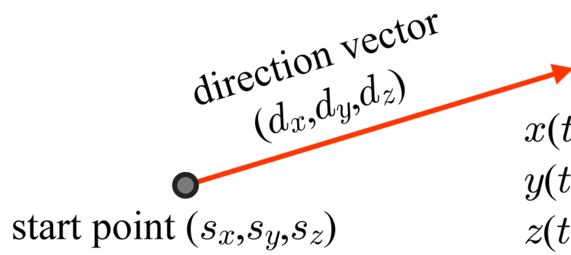
Chapter VI
Parametric Curves and Surfaces

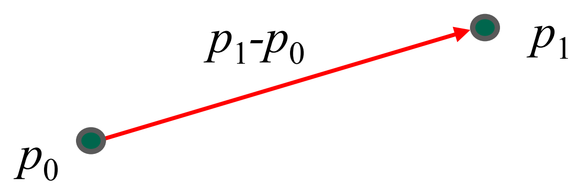
Line Segment

- Recall that in Chapter 3, a ray defined by the start point and the direction vector is represented in a parametric equation.



$$x(t) = s_x + td_x$$
$$y(t) = s_y + td_y$$
$$z(t) = s_z + td_z$$

- Now consider a line segment between two end points, p_0 and p_1. The vector connecting p_0 and p_1, p_1-p_0, corresponds to the direction vector, and therefore the line segment can be represented in the following parametric equation:

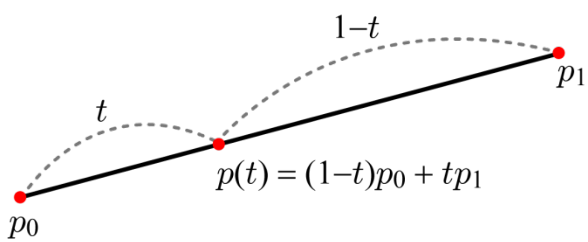


$$p(t) = p_0 + t(p_1 - p_0)$$
$$= (1-t)p_0 + tp_1$$

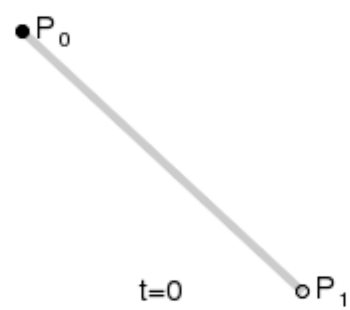
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \\ (1-t)z_0 + tz_1 \end{pmatrix}$$

Line Segment (cont'd)

- A line segment connecting two end points is represented as a *linear interpolation* of the points.
- The line segment may be considered as being divided into two parts by $p(t)$, and the weight for an end point is proportional to the length of the part "on the opposite side," i.e., the weights for p_0 and p_1 are $(1-t)$ and t, respectively.



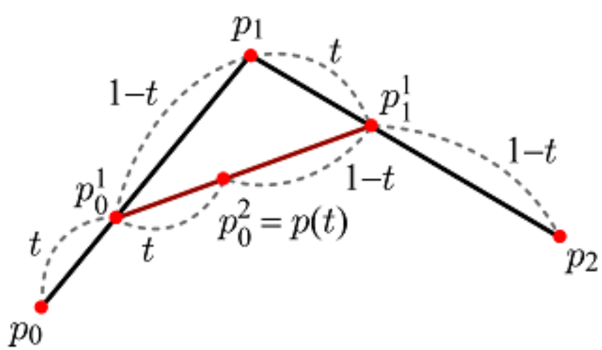
$$p(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} (1-t)x_0 + tx_1 \\ (1-t)y_0 + ty_1 \\ (1-t)z_0 + tz_1 \end{pmatrix}$$

