ZULF'S STANFORD ANALYSIS FALL 2010 QUAL

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1. Problem I.1

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\dot{\|}_Y)$ are Banach spaces. Suppose $Y \subset X$ and the inclusion map $j : Y \to X$ is continuous. Next suppose $T_n \in L(X)$ and for each $x \in X$ we have $T_n x \in Y$ and there exists $C(x)$ such that $\|T_n x\|_Y \leq C(x)$. Show that

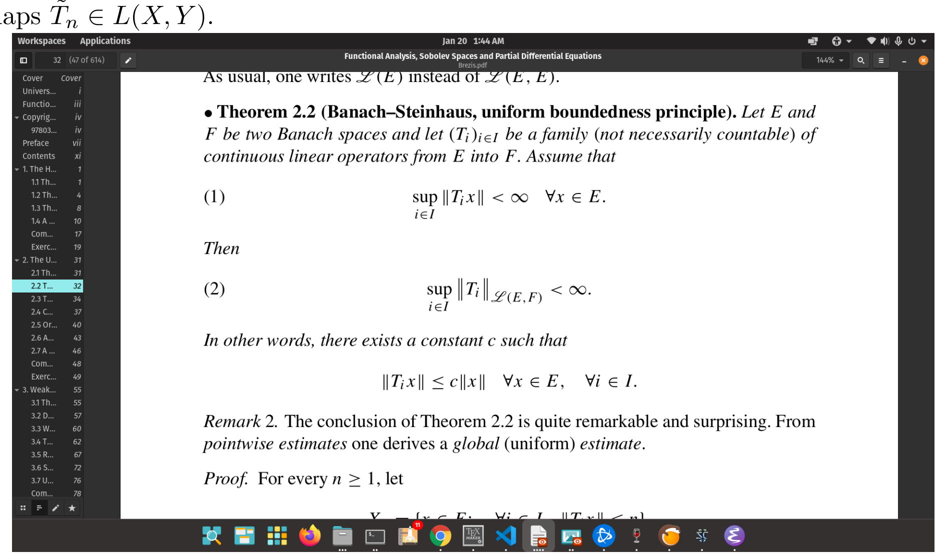
(a) $T_n \in L(X, Y)$

(b) $\|T_n\|_{L(X,Y)} \leq C$ for some $C \in (0, \infty)$.

We are going to be applying here the uniform boundedness principle. And we will be looking at the Haim Brezis version of the Uniform Boundedness Principle to match his particular wording.

First, let us be lax and assume that j^{-1} is defined on $(Y, \|\cdot\|_X)$ defined by $j^{-1}(y) = y$. Even though it's not true we will assume that open mapping theorem applies and tells us that j^{-1} is continuous from $(Y, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$.

This then allows us to define $\tilde{T}_n = T_n \circ j^{-1}$ and being a composition of continuous maps $\tilde{T}_n \in L(X, Y)$.



As usual, one writes $\mathscr{L}(E)$ instead of $\mathscr{L}(E, E)$.

• **Theorem 2.2 (Banach–Steinhaus, uniform boundedness principle).** *Let E and F be two Banach spaces and let $(T_i)_{i \in I}$ be a family (not necessarily countable) of continuous linear operators from E into F. Assume that*

$$\sup_{i \in I} \|T_i x\| < \infty \quad \forall x \in E. \tag{1}$$

Then

$$\sup_{i \in I} \|T_i\|_{\mathscr{L}(E,F)} < \infty. \tag{2}$$

In other words, there exists a constant c such that

$$\|T_i x\| \leq c\|x\| \quad \forall x \in E, \quad \forall i \in I.$$

Remark 2. The conclusion of Theorem 2.2 is quite remarkable and surprising. From *pointwise estimates* one derives a *global* (uniform) *estimate*.

Proof. For every $n \geq 1$, let

Now we apply the Brezis Theorem 2.2 which applies and gives us $\|\tilde{T}_n\|_{L(X,Y)} \leq C$ from the hypotheses.

Special credit for realising that I am showing my *good taste* in choosing Haim Brezis, Michael Reed, and Barry Simon for my reference texts. Now you know that I am aware of who is good in the arena.

Date: January 21, 2022.

1.1. **Why I trust Haim Brezis, Michael Reed and Barry Simon On Functional Analysis.** You see, Haim Brezis works in Partial Differential Equations, and Barry Simon on various sorts of perturbation theory of complicated Schroedinger operators. For these gentlemen, the *use of Banach Space Theory* is a fortunate thing. They are the sort of people if denied Banach space theory so clean and beautiful, would be cutting up their bedsheets into various epsilon-holes until they had totally ruined their entire bedroom and created the menagerie of Lucifer trying to create living things. The FBI would certainly be alerted and they'd join *Hannibal Lechtor* levels of strange behaviour and you'd start seeing Jodie Foster interviewing them sooner or later. So these gentlemen *respect Banach Space Civility* and do not ever make errors on their use. I am 49 years old now, and I know why I trust whom with what. They are great practitioners of their craft, and their feel for what Banach space theory delivers is nuanced and error-free by *natural instinct.* So I trust their methods completely here.

Fine, so what if you had Krzysztof Ki/'eslowski snooping around while you're a French mathematicians completely pre-occupied with doing some difficult analysis problem and you had thought you were discreet and kept all your doors locked and tried to gauge some complicated argument with blue ink on white sheets and started cutting up dozens of sheets with lots of holes. Then he thought to himself, "This man, this man, this man, I can see it perfectly." And he puts his hands in front of him like a camera and says, "Ah, yes. Ah, yes." And then you don't notice.

The next thing you know you're the creepy guy in Rouge II and Irene Jacob is talking to you because she apparently lost her dog and you say, "Ah, Ms. Jacob. No. It's not what you think. It's not at all what you think." And she says, innocently, "Of course, Monsieur, of course."

What will you do then? Huh?

