MTHT 430, Practice Midterm 2

NO CALCULATORS. For full credit, SHOW ALL WORK.

1. (10 points) Complete the following definitions.

(a) A subset $\mathbf{A} \subseteq \mathbb{Q}$ is called a *cut* if and only if ...

(b) Given two cuts $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A} \leq \mathbf{B}$ if and only if ...

(c) The *zero cut* $\mathbf{O}$ is defined by ...

(d) If $\mathbf{A}$ and $\mathbf{B}$ are cuts satisfying $\mathbf{A}, \mathbf{B} \geq \mathbf{O}$ then the *product cut* $\mathbf{A} \cdot \mathbf{B}$ is defined by ...

(e) If S and T are any sets, a function $f : S \longrightarrow T$ is called *injective* if ...

(f) If S and T are any sets, the set S^T is defined to be ...

(g) If S and T are any sets, a function $f : S \longrightarrow T$ is called *bijective* if ...

(h) A set S is called *countable* if and only if ...

(i) A subset $S \subseteq \mathbb{R}$ is called *bounded* if and only if ...

(j) A sequence $x_1, x_2, x_3, \ldots$ of real number is called *increasing* (not strictly) if and only if ...

2. (10 points) For each of the following, mark either "TRUE" or "FALSE".

(a) If the set A is countable and infinite, any infinite subset of A is countable.

(b) There is a bijection $f : \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

(c) The set $\{q \in \mathbb{Q} : q^3 < 5\}$ is a cut.

(d) If a sequence of real numbers converges to a limit, then any subsequence must also converge to the same limit.

(e) Any bounded sequence of real numbers must converge to a limit.

(f) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are each injective functions, then the composition $g \circ f : A \longrightarrow C$ is also injective.

(g) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions such that both g and $g \circ f$ are surjective, then f must also be surjective.

(h) The set $\mathbb{Q} \times \mathbb{Q}$ is countable.

(i) If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing, then f is injective.

(j) If a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly decreasing, then f is surjective.

3. (10 points) (a) Prove that every decreasing sequence of real numbers that is bounded converges to a limit.

(b) Consider the sequence $\sqrt{3}$, $\sqrt{3+\sqrt{3}}$, $\sqrt{3+\sqrt{3+\sqrt{3}}}$, $\ldots$. Prove that it converges to a limit. Can you evaluate the limit it converges to?

4. (10 points) Prove that the function $f : (-1,1) \longrightarrow \mathbb{R}$ given by $f(x) := x/(x^2-1)$ is a bijection. (Hint: by considering the derivative of f, show that f is strictly decreasing on $(-1,1)$, and consider its limits at the endpoints.)

5. (10 points) Let A and B be sets. Given a function $f : A \longrightarrow B$, consider the associated function $f_{\text{pre}} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, defined by $f_{\text{pre}}(S) := f^{-1}(S)$.

(a) In the case $A := \{a,b,c\}$, $B := \{1,2,3,4\}$ and f is defined by $f(a) := 3$, $f(b) := 1$, $f(c) := 4$. Find two subsets $S_1, S_2 \in \mathcal{P}(B)$ with $S_1 \neq S_2$ but $f_{\text{pre}}(S_1) = f_{\text{pre}}(S_2)$.

(b) In general, prove that, if f is not surjective, then f_{pre} is not injective.

6. (10 points) (a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.