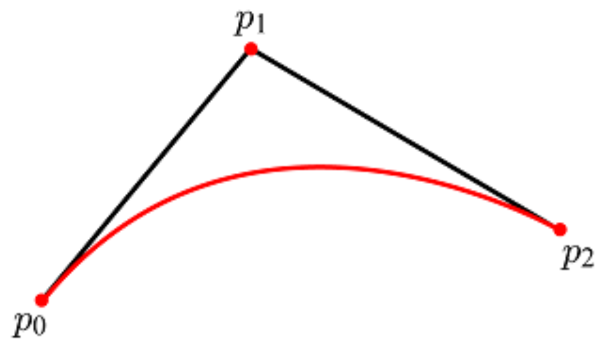


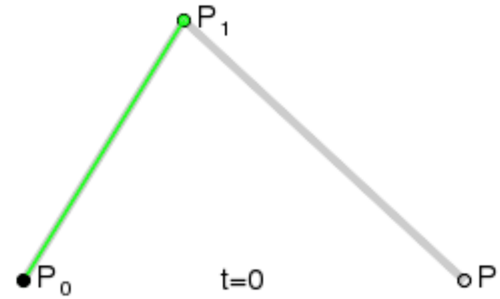
[from Wikipedia]

Quadratic Bézier Curve

- de Casteljau algorithm = recursive linear interpolations for defining a curve
- The quadratic Bézier curve interpolates the end points, p_0 and p_2, and is pulled toward p_1, but does not interpolate it.







[from Wikipedia]

$$p_0 \xrightarrow{1-t} p_0^1, \quad p_1 \xrightarrow{t} p_0^1 = (1-t)p_0 + tp_1$$

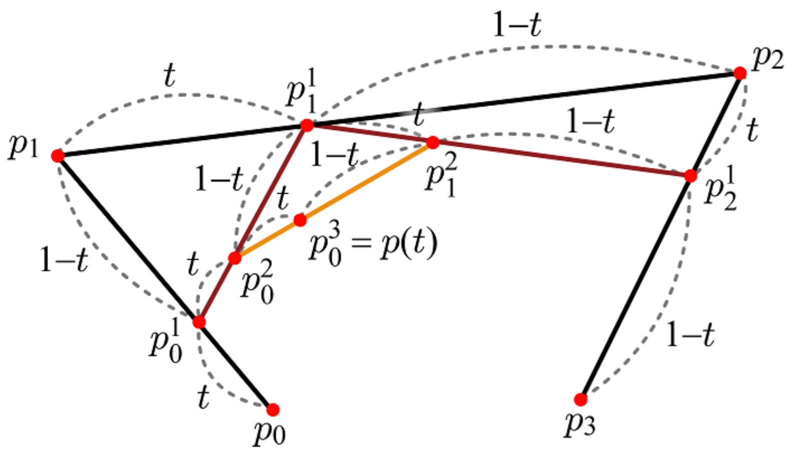
$$p_1 \xrightarrow{1-t} p_1^1, \quad p_2 \xrightarrow{t} p_1^1 = (1-t)p_1 + tp_2$$

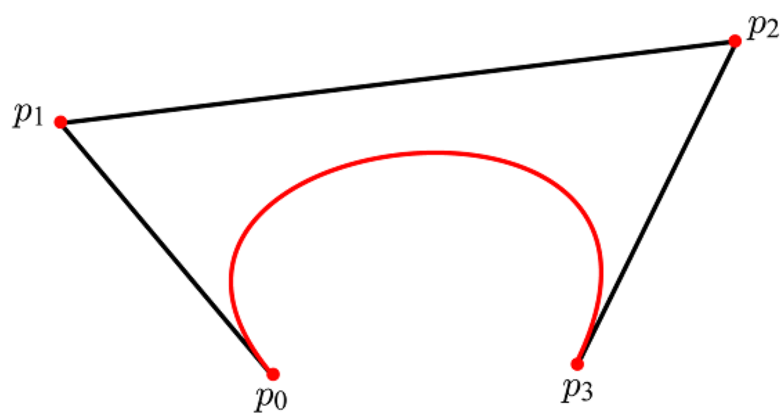
$$p_0^1 \xrightarrow{1-t} p_0^2, \quad p_1^1 \xrightarrow{t} p_0^2 = (1-t)p_0^1 + tp_1^1$$