1.2. **What People Do Not Understand In America.** You see, Krzysztof Ki/'eslowski always impressed me among French directors. I mean I know he's Polish but he's a French director. France was strongly influenced by Charles Baudelaire and Jean-Paul Sartre and Albert Camus, the nihilistic bunch honestly. American literati who put *Dostoevsky* with nihilists have no taste in art at all. Dostoevsky was an *anti-nihilist* not one himself. He was the *least nihilistic writer in history* and American literati's lack of sensitivity to this is just outrageous. Now Krzysztof Ki/'eslowski was a moralist in France which impressed me immediately, as he was then closer to me to Andrei Tarkovsky and Dostoevsky. How is a moralist ever going to be nihilist? The answer can be found in the soul of the American literati that dares to pool them together, as though they will mix easily just oil and water.

No one sane – such as myself– can really comprehend American aesthetic criteria. Are they serious? They literally read the whole of Dostoevsky and decided that the most Christian moral writer in history of Earth is a *nihilist*? Did they actually even read *Notes From The Underground*. This man was all about redemption of the eternal Christian soul through the mysteries of love. Just outrageous. Anyway, Krzysztof Ki/'eslowski is very sharp and a breath of fresh air after the air went stale with Baudelaire. Now Baudelaire did write some good poems but his philosophy was just atrocious.

A word to the wise. Do not call Dostoevsky a nihilist in my presence, because you will then have my unintendended *Le Fleurs du Mal* right in your life. You don't want that, trust me.

2. Problem I.2

Consider $L^p[0,1]$ for $1 \leq p < \infty$. For which p is the unit ball weakly sequentially compact?

We take for granted that $L^q[0,1]$ is the dual of L^p for all the $p \in [1,\infty)$. Let's see now. We have Hölder's inequality

$$\int_0^1 fg dx \leq \|f\|_p \|g\|_q$$

Suppose $f_n \in L^p[0,1]$ are bounded with

$$\|f_n\| \leq B < \infty$$

Then let's consider $f_n - f_m$. We have

$$\int (f_n - f_m) g dx \leq \|f_n - f_m\|_p \|g\|_q \leq 2B\|g\|_q$$

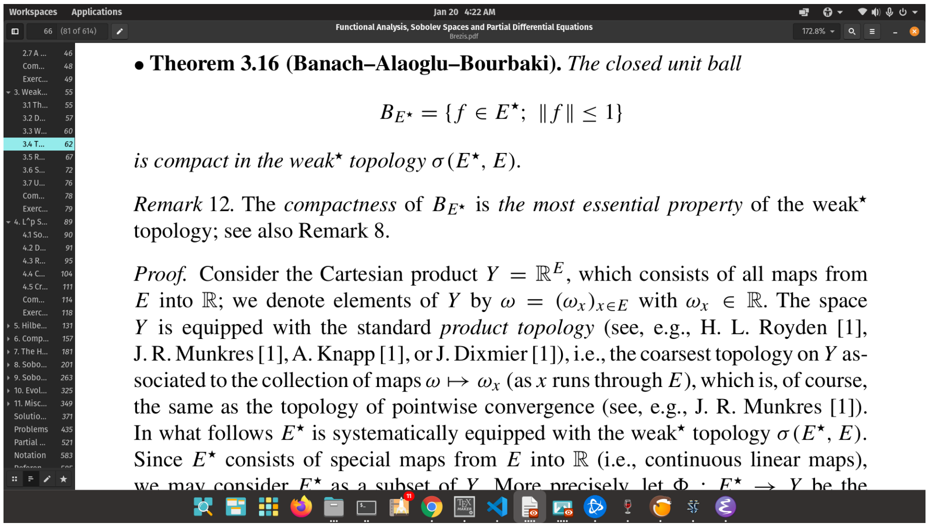
Given $\epsilon > 0$, choose a finite covering of $[0,1]$ so that the centers are $x_1, dots, x_N$ and $J_k = (x_k - \epsilon, x_k + \epsilon)$; we can do this with $N > 2\epsilon^{-1}$. Now let us consider the sequences $f_n(x_k) - f_m(x_k)$ and let $g = 1_{J_k}$. We get

$$\int_{J_k} |f_n(x) - f_m(x)| dx \leq 2B\epsilon$$

Let's take a step back. We want to get a subsequence of f_j such that $f_{j_n}(x_k)$ converges for all the points $x_1, \ldots, x_N$. Ah, we can use measurability and Lusin's theorem to ensure that f_j are continuous at all the points $x_1, \ldots, x_N$. Then we can choose a subsequence that converges at all the points $x_1, \ldots, x_N$ and let's just call this subsequence f_j. Then we have pointwise convergence that we can convert to L^p convergence by dominated convergence.

Let me see, what is wrong here? I shall have to think a bit more. The answer is that all unique ball is weakly convergent for all $p \in (1,\infty)$ certainly.

Hold on. In weak-* topology unit ball is always sequentially compact by the Banach-Alaoglu Theorem.



Functional Analysis, Sobolev Spaces and Partial Differential Equations

• **Theorem 3.16 (Banach–Alaoglu–Bourbaki).** *The closed unit ball*

$$B_{E^\star} = \{f \in E^\star;\ \|f\| \leq 1\}$$

is compact in the weak* topology $\sigma(E^\star, E)$.

Remark 12. The *compactness* of $B_{E^\star}$ is *the most essential property* of the weak* topology; see also Remark 8.

