A GUIDE TO REED-SIMON CHAPTER V FOR PEOPLE INTERESTED IN STANFORD PH.D. QUALIFYING PROBLEMS

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For whatever reasons, Stanford Ph.D. Qualifying Exams have a tendency to demand a strong clear knowledge of material in Reed-Simon Chapter V, *Locally Convex Spaces*.

I love Reed-Simon (volume I) partly for sentimental reasons. I studied functional analysis originally from a course on functional anlysis in 1992-3 given by Peter Sarnak and we used this text for the course. We were not concerned with locally convex spaces then.

I want to go over some of the issues that I had trouble with in the past month and provide a guide for those interested in reaching a level of familiarity to handle problems of the Stanford Mathematics Ph.D. Quals level.

Reed-Simon's motivation for locally convex spaces is excellent and in fact canonical. In a normed topology we have $x_n \to x$ if and only if $\|x_n - x\| \to 0$. In a non-normed locally convex space X instead we have a family of seminorms ρ_a with $a \in A$ and we do not have $\rho_a(x) = 0$ implies $x = 0$ for any particular $a \in A$ but we have instead $\rho_a(x) = 0$ for all $a \in A$ iff $x = 0$.

This is a beautiful observation of Reed-Simon because it gives some sense and meaning for what all these seminorms are doing. You see, I am a normal human being and my mathematical interests were geometric. In geometry, we were always very suspicious of people who took ordinary and beautiful things and turned them into all sorts of horrible things with gazillions of seminorms ρ_a for what seems like totally unjustifiable reasons. I always thought these seminorms were technical mumbo-jumbo introduced by people who really had nothing better to do in their lives than make things that are crystal clear into gobbledy gook that would require entire new civilisations to decipher.

So that is why the Reed-Simon motivation is important. What they are really saying is that all the seminorms are *wiring under the hood*. Most intelligent cultured literate people let technicians handle the wiring. Most *mathematicians* involved in more genteel and civilised problems do not get too involved in topologies generated by seminorms. But for some reason Stanford Mathematics Ph.D. Quals directors want all sorts of innocent young people who have barely begun their lives to be burdened by topologies generated by seminorms.

The first thing to understand is there is no need to panic. Read Chapter 5 of Reed-Simon Volume I thoroughly. You will thank me for this advice.

The key issue to understand is the entire family ρ_a together separates points in contrast to a single norm.

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1. The Nightmare Of Weak Topologies

Unlike in topology of manifolds, where open sets are fairly reasonable, the use of topologies in analysis had always been completely bizarre. Where geometers and low dimensional topologies were civilised and even though a topological space $(X, \mathcal{T})$ is defined by open sets having arbitrary unions, finite intersections, including empty and entire set, geometers and low-dimensional topologists never actually thought that you should then *totally abuse* the definition and suddenly consider bizarre sets that are never even mentioned in the depths of the Underworld in whispers and just *declare* that they are open sets and suddenly they use the word continuity in a context that is totally deceptive.

You see, human psyche is fragile. Human language and understanding is comfortable with some things, and human psyche dislikes sudden changes. Suppose you say you will give someone a diamond ring one day and propose to marry them, then and you my dear readers know this well, and then tell them that you meant it in a *virtual sense* where there won't be any actual diamond rings and no actual marriage but only in a *weak sense*. I can assure you that you will generate murderous fury and could indeed be murdered in the end. And I wouldn't blame the other person. It's like the B. B. King song, who used to make love under the red sunset and used to make promises he was soon to forget, and all that changed when love came to town.

You should not deceive people by saying sets are open and maps are continuous just on formality. And that's unfortunately what is disturbing about analysts' use of 'continuous' and 'open' in the variety of fake topologies that are introduced.

