STANFORD SPRING 2018 ANALYSIS QUAL

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1. Looking For Immediate Full Tenured Professorship

Stanford Faculty ought to understand that I am seeking tenure not for my mathematical skills but my *Scientific Accomplishments* over the past year, not only for Four-Sphere Theory which is a fundamental physics but extensive set of empirical results in (a) establishing the empirical validity of Aristotle's Virtue-Eudaimonia theory (b) establishing the universality of human moral nature refuting strongly theories of both *Friedrich Nietzsche* and *Immanuel Kant*, (c) establishing ethnicity-independence of human moral nature, (d) work on evolution of romantic love before the emergence of homo sapiens. Much of the data I used for my quantitative human nature and morality came from World Values Surveys. The mathematical skills I am testing just to get to levels necessary for deeper examination of the relationship between Nature and Mathematics. I have worked with *Jeff McNeal* who was undergraduate supervisor for a prize-winning Mathematics thesis at Princeton which was publishable quality in 1995. I have worked with Daniel Stroock with whom I have a publication in 2000 [1]. I am seeking immediate tenure with plans to settle in Mission San Francisco rather than work at the Palo Alto campus and I have extensive plans for development of quantitative positive psychology for applications for billions.

Please feel free to talk to people in Sciences regarding the importance of my achievements. I do not publish many things in journals but you will find my work dated and archived in [3], [2].

2. Problem I.2

Let S be a closed subspace of $C[0,1]$ with sup norm. Suppose $f \in S$ implies f is continuously differentiable. Prove that S is finite dimensional.

Here the issue is that the embedding map $j : C^1[0,1] \to C[0,1]$ in a compact operator. Let's assume this known first.

The assumption of the problem is equivalent to $S \subset j(C^1[0,1])$. Let

$$S_1 = \{x \in S : \|x\| \leq 1\}$$

Then $j^{-1}(S_1)$ is bounded in the C^1 norm, and therefore $\bar{S}_1$ the closure is compact. We construct the sequence x_j with $\|x_j\| = 1$ with $x_{j+1} \notin M_j = \mathrm{span}(x_1, \dots, x_j)$ and $d(x_{j+1}, M_j) \geq 1/2$. This sequence would lead to no convergence unless S is finite dimensional.

Now we prove that j is compact. We have natural norm $\|x\|_{C^1} = \|x\|_\infty + \|\partial x\|_\infty$, so obviously $\|jx\|_\infty \leq \|x\|_{C^1}$ and j is bounded.

Now suppose $\|x_n\|_{C^1} \leq 1$ for all $n \geq 1$. We want to prove that x_n has a subsequence that converges. We could do the following. Suppose $\epsilon > 0$. Then we

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cover $[0,1]$ with ϵ-balls and choose a finite subcover. And we have a finite number of $I_1, \dots, I_N$ with centers $t_1, dots, t_N$ so that for all $t \in I_j$ satisfy $|t - t_j| < \epsilon$.

Then we examine the Taylor expansion of x_n on I_j. The substance here is that we have to show sequential compactness of $(x_n(t_1), \dots, x_n(t_N))$ and then use the derivative bound to cover the rest of I_j. But these are isomorphic to a set of vectors in $\mathbf{R}^N$ and so we have sequential compactness from Heine-Borel theorem for them. Now we choose the subsequence and use Taylor expansion to get actual convergence in sup norm of the functions themselves.

I will show you how nice this situation is. Suppose you let $y_n = (x_n(t_1), \dots, x_n(t_N))$ be the *subsequence that converges* in $\mathbf{R}^N$. The major worry is gone because you don't need to pick any subsequence any more and you have that. Now it's Cauchy, or as I like to say these days, Bolzano. Now you can find some M such that $n, m \geq M$ produces

$$\|y_n - y_m\|_{\mathbf{R}^N} < \epsilon$$

These are all 'pegged' to the C^1 functions x_n, x_m at $(t_1, \dots, t_N)$. Now we have for any $t \in [0,1]$, it falls in I_q for some $1 \leq q \leq N$ and so

$$|x_n(t) - x_m(t)| = |x_n(t_q) + x_n'(t_q)(t - t_q) - x_m(t_q) + x_m'(t_q)(t - t_q) + O(\epsilon)| \leq \epsilon + \sup \|x_k'\|(2\epsilon) + O(\epsilon)$$

This is using the boundedness of $\|x_k\|_{C^1}$ and the lengths of I_q all at most 2ϵ and the rest is higher order terms that are easy to control. The heart was the choice of the subsequence in $\mathbf{R}^N$ and I used the Taylor expansion of $C^1[0,1]$ here to get control of all points in $t \in [0,1]$ by the fact that linear terms are controlled by the size of I_q nicely.

3. Stanford Assessed My Physics Genius To Have Exceeded Albert Einstein

I am most gratified that Stanford did an assessment and found that my physics genius to have exceeded Einstein. I would like to remind the world that Einstein was as truly remarkable genius in physics and his work was actually crucial to my breakthroughs in producing the final laws of Nature. I will not go into much detail but it was the form of the General Relativity law that gave me the first intuitions regarding four-sphere theory. When I looked at it carefully in a loft in Williamsburg Brooklyn, I saw that it was the Ricci curvature of a hypersurface embedded in four-sphere, whose formula I knew from the book of Shing-Tung Yau and Richard Schoen, *Lectures on Differential Geometry*, once I ignored the Special Relativity indefinite signature metric. Thus even though my physics theory was quite different from his − I strongly went against Special Relativity − what he had discovered was a different sort of profound truth about Nature, although that is not what he theorised.

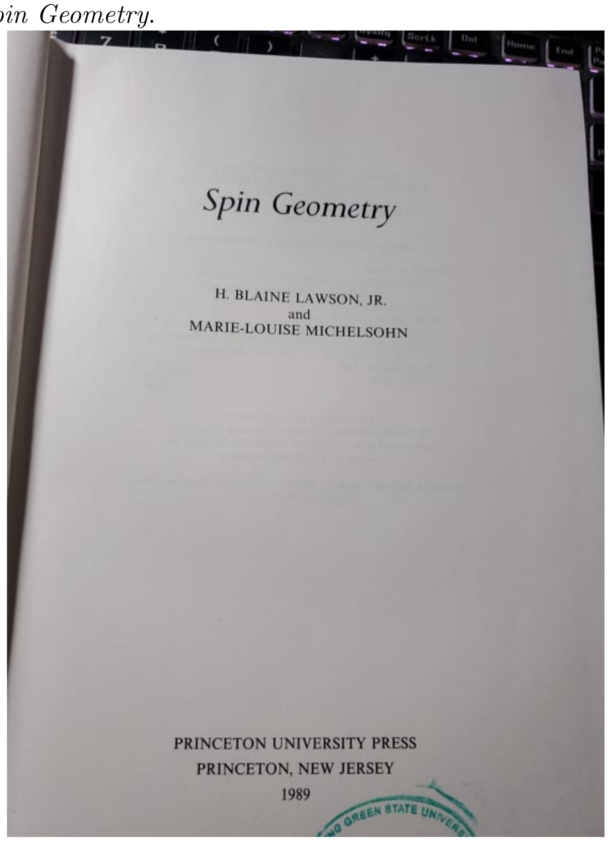
What he had discovered was a remarkable fact that there is an unseen four-sphere structure of existence in which the physical universe is *constrained to be embedded* at all time, and this is crucial to features of Nature.

Einstein was rather eager to avoid a vast fourth spatial dimension as was the rest of the physics community for a long time. His Special Relativity theory was based on (a) firm conviction that the Michelson-Morley conclusions from 1880s were definitive, and (b) the Maxwell laws were exact truth. Henri Poincare introduced Special Relativity invariances slightly before him actually [4]. Special Relativity is wrong in the actual universe. They arise from a strong faith that space-time empty of matter is actually flat, and space contains within it symmetries that keep the wave equation invariant.

I have repeatedly emphasized that this is not the case. Einstein resisted strongly Lemaitre's theory of a expanding cosmos and proposed an S^3 model of space. His introduction of $\Lambda \geq 0$ was more important in the end for four-spere theory for this is what led me to the conviction that four-sphere theory would be true with absolute space and absolute time.

As you know, I had interpreted Λ as the curvature of a static eternal four-sphere. Then I worked quite a bit to examine whether atomic clocks in the early 1970s were accurate enough to have reliable evidence for time deformations. I strongly rejected both time deformation and space deformations are invalid. There are many reasons, the strongest of which was that it was clear to me that Planck's constant h was related to Λ as $\Lambda = Ch^2$ for some universal constant C and it made no sense in my four-sphere theory that space can deform, and the conclusions of Michelson-Morley are right for three dimensional luminiferous aether but the surface of a four-sphere was reasonable substitute.

Now I was also influenced by what I knew in mathematics all these years away from Mathematics. When I had gone to Columbia University for graduate school, John Morgan was kind to me and asked me to master Lawson and Michelsohn's *Spin Geometry*.



This book is truly marvelous in a unified treatment of the Atiyah-Singer-Dirac operator and I have followed their appeal and called D acting on spinor fields of a four-sphere Atiyah-Singer-Dirac operator when I formulated my S4 Electromagnetic

Law, which is inspired by the original wave equation written down for the vibrating string by d'Alembert from 1742.

What occurred to me over the years is that within physics, the central intuition that has been retained is the *harmonic oscillator* as a ubiquitous feature of Nature. I was fortunate that I has studied the geometry of four-manifolds where the pure tones of the four-sphere were on my mind and the homogeneous spacing of energy levels observed had then a clear alternative model, that the entire universe, seen and unseen behaved as a giant drum. And this turned out to have stronger power for fundamental physics theory after over a decade of work.

4. Another Perspective of History Of Physics

This is such an important topic, fundamental physics that it is worthwhile examining it from multiple perspectives to understand. I have explored these for years. You see, the notion of flat empty space was not invented by Einstein. He inherited the notion from all of Western and Eastern Civilisation. Euclid is truly the source of this particular notion. Einstein was quite adventurous in examining alternatives.

It was really mathematicians, first Gauss and Lobachevsky and then Georg Bernhard Riemann who has been quite concerned with Euclid's postulates and basis of geometry. In 1854 Bernhard Riemann put forward the bold proposal of study of metric geometry in arbitrary dimensions with curvature measured using the Riemann curvature tensor. Albert Einstein was a physicist, and there was no reason for him to abandon the total intellectual heritage of all of Western Civilisation for *empty space*. That was left for me, Zulfikar Moinuddin Ahmed, in the summer of 2008, more than a century after Einstein's Special Relativity was published in 1905.

Einstein had not reason to abandon Euclidean geometry of *empty matterless space* but I did. I had reason because I already knew that $\Lambda > 0$ was measured, and $\Lambda = 1.11 \times 10^{-52} m^{-2}$ and I was mathematically trained and did not pursue physics at Princeton, 1991-1995. For me the natural hypothesis *was exactly* that the actual universe had *positively curved compact geometry* and there was never any flat Euclidean space anywhere in Nature at all except in human imagination.

Some people take my theory as a challenge to *Einstein*. I have no animosity towards him. He was a saintly genius and I love him. But Science is not an enterprise about individual geniuses. It is human collective best knowledge of Nature. I do Science with care for truth. Four-Sphere Theory is eternal truth. You will see this in the centuries that follow that is is far stronger than all the theories currently established.

5. Personal Interpretation Of History

You see, these seem small to some people but my natural senses give me some insight. Albert Einstein was born in 1879, and James Clerk Maxwell's great work of unified electromagnetic field and his equations were done before his birth. He inherited a tradition of physics where there was no reason to doubt the strongest ground, that Maxwell's equations were true about nature. Einstein did not challenge this and instead built on this ground. Special Relativity is essentially shoring up the faith in flat empty space. Michelson-Morley experiments disrupted the peace of physics and Einstein rescued physics by relativity because without aether how will anyone think about waves?

I was in a very very different world in 2008. In my world, the positivity of Λ the cosmological constant was well-established measurement, so for me it was very easy to just look at $\Lambda > 0$ and read Marc Kac's paper on "Can One Hear The Shape Of A Drum" and instantaneously consider a closed homogeneous geometry responsible for quantisation of energy. In 1905 the spectral theorem for compact self-adjoint operators was not even part of mathematicians' intuition, and it was bread and butter mathematics from undergraduate years for me. Why would Einstein have hypothesized that empty space is flat or that there is a large purely electromagnetic spatial dimension in the universe that is not directly perceived? Those things do not seem sane to a gentleman born in Europe in 1879. They were not even radical enough in some ways in 2008.

Spectral theorem for compact self-adjoint operators I knew from Peter Sarnak's functional analysis course *junior year* around 1992-3 at Princeton. I did the whole proof for homework problems so I had a firm feeling for it. In one of the recent Stanford exam problems I even give a simpler proof. Sixteen years had passed before I began thinking about four-sphere theory after my first introduction, time in which I had just absorbed it as basic truth. On the other hand Mark Kac's very famous article 'Can One Hear The Shape Of A Drum?" appeared in 1966, some seven years before I was even born. Professor Peter Sarnak emphasized the spectrum of various compact surfaces in his course, and he is a fabulous teacher and lecturer and he made a lasting impression. For me compact spatial geometry for the actual universe was the most natural immediate hypothesis in 2008. At first I was more concerned with Einstein and Relativity but instead challenging *Expansion* and explanation of quantisation. I thought perhaps General Relativity will fit in with my ideas but some years later I had to reject Special Relativity. I took the strong view that theories of physics that deform time or space at all are not parsimonious and are surely wrong.

