SOLUTIONS TO SELECTED HW EXERCISES (HW 5)

1.5.4 (a) Show that $(a,b) \sim \mathbb{R}$ for any interval (a,b).

Pf: As shown in class (see also Practice MT 2, #4) there is a bijection

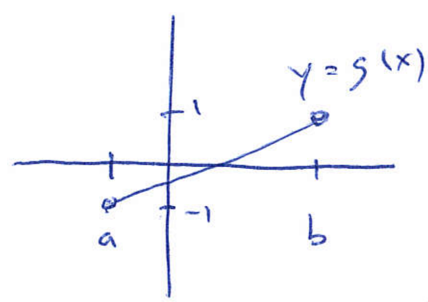
$$f : (-1,1) \longrightarrow \mathbb{R}$$

[we can take either $f(x) = \dfrac{x}{x^2-1}$, or $f(x) = \tan\left(\frac{\pi}{2}x\right)$; there are many other examples of such a bijection].

We now want to find a bijection

$$g : (a,b) \longrightarrow (-1,1).$$

$$g(x) := -1 + \frac{2}{b-a}(x-a)$$



(since $g^{-1}(x) = a + \dfrac{b-a}{2}(x+1)$, we see that g is a bijection.)

Finally, composing these functions, we obtain a bijection $h = f \circ g : (a,b) \longrightarrow \mathbb{R}$.

Concretely,

$$h(x) = \tan\left(\frac{\pi}{2}\cdot\left(-1 + \frac{2}{b-a}(x-a)\right)\right)$$

is a bijection from (a,b) to $\mathbb{R}$.

(b) Show that an unbounded interval like $(a,\infty) = \{x : x > a\}$ has the same cardinality as $\mathbb{R}$ as well.

<u>Soln</u>: Consider the function

$$f : (a,\infty) \longrightarrow \mathbb{R} \quad \text{given by}$$

$$f(x) := \log_e(x-a) = \ln(x-a)$$

Since the function $g : \mathbb{R} \longrightarrow (a,\infty)$ given by

$$g(x) = e^x + a$$

satisfies

$$f(g(x)) = \ln\left((e^x + a) - a\right) = \ln(e^x) = x \quad \text{and}$$

$$g(f(x)) = e^{\ln(x-a)} + a = (x - a) + a = x,$$

We see that, since f is invertible, it is a bijection.

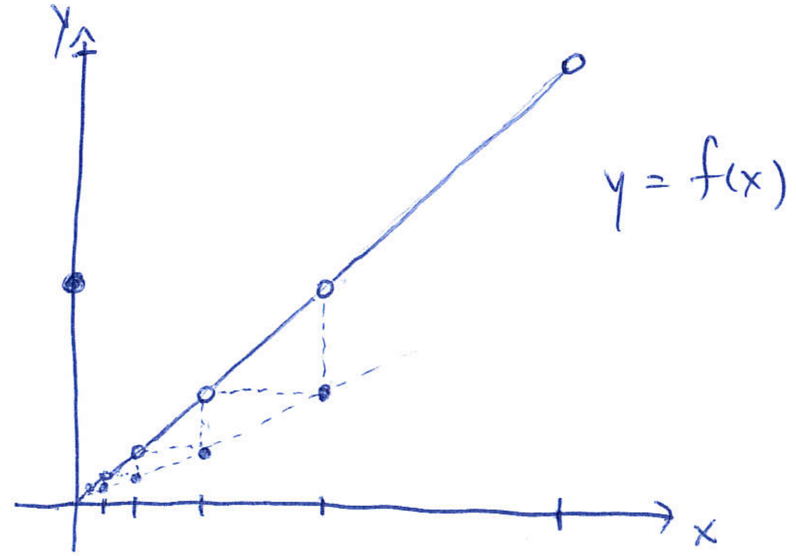
(c) Using open intervals makes it more convenient to produce the required 1-1, onto functions, but it is not really necessary. Show that $[0,1) \sim (0,1)$ by exhibiting a 1-1 onto function between the two sets.

<u>Soln</u>. Define the function $f : [0,1) \longrightarrow (0,1)$

by

$$f(x) = \begin{cases} \frac{1}{2^{n+1}} & \text{if } x = \frac{1}{2^n},\ n \in \mathbb{N} \\ 1/2 & \text{if } x = 0 \\ x & \text{if } x \notin \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots\} \end{cases}$$

Picture



Restricted to the subset

$$\left\{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\} =: S,$$

f does a "shift":

$$\begin{tikzcd}[column sep=small]
\{0, \arrow[dr] & \frac{1}{2}, \arrow[dr] & \frac{1}{4}, \arrow[dr] & \frac{1}{8}, \arrow[dr] & \frac{1}{16}, \arrow[dr] & \frac{1}{32}, \arrow[dr] & \ldots\} \\
\{0, & \frac{1}{2}, & \frac{1}{4}, & \frac{1}{8}, & \frac{1}{16}, & \frac{1}{32}, & \ldots\},
\end{tikzcd}$$

while on the complement of the set S

$$[0,1) \setminus S \subseteq (0,1),$$

f just acts like the identity function.

One checks that f defines a bijection $[0,1) \to (0,1)$.

<u>1.5.9</u> (a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2}+\sqrt{3}$ are algebraic.

<u>Sol'n</u>: Let $f_{\sqrt{2}}(x) := x^2 - 2$. Since $f_{\sqrt{2}}(x)$ is a polynomial with integer coefficients and $f_{\sqrt{2}}(\sqrt{2}) = 0$,

We see that $\sqrt{2}$ is algebraic.

