Permutation and Combinatorics

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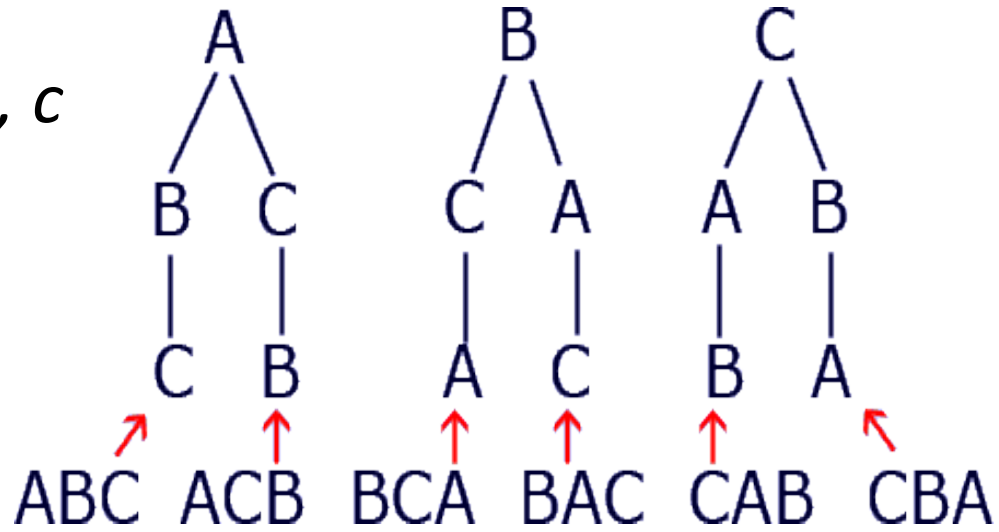
Permutations

A **permutation** is an arrangement of objects in a definite order.

Example :

There are 6 permutations of the letters *a, b, c*

abc, acb, bac, bca, cab, cba.



How many permutations are there of the letters *a, b, c, d*?

Permutations of n elements

- There are n!=n×(n-1) ×(n-2) ----2×1

How many permutations are there in numbers 1,2,3,4?

K permutations of n elements

P(n,k) is the number of k- permutations of n elements, the number of ways to arrange k objective chosen from n distinct objects

P(n,k)=$\frac{n!}{(n-k)!}$

Factorial Representation

Example

a) In how many ways can 6 people be arranged in a row?

According to the **Multiplication Principle**, if one event can occur in m ways and a second event can occur in n ways after the first event has occurred, then the two events can occur in m×n ways. This is also known as the **Fundamental Counting Principle**.

- Example 1

a) Assume that 5 cars are in a race. In how many ways can three cars finish in first, second and third place?

b) How many arrangements are possible if only 3 of them are chosen?

Repetition of an Event

If one event with **n** outcomes occurs **r** times with repetition allowed, then the number of ordered arrangements is $\mathbf{n^r}$

Example 1 What is the number of arrangements if a die is rolled

(a) 2 times ? $\mathbf{6 \times 6 = 6^2}$

(b) 3 times ? $\mathbf{6 \times 6 \times 6 = 6^3}$

(b) r times ? $\mathbf{6 \times 6 \times 6 \times \ldots\ldots = 6^r}$

Repetition of an Event

Example 2

(a) How many different car number plates are possible with 3 letters followed by 3 digits?

(b) How many of these number plates begin with ABC?

(c) If a plate is chosen at random, what is the probability that it begins with ABC?

Repetition of an Event

Example 2

(a) How many different car number plates are possible
with 3 letters followed by 3 digits?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3$

(b) How many of these number plates begin with ABC ?

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

(c) If a plate is chosen at random, what is the probability that it begins with ABC?

Solution: $\dfrac{10^3}{26^3 \times 10^3} = \dfrac{1}{26^3}$

Choose a committee of two people from persons A, B, C, D

1. Find ways to arrange the two people?
2. How many different committee can be chosen?

choose a committee of three people from persons A, B, C, D and E

1. Find ways to arrange the three people?
2. How many different committee can be chosen?

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

a) there are no restrictions?
b) boys and girls alternate?
c) boys and girls are in separate groups?
d) Anne and Jim wish to stay together?

