ZULF RECONSIDERS PROBLEM I.3 OF STANFORD ANALYSIS QUAL 2013 ON DECEMBER 19 2021

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The problem is to prove that a bilinear continous map $B : X \times X \to \mathbf{C}$ can be bounded as

$$B(x,y) \leq C(\rho_{a_1}(x) + \ldots \rho_{a_N}(x))(\rho_{a_1}(y) + \ldots \rho_{a_N}(y))$$

on a complex vector space with seminorms $\{\rho_a : a \in A\}$.

1. Finite Sum of Seminorms?

This problem looks like the sort that is filled with all manner of hidden technical problems, like crocodiles just underneath the water. I know analysts swear by these seminorm mastery. Some analysts even play with seminorms like pianists play Bach and Chopin. Now I am a wise man, and stay away from this sort of thing usually myself and applaud as a spectator. But alas, I do need to understand this myself.

2. Barry Simon Vol I:5.7 Weak Topologies and Locally Convex Spaces

Being a totally ignorant person on the matters of locally convex spaces, the first question on my mind was, "Who the fuck invented this?" And then the answer becomes clear. The term *locally convex space* is due to A. Tychonoff from 1935 and toplogical vetor spaces were introduced by Jon Von Neumann in 1935 [1]

Now why would he do these things?

I see. He wanted to carry over metric completeness over to infinite dimensional vector spaces with some nonstandard topologies. This is quite esoteric to me, so I will just take a relaxed path to it.

First I consider a function $g : X \to \mathbf{C}$ that is continuous *and linear*. For any ball $B(r)$ we have an open set $g^{-1}(B(r))$. Then we hope that Von Neumann's clever topology with families of seminorms and other sorts of technical mumbo-jumbo will tell us that $g^{-1}(B(r))$ being open is equivalent to all sorts of seminorm bounds on $|g(x)|$.

I will tell you honestly, I have no idea why there is a finite set of seminorms in the family $\{rho_a : a \in A\}$ which could be uncountable will give me any bound on $|g(x)|$. I am just hoping that since Von Neumann was a clever boy he put this inside the definition of *locally convex space* and that will solve this problem.

3. The Substantial Issue: How Do You Know What Is Locally Convex?

For finite dimensional vector spaces, there is no need for any seminorm families. To me it seems that the only reason to care about these sorts of things is for

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function spaces of infinite dimensions. What I don't understand is why would we not consider local convexity to the substantial theorem for any given function space with their seminorms?

I don't have enough experience with seminorms that have actual interest in enough examples to evaluate whether this is a purely technical problem or not.

4. Finite Sum Of Seminorms For Bounding Linear Functionals

Let X be a locally convex topological vector space. Let $g : X \to \mathbf{C}$ be a linear functional that is continuous.

Consider what would happen if X were a Banach space. We would get immediately:

$$|g(x)| \leq C\|x\|_X$$

and we'd be done. So I expect that locally convex topological spaces are generalisations of Banach spaces where instead, there exists $a_1, \dots, a_N \in A$ such that

$$|g(x)| \leq C(\rho_{a_1}(x) + \cdots + \rho_{a_N}(x))$$

Then we apply this twice to the bilinear form of the problem and get the result.

References

[1] `https://www.ams.org/journals/tran/1935-037-01/S0002-9947-1935-1501776-7/S0002-9947-1935-1501776-7.pdf`