STANFORD FALL 2018

ZULFIKAR MOINUDDIN AHMED

1. Part I Problem 1

1(a). Define function ν from subsets of $\mathbf{R}$ to $[0, \infty]$ as follows. If A has 0 in its closure then $\nu(A) = \infty$ and otherwise $\nu(A) = 0$. Show that ν is finitely additive not countably.

Proof. First, we claim for any finite collection $A_1, \dots, A_N$ with none having 0 in its closer, the union does not have 0 in its closure either. Since A_j does not have zero in the closure, there exists an $\epsilon > 0$ such that $(-\epsilon_j, +\epsilon_j)$ is disjoint from A_j. Then $\epsilon = \min(\epsilon_1, \dots, \epsilon_N)$ is positive and disjoint from $A_1 \cup \dots \cup A_N$. Then $(-\epsilon, +\epsilon)$ is disjoint from this union and therefore 0 is not in the closure. So here we have

$$\nu(A_1 \cup \dots \cup A_N) = 0$$

and $\nu(A_j) = 0$ so finite additivity holds.

Now suppose that $A_1, \dots A_N$ have at least one with limit point 0. then the union will have limit point zero so both $\nu(A_1 \cup \cdots A_N) = \infty$ and at least one of the summands $\nu(A_k) = \infty$ Therefore finite subadditvity holds.

Now consider $A_j = (1/j, \infty)$ these do not have 0 as a limit point so $\nu(A_j) = 0$ but $\nu(\bigcup_{j=1}^{\infty} A_j) = \infty$ while the sum $\sum_{j=1}^{\infty} \nu(A_j) = 0$. ☐

1 (b) Let m be the Lebesgue measure on $\mathbf{R}$. Let $f : \mathbf{R} \to \mathbf{R}$ be measurable. Let $S = \{x \in \mathbf{R} : m(f^{-1}(x)) > 0\}$ Show that S is countable.

Assume known that any closed set in $\mathbf{R}$ consists of countable unions of closed intervals possible degenerate to a point. Suppose first that f is continuous. Then $f^{-1}(x)$ is closed so it can be expressed as a countable disjoint union of closed intervals and points. For any two distinct points $x, y \in S$ we have disjoint closed sets of positive measure $f^{-1}(x), f^{-1}(y)$. Assume S is uncountable. Then we have an uncountable union of disjoint closed sets of positive measure. We want to show that two of these must intersect. Since these have positive measure, each contains and open interval, O_x. Now we have an uncountable set of open intervals that are disjoint in $\mathbf{R}$. This is impossible since every open set in $\mathbf{R}$ is a countable union of open intervals.

2. Part II Problem 1

(a) Show that a Hilbert space has a countable dense subset if and only if it has a countable orthonormal basis.

(b) Show that in a separable Hilbert space the unit ball is weakly sequentially compact, i.e. every sequence has a weakly convergent subsequence.

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Proof. (a) A finite dimensional vector space has a countable dense subset since $\mathbf{R}$ does. Thus if the Hilbert space has a countable orthonormal basis, then choose for each of the factors a countable basis and then use the fact that countable products of countable sets are countable. Conversely, if there exists a countable dense subset,

(b) Take any sequence with $\|x_n\| = 1$. Now let $e_1, \ldots, e_n, \ldots$ be an orthonormal basis, Then

$$|\langle e_k, x_n \rangle| \leq \|e_k\| \|x_n\| = 1$$

so $g_{kn} = \langle e_k, x_n \rangle$ has s convergent sequence g_{kn_k}. Use this by considering g_{1n_1} then apply this again with $n = n_1$ and e_2 and so on each time gaetting a convergent subsequence for each of the e_k the final sequence will be such that $\langle e_k, x_n \rangle$ will be convergent for all k. This is then a weakly convergent sequence for using x_m we will have convergence of $\langle w, x_m \rangle$ for every $\|w\| \leq 1$. □

3. Problem 3

(a) Let $K \in L^p([0,1] \times [0,1])$ and $Af(x) = \int K(x,y) f(y) dy$. Then Hölder inequality gives $|Af(x)| \leq \|K(x,\cdot)\|_p \|f\|_q$. So $Af(x)$ is well-defined and finite almost everywhere since $\|K(x,\cdot)\|_p$ is in L^p.

