ZONAL SPHERICAL HARMONICS ARE LOCALISED NEAR POINTS

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What I will do in this note is not novel mathematically. It is well-known to various researchers in parts of analysis that zonal harmonics on S^d are localised near a point. However, I am annoyed today, January 2 2021, because I don't remember where I saw the beautiful formula for the width span of a zonal harmonic. As a result, I am quite frustrated because I actually need this for my great campaign to overthrow quantum theory with my immortal genius of four-sphere theory.

Part of my polemical strategy had always been to mock quantum theory's inability to explain particle localisation and wave-particle duality mercilessly and then offer a beautiful and clear alternative theory, i.e. my four-sphere theory, that would provide a satisfying and clear explanation for existence of localised particle. The force of arguments would compel even the hearts of stone and steel when they see that simply the geometry of a sphere forces a large number of zonal harmonics to be localised near a point, and therefore that Dirac eigenspinors exist in vast quantities that are spatially localised. And the whole world would be moved to accept my four-sphere theory because I have given my beloved people the answers to deep questions of Nature.

Unfortunately this strategy has flaws when I cannot find the width formula for zonal harmonics. And I was trying to avoid a detailed analysis of zonal harmonics myself because I prefer that other people do the hard work and I simply take credit for bringing humanity the final laws of Nature with relatively little work. This attitude was put under stress and I decided that perhaps I should do some work here and explain this localisation of zonal harmonics in S^d for $d \geq 2$ directly so that even the quantum theorists and physicists can see with their own eyes that they had strayed from the truth, that they were lost and I offer then Salvation.

1. Two Models Of Nature Compared

I want my dear reader to consider ab initio the following situation. One day you appear in the universe, completely fresh in your mind, and you do not know anything about the sort of place you are in.

Someone comes and says hello and provides you with refreshment. Then you are asked to select between two models of the universe.

The first model is that the universe is Euclidean, flat $\mathbf{R}^3$. The other models is that the universe is $S^4(R)$ with a large radius $R = 3075.69$. But your host says, "I will give you some more information. I will tell you that this universe has *quantisation of energy* and this universe has *localisation of particles*." Which is a better model of Nature now?

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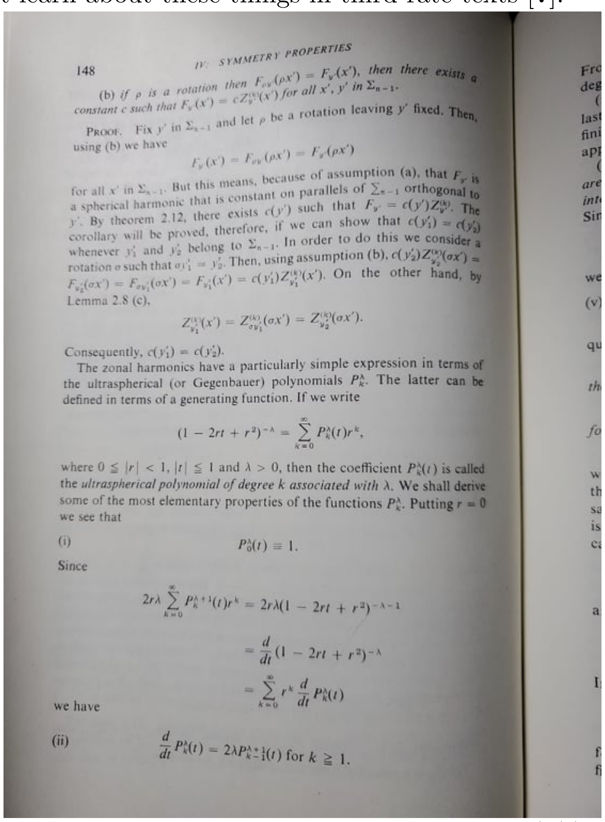
I will help you see that the latter, the $S^4(R)$ is the better model of Nature. And I will show you this by telling you how localised particles are *forced by geometry* in this model and so is quantisation while the geometry of the first model, $\mathbf{R}^3$ does not do these things.

2. Gegenbauer Polynomials And Their Relationship To Zonal Harmonics of S^d

I will only consider unit sphere S^d now. For $\alpha > 0$ the Gegenbauer polynomials $C_n^{(\alpha)}(x)$ are characterised in various ways but the important thing for us to understand is that

$$C_n^{(d-1)/2}(z \cdot z_0)$$

with $z_0 \in S^d$ fixed and $z \in S^d$ are exactly the *zonal harmonics of* S^d [?]. Fine I will also show you that I do have the book of Elias Stein and Guido Weiss, and did not learn about these things in third rate texts [?].



