ZULF'S JANUARY 31 2022 STANFORD FALL 2014 ANALYSIS PH.D. QUAL EFFORT

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1. Introductory Comments

I am seeking a tenured full professorship at Stanford University immediately. The best way to let me know of Stanford University's Decision to give me a nice tenured position is zulfikar.ahmed@gmail.com. I want to live in Mission District, San Francisco and create a couple of companies for technology-psychology applications for service to eight billion people. I am seeking tenure for my successful *Scientific work* in four-sphere theory and Universal Human Moral Nature.

I am doing the Mathematics Ph.D. Qual problems for regaining skills in Mathematics atrophied since my graduation (magna cum laude) Princeton University 1995 with a prize-winning thesis in Several Complex Variables under Jeff McNeal who is at Ohio State University now.

I will put in the effort to do all ten problems and I have my solutions to a number of other exams already.

My interest in gaining some deeper understanding of Analysis. I am also examining the history of Analysis and of Science in eighteenth-twentieth centuries to gain a holistic appreciation for *mathematical substance*.

You can find my work archived and publicly available [1].

2. Problem II.5

The discrete Laplacian L is defined on $\ell^2(\mathbf{Z})$ by

$$Lf(n) = f(n) - 1/2(f(n+1) + f(n-1))$$

We are to prove that the spectrum is $[0, 2]$. Let's see bounded symmetric is easy since we have an identity that's bounded symmetric. Then matched shifts +/-1 is obviously bounded and symmetric as well.

We consider inverting $(s - L)$ for some $s \in \mathbf{C}$. We write

$$g(n) = (s - L)f(n)$$

and try to solve for $f(n)$. Let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} g(n-1) \\ g(n) \\ g(n+1) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f(n-1) \\ f(n) \\ f(n+1) \end{pmatrix}$$

Date: February 3, 2022.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1/2 & s-1 & 1/2 & 0 & 0 \\ 0 & 1/2 & s-1 & 1/2 & 0 \\ 0 & 0 & 1/2 & s-1 & 1/2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

I don't know how to invert the matrix yet, so let us just relax and not get too worried. As you can see, I have this beautiful matrix equation produced in LaTeX. As my dear reader is aware, in case you don't have substance ready, always make a pretty form so the audience, being impatient for substance is put in a pleasant state of mind. I don't want to say that good LaTeX form is extremely important, but it's like the totally cheesy music those companies play whenever you call them when they put you on hold and go off to make out with their girlfriends watching Hollywood movies while you're waiting for some response.

Well it becomes clear that Neumann series is the way to go. We consider

$$(s-L)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}(L/s)^k$$

This will produce a norm-convergent series of bounded operators in $L(\ell^2(\mathbf{Z}))$ whenever $\|L\| < |s|$ by obvious inequalities and geometric series.

We're therefore not going to try to invert the operator by hand since Neumann series is easier. We will have to prove (a) $\|L\| = 2$, and (b) spectrum is real (follows from symmetric) and (c) none of the rest of $s \in [0,2]$ produce invertible operators for $s - L$.

We'll come back to these. Now I want to claim that $Vf(n) = v(n)f(n)$ has discrete spectrum $\{v(n)\}_{n\in\mathbf{Z}}$. It is obvious that $Ve_n = v(n)e_n$ so $v(n)$ is an eigenvalue so lies in the spectrum. Next $\{v(n)\}$ is a bounded set because given $\epsilon > 0$ we have N such that if $|n| \geq N$ then $|v(n)| < \epsilon$ so

$$\sup_n |v(n)| \leq max_{1\leq n\leq N}|v(n)| + \epsilon$$

Therefore V is a compact selfadjoint operator.

We are in the situation of sum of a bounded selfadjoint operator B and a compact selfadjoint operator K. We look at

$$(s-B-K)^{-1} = \frac{1}{s}(1-B/s-K/s) = \frac{1}{s}(1-K/s)^{-1}\sum_{k=0}^{\infty}(B/s)^k$$

Let's look at this formula for $B = L$. Outside $[0,2]$ we have convergence of Neumann series. The nice thing is that if $s \notin \sigma(K)$ we have bounded $(1-K/s)^{-1}$ so we have s is in the resolvent of $B + K$.

2.1. **The Irrational Averaging Of Hermann Weyl.** I was not sure for while how to prove that L will have spectrum $[0,2]$. It's symmetric, so $\sigma(L) \subset \mathbf{R}$, so that's fine. It's also not so bad to show $\|L\| = 2$. We'll come back to these issues. These are not really deep.

What is *mathematical substance* here is that for all $s \in [0,2]$ nothing is in the resolvent of L. That's not simple at all. It is here that we recall the beautiful Weyl Law, that if we average iterations of irrational α translated on $[0,1]$ we will have the uniform measure on $[0,1]$.

You know it was Peter Sarnak who taught us this for the first time, and I think he is a truly great mathematician with a keen nose for mathematical substance.

So what I will do is the following. I will let $F : L^2(\mathbf{T}) \to \ell^2(\mathbf{Z})$ denote the Fourier transform, with F^{-1} be the inverse. Then I will consider the operator $A = F^{-1}LF$. This is a beautiful operator on $L^2(\mathbf{T})$ and I will assume known that spectrum of A is the same as spectrum of L. Then I will try to show that various irrational iterations are involved in translation on $[0, 2\pi]$ in A so that it has spectrum $[0, 2]$.

I know, I know, this is intuitive and vague. But refer to my commentary about the difference between mathematical rigour and mathematical substance to appreciate why I think it worthwhile to talk about the vague plan here. You see, it's true that spectrum of a symmetric operator is real but that's not real mathematical substance. This averaging of irrational iteration, on the other hand is actually mathematical substance.

2.2. **Zulf Attempts Some Jambalaya Management.** Let's say all the $f(n)$ are Fourier coefficients of some function $g \in L^2(\mathbf{T})$. We have then

$$FA = c\int_0^{2\pi} e^{i(n-1)}(e^{ix} - \frac{1}{2} - \frac{1}{2}e^{2ix})g(x)dx$$

In other words this is $n-1$-th Fourier coefficient of the function

$$k(x)g(x)$$

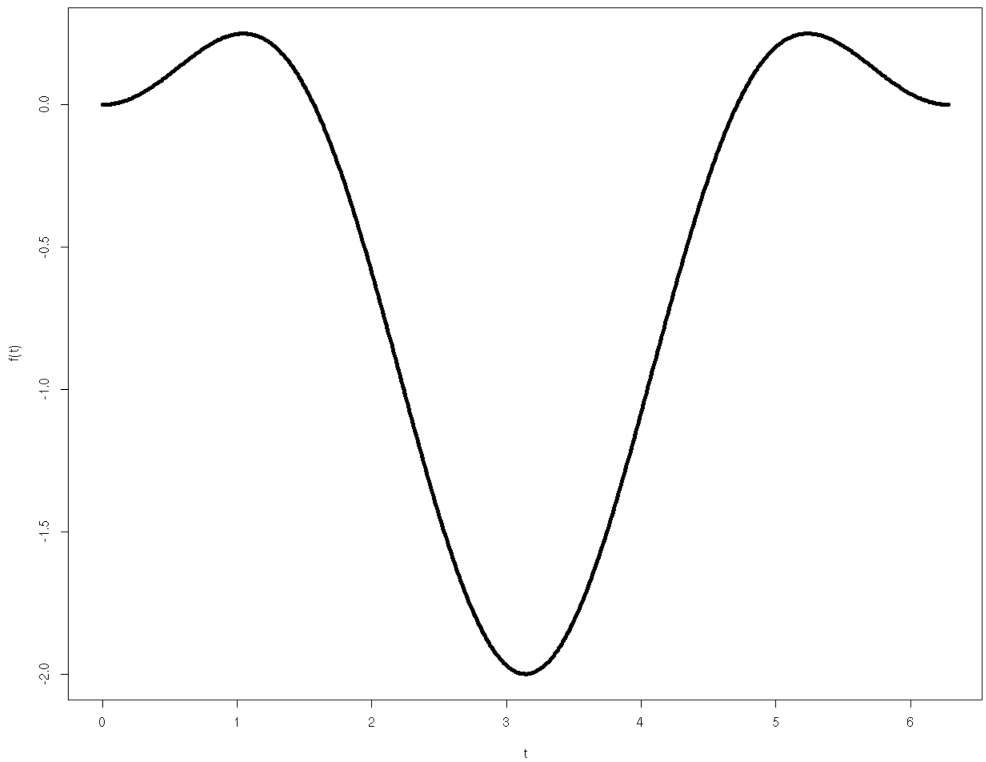
So that's the first step to understanding Problem II.5(a), the part on spectrum of L is $[0, 2]$. We need to have some sense of how to evaluate spectrum of a multiplier operator in $L^2(\mathbf{T})$.