Proof. Consider the Cartesian product $Y = \mathbb{R}^E$, which consists of all maps from E into $\mathbb{R}$; we denote elements of Y by $\omega = (\omega_x)_{x\in E}$ with $\omega_x \in \mathbb{R}$. The space Y is equipped with the standard *product topology* (see, e.g., H. L. Royden [1], J. R. Munkres [1], A. Knapp [1], or J. Dixmier [1]), i.e., the coarsest topology on Y associated to the collection of maps $\omega \mapsto \omega_x$ (as x runs through E), which is, of course, the same as the topology of pointwise convergence (see, e.g., J. R. Munkres [1]). In what follows $E^\star$ is systematically equipped with the weak* topology $\sigma(E^\star, E)$. Since $E^\star$ consists of special maps from E into $\mathbb{R}$ (i.e., continuous linear maps), we may consider $E^\star$ as a subset of Y. More precisely, let $\Phi : E^\star \to Y$ be the

If we apply this theorem, we take each $L^p[0,1]$ for $1 < p < \infty$ as the dual of $L^q[0,1]$ then declare the unit ball of $L^p[0,1]$ as compact in the weak-* topology of $L^q[0,1]$ and then identify the weak-* topology as the weak topology of $L^p[0,1]$. Then we cannot do this for $p = 1$. I was trying to do something without direct application of Banach-Alaoglu theorem but this is not quite so simple.

2.1. **The $p = 1$ Case.** The $p = 1$ case is the interesting case because we don't have Banach-Alaoglu Theorem giving us anything here. Here we can use my original idea. Given $\epsilon > 0$ we start with f_n say. We choose using Lusin's theorem $2^{-n}\epsilon$ so that f_n is continuous outside some set B_n standing for 'bad n' sets where f_n is actually continuous. Then

$$\mu(\bigcup_n B_n) \leq \sum_n 2^{-n}\epsilon = \epsilon$$

Now we just let $B = \bigcup_n B_n$ be the bad set for all of the f_n. This is smaller than ϵ loss, and on $[0,1] - B$ we have all f_n continuous. Now we use compactness to get subsequence convergence by sieving of $x_1, \ldots, x_N$. Then we get pointwise convergence of $f_n(x_k)$ by using real numbers bounded at all the points. Then we get pointwise convergence of subsequence $f_{n_j}(x_k)$ and from that we get pointwise convergence of f_{n_j} and apply dominated convergence to get L^1 strong convergence of f_{n_j} which gives us weak convergence.

3. Trying Out My Late Uncle's Spectacles

I need spectacles as my eyesight is weaker now and I am starting to like my late uncle's style over here.

4. Problem I.3

(a) Let $m_n(f) = \mu\{x : 2^n \leq |f(x)| < 2^{n+1}\}$. Then

$$\sum_{n=-\infty} 2^{np} m_n(f) \leq \int |f(x)|^p \mu(dx) < \sum_{n=-\infty}^{\infty} 2^{(n+1)p} m_n(f)$$

(b) For this problem,

$$\begin{aligned} |Af(x)|^2 &= (\int K(x,y) f(y) \mu(dy))^2 \\ &\leq \int K(x,y)^2 \mu(dx) \|f\|^2 \end{aligned}$$

where we have applied Cauchy-Schwarz and then we integrate both sides to get

$$\|Af\|^2 \leq C^2 \|f\|^2$$

So these problems of single Cauchy-Schwarz I am good at doing, and it looks good with my new spectacles. I look like the sort of cat who knows what's what. Don't be fooled, though. Looks can be deceiving, remember dear readers. I am just experienced with these techniques.

5. Problem I.4

Suppose X is a complex Banach space and $\mathcal{T}$ is its weak topology.

(a) Suppose X is second countable. Show that if $f \in X^*$ then there is a finite set of linear functionals ℓ_j so $f = \sum_{j=1}^{N} a_j \ell_j$.

(b) Suppose X is infinite dimensional; show that $(X, \mathcal{T})$ is not metrizable.

I don't know how to do this problem, so I will refer you to my bespectacled image and go pondering the question looking quite intellectual. Remember dear reader, if you don't know the answer, it is important to seem *thoughtful*; otherwise people get the wrong impression.

5.1. **What Motivates Me Today To Care About Weak Topology Metrizability?** I will admit quite frankly that this is not interesting in its own right at all. I could not give a damn about metrizability of any topology of arbitrary topological vector spaces or what have you. I just don't find it interesting at all.

So why am I doing this? I'll tell you why. You see, I am a product – at least from origins – of a *shame culture* and not a *guilt culture*. If you do not know about this, shame on you. Anyway, I am *this close* to foundations of a mathematical physics that I believe will be *eternal truth*, i.e. four-sphere theory, for which I have sacrificed a great deal of joy in my life. And I will die from shame if I did not do some stupid and inane, totally idiotic weak topology mumbo-jumbo that could have made four-sphere theory the eternal truth for the entire human race because of my childish dislike of some broccoli-like vegetables that I was never convinced was worth eating in the first place. I don't care about how boring and techno-mumbo-jumbo it is. I will do *whatever it takes to make my baby fly*. And so I will do as many of these problems as necessary. I don't do this out of love for finer points of mathematics. I do this because I am sure I need to know these things, for the sake of my baby, four-sphere theory.

5.2. **I Will Just Follow Haim Brezis Chapter 3 For This Problem.** This problem is so unfamiliar to me that I will just work through Exercise 3.8 of [1]. Stanford Math should consider this just a learning exercise for me as I will not attempt to do this unaided.

Let's unravel the definitions and lay out our infrastructure. For (a) the assumption is that $(X, \mathcal{T})$ is first countable. We focus on the neighborhoods of zero in X. There is a countable collection of neighborhoods of zero $\{N_j\} \subset \mathcal{T}$ such that every arbitrary $N \in \mathcal{T}$, we have $N_j \subset N$ for some j.

Since $\mathcal{T}$ is the weak topology, it is the smallest topology such that all the elements $\ell \in X^*$ are continuous. Since $\ell(0) = 0$ for all $\ell \in X^*$ we want to consider N_a in terms of linear functionals and their pull-backs of neighborhoods of zero in $\mathbf{R}$.