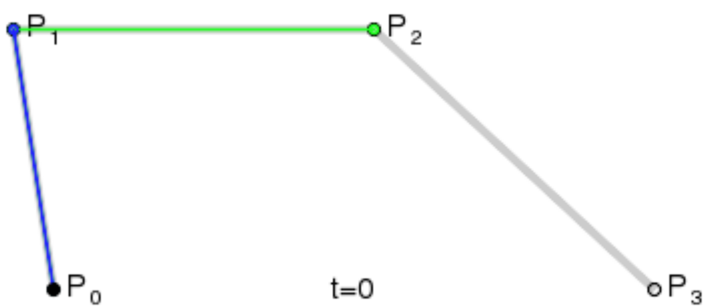
$$= (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2$$

Cubic Bézier Curve

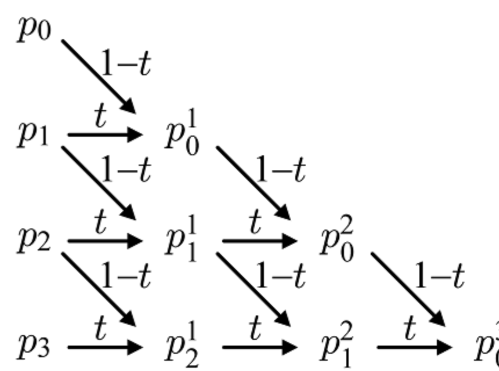
- The cubic Bézier curve interpolates the end points, p_0 and p_3, and is pulled toward p_1 and p_2.







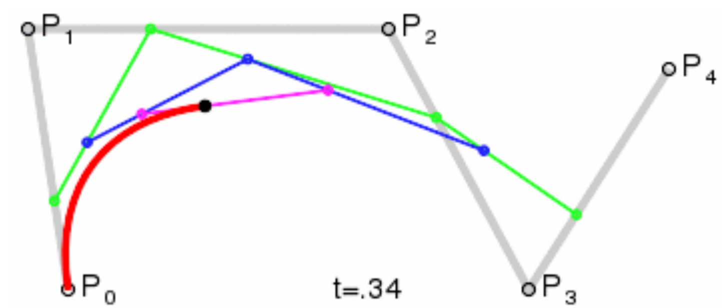
[from Wikipedia]

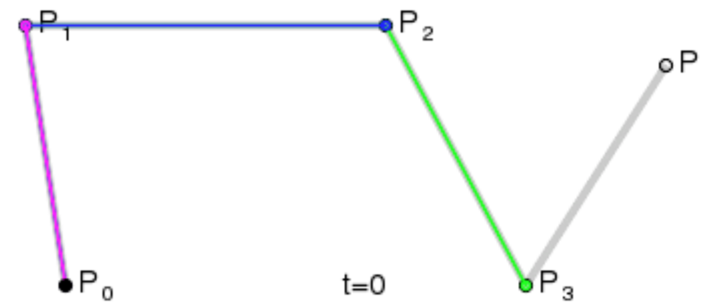


$$p(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t) p_2 + t^3 p_3$$

Quartic Bézier Curve

- The de Casteljau algorithm can be applied for a higher-degree Bézier curve. For example, a quartic (degree-4) Bézier curve can be constructed using five points.

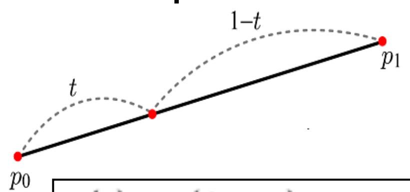




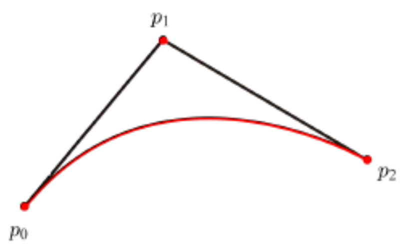
[from Wikipedia]