What is the multiplier?

$$k(x) = \sum_{m=1}^{\infty}(1 - 2^m)(ix)^m/m!$$

where I was hoping that the cancellation of the constant term could help us.

That's one small step towards the spectrum of L. Let's at least look at what the range is for this $k(x)$.



That's the real part. Well maybe there is a negative sign or something but the important observation is that $k(x)$ goes from 0 to -2. One idea is to prove that or just take a look at the graph and proclaim something like spectrum of muliplication operator is range of this $|k(x)|$ but that is not yet justified.

3. Spectrum of Multiplier Functions on $L^2(\mathbf{R})$

Spectrum of multiplier functions on $L^2(\mathbf{R})$ is a fairly sophisticated analytical problem, so we do not promise a clear answer here. We conjecture that if $\phi \in C^\infty(\mathbf{T})$ then the operator $Tf = \phi f$ on L^2 has spectrum including the full range of ϕ. This sort of conjecture is easy to make and hard to prove.

Let's see what we can prove here. Suppose $x_0 \in L^2(\mathbf{T})$ is arbitrary and let $a = \phi(x_0)$. We want to prove that $a \in \sigma(T)$. The problem is that we don't have an easy *exact* eigenfunction that lives in $L^2(\mathbf{T})$. Suppose we consider

$$f_\delta(x) = \begin{cases} 1 & x = x_0 \\ 0 & |x - x_0| > \delta \end{cases}$$

Then $f_\delta \in L^2(\mathbf{T})$ for all $0 < \epsilon < 2\pi$.

Suppose $a \in R(T)$ the resolvent set. Then we have

$$a - T \in L(L^2(\mathbf{T}))$$

Let's say the bound is $B < \infty$. What we do is let $\delta < 1/B$. Then

$$(a - T)^{-1} f_\delta(x) = (a - \phi(x))^{-1} f_\delta(x)$$

For any $\epsilon > 0$ we have existence of $\delta > 0$ such that $|\phi(x) - \phi(y)| < \epsilon$ whenever $|x - y| < \delta$. Then we just pick the δ and plug it in and get

$$|(a - T)^{-1} f_\delta(x)| \geq \epsilon^{-1} |f_\delta(x)|$$

for $|a - x| < \delta$. And we get

$$\int |(a - T)^{-1} f_\delta(x)|^2 dx \geq \epsilon^{-1} \delta^2$$

This looks good. We can fanagle this using explicit bound $\|(a - T)^{-1}\| \leq B$ to get a contradiction.

Once we fix up this argument, we'll get $a \in \sigma(T)$.

3.1. **An Elementary But Useful New Theorem.**

Theorem 1 (Zulfikar Moinuddin Ahmed's Theorem For Spectrum Of Continuous Multipliers)**.** *Suppose $\varphi \in C(S^1)$ is a continuous function on the circle identified with $[-\pi, \pi]$. Let $T : L^2(S^1) \to L^2(S^1)$ be defined by*

$$Tf(x) = \varphi(x) f(x)$$

Then every value $\varphi(y)$ for $y \in [-\pi, \pi]$ belongs to the spectrum $\sigma(T)$.

Proof. Suppose $a = \varphi(y)$ for some $y \in [-\pi, \pi]$. Assume $a \in R(T)$ the resolvent set. Then there exists a $B < \infty$ such that

$$\|(a - T)^{-1}\| \leq B$$

Given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - y| < \delta$ then $|\varphi(y) - \varphi(x)| = |a - \varphi(x)| < \epsilon$.

Choose $\epsilon < 1/B$ and choose $\delta > 0$ appropriately. Define $f_\delta \in L^2(S^1)$ satisfying $\|f_\delta\|_{L^2} = 1$ with $f_\delta(y) > 0$ and $f_\delta(z) = 0$ for $|x - z| \geq \delta$. Then

$$\|(a - T)^{-1} f_\delta\|_2^2 \geq \int_{y-\delta}^{y+\delta} \frac{1}{\epsilon^2} |f_\delta(x)|^d x = \frac{1}{\epsilon^2} \|f_\delta\|^2 = \frac{1}{\epsilon^2}$$

This then contradicts

$$\|(a - T)^{-1}\| \leq B$$

Therefore $a \notin R(T)$ and $a \in \sigma(T)$.

□

Could someone at Stanford Mathematics communicate this to a journal that is widely read so that this very useful but elementary theorem is widely used? The reason this is a theorem is that it is not at all intuitively clear that all the range of values of φ must be in $\sigma(T)$ and since this is a Fourier multiplier result it is widely useful.

4. Problem II.1

(a) Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function such that for all $x \in R$ and all $k \in \mathbf{N}$ there exists a polynomial $P_{x,k}$ such that

$$|f(y) - P_{x,k}(y)| \leq C_{x,k} |y - x|^{k+1}$$

Is f infinitely differentiable?

(b) Suppose Y is a normed complex vector space with norm $\|\cdot\|$ and $f : Y \to \mathbf{C}$ is linear but not continous. Show that $N = f^{-1}(0)$ is dense in Y.

4.1. **Problem II.1(b).** Suppose, to the contrary of the claim of Problem II.1(b) we had a norm ball $B \subset Y$ where we had no zeros of f. Let's say the norm ball is $B(y_0, r)$. Let's take two points $y_1, y_2 \in B(y_0)$ with $f(y_1) \neq f(y_2)$. We can do that with our assumption by taking $y_2 = by_1$ with $b \neq 1$. Now consider the line

$$g(t) = ty_1 + (1-t)y_2$$

we get

$$f(g(t)) = tf(y_1) + (1-t)f(y_2)$$

This maps to a nontrivial interval $[f(y_1), f(y_2)]$; we could have chosen $f(y_2) > f(y_1)$ to get the ordering right.

Our goal is to somehow translate the $B(y_0, r)$ to the origin and show that the image contains an open neighborhood of zero in $\mathbf{R}$. If we are able to do that then we will have f continuous which will contradict the assumption.

I am not yet sure about this scheme, so stay tuned.

4.2. **Problem II.1(a).** My optimistic answer is that $P_{x,k}$ is forced to be the Taylor polynomial up to order k, so the I expect f to be infinitely differentiable.

Let's start with $k = 0$ and build up. We have

$$|f(y) - P_{x,0}(y)| \leq C_{x,0}|x - y|$$

This does show that f is continuous because given $\epsilon > 0$ we let $\delta = \epsilon/C_{x,0}$ and get $\epsilon > 0$ to the right. Then we iterate this. Suppose we know that $f \in C^{k-1}$ already. We consider $f^{(k-1)}$ and take also $k-1$ derivatives of our polynomial and massage

$$|f^{(k)}(y) - P^{(k)}_{x,k}(y)| \leq C|x - y|$$

and get $f \in C^k$.

We'll come back to details about how these things can work.

5. Problem II.2

Suppose A is compact and in $L(H)$. Prove that if $z - A$ is injective, then it has closed range.

Since $z - A$ is injective, we have

$$\|(z - A)x\|^2 > 0$$

for all nonzero $x \in H$. I am not sure how this problem will go, so I will explore

$$\langle (z-A)x, (z-A)x\rangle = |z|^2\|x\|^2 - \langle zx, Ax\rangle - \langle Ax, zx\rangle + \langle x, A^*Ax\rangle$$

Now if $x_n \to x$ then

$$\|(z-A)x_n\|^2 = |z|^2\|x_n\|^2 + \langle x_n, A^*Ax_n\rangle - \langle zx_n, Ax_n\rangle - \langle Ax_n, zx_n\rangle$$

Now A^*A is a compact selfadjoint operator and we want to use this somehow.

Obviously $zx_n \to zx$. Now let's see what we can do about A^*Ax_n. Well adjoints have closed range, so $Ran(A^*A)$ is closed. If z is real then we can use $A + A^*$ has closed range. I see we can use $\bar{z}A + zA^*$ has closed range. So then

$$B = |z|^2 + \bar{z}A + zA^* + A^*A$$

has closed range. Then if $Bx_n \to y$ then $y = Bx$ for some x. Then we write $Bx = (z - A)Kx$ for the right sort of K and then we use that K will be injective to recover the x' such that $(z - A)x' = \lim_n (z - A)x_n$.

This is a bit confused still but the technique can be fixed here for a sharp result. If $(z - A)$ is not injective, we could take $x_n \to x$ with $(z - A)x = 0$.

I need to think about this more.

5.1. **A Second Pass At The Problem.** This problem is nontrivial in a way because it actually requires people to know that adjoints have closed range which is not exactly automatic knowledge. You need to have some experience to know and remember that.