Let's see. I will just go exploring here a bit and not worry about sharp answers. If we had something that told us, counterfactually, that there existed $\varphi_j \in X^*$ that are fixed and the countable basis for neighborhoods of zero are all of the form

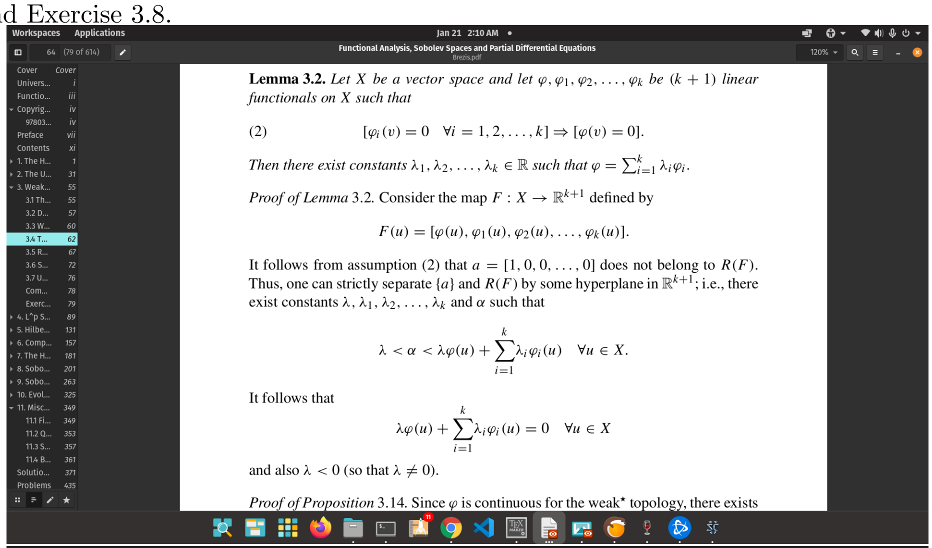
$$N_{a_1,\ldots,a_N;\epsilon} = \{x \in X : \ell_{a_j}(x) < \epsilon \text{ for all } a_j\}$$

Then we can take an arbitrary $\varphi \in X^*$ and find that $\varphi^{-1}(-\epsilon, \epsilon)$ contains one of these $N_{a,\epsilon}$ and then we'd try to get $\ell_{a_j}(v) = 0$ for all $j = 1, \ldots, N$ to imply that $\varphi(v) = 0$ and then apply Haim Brezis Lemma 3.2 that will assure us that φ is a finite linear combination of ℓ_{a_j}.

Yes, I understand this is fruity and not right, but let the man think please. Now let's pretend, counterfactually, we addressed (a).

Then we follow the exercise of Brezis 3.8.

We are far from doing the problem. So let's take a look at Brezis Lemma 3.2 and Exercise 3.8.



Lemma 3.2. *Let X be a vector space and let $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_k$ be $(k+1)$ linear functionals on X such that*

$$(2) \qquad [\varphi_i(v) = 0 \quad \forall i = 1, 2, \ldots, k] \Rightarrow [\varphi(v) = 0].$$

Then there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ such that $\varphi = \sum_{i=1}^k \lambda_i \varphi_i$.

Proof of Lemma 3.2. Consider the map $F : X \to \mathbb{R}^{k+1}$ defined by

$$F(u) = [\varphi(u), \varphi_1(u), \varphi_2(u), \ldots, \varphi_k(u)].$$

It follows from assumption (2) that $a = [1, 0, 0, \ldots, 0]$ does not belong to $R(F)$. Thus, one can strictly separate $\{a\}$ and $R(F)$ by some hyperplane in $\mathbb{R}^{k+1}$; i.e., there exist constants $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_k$ and α such that

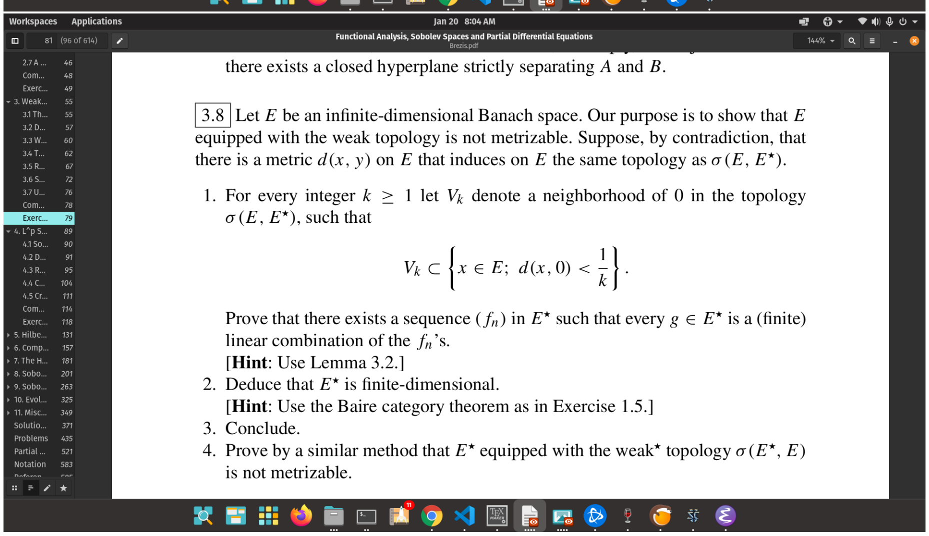
$$\lambda < \alpha < \lambda\varphi(u) + \sum_{i=1}^k \lambda_i \varphi_i(u) \quad \forall u \in X.$$

It follows that

$$\lambda\varphi(u) + \sum_{i=1}^k \lambda_i \varphi_i(u) = 0 \quad \forall u \in X$$

and also $\lambda < 0$ (so that $\lambda \neq 0$).

Proof of Proposition 3.14. Since φ is continuous for the weak* topology, there exists



there exists a closed hyperplane strictly separating A and B.