2. Disclaimer About My Philosophy

My philosophy is that there are some questions that have intrinsic merit and they are questions that have correspondence either to the natural world, or they have intrinsic merit because they are about natural numbers. The questions could be hard to answer or they could be easy. Regardless, their validity as questions is without any contest. For example, we might ask whether some sort of substance will burn with a flame that is some colour. Or we might ask about whether there are integers x, y, z with $x^n + y^n = z^n$ and that was a hard question finally resolved by Andrew Wiles at Princeton while I was an undergraduate there, in 1994.

Between 1870-1940, at least by the reckoning of Ivor Grattan-Guinness the mathematical historian, mathematicians seeking 'foundations' had been impatient to stand on their own definitions of everything. And suddenly things have validity based only on axiomatic foundations. I am from a different philosophical school than these *formalists*. I admit that some of these axiomatic foundations were valuable. But the ultimate measure of whether mathematicians are doing anything valuable is whether in the end people who do not care about the axiomatic definitions of totally esoteric and arcane concepts can evaluate whether the various theorems actually correspond to anything they know independently of all of this technical axiomatic formalism at all.

For example, the Euler characteristic defined by vertices minus edges plus faces is a topological invariant. It's a beautiful theorem and in fact we can check it on ordinary sort of non-mathematical constructions that ordinary people can appreciate. Suppose instead you have a theorem that technical mumbo-jumbo condition $\chi_{a,b,c,1,2,3}$ holds only when quasi-surely $\chi_{e,f,g,4,5,6}$ also holds. I can assure you that *no one cares* and also that *no one should spend their precious moments of life on Earth giving a damn.*

3. Why Locally Convex Spaces Are Important

Locally convex spaces are essentially abstractions of some things that are intrinsically of interest. I have been making the case that $\mathcal{D}'(\Sigma S^4)$, the distributions on spinor fields on the four-sphere are all the matter and other content of Nature. You and I and all the galaxies and their content are made of spinor fields; they are matter fields in Nature.

If you consider the circle S^1 and consider the smooth functions on S^1, you can write

$$C^\infty(S^1) = \bigcap_k C^k(S^1)$$

Now each of the spaces $C^k(S^1)$ is a Banach space, but the smooth functions do not form a Banach space. The dual of $C^\infty(S^1)$ also do not form a Banach space.

Whatever the *mathematical importance* of this particular space of functions and distributions, the S^4 versions are directly part of the best description of Nature, I claim, and so immediately there is extra-mathematical interest and justification for all the mathematics of these spaces.

You see, in the eighteenth century, this connection between Nature and Mathematics was not even doubted. Natural phenomena behaviour often substituted for mathematical proof. And then you consider the relation between pure mathematics and physics or other parts of Science between 1870-2020 and what you will discover is that the severity of the disconnect is frightening and profound. I am from a philosophical disposition that this is a problem. It is bad for Science and it is even worse for Mathematics. Without being anchored to something outside of Mathematics, the field will go off the rails and never recover, wither and die.

4. Return To Families Of Seminorms

Suppose X is a linear space with a family of seminorms ρ_a that separate points. This means that $\rho_a(x) = 0$ for all all $a \in A$ if and only if $x = 0$. This is the situation for $X_0 = C^\infty(S^1)$ with $[-\pi, \pi]$ identified with the circle. In this case we have the seminorms

$$\rho_{k,m}(\phi) = \sup |x^k \partial^m \phi|$$

Since uniform limits of continuous functions are continuous, we get this situation.

Now let us get into some technical matter. We are interested in defining a topology $\mathcal{T}_s$ on X. The topology, let's call it $\sigma(X, \rho_a)$ is not arbitrary but the coarsest topology so that all of the seminorms ρ_a are continuous.

Reed-Simon Chapter V tells us how to examine a base of neighborhoods at 0 of $\mathcal{T}_w$. Sets of the form

$$N(\alpha_1, \dots, \alpha_n, \epsilon_1, \dots, \epsilon_n) = \{x \in X : \rho_{\alpha_j}(x) < \epsilon_j, 1 \le j \le n\} \tag{1}$$

comprise the base of neighborhoods at zero. You have to vary n, all the α_j and all the ϵ_j to obtain all of the elements.

