STANFORD ANALYSIS QUAL 2013 II.1

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1. Stanford Analysis Qual Spring 2012 II.1

On $[a, b]$ with $a < b$ real, we have functions f_n with $n \in \mathbf{N}$ and f. We are given that f_n and f are all *strictly increasing*. We are also given that for all $x \in [a, b]$,

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

We are to prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

almost everywhere.

2. Thoughts

Nineteenth century analysis was obsessed with all manner of pathologies in functions, and analysts take great pride in their fine erudition of all things that can be horrible. It was Karl Weierstrass who really pioneered the cottage industry of the house of horrors in analysis. Thi is one of those sorts of problems. The substance of this problem is how between the cracks of *increasing* all manner of strange and variegated monsters hide.

Let's try to see how bad increasing functions can be. Let's define the following auxiliary function for any increasing function g on $[a, b]$:

$$D(g)(x) = \lim_{y \to x^+} g(y) - \lim_{y \to x^-} g(y)$$

This function will be zero if g is continuous on $[a, b]$ but otherwise it will be non-zero whenever there is a discontinuity. The points of discontinuity of g are

$$A(g) = \{x \in [a, b] : D(g)(x) \neq 0\}$$

At these points $g'(x)$ are not defined. And that is the point of the problem here, to prove that:

$$\mu(\bigcap A(f_n) \cup A(f)) = 0.$$

Let us for convenience define

$$A_T = \bigcap A(f_n) \cup A(f)$$

Given a result that says that $\mu(A(g)) = 0$ we can use the standard Lebesgue measure inequalities to get

$$\mu(A_T) = 0$$

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In the complement $[a,b] - A_T$ we have finiteness and well-definedness of

$$f'(x), f_n'(x)$$

Hold on, don't become pedantic already, let the man think.

Let's assume that there is a small δ such that $f(y), f_n(y)$ are all defined and finite and derivatives are finite and defined for $y \in (x-\delta, x+\delta)$. In this much less pathological situation, we will have for $|h| < \delta$,

$$f(x+h) = \sum_{n=1}^{\infty} f_n(x+h)$$

Now we can definitely pick an $\epsilon > 0$ so that uniformly on $[x-\delta, x+\delta]$, for $n \geq N$ we have

$$|\sum_{k=N+1}^{\infty} f_k(y)| < \epsilon$$

This gives us

$$|f(x+h) - f(x) - \sum_{n=1}^{N}(f_n(x+h) - f(x))| \leq 2\epsilon$$

That's not enough. We need a bit more, which brings in first order Taylor expansion. So we need something like

$$|f(y+h) - f(y) - hf'(y)| \leq Ch^2$$

for $y \in (x-\delta, x+\delta)$. Then the inequality would contain an h in them and we could divide by h and take limit as $h \downarrow 0$.

3. The Problem Is Wrong

Consider

$$f_n(x) = \frac{1}{n^{1+\epsilon}} x^n$$

These are increasing on $[0,1]$ and their sum converges. But the derivatives are

$$f_n'(x) = n^{-\epsilon} x^{n-1}$$

And when you sum them, they diverge near $x \in 1$.

4. Comments About Locally Convex Topological Vector Spaces

You see, in 1935, what Von Neumann was doing on locally convex topological vector spaces is very abstract and frankly not very central to mathematics, like compact manifolds with C^k charts or Lie Groups. Von Neumann was not as good as Hermann Weyl in knowing what is going to be important and central to mathematics, but he was technically a virtuoso. I don't think that he had a serious concern for ubiquity of *examples* that are needed to make sense of some of these things before they become textbook material. Maybe one day in the future, they will matter because function spaces with concrete properties will be more valuable. But these topological vector spaces that are locally convex, these sorts of things matter to very few people even in mathematics. They are far from its center. So they should not have been given the importance they are given. They belong in the middle of technical papers about some function space properties that allow analysis to produce some bounds on things. I don't understand why they are so important. I want to see theorems saying that this and that function space has these specific seminorms and then put local convexity and topologies behind the curtain. I wonder if any career analysts can give me a plethora of examples of function spaces that actually matter and their seminorms, to justify why the concept deserves any attention to all students of mathematics; they are not Lie Groups or compact separable metric spaces. Everyone with any experience can cook up ten examples of these in minutes.