Bézier Curve – Control Points and Bernstein Polynomials

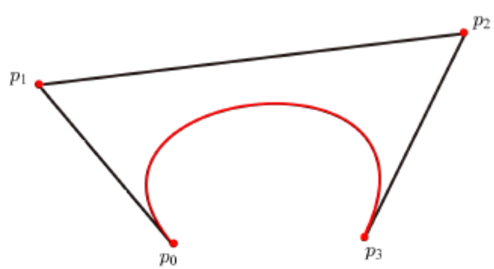
- The points p_is are called *control points.* A degree-n Bézier curve requires $(n+1)$ control points. The coefficients associated with the control point are polynomials of parameter t, and are named *Bernstein polynomials*.



$$\boxed{p(t) = (1-t)p_0 + tp_1}$$



$$\boxed{p(t) = (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2}$$



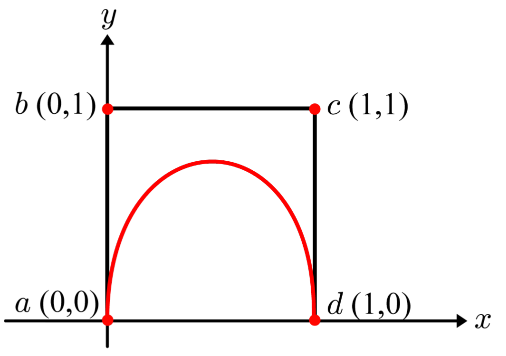
$$\boxed{p(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3}$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

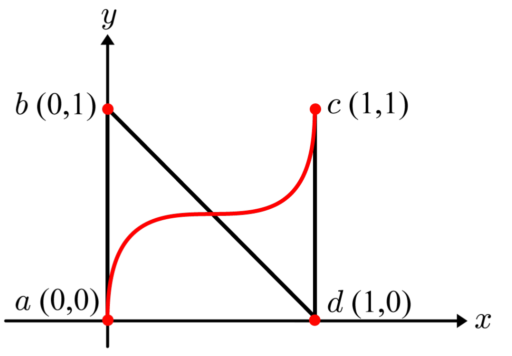
$$p(t) = \sum_{i=0}^{n} B_i^n(t) p_i$$

Bézier Curve – Control Point Order

- Different orders of the control points produce different curves.



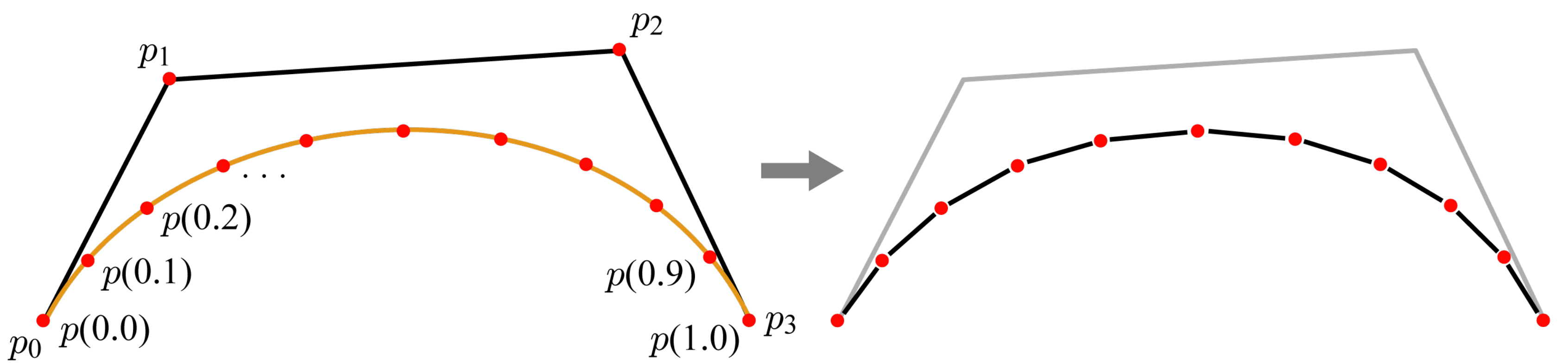
(a) $a \rightarrow b \rightarrow c \rightarrow d$



