SOLUTIONS TO SELECTED HW EXERCISES (HW 4)

8.6.2 Let A be a cut. Show that, if $r \in A$ and $s \notin A$ then $r < s$.

Pf: Let A be a cut, let $r \in A$ and suppose $s \in \mathbb{Q}$ with $s \notin A$. Clearly, $s \neq r$, so either

$r < s$ or $s < r$.

But if $s < r$, then property (c2) of a cut would imply that $s \in A$, a contradiction.

Thus, $r < s$. ∎

8.6.6 (a) Prove that

$$-A := \{ r \in \mathbb{Q} : \exists\, t \notin A \text{ with } t < -r \}$$

defines a cut.

Pf: Since A is a cut, $A \neq \emptyset$.

(c1): If $-A = \mathbb{Q}$, then $\forall\, r \in \mathbb{Q}$, $\exists\, t \notin A$ with $t < -r$.

Taking $r = 1, 2, 3, \ldots$, we see that then, by exercise 8.6.2, $A \subseteq (-\infty, -n)$ for every $n \in \mathbb{N}$,

implying that $A \subseteq \bigcap_{n \geq 1}^{\infty} (-\infty, -n) = \emptyset$,

contradicting that $A \neq \emptyset$. Thus, $-A \neq \mathbb{Q}$.

On the other hand, since $A \neq \mathbb{Q}$, $\exists\ t \in \mathbb{Q}$ with

$t \notin A$. Since $r := -t-1$ satisfies $r \in \mathbb{Q}$ and

$$\underset{t+1}{\overset{\shortparallel}{-r}} > t,$$

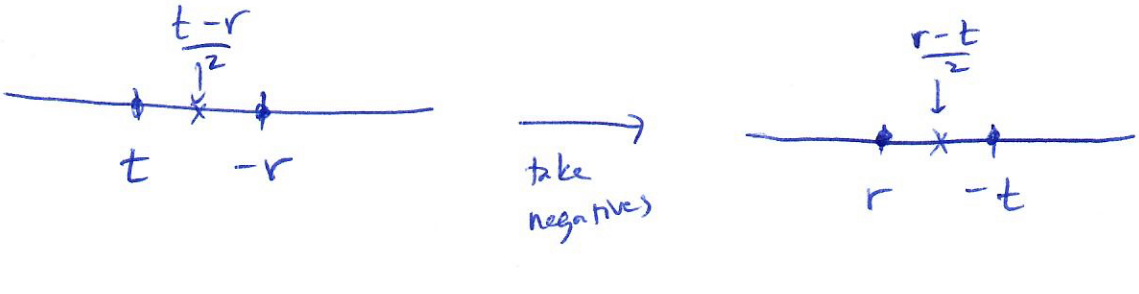
we see that $-t-1 \in -A$, so $-A \neq \emptyset$.

(C2): Suppose $r \in -A$ and $r' \in \mathbb{Q}$ with $r' < r$.

By definition, $\exists\ t \in \mathbb{Q}$ with $t \notin A$ and $t < -r$.

But then $t < -r < -r'$, so $r' \in -A$.

(C3): Let $r \in -A$, so $\exists\ t \in \mathbb{Q} \setminus A$ with $t < -r$.



Since $r = \frac{r+r}{2} < \frac{r-t}{2} < \frac{-t-t}{2} = -t$

we see that $r < \frac{r-t}{2} \in \mathbb{Q}$

and $\frac{r-t}{2} < -t$, i.e. $t < -\left(\frac{r-t}{2}\right)$

Thus, $\frac{r-t}{2} \in -A$, and $\frac{r-t}{2} > r$.

Since $r \in -A$ was arbitrary, we see that $-A$ has no largest element. $\therefore$ $-A$ is a cut.

