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MTHT 430, Practice Midterm 2 SOLUTIONS

NO CALCULATORS. For full credit, SHOW ALL WORK.

1. (10 points) Complete the following definitions.

(a) A subset $\mathbf{A} \subseteq \mathbb{Q}$ is called a *cut* if and only if ... (1) $A \neq \emptyset$ and $A \neq \mathbb{Q}$, (2) $\forall r \in A$ and $\forall s \in \mathbb{Q}$ with $s < r$, $s \in A$, and (3) $\forall r \in A$ $\exists r' \in A$ with $r' > r$.

(b) Given two cuts $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A} \leq \mathbf{B}$ if and only if ...

$A \subseteq B$

(c) The *zero cut* $\mathbf{O}$ is defined by ...

$\{r \in \mathbb{Q} : r < 0\}$

(d) If $\mathbf{A}$ and $\mathbf{B}$ are cuts satisfying $\mathbf{A}, \mathbf{B} \geq \mathbf{O}$ then the *product cut* $\mathbf{A} \cdot \mathbf{B}$ is defined by ...

$\{ab : a \in A, b \in B \text{ and } a \geq 0, b \geq 0\} \cup \{r \in \mathbb{Q} : r < 0\}$

(e) If S and T are any sets, a function $f : S \longrightarrow T$ is called *injective* if ...

$\forall s_1, s_2 \in S$, $f(s_1) = f(s_2) \Rightarrow s_1 = s_2$.

(f) If S and T are any sets, the set S^T is defined to be ...

$\{\text{functions } f : T \longrightarrow S\}$

(g) If S and T are any sets, a function $f : S \longrightarrow T$ is called *bijective* if ...

f is injective (see (e)) and f is surjective, i.e. $\forall t \in T$ $\exists s \in S$ s.t. $f(s) = t$.

(h) A set S is called *countable* if and only if ...

either S is finite or $\exists$ a bijection $f : \mathbb{N} \rightarrow S$.

(i) A subset $S \subseteq \mathbb{R}$ is called *bounded* if and only if ...

$\exists$ $M > 0$ with $S \subseteq [-M, M]$.

(j) A sequence $x_1, x_2, x_3, \ldots$ of real number is called *increasing* (not strictly) if and only if ...

$\forall n \in \mathbb{N}$, $x_{n+1} \geq x_n$.

2. (10 points) For each of the following, mark either "TRUE" or "FALSE".

(a) If the set A is countable and infinite, any infinite subset of A is countable.

TRUE (we proved in class that any subset of a countable set is countable.)

(b) There is a bijection $f : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

FALSE (proved in class that any function $f: \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is not surjective.)

(c) The set $\{q \in \mathbb{Q} : q^3 < 5\}$ is a cut.

TRUE (it's equal to $\{q \in \mathbb{Q} : q < 5^{1/3}\}$, which satisfies the axioms of a cut.)

(d) If a sequence of real numbers converges to a limit, then any subsequence must also converge to the same limit.

TRUE (we proved this in class.)

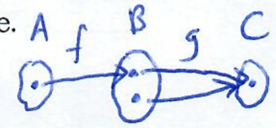
(e) Any bounded sequence of real numbers must converge to a limit.

FALSE (e.g. $1, -1, 1, -1, 1, -1, \ldots$)

(f) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are each injective functions, then the composition $g \circ f : A \longrightarrow C$ is also injective.

TRUE. (if $a_1, a_2 \in A$, then $g(f(a_1)) = g(f(a_2)) \overset{(g \text{ is injective})}{\Rightarrow} f(a_1) = f(a_2) \overset{(f \text{ is injective.})}{\Rightarrow} a_1 = a_2$,

(g) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions such that both g and $g \circ f$ are surjective, then f must also be surjective.

FALSE. (

g, $g \circ f$ each surjective, but f not surjective)

(h) The set $\mathbb{Q} \times \mathbb{Q}$ is countable.

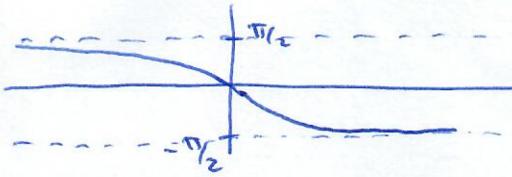
TRUE (proved in class that A, B countable $\Rightarrow A \times B$ countable.)

