THE PICARD-LINDELÖF THEOREM ON EXISTENCE AND UNIQUENESS

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I am a Princeton University graduate who has never taken an Ordinary Differential Equations class in my life. Yes, shameful, I know. So I am looking at the Ernest Lindelöf paper of 1894 when he proves, following Emile Picard, the existence and uniqueness of solutions of first order ordinary differential equations.

What is at stake is quite clear. The method is called *the method of successive approximations*. Let me tell you right now that these mathematicians are totally socially inept and outrageously narcissistic. Who the hell calls something 'the method of successive approximations'? It's like excitedly telling your lover, "darling, that man over there did his that thing and this thing repeatedly" and then laughing until your snot comes out with the expectation that she will have some secret mind-reading abilities and understand uniquely what 'that thing' and 'this thing' means and join you in your humour. Let me tell you right away, that if you cannot actually communicate things that are humorous properly, your most intimate love affair with someone who, for a change, actually loves you deeply, will poof and your love will leave you with the feeling of too much distance between you and so on. I won't go into the sordid details, but it's not personal. It's reasonable frankly. Why didn't you attempt to use language to communicate to her what you meant?

Well 'the method of successive approximations' is not particularly good either, ok? The world is fortunate that these mathematicians, Emile Picard and Ernest Lindelöf had strange and unique names. No, no, no, no, no. Emile Picard. Not Jean-Luc Picard. That's right so I prefer to call the method 'Picard iterations' or 'Picard-Lindelöf iterations'.

The method consists of considering

$$y' = f(x, y)$$

with starting value $y(x_0) = y_0$ and doing this:

$$\varphi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(s, \varphi_k(s))ds$$

Hold on for a moment. Who thinks this ought to be called "method of successive approximations" seriously? You can mark yourselves as linguistically challenged, ok? Get a goddam literature book and learn to communicate properly. It's for your own good. Never call arbitrary techniques like this something that uses only *generic English words* as though you can monopolise English with your trick and then expect people to give you prizes. The 'method of successive approximations' could mean anything at all. It could mean obsessively-compulsively computing new digits of pi and repeating that you're an excellent driver and counting matchsticks

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people drop looking at every odd angle till Dwight Eisenhower orders your murder. Because *that's* what happens to people who do not communicate in a sensible manner. People pop them, whack them, out of undisclosed suspicions.

The convergence happens because you have metric space contractions, i.e.

$$\rho(Ax, Ay) \leq \alpha\rho(x, y)$$

where A is your 'successive approximation' mapping and $\alpha < 1$ so there is some geometric convergence.

Anyway, global uniqueness follows from Gronwall's Lemma. That's the gist of it.

Now for the really important question. How will you know if you have something that has the assumptions on $f(x, y)$? No one really talks about this very much. In fact, I admit that I did not pay attention to every case. What I do know is what I always do when there is an ordinary differential equation in the middle of something that causes problems. And it works like a charm. You look with a sneery face wave your hands dismissively and say "It's an ODE, so it has a unique solution by theory." Most often all of your audience have the same feeling so you can move on without anyone complaining. Some might even test out your particular sneer and style of dismissal. "Pretty good" you might hear.

But you really should check that the Picard-Lindelöf hypotheses hold. I am looking for tenure, ok? Don't ruin it for me.

1. Thank God For Constantin Caratheodory

While this Picard-Lindelöf iteration scheme is beautiful, and from 1894, the truth is that we hate this condition

$$|f(x, y) - f(x, y')| \leq \alpha|y - y'|$$

for all y in our region of interest. We think the iteration scheme is pretty nice to look at but we don't like this condition all that much.

This is where Constantin Caratheory comes to our rescue. You see, mathematicians love to remark on the beauty of a nice argument. But that's all for show. In fact we don't really give a damn about the beauty of an argument – and neither do the mathematicians. We want all sorts of results that we can use, ok? We want delivery, not a pretty face.

Erm, let me modify that for correctness. We do love pretty faces to prove that we exist, too. But in the end we want things that we can use. We don't want all sort of claptrap from mathematicians who are flaunting their flair and giving us things that are useless. We are *nice* about it.

Now let's look at the conditions of Constantin Caratheodory's Existence theorem. So you let $R = \{(t, y) : |t - t_0| < a, |y - y_0| < b\}$ then you let $f(t, y)$ satisfy

- f is continuous in y on R
- f is measurable in t on R
- $f(t, y) \leq m(t)$ for a Lebesgue integrable $m : [t_0 - a, t_0 + a]\mathbf{R}$

And *Constantin Caratheory* gives us existence and uniqueness of a solution.

I have no idea what the proof is like, and frankly I don't really give a damn, because we're cold *users* of the hard labour of the Constantin Caratheory's of the world. We want these fools do the hard labour, work in the rubber plantations and produce results that we coldly use. Of course we say ”shoulders of giants” and all that to keep them happy. But we *like* this result. You've done well, Constantin Caratheodory. This is useful. None of those mamby-pamby sensitive Lipschitz conditions here huh? Good good.

2. Why Caratheodory's Theorem Belongs In Every Real Analysis Course

Let me tell you the enticement of ordinary differential equation theory. There is no enticement for most people. Not many people actually give a damn about a lot of theory about ordinary differential equations. They want to look at an equation:

$$y' = y^5 + sin(t + y^2) + t^5 2$$

You look at this thing and you look sideways, up, down, and $f(x, y)$ is never going to be Lipschitz. You can see it's Lebesgue integrable the moment y is in a bounded interval. Bam, apply Caratheory Theorem, wave your hands and you have a solution $y(t)$ instantly. You're not going to be solving this equation by all sorts of iterative schemes.

When we come out of an elementary real analysis course, we want some thing that tells us that things we want to do can be done, or to be able to tell if it cannot. Ordinary Differential Equations are not all that interesting very often. But we'll need to solve them or claim a solution without work. That's why these existence and uniqueness of solution results are there. They are there so we can *quickly invoke them* when under stress and not have nagging doubts and sleepless nights wondering whether our house of cards will collapse because some neo-Weierstrass shows up with problems with existence of minimizers or whatever. That's *all they are for*.

Now when we have so much luxury in our time that we can savour all sorts of nonlinear dynamical systems that are meant to drive everyone loopy and which defies the laws of everyone's expectations and so on that's very special. Usually we want the thing to have a solution and we don't want some smart alec take us far away from substantial issues in the middle of our high falutin program to solve all the problems of the world with *Erm, how do you know that the messy looking differential equation has a solution?* You are probably like me and don't drag the fuck to a back alley and shoot him fifty times for messing up your rhythm. But you might feel that way, and that's just normal.