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(b) *if* ρ *is a rotation then* $F_{\rho y'}(\rho x') = F_{y'}(x')$, *then there exists a constant* c *such that* $F_{y'}(x') = cZ_{y'}^{(k)}(x')$ *for all* x', y' *in* Σ_{n-1}.

Proof. Fix y' in Σ_{n-1} and let ρ be a rotation leaving y' fixed. Then, using (b) we have

$$F_{y'}(x') = F_{\rho y'}(\rho x') = F_{y'}(\rho x')$$

for all x' in Σ_{n-1}. But this means, because of assumption (a), that $F_{y'}$ is a spherical harmonic that is constant on parallels of Σ_{n-1} orthogonal to y'. By theorem 2.12, there exists $c(y')$ such that $F_{y'} = c(y')Z_{y'}^{(k)}$. The corollary will be proved, therefore, if we can show that $c(y_1') = c(y_2')$ whenever y_1' and y_2' belong to Σ_{n-1}. In order to do this we consider a rotation σ such that $\sigma y_1' = y_2'$. Then, using assumption (b), $c(y_2')Z_{y_2'}^{(k)}(\sigma x') = F_{y_2'}(\sigma x') = F_{\sigma y_1'}(\sigma x') = F_{y_1'}(x') = c(y_1')Z_{y_1'}^{(k)}(x')$. On the other hand, by Lemma 2.8 (c),

$$Z_{y_1'}^{(k)}(x') = Z_{\sigma y_1'}^{(k)}(\sigma x') = Z_{y_2'}^{(k)}(\sigma x').$$

Consequently, $c(y_1') = c(y_2')$.

The zonal harmonics have a particularly simple expression in terms of the ultraspherical (or Gegenbauer) polynomials P_k^λ. The latter can be defined in terms of a generating function. If we write

$$(1 - 2rt + r^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^\lambda(t)r^k,$$

where $0 \leqq |r| < 1$, $|t| \leqq 1$ and $\lambda > 0$, then the coefficient $P_k^\lambda(t)$ is called the *ultraspherical polynomial of degree k associated with* λ. We shall derive some of the most elementary properties of the functions P_k^λ. Putting $r = 0$ we see that

(i) $$P_0^\lambda(t) \equiv 1.$$

Since

$$\begin{aligned} 2r\lambda \sum_{k=0}^{\infty} P_k^{\lambda+1}(t)r^k &= 2r\lambda(1 - 2rt + r^2)^{-\lambda-1} \\ &= \frac{d}{dt}(1 - 2rt + r^2)^{-\lambda} \\ &= \sum_{k=0}^{\infty} r^k \frac{d}{dt} P_k^\lambda(t) \end{aligned}$$

we have

(ii) $$\frac{d}{dt} P_k^\lambda(t) = 2\lambda P_{k-1}^{\lambda+1}(t) \text{ for } k \geqq 1.$$

We will be examining the characteristics of $C_n^{(3/2)}(x)$ for real number $|x| \leq 1$ as $n \to \infty$ in this note. Our hope is to convince our dear reader that one we have shown localisation of these in a small region of $[-1, 1]$ they will yield that we have compelled them that the actual universe is not $\mathbf{R}^3$ but $S^4(R)$ since nothing so clear is possible for localisation of particles in $\mathbf{R}^3$.

3. From A Note Of Laura De Carli

I am not actually interested in the main content of the wonderful paper of Laura de Carli, just the comment of the 'easy to show' asymptotic formula where she refers to Gabor Szego's *Orthogonal Polynomials*. I am rather glad that she and her colleagues think this is easy to see. This reduces my worry that I have to do all sorts of labour.

For n very large, the asymptotic behaviour is

$$\lim_{n\to\infty} \frac{C^{(3/2)}(\cos(x/n))}{C^{(1)}(1)} = \Gamma(2)(\frac{x}{2})^{-1} J_1(x)$$