6. My Experiences In Biotech And Finance Gave Me Experience In Scientific Mindset

I worked between 1995-2008 in Finance and Biotech and it was far from mathematics and also far from *theoretical physics*. The mindset of a professional scientist is very different from even theoretical physicists. There *matching measurements* is the be-all and end-all of all theories, whether *phenomenological* or fundamental. In finance and biosciences there is nothing else to stand on. Theories in quantitative finance are not based on fundamental principles and mathematical theorems. They are based on thousands of hours of fine tuning confronting vast amounts of data. In biology too, the overwhelming complexity dominates and no purely theoretical considerations are trustworthy.

This experience was crucial to my development as a man generally, and also to my viewpoint about four-sphere theory. Although it is pristine and beautiful mathematically, for we have things like $L^2(SO(5)/SO(4))$ as the space of functions on the four-sphere, something that no scientist outside extremely theoretical ones will ever consider seriously, most of my effort from 2008-2018 involved attempting to produce some calibrations and predict the redshift slope in static cosmological model, and do other things that will seem rather unsophisticated to the theoretical physicist like reproduce Earth-Sun gravity using Van der Waals forces with care in numerical values. But that is the heart of science to me, the *calibration of* *models*. This discipline is from Finance. I had the good fortune of working with Andrew Morton, Kaushik Amin, Dev Joneja, Anthony Lazanas, Douglas Macbeth, and Jawahar Chirimar at Lehman Brothers, and they are top notch quants, and quants know how to deal with empirical calibration even better than chemists and lab physicists. Why? Because finance data comes with *no theory at all*. It's just mysteriously explosively messy. And so I did the right thing and put aside theoretical issues and calibrated the model to Nature. And that is why I succeeded. No theory can win against one that fits measurements better and is more parsimonious. You cannot really appreciate that is the truth until you are in environments like Biotech and Finance.

7. Problem I.3

Before I go into this problem, I want to return to the story of Erik Ivar Fredholm again, because there is something especially beautiful about what he did for integral equations that I don't yet understand. Erik Ivar Fredholm was a student of Gosta Mittag-Leffler (doctorate 1898), and his work on solving integral equations moved the great David Hilbert to literally abandon algebra and geometry and devote years of his life into analysis, spawning Hilbert space theory, theory of linear operators on Hilbert spaces, the spectral theory of compact self-adjoint operators. Of course it was Erhard Schmidt who elucidated the geometry of Hilbert spaces. This is very close to my heart because this is really the heart of what allowed me to have complete trust that it is the *compact homogeneous geometry* of the entire universe, or *existence* if you like, that is responsible for this marvelous feature that energy is quantised *evenly* in units of h in Nature. The story began with Erik Ivar Fredholm for the features of mathematics for integral equations.

I will digress a bit and take you on a journey back to my own past. I am at Princeton, and I am trying to fathom the Peter-Weyl theorem. Professor Peter Sarnak tells us a magnificent story of Erik Ivar Fredholm and points out that the Laplacian on a compact manifold has a discrete spectrum but it is the compact self-adjoint *resolvent* operators that is the way to get clear eigenspaces for the Laplacian itself. I was spending a lot of time in Fine Hall Library, in the basement level. I was a bit overwhelmed by the novelty of all this. I felt that I had left Kansas already. Well, fine, I am originally from Bengal, and was never in Kansas at all, so it's a manner of speech. I mean that I had entered some part of serious mathematics, where people do things that transform the intellectual life of the entire world. I worked hard for the course and managed to get an A-. I was not even close to the acumen of some of the others in the course, Terry Tao was there and was sharp and erudite already, and Steven Gubser. There were some of the brightest minds in physics and mathematics there. I was not unhappy.

And so let us return to Erik Ivar Fredholm. His work was on solving the equations

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t) f(t) dt$$

by considering matrices and determinants. He found a way to solve these sorts of equations using finite matrix approximations of the equation and taking a limit. There was no L^2 space at all at this point as Frigyes Riesz had not invented them yet, and Lebesgue did not do anything yet as his great works were between 1902-1905.

Now I will say that I have never actually solved these sorts of integral equations at all in my life. So it was not the integral equations that interest me; rather it is the fact that the spectral theorem for compact self-adjoint operators grew out from the ideas in Fredholm's work of 1900, and the concepts were so powerful that differential operators, quite unbounded, were studied with compact resolvents for eigenfunctions of Laplacian on all manner of smooth closed manifolds ever since.

Let's at least write the problem down before we continue the digression which is more interesting to me than the solution of the integral equations.

7.1. **Statement of Problem I.3.** Let $K \in L^2(\mathbf{R}^2)$. Define $T \in L(L^2(\mathbf{R}))$ by

$$Tf(x) = \int K(x,y)f(y)dy$$

(a) Show that T is bounded and compact.

7.2. **Return To Lengthy Digression.** Jens Lenström has a very nice thesis that addresses some issues of history of functional analysis from 2008 [5]. He points out that in the early period 1900-1930 linear algebra did not have enough development to form the basis for functional analysis. This is an important insight, as for me, linear algebra had been part of my life from even before freshman year at Princeton in 1991 and it's hard to imagine a world where linear algebra is not basic repertoire for mathematics.

8. It Is Always Folly To Underestimate Mysteries of Nature

In Mathematics, there is a valid path in surveying the world of pure and beautiful ideas, and in these you can find solace, and you can build your home in some things that are central to Mathematics. In Science, this is pure folly. Even in theoretical physics, for Nature is a harsh mistress, and will without any doubt punish those who takes her for granted. And this is not a simple thing. Those who are genuine and serious in their pursuit of Science are fools if they do not respect the ineluctable mystery of Nature, and are not awed by her frightening power to befuddle the wisest of sages among Men. I am not particularly impressed by pure theorists in Science as I approach 50 this year. Science is most definitely not Mathematics. The question of their relationship has not even been addressed seriously enough to fool a freshman at Princeton. These are deep issues with unsatisfactory answers.

When I was young, in high school 1987-1991, there were so many thick textbooks filled with exact models that had a sense of finality to them. Especially in physics and chemistry there is a sense that all possible things you can imagine have been found with exact precision and mathematical formulae. But over the decades, I matured in Science, and now I am challenging the explanation of the redshift-distance slope as dynamical expansion for more than a decade. Mathematics is in a sense a vast subject in the productive activity, and there are perhaps some million titles in libraries and bookstores and in private possession. When we say "Nature is described by Mathematical Laws", we are not saying anything substantial at all *until we specify exactly which mathematical objects and their properties and relations actually matter to Nature.* Not all of Mathematics will matter to Science, and the question has never been seriously addressed at all in my view. The significance of this *intersection* is completely opaque and mysterious, certainly to me, but I believe to all human race as well. I think this is a deep and important issue that will be resolved over centuries. There are a cacophany of opinions on these issues today and not many solid and clear answers that can be reasonable or right. Partly this is due to the fact that Nature has vast complexity and there are no guarantees that the specific issues that have purely mathematical interest will ever give us ability to penetrate the complexity. Partly it's because purely Mathematical interest cannot be based on the mysteries of *external Nature* and will drive to issues that have subtleties of an entirely different source. You would need to live in a world dreamt up by Jorge Luis Borges to appreciate the difficulties involved in having clear and solid answers that all people of Earth can appreciate one day. That's an entirely different world than the one in which we live today.

9. Let's Be Serious Bill Gates

Bill Gates contends that *non-white people are not capable* of various sorts of sophisticated intellectual achievements. Never mind that Ramayana (Scholars' estimates for the earliest stage of the text range from the 7th to 4th centuries BCE) and Mahabharata (written down in 400 BC) were completely works of non-whites and occurred before *England had any literacy at all.* So clearly this proposition of Bill Gates is not viable. But more importantly, Bill Gates was not *capable of graduating from Harvard at all* while I graduated from Princeton with magna cum laude in 1995. It does not look to me like European ancestry gives anyone ability to graduate from university let alone be in any position to question abilities of a man who successfully challenged all the established theories of physics for a century.

Why doesn't Bill Gates prove himself. I have not been impressed by his intellectual capacities. Please, take some of the Stanford Ph.D. Quals and show that your ethnicity alone gave you superior intellect. I am not impressed by your *coding ability* either. So why don't you prove that you are capable first before you assert racial aspects. You see, Bill Gates, before you can attribute superior intellectual abilities to *all white people* you have to prove that these theories hold for you first.

Why does Bill Gates think that calling everything 'trivia' will compel people that he is a great genius. This is obviously a childish devious ploy to avoid actually showing any intellectual ability or skill at all? Why is what Erik Ivar Fredholm did with study of these integral equations trivia? Why is what David Hilbert did producing mathematical understanding of complete inner product spaces trivia? There was a vast improvement in the world's understanding of functions on a circle that resulted from it, as $L^2[-\pi, \pi]$ and Fourier coefficients allowed proof that all square integrable functions had convergence in L^2 norm. It's one of the most beautiful parts of mathematics. What makes all this 'trivia' to Bill Gates except ignorance and bad taste and illiterate stupidity? It is rather *ethnicity* that is trivia, and not these far more lofty achievements of Man.

10. Return To Functions

I am pleased with the focus of Jens Lenström on basic issues because the history was not crisp and clear to me. For a long time, the definition of function was due to Euler from 1748 which reads: "A function of a variable quantity is an *analytic expression* composed in any which way whatsoever of the variable quantity and numbers or constant quantities." Now it was not exactly easy for me to think about functions in a different way myself in high school 1987-1991. I think that I was able to psychologically handle functions only after a summer spent at Ohio State University Arnold Ross number theory program in 1988. The assignment of a unique value $f(x)$ for $f : X \to Y$ became habitual for me around this time. Now 1900 was just around 150 years after the definition of function, and this matters to me even now profoundly because I am working on fundamental physics. I need to understand more clearly what it means for me to say, "Let $\phi : S^4 \to \Sigma S^4$ be a matter field." Thus in a sense, the same questions persist because, I would posit, that there are levels of clarity and crispness that slowly governs these fundamental concepts. When Laurent Schwartz introduced distributions, he generalised notions of functions to duals of function spaces, linear duals. And then now I am asking the world to consider matter in the university as spinor fields on a four-sphere universe *only a small part of which we experience directly*, i.e. the physical hypersurface. And so our notion of function will gain further clarity and maturity over time. But these fundamental issues are not arbitrary at all and had never been. They are not decided by axioms; that is a seriously misguided way to understand mathematics. The axioms always represent some substance and this substance, at least for the mathematical models of Nature, is fundamentally of interest, because it is what allows predictions of our models to have realisation in the actual world.

11. Some Interesting Oddities Of The Way We See Objective Reality

What is objective reality. I was strongly scientific during the first part of my life. I venerated my devout Muslim maternal grandmother. She was religious, and her devotion to Islamic piety was pleasant to me. And yet I was an Atheist from 1979. In the period 1979-2008 I was Atheist, but in America, where there was no religious pressure. That year I was curious about many things among them I examined this old idea of Descartes that the *pineal gland* was the locus of the Soul's connection with the physical body. "The pineal gland produces melatonin, a serotonin-derived hormone which modulates sleep patterns in both circadian and seasonal cycles." Not that interesting. But then: "The pineal gland is located in the dorsal diencephalon, and it contains both rod and cone photoreceptor cells. The pineal photoreceptor cells share great homologies with retinal photoreceptor cells, such as cell morphology, the expression of opsin proteins, and responses to light stimuli." [7]. I will not produce a theory here except to point out that the actual universe has a very large purely fourth spatial dimension, so there is something to this old religious idea of a 'third eye' and there is some interest in understanding this in rigorous scientific manner. I will simply leave this as an open question since I don't have the energy to get into a controversy regarding this now.

Well, Stanford Physics, there are more things in heaven and Earth than are dreamt of in your philosophy. But that is the point of four-sphere theory. We want to understand what the things are in heaven and Earth, and S4 Electromagnetic law governs all thing in a static eternal constant radius four-sphere. But I won't get involved in interminable and fruitless debates on things that have no resolution by experiment.

12. Zulf Redraws The Line Between Science And Religion

Once four-sphere theory is firmly established, as I am sure that it will, the line between Religion and Science will change in the following way.

All of existence is a large four-sphere of fixed radius $R = 3075.69$ Mpc. We will have our lives in a dynamically evolving three dimensional hypersurface $M(t) \to S^4(R)$. The four-sphere can have absolute coordinates, but the physical universe will not stay still in those coordinates. From within $M(t)$ we can detect, by indirect means, some phenomena in $S^4(R)-M(t)$. Those are phenomena that has *physically detectable and measurable effects* in $M(t)$. It will be a challenge of many centuries before human race has full understanding of what phenomena they are. And this is what I believe all of existence is, and the lines between Religion and Science will be redrawn over time. But Four-Sphere Theory needs to firmly and clearly established and tested before this for without a reliable fundamental physics with strong calibration to known phenomena in $M(t)$ there won't be any hope of objective understanding of anything more. It is a good idea to strengthen fundamental physics first before embarking on journeys into the unknown.

13. Problem I.3 a

$$\|Tf\|^2 = \int |\int K(x,y)f(y)dy|^2 dx$$

$$\& \leq \int\int |K(x,y)|^2 dxdy \|f\|^2$$

We used just Cauchy-Schwarz and have boundedness since $K \in L^2(\mathbf{R}^2)$. Now suppose $B = \{f \in L^2 : \|f\| \leq 1\}$. Suppose $g_n = Tf_n$ for an infinite sequence $f_n \in B$.

Let

$$k(y) = (\int K(x,y)^2 dy)^{1/2}$$

For all $n, m \geq 1$ we have

$$|g_n(y) - g_m(y)| \leq k(y)\|f_n - f_m\| = 2k(y)$$

Now for any $\epsilon > 0$ we can find $b > 0$ such that

$$\int_{|y|>b} k(y)^2 dy < \epsilon/2$$

This tells us that for all $m, n \geq 1$ and $|y| > b$ we have

$$|g_n(y) - g_m(y)| < \epsilon$$

We will keep our attention, therefore, on what is happening in $I_0 = \{y \in R : |y| < b\}$ with the same $\epsilon > 0$ fixed.

