MTHT 435, Practice Midterm 1 SOLUTIONS

NO CALCULATORS. For full credit, SHOW ALL WORK.

1. (10 points) Complete the following definitions.

(a) If $f : (a,b) \longrightarrow \mathbb{R}$ is a function and $c \in (a,b)$ is a point in the domain of f, we say that f is *continuous at* c if ...

$\lim_{x \to c} f(x) = f(c)$ [i.e. each side exists, and they are equal.]

(b) If $x_1, x_2, x_3, \dots$ is a sequence of real numbers, we say that $\lim_{n\to\infty} x_n = L$ if and only if ...

$\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that $n > N \implies |x_n - L| < \varepsilon$.

(c) If $S \subseteq \mathbb{R}$ is a non-empty subset, we say that $b \in \mathbb{R}$ is an *upper bound for* S if ...

$\forall\ x \in S,\ x \leq b$.

(d) If $S \subseteq \mathbb{R}$ is a non-empty subset, we say that $b \in \mathbb{R}$ is the *supremum of* S (or that b is the *least upper bound for* S) if ...

(1) $\forall\ x \in S,\ x \leq b$, and (2) $\forall\ b' < b,\ \exists\ x \in S$ with $b' < x$.

(e) If $f : (a,b) \longrightarrow \mathbb{R}$ is a function and $c \in (a,b)$, we say that $\lim_{x\to c} f(x) = L$ if and only if ...

$\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$.

(f) If $A_1, A_2, A_3, \dots$ is a sequence of sets, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is defined by ...

$\bigcap_{n=1}^{\infty} A_n := \{x : \forall n \in \mathbb{N},\ x \in A_n\}$

(g) If X and Y are sets, $f : X \longrightarrow Y$ is a function, and $S \subseteq Y$ is any subset, then the *inverse image of* S *under* f is denoted by $f^{-1}(S)$ and defined by ...

$f^{-1}(S) := \{x \in X : f(x) \in S\}$

(h) If $f : (a,b) \longrightarrow \mathbb{R}$ is a function and $c \in (a,b)$, we say that $\lim_{x\to c} f(x) = \infty$ if and only if ...

$\forall\ B > 0\ \exists\ \delta > 0$ such that $0 < |x - c| < \delta \implies f(x) > B$.

(i) If A and B are sets, then the *set difference* $A \backslash B$ is defined to be ...

$A \backslash B := \{a \in A : a \notin B\}$

(j) If $x \in \mathbb{R}$, the *absolute value of* x is denoted by $|x|$ and defined to be ...

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

2. (10 points) For each of the following, mark either "TRUE" or "FALSE".

(a) For real numbers a and b, if $|a-b| < \varepsilon$ for each $\varepsilon > 0$, then $a = b$.

TRUE. (If $a \neq b$ then $\varepsilon := \frac{|a-b|}{2} > 0$ and $|a-b| > \varepsilon$.)

(b) We have $1 \neq 0.99999\ldots$.

FALSE. ($|1 - 0.999\ldots| < \varepsilon$ for any $\varepsilon > 0$; see (a).)

(c) If $x \in \emptyset$, then x is a green-eyed monster.

TRUE. ("If A then B" is true whenever A is false.)

(d) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ such that $n \cdot \varepsilon > 1$.

TRUE. ($n \cdot \varepsilon > 1 \iff n > \frac{1}{\varepsilon}$; can find n since $\lim_{n\to\infty} n = \infty$.)

(e) $\forall \varepsilon > 0\ \exists n \in \mathbb{N}$ such that $n < \varepsilon$.

FALSE. ($\nexists\, n \in \mathbb{N}$ with $n < \frac{1}{2} =: \varepsilon$)

(f) The set $\{\alpha \in \mathbb{Q} : \alpha^2 < 2\}$ has a least upper bound in $\mathbb{Q}$.

FALSE. (We proved in class that $\{\alpha \in \mathbb{Q} : \alpha < \sqrt{2}\}$ has no least upper bound in $\mathbb{Q}$.)

(g) If $a, b \in \mathbb{R}$ satisfy that $a < b + \varepsilon$ for each $\varepsilon > 0$, then $a < b$.

FALSE. (If $a = b = 1$, then $\forall \varepsilon > 0$, $a < b + \varepsilon$ but $a < b$ is false)

(h) If $g(x)$ and $h(x)$ are functions such that $g(x)h(x)$ is continuous at 0 and $g(x) + h(x)$ is continuous at 0, then both $g(x)$ and $h(x)$ must be continuous at 0.

FALSE. Let $g(x) := \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, $h(x) := \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$

$g(x) \cdot h(x) \equiv 0$, $g(x) + h(x) \equiv 1$ but neither is continuous at $x = 0$

(i) If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ is a sequence of rational numbers (written in lowest terms), $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $\lim_{n\to\infty} x_n = \alpha$, then $\lim_{n\to\infty} a_n = \infty$.

TRUE (We verified this in class.)

