STANFORD SPRING 2012 ANALYSIS PROBLEM I.4

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This is my third day working on Stanford Analysis Ph.D. Qual Problems from 2013. I am timing things, but I am relaxed. Recall that I was magna cum laude in Mathematics from Princeton University 1995. I was graduate student in Mathematics at Columbia during 1996-2000 with work with the probability theorist Daniel Stroock 1999-2000 but beyond a joint publication in Journal of Differential Geometry in 2000 I had moved on to industry. I am a bit rusty and 49 so I am not exactly trying to prove myself as a Stanford graduate student taking the exam. I addressed nine problems out of ten, with a beautiful counterexample for I.5 that I am quite proud of since it's very nineteenth century and elegant and gives some insight.

1. Stanford Analysis 2013 I.4

Suppose u is a distribution on $\mathbf{T}$, i.e. an element of the dual of $C^\infty(\mathbf{T})$. The problem is to find a *continuous function* $f \in C(T)$ and a $k \geq 0$ such that

$$u = \frac{d^k}{dx^k} f + c$$

where c is totally constant.

2. My Approach

The approach that makes most sense to be me is to employ Fourier series directly. I want to enumerate the Fourier basis on $\mathbf{T}$ as:

$$\varphi_n(x) = e^{inx}$$

without further comments. Then I want to try to understand conditions on the convergence properties of series expansions in this basis. The hope is that for distributions, the coefficients are not so large in growth that

$$\left|\frac{d^k}{dx^k}(e^{inx})\right|$$

taken out of them would not lead to small enough coefficients for Fourier series to converge to a continuous function for a fixed $k \in \mathbf{N}$.

Without further ado, let's define

$$a_n = u(\varphi_n)$$

and these are some real numbers without any presumed size restrictions. For any smooth $f \in C^\infty$, we can get a Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} b_n \varphi_n(x)$$

and we just use this result without comment.

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Then we find

$$u(f) = \sum_{n=-\infty}^{\infty} a_n b_n$$

which formally looks like an inner product except that a_n has no convergence property to ensure that it's even ℓ^2 let alone continuous.

The problem is to show then that our a_n is of the form

$$n(n-1)\cdots(n-k)d_n$$

where

$$g(x) = \sum_{n=-\infty}^{\infty} d_n \varphi_n(x)$$

is a series that converges to a continuous function.

We are ignoring complex parts. We can do this with positive coefficients and take some inequalities to get them. That's a minor nuisance. The substance here is the size of the modulus of a_n is not growing too fast.

3. Arena For Racial Bias Are Not Primarily in Mathematics Departments

There was an article in New York Times a year or two ago about a black mathematician and his sufferings and trials and tribulations. So there are definitely issues of racial bias and difficulties for nonwhite people in Mathematics in America and Europe.

However, I am broad and understand these issues supremely well now at 49, and do not consider the central problem to be in Mathematics. These are issues in America defined by the *national identity of America as a nation and a people* and history of the American People involved genocide of native Americans; it did involve brutal enslavement of black people and horrible mistreatment and slavery for a long time – on racial grounds so there is no *question* of racial bias in America that is a larger issue and that is geopolitical too. America has had devastating wars in Asia nonstop since 1950 and there is *no question* that there is a collective racial orientation that sees non-white lives as less valuable than white lives. These things are not about Mathematics department. Their substance has history of three centuries and scope that is global. So the arena for those issues for me are *white people in American Government.*

Academic world does not live in a vacuum. It's true that many in academia would love to live in a vacuum, or at least in monastic disregard for the rest of the world. So in Mathematics, there is a secondary and tertiary effects of the mass of the world's sensibilities. These are not things that mathematicians will generally resolve because mathematicians are generally not even educated enough to make any serious contribution to these issues.

So I don't take seriously Stanford Mathematics Department's ability to do all that much about these issues directly. They will be resolved on public sphere, political sphere, cultural sphere, geopolitical sphere.

Look, there was *no hijacking* by anyone at all in those planes on September 11 2001. That's correct. There was *no hijackers* and *no hijacking.* The story was a deception orchestrated by Israel's Mossad. Their goal was to trigger some *classified planned wars in Asia* by neoconservatives in Washington from early and mid 1990s.

