Proof

- $v_1, \ldots, v_m$ spans $V \Rightarrow \Gamma$ is injective

  - Let $\varphi \in V'$ be such that

    $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)) = 0$ — ①

  - Given any $v \in V$, $v = \alpha_1 v_1 + \cdots + \alpha_m v_m$

    since $\text{span}(v_1, \ldots, v_m) = V$

  - $\varphi(v) = \varphi(\alpha_1 v_1 + \cdots + \alpha_m v_m)$

    $= \alpha_1 \varphi(v_1) + \cdots + \alpha_m \varphi(v_m)$

    $\varphi(v) = 0$ by ①

  - $\varphi = 0$

- Γ is injective $\Rightarrow v_1, \ldots, v_m$ spans V

  - Suppose $v_1, \ldots, v_m$ does not span V.

  - There exists $\varphi : V \to \mathbb{F}$ such that

    $\varphi(u) = 0$ for all $u \in \text{span}(v_1, \ldots, v_m)$ but

    $\varphi \neq 0$ (see 3F4, axler)