Proof

- **Big idea**: Let $\varphi \in (V_1 \times \cdots \times V_m)'$

$$= \mathcal{L}(V_1 \times \cdots \times V_m, \mathbb{F})$$

$$V_1 \times \cdots \times V_m \xrightarrow{\varphi} \mathbb{F}$$

- We are interested to see how φ can correspond to

$$\varphi_i \in V_i' = \mathcal{L}(V_i, \mathbb{F}), \quad V_i \xrightarrow{\varphi_i} \mathbb{F}$$

for each $V_1, \ldots, V_m$

- Thus for each $V_1, \ldots, V_m$, if we can find a linear mapping f_i such that $f_i : V_i \to V_1 \times \cdots \times V_m$, we can make the linkage between $\mathcal{L}(V_i, \mathbb{F})$ and $\mathcal{L}(V_1, \ldots, V_m, \mathbb{F})$

$$\begin{tikzcd}
 & \varphi_i \arrow[drr, bend left] & & \\
V_i \arrow[ur] \arrow[r, "f_i"] \arrow[dr] & V_1 \times \cdots \times V_m \arrow[rr, "\varphi"] & & \mathbb{F} \\
 & \varphi \circ f_i \arrow[urr] & &
\end{tikzcd}$$

- Define $f_i : V_i \to V_1 \times \cdots \times V_m$

  by $f_i(v_i) = (0, 0, \ldots, v_i, \ldots, 0, 0)$ $\ast$

- Define $\phi : \mathcal{L}(V_1, \ldots, V_m, \mathbb{F}) \to \mathcal{L}(V_1, \mathbb{F}) \times \cdots \times \mathcal{L}(V_m, \mathbb{F})$

  $\phi(\varphi) : (f_1'\varphi, \ldots, f_m'\varphi)$

  Note that the right hand side is an element of

  $\mathcal{L}(V_1, \mathbb{F}) \times \cdots \times \mathcal{L}(V_m, \mathbb{F})$ since

  $f_i'\varphi = \varphi f_i$ takes an element of V_i to $\mathbb{F}$.

- Next, we show that ϕ is linear

  Given $\varphi, \lambda \in (V_1 \times \cdots \times V_m)'$

$$
\begin{aligned}
f(\varphi + \lambda) &= (f_1'(\varphi + \lambda), \ldots, f_m'(\varphi + \lambda)) \\
&= ((\varphi + \lambda) f_1, \ldots, (\varphi + \lambda) f_m) \\
&= (\varphi f_1 + \lambda f_1, \ldots, \varphi f_m + \lambda f_m) \\
&= (\varphi f_1, \ldots, \varphi f_m) + (\lambda f_1, \ldots, \lambda f_m) \\
&= (f_1'\varphi, \ldots, f_m'\varphi) + (f_1'\lambda, \ldots, f_m'\lambda) \\
&= f(\varphi) + f(\lambda) \quad \checkmark
\end{aligned}
$$

Let $s \in \mathbb{F}$, then

$$
\begin{aligned}
f(s\varphi) &= (f_1'(s\varphi), \ldots, f_m'(s\varphi)) \\
&= (s\varphi f_1, \ldots, s\varphi f_m) \\
&= s(\varphi f_1, \ldots, \varphi f_m) \\
&= s(f_1'\varphi, \ldots, f_m'\varphi) \\
&= sf(\varphi) \checkmark
\end{aligned}
$$

Next, show $\emptyset$ is injective

Let $f(\varphi) = 0$

$$(f_1'(\varphi), \ldots, f_m'(\varphi)) = 0$$

$$(\varphi f_1, \ldots, \varphi f_m) = 0$$

Thus for any $v_i \in V_i$ and $f_i \in \mathcal{L}(V_i, V_1 \times \cdots \times V_m)$

$$\varphi f_i v_i = \varphi(f_i(v_i)) = 0$$

$\Rightarrow \varphi$ is a zero map.

Next, show $\emptyset$ is surjective

Let $(\pi_1, \ldots, \pi_m) \in \mathcal{L}(V_1, \mathbb{F}) \times \cdots \times \mathcal{L}(V_m, \mathbb{F})$

$$
\begin{tikzcd}
 & \pi_i \arrow[drr, bend left] & & \\
V_i \arrow[ur] \arrow[r, "f_i"] \arrow[dr] & V_1 \times \cdots \times V_m \arrow[rr, "\varphi"] & & \mathbb{F} \\
 & \varphi \circ f_i \arrow[urr] & &
\end{tikzcd}
$$

- We want to show for each π_i
  there exist $\varphi \in \mathcal{L}(V_1 \times \cdots \times V_m, \mathbb{F})$
  such that $\varphi \circ f_i = \pi_i$

For arbitrary $v_1 \in V_1, v_2 \in V_2, \ldots, v_m \in V_m$,

denote $\pi_1(v_1) = s_1$, $\pi_2(v_2) = s_2$, ... $\pi_m(v_m) = s_m$

- Select $\varphi \in \mathcal{L}(V_1 \times \cdots \times V_m, \mathbb{F})$ such that

$$\varphi(v_1, 0, \ldots 0) = s_1$$

$$\varphi(0, v_2 \ldots 0) = s_2$$

$$\vdots$$

$$\varphi(0, 0, \ldots v_m) = s_m$$

- Define f_i as $*$
- - Given any $v_i \in V_i$,

$\varphi \circ f_i(v_i) = \varphi(0, \ldots v_i, \ldots 0) = s_i = \pi_i(v_i)$

$\Rightarrow \varphi f_i = \pi_i$, since v_i is arbitrary

$$\Rightarrow \quad (\pi_1, \ldots, \pi_m) = (\varphi f_1, \ldots, \varphi f_m)$$
$$= (f_1' \varphi, \ldots, f_m' \varphi)$$
$$= \emptyset(\varphi)$$