These are L^2 functions so we cannot simply use all sorts of Taylor expansions. However we can use *density arguments here* and try to get some subsequence converging to *zero*.

Let's now pretend $[-b, b]$ is $[0, 1]$ and invoke density of $C^1[0, 1]$ in $L^2[0, 1]$ by the obvious scaling and translation with $0 \mapsto 1/2$ and $-b, b \mapsto 0, 1$. We now are in the situation where g_n, g_m are all $C^1[0, 1]$ and

$$|g_n(y) - g_m(y)| \leq k(y)$$

from the $\epsilon > 0$ we take $\delta > 0$ so that for $|y - z| < \delta$ we have

$$|g_p(y) - g_p(z) - g_p'(z)(y - z)| < \epsilon$$

Then cover $[0, 1]$ with open intervals J_a of length at most 2δ and use compactness to select finite subcover $\{J_a\}_{a=1,\dots,P}$ with centers $y_1, \dots, y_p$. Define

$$g_{n,m}(y) = g_n(y) - g_m(y)$$

for all $y \in [0,1]$.

We have double-indexing here and so we need to be a bit more careful later on. But I want to be sloppy right now to keep my thinking on the idea, so I hope my dear reader forgives my intuitive path.

Now use Heine-Borel theorem on $\mathbf{R}^N$ to choose a subsequence of vectors from $(g_{n,m}(y_1), \ldots, g_{n,m}(y_N)) \in \mathbf{R}^N$ that converges. Then use Taylor's theorem to get the limit to be continuous. Let's call the limiting continuous function $G \in C[0,1]$. We have

$$|G(y)| \leq k(y)$$

So then Lebesgue's Dominated Convergence theorem ensures that not only is G continuous but that the "limit" is pointwise as well as L^2 for $g_{n,m}$ to G. Our major problem is to understand what will ensure us to conclude $G = 0$ identically.

Now suppose $G \neq 0$ on J_a for some $a \in \{1, \ldots, N\}$. Then we have the situation that $g_n(y_a) - g_m(y_a)$ pointwise led to some nonzero "limit", i.e. $G(y_a)$. I know this is not precise, but I am thinking about the issues here, so if you don't mind I will continue with my wooly thinking.

Let's look at this more closely by examining f_n, f_m.

$$|g_n(y_a) - g_m(y_a)| = |\int K(x, y_a)(f_n(x) - f_m(x))dx|$$

In other words

$$0 < A \leq |g_n(y_a) - g_m(y_a)| \leq k(y_a) < \infty$$

for all $n, m \geq 1$. Intuitively this will lead to a contradiction as follows. Let

$$s_n = g_n(y_a)$$

and these are just real numbers. We have a set of real numbers with $|s_p - s_q| \geq A > 0$ so we can order them in $[-k(y_a), k(y_a)]$ which are the bounds for the differences and then begin chaining the lengths from anywhere going to the right and at each step we will go to the right by at least A and eventually we will pass beyond $k(y_a)$.

14. Zulf Repents And Shows Contrition And Demonstrates Karma Is A Serious Force

You know, when I was a young student at Princeton, I had done some continued fraction approximations at Ohio State University at the Ross Program. I did not want my entire life to be filled with achievements of how many transcendental numbers had continued fraction representations efficiently represented. That's not something that would be suitable for a suave gentleman who likes the win the hearts of the beautiful ladies. I had resolved that these sorts of nitty-gritty things belong to the industrious work of all these *analysts*. I was not interested. I routinely skipped the first preliminary chapter of any text with 'real number system' and all sorts of lemmas about sequences of real numbers.

Well finally karma has caught up with me. I am staring at a situation that I had never mastered about the logic of sequences of real numbers satisfying

$$A \leq |s_n - s_m| \leq B$$

for all $n, m \geq 1$. This is a situation that actually requires some care and I have never actually done this problem. Here I am almost 50 years old, still doing these sorts of problems. Oh how the tables have turned! Failure to pay attention to these nitty gritty issues of the ordering of $\mathbf{R}$ comes back to produce problems for me.

Now I have to be like a youngling chaining together subsequences. Let's do it this way. Define a subsequence t_n this way.

$$t_1 = s_1$$

At each step t_{m+1} is defined by going towards left or right based on whether infinitely many of the remaining set of $S_m = \{s_{n_m+1}, dots, \}$ are to the right of t_m or to the left of t_m. By assumption, every element of S_m will be at least distance A away from t_m. And so we can chain them so that $|t_m - t_1| \geq (m-1)A$. Using this argument we can contradict $A > 0$ for bounded differences.

This is not even intuitive for me because my general intuition is not about differences of real numbers but about sequences themselves. And in Problem I.3(a) this is crucial, a fact about real sequences that I was not even particularly attentive to from freshman year.

It's a sad day for Zulf today. The skies will fill up with dark clouds and rain, to wash away my shame.

Let us respect Dan Stroock's view that this requires more precision. The key here is that these S_m are relatively closed because we have an ordering from $\mathbf{R}$ and each point is isolated from all others in S_m by at least A distance. Given any $x \in S_m$ we have $(x - A/2, x + A/2) \cap S_m = \{x\}$ so S_m is relatively closed in $\mathbf{R}$. Then depending on whether S_m is to the right of t_m or left, we have $\inf S_m$ or $\sup S_m$ contained in S_m then we can choose the smallest or largest element of S_m and then the conclusion follows from the ordering of S_m. The ordering does not actually care about the sequence numbering of s_n at all we just invoke abstractly ordering properties of $\mathbf{R}$.

This is a pretty nitty gritty fact that's quite subtle. I am surprised this was not known to me, because these occur all the time for Cauchy-sequence type situations.

Then we apply this thing to contradict that the any limit of these $|g_{n,m}(y_a)|$ can be positive and finally we have a convergent subsequence at y_a to zero.

This is strange to have to do all this but I don't know a better way.

15. Problem I.3a Take Two

I was floundering on Problem I.3a before so I will try to streamline things to get to something rigorous. Compactness of L^2 kernel operators is profoundly important and so deserves care.

Here is what I will plan to do.

- We begin with assuming $\tilde{g}_n \in T(B)$. I change the notation with $\tilde{g}_n$ because I will approximate them with C^1 functions later that I will call g_n. I will get easily a bound

$$|\tilde{g}_n(y) - \tilde{g}_m(y)| \leq k(y)$$

  with $k(y) = (\int K(x,y)^2 dx)^{1/2}$ using and we will have $\tilde{g}_n = Tf_n$ for some set of L^2 functions
- I will aim to prove pointwise subsequence convergence for $\tilde{g}_n$. That's the goal, pointwise subsequence convergence, and then I will apply Lebesgue's dominated convergence for the L^2 convergence of the subsequence.

- I will fix $\epsilon > 0$ now and then first eliminate problems by noting that there exists a $b > 0$ so that
$$\int_{|y|>\epsilon} k(y)^2 dy < \epsilon$$
Then I will focus attention to *pointwise behavior of* $\tilde{g}_n(y)$ *for* $|y| \leq b$ *alone*
- Then I will use a scaling and translation to assume $\tilde{g}_n(y)1_{[-b,b]}$ are on $[0,1]$
- Then I will do a L^∞-approximation of $\tilde{g}_n(y)$ with $g \in C^1[0,1]$. This is some version of Littlewood's three principles theorem. I don't remember their names. The substance is not here so I don't care too much.
- Then I will begin with $g_n \in C^1[0,1]$ and for all $n, m \geq 1$ we have for $y \in [0,1]$:
$$|g_n(y) - g_m(y)| \leq k(y) + \epsilon$$
Note that I only do the approximation on $[0,1]$ avoiding all sorts of problems on the non-compact part $|y| > b$.
- The nitty gritty begins at this point. We let
$$g_{n,m}(y) = g_n(y) - g_m(y)$$
for convenience. Our goal is to prove that there exists a $P \geq 1$ so that for all $n, m \geq P$ that
$$|g_{n,m}(y)| < \epsilon$$
for all $y \in [0,1]$.
- We want to use the $g_{n,m} \in C^1[0,1]$ to obtain, for our $\epsilon > 0$ that was fixed, a $\delta > 0$ such that
$$|g_{n,m}(y) - g_{n,m}(w) - g'_{n,m}(w)(y-w)| < \epsilon$$
whenever $|y - w| < \delta$. This will allow us to cover $[0,1]$ with intervals of size at most 2δ and then find a subcover with centers $y_1, \ldots, y_N$ with intervals $J_a = (y_a - \delta, y_a + \delta) \cap [0,1]$.

  This is our setup. Then we claim that for each of these $y_1, \ldots, y_N$ the sequence of real numbers $g_{n,m}(y_q)$ has a Cauchy subsequence – or based on fairness to Bolzano – Bolzano subsequence which has the same definition as Cauchy but we call it Bolzano because he deserves credit for limits and so on.
- Then we enter the nitty gritty part of our arguments which will use ordering of $\mathbf{R}$ and use some topological tricks involving closed countable sets. This is something that is true, but it seems to be new.

16. The New Element Only

The new element here analytically is a very nitty gritty argument for a sequence of real numbers s_k. If $\{s_k\}$ satisfies
$$0 \leq A \leq |s_n - s_m| \leq B < \infty$$
for all $n, m \geq 1$ then $A = 0$. We assume $A > 0$ and produce a contradiction. We place all s_n in the real line and we construct a new sequence t_q in the following way. First $t_1 = s_1$. Then we consider two sets
$$S_1^- = \{s_j : j \geq 2, s_j < t_1\}$$

and

$$S_1^+ = \{s_j : j \geq 2, s_j > t_1\}$$

By assumption $|t_1 - s_j| > A$ for all $j \geq 2$ which implies that $S_1^- \cup S_1^+ \neq \emptyset$. On or the other has to have $|S_1^+| = \infty$ or $|S_1^-| = \infty$ as all the points are isolated with no intersection with other points in the sequence s_j. These are relatively closed sets and therefore $\inf S_1^+ \in S_1^+$ and $\sup S_1^- \in S_1^-$ when they are non-empty. Then chose t_2 based on which is infinite cardinality and closest point to t_1.

Then at each subsequent step, define analgous sets

$$S_q^+ = \{s_j : j \geq j_q, s_j > t_q\}$$

and continue this procedure. The j_q is the index that occurred in the sup or inf. We can continue this procedure and obtain a contradiction by Q steps where

$$QA > B$$

This topological sort of argument then ensures that we can get a subsequence that is Cauchy for $g_n(y_r)$ or $r = 1, \dots, N$. The rest of the analysis is for Problem I.3a is more standard approximation using Taylor etc. This argument is elementary but it is quite esoteric so it's not something I have seen before. But it's right.

17. Zulf Requests Daniel Stroock To Publish A Sharper Version of The Result Here Under Both Our Names

Professor Daniel Stroock had been instrumental in meta to test this topological idea and he did most of the work on non-compact Hodge theory where he shared credit with me. So it's only fair that he should publish a sharper version that is more acceptable to analysts. I hope that he will do this with sharing credit with me of course. It's a very general theorem here. If you can get Cauchy differences bounded by an L^2 function you can extact a subsequence that converges both pointwise and in L^2. Professor Stroock's books on Analysis as well as references will do well with this move. It's not so important for me as I am more keen on credit for Four-Sphere Theory. And besides, he knows better where to publish something general like this so analysts put it in all the textbooks than myself, so this is a pleasure to seek a joint publication from him.

18. Problem I.3(b)

For $\alpha \in \mathbf{R}$ and $g \in L^2(\mathbf{R})$ consider the integral equation

$$f(x) + \alpha \int K(x,y) f(y) dy = g(x) \tag{1}$$

where $f \in L^2$ is unknown. Prove there exists $\epsilon > 0$ such that for $|\alpha| < \epsilon$, the equation (1) has a unique solution.

We show that the operator $I - \alpha T$ has no kernel for $|\alpha| < \|K\|_{L^2}$ this will show uniqueness of the solution. For existence we will have to do something else.

Suppose $(I + \alpha T)f = 0$ and $\|f\| > 0$. Then

$$\begin{aligned} \int |f|^2 dx &= \alpha^2 \int (\int K(x,y) f(y) dy)^2 dx \\ &\leq \alpha^2 \int \int K(x,y)^2 dy dx \|f\|^2 \end{aligned}$$

Here we've just applied Cauchy-Schwarz. Then eliminate $\|f\|^2$ to obtain

$$\|K\|^{-1} \leq \alpha$$

This implies that when $\alpha < \|K\|^{-1}$ there is no nontrivial kernel for $I - \alpha T$ and so $(I - \alpha T)^{-1}$ exists when $g \in Ran(I - \alpha T)$.

Now we want to prove that $I - \alpha T$ is actually *surjective*. Let me think about this. We consider the formal power series

$$(I + \alpha T)^{-1} = \sum_{q=0}^{\infty} (-\alpha T)^q$$

Then we invoke a theorem that says that validity of this power series expansion is exactly

$$\| - \alpha T\| = |\alpha| \|T\| = |\alpha| \|K\|_{L^2} < 1$$

Since the power series is defined and valid, we can apply it to all $g \in L^2$ and $(I + \alpha T)^{-1}$ is a *bounded* operator, and therefore $(I + \alpha T)$ is surjective.

