END OF THE AGE OF GAUSSIAN CONFUSION

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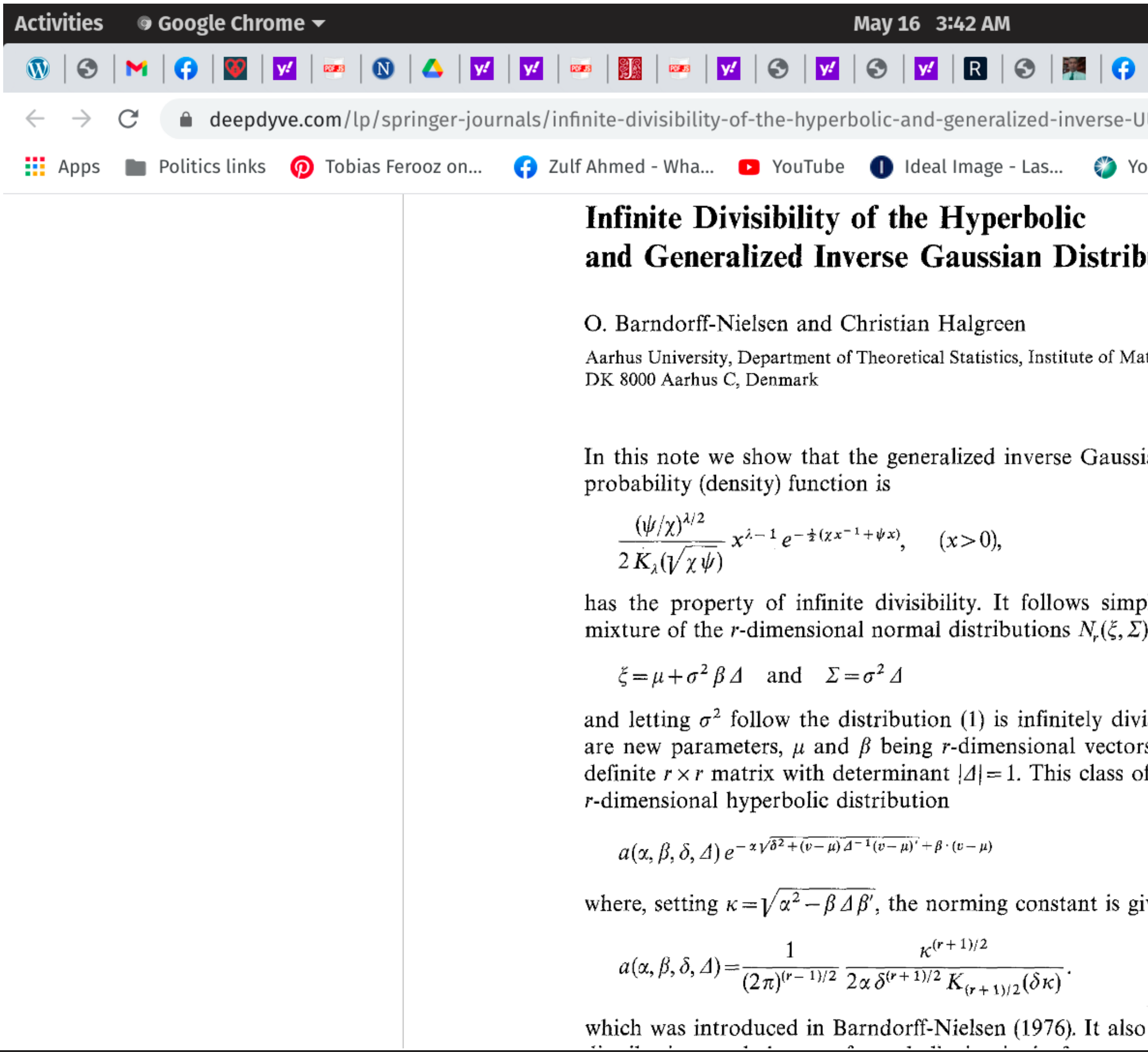
1. Barndorff-Nielsen 1976