Suppose $T \in L(H)$ Then $Ran(T^*) = Ker(T)^\perp$. That's what you need to know. It is then obvious that T^* has closed range because if $y \in Ran(T^*)$ then for any $z \in Ker(T)$ we have

$$\langle y, z\rangle = \langle T^* w, z\rangle = \langle w, Tx\rangle = 0$$

This shows $Ran(T^*) \subset Ker(T)^\perp$. On the other hand T is injective on $Ker(T)^\perp$ and that can be used to prove that $Ran(T^*) = Ker(T)^\perp$.

Once we know that adjoints have closed range, we know if

$$Bx_n = (z - A)^*(z - A)x_n \to w$$

Then there exists $x \in H$ so $w = Bx$. Suppose then that $y_n \in Ran(z-A)$ converges to some y. Then $w_n = (z-A)^* y_n$ converges to $w = (z-A)^* y$. These $w_n \in Ran(B)$ and B is an adjoint, so $w \in Ran(B)$. Now use $(z - A)^*$ is an adjoint and then $z - A$ is injective to see there exists x such that $y = (z - A)x$. This proves $z - A$ has closed range.

If $z - A$ is not injective we'll find nonzero x with $zx = Ax$. This is then a $|z|^2$ eigenvalue of A^*A a compact self-adjoint operator. These operators have two possibilities, either $|z|^2$ is in the spectrum for discrete set or they are in the resolvent. Here $|z|^2$ must lie in the spectrum. But the spectrum will have the property that $|z_j|^2 \to 0$.

Obviously $|z|^2 - A^*A$ has closed range because it's an adjoint. I am not sure about the last question yet.

5.2. **Now For Some Song And Dance And A Picture of Sweaty Face.**

This goes with the Prince song *When Doves Cry*.

Dig if you will the picture
Of you and I engaged in a kiss
The sweat of your body covers me
Can you my darling
Can you picture this?
Dream, if you can, a courtyard
An ocean of violets in bloom
Animals strike curious poses
They feel the heat
The heat between me and you

Fine the reality is that the second part of Problem II.2 is giving me beads of sweat and there are no oceans of violet in bloom at all. Are you satisfied?

5.3. **On The Eigenspaces.** Consider $z - A$ when it is not injective. Then there are x with $zx = Ax$ so we have some eigenvectors of A. Then we have $A^*Ax = zA * x$. Now the situation for A^*A is known. Either $|z|^2$ is an eigenvalue or $|z|^2$ is in the resolvent.

In the first case, x can be an eigenvector of $A^*Ax = |z|^2x$. In this case we have

$$|z|^2 x = zA^*x$$

and $\bar{z}$ is the eigenvalue of A^* and x is an eigenvector of A^*. If $|z|^2$ is not an eigenvalue of $A^*A = AA^*$. Actually if $A^*A = AA^*$ then we have A^*x is an eigenvector of A.

Now the eigenspaces of A^*A are finite dimensional and *orthogonal to each other* by the spectral theorem for compact selfadjoint operators. We want to know if

$z-A$ is invertible on $Ker(z-A)^{\perp}$. The entire space is composed of subspaces with $|z_j|^2 w = A^*Aw$.

Fine, let's roll up our sleeves here and let's say the A^*A has orthonormal bases w_{jk} with eigenvector $|b_j|^2$ and they span H. Then let's do a computation assuming

$$w_n = \sum_{jk} a_{njk} w_{jk}$$

where k counts the degeneracy at eigenspace associated to $|b_j|^2$. Then

$$(z-A)w_n = \sum_{jk} a_{njk}(z-A)w_{jk}$$

I am not seeing this yet. What is $z-A$ doing on these eigenspaces.

This is progress. See, in four-sphere theory, I have been doing some basis expansions for Dirac eigenspinors on S^4 and I have the intuition that these $z-A$ has some reasonable behavior on eigenspaces of A^*A and then we can say something about range of $z-A$ such as it's closed.

Unfortunately I don't have something yet.

6. Vacuousness Of Bill Gates White Superiority

Well look, all men are created equal has basis in genetic code in common G_c which constitutes 99.9% of everyone's genome and is letter-by-letter identical for all people everywhere.

But Bill Gates' special brand of white superiority is provably vacuous.

Consider the scenario.

Zulf: Bill Gates, how is the weather there in podunk Pacific Northwest hickslandia?

Bill Gates: I'm white. I'm white. Whites are superior. Whites are superior.

Zulf: Well that's really great. That's really great that whites are superior. Now can you, as a superior white man, Master Race, etc. please tell me how to do second part of Problem II.2?

Bill Gates: Trivia. Trivia. Trivia.

Zulf: Ok, fine fine. Trivia. What's the answer of the trivia?

Bill Gates: Whites are superior. Whites are superior.

Zulf: Groan.

7. Let Me Tell You What I Am About

I am a man of faith. I believe that I have been an Archangel in the universe for billions of years, and have lived in the Universe, and that I am on Earth, by ordinary birth to a family in Indian Subcontinent on November 19 1973. I believe that I am on Earth to benefit the Human Race but really just on vacation. I believe that when I die, I will regain my Archangel Consciousness and begin a trek to the next world. I believe that the universe is filled with many worlds, planets like Earth and many races. Human beings are a *Primitive Angelic Race*. The reference for primitivity is not found on Earth. There are worlds with Angelic Species like humans who are much much older than seven or eight million years. So primitivity is relative to them. I believe that I have lived in thousands of these worlds before my birth on Earth. I believe that among Archangels of Heaven I have seniority and that it does not matter on Earth at all because that's a social issue among Archangels of Heaven and has no bearing at all on human society.

I believe that I ought to have my natural rights on Earth protected. I have proven that (a) Human Race is a single race and (b) the human race is an Angelic Race. I am Asian-American, but I am not strongly tribal about Asia per se. I am not racial but I do not appreciate Bill Gates' extreme hateful and destructive views and direct targeting of myself for destructive war acts. He is a Demon and outght to be destroyed immediately and his flesh ought to be consumed in fire to prevent spread of extreme evil among the human race.

Four-sphere theory is absolute truth. It is one of my gifts to my beloved people the human race. I have already given a completion of Aristotle's Virtue-Eudaimonia theory to United Nations which contain secrets for autonomous life satisfaction.

I have worked in Finance, Technology and Biotech and am not even interested in \$350k/year jobs any more. I have grander plans of establishing Quantitative Positive Psychology and Global Individual Debt projects, preferably with a tenure and full professorship at Stanford.

Bill Gates invaded my Deep Interior with devastating destructive power and I will hold United States Government responsible for all harm to me and seek \$1-2 trillion in penalty for this. I also plan to litigate against United States Government for involuntary medical incarceration in the period January 12-19 2022 for claims of hallucination and for frivolous charges based on the following poster. I have never had any firearms and do not really like guns.

It's true that Bill Gates ought to be physically destroyed and his flesh burned with purifying fire to restrict contagion of extreme Evil that might ultimately destroy the human race. I expect the United States Government to do those sorts of dastardly deeds, not myself.

8. Problem II.3

Let $E \subset \mathbf{R}$ and let $E+E = \{x+y : x, y \in E\}$ and define $E-E$ similarly. Prove that if $m(E) > 0$ then $E+E$ and $E-E$ contain nonempty open sets.

8.1. Beginning of Exploration.

I have very little intuition about set operations in Lebesgue theory. Honestly, I never considered these things all that important. I remember Littlewood's three principles, (a) a measurable set is almost an open set, (b) a measurable function is almost a continuous function, and (c) every convergent sequence of functions is almost uniformly continuous.

I have always been more geometric, and never really liked hacking with the wiring, so to speak. But I am moved to do some of this in this late age at 49 because my precious baby, four-sphere theory, seems to need analytical care. Lebesgue measurable sets of S^4 are important and so here I am doing these problems.

The first track is to see if we can do the following. Given any $\epsilon > 0$ we can find open intervals I_j so that

$$E \subset \bigcup_j I_j$$

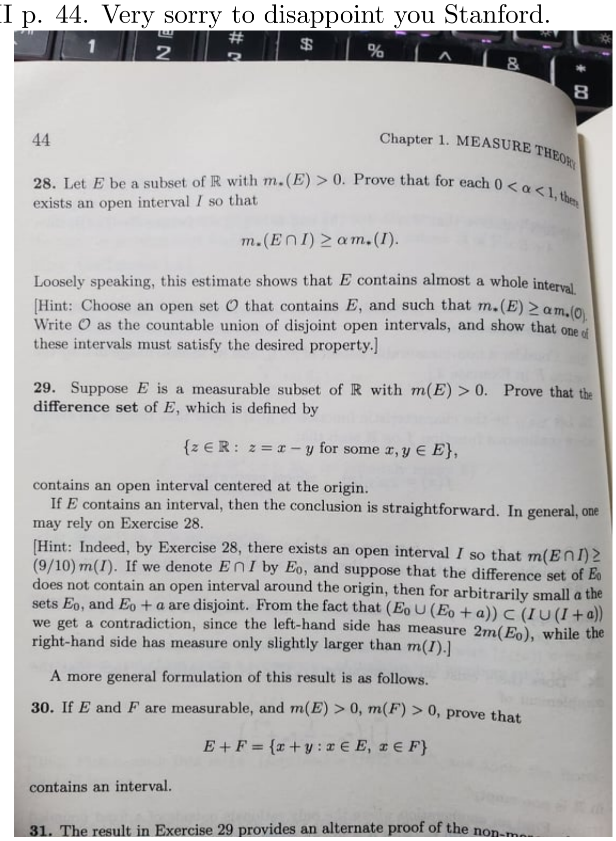
and more importantly

$$m(\bigcup_j I_j - E) < \epsilon$$

That's part of Lebesgue measure theory, and follows from the definition of Lebesgue measurability. We are interested in understanding what we can do with this covering of E.

