ZULF'S MOTIVATION FOR PUTNAM REAL ANALYSIS PROBLEMS

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I just had a rough completion of Stanford Analysis Ph.D. Qual for Spring 2018 and I have something for all ten problems. I feel that I had neglected real variable theory for too long, and I need to get practice. I obtained the book of Razvan Gelca and Titu Andreescu *Putnam And Beyond* to train for some of the problems. I am interested in Putnam problems in real analysis. I will use their numbering. I am not worried about failures.

1. Problem 363

Find a formula for the general term of the sequence

$$1, 2, 2, 3, 3, 3, \ldots$$

Well the total sum of the series is

$$T(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

so I think we can do something like

$$a_{n(k)} = k$$

Let's see now if we can determine the formula for the sum

$$1 + 4 + 9 = 14 = 3 \cdot 5$$

and

$$1 + 4 + 9 + 16 = 30 = 3 \cdot 5 \cdot 2$$

Let's see if cubing things would do something. We have $3^3 = 27$ and our value is what was it? Yes, 15. What about $4^3 = 16 \times 4 = 64$ and our value is what? It's 30. So I would vote for the formula

$$t(k) = (k^3 - k)/2$$

So the sort of formula could be

$$a_n = k$$

for

$$t(k-1) \le n \le t(k)$$

with

$$t(k) = \frac{1}{2}(k^3 - k)$$

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2. Problem 364

Find a formula for general term of $x_1 = 1$ and $x_n = x_{n-1} + n$ if n is odd and $x_n = x_{n-1} + n - 1$ if n is even.

For this my thought is why not just get a formula first for $y_1 = 1$ and $y_n = y_{n-1} + n$ without worry about whether n is even or odd?

For this easier problem,

$$y_n = 1 + 2 + \cdots + n = n(n+1)/2$$

Now the other part is

$$w_n = \begin{cases} w_{n-1} - 1 & n \text{ odd} \\ w_{n-1} & n \text{ even} \end{cases}$$

Now I always get boundary cases wrong, so I write down the first few terms so I can track them. I start at $n = 2$

$$w_2 = 0, w_3 = -1, w_4 = -1, w_5 = -2$$

I want to see if it's $-[(n-1)/2]$. This looks okay, so my answer is

$$x_n = n(n+1)/2 - [(n-1)/2]$$

3. Problem 365

Let $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 6$ and

$$a_{n+4} = 2a_{n+3} + a_{n+2} - 2a_{n+1} - a_n$$

Prove that n divides a_n for all n.

Let me consider a simpler sequence b_n to get some feel for this situation. Let us say $b_0 = 1$ and $b_1 = 1$ and

$$b_{n+2} = 2b_{n+1} + b_n$$

In this case, we have

$$b_3 = 2 + 1 = 3, b_4 = 6 + 1 = 7, b_5 = 14 + 3 = 17,$$

Let me try something else.

$$a_{n+6} = 2a_{n+5} + a_{n+4} - 2a_{n+3} - a_{n+2}$$

So

$$a_{n+6} - a_{n+4} = 2a_{n+5} - 2a_{n+3} - a_{n+2} = 2(a_{n+5} - a_{n+3}) - a_{n+2}$$

Now I let

$$b_n = a_{n+2} - a_n$$

and I get

$$b_{n+4} = 2b_{n+3} - b_n + a_n$$

I don't think I have a good idea here yet.

I see there is a nice theory of recurrence relations by matrix powers. I will skip this for now.

4. Problem 381

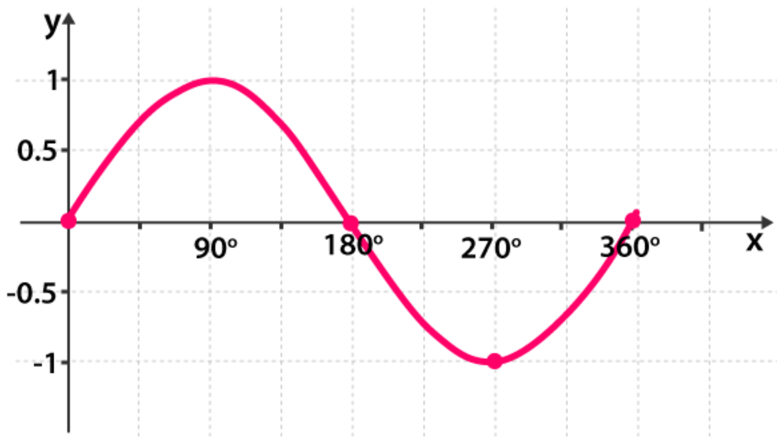
Compute

$$\lim_{n\to\infty} |\sin(\pi\sqrt{n^2+n+1})|$$

Let us see what happens with completing the square

$$\lim_{n\to\infty} |\sin(\pi\sqrt{(n+1/2)^2+3/4})|$$

Let us remember the graph of sine from high school trigonometry.