$\boxed{3.8}$ Let E be an infinite-dimensional Banach space. Our purpose is to show that E equipped with the weak topology is not metrizable. Suppose, by contradiction, that there is a metric $d(x, y)$ on E that induces on E the same topology as $\sigma(E, E^\star)$.

1. For every integer $k \geq 1$ let V_k denote a neighborhood of 0 in the topology $\sigma(E, E^\star)$, such that
$$V_k \subset \left\{ x \in E;\ d(x, 0) < \frac{1}{k} \right\}.$$
Prove that there exists a sequence (f_n) in $E^\star$ such that every $g \in E^\star$ is a (finite) linear combination of the f_n's.
[**Hint**: Use Lemma 3.2.]
2. Deduce that $E^\star$ is finite-dimensional.
[**Hint**: Use the Baire category theorem as in Exercise 1.5.]
3. Conclude.
4. Prove by a similar method that $E^\star$ equipped with the weak* topology $\sigma(E^\star, E)$ is not metrizable.

5.3. **Zulf Gives Himself Pep-Talk To Master Weak And Weak-* Topologies.** One day, not in the distant future, I will have to consider solvability and regularity of solutions of all manner of partial differential equations. Weak and Weak-* topologies and all manner of seminormed topologies will storm my room and I will be wailing "Restless soul enjoy your youth, like Muhammad hit's the truth, small my table, sits just two, got so crowded can't make room, where did they come from stormed my room, this is not for youuuuuu. This is not for youuuu". And all the wailing won't make the need for weak and weak-* topologies go away. And what will a man do then? I can keep wailing about "This is not for youuuuu". But will they listen? I doubt it. I doubt it very much. So I better get ready for the storm.

6. Problem I.5

We are given $A : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ defined by

$$(Ax)_k = x_{k-1} - 2x_k + x_{k+1}$$

and asked to show that it is symmetric and bounded. This is done easily since

$$\langle (Ax)_m, x_n \rangle = \delta_{m-1,n} - 2\delta_{m,n} + \delta_{m,n+1}$$

showing symmetry by $\delta_{a,b} = \delta b, a$ and bounded since it's a sum of two shifts and a scaled identity.

Then we are told that the Fourier series map

$$Tx = \frac{1}{\sqrt{2\pi}} \sum_k x_k e^{ikt}$$

will have the multiplication property

$$(TAT^{-1}f)(t) = m(t)f(t)$$

Let's write the Fourier series out. Suppose

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_k c_k e^{ikt}$$

with

$$c_k = \int_{-\pi}^{\pi} f(t)e^{-ikt}dt$$

Then

$$(AT^{-1}f)_k = \int_{-\pi}^{\pi} f(t)g(t)dt$$

where

$$g(t) = e^{-i(k-1)t} - 2e^{-ikt} + e^{-i(k+1)t}$$

Now we have to play a game of Fourier inversion. Let's see. Oh I see, the +1 shift cancels the -1 shift and vice-versa and we are left with $m(t) = -2$. The spectrum and eigenvalues of A are $\sigma(A) = \{-2\}$.

We'll leave this as the answer and return to this later.

7. Problem II.1

(a) Suppose $f \in C_0(\mathbf{R}^n)$ so then $\hat{f}(\xi)$ is analytic, and so if it vanishes on an open set it must be identically zero.

(b) We must show trigonometric polynomials are dense in $L^p(\mathbf{T})$ for $1 < p < \infty$. For $p = 2$ the trigonometric monomials for a basis of $L^2(\mathbf{T})$, therefore Riesz-Fischer theorem tells us that the trigonometric polynomials are dense. We could use the finiteness of the measure to prove for $p > 2$. Cauchy-Schwarz gives us $L^4(\mathbf{T})$,

$$\int f^2 dx \leq (\int f^4 dx)^{1/2} m(\mathbf{T})^{1/2}$$

So $f \in L^4(\mathbf{T})$ implies $f \in L^2(\mathbf{T})$. This means $L^4(\mathbf{T}) \subset L^2(\mathbf{T})$. We can use Hölder's instead to ensure $L^p(\mathbf{T}) \subset L^2(\mathbf{T})$ and so density of trigonometric polynomials for $L^p(\mathbf{T})$ can be arranged for $p > 2$. Then we have to use duality to get density of trigonometric polynomials are dense in $L^p(\mathbf{T})$ for $p \in (1, 2)$.

8. Problem II.2

Let X consist of sequences a_n with

$$\sum_{n=1}^{\infty} n|a_n|^2 < \infty$$

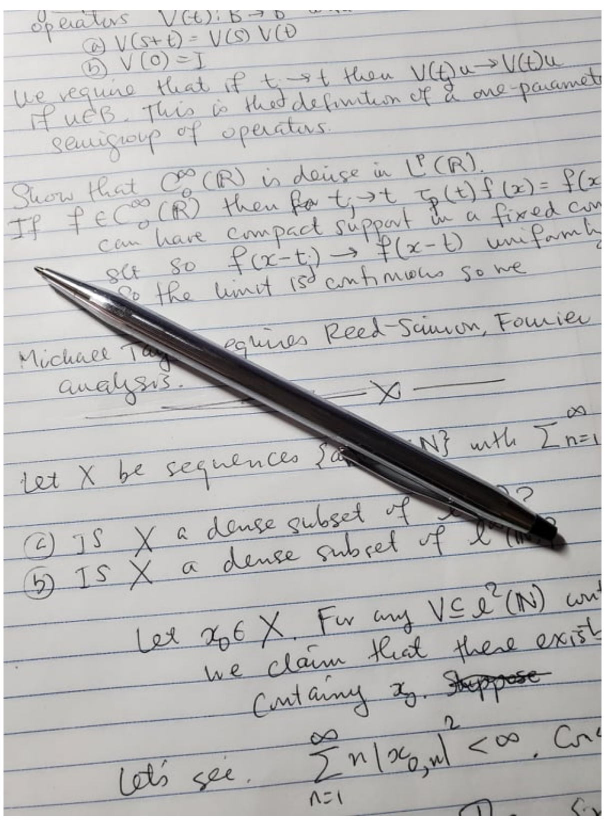
We claim that for any $a \in X$ for any $\ell^(\mathbf{Z})$ neighborhood V of a there is another open set $U \subset V$ in $\ell^2(\mathbf{Z})$ not containing a. Take any ℓ^2-norm open V containing a. Then there is an $\epsilon > 0$ such that $B(a, \epsilon) \subset V$. Now consider the sequence