They saw opportunity in the wars being in Israeli national interest so they remote-controlled some planes and demolished twin towers. Then they faked hijacking by artificial telephone calls and other paraphenalia.

But the American people believed the ruse, and so you have the entire American people believe that Muslims did it, and so large numbers of Americans thought the wars of two decade had some justification as retaliation. When this sort of thing is blasted and supported by New York Times and Economist, where is the basis of racist attitudes? It's obviously policy at larger scales than whether one non-white person's work in mathematics was overlooked and other small petty things of this sort.

Well, just as mathematics requires time and effort to gain feel for some things, so does geopolitics, to know clearly who will lie and how large the lie will be for what *rational purpose*. That requires years just like mathematics. I have developed these talents mostly out of necessity more than professional interest. I am actually very talented in this area. I consider myself better than George Kennan and better than Dean Acheson on these things.

4. Returning to Mathematical Issues

We're going to try to prove the following: given a element u of the dual of $C^\infty(\mathbf{T})$, there exists a $k \in \mathbf{N}$ such that

$$|u(\varphi_n)| \leq n^k$$

This will allow us to get the continuous function with

$$|d_n| \leq |n|^{-2}$$

via

$$g(x) = \sum_{n=-\infty}^{\infty} d_n \varphi_n(x)$$

which is bounded in absolute value by

$$h(x) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \varphi_n(x)$$

by Fourier series convergence theorems. It's obviously ℓ^2 but we punt right now on how to prove continuity; it's continous, for now because Zulf says so.

Now the dual of $C^\infty(\mathbf{T})$ are bounded linear functionals. Oh this might be easier than I thought. We have a single constant $C > 0$ such that

$$|u(\varphi_n)| \leq C\|varphi_n\|_2 = C$$

Well, that's pretty good then, we're done now, because we can do $k = 3$ for example and get the result now.

5. Gratitude To Stanford Analysis 2013 Problem Composers

I did take quite a few courses during my undergraduate years at Princeton, but I was never extremely sharp in my understanding of *distributions*, tempered or ordinary. I had various prejudices about them, mostly that they are wild and crazy and pathological and suitable not for me– for I was too special and geometrical – but for various extremely industrious *analysts*. Already between 1991-1995 I was quite posh in always telling analysts that we all stand on their giant shoulders and so on, with the ulterior motive of getting *them* to do all the laborious work so we could live off their backs. I admit that this was wrong today, after I 49, and I will show contrition. I was wrong, analysts. I was wrong to think that I could live off your work without compensation at rubber plantations while I live in luxury. Wait, that's not what I meant. Anyway, I will show contrition. I will repent, analysts. I will promise a new start. I will appreciate your work for its own sake and not how I will exploit it with a feeling of entitlement.

I am grateful for the work of Stanford for this problem, for it suddenly made me realise that I could have been somebody. I could have had some sense that Fourier coefficients of distributions cannot grow but must remain bounded. I suddenly felt that these distributions, they are not so bad after all. Marcel Riesz' comments to Laurent Schwartz were unwarranted. I see the light, analysts. I am a man reborn!

6. How Analysis Looked Like To Me During Youth

Don't misunderstand, I studied with Elias Stein and Peter Sarnak and Joe Kohn so I knew some analysis including Bers, Schechter, John's book on Partial Differential Equations. But I was not an analyst by training. I hung around Peter Oszvath and others thinking about geometry and topology.

The anatomy of an analysis theorem was schematic to me as follows. First you take the entire Roman alphabet. The you say there exists $A > 0, B > 0, C > 0, D > 0, \dots$ then you take the entire Greek alphabet, and then you put conditions like $\alpha(A, B) < \epsilon$, $\pi(K, M) < 5\epsilon^{1/2}$, and so on. Then you mention all sorts of esoteric spaces, locally rectangular Orlicz-spaces with separability but not uniformity and convexity that is not pseudo but quasi, and more technical mumbo-jumbo. Then the conclusion of the theorem is nothing to write your mom about.

You see, we were aware that there were great analysts, but we didn't want to live our lives like that. We wanted to have theorems that gave insight about things and I did not see what insights could have such horrible form. So I was careful not to wade too deep into this.