(b) Suppose for every $f \in L^q$ and $Af(x) = 0$ for all x. Show $K = 0$.

For any given x define $f_{x_0}(y) = K(x_0, y)$ Then $Af_{x_0}(x) = 0$ for all x and therefore for $x = x_0$. This implies $\int K(x_0, y)^2 dy = 0$ which proves $K(x_0, y) = 0$ for almost every y. Since x_0 was arbitrary, we see $K(x,y) - 0$ for almost all x and y.

4. Problem 4

Suppose $T : X \to Y$ is a Fredholm map so both $Ker(T), Y/Ran(T)$ are finite dimensional. (a) Show there exists m, n so that $X \oplus \mathbf{C}^n$ and $Y \oplus \mathbf{C}^m$ are isomorphic as Banach spaces. (b) Prove that X is separable if and only if Y is separable.

(a) Let n be the dimension of the cokernel $Y/Ran(T)$ and let m be the dimension of the kernel. Let $phi : Ker(T) \to \mathbf{C}^n$ and $\psi : Y/Ran(T) \to \mathbf{C}^m$ be isomorphisms. Then consider the map $U : onX \oplus \mathbf{C}^n$ defined as T on $X/Ker(T)$, ϕ on $Ker(T)$ and ψ^{-1} on the additional $\mathbf{C}^n$. On each of these pieces it is a bijection. Therefore it will be a Banach space isomorphism.

(b) Separability of $X \oplus \mathbf{C}^n$ and $Y \oplus \mathbf{C}^m$ are equivalent because of the Banach space isomorphism of (a). And separability of X and Y are equivalent to adding finite dimensional spaces to them since these are closed subspaces.

5. Problem 5

Let $\mathbf{T}$ be the unit circle. Show that $I(r) = \int_T e^{ircos(t)} \phi(t) dt$ satisfies

$$|I(r)| \leq C r^{-1/2}$$

Proof. Change variables to $u^2 = \cos(t)$. And not worrying about the limits of integration obtain

$$I(r) = \int e^{iru^2} \phi(\arccos(u^2)) \frac{2u}{\sqrt{1-u^2}} du$$

Now make change of variables again to $w = \sqrt{r} u$ so

$$I(r) = \frac{1}{\sqrt{r}} \int e^{iw^2} \phi(\arccos(w^2/r)) \frac{2w}{\sqrt{r - w^2}} dw$$

Now use sup-norm bounds on the integrand to get the integral bounded by C. Conclude $I(r) \leq Cr^{-1/2}$.

$\square$

6. Part I Problem 2

Let $f \in \mathcal{S}(\mathbf{R})$. Define $\|f\|_{H^s} = \|(1+|\xi|^2)^{1/2}\hat{f}\|_2$.

(a) Show that $\|f\|_\infty \leq C\|f\|_{H^{s/2}}$.

Define an easier equivalent norm for Sobolev norm $s = 1/2$ as follows.

$$\|f\|_{W^1} = \|f\|_2 + \|f'\|_2$$

Since $\|\xi\hat{f}\|_2 = \|f'\|_2$ by the Plancherel theorem, we easily see

$$c_1\|f\|_{W^1} \leq \|f\|_{H^{1/2}} \leq C_1\|f\|_{W^1}$$

Now since for any $s > 1/2$ we have

$$\|f\|_{H^{1/2}} \leq C\|f\|_{H^{s/2}}$$

we will be done if we can prove

$$\|f\|_\infty \leq \|f\|_{W^1}$$

Now we can find an interval $I = [a,b] \subset \mathbf{R}$ so that

$$\sup_{\mathbf{R}} |f| = \sup_I |f|$$

because for Schwartz functions we have rapid decay outside a sufficiently large interval so we can focus our attention on this interval.

Now use fundamental theorem of calculus:

$$f(x) = f(m) + \int_m^x f'(t)dt$$

From here,

$$|f(x)| \leq \frac{1}{|I|}\|f\|_{L^1(I)} + \|f'\|_{L^1(I)}$$

Then apply Cauchy-Schwarz on both summands and obtain the sup-norm bound

$$\|f\|_\infty \leq C\|f\|_{W^1}$$

This solves the problem.