$$b_n = a_n + \epsilon/2n$$

This has the property that $\sum_{n=1} b_n^2 < \infty$ but $\sum_{n=1}^{\infty} n|b_n|^2 = \infty$, so $b \notin X$. Now we claim $B(b, \epsilon/2) \subset B(a, \epsilon)$ but $B(b, \epsilon/2) \cap X = \emptyset$. This shows X is nowhere dense in $\ell^2(\mathbf{N})$ and so X is Baire second category and nowhere dense in $\ell^2(\mathbf{N})$. This answers (a).

A similar argument with the constant point $b = (\epsilon, \epsilon, \ldots,)$ with *epsilon* $> \sup_n |a_n|$ will give that it is nowhere dense in $\ell^\infty(\mathbf{N})$.

9. Zulf Admires My Beautiful Silver Cross Pen



Let me assure you that I am cross with Bill Gates because he's been exercising every possible illegitimate power against me in my sleep because the little snot-nosed shit literally believes that he will destroy me and kill me by his black magic and other powers soon. The nasty little shit has another thing coming. You see the pen over here, dear readers, this is mightier than Excalibur, and he's a peasant knave, sooner or later I'll get the United States Government to destroy and obliterate the worthless little dipshit. It's just a matter of time.

Yes, you're right. I am dilly-dallying and I am twiddling with my navel instead of swiftly engaging with various matters analytical. But you will have to admit that it's quite annoying when a worthless illiterate scumbag dares to annoy a great immortal genius – and one with a shiny new Cross pen no less – while his abilities in mathematics does not move even the village idiots of Bangalore who have finer souls than he will in a trillion years.

10. Problem II.3

Suppose $T \in L(H)$ for a Hilbert space H and T^* is the adjoint.

(a) Show $H = Ker(T) \oplus \bar{Ran}(T^*)$

(b)

(c)

10.1. Problem II.4(a). Let $V = Ker(T)$. If $x_n \in V$ and converges in H to x then since T is continuous we have

$$T(x) = \lim_{n\to\infty} T(x_n) = 0$$

since $x_n \in V = Ker(T)$ so $x \in V = Ker(T)$ and so V is closed. It is then clear that $H = V \oplus V^{\perp}$.

Next we show that $Ran(T^*)$ is orthogonal to $V = Ker(T)$. Suppose $w \in Ran(T^*)$ we have $y \in H$ with $w = T^*y$. For any $x \in V$, then

$$\langle w, x\rangle = \langle T^*y, x\rangle = \langle y, Tx\rangle = 0$$

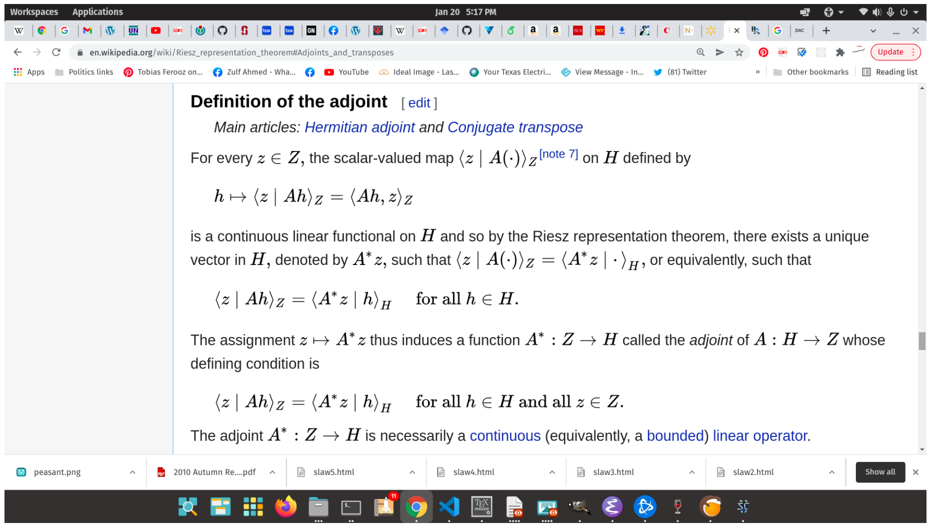
where we used $Tx = 0$ so $Ran(T^*)$ is orthogonal to V. This shows $Ran(T^*) \subset V^{\perp}$. We still have to show that $Ran(T^*)$ is dense in $V^{\perp}$.

So that is the heart of (a), and here we invoke the intuition that injectivity of T and surjectivity of T^* and vice versa is the intuitive duality here.

We're going to do the following. Let's assume that $Ker(T) = 0$ now and we'll show that T^* is surjective.

It is at this point, when the real difficulty shows up, when Zulf quickly asks the dear reader to kindly pay attention to my thoughtful intellectual spectacles while I ponder the situation a bit more.

As I suspected one has to examine the way in which adjoint is defined for the hard part here.



