Proof

- Note that given arbitrary $v_i \in \{v_1, \ldots, v_n\}$

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)v_i$$

$$= \psi(v_1)\varphi(v_i) + \cdots + \psi(v_n)\varphi_n(v_i)$$

$$= 0 + \cdots + \psi(v_i)\varphi_i(v_i) + \cdots + 0$$

$$= \psi(v_i)\varphi_i(v_i)$$

$$= \psi(v_i)$$

because $v \in \operatorname{span}(v_1, \ldots, v_n)$

- Thus given arbitrary $v \in V$, $v = a_1 v_1 + \cdots + a_n v_n$,

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)v$$

$$= \psi(v_1)\varphi_1(v) + \cdots + \psi(v_n)\varphi_n(v)$$

$$= \psi(v_1)a_1 + \cdots + \psi(v_n)a_n \quad \text{by part 1 above}$$

$$= \psi(a_1 v_1 + \cdots + a_n v_n) \quad \text{by linearity}$$

$$= \psi(v)$$

- $\therefore \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n = \psi$