7. Continuity Of Bounding Function

I will now clear up the following issue. Let

$$g(x) = \sum_{n=-\infty}^{\infty} \frac{1}{n^2} \varphi_n(x)$$

I will prove that this is a continuous function. There are various other parts of this problem that can be improved. The key to this is that *uniformly* convergent sequence of continuous functions is continuous. The uniformity is necessary for the conclusion.

Now here what happens is that for $\epsilon > 0$ we have an N such that for $n \geq N$ with

$$|\sum_{|n| \geq N} n^{-2}| < \epsilon$$

which is totally elementary. We just use the bound on the functions.

$$\begin{aligned}
|\sum_{|n|\geq N} n^{-2}\varphi_n(x)| &\leq \sum_{|n|\geq N} n^{-2}|\varphi_n(x)| \\
&\leq \sum_{|n|\geq N} n^{-2}\sup_x |\varphi_n(x)| \\
&= \sum_{|n|\geq N} n^{-2} \\
&< \epsilon
\end{aligned}$$

This gives *uniform convergence* of the partial sums and evidently the finite sums being smooth are continuous. Therefore the limit is as well. This technique gives continuity for

$$\sum_{n=-\infty}^{\infty} d_n\varphi_n(x)$$

and that's a key part of the problem. We manage to prove that $k = 3$ is enough for all ordinary distributions.

8. Slow Maturity For Zulf

You see, I was well-regarded as an undergraduate mathematics student at Princeton, and treated with respect. I knew I was good, and I did well in many courses in Mathematics. I was very spoiled, though, since I was treated with respect as intellectually gifted by Lys K. Waltien in high school, and did well in my Mathematics courses at Princeton, was surrounded by truly great people like Peter Oszvath and Shin Mochizuki, who were very talented and very kind to me. They always were nice about explaining all manner of things about meaning of characteristic classes and Yang-Mills instantons, or algebraic topology issues and so on. They were exceedingly talented. I was socially immature, and did not know that you have to prove yourself to people who did not know all your homework problems and courses at Princeton. So when I went to Columbia, I was in trouble because they did not know anything about my work at Princeton at all, and when I went to work with Dan Stroock I never made any effort to get him to examine my record at all. This was a very foolish error, and it was not a mathematical error. It is a far more basic error of not being aware of how social world works. And so I was in a problem as Dan Stroock was not happy with me and I was not aware of this at all. I was not concerned about it because of pure *social immaturity* and also I was recovering from my ex-wife leaving me. It took me two decades to even realise the problem, and frankly the realisation only came after 2018 when I had achieved phenomenal success with Four-Sphere Theory but no one in physics even sent me a congratulatory note even by email. It was deadly silence.

Suddenly I say to myself, "Ah, they don't even read Four-Sphere Theory notes because they don't know who I am." Suddenly a lot of things clicked into place. People are busy with their own narcissistic concerns and that is part of human life on Earth. No one will care about who the hell you are till they have some actual self-interest in doing so! Suddenly, I had learned something about the world I did not know in 1996-2000 or even 2000-2008 and that allowed me to evolve as a person.

This is not a personal thing with Dan Stroock at all. He is *normal.* I was not. He was kinder to me than all the physicists in the end.

9. The Every Man For Himself In America

United States of America is a country where you cannot even rely on the American Government to secure your *basic natural rights* when a racial murderer tycoon assaults you in your blood meta and deep interior with extremely hateful and malevolent intention to deny all your legitimate earnings and destroy your life. Now this was not ever clear to me when I was younger. So the truth is that this is the sort of place we live in, and so it is not very surprising that everyone has to ensure that their accolades and accomplishments need to be presented in an unabashed way so that people who need to know them, such as my prize for most original thesis of year 1995 in Mathematics, needs to be *advertised explicitly* because no one has any obligation to know these things at all. They are not automatic. You need to ensure that everyone always gets updated. And that's just part of life in America. And I was not prepared for this when I was younger. Now I don't give a damn who I am bothering. They won't read my emails anyway if they don't feel like it. This is very important for Asians to understand. United States of America is not Asia. No one has any time for you in America and it's okay to annoy anyone because they have their tools for not being interested; you don't need to worry about that.