(i) If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing, then f is injective.

TRUE. (if $x_1 \neq x_2$, WOLOG $x_1 < x_2$, which implies $f(x_1) < f(x_2)$ so $f(x_1) \neq f(x_2)$)

(j) If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly decreasing, then f is surjective.

FALSE (e.g. $f(x) = -\arctan x$.
strictly decreasing but
e.g. $\pi \notin \mathrm{Im}(f)$)



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(See Attached page)

3. (10 points) (a) Prove that every decreasing sequence of real numbers that is bounded converges to a limit.

(b) Consider the sequence $\sqrt{3}$, $\sqrt{3+\sqrt{3}}$, $\sqrt{3+\sqrt{3+\sqrt{3}}}$, ... Prove that it converges to a limit. Can you evaluate the limit it converges to?

4. (10 points) Prove that the function $f : (-1, 1) \longrightarrow \mathbb{R}$ given by $f(x) := x/(x^2 - 1)$ is a bijection. (Hint: by considering the derivative of f, show that f is strictly decreasing on $(-1, 1)$, and consider its limits at the endpoints.)

5. (10 points) Let A and B be sets. Given a function $f : A \longrightarrow B$, consider the associated function $f_{\text{pre}} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, defined by $f_{\text{pre}}(S) := f^{-1}(S)$.

(a) In the case $A := \{a, b, c\}$, $B := \{1, 2, 3, 4\}$ and f is defined by $f(a) := 3$, $f(b) := 1$, $f(c) := 4$. Find two subsets $S_1, S_2 \in \mathcal{P}(B)$ with $S_1 \neq S_2$ but $f_{\text{pre}}(S_1) = f_{\text{pre}}(S_2)$.

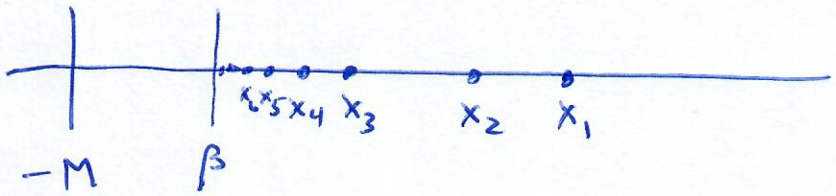
(b) In general, prove that, if f is not surjective, then f_{pre} is not injective.

6. (10 points) (a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

3 (a) Prove that every decreasing sequence of real numbers that is bounded converges to a limit.

Picture:



Proof: Let $x_1, x_2, x_3, \ldots$ be a decreasing sequence of real numbers (i.e. $\forall n \in \mathbb{N}$, $x_{n+1} \leq x_n$.)

Let $S := \{x_1, x_2, x_3, \ldots\}$, so that $S \neq \emptyset$.

By assumption, S is bounded, i.e. $\exists\ M > 0$ with $S \subseteq [-M, M]$.

In particular, $\forall\ n \in \mathbb{N}$, $x_n \geq -M$.

Let $\beta :=$ the greatest lower bound of S.

We will prove that $\lim_{n \to \infty} x_n = \beta$.

Let $\varepsilon > 0$ be given.

Since $\beta + \varepsilon$ is not a lower bound for S,

$\exists\ N \in \mathbb{N}$ with $\beta \leq x_N < \beta + \varepsilon$.

Since $(x_n)_{n \in \mathbb{N}}$ is decreasing, $\forall\ n \geq N$ we have

$x_n \leq x_N$.

Thus, $\forall\ n \geq N$ we have

$$\beta \leq x_n \leq x_N < \beta + \varepsilon .$$

Therefore $\forall\ n \geq N,\quad 0 \leq x_n - \beta < \varepsilon,$

so $|x_n - \beta| < \varepsilon .$

Since $\varepsilon > 0$ was arbitrary, we see that

$$\lim_{n \to \infty} x_n = \beta . \qquad \blacksquare$$

(b) Consider the sequence $\sqrt{3}, \sqrt{3+\sqrt{3}}, \sqrt{3+\sqrt{3+\sqrt{3}}}, \ldots$

Prove that it converges to a limit. Can you evaluate the limit it converges to?