(j) If $f : (a,b) \longrightarrow \mathbb{R}$ is a function, $c \in (a,b)$ and there exists a sequence $x_1, x_2, x_3, \ldots$ of points $x_n \in (a,b)$ satisfying $x_n \neq c$ for each n, $\lim_{n\to\infty} x_n = c$, and $\lim_{n\to\infty} f(x_n) = f(c)$, then f must be continuous at c.

FALSE. (Take $f(x) = \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$. $x_n := 1 - \frac{1}{n}$, $c = 1$

$\forall n,\ x_n \neq c$ ✓ $\lim_{n\to\infty} x_n = c$ ✓ $\lim_{n\to\infty} f(x_n) = f(c)$ ✓

but f not continuous at 1.)

3. (10 points) Prove, using the ε-δ definition of limit, that $\lim_{x \to 2} 3x - 5 = 1$.

(See attached sheet)

4. (10 points) Let X and Y be sets and $f : X \longrightarrow Y$ be a function from X into Y. If $S, T \subseteq Y$ are any subsets, prove that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

" —— "

5. (10 points) Prove that there exists a real number x_0 satisfying $x_0 = \cos x_0$.

" —— "

6. (10 points) Prove that $\sqrt[3]{2} \notin \mathbb{Q}$.

" —— "

3. Prove, using the ε-δ definition, that $\lim_{x \to 2} 3x-5 = 1$.

Proof: Let $\varepsilon > 0$ be given

and define $\delta := \varepsilon/3$.

Note that $\delta > 0$.

If $0 < |x-2| < \delta = \varepsilon/3$

then $3|x-2| < \varepsilon$,

so $|3x-6| < \varepsilon$,

so $|(3x-5)-1| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, this proves that $\lim_{x \to 2} 3x-5 = 1$

$\blacksquare$

Scratchwork:

Want $|3x-5-1| < \varepsilon$

$|3x-6| < \varepsilon$

$|3|\,|x-2| < \varepsilon$

$|x-2| < \varepsilon/3$

4. Let X and Y be sets and $f: X \longrightarrow Y$ a function from X into Y. If S, T are any subsets, prove that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

Proof: We will first show that $\boxed{\overset{*}{f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)}}$.

Let $x \in f^{-1}(S \cap T)$. Then we have $f(x) \in S \cap T$,

which means that $f(x) \in S$ and $f(x) \in T$.

$\therefore$ by definition of $f^{-1}(S)$ and $f^{-1}(T)$, we have

$x \in f^{-1}(S)$ and $x \in f^{-1}(T)$,

so $x \in f^{-1}(S) \cap f^{-1}(T)$. Thus, $*$ holds.

Now we'll show that $\boxed{\overset{**}{f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)}}$.

Let $x \in f^{-1}(S) \cap f^{-1}(T)$. By definition of intersection,

$x \in f^{-1}(S)$ and $x \in f^{-1}(T)$.

Thus, $f(x) \in S$ and $f(x) \in T$,

so $f(x) \in S \cap T$. Therefore $x \in f^{-1}(S \cap T)$, so

$**$ holds. Having established $*$ and $**$, we have thus shown that $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. $\blacksquare$

5. Prove that there exists a real number x_0 satisfying

$$x_0 = \cos x_0 .$$

Proof: Since $f(x) := \cos x$ and $g(x) := x$ are continuous,

we may consider the function $h(x) := \cos(x) - x$,

which is also continuous. Note that

$$\cos(x_0) = x_0 \iff h(x_0) = 0,$$

So we want to prove that $\exists\, x_0 \in \mathbb{R}$ with $h(x_0) = 0$.

Now, $h(0) = \cos(0) - 0 = 1$ and

$$h(\pi/2) = \cos(\pi/2) - \pi/2 = -\pi/2 .$$

Since 0 is a value in between $h(0) = 1$ and $h(\pi/2) = -\pi/2$

by the Intermediate Value Theorem, $\exists$ a point

$$x_0 \in (0, \pi/2) \quad \text{with} \quad h(x_0) = 0, \text{ i.e.}$$

with $x_0 = \cos(x_0)$.

∎

6. Prove that $\sqrt[3]{2} \notin \mathbb{Q}$.

Proof: Assume that $\sqrt[3]{2} \in \mathbb{Q}$ and write

$$\sqrt[3]{2} = \frac{a}{b} \quad \text{where} \quad a, b \in \mathbb{Z} \quad \text{with} \quad b \neq 0$$

$$\text{and} \quad \gcd(a,b) = 1.$$

Then $2 = \frac{a^3}{b^3}$, so $2b^3 = a^3$.

Since 2 divides $2b^3$, 2 divides a^3, and so 2 divides a. Writing $a = 2a'$, we then have

$$2b^3 = (2a')^3 = 8(a')^3.$$

Cancelling 2, we get $b^3 = 4(a')^3$, so 4 divides b^3.

Thus, b^3 is even, so b is even.

We conclude that 2 divides both a and b, contradicting our assumption that $\gcd(a,b) = 1$.

Thus, $\sqrt[3]{2} \notin \mathbb{Q}$, as asserted. ∎