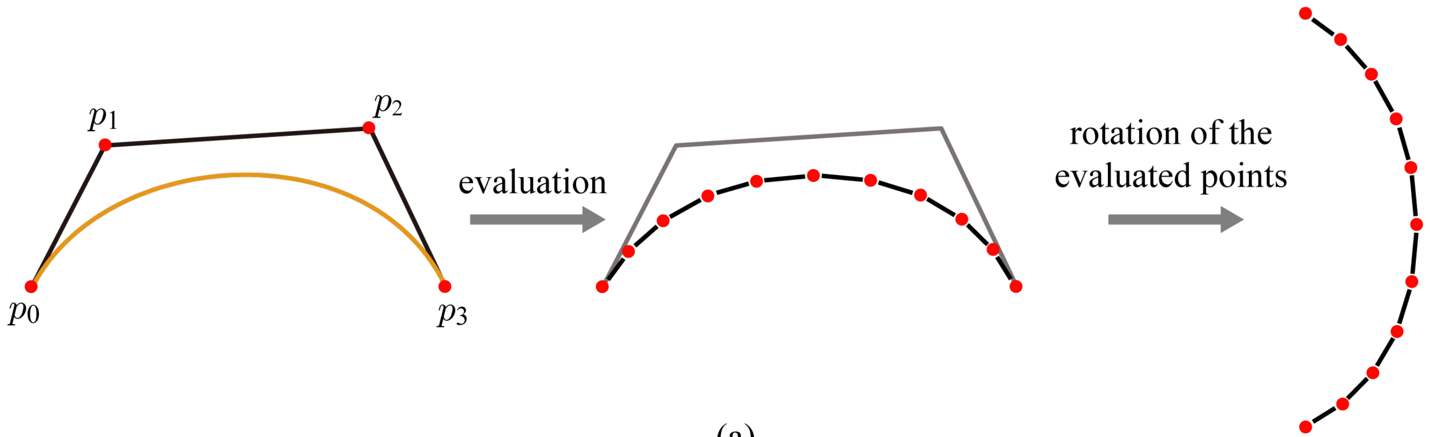
(b) $a \rightarrow b \rightarrow d \rightarrow c$

Bézier Curve – Tessellation for Rendering

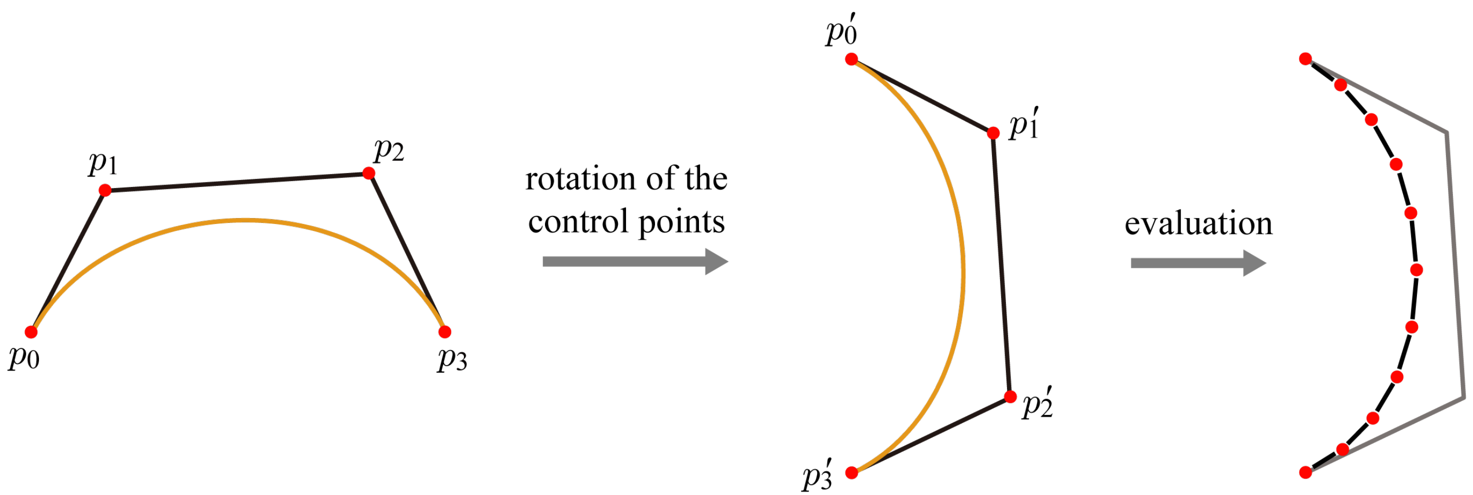
- The typical method to display a Bézier curve is to approximate it using a series of line segments. This process is often called *tessellation*. It evaluates the curve at a fixed set of parameter values, and joins the evaluated points with straight lines.



