STANFORD ANALYSIS 2014 PROBLEM I.1

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1. Stanford Analysis 2014 Problem I.1

For $0 < \alpha < 1$ let $C^{\alpha}([0,1]) \subset C([0,1])$ be defined by finiteness of norm

$$\|f\|_{\alpha} = \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}$$

(a) Show that the closure of the unit ball of $C^{\alpha}([0,1])$ is compact in $C([0,1])$
(b) Show that $C^{\alpha}([0,1])$ is of first category in $C([0,1])$.

2. A Small Tribute To Elias Stein

I took several classes with Professor Elias Stein in 1991-1995 and at the time I was simply not experienced enough to truly grasp how great a mathematician he was. I liked his lectures which were always enlightening but he was a master of marvelous *taste* in mathematics and that would only be clearer to me from studying his books over time. With great reverence, I want to go through his last great teaching extravaganza for *undergraduates* from Stein-Shakarchi *Princeton Lectures in Analysis III: Measure Theory, Integration, and Hilbert Spaces*.

What becomes clear from studying Elias Stein and Rami Shakarchi's book is for me at least, the remarkable focus and clarity in what is central and important to people who are interested in mathematics, without any clutter and nonsense.

Take for example 4.6 Compact Operators, pp. 188-193. A set $X \subset H$ of a Hilbert space is compact if for every sequence f_n there is norm-convergent subsequence f_{n_k} to an element of X. A bounded linear operator $T : H_1 \to H_2$ is called compact if and only if $T(B)$ has a compact closure in H_2 where

$$B = \{x \in H_1 : \|x\| \leq 1\}$$

To see that B is not compact, note

$$\|e_n - e_m\|^2 = 2$$

by Pythagorean theorem, so no subequence of e_n can converge.

Stein-Shakarchi present some criteria for compactness that are valuable, examples, and the spectral theorem for compact self adjoint operators.

This is just perfect, just what is substantial to know and nothing more and nothing less with perfect clarity. For me, who just worked on a novel proof of the spectral theorem for compact self-adjoint operators yesterday for 2013 Stanford Analysis Qual, this is truly magnificent the level of clarity of the great master.

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3. Compactness of Embedding

For (a) we need to prove that the embedding $j : C^\alpha([0,1]) \to C([0,1])$ is a compact operator.

4. The Fourier Coefficients Approach

There might be some extremely well-understood estimates that yield the compactness of the embedding for $[0,1]$. But I did not feel comfortable enough about understanding the direct approach. I will show compactness of embedding of something similar. I consider the torus case instead. I define

$$H^\alpha(\mathbf{T}) = \{f \in C(\mathbf{T}) : (1 + |n|^\alpha)|\hat{f}(n)| < \infty\}$$

Then then attempt to show $H^\alpha(\mathbf{T})$ is compact in $C(\mathbf{T})$. For this I will assume that

$$D^\alpha f(x) = \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

exists for all $x \in \mathbf{T}$ for $f \in H^\alpha$ and

$$|F(D^\alpha f)(n)| = |n|^\alpha |Ff(n)|$$

Then we have, for bounded sequence f_k in $C(\mathbf{T})$ of elements of $H^\alpha(\mathbf{T})$, the Fourier coefficients satisfying

$$|Ff_k(n)| \leq C|n|^{-\alpha}$$

And that is the key here, that the derivative bound ensures that higher frequency Fourier coefficients are uniformly small as $n \to \infty$. Then the projection of $C(\mathbf{T})$ to span of $e_{|n| \leq N}$ where e_n is the Fourier basis $e_n = e^{inx}$ contains all $j(f)$ for $f \in H^\alpha$ with $\|j(f)\| \geq N^{-alpha}$. And that is the reason for the compactness of the embedding $j : H^\alpha(\mathbf{T}) \to C(\mathbf{T})$.

The Fourier basis makes this problem clearer in terms of what happens to elements of different frequencies, so once it is approximately finite rank, we're done.

5. Analytical Approaches

See, for me, it is easy to read off what is happening in the Fourier basis. The mathematical substance is there, made concrete. The analytical approach has to somehow replicate the same substance by other means, and I do not have facility at the moment to do that.

6. First Category

First category is countable union of nowhere dense sets. Using the Fourier basis, I can take some $n > N$, and look at e_n and use it to get some neighborhood in $C(\mathbf{T})$ that is untouched by the closure of H^α but whose distance to it is as small as I like. I won't go into details for this. The important issue is how to understand (a) with the Fourier basis and then relate that to analytical methods for proving (a).

This is a very serious mathematical problem so I will return to it later. I don't understand the analytical approaches.