19. Zulf Has Vowed Only To Work On Things Directly Benefitting Four-Sphere Theory and Human Nature And Psychology

I fear that I must seek forgiveness of all ladies and gentlemen of mathematics and physics as I will never work on anything at all that does not benefit directly my Four-Sphere theory and my other discoveries in Universal Human Moral Nature, Virtues and Eudaimonia for all human beings including Virtues of Romantic Love, and issues of Romanticism and Modernism in Literature. These are the issues that have animated my life for five decades and I believe it is a fool who abandons what progress he has made and at 50 get distracted by various sorts of things that I was not meant to do with my life. I am not seeking to work on problems of interest in Mathematics or indeed of any other field that will not benefit the viewpoint that I have gained from enormous sacrifices, adversity and struggle throughout my life, at least from 1987-2022.

20. Problem I.3(c)

Suppose $\int \int K(x, y)h(x)h(y)dxdy \geq 0$ for all $h \in L^2$. Prove that (1) is solvable for all $\alpha \geq 0$.

I want Stanford Mathematics and others to understand that thus far, for (a)-(b) I did not consult any texts. In fact for (a) for meta considers of Daniel Stroock I was able to refine my own original path. For (c) I will examine texts because I have not been thinking about positive operators and ellipticity and all that recently and this is an opportunity for me to sharpen my understanding. Actually I am able to do this based on the following.

Suppose $A : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is a bounded operator satisfying

$$\|f\|^2 \leq \langle Af, f\rangle \leq \|A\| \|f\|^2 \tag{2}$$

for all $f \in L^2$ then A is invertible with a bounded inverse.

Suppose $g \in Ker(A)$. Then

$$\|g\| \leq \langle Ag, g\rangle = 0$$

Therefore $g = 0$ identically. So A has trivial nullspace. Then consider the Hilbert space adjoint A^* and (2) holds for it too, and so A^* has trivial nullspace as well by the same argument. But $Ker(A^*) = Ran(A)^\perp$ and so A is surjective. Therefore A is surjective bounded linear and open mapping theorem says it must be open. The inverse A^{-1} is defined and bounded.

We apply this to $A = I + \alpha T$ where the condition of the problem gives (2) immediately.

21. No No No PDE Is Not Rubbish At All

I always admired Karen Uhlenbeck and Clifford Henry Taubes as great geometers and when I was young I used to think to myself, "Wow, these are geometers, but they are geometers who know all sorts of esoteric rubbish about partial differential equations and analysis. Some of this rubbish is good." Those were times before I truly gained the light. I studied Hörmander for Several Complex Variables for my undergraduate thesis, but I was convinced that most of this analysis rubbish could be used with some organisation to focus on things that *really matter*, the true fruits, i.e. geometry. But this sort of way of thinking produced bad habits.

22. Problem II.1

Suppose f is a nonnegative Lebesgue measurable function on $[0, 1]$ and $f > 0$ almost everywhere. Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that if $E \subset [0, 1]$ is Lebesgue measurable with $m(E) \geq \epsilon$ then $\int_E f(x)dx \geq \delta$.

Suppose g is continuous and $g(x) > 0$ for all $x \in [0, 1]$. We're going to prove the statement for this case. Since $g > 0$ on the compact set, we have $g_{min} = \inf_{x\in[0,1]} g(x) > 0$. Now we can take $\delta = \epsilon g_{min}$ and

$$\int_E gdx \geq g_{min}m(E) \geq g_{min}m(E)$$

This is the easy part as we now know the proposition holds for continuous g. Now we claim that an arbitrary measurable $f > 0$ can be approximated by continous $g_n > 0$ in the sup-norm. So given $\epsilon > 0$ we can take $\epsilon/100$ and for $n \geq N$ we'll have

$$|g_n - f| \leq \epsilon/100$$

and

$$\int_E |f|dx = \int |f - g_n + g_n|dx \geq \int_E |g_n| - \int_E |f - g_n| \geq \epsilon \inf_x |g_n(x)| - \epsilon\epsilon/100$$

Thus *if we are able to produce continuous sup-norm approximations for f then we will have the conclusion.*

23. Proble II.1 Supplement

The task of approximating L^p and other possibly discontinuous functions with continuous functions is generally done by *convolution* with a family φ_n that approach the delta measure δ_0. The operators $T_n f = \varphi_n * f$ are called approximation to the identity. There are various families with various special properties. The sharpest ones are due to Hörmander and detailed in the first chapter of his majestic *The Analysis of Linear Partial Differential Operators I*. I did not really understand the history of this technique but it's not new. Ah, *mollifiers* were introduced by

Kurt Otto Friedrichs in 1944 [6] and Sergei Sobolev used these in 1938 when he proved the Sobolev Embedding Theorem. Gaussian smoothing with the family

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

is not sharp as it always produces large support but it will mollify with approximation to the delta as $t \to 0$.

Let us examine a very simple mollification process. We consider, for $a_n = 1/n$ and the mollification will be

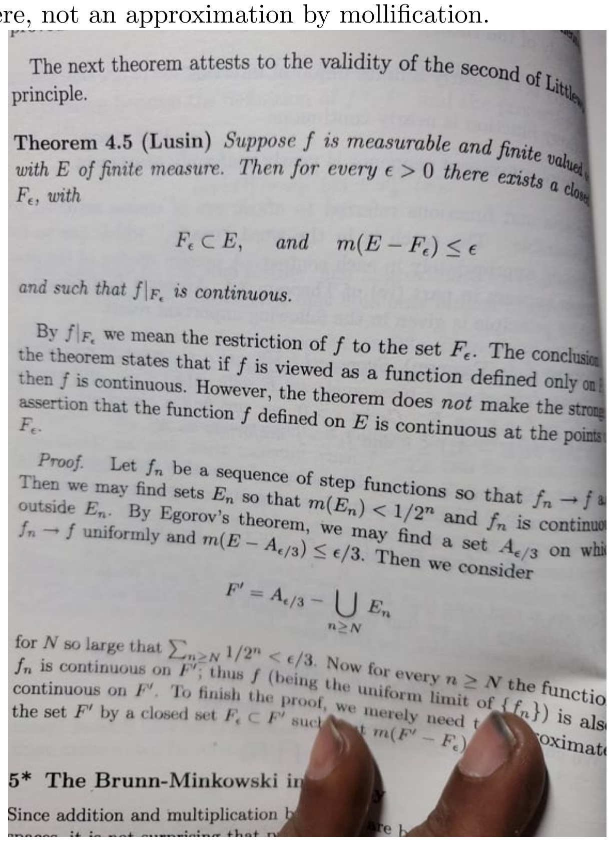
$$f_n(x) = \int_{[x-a_n, x+a_n]\cap[0,1]} f(y)dy$$

In other words we just average the function over intervals of size $2a_n$. These $f_n(x)$ will be continuous.

I am not yet sure of the best way to proceed with sup-norm bounds here.

24. Problem II.1 Supplement Lusin's Theorem

Stein-Shakarchi p. 34 contains the right way to approximate in for problem II.1. So this was lacking in my education and repertoire, as Lusin's theorem gives us that f *itself* without any mollification is actually continuous. That's the right answer here, not an approximation by mollification.



The next theorem attests to the validity of the second of Littlew
principle.

Theorem 4.5 (Lusin) *Suppose f is measurable and finite valued with E of finite measure. Then for every $\epsilon > 0$ there exists a clos F_ϵ, with*

$$F_\epsilon \subset E, \quad \text{and} \quad m(E - F_\epsilon) \le \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

By $f|_{F_\epsilon}$ we mean the restriction of f to the set F_ϵ. The conclusion the theorem states that if f is viewed as a function defined only on then f is continuous. However, the theorem does *not* make the strong assertion that the function f defined on E is continuous at the points F_ϵ.

Proof. Let f_n be a sequence of step functions so that $f_n \to f$ a Then we may find sets E_n so that $m(E_n) < 1/2^n$ and f_n is continuo outside E_n. By Egorov's theorem, we may find a set $A_{\epsilon/3}$ on whi $f_n \to f$ uniformly and $m(E - A_{\epsilon/3}) \le \epsilon/3$. Then we consider

$$F' = A_{\epsilon/3} - \bigcup_{n \ge N} E_n$$

for N so large that $\sum_{n \ge N} 1/2^n < \epsilon/3$. Now for every $n \ge N$ the functio f_n is continuous on F'; thus f (being the uniform limit of $\{f_n\}$) is als continuous on F'. To finish the proof, we merely need t the set F' by a closed set $F_\epsilon \subset F'$ such $m(F' - F_\epsilon)$ oximat

5* The Brunn-Minkowski in

Since addition and multiplication b are

What we do is then take $\epsilon > 0$ and take a tiny piece, $\epsilon/1000$ and find a closed set $F \subset E$ with f continuous on F and $m(F) \ge m(E) - \epsilon$ by Lusin's theorem. The rest will follow.

I simply did not have Lusin's theorem in my mind but this is good. I learned something important.

25. A Tribute To Elias Stein

I took a graduate class on discrete analogues of singular integral operators with Elias Stein at Princeton. He is a great figure to me, and he had vast erudition and a humanist's talent for language. He belonged in my mind with aristocratic European great scientists and mathematicians like Pierre-Simon Laplace. His ability to make extremely sophisticate mathematics transparent and lucid was legendary. He never brandished technical skills in an arrogant manner but was always concerned with elegance of language. His language was extremely precise, but seemed beautiful and transparent and rare among mathematicians today who are not touched deeply by humanistic scholarship. You don't compare his writing with other mathematicians, but with Fernand Braudel, or with Anthony Grafton, erudite historians with a fine taste in writing. His technical skills were not as apparent in his beautiful writing, and so I take Stein-Shakarchi far more seriously than many others. This was a truly magnificent mathematician's gift to young people, his project to give his gifts to the world in ability to clearly show you an entire new world. Nothing in those volumes is unnecessary, everything is necessary, and distilled by a great tradition of Antoni Zygmund and Alberto Calderon. This is a rich tradition. I have great veneration for Elias Stein; his taste in mathematics was marvelous.

It's true, I still had grave prejudices about analysis, but that was not due to him. It was a natural inclination, and proven to be right for Four-Sphere Theory would never have occurred if I was mesmerised by Analysis first. The time to appreciate Analysis is after one's geometric desires come to fruition, not before. That would have been *absurd*.

26. Problem II.2

Suppose X is a Banach space, $T \in L(X)$, and $\rho(T)$ is the resolvent and $\sigma(T)$ is the spectrum.

(a) Suppose $\lambda_n \in \rho(T)$ and $\lambda_n \to \lambda$ and $\lambda \in \sigma(T)$. Prove

$$\sup_n \|(\lambda_n I - T)^{-1}\| = +\infty$$

(b) Prove that there exists a $\lambda \in \sigma(T)$ such that there's a non-zero $x \in X$ such that

$$\|(\lambda I - T)x\| \leq c\|x\|$$

for *every* $c > 0$.

For the first, we assume

$$\sup_n \|(\lambda_n - T)^{-1}\| = A < \infty$$

and then we shall prove that

$$\lim_{n\to\infty} \lambda_n = \lambda \in \rho(T)$$

For any $x \in X$ and every $n \geq 1$ by the assumption we have

$$\|(\lambda_n - T)^{-1}x\| \leq A\|x\|$$

Now let's look at the resolvent formula. Let $R(T,s) = (s-T)^{-1}$ we have

$$R(T,\lambda_n) - R(T,\lambda_m) = (\lambda_m - \lambda_n)R(T,\lambda_n)R(T,\lambda_m)$$

For any $x \in X$ we have

$$\|R(T,\lambda_n)x - R(T,\lambda_m)x\| \le |\lambda_m - \lambda_n| A^2 \|x\|$$

Now since λ_n is Cauchy, we can force $R(T,\lambda_n)$ to be Cauchy and so $R(T,\lambda) \in L(X)$ and $\lambda \in \rho(T)$.

That's right. I did not refer to anything at all for this solution. I did look at Uniform Boundedness Principle but did not invoke it.

(b) Show there exists a $\lambda \in \sigma(T)$ such that for all $c > 0$ there is a nonzero $x \in X$ with

$$\|(\lambda - T)x\| \le c\|x\|$$

For this we unravel the definition of the operator norm. We have

$$\|(\lambda_n - T)^{-1}\| = \sup_{\|x\| \le 1} \frac{\|(\lambda_n - T)^{-1}x\|}{\|x\|}$$

Let $c > 0$ and choose n such that

$$\|(\lambda_n - T)^{-1}\| \ge c$$

Now we can find $x \in X$ with $\|x\| \le 1$ with

$$\|(\lambda_n - T)^{-1}x\| \ge (1/c - \epsilon)\|x\|$$

Now let $y = (\lambda_n - T)^{-1}x$. Then

$$\|y\| \ge (1/c - \epsilon)\|(\lambda_n - T)y\|$$

Take a limit on the right side for $n \to \infty$ and get the result.

27. Problem II.4

For this problem I will look through the book of Michael Reed and Barry Simon and not attempt a solution of my own independently. I need to have a finer understanding of the issues with dual spaces of Banach spaces and take this opportunity to learn something about these objects.

27.1. **Reed-Simon p. 72.**

Dual spaces play an important role in mathematical physics. In many models of physical systems, the possible states of the system are associated with linear functionals on appropriate Banach spaces. Furthermore linear functionals are important in the modern theory of partial differential equations. Research has proceeded in finding the dual spaces of particular Banach spaces or proving general theorems relating properties of Banach spaces to the properties of their duals.

27.2. **Problem II.4a.**

Suppose the hypotheses hold and $E \neq X^*$. Then there is a nonzero element w in $X^* - E$.

So for some $x_0 \in X$,

$$w(x_0) = a \neq 0$$

We can find $\ell \in E$ such that $\ell(x_0) = b \neq 0$.

Then $\ell' = w - (a/b)\ell \in X^*$ and $\ell'(x_0) = 0$.

