1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific counterexample.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

Sol'n: This is FALSE. For example: $A_n = \{n, n+1, n+2, \ldots\}$
$= \{m \in \mathbb{N} : m \geq n\}$.

Then $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$

and each A_n is infinite ✓.

On the other hand,

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

To see this, fix any $m \in \mathbb{N}$ and note that

$m \notin A_{m+1}$. Thus, $m \notin \bigcap_{n=1}^{\infty} A_n = \{i \in \mathbb{N} : i \in A_n \text{ for every } n \in \mathbb{N}\}$.

Since $m \in \mathbb{N}$ was arbitrary, we see that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

<u>Sol'n</u>: This is <u>true</u>.

It is clear that $\bigcap_{n=1}^{\infty} A_n$ is finite, since, e.g.

$\bigcap_{n=1}^{\infty} A_n \subseteq A_1$, and a subset of a finite set is finite.

To see that $\bigcap_{n=1}^{\infty} A_n$ is non-empty, consider the decreasing sequence

$$* \quad \boxed{\#A_1 \geq \#A_2 \geq \#A_3 \geq \dots}$$

of positive integers. Now, either

$$** \quad \boxed{\exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ \#A_{n+1} = \#A_n \text{ (i.e. the sequence } * \text{ stabilizes at some point)}}$$

or $\forall\, N \in \mathbb{N}\ \exists\, n \geq N$ with $\#A_{n+1} > \#A_n$.

(i.e. the sequence $*$ decreases infinitely often).

Since an infinite, strictly decreasing sequence of positive integers does not exist, we see that

** must hold.

Thus, $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=N}^{\infty} A_n = A_N$,

and it is thus nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

Sol'n: This is FALSE.

For example, $A = \{1\}$, $B = \{2\}$, $C = \{1,3\}$.

Then $A \cap (B \cup C) = \{1\} \cap \{1,2,3\} = \{1\}$

and $(A \cap B) \cup C = \emptyset \cup \{1,3\} = \{1,3\}$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

Sol'n: This is TRUE.

For any x, $x \in A \cap (B \cap C) \overset{\text{def}}{\iff} x \in A$ and $x \in B \cap C$

$\iff x \in A$ and ($x \in B$ and $x \in C$)

$\iff$ ($x \in A$ and $x \in B$) and $x \in C$

$\iff x \in A \cap B$ and $x \in C$

$\iff x \in (A \cap B) \cap C$.

Thus, $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Sol'n: this is TRUE.

For any x, we have

$$
\begin{aligned}
x \in A \cap (B \cup C) &\iff x \in A \text{ and } x \in B \cup C \\
&\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\
&\iff (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
&\iff x \in A \cap B \text{ or } x \in A \cap C \\
&\iff x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.2.5 (c). Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Proof: To see $(A \cup B)^c \subseteq A^c \cap B^c$:

Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$, so

$x \notin A$ and $x \notin B$, and thus $x \in A^c$ and $x \in B^c$.

(if $x \in A$ or $x \in B$ then $x \in A \cup B$)

Thus, $x \in A^c \cap B^c$. $\therefore (A \cup B)^c \subseteq A^c \cap B^c$.

To see that $A^c \cap B^c \subseteq (A \cup B)^c$:

Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$,

i.e. $x \notin A$ and $x \notin B$.

Thus, it is not the case that $x \in A$ or $x \in B$.

$\therefore$ $x \notin A \cup B$,

So $x \in (A \cup B)^c$.

$\therefore$ $A^c \cap B^c \subseteq (A \cup B)^c$. Having proved the double inclusion, we see that $(A \cup B)^c = A^c \cap B^c$. ∎

4.5.6 (a) Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous with $f(0) = f(1)$.

(a) Show that $\exists$ $x, y \in [0,1]$ satisfying $|x-y| = \frac{1}{2}$ and $f(x) = f(y)$.

Pf: Consider the function $g : [0, \frac{1}{2}] \longrightarrow \mathbb{R}$

defined by $g(x) = f(x + \frac{1}{2}) - f(x)$.

Note that $f(\underbrace{x_0 + \frac{1}{2}}_{y_0}) = f(x_0) \iff g(x_0) = 0$.

So we'd like to show that

$$\exists \; x_0 \in [0, 1/2] \quad \text{with} \quad g(x_0) = 0.$$

If $g(0) = 0$ or if $g(1/2) = 0$ then we are done

$\Updownarrow$

since then $f(0) = f(1/2)$ or $f(1) = f(1/2)$.

Now assume that $g(0) \neq 0$ and $g(1/2) \neq 0$.

Note that, since $f(0) = f(1) =: C$ (by hypothesis),

we have
$$g(1/2) = f(1) - f(1/2) = C - f(1/2)$$
$$g(0) = f(1/2) - f(0) = f(1/2) - C,$$

So $g(1/2) = -g(0)$, and so they

have opposite sign: one is positive and the other is negative.

Also note that g is <u>continuous</u>.

By the IVT, $\exists \; x_0 \in (0, 1/2)$ with $g(x_0) = 0$,

i.e. with $f(x_0 + 1/2) = f(x_0)$.

Setting $y_0 := x_0 + 1/2$, we have $|x_0 - y_0| = 1/2$ and $f(x_0) = f(y_0)$.