SOLUTIONS to SELECTED HW Exercises (HW 2)

4.2.5 (c). Prove that $\lim_{x\to 2} x^2+x-1 = 5$.

Proof: Let $\varepsilon > 0$ be given, and set $\delta := \min\{1, \varepsilon/6\}$.

If $0 < |x-2| < \delta$, then (since $\delta \le 1$)

$$|x-2| < 1.$$

Thus, $-1 < x-2 < 1$,

so $4 < x+3 < 6$,

so $|x+3| < 6$.

We thus have, if $0 < |x-2| < \underset{\varepsilon/6}{\overset{}{\delta}}$ ($\delta \le \varepsilon/6$),

$$|x^2+x-1-5| = |x^2+x-6|$$

$$= |(x-2)(x+3)|$$

$$= |x-2|\cdot|x+3| < \varepsilon/6 \cdot 6 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\lim_{x\to 2} x^2+x-1 = 5$. ∎

SCRATCHWORK

Want

$$|x^2+x-1-5| < \varepsilon$$

$$|x^2+x-6| < \varepsilon$$

$$\boxed{* \quad |(x-2)(x+3)| < \varepsilon.}$$

Suppose $\delta \le 1$, so

$$|x-2| < \delta \Rightarrow |x-2| < 1$$

so

$$\begin{array}{ccccc} -1 & < & x-2 & < & 1 \\ +5 & & +5 & & +5 \\ \hline 4 & < & x+3 & < & 6 \end{array}$$

$\therefore\ |x+3| < 6$

So $*$ has, if $\delta \le 1$,

$|x-2|<1$

$\Rightarrow |x-2||x+3| < |x-2|6 < \varepsilon$

$\Updownarrow$

$|x-2| < \varepsilon/6$

So put $\delta := \min\{1, \varepsilon/6\}$.

4.2.5 (d) Prove that $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Proof: Let $\varepsilon > 0$ be given, and set $\delta := \min\{1, 6\varepsilon\}$.

If $0 < |x-3| < \delta$, then (since $\delta \le 1$)

$|x-3| < 1$,

so $-1 < x-3 < 1$,

so $2 < x < 4$

so $6 < 3x < 12$

so $\frac{1}{6} > \frac{1}{3x} > \frac{1}{12}$,

and thus $\frac{1}{|3x|} < \frac{1}{6}$.

Therefore, if $0 < |x-3| < \delta \le 6\varepsilon$,

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| = \overset{(< 6\varepsilon)}{|x-3|} \cdot \overset{(< 1/6)}{\frac{1}{|3x|}}$$

$$< 6\varepsilon \cdot 1/6 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

SCRATCHWORK

Want

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$$

$$\left|\frac{3-x}{3x}\right| < \varepsilon$$

$$|x-3| \cdot \frac{1}{|3x|} < \varepsilon.$$

Suppose $\delta \le 1$, so

$|x-3| < \delta \Rightarrow |x-3| < 1$

so $-1 < x-3 < 1$

$+3 \quad +3 \quad +3$

$2 < x < 4$

$6 < 3x < 12$

$\frac{1}{6} > \frac{1}{3x} > \frac{1}{12}$.

So $\frac{1}{|3x|} < \frac{1}{6}$.

So $|x-3| \cdot \frac{1}{|3x|} < |x-3| \cdot \frac{1}{6}$

$< \varepsilon$

if $|x-3| < 6 \cdot \varepsilon$.

So put $\delta := \min\{1, 6\varepsilon\}$

4.2.6 Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ε challenge, then any smaller positive δ will also suffice.

Sol'n: this is TRUE.

Suppose $\varepsilon > 0$ and that $\delta > 0$ has been found so that, whenever $0 < |x-c| < \delta$, we have $|f(x) - L| < \varepsilon$.

Pick any δ' with $0 < \delta' < \delta$ and

Suppose $0 < |x-c| < \delta'$.

Then $0 < |x-c| < \delta$, so $|f(x)-L| < \varepsilon$.

Thus, any smaller δ' will also suffice. ✓

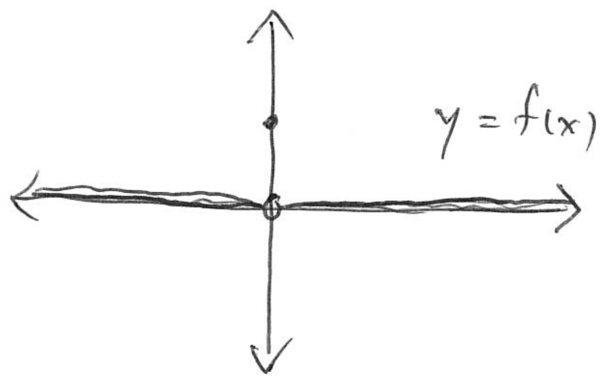
(b) If $\lim\limits_{x\to a} f(x) = L$ and $a \in \mathrm{dom}(f)$ then $L = f(a)$.

Sol'n: this is FALSE.

(this has to do with the "$0<$" in

"$0<|x-c|<\delta \Rightarrow |f(x)-L| < \varepsilon$")

For example, let $f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$



Then $\displaystyle\lim_{x \to 0} f(x) = \overset{\text{"}L\text{"}}{0}$

and $0 \in \operatorname{dom}(f)$

but $0 \neq f(0) = 1$.

(c). If $\displaystyle\lim_{x \to a} f(x) = L$ then $\displaystyle\lim_{x \to a} 3\,[f(x) - 2]^2 = 3(L-2)^2$.

Sol'n: This is <u>TRUE</u>.

It follows from "limit of a product is the product of the limits"

and "limit of a sum is the sum of the limits"

$$\lim_{x \to a} 3\,[f(x) - 2]^2 = 3 \cdot \left[\lim_{x \to a} f(x) - 2\right]^2 = 3(L-2)^2.$$

(d) If $\displaystyle\lim_{x \to a} f(x) = 0$, then $\displaystyle\lim_{x \to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

Sol'n: this is <u>FALSE</u>.

For example, let $a = 0$ and $f, g : \mathbb{R} \to \mathbb{R}$

$$f(x) = x, \qquad g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0$ but $f(x) \cdot g(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

So $\lim_{x \to 0} f(x) g(x) = 1 \neq 0$.

4.2.11 (the SQUEEZE Theorem)

Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$

for all x in some common domain $A = (a, b)$

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, (for some $c \in (a, b)$).

show that $\lim_{x \to c} g(x) = L$ as well.

Proof: First note that, for any $x \in (a, b)$, we

have
$$\begin{aligned} |g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &\leq |g(x) - f(x)| + |f(x) - L| \end{aligned}$$

(we used the triangle inequality $|A+B| \le |A| + |B|$ repeatedly here...)

$$
\begin{aligned}
&\le |h(x) - f(x)| + |f(x) - L| \\
&= |h(x) - L + L - f(x)| + |f(x) - L| \\
&\le |h(x) - L| + |L - f(x)| + |f(x) - L| \\
&= |h(x) - L| + 2|f(x) - L|.
\end{aligned}
$$

Let $\varepsilon > 0$ be given. Since $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$,

$\exists\ \delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon/3 \text{ and } |h(x) - L| < \varepsilon/3$$

Since $|g(x) - L| \le |h(x) - L| + 2|f(x) - L|$,

we then have

$$0 < |x - c| < \delta \implies |g(x) - L| < \varepsilon/3 + 2 \cdot \varepsilon/3 = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have proved

that $\lim_{x \to c} g(x) = L$. ∎