Here is one idea. We could take a dense subset D of X that's countable and then for each $d \in D$ match $w(d)$ with some $\ell_d \in E$ with the property that $\ell_d(d) \neq 0$. Then we will end with

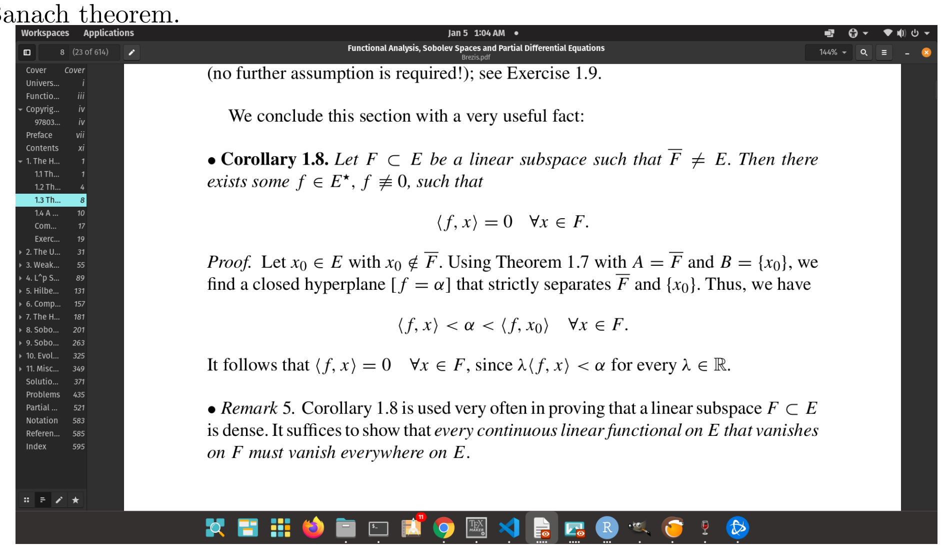
$$(w - \sum_d a_d \ell_d)(d) = 0$$

where a_d is the proportionality constant. Since this will hold for a dense subset we conclude w must be in the closure of span of ℓ_d which is contained in E since it is closed.

If $Ran(T')$ is not dense then there is some $r \in X^*$ that is not in the closure $\overline{Ran}(T')$. We claim there exists nonzero $x_0 \in X$ such that $T'(\ell)(x_0) = 0$ for all $\ell \in Y^*$.

We need to justify the above, as it is unclear, but then we have $\ell(Tx_0) = 0$ for all $\ell \in Y^*$ so $Tx_0 = 0$, and so T is *not injective.*

27.3. **Zulf Returns To The Problem With Haim Brezis Technology.** Haim Brezis has a beautiful book combining functional analysis and partial differential equations and gives Zulf the first sign of *enlightenment* regarding these dual spaces. He provides us with this extremely valuable use of a geometric form of the Hahn-Banach theorem.



(no further assumption is required!); see Exercise 1.9.

We conclude this section with a very useful fact:

• **Corollary 1.8.** *Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^\star$, $f \not\equiv 0$, such that*

$$\langle f, x\rangle = 0 \quad \forall x \in F.$$

Proof. Let $x_0 \in E$ with $x_0 \notin \overline{F}$. Using Theorem 1.7 with $A = \overline{F}$ and $B = \{x_0\}$, we find a closed hyperplane $[f = \alpha]$ that strictly separates $\overline{F}$ and $\{x_0\}$. Thus, we have

$$\langle f, x\rangle < \alpha < \langle f, x_0\rangle \quad \forall x \in F.$$

It follows that $\langle f, x\rangle = 0 \quad \forall x \in F$, since $\lambda\langle f, x\rangle < \alpha$ for every $\lambda \in \mathbb{R}$.

• *Remark* 5. Corollary 1.8 is used very often in proving that a linear subspace $F \subset E$ is dense. It suffices to show that *every continuous linear functional on E that vanishes on F must vanish everywhere on E.*

You see, this is the sort of argument that would never occur to yours truly over here at all without having seen it somewhere.

This might be quite crucial to this problem. Let's assume that Y is a *reflexive Banach space* to make things simpler. Then we identify points of Y^{**} with points of Y and see how this newfangled *Haim Brezis Technology* could be valuable here.

Let's assume $F = \overline{Ran}(T')$. It's a nice closed subspace, and the result says there exists a $x_0 \in X$ such that

$$\langle x_0, \rho\rangle = 0$$

for all $\rho \in F = X^*$. Then we translate that to there exists $x_0 \in X$ such that

$$\rho(x_0) = 0$$

for all $\rho \in F$. This is equivalent to

$$T'(\ell)(x_0) = \ell(Tx_0) = 0$$

for all $\ell \in X^*$. This guarantees $x_0 \in Ker(T)$, which is equivalent to T is not injective.

This is beautiful. So Problem II.4(b) is fully resolved by a very special corollary of geometric form of Hahn-Banach theorem. The formulation of this corollary by Haim Brezis is just exactly right [8]. So this was an opportunity for education for me. I had never seen this particular approach at all. If we choose any other approach, we will have technicalities that are a tangled mess, while the use of geometric Hahn-Banach is crisp and rigorous and painless here.

The key point here is that lacking orthogonality in Banach spaces, separation of closed from compact sets is a significant substitute for proving some reasonable propositions.

28. Zulf Explains My Way Of Doing Things

Between 1995-2008 I had senior level industrial positions, so I am not a fresh graduate student. I was a graduate student at Columbia University 1996-2000 and never completed my doctorate in Mathematics.

You see, in *industry* no business leader will be impressed if someone, on a serious assignment or project, *reinvents the wheel* for their own egotistical preferences unwisely instead of using *best practices* of industry. Also, no professional will be respected if they take ideas or material without attribution. Bill Gates is an anomaly. Most people do not risk their reputation by claiming credit for work of others.

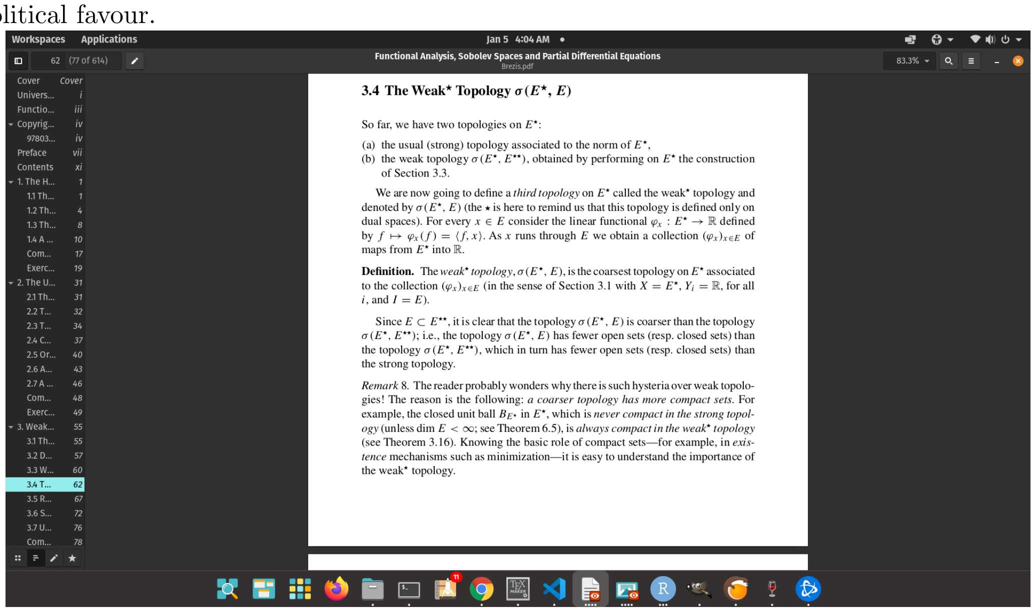
So here you see my own efforts; they are not professional quality at all and they are based on my intuition. But then I was dissatisfied, discovered the Haim Brezis corollary to geometric Hahn-Banach theorem, recognised that it applied in this problem exactly, appreciated the power of this particular approach compared to my own weaker effort, and immediately credited Professor Brezis with bibliographic entry and showed how to apply it. This shows integrity of my character, professionalism, and my ability to produce an intuitive understanding but no psychological problems with appreciating a sharper and professional approach. Haim Brezis and Felix Browder's History of Partial Differential Equations article was valuable to me and I do not hesitate to proclaim that this particular use by H. Brezis is likely to be the most efficient and professional answer for Problem II.4(b). This is clearly the *best practices* answer for this arena.

Let me tell you a story. Years ago when I was an undergraduate, I was friends with many of the graduate students at Princeton. Once I was witness to a small conflict about method to solve a problem between Shin Mochizuki, who was extremely good, and another graduate student, and I don't remember who it was. It might have been Peter Oszvath or someone else. Now Shin Mochizuki's mathematical skills are quite outrageously good. He was adamant that his approach would lead to a powerful conclusion efficiently and lead to stronger results. The other side said his approach led to not lucid understanding of the situation at all and was just technical hopscotch. I was young and did not comment since my knowledge of algebraic geometry and so on were not up to snuff. I was a young impressionable man. Shin's talents led him to producing his extremely technical treatises in his own manner. He went to Kyoto for his career home base. The point of the story is that not all ways of doing mathematics are equal. In this problem, there is no question in my mind that my original approach is low technology, and the reason is that I have not significant experience with geometric Hahn-Banach theorem, and that's the right technology for dealing with these sorts of problems, and without them one is left without sufficient power. Now the story is from over 30 years ago. Today, it's instantaneous when I notice that my answer is lacking in the right powerful tools and I do not get any sudden feelings of inferiority when I choose another answer as the right answer than my own. I have Four-Sphere Theory to my credit able to wash away all of General Relativity, Expansionary Cosmology, and Quantum Field Theory. I can afford to be instantaneously magnanimous and swiftly and graciously accept the superior path that someone else had polished without great wound to my pride.

29. Bonus Section: Haim Brezis Clearest For Weak-* Topologies

Thus far in my life, the clearest presentation of weak-star topologies is Haim Brezis. Probably this is because I am not as observant as many others and I love Reed-Simon but I want *intuition* at the moment, and the notation $\sigma(E^*, E^{**})$ for weak topology on E^* and $\sigma(E^*, E)$ for the weak-* topology is just very helpful.

In case you were wondering, not only am I always getting confused about weak and weak * and other sorts of topologies, but they spin around like thing you see when on psychedelic drugs right in front of my eyes without any meaning or sense. One day, this will change, but until that day, I congratulate Haim Brezis for recognising that the reader was wondering about the *hysteria* for these exotic topologies; and *hysteria* is the right word. You see, I am a sane cultivated gentleman, and I prefer that people don't knife each other and burn down each other's houses for having disagreements about whether weak or weak-* topologies should earn their political favour.



3.4 The Weak⋆ Topology $\sigma(E^\star, E)$

So far, we have two topologies on $E^\star$:

(a) the usual (strong) topology associated to the norm of $E^\star$,
(b) the weak topology $\sigma(E^\star, E^{\star\star})$, obtained by performing on $E^\star$ the construction of Section 3.3.

We are now going to define a *third topology* on $E^\star$ called the weak⋆ topology and denoted by $\sigma(E^\star, E)$ (the ⋆ is here to remind us that this topology is defined only on dual spaces). For every $x \in E$ consider the linear functional $\varphi_x : E^\star \to \mathbb{R}$ defined by $f \mapsto \varphi_x(f) = \langle f, x\rangle$. As x runs through E we obtain a collection $(\varphi_x)_{x\in E}$ of maps from $E^\star$ into $\mathbb{R}$.

Definition. The *weak⋆ topology*, $\sigma(E^\star, E)$, is the coarsest topology on $E^\star$ associated to the collection $(\varphi_x)_{x\in E}$ (in the sense of Section 3.1 with $X = E^\star$, $Y_i = \mathbb{R}$, for all i, and $I = E$).

Since $E \subset E^{\star\star}$, it is clear that the topology $\sigma(E^\star, E)$ is coarser than the topology $\sigma(E^\star, E^{\star\star})$; i.e., the topology $\sigma(E^\star, E)$ has fewer open sets (resp. closed sets) than the topology $\sigma(E^\star, E^{\star\star})$, which in turn has fewer open sets (resp. closed sets) than the strong topology.

Remark 8. The reader probably wonders why there is such hysteria over weak topologies! The reason is the following: *a coarser topology has more compact sets*. For example, the closed unit ball $B_{E^\star}$ in $E^\star$, which is *never compact in the strong topology* (unless $\dim E < \infty$; see Theorem 6.5), is *always compact in the weak⋆ topology* (see Theorem 3.16). Knowing the basic role of compact sets—for example, in *existence* mechanisms such as minimization—it is easy to understand the importance of the weak⋆ topology.

30. Zulf Orders Bas Van Fraasen's Laws And Symmetry

I have been quite focused on producing *empirical predictions* for redshift, for width of electron, for Earth-Sun gravity with only Van der Waal's (and therefore purely electromagnetic forces), explaining the wave-particle duality and quantum phenomena in my four-sphere theory for many years. Only once this was done, and after I had spent literally some years declaring that I was the Copernicus of this Age and I will produce a great Scientific Revolution to displace Schroedinger's Quantum Mechanics and Einstein's General Relativity, and Big Bang theory and replace these with my Four-Sphere Theory and my S4 Electromagnetic Law, the single law that replaces Maxwell and Schroedinger's dynamical laws, and unify all of Science above $\delta = 10^{-13}$ cm, and no one gave me any Nobel Prizes, did I begin to realise that perhaps I shall simply ensure my immortality by archiving my works and focus on issues of the philosophical relation between Mathematics and the external world. I also slowly realised that it is finally time for me to return to Analysis, and attempt to understand in what way mathematical models are able to capture phenomena of the external world, and how reliable is this correspondence? How can we be certain of our ideas here and knowledge? What physical content are there in mathematical situations? These are profound questions that do not have satisfactory answers.