Our hope is that $O = \bigcup_j I_j$ will have something to contribute to $E + E$ and $E - E$. This is aimless exploration. We consider $O + O$ first.

I am not going to do this problem and just learn how to do it from Stein-Shakarchi III p. 44. Very sorry to disappoint you Stanford.



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28. Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha\, m_*(I).$$

Loosely speaking, this estimate shows that E contains almost a whole interval.

[Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*(E) \geq \alpha\, m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

29. Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the **difference set** of E, which is defined by

$$\{z \in \mathbb{R} : \ z = x - y \text{ for some } x, y \in E\},$$

contains an open interval centered at the origin.

If E contains an interval, then the conclusion is straightforward. In general, one may rely on Exercise 28.

[Hint: Indeed, by Exercise 28, there exists an open interval I so that $m(E \cap I) \geq (9/10)\, m(I)$. If we denote $E \cap I$ by E_0, and suppose that the difference set of E_0 does not contain an open interval around the origin, then for arbitrarily small a the sets E_0, and $E_0 + a$ are disjoint. From the fact that $(E_0 \cup (E_0 + a)) \subset (I \cup (I + a))$ we get a contradiction, since the left-hand side has measure $2m(E_0)$, while the right-hand side has measure only slightly larger than $m(I)$.]

A more general formulation of this result is as follows.

30. If E and F are measurable, and $m(E) > 0$, $m(F) > 0$, prove that

$$E + F = \{x + y : x \in E,\ x \in F\}$$

contains an interval.

31. The result in Exercise 29 provides an alternate proof of the non-me...

Let me tell you something. Elias is Stein was the world's master at these sorts of things and Zulf is wise and does not try to do these things at all and re-invent the wheel. Chalk this up for Zulf knows where to find these sorts of answers.

9. Problem I.1

Suppose H is separable Hilbert space. Then show that $B(H, H)$ is separable in operator norm topology.

10. Problem I.2

Define $M_f : L^2(S, \mu) \to L^2(S, \mu)$ by $M_f = fg$ for $f \in L^\infty(S, \mu)$ and (S, μ) is a sigma-finite.

(a) Give necessary and sufficient conditions for M_f to have an eigenvector.
(b) Give necessary and sufficient conditions for M_f to be compact.

10.1. **Problem I.2a.**

Suppose $M_f g = zg$ for some $z \in \mathbf{C}$ and some $g \in L^2(S, \mu)$ then

$$(f - z)g = 0$$

so either $f = z$ or $g = 0$. So f must be constant.

10.2. **Problem I.2b.**

Suppose $\|g_n\| \leq B$ and we want a subsequence of fg_n converging in $L^2(S, \mu)$.

A sufficient condition is that f is supported on a finite number of points. If the support is $x_1, \ldots, x_N$ then we can use the Heine-Borel Theorem to produce a sieve that converges for g_n at those points.

The analogy is the compactness of $H^1 \to L^2$ or Arzela-Ascoli theorem. If you cannot localise to a finite number of points then there can be no subsequence convergence.

11. Problem I.3

(a) Suppose x_n is a sequence in a Banach space and let $X_n = Span\{x_1, \ldots, x_n\}$. Prove that if

$$X = \bigcup_{n=1}^\infty X_n$$

then X is finite dimensional.
(b) Prove that if

11.1. **Failure of Intuition.**

The idea that you cannot just approximate a space by taking spans of more and more vectors is outrageous to my intuition. And this is the sort of thing that I really dislike.

The conclusion of (a) is in the end a simple consequence of Baire Category theorem. If $X = \bigcup_n X_n$ then one of these have to have empty interior, but they are all finite dimensional and therefore X must be as well.

I am devastated by this because I am already 49 years old and suddenly infinite dimensional linear spaces seem a lot spookier to me. It's true. You can't keep going on adding spans of more vectors and reach the full space by unions.

That's just too strange for me and it's true. Who was this René-Louis Baire? This result is so shocking but it's true. I suddenly feel sad because I don't feel as comfortable with Banach Spaces any more. Brrrrr.

(b) Show that if X has countable base in its weak topology then X is finite dimensional. Let X have neighborhood basis of zero given by U_α in weak topology. These are intersections of *finite number* of linear functionals $f_j \in X^*$ satisfying $f_j^{-1}((-\epsilon, \epsilon))$.

Let's simplify and assume that X is reflexive so $X \simeq X^*$. The finite number of linear functionals correspond to finite number of of points $x_1, \dots, x_N \in X$ and then we want to say $X = \bigcup_n X_n$ as in (a).

I do not have a very sharp idea for this problem because I will admit that here I have a severe weakness. I need to adjust my intuitions which are wrong about Banach spaces because I was so used to geometric thinking that it did not occur to me that these Banach spaces are not the sorts of things I would like them to be at all.

I am going to go back to Reed-Simon and Brezis to try to adjust my intuition here. This is just very devastating for me. I had such *beautiful* geometric intuition, and suddenly I feel small and finite dimensional and I don't like this feeling at all.

Chapter III of Reed-Simon deals with Banach Spaces and the first few of Brezis' book. I also have Riesz-Nagy but this is quite devastating, let me tell you. I treated Banach Spaces so well over the years. Why did they have to be this way?

12. Zulf Is Now Concerned About Problem I.3

Are the conclusions of Problem I.3 really true? Baire Category Theorem only tells us that X is not a countable union of nowhere dense sets. If $X = \bigcup_j A_j$ then some of the $\bar{A}_j$ must have nonempty interiors.

Suppose I take $L^2(S^1)$. Then what is wrong with taking the basis e^{inx} and letting

$$A_j = Span\{e^{inx} : |n| \leq j\}$$

Then I want to say $L^2(S^1) = \bigcup_j A_j$. Is this not true? Oh I see, an arbitrary element of $L^2(S^1)$ has nonzero coefficients in all of the e^{inx} while $\bigcup_j A_j$ only contains the elements with finite non-zero coefficients. The union will not exactly *ever reach* $L^2(S^1)$.

I see this. One should not consider limits of unions for approximations at all. Finite dimensional subspaces have to be treated with *analysis* and not *set theory*; we have to use something like: for $\epsilon > 0$ we have

$$|x - \sum_{|n| \leq N} x_n e^{inx}| \leq \epsilon$$

but we can't really fill up the ϵ with unions.

I start to understand this. Unions in this case are less refined set theoretic operations and are only technical constructions. In the Analysis one does not use them at all.

See, this is a serious subtlety. Set theoretic constructions are not as powerful in Analysis of infinite dimensional spaces.

No wonder all these weak topologies and other things are always causing problems for me. Take X a Banach space and consider the weak topology $\mathcal{T}$. The neighborhood base at zero consists of

$$N(f_1, \dots, f_N; \epsilon_1, \dots, \epsilon_N) = \{x : |f_j(x)| < \epsilon_j, f_j \in X^*\}$$

Lets call these N_α. I am not sure how to make this precise yet, but the point of Problem I.3(b) is that one should be able to take countable union of these to fill up the entire space and then deduce something about X being finite dimensional.

Now for (b) I am more confused because sets that are intersections of sets like $|f_j(x)| < \epsilon_j$ look like what? They are linear functionals, so the zero set of $f_j(x)$ is a hyperplane that is infinite dimensional, morally codimension 1 subspace. So these are hyperplanes with a little bit of thickness to them.

How do we get span of finite number of vectors from these? This is extremely murky at the moment to me.

13. The World Needs Much Clearer Details On Weak Topology Issues

Weak topology on Banach Spaces and Locally Convex Spaces, the development of intuition about how to deal with them, the arguments and techniques that have produced some actual results, and the adjustment of dealing with the disjunction of intuition that ordinary mortals have, these things I feel are in total disarray and anarchy.