Bézier Curve – Affine Invariance



(a)



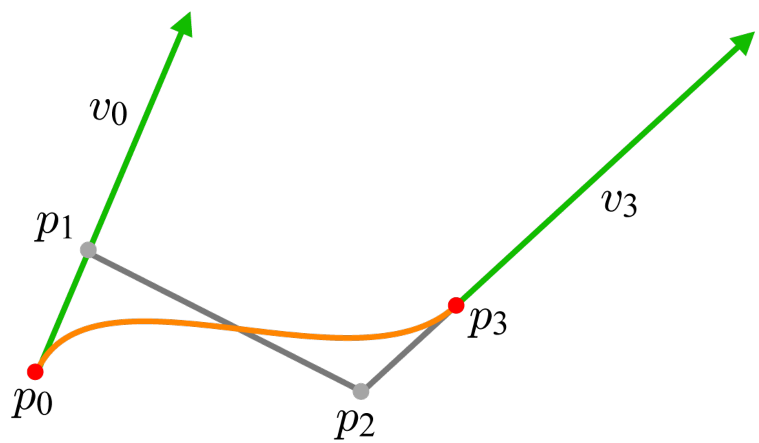
(b)

Hermite Curve

$$\dot{p}(t) = \tfrac{d}{dt}[(1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3]$$
$$= -3(1-t)^2 p_0 + [3(1-t)^2 - 6t(1-t)]p_1 + [6t(1-t) - 3t^2]p_2 + 3t^2 p_3$$

$$v_0 = \dot{p}(0) = 3(p_1 - p_0) \Rightarrow p_1 = p_0 + \frac{1}{3}v_0$$

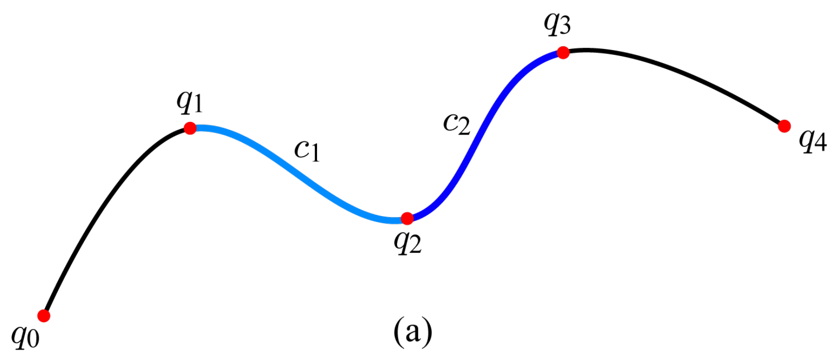
$$v_3 = \dot{p}(1) = 3(p_3 - p_2) \Rightarrow p_2 = p_3 - \frac{1}{3}v_3$$