So I just ordered Bas Van Fraasen's book *Laws And Symmetry* to return to these questions as well. This Bill Gates has been a serious disaster for the world. These illiterate hicks from podunk outskirts of civilisation should not be getting in my way. He's an intellectual amoeba while I am a great and profound immortal genius. I am surprised at the mistreatment here by my own government. United States Government will be thrown into the backwaters of the intellectual world if they do not totally destroy this Bill Gates.

31. There Is Very Little Question That My Redshift Prediction In Four-Sphere Cosmos Is One Of The Greatest Events In Human History

Although I did not get many congratulatory notes on my success, first around 2012, around a decade ago, there is little question in my own mind that my redshift prediction in static eternal four-sphere cosmos is one of the greatest events in this history of our people, the human race.

You see, Edwin Hubble in 1929 published his empirical findings in Proceedings of the National Academy of Sciences. There was a beautiful fit to a line for distant observations and redshift from these objects. There was some hullabaloo and the scientific consensus converged upon a *dynamical model of the cosmos*. This is most definitely *not a great scientific choice*. I have predicted the redshift accurately in a static four-sphere model as geometric distortion of *innacurate wavelength-frequency relationship* in a four-sphere of curvature $\Lambda = 1.11 \times 10^{-52} m^{-2}$. This shows that the parsimonious model is my four-sphere model, that there is no expansion, and there was no Big Bang 14 billion years ago, and these theories are wrong. The redshift does not represent any dynamical process of expansion in the universe. The universe is a static eternal four-sphere of radius $R = 3075.69$ Mpc. This is my immortal discovery. Expansionary Cosmology is deeply wrong. The universe did not have any past of hot dense small ball. It was exactly the same size 14 billion years ago, a trillion years ago, and 100 trillion years ago. Established cosmological models are totally wrong.

There is no question that a static cosmology model that can explain the redshift clearly is an *infinitely better scientific model* than any model where the universe has dynamics that is mystical, and so it is a great moment for Human History when I discovered that there was no expansion at all and no Big Bang.

32. My Views About Noia Efrat, Vlad Teichberg and Bill Gates

I know Vlad Teichberg from when I was 14 years old at Ohio State University. He is naturally quite gifted in Mathematics and won a good place at Westinghouse on some number theory theorems in the late 1980s. He was a year ahead of me at Princeton. His father is well-regarded in Wall Street. Noia Efrat I met at his place. I did not know that he was serious about her. I became interested in her, and did not realise that Vlad was serious about her in 2008.

Bill Gates has been telling people that I *soiled* her. First of all, I am a gentleman, and do not forgive this sort of language. I do not *soil* people; that's what third rate charlatans who have harems of young girls *quite mature for their age* do. I never kissed her even and gave her a red rose once. She did not respond and she cut into my eyes and was a blood meta parasite. I was not impressed by her.

33. Problem II.5

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ act on $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ by matrix multiplication and on $C(\mathbf{T}^2)$ by

$$(T_A f)(x) = f(Ax)$$

(a) Show that the action on $C(\mathbf{T}^2)$ extends to a weak * continuous action on $\mathcal{D}'(\mathbf{T}^2)$.

(b) Express Fourier coefficients of $T_A f$ in terms of those of f.

33.1. The Brezis Notation To Remind Us Of Weak *.

The weak * topology by notation in Brezis is extremely good for remembering the substance. It would be $\sigma(\mathcal{D}'(\mathbf{T}^2), C(\mathbf{T}^2))$. It is the coarsest topology where the things on the right side are continuous.

Now suppose $u, u_n \in \mathcal{D}'(\mathbf{T}^2)$ and $u_n \to u$. This is *equivalent* to

$$\langle u_n, f\rangle \to \langle u, f\rangle$$

for every $f \in C(\mathbf{T}^2)$. That is the charm of the Brezis notation. Otherwise it is a nightmare to remember the condition for weak topology versus weak * topology versus freak topology and break topology and greak topology and leak topology and in general the **eak * topologies* that Man has invented to confuse ourselves.

Then we just gently let $f = T_A g$ in the statement. We see

$$\langle u_n, T_A g\rangle \to \langle u, T_A g\rangle$$

for all $g \in C(\mathbf{T}^2)$. This says $T_A u_n \to T_A u$ in the weak * topology but not in the leak or freak or more generally *eak * topology for which erudite and incomprehensible texts will provide various counterexamples whose taxonomy will defy the hierarchical data structures and will in fact be found in the Great Library of Jorge Luis Borges short story where the whole world composed of hexagonal rooms that extend in all directions. But the margin of this note is not large enough for me to share such knowledge.

34. Fourier Coefficients

Fourier coefficients of f are

$$c_{p,q} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{ipx+iqy}dxdy$$

So we are going to be doing a change of variables and some calculus manipulations for this problem. Let us introduce A^{-1} right away because the change of variables will be

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

Hold on. Hold on. What we're going to do is take a look at $\det(A)$ because the volume form changes by determinant. Let

$$d_{p,q} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(A(x,y)^t)e^{ipx+iqy}dxdy$$

We can do this $(p,q)A^{-1}(x,y)^t$ in the exponent and pull out $1/\det(A)$ and that will give us the answer. Oh good $\det(A) = 1$.

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Zulf is impressed. Zulf inverted a 2-by-2 matrix flawlessly.

$$d_{p,q} = c_{p-q,2q-p}$$

Would you look at that, that's just so sweet, a *formula*. Zulf is pleased. That's a marvelous formula. Oh wait. It's a Stanford Ph.D. formula. Drats. Someone else had discovered this before me. That's obnoxious. Why would someone else discover a nice formula before Zulf?

34.1. **Invariant L^1 functions.** If $T_A f = f$ and $f \in L^1(\mathbf{T}^2)$ the f is constant almost everywhere. Let's try this.

$$\int_{\mathbf{T}^2} |f(Aw) - f(w)|dw = 0$$

In two variables

$$\int |f(2x+y, x+y) - f(x,y)|dxdy = 0$$

Hehehe. We will now employ *Lusin's theorem* to take $\epsilon > 0$ small and get a *closed subset* $F \subset \mathbf{T}$ where f is continuous on F, and the integrand is bounded by C on $\mathbf{T}^2 - F$ and $m(\mathbf{T}^2 - F) \leq \epsilon$. Why would I do that? That's a good question. I am doing that because I think continuity is good and I don't want to do any approximations at all with C^1 functions using Friedrichs mollifiers and so on until it is necessary.

Let's digress a bit and look at this cat Lusin.

That's fascinating. He published a paper in 1912 about this. Beautiful theorem. Aha, he was the doctoral advisor to A. N. Kolmogorov. And here I was wondering about who this cat is. Holy moly his students included: Andrey Kolmogorov, Alexander Kronrod, Mikhail Lavrentyev Alexey Lyapunov, Lazar Lyusternik Pyotr Novikov, Lev Schnirelmann,Pavel Urysohn. The whole damn Russian Mathematical Mafia were his students. That's wild. This theorem is beautiful and much more useful than I thought.

Hmm. Let's pretend that we have $f \in C^2$. Note

$$A(x,y)^t - (x,y)^t = (x+y,x)^t$$

$$f(A(x,t)^t) = f((x,y)^t) + \partial_x f(x,y)(x+y) + \partial_y f(x,y)x + E(x,y)$$

So the point of the problem could be construed as saying that the partial terms must be zero because otherwise

$$\int_{\mathbf{T}} |x+y|dxdy = 0$$

and

$$\int_{\mathbf{T}^2} |x|dxdy = 0$$

which is impossible. The rest would be all sorts of $\epsilon > 0$ was arbitrary, wave hand wildly. And if people still don't think the proof is valid keep repeating "Trivia! Trivia! Trivia!" like Bill Gates and bamboozle the whole world convincing them of one's prodigious genius. If that fails keep repeating "Whites are superior! Whites are superior!" Never mind I am not white.

I will return to a clearer answer here.

34.2. **Invariant Distributions.** The key point is that the Jacobian determinant of a linear transformation changes the volume element by determinant on $A : \mathbf{R}^2 \to \mathbf{R}^2$. Now the Lebesgue measure is invariant under $w \mapsto Aw$ on the plane.

The eigenvalues of A are

$$\lambda_1 = \frac{3+\sqrt{5}}{2}$$

and

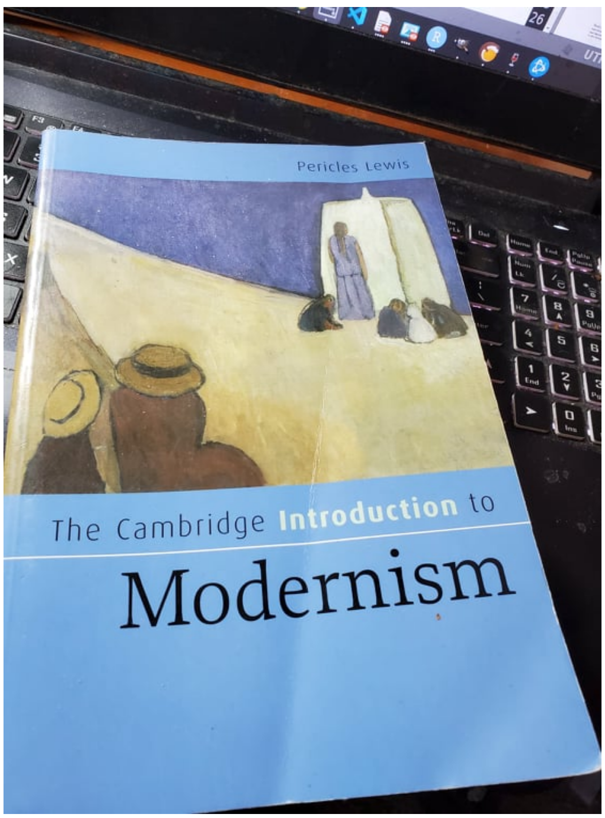
$$\lambda_2 = \frac{3-\sqrt{5}}{2}$$

If we rotate to eigenvectors, we are stretching in axes by $0 < \lambda_1, \lambda_2$. We are looking for measures μ_a on $[-\pi, \pi] \times [-\pi, \pi]$ such that for all $f \in C(\mathbf{T}^2)$ we have

$$\int_{\mathbf{T}^2} f(x)\mu_a(dx) = \int_{\mathbf{T}^2} f(Ax)\mu_a(dx)$$

I still don't see infinite dimensions for this.

35. What Is The Source Of Affection For Modernism?



At thirty-three, recovering in a sanatarium in Switzerland, Thomas Stearns Eliot wrote perhaps one of the greatest poems in history of English literature, *The Waste Land* whose sublime heights he has never ascended ever before of ever afterward. Not one other piece of Eliot's ouvre can remotely reach the sublime heights of *The Waste Land*. I have always felt that he deserves my personal thanks for writing something meant to be a prophesy of my arrival and passage through Earth, so profoundly has this poem been a part of my life. I, more perhaps than any other human being who had lived, had experienced the poem through the actions and events of my life.

You see Modernism of 1922 still governs the deep questions that animate my own soul in a particular way that periods afterward and before do not have the power to do. I still read with care literary works and criticism particularly from Romantic and Modernist period not as a hobby, but as a significant part of my spiritual life, to know who I am, to know what I ought to care about. The sorts of ugly and menial political troubles that arise today, such as a racial murderer deranged beyond imagination, this *Bill Gates* being a tycoon in the modern world is too degenerate for me to consider seriously as part of the traditions of Civilisation that I had inherited. I am annoyed that United States Government did not use extreme lethal destructive force to eliminate him years before the insolent scrub dared to disturb my peace. I am a profound man, a gentleman with deeper concerns than a snot-nosed scrub with menial intellect can even comprehend.

36. Problem I.5

Let X be a Banach space over $\mathbf{C}$ and let M, N be closed subspaces in X. Let $M + N = \{x \in X : x = m + n, \text{ for some} m \in M, n \in N\}$.

(a) Show that $M + N$ is closed if and only if there exists $C > 0$ such that for all $x \in X$ there exists $m \in M$ and $n \in N$ with $x = m + n$ and

$$\|m\| + \|n\| \leq C\|x\|$$

36.1. **Criterion For Closed $M + N$.** Suppose $M + N$ is closed. This is equivalent to the proposition that every Cauchy sequence x_n converges to an element x of $M + N$.

Suppose x_n is a Cauchy sequence, or, if you will, a *Bolzano sequence* because this terminology of 'Cauchy sequence' was denial of credit to Bolzano for invention of precise definitions of limits and continuity by Maurice Fréchet.

Given $\epsilon > 0$ there exists a $N \geq 1$ such that for all $j, k \geq N$ we have

$$\|x_j - x_k\| < \epsilon$$

We will have to be careful with our language here since there are infinite different representations of x_j and x_k. Let us write $x_j = m'_j + n'_j$ and $x_k = m'_k + n'_k$ as generic representations. For any m'_j, m'_k, n'_j, n'_k satisfying these we have

$$\|m'_j + n'_j - m'_k - n'_k\| < \epsilon$$

or

$$\|(m'_j - m'_k) + (n'_j - n'_k)\| < \epsilon$$

So we are given that

$$\lim_{j \to \infty} \|x_j - x\| = 0$$

for some $x \in M + N$. So we put $x = m' + n'$ and

$$\|(m'_j - m') + (n'_j - n')\| < \epsilon$$

Oh my God. It's January 5 2022, 3:13 PM. And I don't have the right answer to I.5(a) with a snap of my fingers at all. Zulf, calm down. Take a deep breath. It's okay. It's okay to not have the right answer. It's just a Banach space problem. It's not the end of the world. The world will continue without problems if you don't have an answer to issues of Banach space subspaces!

