STANFORD SPRING 2012 ANALYSIS PROBLEM I.3

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1. Stanford Spring 2012 Analysis Problem I.3

We have a normed vector space $(X, \|\cdot\|_X)$, and two subspaces with other norms specific to them with identity map being continuous. $M, N \subset X$ and $(M, \|\cdot\|_M) \to (M, \|\cdot\|_X)$ and $(N, \|\cdot\|_N) \to (N, \|\cdot\|_X)$ are continuous. Neither of M, N are necessarily closed.

Define

$$\|x\|_{M+N} = \inf\{\|m\|_M + \|n\|_N : x = m + n\}$$

(a) Show $\|\cdot\|_{M+N}$ is a norm.

(b) Show that if M and N are complete, so is $(M + N, \|\cdot\|_{M+N})$.

For the scaling property of norm,

$$\|cx\|_{M+N} = \inf\{\|m\|_M + \|n\|_N : cx = m + n\}$$

Rewrite the inf over $\|cm'\|_M + \|cn'\|_N$ with $cx = cm' + cn'$. Now pull out the $|c|$ from inside the inf and get scaling $\|cx\|_{M+N} = |c|\|x\|_{M+N}$.

For triangle inequality,

$$\|x + y\|_{M+N} = \inf\{\|m\|_M + \|n\|_N : x + y = m + n\}$$

Consider arbitrary m, n satisfying $x + y = m + n$. Since $x, y \in M + N$, there exists m', n' so $x = m' + n'$ and define $m'' = m - m'$ and $n'' = n - n'$.

Now $\|m' + m''\|_M \leq \|m'\|_M + \|m''\|_M$ and $\|n' + n''\|_N \leq \|n'\|_N + \|n''\|_N$.

This gives us

$$\|x + y\|_{M+N} \leq \|m'\|_M + \|m''\|_M + \|n'\|_M + \|n''\|_N.$$

and we have $x = m' + m''$ and $y = n' + n''$. Take two infima on the right side to get the triangle inequality.

For $\|x\|_{M+N} = 0$ iff $x = 0$, note this holds for $\|m\|_M = 0$ iff $m = 0$ and $\|n\|_N = 0$ iff $n = 0$. Since $0 \in M, N$ and there is an inf over things on the right, it follows.

2. Completeness

Suppose x_k is a Cauchy sequence in $M + N$. For $\epsilon > 0$ and $p, q \geq K$ we'll have

$$\|x_p - x_q\|_M < \epsilon.$$

Let m_k, n_k be *arbitrary* elements of M, N with $x_k = m_k + n_k$. So x_k has the property

$$\|x_p - x_q\|_{M+N} \leq \|m_p - m_q\|_M + |\|n_p - n_q\|_N$$

So we need to just not take arbitrary elements, but choose m_k, n_k to be such that $\|m_p - m_q\|_M < \epsilon$ and $\|n_p - n_q\|_N < \epsilon$. Then we'll get convergence for m_k, n_k hence x_k.

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