Permutations with Restrictions

Eg. In how many ways can 5 boys and 4 girls be arranged on a bench if

c) boys and girls are in separate groups?

Solution : Boys & Girls or Girls & Boys

$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^5P_5 \times {}^4P_4 \times 2$$

d) Anne and Jim wish to stay together?

Solution : (AJ) _ _ _ _ _ _ _

$$= 2 \times 8! \text{ or } 2 \times {}^8P_8$$

Arrangements with Restrictions

From the digits 2, 3, 4, 5, 6

a) how many numbers greater than 4 000 can be formed?

b) how many 4-digit numbers would be even?

Arrangements with Restrictions

- Eg 3. From the digits 2, 3, 4, 5, 6

Solution : **5 digits (any) = $^5P_5$**

- a) how many numbers greater than 4 000 can be formed?

4 digits (must start with digit $\geq$ 4) = $^3P_1 \times {}^4P_3$

Total = $^5P_5 + {}^3P_1 \times {}^4P_3$

b) how many 4 digit numbers would be even?

Even (ends with 2, 4 or 6) = _ _ _ $^3P_1$

= $^4P_3 \times {}^3P_1$

Arrangements with Repetitions

If we have **n** elements of which ***x*** are alike of one kind, ***y*** are alike of another kind, ***z*** are alike of another kind, then the number of ordered selections or permutations is given by:

$$\frac{n!}{x!\,y!\,z!}$$

Arrangements with Repetitions

Eg.1 How many different arrangements of the word **PARRAMATTA** are possible?

Solution :

10 letters but note repetition (4 A's, 2 R's, 2 T's)

P

A A A A

R R

M

T T

$$\text{No. of arrangements} = \frac{10!}{4!\,2!\,2!}$$

$$= 37\,800$$

- How many different arrangements of the word
  - MATHEMATICS are possible?
  - MATARA are possible?

The number of permutations of n people taking r at a time is P(n,r) and the number of ways to rearrange the people chosen is r!. Putting these together we

$$\frac{n!}{\#\ ways\ arrange\ the\ r\ items} = \frac{p(n,r)}{r!}$$

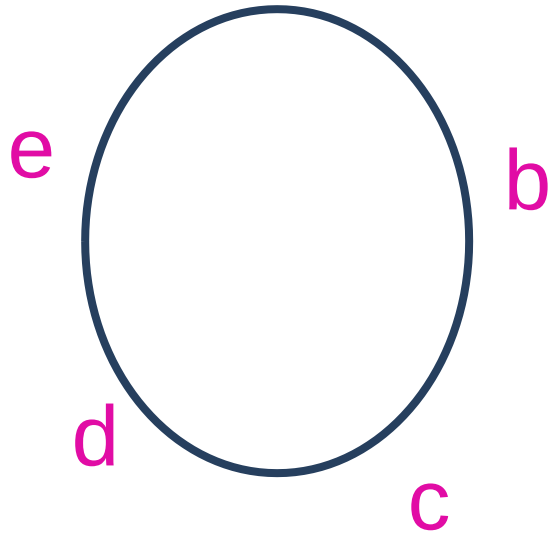
$$= \frac{\frac{n!}{(n-r)!}}{r!}$$

$$= \frac{n!}{(n-r)!.r!}$$

Circular Arrangements

Circular arrangements are permutations in which objects are arranged in a circle.

Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements



abcde

bcdea

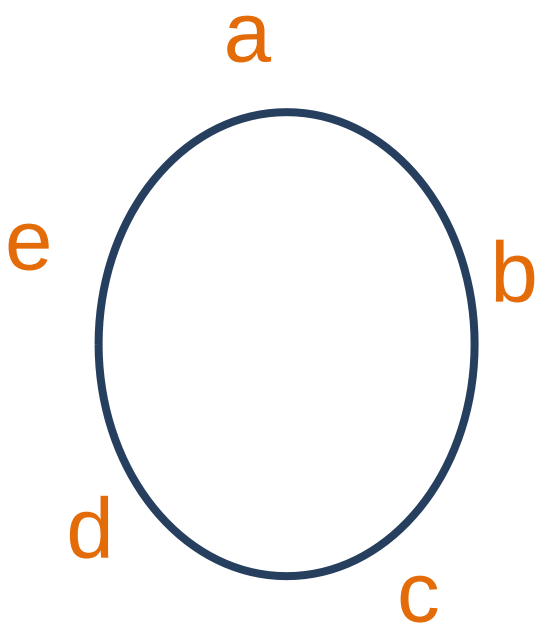
cdeab

deabc

eabcd

are different in a line, but are **identical** around a circle.

