Proof

Suppose rank A = 1

- Since rank $A = 1$, columns $2, 3, \dots, n$ of matrix A are multiple of column 1.

- Let the first column of A be $A_1 = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$, for some $c_1, \dots, c_m \in \mathbb{F}$

- Then columns $2, 3, \dots, n$ are

$$A_2 = d_2 A_1 ; \quad A_3 = d_3 A_1 , \dots , \quad A_n = d_n A_1$$

for some $d_2, \dots, d_n \in \mathbb{F}^n$

- Thus $A = \begin{bmatrix} A_1 & d_2 A_1 & \cdots & d_n A_1 \end{bmatrix}$

$$= \begin{bmatrix} 1.c_1 & d_2 c_1 & \cdots & d_n c_1 \\ \vdots & \vdots & & \vdots \\ 1.c_m & d_2 c_m & & d_n c_m \end{bmatrix}$$

Thus letting $d_1 = 1$,

$A_{j,k} = c_j d_k$ for

every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

Suppose there exist $(c_1, \dots, c_m) \in \mathbb{F}^m$ and

$(d_1, \dots, d_n) \in \mathbb{F}^n$ such that $A_{j,k} = c_j d_k$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 d_1 & \cdots & c_1 d_n \\ \vdots & & \vdots \\ c_m d_1 & \cdots & c_m d_n \end{bmatrix}$$

Since $A \neq 0$, there is at least one of

$d_1, \dots, d_n$ that is not zero. WLOG, suppose $d_1 \neq 0$

let the first column be $A_1 = \begin{bmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{bmatrix}$

The second column $A_2 = \frac{d_1}{d_1} A_2 = \frac{d_1}{d_1} \begin{bmatrix} c_1 d_2 \\ \vdots \\ c_m d_2 \end{bmatrix} = \frac{d_2}{d_1} \begin{bmatrix} c_1 d_1 \\ \vdots \\ c_m d_1 \end{bmatrix} = \frac{d_2}{d_1} A_1$

Similarly, the third column $A_3 = \frac{d_3}{d_1} A_1$,

Proceed for all columns;

and thus we see that every column is a multiple of the first column.

Thus, rank $A = 1$.