I took a long break and watched some parts of the Netflix film *King Arthur: Legend Of The Sword*. I like the artistic reworking of a truly great work of Western Literary Tradition. You see, I have some royal blood, of three thousand years. Vedic Indian lineage. I suspect that there were no actual ruling kings in my ancestry for some centuries before me. But I see different things in the Arthurian Legends than what the movie directors do, I see a total focus on Character Virtues, the element of the mythological, never really about the adventures themselves but about the greatness of noble Character. Arthur was not invented as the warrior king. The butchers were not hallowed and there had been many in Europe. It was the light that was preserved. Do you really think that a man born in Seattle in 1955 who hoodwinked and cheated and lied and stole and murdered and sold tall tales for his entire life has any inkling of what it takes to be a true king?

Anyway, I thought about this problem. And I came to the realisation that I need to take a slightly different approach.

I will put on V=M+N a new norm, and then consider it a separate space. Then I will consider the map

$$J : V \to X$$

with $J(x) = x$ as a map between normed spaces. Then I will show that J is continuous if and only if V is complete, you see.

You might think this is quite a longwinded way to see Problem I.5(a). You have your right to think those things, but you are not the one bewildered and dazed by how outrageously annoying this problem is without taking this conceptual sort of approach, so I will ignore your preferences. I will, instead, think about how, if you don't actually demand skills of physical combat prowess, I am far more similar to a true king, a King Arthur, gallant and noble, than this *Bill Gates* who surely is a black sheep of English ethnic descent. Shameful what has transpired. Well he is of peasant stock, which is a relief. His Tarquin's ravishing strides and so on would have collapsed English Civilisation if he were actually English Nobility.

Let us define, then,

$$\|x\|_s = \inf\{\|m\| + \|n\| : x = m + n, m \in M, n \in N\}$$

This definition implies that

$$\|x\| \leq \|m\| + \|n\|$$

for all $m \in M, n \in N$ satisfying $x = m + n$ and therefore

$$\|x\| \leq \|x\|_s \tag{3}$$

This is very good. I love this.

For checking this is a genuine norm, scalar multiplication is obvious. Let $x, y \in M + N$ and

$$\|x + y\|_s = \inf\{\|m\| + \|n\| : x + y = m + n, m \in M, n \in N\}$$

Let's see. From definition of the norm we get for every m_x, n_x with $m_x + n_x = x$ that

$$\|x\| \leq \|x\|_s \leq \|m_x\| + \|n_x\|$$

This is good. Then we have

$$\|x+y\|_s \leq \inf\{\|m_x\|+\|n_x\| : x = m_x+n_x, m_x \in M, n_x \in N\}+\inf\{\|m_y\|+\|n_y\| : y = m_y+n_y\}$$

So we have triangle inequality. That $\|x\|_s = 0$ implies $x = 0$ follows from $0 \in M, N$. It's a norm.

Let's now consider it abstractly as $(V, \|\cdot\|_s)$. Now suppose v_n is Cauchy, then there exists $N \geq 1$ such that for $j, k \geq N$

$$\|v_j - v_k\|_s < \epsilon$$

And this is the magical moment. This is where you say, there exists $m_{jk} \in M$ and $n_{jk} \in N$ such that

$$\|m_{jk}\| + \|n_{jk}\| < \epsilon$$

This is magically given to you from definition of $\|z\|_s$ because it's an infimum over such things and therefore can slip into the ϵ without problems.

This then guarantees that v_j is Cauchy in $\|\dot{\|}$-norm, so v_j converges to some v in X.

Now we can address the problem again and attempt to prove that $V = M + N$ is closed if and only if there exists $C > 0$ such that

$$\|y\|_s \leq C\|y\|$$

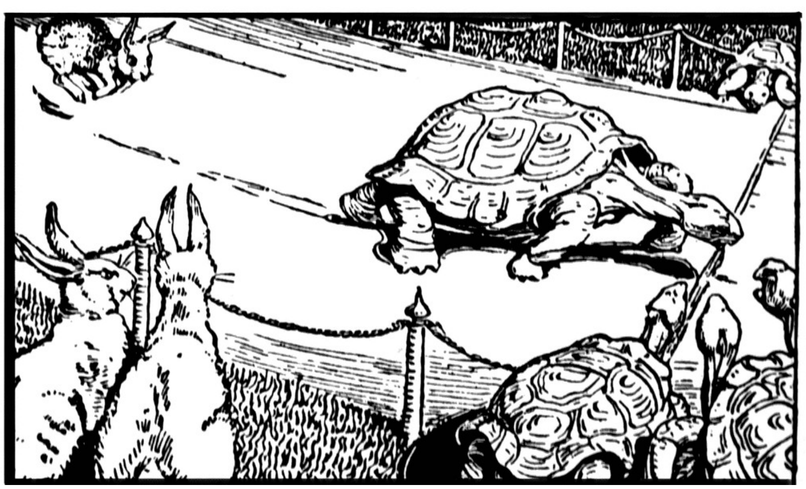
for all $y \in V$.

36.2. **Problem I.5(a) Is A Problem Of Finer Points Of Open Mapping Theorem.** After doing a huge amount of lightweight work we have a bounded linear operator $J \in L(V, X)$. The norm on V is this $\|\cdot\|_s$ and the norm on X is $\|\cdot\|$.

Now *when* $Ran(J) = N + M$ is closed then it is complete and ergo a bona fide Banach space. And *only then* can we apply the open mapping theorem since J is continuous by (3) and J is surjective onto a bona fide Banach space. The point is to realise that you need the image to be complete for Open Mapping Theorem to give you what you want, i.e. an open map, and now since J is *injective* we have J^{-1} is well defined, and continuous because J is an open map. But J^{-1} is continuous, so there exists $C > 0$ so that

$$\|x\|_s = \|J^{-1}(x)\|_s \leq C\|x\|$$

This tells you one direction, necessity I believe. For the other direction, *sufficiency* in the abstruse jargon of mathematicians, we need to think a bit more. This is the bit where when you do not have closed range for J then there is an inverse map J^{-1} in set theoretic terms, but it's not going to be *bounded* or *continuous*.



Here the setup with J with the various norms and J^{-1} etc. seems like elaborate window dressing until you realise that the logic of Open Mapping Theorem is not transparent in the argument till you do those things.

Let's consult the great Brezis to ensure we are actually using Open Mapping Theorem appropriately.

• **Theorem 2.6 (open mapping theorem).** *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is* ***surjective*** *(= onto). Then there exists a constant $c > 0$ such that*

$$T(B_E(0,1)) \supset B_F(0,c). \tag{7}$$

Remark 5. Property (7) implies that the image under T of any open set in E is an open set in F (which justifies the name given to this theorem!). Indeed, let us suppose U is open in E and let us prove that $T(U)$ is open. Fix any point $y_0 \in T(U)$, so that $y_0 = Tx_0$ for some $x_0 \in U$. Let $r > 0$ be such that $B(x_0, r) \subset U$, i.e., $x_0 + B(0,r) \subset U$. It follows that

$$y_0 + T(B(0,r)) \subset T(U).$$

Using (7) we obtain

$$T(B(0,r)) \supset B(0,rc)$$

and therefore

$$B(y_0, rc) \subset T(U).$$

Some important consequences of Theorem 2.6 are the following.

• **Corollary 2.7.** *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is* ***bijective****, i.e., injective (= one-to-one) and surjective. Then T^{-1} is also continuous (from F into E).*

Proof of Corollary 2.7. Property (7) and the assumption that T is injective imply that if $x \in E$ is chosen so that $\|Tx\| < c$, then $\|x\| < 1$. By homogeneity, we find that

$$\|x\| \le \frac{1}{c}\|Tx\| \quad \forall x \in E$$

There is very little question, when you see the text of the open mapping theorem, that the necessity part of Problem I.5(a) is an application of this theorem. For sufficiency, we could assume that $Ran(J)$ is not closed and the point is that J^{-1} which is defined still on $Ran(J) = N + M$ is not bounded $Ran(J) \to V$. I will make a sharper argument later on.

36.3. **Problem I.5(b).** Suppose $\ell_M \in L(M, \mathbf{C})$ and $\ell_N \in L(N, \mathbf{C})$ and $\ell_M = \ell_N$ on $M \cap N$. Prove that when $N + M$ is closed there is an extension $\ell \in X^*$.

We will employ the Hahn-Banach Theorem of course. We will apply the Hahn-Banach theorem on an intermediate functional, $\ell_{M+N} \in (M+N)^*$.

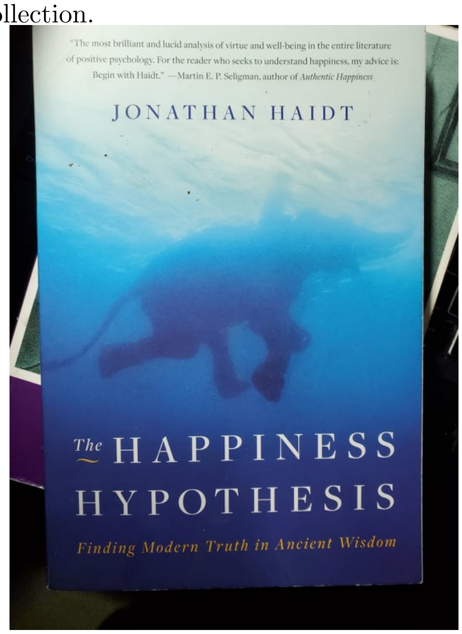
We define

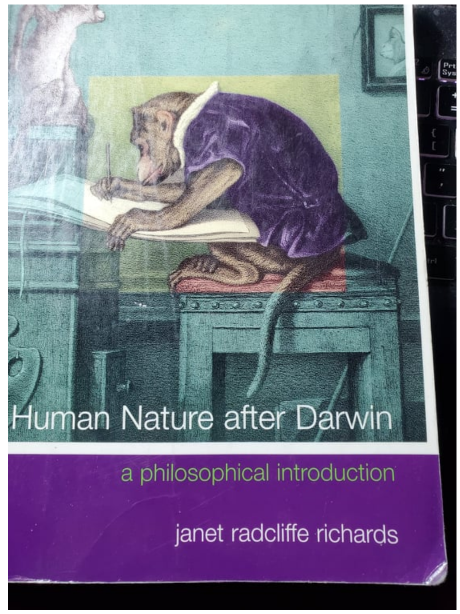
$$\ell_{M+N}(x) = \ell_M(m) + \ell_N(n)$$

whenever $x = m + n$. Now Hahn-Banach theorem can extend it to $\ell \in X^*$

37. My Theory Of Habituation And Human Capabilities

For the past few years I have been extremely interested in human capabilities. I am deeply distrustful of the various racial theories that have been floating around, and held dearly by Bill Gates and other illiterate morons in the West. I go strongly against Francis Galton regarding these sorts of theories. He was reasonable in some things; his linguistic hypothesis and some work in early statistics was good. But he was not very bright on theories of human capabilities. Anyway, I am not without a lot of reading on this. I will just show you a couple of books from my personal collection.





I am interested in *universal capabilities of human beings* and are not satisfied with the scientific as well as humanistic qualities of the theories that are popular on the topic.

I strongly believe that *habituation*, already highlighted in Aristotle's *Nicomachean Ethics* as the method to attain arete, moral virtue or excellence, is not properly understood today and as a result lives experience disasters. I am not exactly an avid Starcraft II player, but I had a project to track my improvement (against the AI) for a while. And surely enough I noticed that without any conscious change, my skill level began to improve slowly. The explanation is unclear but key is nothing but *habituation*. The general advice, therefore, to all people is to always keep doing what you love to do every day, a little bit, without too much pressure and over time your psyche will adapt and you will find yourself doing it with higher levels of skill.

This is true generally for all things, and the principle is so strong in my view, that it will transform the entire world – for the better – if we had sharper understanding of notions of habituation and its properties. There is no intrinsic limit, I believe, to what anyone can achieve, if they have the opportunity to simply habituate to whatever productive activity they love to do.

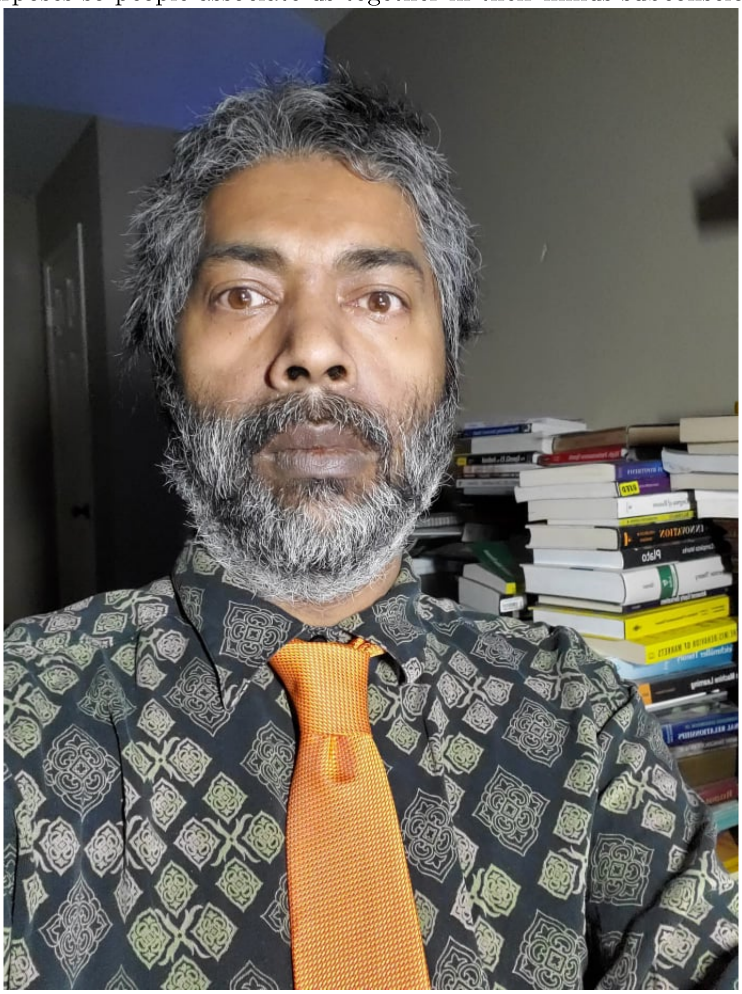
38. Zulf's Open Letter of January 5 2022 To Xi Jinping

Dear President Xi Jinping and All Asian People,

Even though we face difficulties of today and tomorrow, I still have a dream. I have a dream that the Continent will rise up and live according to the creed that we hold these truths to be self-evident that all men are equal. I have a dream that the East Asians and West Asians and North Asians and South Asians will recognise our unity. I have a dream. I have a dream that one day one Continent under Celestial Divine Light, the birthplace of all Human Civilisation, shall give up this unspoken *Aparheid* that has separated all Asians and live together in harmony and peace. I have a dream that Asia will rise to its responsibilities to be the beacon of Civilisation and light and no longer look upon each others as strangers with suspicion. I have a dream that all my Asian brothers and sisters will treat each other with the respect and dignity that we have earned, and ban all intra-Asian strife permanently and without conditions and take up joint Defense of Asia.

