Ph.D. Qualifying Exam, Real Analysis

Fall 2016, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1 Suppose X is a Banach space and $A \in \mathcal{L}(X)$ is a bounded linear operator on it. Show that the spectrum of A is a closed, bounded subset of $\mathbb{C}$.

2 Recall that $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$, and $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions. Show that there exists no $u \in \mathcal{S}'(\mathbb{R})$ such that for $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}\phi \subset (0,\infty)$, $u(\phi) = \int e^{1/x}\phi(x)\,dx$. (Hint: if such a distribution u existed, it would satisfy an estimate!)

3 Let μ be a non-negative Borel measure on $\mathbb{R}^n$ such that $\mu(A) < \infty$ for each bounded Borel subset $A \subset \mathbb{R}^n$.

a. Setting $\overline{B}_\rho(x) = \{y \in \mathbb{R}^n : |y-x| \leq \rho\}$, prove that for each $\rho > 0$, $x \mapsto \mu(\overline{B}_\rho(x))$ is an upper semi-continuous function on $\mathbb{R}^n$. (A real-valued function θ on $\mathbb{R}^n$ is upper semi-continuous if for all $x \in \mathbb{R}^n$, $\theta(x) \geq \limsup_{y\to x}\theta(y)$.)

b. Give an example of a Borel measure μ as above and $\rho > 0$ such that $x \mapsto \mu(\overline{B}_\rho(x))$ is not continuous.

4 Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, e^{-x^2}\,dx)$, and let $\phi_n(x) = x^n$ for $n \geq 0$ integer, so $\phi_n \in \mathcal{H}$.

a. Let $e_\xi(x) = e^{i\xi x}$ for $x \in \mathbb{R}$. Prove that $\sum_{n=0}^k \frac{(i\xi)^n}{n!}\phi_n$ converges to $e_\xi \in \mathcal{H}$ in the norm topology as $k \to \infty$.

b. Using (a) or otherwise show that if $\phi \in \mathcal{H}$ and $\langle \phi_n, \phi\rangle_{\mathcal{H}} = 0$ for all n then $\phi = 0$. (Hint: show that the Fourier transform of $e^{-x^2}\phi$ vanishes!)

c. Show that there is an orthonormal basis $\{\psi_n\}_{n=0}^\infty$ of $\mathcal{H}$ such that for all n

$$\operatorname{span}\{\phi_0, \phi_1, \ldots, \phi_n\} = \operatorname{span}\{\psi_0, \psi_1, \ldots, \psi_n\}.$$

5 Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be the unit circle, and for $m > 0$ let $H^m(\mathbb{T})$ be the Sobolev space consisting of $L^2(\mathbb{T})$-functions f whose Fourier coefficients $\hat{f}(n)$ satisfy $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2(1+n^2)^m < \infty$. Suppose $A : H^m(\mathbb{T}) \to L^2(\mathbb{T})$ is a continuous linear map, and there is $B \in \mathcal{L}(L^2(\mathbb{T}), H^m(\mathbb{T}))$ such that $AB - I = E$ and $BA - I = F$ are compact on $L^2(\mathbb{T})$, resp. $H^m(\mathbb{T})$, [and in fact $E^* \in \mathcal{L}(L^2, H^m)$]. Suppose also that $\langle Af, g\rangle_{L^2} = \langle f, Ag\rangle_{L^2}$ when $f, g \in H^m(\mathbb{T})$, [and $\langle Bf, g\rangle_{L^2} = \langle f, Bg\rangle_{L^2}$ for $f, g \in L^2(\mathbb{T})$]. Show that $A - \lambda I : H^m \to L^2$ is invertible when $\lambda \notin \mathbb{R}$.

Ph.D. Qualifying Exam, Real Analysis

Fall 2016, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Show that if μ is a σ-finite measure on a measurable space $(X, \mathcal{A})$ (i.e. on $\mathcal{A}$, where $\mathcal{A}$ is a σ-algebra of subsets of X), then there is a *finite* measure ν on $(X, \mathcal{A})$ with $\nu << \mu$ and $\mu << \nu$.

2 Suppose $K \in L^2(\mathbb{R}^{2n})$, and define $(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$. Show that $T \in \mathcal{L}(L^2(\mathbb{R}^n))$, and T is compact.

3 Recall that $\mathcal{S}(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$, and $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions. Let $\phi_n \in \mathcal{S}(\mathbb{R})$, $n \geq 1$. Suppose that $u \in \mathcal{S}'(\mathbb{R})$, and the distributions corresponding to ϕ_n, namely $u_n(\psi) = \int \phi_n \psi$ for $\psi \in \mathcal{S}(\mathbb{R})$, converge to u in $\mathcal{S}'(\mathbb{R})$.

a. Suppose that there exists $C > 0$ such that $\|\phi_n\|_{L^2} < C$ for all n. Show that there exists $\phi \in L^2$ such that $u(\psi) = \int \phi\psi$, $\psi \in \mathcal{S}(\mathbb{R})$ and ϕ_n converge weakly to ϕ in L^2.

b. Show that the analogous statement is not true if L^2 is replaced by L^1, namely show that there exist ϕ_n and u with $\|\phi_n\|_{L^1} < C$ such that u is not a distribution given by an L^1 function ϕ.

4 Suppose $X \subset Z$, $Y \subset V$, with X, Y, Z, V Banach spaces, and with both inclusions continuous with respect to their respective norms. Suppose that $P : Z \to V$ continuous linear and has the property that $u \in Z$, $Pu \in Y$ implies $u \in X$. Show that there exists $C > 0$ such that for all $u \in Z$ satisfying $Pu \in Y$, one has

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_Z).$$

5 Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

a. Assume that for a given function $\phi \in L^1([0,1])$ there exists an irrational number α such that $\phi(x) = \phi(x + \alpha)$ for almost all $x \in [0,1]$, where $+$ is addition modulo $\mathbb{Z}$. Show that $\phi(x)$ equals to a constant for almost all $x \in [0,1]$.

b. Given an irrational number α, consider the equation

$$g(x+\alpha) - g(x) = p(x),\ x \in \mathbb{S}^1,$$

for an unknown function $g(x)$, with a given function $p \in C^\infty(\mathbb{S}^1)$, such that

$$\int_{\mathbb{S}^1} p(x) dx = 0.$$

Give a condition on α that would guarantee that $g \in C^1(\mathbb{S}^1)$ for any such function p.