Proof :

Since V/U is finite dimensional, there exists basis of V/U,

$v_1 + U, \ldots, v_m + U$

s.t $v_1, \ldots, v_m \notin U$

Thus given any $v \in V$,

$$v + U = a_1(v_1 + U) + \cdots + a_m(v_m + U)$$

$$= (a_1 v_1 + \cdots + a_m v_m) + U$$

$\Rightarrow v - (a_1 v_1 + \cdots + a_m v_m) \in U$

by 3.85

$\ast$

- Define $W = \text{span}(v_1, \ldots, v_m)$. **

- We want to show $v_1, \ldots, v_m$ is basis of W; ie $v_1, \ldots, v_m$ is linearly indep.

  Suppose there exists $a_1, \ldots, a_m$ not all are zero such that

  $a_1 v_1 + \cdots + a_m v_m = 0$.

  then

$$a_1 v_1 + U + \cdots + a_m v_m + U$$

$$= (a_1 v_1 + \cdots + a_m v_m) + U$$

$$= 0 + U$$

$$= 0_{V/U}$$

contradicting that

$v_1 + U, \ldots, v_m + U$ is independent

(therefore a basis) in V/U.

▶ Thus $v_1, \ldots, v_m$ is a basis of W.

$\Rightarrow \dim(W) = m = \dim(V/U)$

▶ Next, we show $V = W \oplus U$

Given any v,

$$v = \left(v - (a_1 v_1 + \cdots + a_m v_m)\right) + (a_1 v_1 + \cdots + a_m v_m)$$

$\in U + W$

by $*$ and $**$

Thus $V \subseteq U + W$.

Moreover, $U + W \subseteq W$ by 1.39 Axler.

Therefore $U + W = W$.

Next, we show $U \cap W = \{0\}$.

Let $x \in U \cap W$.

Since $x \in W$, $x = b_1 v_1 + \dots + b_m v_m$

Moreover, observe that $x - 0 = x \in U$.

By 3.85, $x + U = 0 + U$

Thus, $0 + U = x + U$

$$= (b_1 v_1 + \dots + b_m v_m) + U$$

$$= (b_1 v_1 + U) + \dots + (b_m v_m + U)$$

since $v_1 + U, \dots, v_m + U$ is linearly indep.,

$b_1 = \dots = b_m = 0 \Rightarrow x = 0$

$\Rightarrow U + W \subseteq \{0\}$

Also, $\{0\} \subseteq U+W$ since $U+W$ is a subspace.

Thus $U+W = \{0\}$.