will have $\hat{p}(\xi)^n$ converge to a set of delta measures.

I will return to this later. This is possible to accomplish for me.

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