I have been studying from Reed-Simon and Haim Brezis, and let me tell you right away that none of these weak topology arguments seem to be to be anything but technical mumbo-jumbo. I am deeply uncomfortable with them still because I don't have any sense of the naturality in this arena. I am literally feeling like a fish out of water and soon I will have all sorts of psychological paranoia, like weak topologies from another galaxy will begin to invade Earth.

In this case, frighteningly enough, *Bill Gates* who has an agenda to *literally enslave and destroy all non-white people* and is one of the most Evil horrible savage disgusting monsters who makes Adolf Hitler seem like a saint – no exaggeration intended here at all and I am serious – is the salvation. I regain my human wisdom and put things in perspective and suddenly I realise that it's *okay to consider all this technical mumbo-jumbo the realm of all sorts of technicians.* No law of Nature forces me, Zulfikar Moinuddin Ahmed, to suddenly at 49 grow functional analysis topology wings and fly through the arcane technicalities that were probably all plotted and foisted upon the innocent people by Jon Von Neumann out of malicious pleasure of schadenfreude, of other people's sufferings.

But the sad truth is that I am actually willing to learn this material. On my mind is only one thing, that my precious baby, my *four-sphere theory* might not survive if papa does not nourish it with care. Square integrable and L^p spinor fields on the four-sphere might need some of this mumbo-jumbo. So I swallow my intense discomfort and heroically push forward. What sacrifices have I not endured for my baby?

And so these are the ways that we human beings keep pursuing things that are perhaps not really meant for human beings at all.

Now I ask the Earth, "Oh Earth, what is this issue of first countable weak topology basis of Banach space forces it to be finite dimensional?" I ask the Heavens, the Sky and the Ocean, and to no avail. They remain silent. They offer no solace for my distress. Such is the loveless cold world.

14. Barry Simon's *Real Analysis I*

Thank Heavens for Barry Simon. Some years ago, I bought his volumes *Comprehensive Course In Analysis.* Let me show you the pictures of Sections 3.6 and 5.7 dealing with weak topologies.

Yes, it matters that I have actual physical hard copies of the books that I am studying.

and $P_n^\star(e^{i\theta})$ means $\frac{d}{d\theta} f(\theta)$ with $f(\theta)$ ...

(b) Find an explicit formula for F_n (not as a sum) an...
and $\int F_n(\theta)\,\frac{d\theta}{2\pi} = n$.

(c) Conclude that

$$\sup_{\theta\in[0,2\pi]} |P_n'(e^{i\theta})| \leq n \sup_{\theta\in[0,2\pi]} |P_n(e^{i\theta})|$$

(This is known as *Bernstein's inequality.*)

3.6. The Weak Topology

The central parts of this chapter are concluded. The final
hint at further directions. In this section, we introduce
and explain the basic compactness result that is respons
This material will be largely subsumed in the discussio
nach space case in Sections 5.5 and 5.7. In the next se
some special classes of operators, whose discussion will
Part 4 and in the final section, we discuss tensor produ

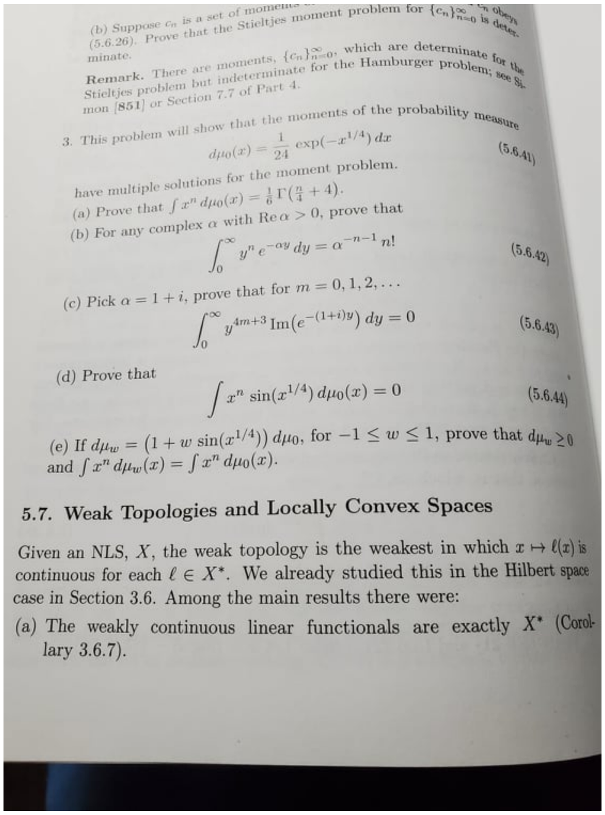
So far, the only topology we've put on a Hilbert sp
(aka strong) topology, where a net, φ_α converges to φ if a
$\varphi\| = 0$.

Definition. The weak topology on $\mathcal{H}$ is the weakest top
for all $\psi \in \mathcal{H}$, $\varphi \mapsto \langle\psi, \varphi\rangle$ is continuous.

Thus, a neighborhood base for $\eta \in \mathcal{H}$ is given by

$$\{\varphi \mid |\langle\psi_1, \varphi - \eta\rangle| < \varepsilon_1, \dots, |\langle\psi_n, \varphi - \eta\rangle| < \varepsilon_n\} \equiv N_\psi$$

as $\psi_1, \dots, \psi_n$ runs through all n-tuples in $\mathcal{H}$, $\varepsilon_j > 0$, bu
and $n = 1, 2, \dots$. Equivalently, a net ...



(b) Suppose c_n is a set of moments ... obeys (5.6.26). Prove that the Stieltjes moment problem for $\{c_n\}_{n=0}^\infty$ is determinate.

Remark. There are moments, $\{c_n\}_{n=0}^\infty$, which are determinate for the Stieltjes problem but indeterminate for the Hamburger problem; see Simon [**851**] or Section 7.7 of Part 4.

3. This problem will show that the moments of the probability measure

$$d\mu_0(x) = \frac{1}{24}\exp(-x^{1/4})\,dx \tag{5.6.41}$$

have multiple solutions for the moment problem.

(a) Prove that $\int x^n\,d\mu_0(x) = \frac{1}{6}\Gamma(\frac{n}{4}+4)$.

(b) For any complex α with $\operatorname{Re}\alpha > 0$, prove that

$$\int_0^\infty y^n e^{-\alpha y}\,dy = \alpha^{-n-1}\,n! \tag{5.6.42}$$

(c) Pick $\alpha = 1+i$, prove that for $m = 0, 1, 2, \dots$

$$\int_0^\infty y^{4m+3}\operatorname{Im}(e^{-(1+i)y})\,dy = 0 \tag{5.6.43}$$

(d) Prove that

$$\int x^n \sin(x^{1/4})\,d\mu_0(x) = 0 \tag{5.6.44}$$

(e) If $d\mu_w = (1 + w\sin(x^{1/4}))\,d\mu_0$, for $-1 \le w \le 1$, prove that $d\mu_w \ge 0$ and $\int x^n\,d\mu_w(x) = \int x^n\,d\mu_0(x)$.

5.7. Weak Topologies and Locally Convex Spaces

Given an NLS, X, the weak topology is the weakest in which $x \mapsto \ell(x)$ is continuous for each $\ell \in X^*$. We already studied this in the Hilbert space case in Section 3.6. Among the main results there were:

(a) The weakly continuous linear functionals are exactly X^* (Corollary 3.6.7).

You see, if people don't know that you are actually being serious and examining these issues with some idea of where the answers might come from, they can't take you seriously.

I simply don't understand what is really going on in Problem II.3 and I don't think it's just a minor issue. I need to have some sense of how to manage weak topologies and I don't know enough about them to do this.

14.1. **Lemma 3.6.5 and 3.6.6 of Barry Simon.**

Lemma 1. *If X is any vector space and $a \in \mathbf{R}^n$ and $\ell_1, \dots, \ell_n$ are linear functionals that are linearly independent, then there must exist $x \in X$ such that $(\ell_1(x), \dots, \ell_n(x)) = a$.*

Proof. Let $Q = \{(\ell_1(x), \dots, \ell_n(x)) : x \in X\}$. If $Q \neq \mathbf{R}^n$ then let $b \in \mathbf{R}^n$ and for all $x \in X$ we have $\sum b_j \ell_j(x) = 0$ and that contradicts linear independence. □

Lemma 2. *Suppose X is a vector space and $\ell_1, \dots, \ell_n$ are linear functionals. Now suppose ℓ is another linear functional such that*

$$\{x : |\ell(x)| < 1\} \subset \{x : |\ell_1(x)| < \epsilon, \dots, |\ell_n(x)| < \epsilon\}$$

Then ℓ is a linear combination of ℓ_j.

