- $b \to a$

Suppose columns of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$

$\Leftrightarrow$ The only choice of $c_1, \dots, c_n \in \mathbb{F}$ such that

$c_1 \mathcal{M}(T)_{\cdot,1} + \cdots + c_n \mathcal{M}(T)_{\cdot,n} = 0$

is $c_1 = \cdots = c_n = 0$ (by definition of independence)

$\Leftrightarrow$ The only soution for $\mathcal{M}(T) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$ is $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$ ** (by 3.52)

Let $v = c_1 v_1 + \cdots + c_n v_n$. Then, this is equivalent to

$\Leftrightarrow$ $\mathcal{M}(T)\mathcal{M}(v) = \mathcal{M}(Tv) = 0$ where $\mathcal{M}(v) = 0$, (by definition of $\mathcal{M}(v)$)

(I think this can be deleted)

$\Rightarrow$ Consider $Tv = 0$ *

If v can only be $v = 0$, then T is injective, and by 3.69, T is invertible and we are done.

Suppose $v \neq 0$. Then

$$\underbrace{\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)}_{\text{by 3.65}} \underbrace{= \mathcal{M}(0)}_{\text{by linearity and *}} \underbrace{= 0}_{\text{by 3.11}};$$

$\Rightarrow$ $\mathcal{M}(T) \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix} = 0$ where at least one c' not zero since $v \neq 0$

A contradiction. **