Infinite Divisibility of the Hyperbolic and Generalized Inverse Gaussian Distrib

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In this note we show that the generalized inverse Gaussi
probability (density) function is

$$\frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} x^{\lambda-1} e^{-\frac{1}{2}(\chi x^{-1}+\psi x)}, \qquad (x>0),$$

has the property of infinite divisibility. It follows simp
mixture of the r-dimensional normal distributions $N_r(\xi,\Sigma)$

$$\xi=\mu+\sigma^2\beta\Delta \quad \text{and} \quad \Sigma=\sigma^2\Delta$$

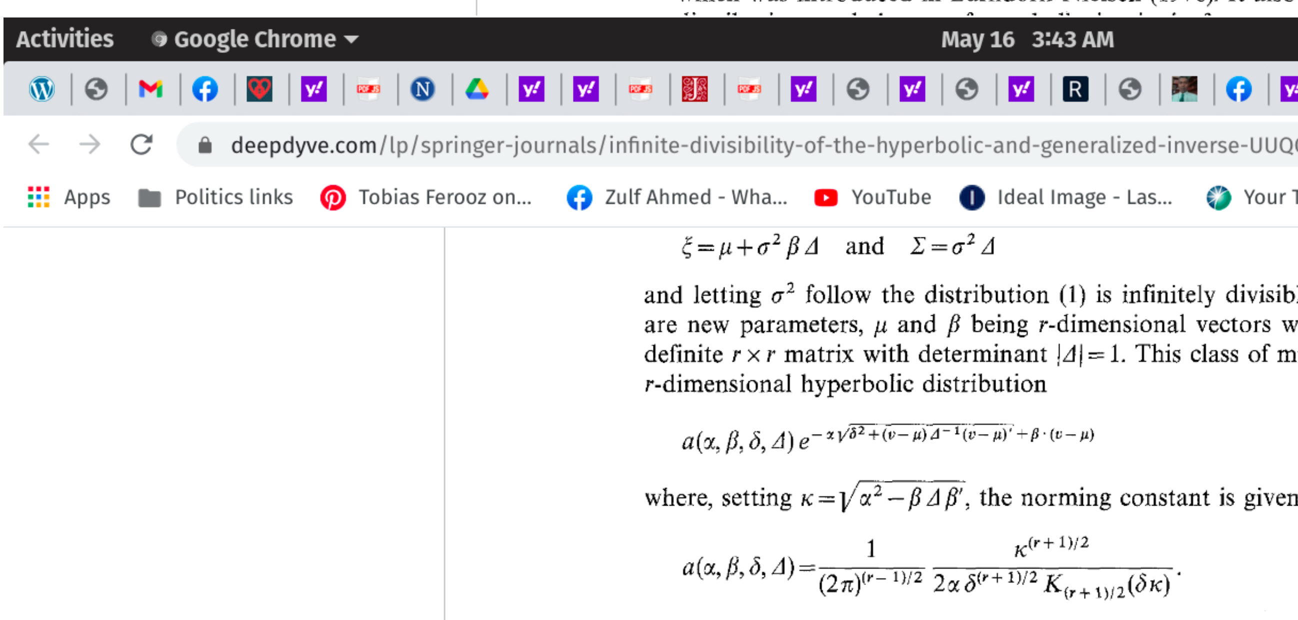
and letting σ^2 follow the distribution (1) is infinitely divi
are new parameters, μ and β being r-dimensional vectors
definite $r\times r$ matrix with determinant $|\Delta|=1$. This class of
r-dimensional hyperbolic distribution

$$a(\alpha,\beta,\delta,\Delta)\, e^{-\alpha\sqrt{\delta^2+(v-\mu)\Delta^{-1}(v-\mu)'}+\beta\cdot(v-\mu)}$$

where, setting $\kappa=\sqrt{\alpha^2-\beta\Delta\beta'}$, the norming constant is gi

$$a(\alpha,\beta,\delta,\Delta)=\frac{1}{(2\pi)^{(r-1)/2}}\,\frac{\kappa^{(r+1)/2}}{2\alpha\,\delta^{(r+1)/2}K_{(r+1)/2}(\delta\kappa)}.$$

