3.106 Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

Proof

Let $u_1, \ldots, u_m$ be basis of U

Extend the list to be basis of V, $u_1, \ldots, u_m, u_{m+1}, \ldots, u_n$

Let $\varphi_1, \ldots, \varphi_m, \varphi_{m+1}, \ldots, \varphi_n$ be dual basis of V'.

We will show that $U^0 = \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n)$

$\operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n) \subset U^0$

Let $\varphi \in \operatorname{span}(\varphi_{m+1}, \ldots, \varphi_n)$.

Then there exist $c_{m+1}, \ldots, c_n$ such that

$$\varphi = c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n$$

Let $u \in \operatorname{span}(u_1, \ldots, u_m) = a_1u_1 + \cdots + a_mu_m$ for some $a_1, \ldots, a_m \in \mathbb{F}$.

Thus $\varphi(u) = (c_{m+1}\varphi_{m+1} + \cdots + c_n\varphi_n)(a_1u_1 + \cdots + a_mu_m)$

$= 0$

$\Rightarrow \varphi \in U^0$