(b) What goes wrong if we set

$$\boxed{* \quad -A := \{r \in \mathbb{Q} : -r \notin A\}} \ ?$$

Condition (C3) isn't always satisfied, e.g. if $A = 0 = \{r \in \mathbb{Q} : r < 0\}$

Then $-A$ as defined by $*$ would be

$$\begin{aligned} -A &= \{r \in \mathbb{Q} : -r \notin A\} \\ &= \{r \in \mathbb{Q} : -r \geq 0\} \\ &= \{r \in \mathbb{Q} : r \leq 0\}, \end{aligned}$$

which has a largest element, namely 0.

(c) If $a \in A$ and $r \in -A$, show that $a+r \in \mathcal{O}$.

Now, finish the proof of property (f4) for <u>addition</u> in Definition 8.6.4.

<u>Sol'n</u>: Let $a \in A$ and $r \in -A$.

Thus, $\exists\, t \in \mathbb{Q} \setminus A$ with $t < -r$, so $r < -t$

By exercise 8.6.2, $a < t$.

Thus, $a + r < t + (-t) = 0$, so $a + r \in \mathcal{O}$.

It follows that $A + (-A) \subseteq \mathcal{O}$.

Conversely, we now show that $\mathcal{O} \subseteq A + (-A)$.

Let $q \in \mathcal{O}$, so $q < 0$. Then $-q > 0$.

We claim that $\exists\, a \in A$ with $a - \frac{q}{2} \notin A$.

Otherwise, $\forall\, a \in A$, $a - \frac{q}{2} \in A$. But then the sequence

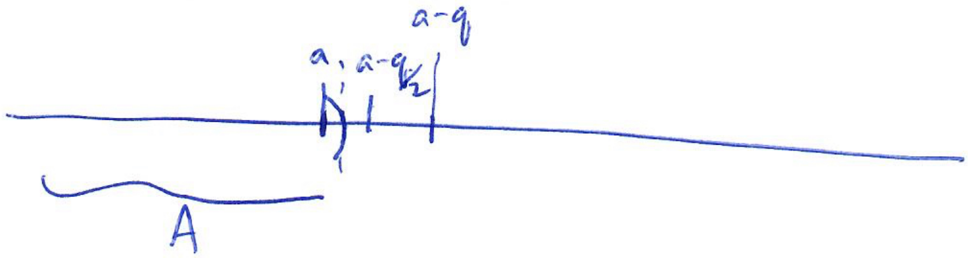
$a, a - \frac{q}{2}, a - 2\frac{q}{2}, a - 3\frac{q}{2}, \ldots$ satisfies $a - n\frac{q}{2} \in A \;\forall\, n \in \mathbb{N}$

and $\lim_{n \to \infty} a - n(\frac{q}{2}) = \infty$, since $-\frac{q}{2} > 0$.

But then, by (C2), we would have

$\mathbb{Q} \cap (-\infty, a - n\frac{q}{2}) \subseteq A$ $\quad \forall n$, so then $A = \mathbb{Q}$, a contradiction. Thus, $\exists\ a \in A$ with $a - \frac{q}{2} \notin A$

(Picture:



)

But then $-r = a - q$ satisfies $-r > t \notin A$

$$\underset{a-q}{\underset{\shortparallel}{-r}} > \underset{a-q/2}{\underset{\shortparallel}{t}} \notin A$$

$\therefore\ r \in -A$, so $q = a + r \in A + (-A)$,

and so $\theta \subseteq A + (-A)$. $\therefore\ \theta = A + (-A)$

This finishes the proof of the <u>addition</u> part of (f4).

<u>8.6.7</u> (a) Show that (when $A, B \geq \theta$)

$$\boxed{AB := \{ab : a \in A,\ b \in B \text{ with } a, b \geq 0\} \cup \overbrace{\{q \in \mathbb{Q} : q < 0\}}^{\theta} \quad **}$$

is a cut and that property (o5) holds.

<u>Soln</u>: Since $-1 \in AB$, $AB \neq \emptyset$.

Furthermore, since $\exists\ r \in \mathbb{Q} \setminus A$ and $\exists\ s \in \mathbb{Q} \setminus B$

By exercise 8.6.2, $\forall\, a \in A,\ a < r$

and $\forall\, b \in B,\ b < s$.