Circular Arrangements



To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining (n-1) objects can be arranged as if they were on a straight line in (n-1)! ways.

i.e. the number of arrangements = (n – 1) ! in a circle

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

a) there are no restrictions

b) men and women alternate

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

a) there are no restrictions

Solution :

$(12 - 1)! = 11!$

b) men and women alternate

Solution : **$(6 - 1)! \times 6! = 5! \times 6!$**

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

c) Ted and Carol must sit together

Solution : (TC) & other 10 = 2! × 10!

d) Bob, Ted and Carol must sit together

Solution : (BTC) & other 9 = 3! × 9!

Circular Arrangements

Eg 1. At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

d) Neither Bob nor Carol can sit next to Ted.

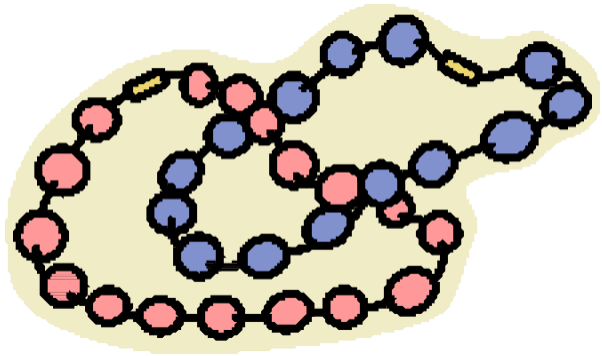
Solution : Seat 2 of the other 9 people next to Ted in (9×8) ways or ${}^{9}P_{2}$

Then sit the remaining 9 people (including Bob and Carol) in $9!$ ways

Ways = $(9 \times 8) \times 9!$ or **${}^{9}P_{2} \times 9!$**

Circular Arrangements

Eg 2. In how many ways can 8 differently coloured beads be threaded on a string?



Solution :

As necklace can be turned over, clockwise and anti-clockwise arrangements are the same

$= (8-1)! \div 2 \quad = 7! \div 2$

Unordered Selections

The number of different **combinations** (i.e. unordered sets) of **r** objects from **n** distinct objects is represented by :

$$\text{No. of Combinations} = \frac{\text{number of permutations}}{\text{arrangements of r objects}}$$

and is denoted by

$$^{n}C_{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!\,(n-r)!}$$

Combinations

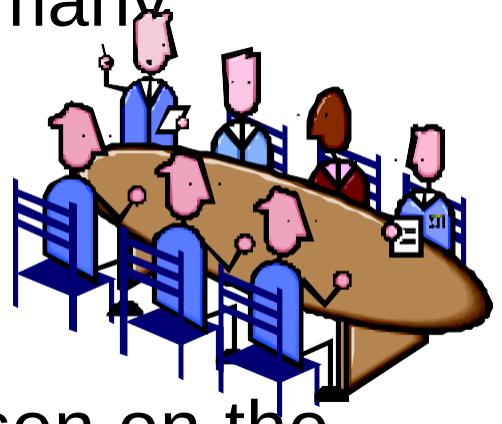
Eg 1. How many ways can a basketball team of 5 players be chosen from 8 players?

Solution :

$^{8}C_{5}$

Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if



a) there are no restrictions?

Solution : $^{10}C_5$

b) one particular person must be chosen on the committee?

Solution : $\underline{1} \times {}^{9}C_4$

c) one particular woman must be excluded from the committee?

Solution : $^{9}C_5$

Combinations

Eg 2. A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

d) there are to be 3 men and 2 women?

Solution : **Men & Women = $^{6}C_{3} \times {}^{4}C_{2}$**

e) there are to be men only?

Solution : **$^{6}C_{5}$**

f) there is to be a majority of women?

