ZULF RECONSIDERS PROBLEM II.5 OF STANFORD ANALYSIS QUAL DECEMBER 19 2021

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I have six problems from Stanford Spring 2013 Analsis Quals that I do not have, and I do want to understand these quite a bit better. Now II.5 is about elliptic operators on $\mathbf{T}^n$. This is an important problem and I want to gain sharper understanding of this situation.

Let me recall the problem using slightly different notation so it's clearer. We consider polynomials

$$p(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$$

The polynomials are called *elliptic* if $\xi \neq 0$ then $p(\xi) \neq 0$.

Then we consider differential operators associated to the polynomial p, i.e. $P = p(D)$. Then we consider these operators acting on Sobolev space $H^m = H^m(\mathbf{T}^n)$ defined in terms of Fourier coefficients.

The problems are to prove the following.

(a) The nullspace of $P : H^m \to L^2$ is finite dimensional and consist of only smooth functions.

(b) P is invertible if and only if it is injective.

My attempts yesterday to address the problem had been unsuccessful. Today I want to try to examine the way that Fredholm's 1903 works worked in order to gain some understanding. So Fredholm's work led to the notions of compact operators.

The thought I had is that we could just use ellipticity to define $Q = p(D)^{-1}$ as another operator defined the symbol and examine what happens. Fourier transform F will be an isomorphism on L^2, and we could use that here.

Let $m(p)$ be the operator on $\ell^(\mathbf{Z}^n)$ given by

$$m(p)(a_k) = p(k)a_k$$

Now consider $m(p^{-1})$ as well. Then

$$m(p^{-1})m(p) = 1$$

for $k \neq 0$.

Write $P = F^{-1}m(p)F$ and let $Q = F^{-1}m(p^{-1})F$. The idea is that $I - PQ$ might be a compact operator so that nullspace of P would be in the range of $I - PQ$ and therefore finite dimensional.

1. On Torus Fourier Coefficients Have Specific Structure

On the torus $\mathbf{T}^n$, the Fourier coefficients are indexed by $\mathbf{Z}^n$, and that associated with $0 \in \mathbf{Z}^n$ are the constant functions on $\mathbf{T}^n$. Our approach tells us that the nullspace of P is actually either one or zero dimensional depending on whether P annihilates constants.

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This allows us to conclude a stronger version of (b) which is that P is invertible if and only if P does not annihilate constant functions on $\mathbf{T}^n$.

In general one could laboriously apply the inverse function theorem but in this case we don't have to since our approximate inverse Q gives us stronger information.

2. Why I Adore My Answer

This answer is extremely Michael Atiyah-Michael Atiyah type answer, honouring the long tradition of topologists and geometers of not doing rough grueling work and obtaining insights with a rather je ne sais quois, a certain grace and lack of any sign of industriousness. I adore this answer because it totally avoids doing arduous back-breaking labour.