RUDIN REAL AND COMPLEX CHAPTER I

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1. Problem 1

Does there exist an infinite σ-algebra with only countably many members?

2. Problem 12

Suppose $f \in L^1(\mu)$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

We consider $\delta = \epsilon/\|f\|_\infty$ and reduce to proving that if $f \in L^1(\mu)$ then $\|f\|_\infty < \infty$. Suppose $\|f\|_\infty = \infty$ then on a set of positive measure $|f|$ is unbounded which integrates to infinity on the set contradicting $f \in L^1(\mu)$.

3. Problem 10

Suppose $\mu(X) < \infty$, that f_n are bounded complex measurable, and $f_n \to f$ uniformly. Then show

$$\lim_n \int |f_n| d\mu = \int |f| d\mu$$

The uniformity in convergence implies for $\epsilon > 0$ there exists N so that for $n \geq N$:

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in X$. So

$$\int |f_n| - \int |f| < \epsilon \mu(X)$$

This shows for $n \geq N$ the sequence

$$\frac{1}{\mu(X)} \left(\int |f_n| - \int |f| \right) < \epsilon$$

4. Problem 7

Suppose $f_1 \in L^1(\mu)$ and $f_1 \geq f_2 \geq f_3 \leq \cdots \geq 0$ and $f_n \to f$, and f_n, f measurable. then

$$\int f_j \to \int f$$

Since $f_1 \in L^1(\mu)$ the rest including the limit are. We can apply the dominated convergence theorem since f_1 dominates the sequence.

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5. Problem 5

(a) Suppose f, g measurable implies $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

The function $h = f - g$ is measurable, and $h^{-1}((-\infty, 0))$ is inverse of open set; this is the first . Similarly $h^{-1}(\{0\})$ is preimage of a closed set, the second set, so measureable.

(b) the set of points where a sequence of measurable functions converge to a finite limit is measurable

I will do this later.