Proof :

a) Let $S, S' \in \mathcal{L}(V/U, W)$.

$\Gamma(S + S') = (S + S') \circ \pi$

$= S \circ \pi + S' \circ \pi$ (by theorem 3.9 Axler)

$= \Gamma(S) + \Gamma(S')$

Thus satisfy additivity. ✓

Let $\lambda \in \mathbb{F}$. $\Gamma(\lambda S) = (\lambda S) \circ \pi$

$= \lambda (S \circ \pi)$ by 3.6 Axler

$= \lambda \Gamma(S)$

Thus satisfy homogeneity ✓

b). Let S such that $\Gamma(S) = 0$ ← zero map from $V \to W$.

$\Rightarrow S \circ \pi(x) = 0$ for all $x \in V$

$\Rightarrow S(x + U) = 0$ for all $x \in V$

$\Rightarrow S$ is a zero map from $V/U \to W$.

c) (version 1)

Range Γ = the set of $\Gamma(S)$,

for all $S \in \mathcal{L}(V/U \to W)$

= the set of $S \circ \pi$

for all $S \in \mathcal{L}(V/U \to W)$

(by statement definition)

= the set of $T = S \circ U$

for all $S \in \mathcal{L}(V/U \to W)$

(by labelling)

= the set of T such that

$T(u) = 0$ for all $u \in U$

(by 3E18, which states that

$U \subset \text{null } T$)

$= \{T \in \mathcal{L}(V, W) : Tu = 0 \text{ for every } u \in U\}$

c) version 2.

$$\boxed{\{T \in \mathcal{L}(V,W) : Tu = 0 \text{ for every } u \in U\} \subseteq \operatorname{range} \Gamma}$$

- let $T \in \{T \in \mathcal{L}(V,W) : Tu = 0 \text{ for every } u \in U\}$.

want to show $T \in \operatorname{range} \Gamma$ ie there exists $S \in \mathcal{L}(V/U, W)$ s.t $\Gamma(S) = S \circ \pi = T$.

$\Gamma : \mathcal{L}(V/U, W) \to \mathcal{L}(V, W)$ by

$\Gamma(S) = S \circ \pi$.

- By 3E18 we know that given any $T \in \mathcal{L}(V,W)$, $T(u) = 0$ for all $u \in U$, there exist S such that $S \circ \pi = y$.

- Thus $y \in \operatorname{range} \Gamma$,

$$\boxed{\operatorname{range} \Gamma \subseteq \{T \in \mathcal{L}(V,W) : Tu = 0 \text{ for every } u \in U\}}$$

- given $T \in \operatorname{range} \Gamma$, $T = S \circ \pi$ we want to show $T \in \mathcal{L}(V,W)$ (true by definition) and $Tu = 0$ for every $u \in U$. This is true by 3E18, as such T exists iff $U \subseteq \operatorname{null} T$