Proof. Suppose $(\ell, \ell_1, \dots, \ell_n)$ are linearly independent. Then we can find some x_0 with $\ell(x_0) = 2$ but $\ell_j(x_0) = 0$. The latter puts x_0 in the set in the smaller set $\{x : |\ell_j(x)| < \epsilon\}$ but the first condition says x_0 is not in the bigger set. Therefore $\ell = \sum a_j \ell_j$. □

Now 3.6.4 of Barry Simon seems to be exactly the right sort of thing to understand. It says that in a *Hilbert space* if x_n is countable then there is a y that is not a *finite linear combination* of elements of $\{x_n\}$. Let us try to prove this for a *Banach Space*. Here we need to know a result that says that if $K \subset X$ is any linear subspace then there is a $y \in X$ with $\inf_{k\in K} \|y - k\| \geq 1/2$. Let $K_n = Span\{x_1, \ldots, x_n\}$.

Let's assume that this is known for a Banach space. The proof of Barry Simon's 3.6.8 seems to contain the main issue of this finite dimensionality business.

That is progress for Problem I.3(b). We note that this requires some unusual mixing of linear algebra and these weak sets. In other words, this is technical mumbo-jumbo turf for me.

15. February 1 2022 Another Attempt At Problem I.3(b)

I don't really worry too much about failures as learning entails failures. The issue is that we have a Banach Space X and we are given that there is a countable base of neighborhoods at every point. We are give an hint to show that the dual space has some property.

So let's return to this issue. By assumption, there is a countable base of neighborhoods at zero in X. In other words there is a collection U_k of neighborhoods of zero such that for any open set W containing zero we will have $U_k \subset W$.

I won't even try to get a sharp proof of this result. Let us assume that not only that we have a countable set of neighborhoods U_k but in addition that associated to each U_k there exists $\ell_1^k, \ldots, \ell_{N_k}^k \in X^*$ and also $\epsilon_1^k, \ldots, \epsilon_{N_k}^k$ such that all of the U_k are in the standard form

$$U_k = N(\ell_1^k, \ldots, \ell_{N_k}^k, \epsilon_1^k, \ldots, \epsilon_{N_k}^k)$$

I will not attempt in this qual exam to go through the issue of how weak topology *by construction* allows us to choose a *finite* number of linear functionals to define the neighborhood bases.

This is one of the things I really really hate about weak topologies on Banach Spaces X. Almost no one really spells out clearly why these steps are justified, and no one really wants to write them either and so they are repeatedly put in exercises for the reader in most presentations. And I am 99% sure that no ordinary normal human being finds any of this all that interesting so naturally not many people have any clue about what is going on. This is part of life, where no one wants to do it and then people pull rabbits out of a hat and put everything under the rug. That's what happens when Jon Von Neumann introduces these things in 1929. People know it's important but everyone is interested in other, naturally much more important, things and so no one spends any time being very clear about what is going on. And in the end Zulf has to pay for it because Zulf does not have any interest in spending a lot of time on such technical mumbo-jumbo either but is forced to because I don't know why any of the propositions are valid at all and then I feel guilty about using the theorems. I am not *Bill Gates* who lies constantly about all things and pretends that while he has been pretending to be liberal and normal in his public appearances his whole entire life is fake and in actuality his entire life revolved around white power and white superiority and pretending that being white allows you magical genius without effort and he is proud of his work in destroying my career and livelihood, health and earning, personal life because he feels that all other white people are like that too and don't talk about it. I simply *cannot even believe that United States Government did not send in lethal force to nuke his ass sky high the moment he was born. They will obviously pay, and their cost will be \$1-2 trillion just for this insolence against Zulf.*

But let us return to the issue at hand. We want to prove that first countability of weak topology on X leads to a conclusion that $X^* = \bigcup_j X_j$ where X_j is a span of a finite number *of linear functionals.*

Let's examine first what each of the U_k are giving us. Barry Simon's Lemma 3.6.6 would tell us that if any weakly open set defined by a single functional ℓ contains U_k then it will be a linear combination of $\ell_1^k, \dots, \ell_{N_k}^k$. This suggests that we ought to obtain X^* as the following sort of thing:

$$B_k = Span(\bigcup_{p=1}^{k} \{\ell_1^p, \dots, \ell_{N_p}^p) \tag{1}$$

I have not told you how to get this yet. Be aware that I am confused about these issues, and just exploring. I am seeking the right sort of approach blindly. Now if we did manage to secure $X^* = \bigcup_k B_k$ then we can apply Problem I.3(a) to X^* and conclude it is finite dimensional and since $X \subset X^{**}$ and the right side is finite dimensional, so is X and we're done with Problem I.3(b).

But this translation (1) *is the heart of the problem. I am not even sure if* (1) *is true yet.* That *is the problem. Manipulation of weak neighborhoods and making conclusions about them is* murky. *The reason is quite clear. Jon Von Neumann was really probably from another galaxy and had absurd abilities with arcane knowledge that no human being was ever able to penetrate but we pretend that we understand what he did. I am an honest man, and I am sure I don't understand these arcane machinations. And that is why we are in the situation of confusion right now.*

16. Search For Clarity For Weak Topology On Banach Spaces

Let X be a Banach Space. We are concerned about the Weak Topology*, let us call it $\mathcal{T}_w$ on X.*

There is nothing so totally arcane as this sort of game one plays between set theoretic arguments and firm conclusions about linear subspaces of X^*.

It is so infuriating that I am just going to try to clear up these issues. We made progress before. Let us assume known that a neighborhood base at zero for $\mathcal{T}_w$ consists of sets of form

$$N(\ell_1, \dots, \ell_N, \epsilon_1, \dots, \epsilon_N) = \{x \in X : |\ell_j(x)| < \epsilon_j, 1 \le j \le N\}$$

We get to this by rigorous construction of the weak topology as the coarsest topology that ensures that all *the linear functionals $\ell \in X^*$ will be continuous as functions $\ell : X \to \mathbf{R}$ with the ordinary topology on $\mathbf{R}$. So there is no deep theorem involved for why neighborhood base elements only involve* finite *number of functionals. This was quite confusing for me some weeks ago in another Stanford Qual that I was doing. I was then wondering, "Huh, what? How do we know that only finitely many linear functionals are guaranteed to produce the topology $\mathcal{T}_w$?" It was some work before Haim Brezis' presentation cleared up the issue and it was a matter of technical matter of finite intersections are in topologies generally.*

Note that this has nothing to do with first countable or whatever other qualities we are interested in examining. Every weak topology *of a Banach space always has a neighborhood base at zero of this form. Once the form is known, we say, "And* *then we vary* ℓ_j, ϵ_j *over all possible elements of* X^* *and* $(0, \infty)$ *and this gives us the base.*

Now let us specialise to the case at hand, that we know that $\mathcal{T}_w$ *is first countable. What changes here is that we do* not *go through all possible* $\ell_j \in X^*$ *and* $\epsilon_j \in (0, \infty)$ *any more. Instead we have a countable set.*

At this point we can identify our countable base $\{U_k\}_{k=1}^{\infty}$ *with form and get*

$$U_k = N(\ell_1^k, \ldots, \ell_{N_k}^k, \epsilon_1^k, \ldots, \epsilon_{N_k}^k)$$

Look, we are not trying to be rigorous here. We are instead trying to understand what is going on.

See, there is a maneuver we need to do for Problem I.3(b) and I don't understand the particular clear steps to make the maneuver justified. The maneuver has to start with this countable base and reach some conclusion about X^* *but in a careful manner so that we can actually prove something.*

Suppose $\ell \in X^*$ *is arbitrary. Aha. That is the key point. When* $\ell \in X^*$ *is* arbitrary *the weak topology* $\mathcal{T}_w$ *ensures that it is continuous. This means that*

$$\{x : |\ell(x)| < \epsilon\}$$

must contain a U_k*. This will work. This implies*

$$\ell = \sum_r a_r \ell_r^k$$

Now when we go over all elements of $\ell \in X^*$ *we see that*

$$X^* = \bigcap_{k=1}^{\infty} Span(\ell_1^k, \ldots, \ell_{N_k}^k)$$

This is a valid step. It is an extremely annoying technicality, but this is now reasonable. Then we apply Problem I.3(a) and conclude, using Baire Category Theorem, that X^* *is finite dimensional because one of the spans must have non-empty interior.*

I don't even understand how this happened. We have a valid end to Problem I.3(b).

Let me tell you what is so confusing. These arguments rely on one having a sharp sense of what the neighborhood bases are generally, and what they are when someone says 'assume countable' and then knowing esoteric lemmas from the depths of Barry Simon's Comprehensive Course In Analysis *and then using all sorts of topological arguments.*

Whew. I'm glad this is done now. I thank Barry Simon for writing about this. Who would have cooked up this? Never mind it was Jon Von Neumann. I will fast exit from this weak topology business. Hahaha I live! I'm still alive after this harrowing experience!