which was introduced in Barndorff-Nielsen (1976). It also



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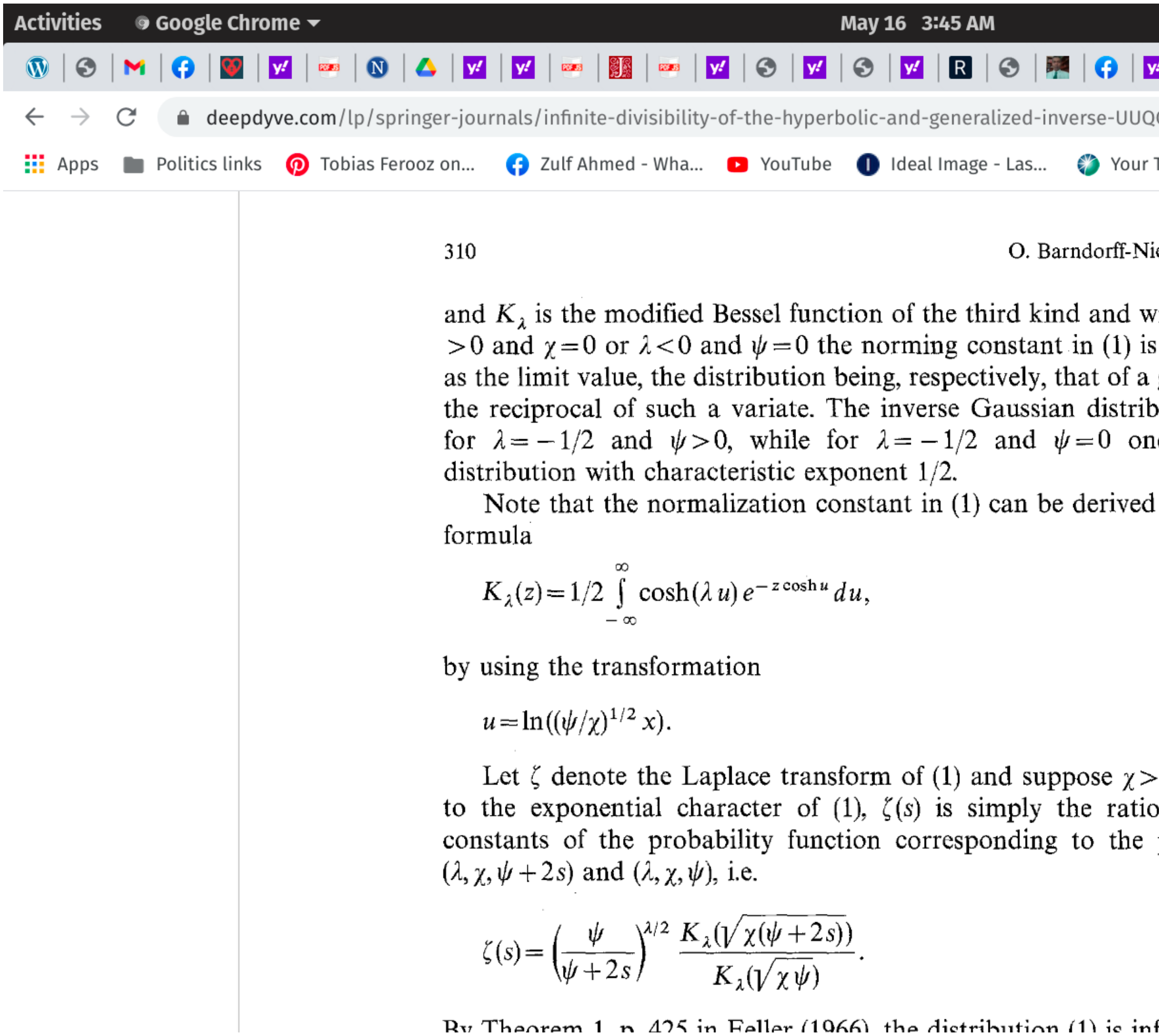

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and K_λ is the modified Bessel function of the third kind and w
>0 and $\chi=0$ or $\lambda<0$ and $\psi=0$ the norming constant in (1) is
as the limit value, the distribution being, respectively, that of a
the reciprocal of such a variate. The inverse Gaussian distrib
for $\lambda=-1/2$ and $\psi>0$, while for $\lambda=-1/2$ and $\psi=0$ one
distribution with characteristic exponent $1/2$.

