DECEMBER 29 2021 NOTES ON THE INVERSE MAPS FOR INJECTIVE BOUNDED LINEAR OPERATORS

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I do not really have a strong understanding of something that I hold intuitively to be true and invoke without worry. Let X, Y be Hilbert spaces and let $T \in L(X, Y)$. I want to have firm clarity regarding the following.

There is a decomposition of X into a direct sum $X = V \oplus W$ where $Tx = 0$ iff $x \in W$ and T is injective on V and has a bounded inverse $T^{-1}|Ran(T)$.

First question that arises is whether these are *actually true* or whether they have exceptions. The second question is whether, if true, there is a clear proof. They are important to me because for me they seem intuitively right and I have invoked them in Stanford Analysis 2015 Quals and elsewhere and I want to be sure that they are right.

Let's start with some simple things. The nullspace is a linear subspace that is closed. Both are clear because $\{0\} \subset Y$ is closed and the nullspace is $T^{-1}(\{0\})$ and T is continuous. Linearity is easy.

Now let's try to prove something general. Suppose generally $W \subset X$ is any closed linear subspace. Let $V = W^{\perp}$ be defined as the orthocomplement.

$$W^{\perp} = \{x \in X : (x, w) = 0 \text{ for all } w \in W\}$$

Then $V = W^{\perp}$ is closed. Suppose $x_j \to x$ with $x_j \in V$. Then for every $w \in W$ we have

$$\lim_{j\to\infty} (x_j, w) = (\lim_{j\to\infty} x_j, w)$$

This requires $(, w)$ is a continuous functional. The limit on the left is 0.

Direct sum. Any $x \in X$ we want a representation $x = x_W + x_V$ using projections. It is the uniqueness of this representation that allows us to see that $T|_V$ is injective.

All the really easy things are done. Now we apply the Open Mapping Theorem of Banach to $T|_V$ now where *injectivity* is assured and surjectivity to $Ran(T)$ is by definition. Then every open set of V has an open image $T(V)$ and therefore the set theoretic function T^{-1} is actually continuous.

1. Intuition: Closed Range Is Not Guaranteed

My intuition is from John Conway's Linear Algebra course from freshman year at Princeton during 1991-2. Then later from Functional Analysis taught by Peter Sarnak around 1993. Finally for thesis work of 1994-5 supervised by Jeff McNeal.

One of the intuitions is that while injectivity of an arbitrary $T \in L(X, Y)$ on the orthocomplement of the nullspace of T produces a bounded inverse, the range will not generally be closed and is special. This might seem obvious to some because not all linear operators have closed range so this sort of scheme cannot have magical properties of closing the range of arbitrary bounded operators T. If you ever

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discover any operator without a closed range, no amount of tricks cutting up the domain will change this.

This is now clear. Bounded operators always have a *closed graph* but the range is not always closed. You need something more than boundedness of T to obtain closed $Ran(T)$. This is good intuition, for when you have a bounded operator with closed range then there is some special analytical situation; it's not just a matter of linearity and boundedness.

These intuitions allow me to separate the totally generally true things from special aspects of a situation.

Ah, Stefan Banach has some equivalences from 1932 for closed range [1]. See, I am the sort of man who looks at a theorem of Stefan Banach about *conditions* for closed range and immediately infer that it's not something easy and automatic. These Stefan Banach types don't prove all sorts of technical equivalences unless they don't have a general theorem. That's the other way of seeing that closed range is never easy.

2. People Who Think They Will Prove General Theorems In Hilbert or Banach Spaces Are Deluded

You see among Stefan Banach, Frigyes Riesz and Marcel Riesz, they were such great geniuses, they basically *vacuum cleaned* all the goodies of general theorems all by themselves. Anyone who thinks they left any general theorem for them to discover is sadly deluded. There's nothing left. You'll end up proving something far away from the mainstream if you go into this delusion.

The only sort of cat that ever surprised these men was *Laurent Schwartz* and he did this spectacularly successfully because Marcel Riesz was too fraidy cat for horribly misbehaved dual elements of C^∞ and this was why Laurent Schwartz had succeeded. I don't know if he was wise to go with Trotskyism but he was a super-genius to realise that he ought to stick to his guns when Marcel Riesz thought general theorems about constant coefficient PDEs could be solved. People look at the negative comments of Marcel Riesz as bad. They were fabulous for Laurent Schwartz. The right way to read the comments is my translation: "We're too fraidy cats for these horrible things. So we will dissuade you from taking over the world." And that's exactly what Schwartz did. Distribution theory produced new geniuses like Lars Hormander.

3. Racial Biases Against Non-Whites In America Is Very Strong

You see people think that racial bias against non-whites is not so bad. That is relative to when it was okay to just murder all the native Americans and take their things. The reduction of racial bias is epsilon from the routine genocide by comparison to what would be reasonable to me. These are strong tendencies spanning centuries. They don't disappear in a snap. Things are better glacially. And that's just life. It's not going to disappear by wishing for it to disappear. It will reduce eventually to a tolerable point, but maybe in a century.

References

[1] https://en.wikipedia.org/wiki/Closed_range_theorem