What is not made very clear even in Reed-Simon is how these arise. What is going on is that these are finite intersections

$$\rho_a^{-1}(-\epsilon, \epsilon)$$

and so they are forced to belong to $\mathcal{T}_s$ because it is defined as the weakest topology that ensures ρ_a are all continuous.

Then you will want to have some firm assurance that taking finite number of seminorms here is sufficient and this is something that is routinely glossed over. Haim Brezis says Folland's book on Real Analysis covers it. Not many analysts really talk about these things very often at all. You have to do it yourself unfortunately. The important thing to know is that there is nothing deep lurking around here. It's just part of the detail of construction of the weakest topology that ensures ρ_a are all continuous.

I was totally frustrated because I felt that I just did not understand some of these things. I wasn't sure if I was missing some deep theory. What I discovered is that *everyone is lazy and everyone glosses over these issues* and so you need to forgive everyone because you know why they are lazy. This is totally abstract nonsense. But since you are not a deceptive charlatan, you will spend the time to make sure that it's true. Once you are sure, always just write down the form of the base at zero immediately, claim that they are a base and generate the whole topology $\mathcal{T}_w$ and then start doing your hookey-dookey to prove absolutely techno mumbo-jumbo propositions that Stanford Mathematics Ph.D. Qual Directors will demand from you by their power tripping convictions that even they won't do this themselves, all young people who don't will fail Ph.D. Quals without doing these. You have to accept the world as it is and not the world as you would like it to be, and jump through the hoops to move on to more important issues.

5. Paper Chase Through Brezis and Reed-Simon

Haim Brezis is charming about his punting of the issue in Lemma 3.1. Let me tell you something: watch out for these Frenchmen. Always keep in mind that Marquis de Sade and Charles Baudelaire were French. They are not as uncouth as Americans who punt on issues pretending that they did you a favour.

Haim Brezis at least tells you that what's left to you is a 'charming exercise in set theory' and refers you to Folland. France has produced a more refined culture here. An American mathematician might just say 'trivia trivia' while you are left as confused as ever.

6. What Haim Brezis does tell us

I don't have an infinite budget, and I lost 3000+ books when I abandoned all my material possessions in 2008 when I did have a copy of Gerald B. Folland's *Introduction to Partial Differential Equations*. So let me try to do this problem of making very explicit how the seminorm base (1) actually form a base, i.e. there are no open sets in $\mathcal{T}_s$ that cannot be expressed as unions of these sets.

As you can expect, everyone want to avoid these exercises because they are technical. However, Haim Brezis was kind enough to make some comments about these.

Brezis says one begins with arbitrary open sets $\omega_b \subset \mathbf{R}$ and then consider $\rho_a^{-1}(\omega_b)$. Following Brezis let's call $\lambda = (a, b)$ and

$$U_\lambda = \rho_a^{-1}(\omega_b)$$

Now we have a collection of sets $(U_\lambda)_{\lambda\in\Lambda}$ in X. The weakest topology in which ρ_a must necessarily include these. Then the question is what other sets we have to add to $\mathcal{T}_w$ beyond (U_λ)?

Let Φ be the sets that are finite intersections of U_λ. And then $\mathcal{F}$ be the family generated by arbitrary unions of elements of Φ. Then he refers to Folland to ensure $\mathcal{F}$ is stable under finite intersections. Then we see $\mathcal{T}_s = \mathcal{F}$.

The point of all this hookey-dookey (I love this phrase), or rigmarole, if you will, is to have some certainty that when you write down (1) and declare them to be a neighborhood basis you have some firm confidence that you are not just being a glib charlatan.

