Proof

- $v_1, \ldots, v_m$ spans $V \Rightarrow \Gamma$ is injective

  - Let $\varphi \in V'$ be such that

    $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)) = 0 \quad - (1)$

  - Given any $v \in V$, $v = \alpha_1 v_1 + \cdots + \alpha_m v_m$

    since $\text{span}(v_1, \ldots, v_m) = V$

  - $\varphi(v) = \varphi(\alpha_1 v_1 + \cdots + \alpha_m v_m)$

    $= \alpha_1 \varphi(v_1) + \cdots + \alpha_m \varphi(v_m)$

    $\varphi(v) = 0$ by (1)

  - $\varphi = 0$

- $\Gamma$ is injective $\Rightarrow v_1, \ldots, v_m$ spans $V$

  - Suppose $v_1, \ldots v_m$ does not span $V$.

  - There exists $\varphi : V \to \mathbb{F}$ such that

    $\varphi(u) = 0$ for all $u \in \text{span}(v_1, \ldots, v_m)$ but

    $\varphi \neq 0$ (see 3F4, axler)

- $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)) = 0$

  but $\varphi \neq 0$

- $\Gamma$ is not injective

▶ $v_1, \ldots, v_m$ is linearly independent $\Rightarrow$ $\Gamma$ is surjective.

- Let $(a_1, \ldots, a_m) \in \mathbb{F}^m$

- Extend $v_1, \ldots, v_m$ into basis of $V$.

- By 3.5 Axler, there exists $\varphi : V \to \mathbb{F}$

  such that

  $\varphi(v_i) = a_i$ for $i = 1, \ldots, m$ and

  $\varphi(v_i) = 0$ for $i > m$.

- Thus $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$

  $= (a_1, \ldots, a_m)$

- $\Gamma$ is surjective.

- $\Gamma$ is surjective $\Rightarrow$ $v_1, \dots, v_m$ is linearly independent

  - Suppose $v_1, \dots, v_m$ is linearly dependent

    There exist $\alpha_1, \dots, \alpha_m$, not all zero such that

    $0 = \alpha_1 v_1 + \dots + \alpha_m v_m$

    Suppose $\alpha_i \neq 0$, thus $v_i$ can be expressed as a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots v_m$.

  - claim: $(0, \dots, 1, \dots, 0) \in \mathbb{F}^m$, where the element is zero everywhere, except 1 is in the $i$'th position, is not in range $\Gamma$.

    ↳ Proof:

    Suppose there is $\varphi$ such that

- $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_i), \dots \varphi(v_m)) = (0, \dots, 1, \dots, 0)$ — ①

- $v_i$ is a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m$

$$v_i = \frac{-\alpha_1 v_1 - \cdots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} - \cdots - \alpha_m v_m}{\alpha_i}$$

$$= \lambda_1 v_1 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_{i+1} v_{i+1} + \cdots + \lambda_m v_m \quad \text{— (2)}$$

- $\varphi(v_i) = 1$ from (1)

$= \varphi(\lambda_1 v_1 + \cdots + \lambda_{i-1} v_{i-1} + \lambda_{i+1} v_{i+1} + \cdots \lambda_m v_m)$

from (2)

$= \lambda_1 \varphi(v_1) + \cdots + \lambda_{i-1} \varphi(v_{i-1}) + \lambda_{i+1} \varphi(v_{i+1})$

$+ \cdots \lambda_m \varphi(v_m)$ by linearity

$= 0$ from (1)

- Not possible. $\Gamma$ is not surjective.