Note that the normalization constant in (1) can be derived
formula

$$K_\lambda(z)=1/2\int_{-\infty}^{\infty}\cosh(\lambda u)\,e^{-z\cosh u}\,du,$$

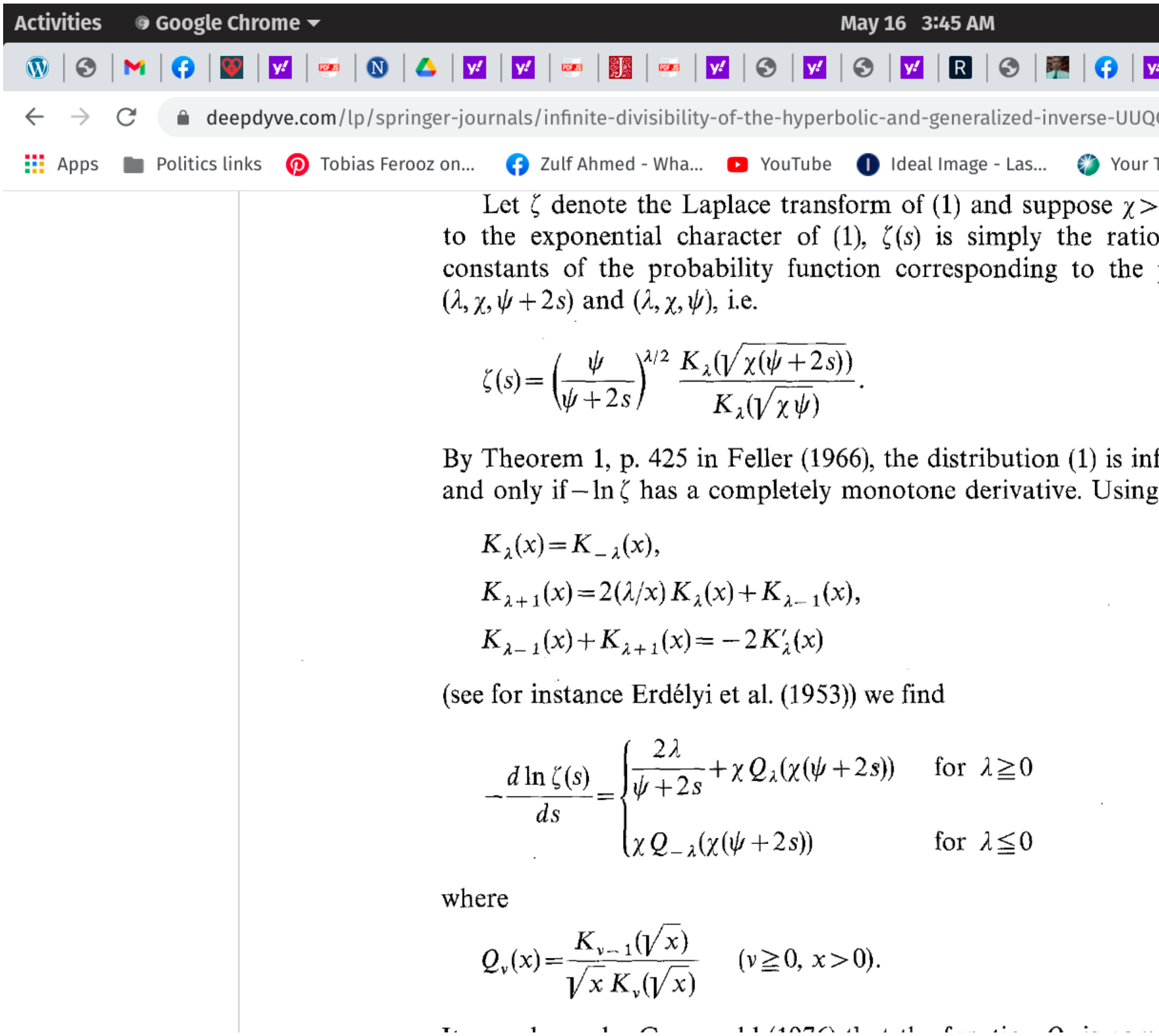
by using the transformation

$$u=\ln((\psi/\chi)^{1/2}x).$$

Let ζ denote the Laplace transform of (1) and suppose $\chi>$
to the exponential character of (1), $\zeta(s)$ is simply the ratio
constants of the probability function corresponding to the
$(\lambda,\chi,\psi+2s)$ and (λ,χ,ψ), i.e.

$$\zeta(s)=\left(\frac{\psi}{\psi+2s}\right)^{\lambda/2}\frac{K_\lambda(\sqrt{\chi(\psi+2s)})}{K_\lambda(\sqrt{\chi\psi})}.$$