Now consider the issues frankly. I can be quite confident that Stanford Mathematics Ph.D. Qual Directors think that this level of going through details on something that does not lead to all sorts of great accolades is *beneath their dignity*. And yet they will use their authority and power to ensure all the innocent vulnerable graduate students must go through this just to pass the qualifying exam. Is this really fair and equitable? Of course not. But that's just the way the world works. I cannot even get the goddam United States Government to eliminate a *habitual racial murderer with explicit plans to destroy Asian-Americans, who has contempt for natural rights, and is plotting ethnic cleansing of all non-whites from United States industry*. You can't solve all the injustices of the world. You have to just go through this hookey-dookey and plaster it in the Stanford Mathematics Ph.D. Qual Exam and pick more important battles in your life.

7. Grand Locally Convex Spaces Conjecture

My feeling is that locally convex topological vector spaces are not extensive. I will conjecture that there is a fairly reasonable equivalence or isomorphism notion that is unknown still that ensures that besides $\mathcal{D}$ and $\mathcal{D}'$ for some reasonable model spaces, there are no other locally convex spaces at all.

I will implore mathematicians to consider this. There is a gigantic amount of technical theory for locally convex topological vector spaces that are most likely just convenience for smooth functions and distributions and there will never be any non-isomorphic example possible at all.

These are not as good definitions as Hilbert spaces, Banach Spaces, or Metric Spaces. I do not believe it is worthwhile having entire fields in Mathematics developing theorems about them in the abstract. All the theory could be refactored into distribution theory and the entire field can be made obsolete and that is better in my view. It introduces burden of a lot of new concepts that are actually just applicable in fairly small (albeit important) examples and nowhere else.

Now you might say well why have Hilbert spaces then because they are all isomorphic. That's because there is actual mathematical substance in being able to take operators in $L^2(S^1)$ and analysing them by Fourier conjugation on $\ell^2(\mathbf{Z})$ sequences. In fact this is one of the most powerful tools in analysis and the techniques are different.

Maybe someone will produce extremely important set of locally convex spaces that radically alter mathematical landscape, but I am pragmatic. These developed mostly after 1929 when Von Neumann originated weak topologies and then of course were important with Laurent Schwartz' work in 1940s. But it is 2022. Where are the locally convex topological spaces that differ substantially from the standard examples? I don't think they exist at all.

8. The Prejudices of Analysts Toward All Other Fields

I know exactly how Analysts have their prejudices about all other fields of mathematics. You see Analysis is really old. Newton and Leibniz began modern analysis in 1660s. Analysis is so old that analysts think they are the main people in mathematics. They go around being generally nice to people but with the feeling that they are the rulers of mathematical empire. They are always saying, "Oh you do Number Theory? That's very nice. It's quite fascinating, and so many neat little theorems you have over there. Keep up the good work." Or, "ah another cozy little field over there, specialist material, topology of manifolds you say? Ah that's quaint. That's really quaint. Good. Good. We need more of your type of people who do good work on various arcane little specialties. It adds colour to mathematics, gives it a certain colour and life. Good. Oh you have some charming little theorems about these special objects I see. Marvelous. Marvelous."

You see Analysts think what they do is *central mainstream important mathematics*. They don't actually think anything else is all that important. They think everything else is quaint and charming and analysis is the main dish for all of mathematics. You can't really blame them. Everyone needs real variable theory to survive. Even algebra and number theory. They look at Analytic Number Theory and say, that's really nice, some serious work these chaps are doing, look at all these *estimates*. Some of these specialty topics people do some serious work too!"

9. Example From Fall 2014 Stanford Analysis Qual

The problem is to consider seminorms

$$\rho_{k,m}(\phi) = \sup_x |x^{-k}\partial^m \phi(x)|$$

on smooth functions that vanish with all their derivatives at 0 defined on $[0,1]$.