In particular, since $0 \in A$ and $0 \in B$, we see that

$r, s \geq 0$.

Also, it follows from $**$ that $\forall\, c \in AB$,

$c < r \cdot s$ (if $c < 0$ it's obvious, and if $c \geq 0$ then $c = a \cdot b$ for some $a \in A$, $b \in B$ with $a, b \geq 0$, so then follows from).

In particular, $r \cdot s \notin A \cdot B$, so $AB \neq \mathbb{Q}$. (Thus (c1) holds for AB) ✓

Now fix $c \in AB$, and let $\underset{\mathbb{Q}}{\overset{\cap}{c'}} < c$.

If $c < 0$, then $c' < 0$ so $c' \in AB$, by $**$.

If $c \geq 0$ and $c' \geq 0$ then $c = a \cdot b$ for some $a \in A$, $b \in B$ with $a, b \geq 0$.

Thus, $c' < c = a \cdot b$, so if a or b is zero, then $c' < 0$ so again $c' \in AB$.

If $a \cdot b > 0$, then each of $a, b > 0$ so

$c'/a < b$, so $0 < c'/a \in B$ by (c2).

Thus, $c' = \underbrace{\left(\frac{c'}{a}\right)}_{(\in B \text{ and } > 0)} \cdot \underbrace{a}_{(\in A \text{ and } > 0)} \in AB$, by $**$.

$\therefore$ AB satisfies (c2). ✓

Finally, let $c \in AB$. If $c < 0$ then $0 > c/2 > c$ and

$c/2 \in AB$.

If $c = 0$, then $0 \in A$ and $0 \in B$

(if either $0 \notin A$ (or $0 \notin B$), then $A = \emptyset$ (or $B = \emptyset$) and so $AB = \emptyset$ by $**$ in this case)

so $\exists$ $a \in A$ with $a > 0$
$\exists$ $b \in B$ with $b > 0$, but then $ab > 0$ and $ab \in A \cdot B$.

Finally, if $c > 0$ then $c = a \cdot b$ for some $a \in A$, $b \in B$ with $a, b > 0$. By (c3) $\exists$ $a' \in A$, $b' \in B$ with

$a' > a$, $b' > b$. But then

$$a'b' \in AB \quad \text{and} \quad a'b' > ab = c.$$

We've shown in all cases that $\forall\ c \in AB$,

$\exists\ c' \in AB$ with $c' > c$, verifying condition (C3). ✓

(O5), in the language of cuts, says for cuts A, B that

$$\theta \subseteq A \text{ and } \theta \subseteq B \implies \theta \subseteq A \cdot B.$$

This follows immediately from the definition $**$, which holds for all A, B satisfying $\theta \subseteq A$ and $\theta \subseteq B$.

(b) Propose a good candidate for the multiplicative identity on $\mathbb{R}$ and show that this works for all cuts $A \geq \theta$.

<u>Soln</u>: Define $I := \{q \in \mathbb{Q} : q < 1\}$.

Fix any cut A with $\theta \subseteq A$ and consider

$$I \cdot A = \{i \cdot a : a \in A \text{ and } i \in I \text{ with } a, i \geq 0\} \cup \theta.$$

$$\boxed{\{i \in I : i \geq 0\} = \mathbb{Q} \cap [0,1). \ @}$$

Thus, $\forall\ a \in A$ with $a \geq 0$ and $\forall\ i \in I$ with $i \geq 0$,

$a \cdot i < a$, so $a \cdot i \in A$.

Therefore $I \cdot A \subseteq A$, since by assumption $0 \leq A$.

Conversely, let $a \in A$.

If $a < 0$ then $a \in I \cdot A$.

If $a \geq 0$, we reason as follows:

$\exists$ $a' \in A$ with $a' > a$

By @, we then have $\frac{a}{a'} \in \{ i \in I : i \geq 0 \}$,

and thus $a = a' \cdot \frac{a}{a'} \in A \cdot I$.

Since $a \in A$ was arbitrary, we see that $A \subseteq I \cdot A$.

We've shown that $A = A \cdot I$.

