Proof:

- $\dim V/U = 1 \iff$ isomorphic to $\mathbb{F}$

(by 3.59 Axler)

There exists invertible linear map $\pi_1 : V/U \to \mathbb{F}$

(by definition of isomorphism, 3.58 Axler)

- Meanwhile, there exists a quotient map

$\pi_2 : V \to V/U$

$\pi_2(v) = v + U$, which is linear

(see 3.88)

and $\operatorname{null} \pi_2 = U$

- Thus, define $\pi = \pi_1 \circ \pi_2 : V \xrightarrow{\pi_2} V/U \xrightarrow{\pi_1} \mathbb{F}$

- We need to show that π is linear and $\operatorname{null} \pi = U$

- π is linear

composition of two linear maps is again linear, we are done. (true by page 56 Axler, line 3–4) ✓

- $U \subseteq \text{null}\, \pi$

$$\pi(U) = \pi_1 \circ \pi_2 (U)$$
$$= \pi_1(U) \quad \text{(by definition of } \pi_2\text{)}$$
$$= 0 \quad \text{(need to define } \pi_1\text{)}$$

Need to construct $\pi_1 : V/U \to V$ such that

π_1 is linear and $\pi_1(U) = 0$.

- Since $\dim V/U = 1$, there exists basis $v_1 + U$ of V/U, such that $v_1 \notin U$.

- Define $\pi_1 : V/U \to \mathbb{F}$ by:

$$\pi_1(\lambda v_1 + U) = \lambda, \quad \lambda \in \mathbb{F}.$$

- π_1 is linear: Let $x, y \in V/U$

Then $x = \lambda_1 v_1 + U$ and

$y = \lambda_2 v_1 + U$, for some $\lambda_1, \lambda_2 \in \mathbb{F}$.

Thus $\pi_1(x+y) = \pi_1(\lambda_1 v_1 + U$

$+ \lambda_2 v_1 + U)$

$= \pi_1((\lambda_1 + \lambda_2) v_1 + U)$

$= \lambda_1 + \lambda_2$

$= \pi_1(x) + \pi_1(y)$

$\Rightarrow$ satisfy addivity. ✓

$\pi_1(\lambda x) = \pi_1(\lambda[\lambda_1 v_1 + U])$

$= \pi_1(\lambda \lambda_1 v_1 + U)$

$= \lambda \lambda_1$

$= \lambda \pi_1(x)$

$\Rightarrow$ satisfy homogeneity ✓

- $\pi_1(U) = 0$ (by definition)

$V \supseteq \text{null}\ \pi$

Let $k \in V$ such that $\pi(k) = 0$

$\Rightarrow \pi_1 \circ \pi_2 (k)$

$= \pi_1 (k + U)$ by definition of π_2

$= 0k + U$ by definition of π_1, with $\lambda = 0$

$= U$

$\Rightarrow \text{null}\ \pi \subseteq U$. ✓