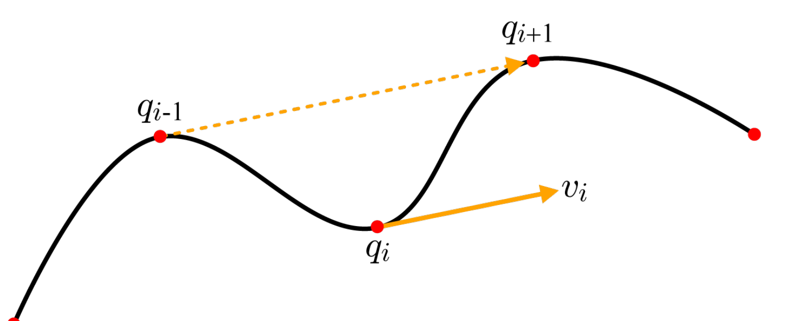
$$p(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3$$
$$= (1-t)^3 p_0 + 3t(1-t)^2 (p_0 + \tfrac{1}{3}v_0) + 3t^2(1-t)(p_3 - \tfrac{1}{3}v_3) + t^3 p_3$$
$$= (1 - 3t^2 + 2t^3)p_0 + t(1-t)^2 v_0 + (3t^2 - 2t^3)p_3 - t^2(1-t)v_3$$

Catmull-Rom Spline

- A spline (piecewise curve) composed of cubic Hermite curves passes through the given points q_is. Two adjacent Hermite curves should share the tangent vector at their junction.
- The tangent vector at q_i is parallel to the vector connecting q_{i-1} and q_{i+1}.



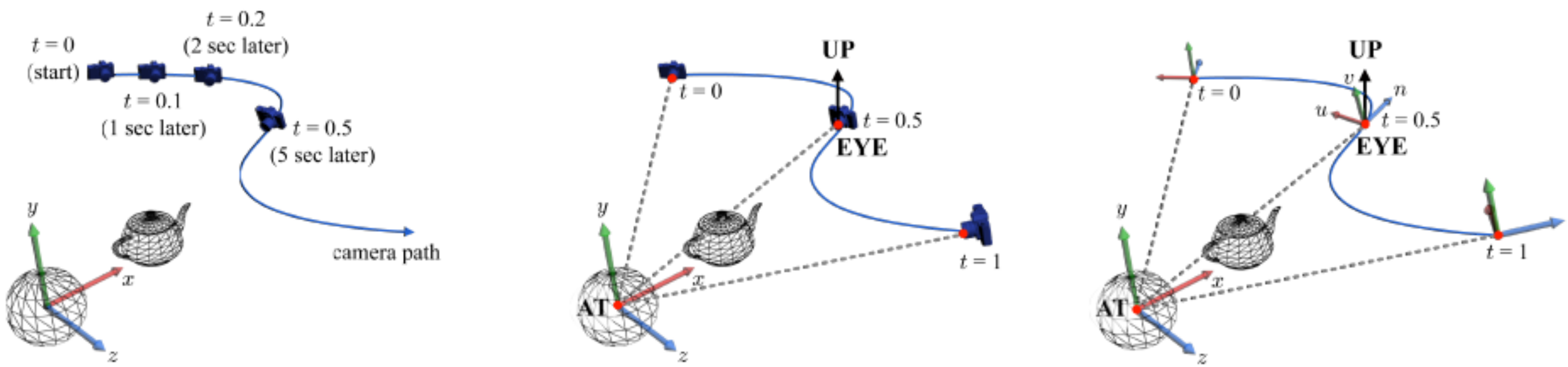
(a)