Solution :

3 Women & 2 men Or 4 Women & 1 man

= $^{4}C_{3} \times {}^{6}C_{2} + {}^{4}C_{4} \times {}^{6}C_{1}$

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

(i) What is the total possible number of hands if there are no restrictions?

Solution :

$^{52}C_5$

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

a) 4 Kings?

Solution : ${}^{4}C_{4} \times {}^{48}C_{1}$ or 1×48

b) 2 Clubs and 3 Hearts?

Solution : ${}^{13}C_{2} \times {}^{13}C_{3}$

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

c) all Hearts?

Solution : $^{13}C_5$

d) all the same colour?

Solution : **Red or Black** $^{26}C_5 + {}^{26}C_5 = 2 \times {}^{26}C_5$

=

Combinations

Eg 3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

ii) In how many of these hands are there:

e) four of the same kind?

Solution :

$$^{4}C_{4} \times {}^{48}C_{1} \times 13 = 1 \times 48 \times 13$$

f) 3 Aces and two Kings?

Solution : $^{4}C_{3} \times {}^{4}C_{2}$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf:

a) If there are no restrictions?

Solution : ${}^6C_4 \times {}^5C_3 \times 7!$

c) If the 4 Maths books remain together?

Solution : **= (MMMM) _ _ _**

= ${}^6P_4 \times {}^5C_3 \times 4!$ or $({}^6C_4 \times 4!) \times {}^5C_3 \times 4!$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

c) a Maths book is at the beginning of the shelf?

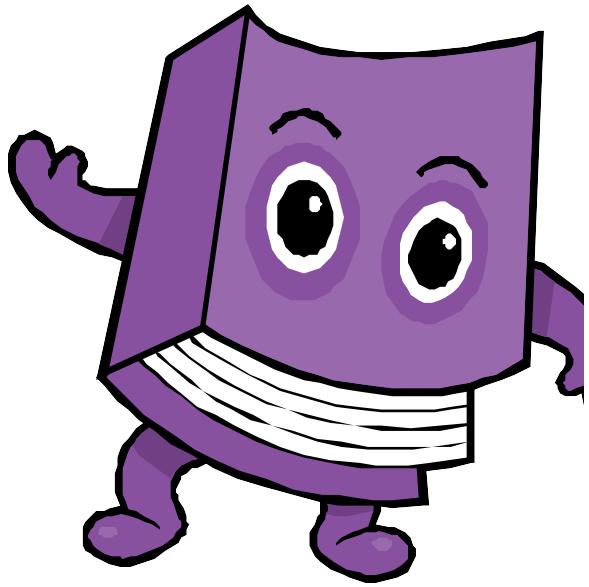
Solution : $= M\_\_\_\_\_\_$

$$= 6 \times {}^5C_3 \times {}^5C_3 \times 6!$$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

d) Maths and English books alternate



Solution : $= M\ E\ M\ E\ M\ E\ M$

$= {}^{6}P_4 \times {}^{5}P_3$

Further Permutations and Combinations

Eg.1 If 4 Maths books are selected from 6 different Maths books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:

e) A Maths is at the beginning and an English book is in the middle of the shelf.

Solution : **M _ _ E _ _ _**

$$= 6 \times 5 \times {}^{5}C_{3} \times {}^{4}C_{2} \times 5!$$

Further Permutations and Combinations

Eg 2. (i) How many different 8 letter words are possible using the letters of the word SYLLABUS ?

Solution : **2 S's & 2 L's**

$$\text{Words} = \frac{8!}{2! \times 2!}$$

$$= 10\,080$$

Further Permutations and Combinations

SYLLABUS **= 10 080 permutations**

(ii) If a word is chosen at random, find the probability that the word:

a) contains the two S's together

Solution : **(SS)** _ _ _ _ _ _ **(Two L's)**

Words = $\frac{7!}{2!}$ **= 2520** **Prob =** $\frac{2520}{10080} = \frac{1}{4}$

b) begins and ends with L

Solution : **L** _ _ _ _ _ _ **L** **(Two S's)**

Words = $\frac{6!}{2!}$ **= 360** **Prob =** $\frac{360}{10080} = \frac{1}{28}$