Proof: The sequence may be described recursively as follows:

$$x_1 = \sqrt{3}, \quad \text{and} \quad \forall\ n \geq 1,\ x_{n+1} = \sqrt{3 + x_n} .$$

It is "clear" that, if $\lim_{n \to \infty} x_n = \alpha$ exists, then

we will have $\alpha = \sqrt{3 + \sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}}$,

so $\alpha = \sqrt{3+\alpha}$.

$\therefore \quad \alpha^2 = 3+\alpha$, so $\alpha^2 - \alpha - 3 = 0$.

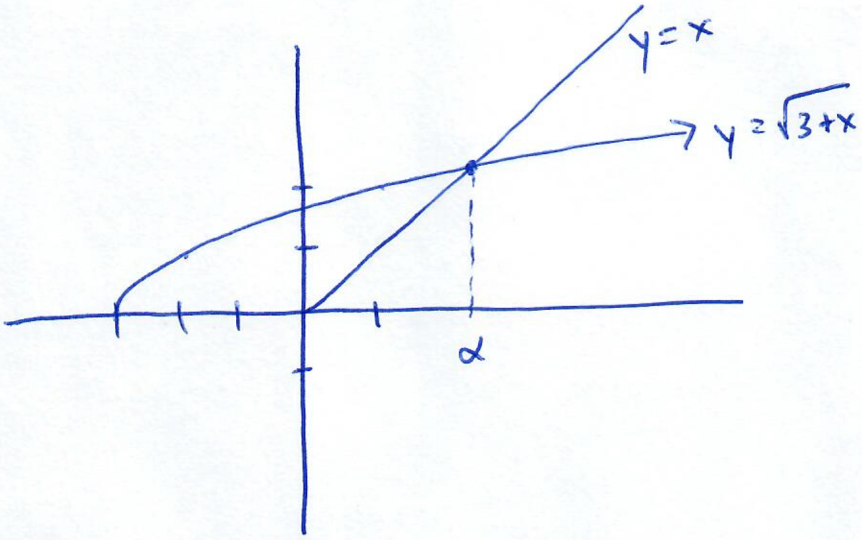
By the quadratic formula, we have

$$\alpha = \frac{1 \pm \sqrt{1+12}}{2}, \quad \text{and} \quad \alpha > 0, \text{ so}$$

$$\alpha = \frac{1+\sqrt{13}}{2}.$$ (We've evaluated the limit it converges to ...)

To prove that the sequence $(x_n)_{n\in\mathbb{N}}$ converges, we'll prove that it is increasing and bounded.

Picture



See: $\forall \underset{0 \le}{x} < \alpha, \quad x < \sqrt{3+x}$.

We claim that $\forall n \in \mathbb{N}, \quad 0 \le x_n < \alpha$.

It will then follow that $\forall n \in \mathbb{N}, \ x_n < \sqrt{3+x_n} = x_{n+1}$

and so the sequence $(x_n)_{n\in\mathbb{N}}$ will be increasing and bounded above (by α).

To prove that $\forall\, n \in \mathbb{N},\ 0 \le x_n < \alpha$, we argue by induction.

Base case $n=1$: $$0 \overset{\checkmark}{\le} \sqrt{3} \overset{(?)}{<} \frac{1+\sqrt{13}}{2} = \alpha$$
$$\Updownarrow$$
$$2\sqrt{3} < 1+\sqrt{13}.$$

But $(2\sqrt{3})^2 = 12 < 13 = (\sqrt{13})^2 < (1+\sqrt{13})^2$,

so $2\sqrt{3} < 1+\sqrt{13}$ ✓

Now fix $n \ge 1$ and assume that $0 \le x_n < \alpha$

Recalling that $\alpha^2 - \alpha - 3 = 0$, we have $\alpha = \alpha^2 - 3$.

So $0 \le x_n < \alpha^2 - 3$, so $x_n + 3 < \alpha^2$,

so $\underbrace{0 \le \sqrt{x_n + 3}}_{= x_{n+1}} < \alpha$

By induction, it follows that, $\forall\, n \in \mathbb{N}$, $0 \le x_n < \alpha$.

We've established the claim. Since $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded above, it converges to a limit. ∎

4. Prove that the function $f : (-1, 1) \longrightarrow \mathbb{R}$ given by $f(x) = \dfrac{x}{x^2 - 1}$ is a bijection.

Proof: We have $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to (-1)^+} \dfrac{x}{x^2 - 1} = \dfrac{-1}{\text{"}-0\text{"}}$

$= +\infty$

and $\lim\limits_{x \to +1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{x}{x^2-1} = \dfrac{1}{\text{"}-0\text{"}} = -\infty$.