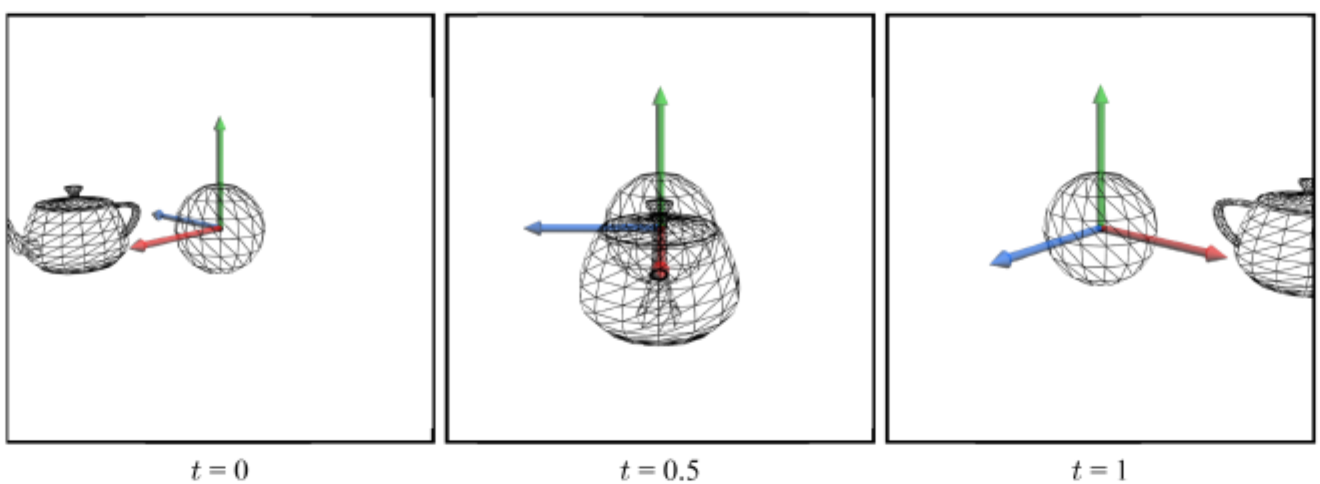


(b)

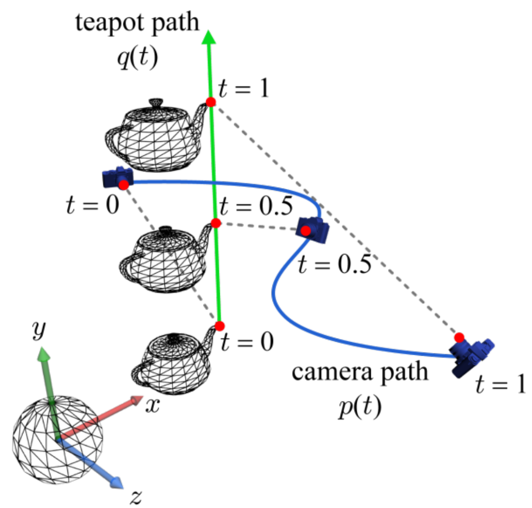
$$\boxed{v_i = \tau(q_{i+1} - q_{i-1})}$$

Application

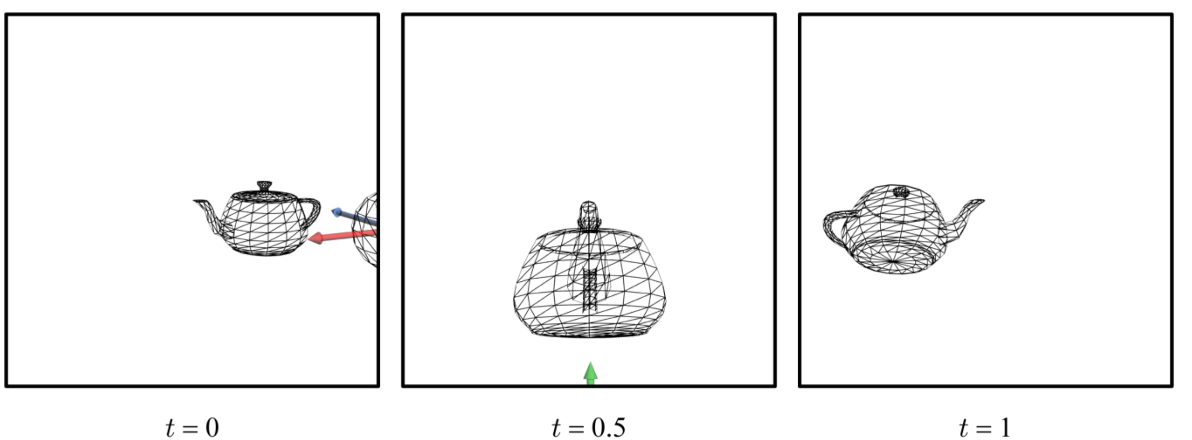




Application (cont'd)



(a)



(b)