

Proof

▶ Big idea: let $\varphi \in (V_1 \times \dots \times V_m)'$
 $= \mathcal{L}(V_1 \times \dots \times V_m, \mathbb{F})$

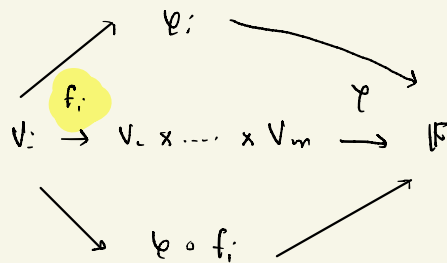
$$V_1 \times \dots \times V_m \xrightarrow{\varphi} \mathbb{F}$$

▶ We are interested to see how φ can correspond to

$$\varphi_i \in V_i' = \mathcal{L}(V_i, \mathbb{F}), \quad V_i \xrightarrow{\varphi_i} \mathbb{F}$$

for each V_1, \dots, V_m

▶ Thus for each V_1, \dots, V_m , if we can find
a linear mapping f_i such that $f_i: V_i \rightarrow V_1 \times \dots \times V_m$,



we can make the
linkage between
 $\mathcal{L}(V_i, \mathbb{F})$ and
 $\mathcal{L}(V_1, \dots, V_m, \mathbb{F})$

Define $f_i : V_i \rightarrow V_1 \times \dots \times V_m$

by $f_i(v_i) = (0, 0, \dots, v_i, \dots, 0, 0)$ *

Define $\phi : \mathcal{L}(V_1, \dots, V_m, \mathbb{F}) \rightarrow \mathcal{L}(V_1, \mathbb{F}) \times \dots \times \mathcal{L}(V_m, \mathbb{F})$

$$\phi(\psi) : (f_1' \psi, \dots, f_m' \psi)$$

Note that the right hand side is an element of $\mathcal{L}(V_1, \mathbb{F}) \times \dots \times \mathcal{L}(V_m, \mathbb{F})$ since

$f_i' \psi = \psi f_i$ takes an element of V_i to \mathbb{F}

Next, we show that ϕ is linear

Given $\psi, \lambda \in (V_1 \times \dots \times V_m)'$

$$\begin{aligned} \phi(\psi + \lambda) &= (f_1'(\psi + \lambda), \dots, f_m'(\psi + \lambda)) \\ &= ((\psi + \lambda)f_1, \dots, (\psi + \lambda)f_m) \\ &= (\psi f_1 + \lambda f_1, \dots, \psi f_m + \lambda f_m) \\ &= (\psi f_1, \dots, \psi f_m) + (\lambda f_1, \dots, \lambda f_m) \\ &= (f_1' \psi, \dots, f_m' \psi) + (f_1' \lambda, \dots, f_m' \lambda) \\ &= \phi(\psi) + \phi(\lambda) \quad \checkmark \end{aligned}$$

Let $s \in \mathbb{F}$, then

$$\begin{aligned} f(s\psi) &= (f_1'(s\psi), \dots, f_m'(s\psi)) \\ &= (s\psi f_1, \dots, s\psi f_m) \\ &= s(\psi f_1, \dots, \psi f_m) \\ &= s(f_1'\psi, \dots, f_m'\psi) \\ &= sf(\psi) \quad \checkmark \end{aligned}$$

Next, show ϕ is injective

$$\text{Let } f(\psi) = 0$$

$$(f_1'(\psi), \dots, f_m'(\psi)) = 0$$

$$(\psi f_1, \dots, \psi f_m) = 0$$

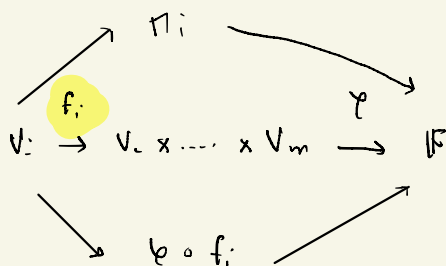
Thus for any $v_i \in V_i$ and $f_i \in \mathcal{L}(V_i, V_1 \times \dots \times V_m)$

$$\psi f_i v_i = \psi(f_i(v_i)) = 0$$

$\Rightarrow \psi$ is a zero map.

Next, show ϕ is surjective

$$\text{Let } (\eta_1, \dots, \eta_m) \in \mathcal{L}(V_1, \mathbb{F}) \times \dots \times \mathcal{L}(V_m, \mathbb{F})$$



► We want to show for each π_i

there exist $\varphi \in \mathcal{A}(V_1 \times \dots \times V_m, \mathbb{F})$

such that $\varphi \circ f_i = \pi_i$

For arbitrary $v_1 \in V_1, v_2 \in V_2, \dots, v_m \in V_m$,

denote $\pi_1(v_1) = s_1, \pi_2(v_2) = s_2, \dots, \pi_m(v_m) = s_m$

► Select $\varphi \in \mathcal{A}(V_1 \times \dots \times V_m, \mathbb{F})$ such that

$$\varphi(v_1, 0, \dots, 0) = s_1$$

$$\varphi(0, v_2, \dots, 0) = s_2$$

$$\varphi(0, 0, \dots, v_m) = s_m$$

► Define f_i as *

► - Given any $v_i \in V_i$,

$$\varphi \circ f_i(v_i) = \varphi(0, \dots, v_i, \dots, 0) = s_i = \pi_i(v_i)$$

$\Rightarrow \varphi \circ f_i = \pi_i$, since v_i is arbitrary

$$\begin{aligned}
 \Rightarrow (\pi_1, \dots, \pi_m) &= (\varphi f_1, \dots, \varphi f_m) \\
 &= (f_1' \varphi, \dots, f_m' \varphi) \\
 &= \varnothing(\varphi)
 \end{aligned}$$