By Theorem 1, p. 425 in Feller (1966), the distribution (1) is inf



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$$\zeta(s) = \left(\frac{\psi}{\psi + 2s}\right)^{\lambda/2} \frac{K_\lambda(\sqrt{\chi(\psi + 2s)})}{K_\lambda(\sqrt{\chi\psi})}.$$

By Theorem 1, p. 425 in Feller (1966), the distribution (1) is inf
and only if $-\ln\zeta$ has a completely monotone derivative. Using

$$K_\lambda(x) = K_{-\lambda}(x),$$

$$K_{\lambda+1}(x) = 2(\lambda/x)\,K_\lambda(x) + K_{\lambda-1}(x),$$

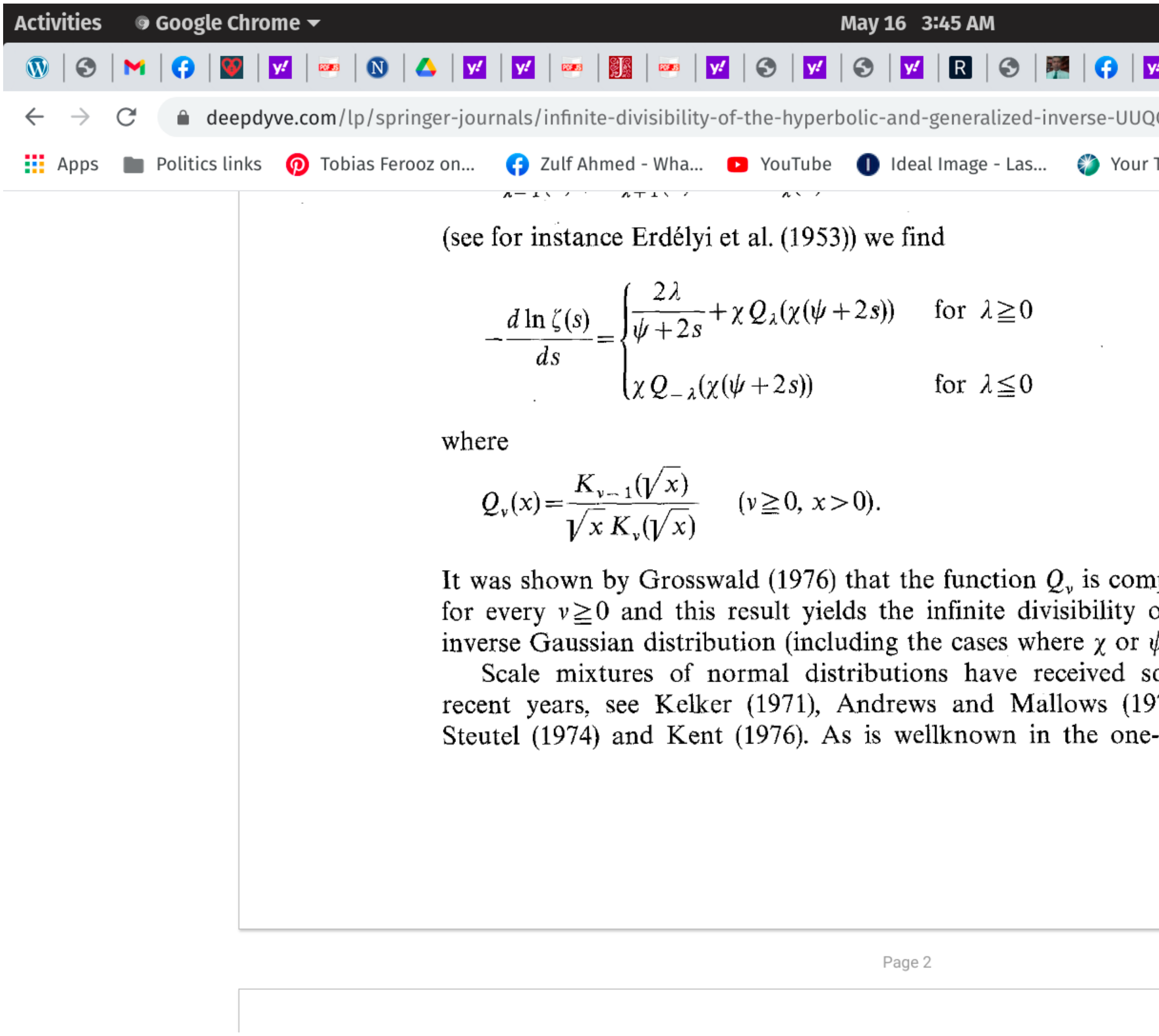
$$K_{\lambda-1}(x) + K_{\lambda+1}(x) = -2K'_\lambda(x)$$