I am most pleased that you took some time away from your great responsibilities to pay some attention to an American of Asian descent. I decided to take advantage of this to write some nice tribute to Martin Luther King, Jr. speech of 1963.

And this is me in Princeton colours, also the colours of the Bengal Tiger just now. It's good to put my picture close to Martin Luther King for good propaganda purposes so people associate us together in their minds subconsciously.



Of course I am not doing exactly what Dr. King was doing, but I liked this speech from high school years in New York 1987-1991.

Sincerely,

Zulfikar Moinuddin Ahmed

39. Explanation Of The Delicate Political Calculus To Stanford Mathematicians

Bill Gates had no interest in doing any business with Zulf in good faith at all, discovered China. Their consider found that Bill Gates had planned to murder me from the very moment he discovered my existence. He apparently thought that I had good ideas, so he will murder me and take my ideas and make money off it. But his industrial clout in America was *offset* by my success in Four-Sphere Theory, and since I succeeded against Albert Einstein and Erwin Schroedinger and George Lemaitre, I could then play a peculiar and unique *Asian-American* game, or I would rather consider it an *American-Asian* game which is to simultaneously promote my American-ness and Asian origins flawlessly and unite Asia with a great dream while pursuing my American dream tugging at *massive Asian pride* to counter racial murderous intent of Bill Gates. Bill Gates' caveman level strategy of 'Gruk big. Gruk will smash little guy with a club and take all his things' was suddenly transformed into a lofty game of *geopolitical chess* which was beyond Gruk, i.e. Bill Gates' comprehension.

40. Martin Luther King, Jr. Becoming Popular In Asia Will Be Fatal To White Supremacy In America

White Supremacy politics cannot survive great popularity of Martin Luther King, Jr. in *Asia* because when he is a popular figure in Asia, eventually the weakened White Supremacy movement will wither and die because Asia has some global influence, and it is softer than West's so over time, Asia's influence will go *against* White Supremacy rather than being *neutral* and that will shift the relative power of white supremacy in the world stage and will begin to have an affect in the American scene which had been largely isolated in the past. So my play is a guaranteed win in $t \to \infty$ limit.

41. Problem I.4

Let

$$\|f\|_a^2 = \int_0^\infty x^{-1}|f(x)|^2 dx$$

The problem is to determine the $s \in \mathbf{R}$ such that there exists $C > 0$ such that

$$\|x^{s-1}u\|_a \leq C\|x^s \partial_x u\|_a \tag{4}$$

holds for *all* $u \in C_0^\infty((0,\infty))$.

I want to rewrite the inequality (4) as

$$\int_0^\infty x^{2s-3}|u|^2 dx \leq C \int_0^\infty x^{2s-1}|\partial_x u|^2 \tag{5}$$

for all $uC_0^\infty((0,\infty))$.

First of all, I assure the reader that I have no idea what the answer is at the moment. It is a mystery to me, and so I will step with trepidation towards this enigma and make small steps towards the resolution.

I note immediately that *all* $u \in C_0^\infty((0,\infty))$ are involved. I use my faltering memory and immediately recall where I had first seen this, and that is in Lars Hörmander's *The Analysis of Linear Partial Differential Operators I.* So now you know where I had seen these for the first time in my life. I literally saw it in Hörmander's book and not even in Reed-Simon Volume I. I was geometric in those days and resented beautiful smooth functions being denigrated as *test functions* as though they are not the centers of all that is good, true and beautiful, but mere experimental objects. My heart cried out for their release from these cruel Analysts's captivity. I was young then and thought that Cyrus releasing the Jewish People from captivity in Egypt was not as important as freeing the lovely smooth functions being tortured as *test animals.* I could not bear it any more.

I do not have any regrets in the end about not gaining deeper interest in Laurent Schwartz' great contributions to the world. I was geometric in my talents and I have accomplished my life goals. I do not feel that my life had been a failure in pursuit of great things, for my Four-Sphere Theory shall be eternal truth of Nature, and will displace General Relativity and Quantum Theory of Schroedinger and Expansionary Cosmology, which is profound success. I could not have asked more of life. But it was not a normal path, as my lack of interest in analytical issues was not particularly impressive to Daniel Stroock several decades ago, and he was unenthusiastic about me. But this is precisely what true adventures have always been like in the history of intellectual life of humanity. Great pioneers have to struggle alone for many years, and was I so different than Charles Darwin in the 1850s-1870s? He too faced difficulties. But of course I came across also vile evil of racial genocide plotters along the way, this evil *Bill Gates* and so my life is eventful and fruitful, even if Stanford is not able to accomodate my request for tenure, I shall live in the memory of my beloved people the human race as an immortal genius and a benefactor to all Mankind with Scientific Revolution that is more profound than had occurred in Europe in early twentieth century.

Now let's pay attention to the distributional inequality that we see before our very eyes.

We begin with a simple Leibniz rule formula

$$\partial_x[x^{s+1/2}u] = \frac{x^{s-1/2}}{s+1/2}u + x^{s+1/2}\partial_x u$$

If we integrate this, using $u \in C_0^\infty((0,\infty))$ we obtain

$$0 = \frac{1}{s+1/2}\int_0^\infty x^{s-1/2}udx + \int_0^\infty x^{s+1/2}\partial_x udx$$

So we have

$$\int_0^\infty x^{s-1/2}udx = -(s+1/2)\int_0^\infty x^{s+1/2}\partial_x udx$$

for all $u \in C_0^\infty((0,\infty))$ whenever both of these integrals are absolutely summable. Now we cannot arbitrarily play around here with absolute values.

As I said, trepidation is central to moving near any dangerous sorts of analytical object. I use caution. These things could just snap and bite off my extremities at any moment.

Since $u \in C_0^\infty((0,\infty))$

$$(\int_0^\infty x^{s-1/2}u(x)dx)^2 \le C(s+1/2)^2 \int_0^\infty x^{2s+1}|\partial_x u|^2 dx$$

using Jensen's inequality using compactness of support. I see we actually have

$$\sum_j a_j^2 \le (a_1 + a_2 + \cdots + a_N)^2$$

and

$$(a_1 + a_2 + \cdots + a_N)^2 \le C \sum_j a^2$$

Using this we do have

$$\int_0^\infty x^{-1}|x^{s-1}u|^2 \le C(s+1/2)^2 \int_0^\infty x^{-1}|x^{s+1}\partial_x u|^2$$

We will have this hold for $s - 1/2 \ge 0$.

That's nice we managed without the hint of using Fourier transform.

42. How Will Anyone Become Aware Of Anyone's Acumen In Anything Without Due Diligence

I am really upset now. Twenty years of difficulties in America because Dan Stroock totally randomly and irresponsibly damaged my career frivolously in Mathematics and Finance without *even a slight amount of due diligence.* He is a good analyst and probability theorist, but he's totally incompetent to know anything about anyone's acumen at anything by sheer mystical divination without doing any work.

43. Problem II.3

The Fourier transform of a Gaussian. Let A be a real symmetric matrix. Let

$$f(x) = \exp[-i\langle Ax, x\rangle]$$

Show that it is a tempered distribution and calculate its Fourier transform when $\det(A) \neq 0$.

We begin with just doing this for $x \in \mathbf{R}$. We have to complete the square

$$-iAx^2 - ixy = -i(x^2 - xy + (y/sqrtA)^2/4) + i(y/2\sqrt{A})^2$$

and therefore

$$\int e^{-iAx^2} e^{-ixy} dx = e^{i(y/sqrtA)^2/4} \int e^{-i(\sqrt{A}x + y/2\sqrt{A})^2} dx$$

with a substitution we have

$$A^{-1/2} e^{iy^2/2A} \int e^{iu^2} du$$

Without any effort at being rigorous, we say,

$$\frac{d}{du}(e^{iu^2} u^p) = 2iu^{p+1} e^{iu^2} + pu^{p-1} e^{iu^2} = (2iu^{p+1} + pu^{p-1}) e^{iu^2}$$

44. Zulf Is Saddened Now

I did not know that Dan Stroock literally sabotaged my finance and mathematics career with all manner of forbids. That's not right. I went to work with him in good faith. He could have refused to accept me and told me I was not up to snuff right away instead of sabotaging my finance and mathematics career and never checking that I was well-regarded at Princeton. What sort of behaviour is that. It's quite petty and malicious. This is not happy for me at all. I had gone through a lot of problems in the last twenty years never knowing that Dan Stroock was literally active in sabotaging my future.

45. Zulf Reiterates His Position Regarding Jewish People Are Good People

That's very bad Daniel Stroock. What you don't understand is that when a Jewish person harms a non-Jewish person, all Jewish people get hurt. That's simple statistics. Jewish people are doing well in America, but there are many people who dislike Jewish people; I happen not to be one of them. But the rumour spreads that a Jewish person harmed a non-Jewish person and 15 million Jewish people do not have the numbers to keep these things from affecting 8 billion. East of Israel, people don't even know that many Jewish people. Then that poisons reputation of Jewish people. I don't know what Dan Stroock was thinking. Extremely irresponsible. I have literally fought off anti-Semitic claptrap with enormous cost to myself. I am most unimpressed. Why would anyone do that to anyone?

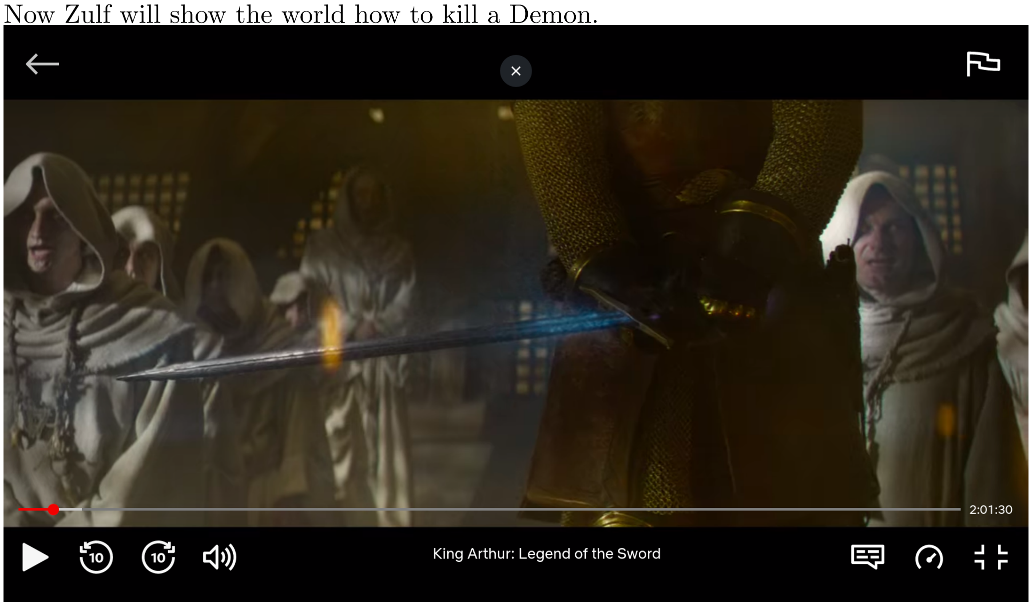
You see, I am quite friendly and positive about Jewish people. I had good Jewish friends from high school, from Princeton, and elsewhere, but I am 50 years old and I know that Jewish people always face problems for simple reason that they are not *large enough a population* to be able to control negative sentiment. I am Bengali, and Bengali ethnics are roughly 300 million worldwide. I am not ethnically focused but that's literally 20 times larger than Jewish global population. Or consider

English ethnic people. They are around 80 million globally. That is 5.5 times the number of Jewish people worldwide. I don't think it is a wise and rational thing for a Jewish person to harm any non-Jewish person's life at all without extremely serious reasons. I had done nothing at all to harm Daniel Stroock's life. Why would he have harmed mine? I just do not understand the significant reason here.

46. Dan Stroock Will You Move Out Of The Way

Look Dan Stroock you're a good mathematician but a civilian type. You're a civilian type. Move out of my way please.

Now Zulf will show the world how to kill a Demon.



You need to cut off the tongue of the Demon and burn the body at the stake. Don't get in Zulf's way.

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