ZULF RUMINATES ON MY UNDERGRADUATE FUNCTIONAL ANALYSIS AND ANALYSIS EDUCATION WITH STANFORD SPRING 2012 ANALYSIS PROBLEM 4

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1. Stanford Analysis Qual Spring 2012 II.4

Suppose X is a $\mathbf{C}$-vector space and $\mathcal{F}$ a collection of linear functionals. Equip X with $\mathcal{F}$-weak topology, i.e. weakest topology for which all $f \in \mathcal{F}$ are continuous.

(a) Show that, with product topologies on $X \times X$ and $\mathbf{C} \times X$, addition and scalar multiplication are continuous.

(b) Suppose $\rho : X \to [0, \infty)$ is a seminorm. Show there exists a finite set $\ell_1, \dots, \ell_N \in \mathcal{F}$ with

$$\rho(x) \leq C \sum_{k=1}^{N} |\ell_k(x)|$$

2. Thoughts

Suppose $O \subset X$ is an arbitrary open subset of X; then for every $f \in \mathcal{F}$, there exists an open $U_f \subset \mathbf{C}$ such that $f^{-1}(U_f) = O$.

Let $p : X \times X \to X$ be addition and $d : \mathbf{C} \times X \to X$ be scalar multiplication.

For (a) we want to prove $p^{-1}(O)$ is open and $d^{-1}(O)$ is open in the product topologies.

Now the product topology is the weakest topology where $\pi_1, \pi_2 : X \times X \to X$ is continuous.

Let's see if we can get somewhere easily, and focus on $p^{-1}(O)$. Write:

$$p^{-1}(O) = \{(x, y) \in X \times X : x + y \in O\}$$

Ah, I see, the trick is to use the fact that translation and multiplication by constant complex numbers preserves open sets in $\mathbf{C}$. That will do this.

3. Seminorm Bounds

This one is more nontrivial, we want to bound an arbitrary seminorm

$$\rho(x) \leq C \sum_{k=1}^{N} |\ell_k(x)|$$

Seminorms are characterised by

$$\rho(x + y) \leq rho(x) + \rho(y)$$

and

$$\rho(sx) = |s| rho(x)$$

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I am not sure how to do this yet. We'll get there, not to worry.