(see for instance Erdélyi et al. (1953)) we find

$$-\frac{d\ln\zeta(s)}{ds} = \begin{cases} \dfrac{2\lambda}{\psi + 2s} + \chi\, Q_\lambda(\chi(\psi + 2s)) & \text{for } \lambda \geqq 0 \\ \chi\, Q_{-\lambda}(\chi(\psi + 2s)) & \text{for } \lambda \leqq 0 \end{cases}$$

where

$$Q_\nu(x) = \frac{K_{\nu-1}(\sqrt{x})}{\sqrt{x}\,K_\nu(\sqrt{x})} \qquad (\nu \geqq 0,\ x > 0).$$



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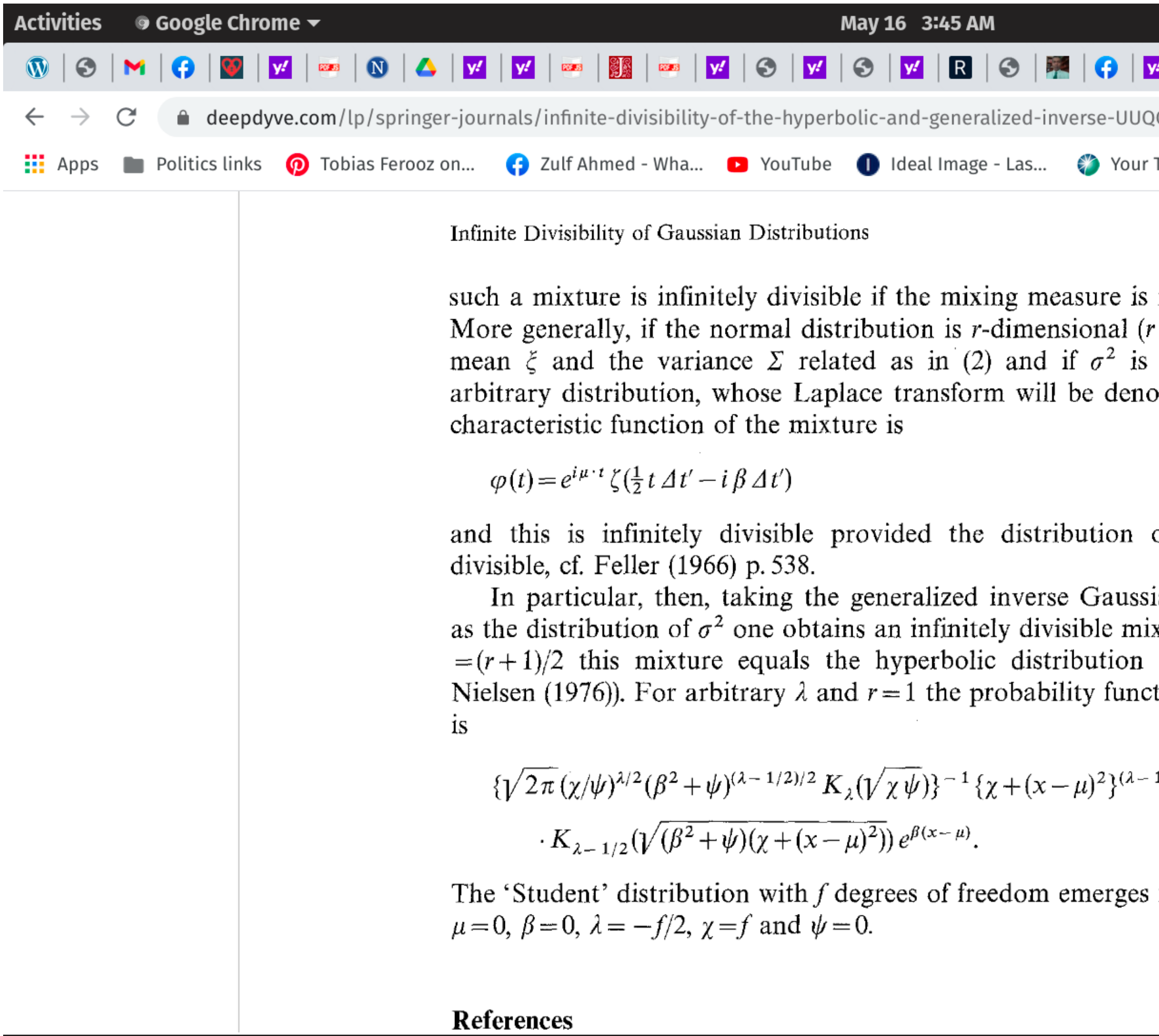
where

$$Q_\nu(x)=\frac{K_{\nu-1}(\sqrt{x})}{\sqrt{x}\,K_\nu(\sqrt{x})}\qquad (\nu\geqq 0,\ x>0).$$

It was shown by Grosswald (1976) that the function Q_ν is com
for every $\nu\geqq 0$ and this result yields the infinite divisibility o
inverse Gaussian distribution (including the cases where χ or ψ

Scale mixtures of normal distributions have received so
recent years, see Kelker (1971), Andrews and Mallows (19
Steutel (1974) and Kent (1976). As is wellknown in the one-



Infinite Divisibility of Gaussian Distributions

such a mixture is infinitely divisible if the mixing measure is
More generally, if the normal distribution is r-dimensional (r
mean ξ and the variance Σ related as in (2) and if σ^2 is
arbitrary distribution, whose Laplace transform will be deno
characteristic function of the mixture is

$$\varphi(t) = e^{i\mu\cdot t}\,\zeta(\tfrac{1}{2}t\,\Delta t' - i\,\beta\,\Delta t')$$