Definition of the adjoint [edit]

Main articles: Hermitian adjoint and Conjugate transpose

For every $z \in Z$, the scalar-valued map $\langle z \mid A(\cdot)\rangle_Z$ [note 7] on H defined by

$$h \mapsto \langle z \mid Ah\rangle_Z = \langle Ah, z\rangle_Z$$

is a continuous linear functional on H and so by the Riesz representation theorem, there exists a unique vector in H, denoted by $A^* z$, such that $\langle z \mid A(\cdot)\rangle_Z = \langle A^* z \mid \cdot\,\rangle_H$, or equivalently, such that

$$\langle z \mid Ah\rangle_Z = \langle A^* z \mid h\rangle_H \quad \text{for all } h \in H.$$

The assignment $z \mapsto A^* z$ thus induces a function $A^* : Z \to H$ called the *adjoint* of $A : H \to Z$ whose defining condition is

$$\langle z \mid Ah\rangle_Z = \langle A^* z \mid h\rangle_H \quad \text{for all } h \in H \text{ and all } z \in Z.$$

The adjoint $A^* : Z \to H$ is necessarily a continuous (equivalently, a bounded) linear operator.

So let's see here. The adjoint T^*x is defined as the *unique element* of H such that

$$\langle x, T^*y\rangle = \langle Tx, y\rangle$$

for all $y \in H$. The uniqueness is guaranteed by the Riesz Representation Theorem. Injectivity of T implies that if $x_1 \neq x_2$ then $T(x_1 - x_2) \neq 0$, so there exists $y \in H$ such that

$$\langle T(x_1 - x_2), y\rangle \neq 0$$

so

$$\langle x_1 - x_2, T^*y\rangle \neq 0$$

Now we have to translate this to every nonzero element of H is in the range T^*.

Fine, I am not sharp here but getting closer.

10.2. **Problem II.4(b).** Our assumption is there is a $C > 0$ so

$$\|x\| \leq C\|Tx\|$$

The relation implies that $Ker(T) = 0$. Now part (a) says $Ran(T^*)$ is dense in H. But we already showed that T^* is surjective in this case so the range is the entire space H.

10.3. **Problem II.4(c).** We are given $T^*T = TT^* = I$ and we have to show $T - \lambda I$ is invertible whenever $|\lambda| \neq 1$ and

$$\|(T - \lambda I)^{-1}\| \leq |1 - |\lambda||^{-1}$$

We claim

$$(T - \lambda)^{-1} = T(I - \lambda T^*)$$

when $|\lambda| \neq 1$. This is proven by algebraic manipulation. The right side is manifestly bounded when $|\lambda| \neq 1$.

11. Problem II.3

12. Problem II.5

13. H. L. Royden's Book Is Quite Good

I remember studying real analysis from H. L. Royden's book at Princeton. I just looked through it now and realise that it is quite a bit more valuable than I had realised. I will study through it more carefully. It's a good book. I think only time and experience tells you about these things. I think one needs to have several books to gain deeper familiarity with the issues, but Royden's level is good. I think that Stein-Shakarchi level is better for a primary text for people getting into the topics, and Royden's book is a good educated presentation that is good as a secondary text for further development. And I think Haim Brezis book is very good as well for functional analysis parts. If I had to get a rounded view, I would put Stein-Shakarchi, Royden, Brezis, and Reed-Simon all in the course texts because these are fairly difficult issues without revisiting issues from multiple perspectives.

14. Return To Problem I.4(a)

We return to this issue because there are many things that are mysterious to me about these weak topologies. We have a second countable normed linear space X and we consider the *weak topology* for X. I am not attempting here to show my great mastery of these issues at the moment because there is not much great mastery to be shown. I am trying to learn from the great master Haim Brezis.

You see, I *think* the neighborhood basis of a point $x_0 \in X$ in this case *ought to be of the form*:

$$N(\varphi_1, \dots, varphi_K; \epsilon) = \{x \in X : |\langle \varphi_j, x - x_0 \rangle| < \epsilon, j = 1, \dots, K\}$$

and a neighborhood basis forms when we let these sorts objects vary by changing ϵ and K and the various φ_j. Now that seems so nice, so neat, and so perfect, doesn't it? It seems so posh and presentable.

There is a problem. I have *absolutely no idea* why this should be the case at all. Why should only finitely many of these φ_j occur for the neighborhood bases? Why not an infinitely many of them?

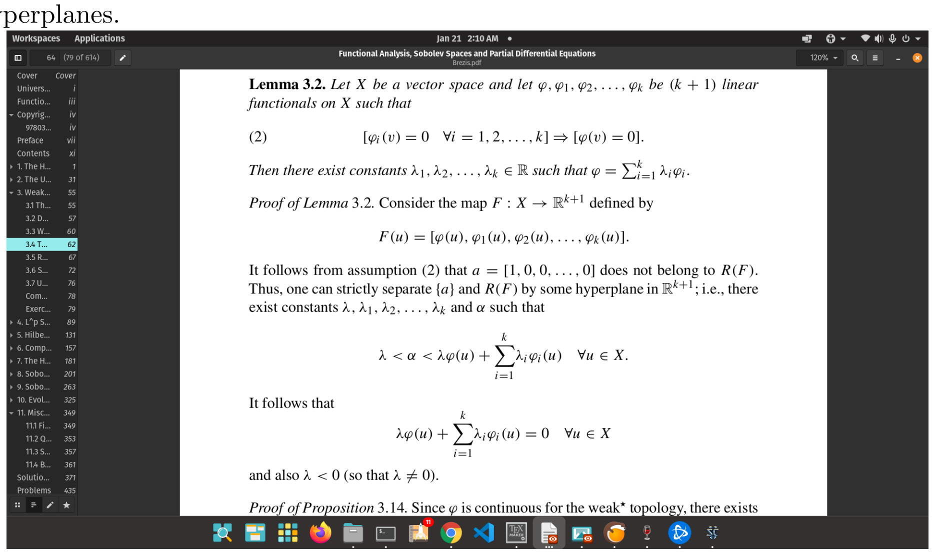
You see, I don't have any shame following Haim Brezis because I don't have any sense for these techno-mumbo-jumbo weak topology things. I understand, I understand completely that I need them, and one day I will be dying from gratitude for their blessed existence, but today, I am just glad for Haim Brezis' existence, bless the man's heart. He is attempting to tell me something in his book, if I could only *listen* to what the man is trying to say.