(c) Show that the distributive property (f5) holds for all non-negative cuts.

<u>Sol'n</u>: We want to show that

$$\boxed{☺ \quad A \cdot (B + C) = A \cdot B + A \cdot C},$$

assuming $0 \subseteq A$, $0 \subseteq B$ and $0 \subseteq C$.

In case any of A, B or C are 0,

it follows from "$X \cdot 0 = 0 \;\; \forall \, X \geq 0$"

that ☺ holds in that case.

(BTW, " $\forall x$ with $x \geq \theta$, $x \cdot \theta = \theta$ " follows immediately from the definition $**$.)

We may then henceforth assume that none of A, B or C is θ.

Notation: $A_{\geq 0} := \{a \in A : a \geq 0\}$ & similarly for B, C.

Thus, $A \cdot B = \{a \cdot b : a \in A_{\geq 0}, b \in B_{\geq 0}\} \cup \theta$.

$A \cdot C = \{a' c : a' \in A_{\geq 0}, c \in C_{\geq 0}\} \cup \theta$

$$\Rightarrow A \cdot B + A \cdot C = \{a \cdot b + a' \cdot c : a, a' \in A_{\geq 0}, b \in B_{\geq 0}, c \in C_{\geq 0}\}$$
$$\cup \{ab + z : a \in A_{\geq 0}, b \in B_{\geq 0}, z \in \theta\}$$
$$\cup \{ac + z : a \in A_{\geq 0}, c \in C_{\geq 0}, z \in \theta\}$$
$$\cup \overbrace{(\theta + \theta)}^{\theta}$$
$$= \{a \cdot b + a' \cdot c : a, a' \in A_{\geq 0}, b \in B_{\geq 0}, c \in C_{\geq 0}\} \cup \theta;$$

[to see " $\subseteq$ ": RHS obviously contains 1^{st} and 4^{th} sets on LHS. for $ab + z \in 2^{nd}$ set, if $ab + z < 0$ it belongs to RHS, and if $ab + z \geq 0$, then write $ab + z = a(b + z/a) + 0 \cdot c$

Note that $a, b > 0$ since otherwise $ab + z < 0$ when $z < 0$;

also $b + z/a \in B_{\geq 0}$ since B is a cut (and $ab + z \geq 0$),

also $0 \in A$ since $A \geq 0$ and $A \neq O$.

thus $\{ab + z : a \in A_{\geq 0}, b \in B_{\geq 0}, z \in O\} \subseteq RHS$,

and similarly for the 3rd set on LHS.

RHS is obviously a subset of LHS.

Now any $a \cdot b + a' \cdot c$ can be written as

(WLOG suppose $a \geq a'$) $\quad a \cdot b + a \cdot (\underbrace{\tfrac{a'}{a} \cdot c})$

$$= a \cdot b + a \cdot c' \qquad c' \in C,$$

So $AB + AC = \{a \cdot (b+c) : a \in A_{\geq 0}, b \in B_{\geq 0} \text{ and } c \in C_{\geq 0}\} \cup O$.

On the other hand,

$$A(B+C) = \{a \cdot (b+c) : a \in A_{\geq 0}, b \in B, c \in C, b+c \geq 0\} \cup O.$$

For any expression $b + c \geq 0$ with $b \in B$, $c \in C$,

may re-write it as
$$b+c = \begin{cases} \overset{B}{\underset{\cup}{(b+c)}} + \overset{C}{\underset{\cup}{0}} & \text{if } c < 0 \\ \underset{B}{\overset{\cap}{0}} + \underset{C}{\overset{\cap}{c+b}} & \text{if } b < 0 \end{cases}$$

We see that any $b+c$ as in $\}$ may be re-written

as $b + c = b' + c'$ where $b' \in B_{\geq 0}$ and $c' \in C_{\geq 0}$.

thus, $A(B+C) = \{a(b+c) : a \in A_{\geq 0}, b \in B_{\geq 0}, c \in C_{\geq 0}\} \cup O = A \cdot B + A \cdot C$ ∎