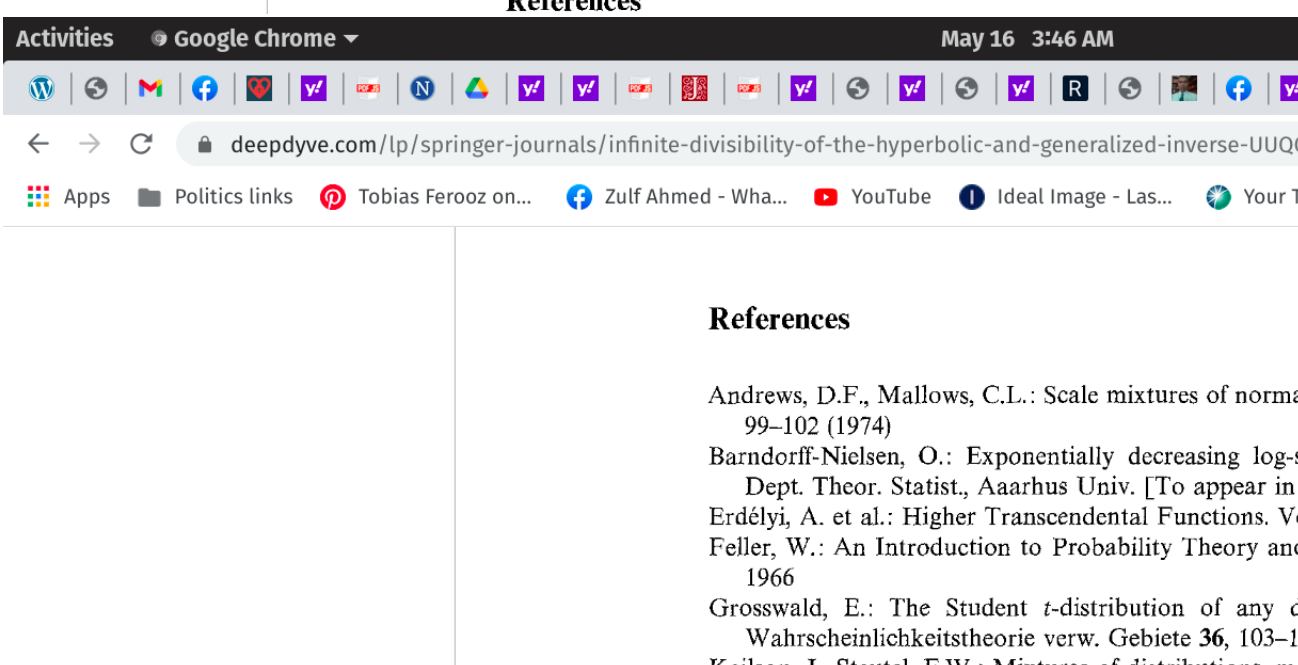
and this is infinitely divisible provided the distribution o
divisible, cf. Feller (1966) p. 538.

In particular, then, taking the generalized inverse Gaussi
as the distribution of σ^2 one obtains an infinitely divisible mix
$=(r+1)/2$ this mixture equals the hyperbolic distribution
Nielsen (1976)). For arbitrary λ and $r=1$ the probability funct
is

$$\{\sqrt{2\pi}\,(\chi/\psi)^{\lambda/2}(\beta^2+\psi)^{(\lambda-1/2)/2}\,K_\lambda(\sqrt{\chi\psi})\}^{-1}\{\chi+(x-\mu)^2\}^{(\lambda-1}$$
$$\cdot K_{\lambda-1/2}(\sqrt{(\beta^2+\psi)(\chi+(x-\mu)^2)})\,e^{\beta(x-\mu)}.$$

The 'Student' distribution with f degrees of freedom emerges
$\mu=0$, $\beta=0$, $\lambda=-f/2$, $\chi=f$ and $\psi=0$.

References



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2. Two Centuries of Gaussian Confusion In Science

The pristine beauty of Gaussian distributions and wonderful theory had taken us far. And it is time to give up our childhood in Science regarding what Noise is in the actual universe rather than in mathematical theory. And it was in 1976 that Ole Barndorff-Nielsen had proved the *infinite divisibility* of one of the deepest secrets of the universe, the Generalised Hyperbolic Distribution, which would one day completely replace all use of Gaussian Distributions in all of Science. You see Gaussian Distributions were our childhood toys. They allowed the possibility to produce a coherent mathematical body for Brownian motion.

But ever since the sixties, if not even before then, there were phenomena that had various non-Gaussian noise. The terminology developed around Gaussian, and it is dreadful terminology. ”Heavy-tailed” distributions. As though Nature ought to have followed Gaussian Distribution but did not comply. No this is *wrong. Gaussian Distribution and Processes are toys*!! Let there be a new and clearer Age of Science. Let us celebrate the great discovery of Ole Barndorff-Nielsen and replace all Gaussian Distributions and Processes with these GHD, and obviously I am going to consider the names to be Barndorff-Nielsen Distribution and Nature's fundamental processes to be Barndorff-Nielsen Processes because he is Dane and would not propose it. Let us celebrate this great transition of an Age of Science.