Similarly, put $f_{\sqrt[3]{2}}(x) := X^3 - 2$, which also has integer coefficients and satisfies $f_{\sqrt[3]{2}}(\sqrt[3]{2}) = 0$.

Thus, $\sqrt[3]{2}$ is also algebraic

To find a minimal polynomial for $\sqrt{3}+\sqrt{2}$, let

$\alpha := \sqrt{3}+\sqrt{2}$. Then
$$\alpha^2 = 3 + 2\sqrt{6} + 2$$
$$= 5 + 2\sqrt{6},$$

So $\alpha^2 - 5 = 2\sqrt{6}$.

$\therefore$ $(\alpha^2-5)^2 = 24$, so
$$(\alpha^2-5)^2 - 24 = 0$$
$$\alpha^4 - 10\alpha^2 + 25 - 24 = 0$$

So we set $f_{\sqrt{3}+\sqrt{2}}(x) := X^4 - 10X^2 + 1$.

We've seen that $f_{\sqrt{3}+\sqrt{2}}(\sqrt{3}+\sqrt{2}) = 0$, so

$\sqrt{3}+\sqrt{2}$ is algebraic.

[In general, the sum and product of any two algebraic numbers is algebraic ...]

(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a finite number of roots, show that A_n is countable.

<u>Soln</u>: As proved in class, if a set B is countable, then $\underbrace{B \times B \times B \times \dots \times B}_{n \text{ times}}$ is also countable. ($n \in \mathbb{N}$ fixed)

In particular, $\mathbb{Z}^{n+1} = \overbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}^{n+1 \text{ times}}$ is countable.

But the function

$$\underset{\text{Pol}_{\mathbb{Z}}(n)}{\overset{\shortparallel}{\left\{\begin{matrix}\text{polynomials } f(x) \text{ with integer} \\ \text{coefficients of degree } n\end{matrix}\right\}}} \xrightarrow{\psi} \mathbb{Z}^{n+1}$$

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \longmapsto (a_n, a_{n-1}, \dots, a_1, a_0)$$

is clearly <u>injective</u>.

Since $\text{Pol}_{\mathbb{Z}}(n) \sim \psi(\text{Pol}_{\mathbb{Z}}(n)) \subseteq \mathbb{Z}^{n+1}$,

we see that $\text{Pol}_{\mathbb{Z}}(n)$ is countable, being ~~the~~ a subset of a countable set.

Finally, for each $f(x) \in Pol_{\mathbb{Z}}(n)$, f has at most n roots, so

$$A_n = \bigcup_{f(x) \in Pol_{\mathbb{Z}}(n)} \{\text{roots of } f(x)\}$$

is a countable union of finite sets.

By Theorem 1.5.8 (ii) of our textbook, A_n is then countable, being a countable union of countable sets.

(c) Now, argue that the set of all algebraic numbers is countable. What can we conclude about the set of all transcendental numbers?

<u>Sol'n</u>. Since $\{\text{algebraic numbers}\} = \bigcup_{n=1}^{\infty} A_n$,

We see that it is a countable union of countable sets (by part (b)). Again using

Theorem 1.5.8 (ii), we see that

$\{$algebraic numbers$\}$ is countable. ✓

Finally, since

$$\mathbb{R} = \{\text{algebraic numbers}\} \cup \{\text{transcendental numbers}\},$$

it follows that the set of transcendental numbers is uncountable (otherwise $\mathbb{R}$ would be countable).

1.6.5 (a) Let $A = \{a, b, c\}$. List the eight elements of $\mathcal{P}(A)$.

Sol'n: $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

(b) If A is finite with n elements, show that $\mathcal{P}(A)$ has 2^n elements.

Sol'n: We induct on $n \geq 0$.

Base case $n = 0$. Then $A = \emptyset$, and $\mathcal{P}(A) = \{\emptyset\}$, So $\#\mathcal{P}(A) = 1 = 2^0 = 2^{\#A}$ ✓

Now fix $n \geq 0$ and assume that, $\forall$ set A satisfying $\#A = n$, $\#\mathcal{P}(A) = 2^n$.

Let B be a set with $\#B = n+1$ and pick $x \in B$, and set $A := B \setminus \{x\}$, so $x \notin A$ and $\#A = n$.

We now partition $\mathcal{P}(B)$ into

$$\mathcal{P}_{\in x}(B) \quad \text{and} \quad \mathcal{P}_{\notin x}(B),$$

where $\mathcal{P}_{\in x}(B) := \{ C \subseteq B : x \in C \}$

and $\mathcal{P}_{\notin x}(B) := \{ C \subseteq B : x \notin C \}$

We have $\mathcal{P}(B) = \mathcal{P}_{\in x}(B) \sqcup \mathcal{P}_{\notin x}(B)$

(disjoint union)

Note that $\mathcal{P}_{\in x}(B) \sim \mathcal{P}(A)$

$$C \longmapsto C \setminus \{x\}$$

$$D \cup \{x\} \longleftarrow\!\shortmid D$$

and $$\mathcal{P}_{\notin x}(B) = \mathcal{P}(A)$$

$$\left(\begin{array}{l} \forall\, C \in \mathcal{P}(B), \\ x \notin C \iff C \subseteq A \end{array} \right)$$

Thus, $$\# \mathcal{P}(B) = \# \mathcal{P}_{\in x}(B) + \# \mathcal{P}_{\notin x}(B)$$

$$= \# \mathcal{P}(A) + \# \mathcal{P}(A)$$

$$= 2^n + 2^n = 2^{n+1}.$$

This completes the induction step, establishing that

$\forall$ finite set A, $\# \mathcal{P}(A) = 2^{\# A}$. $\blacksquare$