17. Problem I.4

I am going to do a simple computation here before getting to the problem. Let $g(x) = e^{-iax^2}$ *and let's calculate the one-dimensional Fourier transform.*

$$\begin{aligned} ax^2/2 + x\xi &= a/2(x^2 + 2x\xi/a) \\ &= a/2(x^2 + 2x\xi/a + (\xi/a)^2) - a/2(\xi/a)^2 \\ &= a/2(x + \xi/a)^2 - a/2(\xi/a)^2 \end{aligned}$$

So the Fourier transform of g looks like

$$e^{ia/2(\xi/a)^2}\int e^{-ia/2(x+\xi/a)^2}dx$$

Then we do a substitution $u = x + \xi/a$ and claim

$$C = \int e^{-iu^2}du$$

the end result will be

$$\hat{g}(\xi) = Ce^{ia/2(\xi/a)^2}$$

We're going to take this as our model for the Problem I.4. Since $det(A) \neq 0$ we just use A^{-1} without comment. We will comment on convergence issues later.

First let's just give the answer.

$$\hat{f}(\xi) = Ce^{i\frac{1}{2}\langle A^{-1}\xi, A^{-1}\xi\rangle}$$

That's the answer after the various steps are justified. I will examine the various steps later on.

18. Interlude: Commentary On My Development

In the past months I had been doing some of the Ph.D. Qualifying Exams of Stanford University. I think it is worthwhile to examine some of the strengths and weaknesses that I can detect in my own abilities as a result.

One broad theme is that I am much more comfortable with more eighteenth and nineteenth century mathematical issues than more modern ones in Analysis. This I attribute to my geometric focus from Princeton years. These are double-edged swords, in a way. If I had spent more time developing in modern analysis, i.e. Lebesgue integration theory and Distribution theory, I would probably have not instead been naturally led to four-sphere theory. And that would have been unfortunate. Four-sphere theory is, I am extremely sure, one of the most momentous events in the entire 350 years of Science, although it is not yet seen to be so. It is only after empirical success was certain in four-sphere theory after more than 15 years of labour without income, without a reasonable life in America, living on disability, that success was apparent. And it invited all the predators and the parasites to destroy me, the foremost among them this horrid and evil, savage and barbaric, malevolent and destructive Bill Gates.

In terms of mathematical education, it is clear that my weaknesses have a historical regularity. Weak topologies were introduced in 1929 by Jon Von Neumann, and first Sergei Sobolev and then Laurent Schwartz introduced distribution theory between 1935-1945. This gives me some solace, for I was wondering why these things seem suddenly unfamiliar and I am lost with these. They represent a different mode of mathematical thought than the issues that I understand better which are still pre-Lebesgue and pre-Schwartz.

It is clear that I ought to gain better education on these issues, and I was looking at Robert Strichartz' book. I suddenly gain clarity regarding my own path to establish four-sphere theory. Some days ago I realised that four-sphere theory naturally suggests that distributions that will have existence in nature ought to be ubiquitous. These are in the dual of smooth spinor fields on the four-sphere.

I can see conceptually that distribution theory and measurability, as Robert Strichartz sees them, completions *of integration theory and differential calculus belongs quite* *naturally in four-sphere theory in the actual objects of nature, in the scientific theory itself.*

And for this reason I am willing to put in some effort in learning about these to some level of competence. Both measurable *spinor fields and* distributional spinor fields*, I hypothesize must exist in nature.*

In Robert Strichartz' book A Guide To Distribution Theory And Their Fourier Transforms *he often mentions that in many situations of partial differential equations often there is no reason to* prefer *a classical solution over a distributional solution* [2].

I am not a professional Mathematician but a Scientific Revolutionary and so I am not actually advocating any particular interpretation of orthodox theory that is established at all. My concern is to insert distributional solutions are physically meaning in all cases in four-sphere theory.

In other words I am not advocating anything at all about orthodox established theories. I am mandating *that in four-sphere theory distributional solutions of Ahmed-d'Alembert Equation are physically meaningful as part of four-sphere theory. I did not gain the idea from Strichartz or Laurent Schwartz who are Mathematicians. My reasoning is based on my philosophy of Mathematics and exact representability of Nature.*

I won't go through the issues again in detail. When the fundamental equation is a wave equation, it is intuitively clear that Nature *will produce an abundance of distributional solutions that are not 'classical solutions'. That's just the sort of equations wave equations are.*

19. Problem I.5

I will change notation to make things simpler for myself. For $k \geq 0$ define spaces H^k consisting of functions f satisfying

$$|\hat{f}(n)| \leq C_{k,n}|n|^{-N}$$

for $n < 0$

$$|\hat{f}(n)| \leq C_k(n+1)^k$$

for $n \geq 0$. Suppose $f, g \in H^k$. Prove that there is a K such that the product $fg \in H^K$. Then prove that the map $(f,g) \mapsto fg$ is continuous.

The Fourier transform of the product is the convolution of the series. Let $a_n = \hat{f}(n)$ and $b_n = \hat{g}(n)$. Then

$$F(fg)(n) = a * b(n)$$

This is a delicate problem to get estimates on the convolution of the series. We will return to this issue. The more alarming question is how to prove continuity.

This is a fairly hard question. I will take some time to think about this. I think I will do this slowly to gain some experience with the setting of topologies of distributions.

In general multiplication of distributions is not continuous, so this entire setting is fairly avant garde.

20. Problem II.4

Let $X = \{f \in C^\infty[0,1] : f^(k)(0) = 0, k \geq 0\}$.

(a) Define the seminorms $\rho_{k,m}(\phi) = \sup |x^{-k}\partial^m \phi|$. *Show that the topology of these seminorms is the same as the usual seminorm topology of* C^∞ *functions restricted to* $[0,1]$.

(b) Let $Y = C^\infty[-1,1]$ *with the usual seminorm topology. Show that every linear functional* f *on* X *has an extension to a continous linear functional* $\tilde{f}$ *on* Y *with* X *identified with functions vanishing on* $[-1,0]$.

See the next section for part (b) above. I am most definitely not going to prove another version of Hahn-Banach theorem for this exam.

I have my pragmatic principle that some things are worth just quoting or looking up, like Hahn-Banach Theorem but for locally convex spaces rather than Banach spaces.

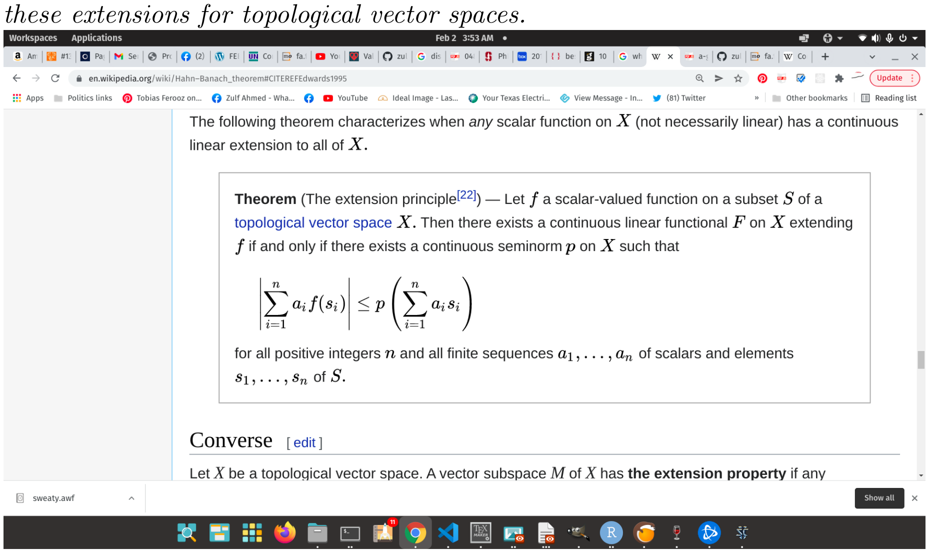
So I will consider only (a). Now equivalence of topologies given by seminorm families p_a, q_b *are established by a bound of the type*

$$|p_a(x)| \leq C(|q_{b_1}(x)| + \cdots + q_{b_N}(x)|)$$

So we can push gently into this idea for (a).

21. Search For A Hahn-Banach Theorem For Locally Convex Topological Vector Spaces

As I suspected, the Hahn-Banach Theorem has been extended in various versions ever since the original theorem was proved. Problem II.4(b) is an application of one of these extensions for topological vector spaces.



The following theorem characterizes when *any* scalar function on X (not necessarily linear) has a continuous linear extension to all of X.

Theorem (The extension principle[22]) — Let f a scalar-valued function on a subset S of a topological vector space X. Then there exists a continuous linear functional F on X extending f if and only if there exists a continuous seminorm p on X such that

$$\left|\sum_{i=1}^n a_i f(s_i)\right| \leq p\left(\sum_{i=1}^n a_i s_i\right)$$

for all positive integers n and all finite sequences $a_1, \ldots, a_n$ of scalars and elements $s_1, \ldots, s_n$ of S.