Now I will draw this limiting expression on the whole range $x \in [-1, 1]$. I want you to actually see these things up close and personal. In fact I will tell you how to draw it in R. That way, you will be quite compelled that there is nothing hidden here.

```
> x<-seq(-1,1,by=0.001)
> y<-2*gamma(2)*besselJ(abs(x),3/2)/abs(x)
> plot(x,y,type='l',col='darkblue',lwd=5,main='limit of S4 Gegenbauer')
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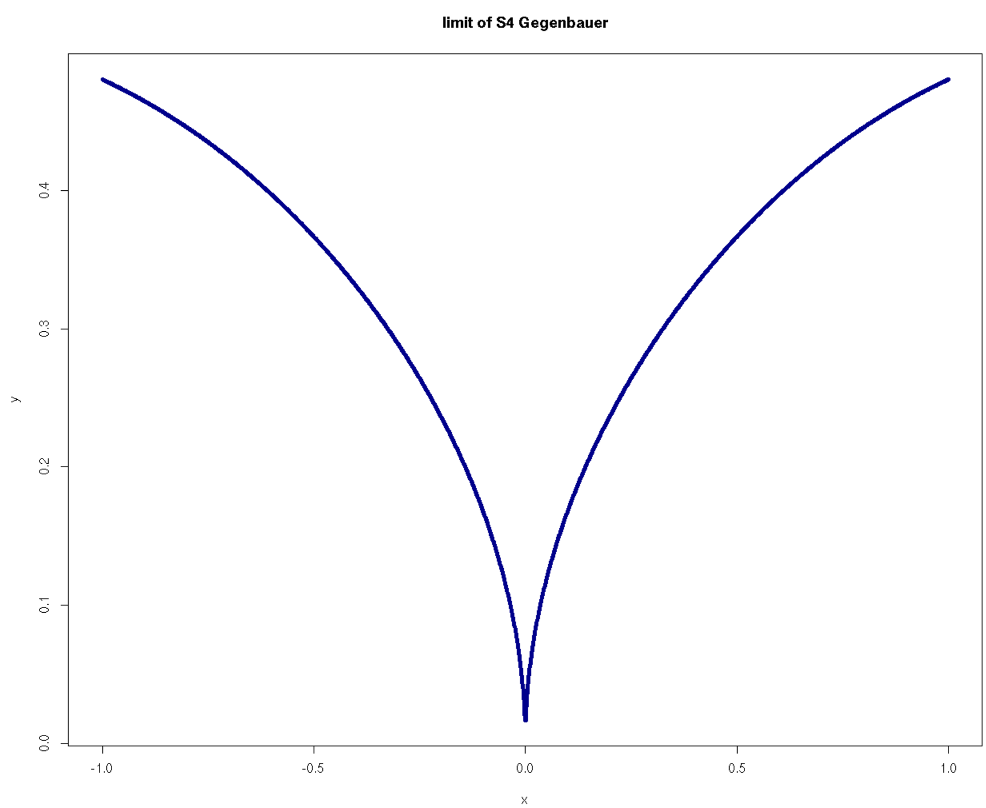
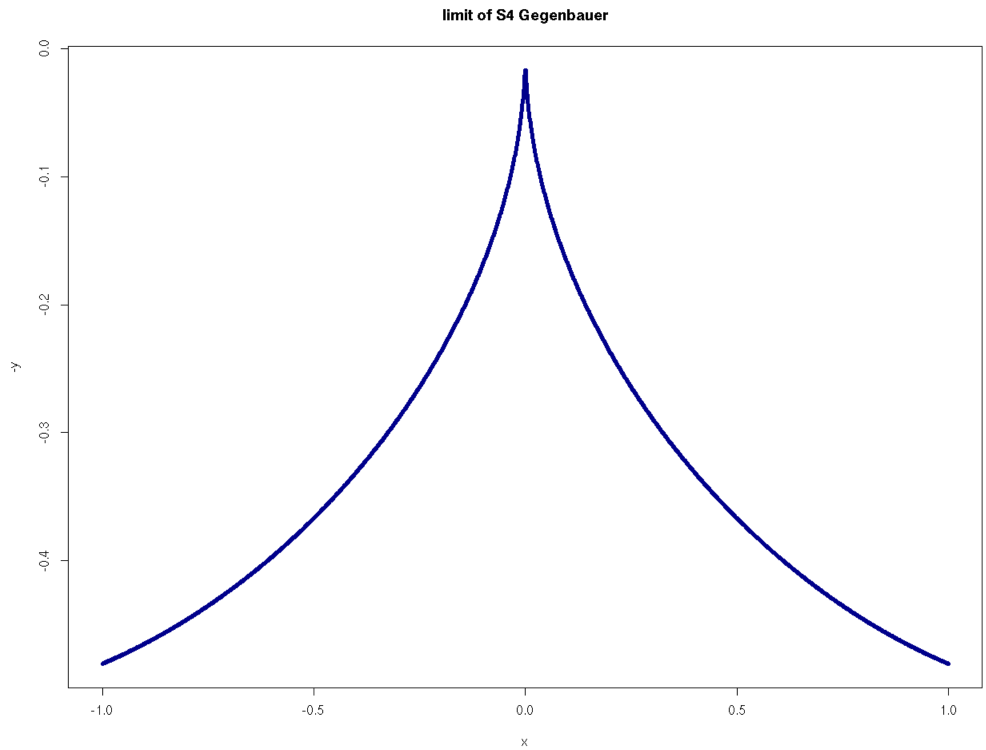


Fine, you don't like the pointing down thing. We'll draw the negative of this thing.



Now you note the limit is for x/n in the left side, so what actually happens is these peaky shapes get squeezed by $1/n$ roughly as $n \to \infty$.

In four-sphere theory, $n \sim 10^30$ or higher before anything remotely particle-like occurs in Nature, so in the physical world we start getting subatomic point localisation of some spinor fields, and they are our electrons and so on.

Note that zonal harmonics are scalar, and Dirac eigenspinors will be of the form $\sigma_1 h_1 + \ldots \sigma_{16} h_{16}$ where σ_j are Killing spinor fields globally defined on S^4 and h_j are general harmonics, but for the *zonal harmonics* they are going to be localised like above and then this will *force* particle localisation in four-sphere theory.

4. Why Don't You Leave The Cat Alone Already?

This cat has suffered tremendously over a century. Why don't we leave the cat alone? I don't like animal cruelty and all that. Leave the cat alone. Poor cat.

There is nothing strange and 'quantum mechanical' in four-sphere theory. There are no probabilities involved. You see in four-sphere theory the electron is a field with point localisation that is fixed. It's not a probability distribution. It's just this way, with a peak near a point. There is nothing 'holding it' near the point. It's a eigenspinor of the Atiyah-Singer-Dirac operator and its intensity distribution is a consequence of geometry of four-sphere.

5. Strong Views Against Coercion