That's just totally dirty rotten scoundrelling, abstract topological nonsense issue. So it's hidden in the construction of the *weakest topology* that makes all the linear functionals continuous. The *finiteness* of the linear functionals comes from the fact that when we construct the weakest topology we start with sets $\varphi^{-1}((a - \epsilon, a + \epsilon))$ and make the topology $\mathcal{T}$ from *finite intersections* of these sets to form the neighborhoods. I want to thank Haim Brezis because people don't tell you these things. They never tell you that *Ahem, ahem, you will only get finite number of linear functionals in your neighborhood bases because that's a totally artificial general nonsense thing that goes into manufacturing the weak topology so you can just mark this as by standard construction.* They don't tell you this important piece of information and then suddenly you start thinking that there is some Borbaki VII Chapter 37 footnote that invokes juggernaut-theorem-from-hyperspace that has Riemann Hypothesis proven in the footnote but no one has been able to decipher the argument yet that leads to the conclusion that only finite number of linear functionals shall arise. See, that's just very annoying to me. In fact, I don't think anyone even talks about this besided Haim Brezis in everyone's breezy and silent surprise that finiteness comes with the standard construction of these weak topologies.

At least that's good, that it's not a deep theorem being invoked. I don't think these are trivial issues at all. It's fairly basic but it's not discussed clearly almost anywhere.

14.1. **Clear Argument For Form Of Neighborhood Basis.** The basis of neighborhoods in $\mathbf{R}$ are intervals $(a-\epsilon, a+\epsilon)$. So for the weak topology of X, we have a basis of neighborhoods of $x_0 \in X$ consisting of finite intersections of $\varphi_j^{-1}((a-\epsilon, a+\epsilon))$. And the finiteness is just from the construction of the topology which only allows finite intersections. If U is an arbitrary neighborhood of $x_0 \in X$ in the weak topology, then we can find a neighborhood that is smaller $V \subset U$ so that $V = \cap_{\text{finite} i \in I} \varphi_j^{-1}(J_j)$ where J_j are open sets containing $\varphi_j(x_0) = a_j \in \mathbf{R}$. Then we just take smaller intervals $(a_j - \epsilon_j, a_j + \epsilon_j) \subset J_j$ and get $V = \cap_j \varphi_j^{-1}(a_j - \epsilon_j, a_j + \epsilon_j)$ and now we have some finite number of (φ_j, ϵ_j) characterising the basis neighborhood $V = V(\varphi_1, \ldots, \varphi_K, \epsilon_1, \ldots, \epsilon_K)$ of $x_0 \in X$.

Now from this we need to use the Haim Brezis linear algebra Lemma 3.2 that says that if $\varphi, \varphi_1, \ldots, \varphi_K$ are linear functionals such that $\varphi_k(v) = 0$ for all $k = 1, \ldots, K$ implies $\varphi(v) = 0$, then $\varphi = \sum_k b_k \phi_k$ and that's what is needed for I.4(a). This is not extremely easy either, since it uses Hahn-Banach geometric form for separating hyperplanes.



Lemma 3.2. *Let X be a vector space and let $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_k$ be $(k+1)$ linear functionals on X such that*

(2) $$[\varphi_i(v) = 0 \quad \forall i = 1, 2, \ldots, k] \Rightarrow [\varphi(v) = 0].$$

Then there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ such that $\varphi = \sum_{i=1}^k \lambda_i \varphi_i$.

Proof of Lemma 3.2. Consider the map $F : X \to \mathbb{R}^{k+1}$ defined by

$$F(u) = [\varphi(u), \varphi_1(u), \varphi_2(u), \ldots, \varphi_k(u)].$$

It follows from assumption (2) that $a = [1, 0, 0, \ldots, 0]$ does not belong to $R(F)$. Thus, one can strictly separate $\{a\}$ and $R(F)$ by some hyperplane in $\mathbb{R}^{k+1}$; i.e., there exist constants $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_k$ and α such that

$$\lambda < \alpha < \lambda\varphi(u) + \sum_{i=1}^k \lambda_i \varphi_i(u) \quad \forall u \in X.$$

It follows that

$$\lambda\varphi(u) + \sum_{i=1}^k \lambda_i \varphi_i(u) = 0 \quad \forall u \in X$$

and also $\lambda < 0$ (so that $\lambda \neq 0$).

Proof of Proposition 3.14. Since φ is continuous for the weak* topology, there exists

Once we append the Brezis Lemma 3.2 we obtain a valid answer to I.4(a).

I am *extremely glad* that I did not spend days attempting to put this together myself. This is outrageously difficult a process for any fair examination. Who would do this at all. I've never seen anything so outrageously esoteric in even ordinary problems. I wonder if anyone has ever actually produced a valid solution to either I.4(a) or I.4(b) since 2010 in ordinary test conditions.

15. Why Is It Mathematically Interesting If Exotic Weak Topologies Are Not Metrizable?

This is a serious issue. We've known these sorts of metrizability issues for exotic coarse topologies for ages, from the 1930s at least. Who cares if the weak topology is not metrizable? Why is this mathematically interesting? We need to use them for what they are good for. Why do we care to spend huge effort knowing something that never mattered in the first place? I don't understand why anyone cares. No one *uses* metrizability as a go-to condition for weak topologies anyway. They give more compact subsets and they give continuity for things like maps from $\mathbf{R}^N$, parametrized families of linear operators etc. I think this is just a wrong direction overall. Why don't mathematicians spend more time on things that we can use and less on totally arcane trivia that tell us that we can't do things that we were not trying to do in the first place, and even that at great labours. Why is this an interesting theorem, besides requiring Olympic Gymnastic skills in mathematical ingenuity?

References

[1] Haim Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer 2010