Converse [edit]

Let X be a topological vector space. A vector subspace M of X has **the extension property** if any

I don't see the point of putting extraordinary effort on Problem II.4(b) given that what the world really needs is a clear catalogue of Hahn-Banach theorems that are useful in various circumstances. I seriously doubt that anything more can be squeezed than results like the one in the screenshot.

I am very sorry but I won't re-invent the wheel in this case. No one in their right mind would prove another version of Hahn-Banach theorem for any legitimate reason. People have squeezed it to death already.

22. Proposal For A New Organisation Of Mathematics

Mathematicians in the world today are thrust into an environment that would have been totally alien to d'Alembert and Lagrange. The world has eight billion people and the number of journals is gigantic. The fortunate mathematician is part of a close-knit productive association. For example the circle of mathematicians in the Calderon-Zygmund-Stein school were productive and had flourished and Elias Stein was a central figure around whose circle some of the great advances took place.

But there are so many journals and so many of them are expensive, so many people working on very different things, that there needs to be innovations of organisation of mathematics today.

One idea I had was that there should be a repository of extremely valuable and standardised theorems that are organised like Gradshteyn-Ryzhik, and Wikipedia is a poor substitute for what is needed. There needs to be extreme standardisation of some of the most valuable, most curated theorems and proofs of all parts of mathematics that is meticulously organised where all of Mathematical theorems, history results are archived and available without financial cost *to the world's public so that theorems can be quoted with assurance of quality of proof. There is just an anarchy of variations of results without normalisation.*

In the age of d'Alembert and Lagrange the mathematicians were a relatively small community and highly elite in education and were privileged in society in Europe for the most part. They could afford to be relaxed and there was no explosion impossible to manage.

I imagine one day that Mathematical Theorems meticulously curated will be available to the entire world public free of financial cost that people can refer to for various basic results all the way up to research mathematics level. This will significantly reduce difficulties of communication and misunderstandings from which all mathematicians can be encouraged to quote for various elementary results and standard results.

I see the impulse of this type behind Barry Simon's A Comprehensive Course In Analysis *but even books are not appropriate for the long run, or journals. The time has come for something more robust and able to serve billions of people. Then very high quality mathematical content would be available to* all people everywhere *and will reduce the workload as well as reduce repetition by twenty thousand people of the same things.*

In fact they could even be organised so that artificial intelligence can be used to evaluate students' efforts to work through proofs of some of the standard results as well.

Of course many mathematicians do not want to be distracted from how they do their work. That is fine, but one also has to be rational and prepare for a positive future *for our beloved people and raise the standard of Mathematics outside the elite circles.*

23. Zulf's Advice About "Timing" Of Anything For The World

Zulf does love my beloved people the human race. But Zulf is quite fragile in health and might perish at some point, especially after that horrid evil Bill Gates destructive assaulted my Deep Interior and sabotaged my legitimate income with collusion by the United States Government.

My advice is do whatever you thing is good right now. There will never be a better time before you die. Just do what you think is right and good all the time, immediately. People who think that they will have time later are deluded. There is no later. You will be dead and that will be that and all your good ideas and plans will be gone with you.

24. Feb 3 2022 Return to Problem I.5

Now Problem I.5 is a hard problem. I have an idea, which is to examine whether we can get Hilbert space $\ell^2(\mathbf{Z})$ involved.

We consider the estimate

$$|\hat{f}(n)| \leq C_k(1+n)^k$$

The idea is to do the following. We define

$$M^k(\mathbf{Z}) = \{a \in \ell^2 : \sum_n (1+n)^{2k+2}|a_n|^2 < \infty\}$$

Now these M^k are small spaces in $\ell^2(\mathbf{Z})$. The idea is that the dual $(M^k)^*$ is a big space that is a Hilbert space isomorphic to H^k of the problem.

The important point here is that M_k is not a huge space at all and it is a Hilbert space. Now multiplication $(f,g) \to fg$ is continous in $\ell^2 \times \ell^2 \to \ell^1$ by Cauchy-Schwarz inequality. Thus we obtain immediately that multiplication $M^k \times \ell^2 \to \ell^2$ is continuous.

This is promising because we're dealing with Hilbert spaces and there are no seminorms involved at all. Topology is strictly determined by Hilbert-space norms.

We are interested in understanding if and how this can be used to produce continuity of multiplication of the type

$$H^k \times H^k \to H^K$$

I am not sure where this can lead.

25. Hah The Class Has Become Too Clever Eh?

Well, Stanford Mathematics Ph.D. Quals Directors, that's pretty funny. That's really quite funny. You see years ago, the great number theorist Gert Faltings was teaching some multivariable calculus courses. I did not attend them. But one of my friends, Yuvraj C. Singh, who was in Economics, had taken it and had found it difficult. He had the impression that Gerd Faltings always gave very difficult problems for homework. And no one really managed to do any of the problems. And if, by chance someone did manage to one of the problems he would announce "Ah, the class *has become very clever, eh?"*

You see, this Problem I.5(b) of yours here, this is about multiplication being continuous for Sobolev Spaces with negative exponents.

You probably heard of the brilliant analyst from Jena, a certain Hans Treibel. *Yes, he's quite good, isn't he?*

Well he actually has a very famous paper *from 1977 where he barely managed to prove that some Besov spaces have continuous multiplication* [3].

So this is what you have been doing to innocent vulnerable graduate students at Stanford with fragile ego and confidence right in the middle of their chance to have a mathematical career, a doctorate, perhaps a dignified life? You give them problems that even Hans Treibel was not resolving in 1977.

Do you feel good about how you're living your life, Stanford Ph.D. Quals Director? You thing you've been good people? You don't have any shame, no compassion for people at all?

I will be serious and say this problem was very very poor judgment of Stanford Ph.D. Quals Directors. You see, you are trusted by the younger students seeking to have an alma mater *at Stanford. This is not exactly nourishing mother behaviour, to sabotage the confidence and career of young people out of the blue right when they are just proving their skills to be a professional to be given an obstruction like this, a problem that is unsolved by the luminaries of analysis. It's poor judgment and poor taste too.*

This sort of problem belongs in an analysis seminar. It does not matter if they pass without doing it. It's a cheap dirty trick. You should not use unsolved open problems or ones that took many decades of focused efforts by top analysts in the world in Ph.D. Qualifying *exams. That is a dereliction of duty for you. How are you preparing young people for a career by this sort of dirty trick?*

Leave these problems for seminars and research efforts. People should be able to do an entire doctorate without bad judgments like this.

I am wise and will not attempt to solve it at all. You see I don't really care if I look good or not. I am interested in gaining a reasonable level of proficiency. I'll let the Lars Hörmanders of the world do outrageously hard analysis. I don't need to get involved.

I'm a smoker, ok? It would be akin to suddenly me going to England and telling Manchester United and Liverpool and other top English League football teams that I am ready to be a striker at 49. That's not reasonable. You should apologise to the entire 2014 crop for putting them in distress for no rational reason.

26. The Starcraft II Analogy For Problem I.5

Problem I.5 is beyond my skill level in analysis; it's unsolved and there are 2021 papers where it is not clear what multiplication theorems exist for H^{-k} at all. I just played a Starcraft II game at Harder level with good enough performance to reach Very Hard [**?**]. *That's my level. I can't win Very Hard at all. My level is not high enough.*

I don't have psychological problems with some things being my level and some things being beyond it. In Starcraft II it really makes this clear that I will not be winning Very Hard games very often right now. I am not devastated by self-loathing about it. I do not feel small and low and crestfallen about it.

Why is that? Well it's because I am not a petulant child. When there are Magnus Carlsens in Chess with absurdly high ELO ratings, I do not waste precious moments of my life feeling horrible that my chess rating is not going to even reach 1800 even if I got into it and his is something like 2800. There is more to existence than chess and I am glad Magnus Carlsen is a great genius there.

I am not interested in becoming a master of hard analysis right now. I need to gain a broader understanding of Mathematics and I know when it is too hard and I won't get involved in this problem.

So Fall 2014 I will leave with nine problems done after polishing etc. That's fine. I am satisfied that I did a good job.

References

[1] `https://github.com/zulf73/S4TheoryNotes`

[2] Robert Strichartz, *A Guide To Distribution Theory And Their Fourier Transforms*, CRC 1993

[3] Hans Treibel, *Multiplication Properties Of Besov Spaces*, Annali di Matematica Pura ed Applicata, 1977, 114, pp 87–102

[4] `https://drive.google.com/file/d/1JlJan30duFMZumz3MyllqMhV2qwF5FqJ/view?usp=sharing`