I want to make this point very clear. Genius work in any field comes from one thing and one thing above all, and that is love. All great genius is rooted in love of the work, love of the people surrounding the work, love of service to our people the human race. Love cannot be coerced, and nothing good will ever come from coercion. I believe in security of natural rights of liberty for all people of the world, and I detest, despise and want to totally destroy these people like Bill Gates who are obsessed by power and are totally uncultivated and deserve to be beaten to a pulp and burned at the stake for their insolence. These are people who do not deserve to live at all, these evil people who violate other people's natural rights and expect impunity.

The consequence is that I do not believe scientists ought ever to have coercive pressures. They will never do great work that way. They ought to do whatever they believe is the right science. Now I do believe that if scientists choose to dismiss my four-sphere theory, they will not fare well in posterity, as all other theories will be displaced by four-sphere theory for it is eternal truth, and it will stand for a trillion more years without faltering.

6. Consider the Compton Wavelength Of The Electron

We consider the Compton wavelength of the electron and it is

$$\lambda_e = 2.426 \times 10^{-12} m$$

We want to see if the four-sphere theory can provide some independent prediction to this that is not nontrivial, i.e. not a matter of tautology.

First, we note that we can numerically estimate the full-width at half height for the graph above and it is 0.442 compared to the scale of length of the interval $[-1,1]$, a ratio of $w = 0.221$.

What would be nice if if we could determine the eigenvalue number k_e associated with the electron and then check if the following expression yields something close to the Compton wavelength that is measured.

$$\lambda_{theoretical} = 0.221 \frac{2\pi R_{universe}}{k_e^2} \tag{1}$$

So here is what I can do quickly. I can take the electron mass

$$m_e = 9.109 \times 10^{-31} kg$$

Then I calculate the frequency by Planck formula

$$\nu_e = E/h = m_e c^2/h = 1.237^{10} Hz$$

Then I do a fudge that I will worry about justifying later

$$k_e = \nu_e \times c$$

Then I get

$$\lambda_{pred} = 0.221 \frac{2\pi R_{universe}}{k_e^2} = 9.57 \times 10^{-12} m$$

This does not give us agreement, but it is interesting because we get close

$$\frac{|\lambda_e - \lambda_{pred}|}{\lambda_e} = 2.943$$

We are off by a factor of 3. That's actually not very bad at all. We're in the $10^{-12}m$ range, and that's reasonable because $R_{universe} = 3075.69$ Mpc could have precision issues anyway.

I would chalk this down as quite good success.