By the theory I outlined in this note, the first thing you should do is write down a base of neighborhoods at 0. So you set

$$\alpha = (k,m)$$

and a base of neighborhoods is

$$N(\alpha_1,\dots,\alpha_n;\epsilon_1,\dots,\epsilon) = \{\phi \in C^\infty([0,1]) : \rho_{\alpha_j}(\phi) < \epsilon_j, 1 \le j \le n\}$$

What you have done is apply the entire rigmarole that guarantees that these explicit sets of functions generate the entire natural topology for the seminorms and you don't have to worry about the arbitrary open sets. In one fell swoop you have transformed an arbitrarily impossible problem with various nebulous terms into something where you have some hope of pushing and prodding and getting somewhere.

What is your goal here? It is to show that this exact neighborhood could consist of a neighborhood basis on all functions in $C^\infty[0,1]$ with the seminorms

$$\sigma_\beta(\phi) = \sup |\partial^\beta \phi|$$

You do have to put the finite number of α_j in the base of neighborhoods. But once you have written it down, then you can look at it and say to yourself, "I don't even have to worry about the finite number because if I can get one of these $\rho_\alpha(\phi) =$

$\sigma_\beta(\psi)$ with ϕ satisfying all the zero conditions at $x = 0$ and ψ being a general smooth function without conditions then I have proven that these neighborhoods are the same as σ_β neighborhoods and we do not have an exotic topology here at all."

This is a good fruitful step because you managed to avoid all the hookey-dookey and reduced the problem to manipulating some derivatives and their bounds. That's pretty good divident for doing things Zulf's Way, huh?

Other examples have the same feature. Always write down the neighborhood basis and reduce whatever arcane problem that they give into just manipulating a finite explicit number of seminorms with explicit ϵ_j etc. That reduces the problem to manipulating concrete quantities without too much cost.

Also, never think about what got to the explicit neighborhood; that's a total waste of time because all the hookey-dookey does work and the methods for those have nothing to do with the analytic substance of any actual question of importance.

Now let us get to the mathematical substance of the problem. The central example will be $\phi = x\psi$. For this function

$$\rho_{1,m}(\psi) = \sup |x^{-1}\partial^m(x\phi)|$$

Then computing this gives

$$x^{-1}\partial^m(x\phi) = \partial^m\phi + x^{-1}\partial^{m-1}\phi$$

In other words, we can consider $x^k\psi$ for arbitrary $\psi \in C^\infty[0,1]$ and obtain functions in our space of functions that vanish at zero. Then we can prove something like

$$\rho_{x,m}(x^k\phi) = \sigma_m(\phi) + E$$

where E involves other controllable terms σ_r. This is not a precise argument yet. We are trying to examine equivalence of families of seminorms. The first term already tells us that this approach will yield something reasonable.

Let me see. This is Problem II.4(a) of Fall 2014 Analysis Qual.

10. Request To Stanford University To Thoroughly And Accurately Assess Any Spark At All In Bill Gates

I will request that Stanford University use a thorough examination of the following question: Let $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$ be Bill Gates' acumen in Mathematics, Science, Coding, and Finance. Between time $t \in [1955, 2022]$ in years. Could you please assess in detail if possible the level of his acumen in three areas in the entire period?

I claim that he did not achieve a 1590 in his SAT and I have heard it was 1290. We know he does not have a college degree at all. So he has no significant evidence of higher postsecondary level abilities at all. I claim he was not even good at the high school levels at all.

He is stupid, has dumbfuck ideas like "Hammurabi was white" and "Egypt came before Mesopotamia" and dull thought like "I don't what this guy being the first non-white tenured professor at Harvard and Stanford" and "I am preserving White Science by destroying this guy" and such. Much obliged. I am interested in getting I.5 and II.4 in much better order and spent the day just examining one of the most annoying things about locally convex spaces, which is the issue of firm confidence that writing down a neighborhood base at 0 is valid for all problems.

Further progress on I.5 is actually hard and not so bad in II.4(a). But Bill Gates says he does not detect any 'spark' in me. How would he know? He's really not a very intelligent man on intellectual matters. He's a total charlatan.