We will show that $f'(x) < 0 \quad \forall\, x \in (-1, 1)$.

It follows from this that f is injective, since f is then strictly decreasing.

$$f'(x) = \frac{x^2 - 1 - x(2x)}{(x^2-1)^2} = \frac{-x^2 - 1}{(x^2-1)^2} = - \overbrace{\boxed{\frac{x^2+1}{(x^2-1)^2}}}^{\text{strictly positive}}$$

See: $\forall\, x \in (-1, 1)$, $f'(x) < 0$, so f is injective.

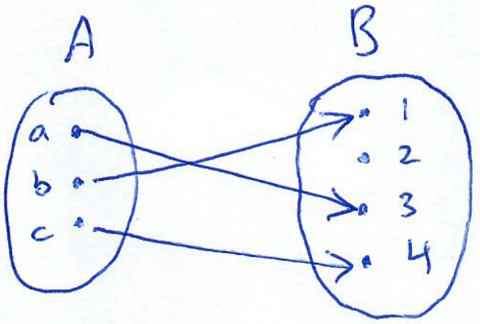
On the other hand, since f is continuous and $\lim\limits_{x \to -1^+} f(x) = +\infty$, $\lim\limits_{x \to 1^-} f(x) = -\infty$, f is surjective. $\blacksquare$

5. Let A and B be sets. Given a function $f: A \to B$, consider the associated function $f_{pre}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, defined by $f_{pre}(S) := f^{-1}(S)$.

(a) In the case $A := \{a, b, c\}$, $B := \{1, 2, 3, 4\}$ and f is defined by $f(a) := 3$, $f(b) := 1$, $f(c) := 4$.

Find two subsets $S_1, S_2 \in \mathcal{P}(B)$ with $S_1 \neq S_2$ but $f_{pre}(S_1) = f_{pre}(S_2)$.

Picture:



$$f_{pre}(\{1,2\}) = f^{-1}(\{1,2\}) = \{b\}$$
$$f_{pre}(\{1\}) = f^{-1}(\{1\}) = \{b\}$$

$S_1 := \{1, 2\}$

$S_2 := \{1\}$

$f_{pre}(S_1) = f_{pre}(S_2)$

but

$S_1 \neq S_2$.

(b) In general, prove that, if f is not surjective, then f_{pre} is not injective.

Proof: Suppose f is not surjective and let $b \in B$

be any element s.t. $\nexists\, a \in A$ with $f(a) = b$

(i.e. b is not in the image of f).

Then $f_{pre}(\{b\}) = f^{-1}(\{b\}) = \emptyset$

and $f_{pre}(\emptyset) = f^{-1}(\emptyset) = \emptyset$,

and so f_{pre} is not injective.

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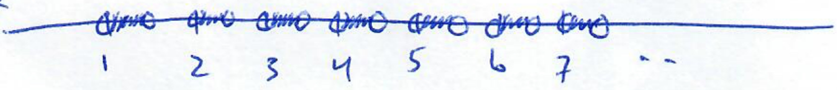
6 (a) Give an example of a countable collection of disjoint open intervals.

Example: $\{(n, n+\frac{1}{2})\}_{n \in \mathbb{N}} = \{(1, \frac{3}{2}), (2, \frac{5}{2}), (3, \frac{7}{2}), \ldots\}$

is an infinite, countable collection of disjoint open intervals. Picture



(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Claim: No uncountable collection of disjoint open intervals exists.

Pf: Let $\mathcal{U}$ be a collection of disjoint open intervals. $\forall\, (\alpha,\beta) \in \mathcal{U}$, pick a rational number $r_{(\alpha,\beta)} \in (\alpha,\beta)$. The function

$R := \{r_{(\alpha,\beta)} \in \mathbb{Q} : (\alpha,\beta) \in \mathcal{U}\} \subseteq \mathbb{Q}$ is countable, and

$$\begin{aligned} \mathcal{U} &\longrightarrow R \\ (\alpha,\beta) &\longmapsto r_{(\alpha,\beta)} \end{aligned}$$

is a bijection. Thus, $\mathcal{U}$ is countable. ∎