You see I don't care too much if things are elementary. I look at the graphs to remember things. It look to me like $\sin(\pi) = 0$. That's good, and what about $\sin(\pi/2)$? that looks to me like 1. I am hoping now to see whether

$$\sqrt{(n+1/2)^2+3/4}$$

modulo 1 gives me something useful.

Let us do something else. We note

$$n \leq \sqrt{n^2+n+1} \leq n+1$$

This is valuable. Then we note that although $\sin(x)$ has period 2π, the absolute value $|\sin(x)|$ has period π. See, that's what I am talking about, because now

$$|\sin(\pi\sqrt{n^2+n+1})| = |\sin(\pi(\sqrt{n^2+n+1}-1))|$$

That is the key to the problem methinks. Now we do the rigmarole. And this is streamlined rigmarole too. We let

$$A = \sqrt{n^2+n+1} - n$$

Then

$$A(\sqrt{n^2+n+1}+n) = n+1$$

Then

$$A = \frac{n+1}{\sqrt{n^2+n+1}+n}$$

Then we divide top and bottom by n and let $n \to 0$. We magically add index A_n to the formula and obtain

$$\lim_{n\to 0} A_n = \frac{1}{2}$$

Then we declare

$$\lim_{n\to\infty} |\sin(\pi A_n)| = |\sin(\pi/2)| = 1$$

That required some work. This is physical labour over here. Not as interesting as assuring the human race that my Ahmed-d'Alembert law is the final law of Nature, and will outlast James Clerk Maxwell's law by fifty thousand years as the final law of Nature.

5. Microsoft Word And Excel Are Far More Trivia Than Putnam Problems

This criminal Bill Gates literally thinks the horrible design and horrible code of Microsoft, in Word and Excel are less trivia than Putnam Problems. Such a stupid illiterate ignoramus the world has never seen. Microsoft code is some of the worst bloated, badly designed code the world has ever seen. The solution I provided to the problem of Razvan Gelca and Titu Andreescu's book 381 is far less trivia than all the code in Word and Excel. Why does Bill Gates think sharp and elegant mathematics is trivia when his horribly badly designed code in Word and Excel and his third rate Windows Operating System are not trivia?

I have coded professionally and I can assure you that Microsoft Windows, Word, Excel are paradigms of horribly bad code. Good code is in Meteor and MongoDB. They have taste. Microsoft code is third rate.

6. Problem 453

Prove for all n

$$\sqrt[n]{3} + \sqrt[n]{7} > \sqrt[n]{4} + \sqrt[n]{5}$$

I rearrange the inequality to

$$\sqrt{n}7 - \sqrt{n}5 > \sqrt[n]{4} - \sqrt[n]{3}$$

Now I use a two term Taylor expansion. Let

$$f(x) = x^{1/n}$$

so

$$f'(x) = \frac{1}{n\sqrt[n]{x^{n-1}}}$$

We will get the above inequality from

$$2 \cdot 6^{n-1} > 5^{n-1}$$

This is clearly true for $n \geq 1$. We just plug in the one term Taylor expansion and obtain the need to prove

$$\frac{2}{n\sqrt[n]{5^{n-1}}} > \frac{1}{n\sqrt[n]{3^{n-1}}}$$

This turns is equivalent to the clearly true inequality after some manipulation. The error terms can be handled with higher Taylor expansion but I skip that here.

7. There Is No Physical Meaning To Semiclassical Limits At All

I consider it my sacred duty to bring enlightenment to the world regarding how Nature actually works. For more than a century, this world has been confused, and darkness and ignorance had dominated the entire world regarding the true laws of Nature and how Nature works.

A false paradigm had grown from the work of Max Planck. My beloved people, the human race had been led astray, and so there developed this idea that there is a *Classical World* and a *Quantum World* and these are mediated by the parameter $h \to 0$.

I wish to assure my good people, my beloved human race, that this is folly. This thing called 'semiclassical limit' is physically meaningless. In fact h is equivalent to *global geometric quantity* Λ the so-called Cosmological constant

$$\Lambda = c_0 h^2$$

I don't want to fiddle with c_0. It's a universal constant. The point is that global geometry of the four-sphere is always present and there is nothing meaningful in this 'semiclassical limit'. The actual fundamental law of Nature is my S4 Electromagnetic Law, or Ahmed-d'Alembert Law, and it is a deterministic wave equation in both small and large scales. In macroscopic objects nothing happens to h at all. Instead the issues are about managing complexity in spinor field basis, large numbers of particles and components produce macroscopic measurements *without any $h \to 0$ limit.* There is nothing interesting physically by simply letting $R \to \infty$. The R is constant in the universe, and so macroscopic behaviour of physics is *not available by $h \to 0$.* This sort of thing leads to some sort of mathematical model that is not actually semi-classical but simply meaningless mathematical wonderland without physical sense.