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Stochastic Models: Exercise 5

1. Let $\{X_n : n \geq 0\}$ be an irreducible Markov chain with period $d \geq 1$ thus for any state i

$$d = \gcd\{n \geq 1 : P[X_n = i \mid X_0 = i] > 0\}$$

Suppose $\{X_{nd} : n \geq 0\}$ is not aperiodic, thus for some integer $k > 1$, we have

$$k = \gcd\{n \geq 1 : P[X_{nd} = i \mid X_0 = i] > 0\}$$

for every state i. This implies that, for all states, the number of transitions needed to return to state i given that it starts from i in $\{X_n : n \geq 0\}$ is of the form lkd where l is a positive integer. This contradicts that $\{X_n : n \geq 0\}$ is a Markov chain with period d since for any integer l, $lkd > d$, thus $k = 1$ as required and $\{X_{nd} : n \geq 0\}$ as aperiodic.

If we consider the states accessible to $\{X_{nd} : n \geq 0\}$, it is irreducible. Else it is not. Consider the simple symmetric random walk where $\{X_{2n} : n \geq 0\}$ is aperiodic and irreducible for the even states. But if we consider both even and odd states, it is not irreducible as it cannot visit an odd state in even number of steps.

2. Suppose $i \leftrightarrow j$ and let i be positive recurrent, thus $\sum_{n=1}^{\infty} n f_{ii}^n < \infty$.

$$\infty > \sum_{n=1}^{\infty} n f_{ii} \geq \sum_{n=1}^{\infty} n f_{ji} f_{ii}^n f_{ij} \geq \sum_{n=1}^{\infty} n f_{jj}$$

3. Let $\{X_n : n \geq 0\}$ be an irreducible and aperiodic Markov chain. The chain is doubly stochastic, thus $\sum_i P_{ij} = 1$. For any two states i and j, we have $i \leftrightarrow j$ since the Markov chain is irreducible and together with the aperiodicity, we have $\lim_{n\to\infty} P_{ij}^n = 1/\mu_{jj}$. We can prove by induction that $\sum_{i=0}^{k} P_{ij}^n = 1$. Thus

$$1 = \lim_{n\to\infty} \sum_{i=0}^{k} P_{ij}^n = \sum_{i=0}^{k} \lim_{n\to\infty} P_{ij}^n = (k+1)/\mu_{jj}$$

Thus $\mu_{jj} = k + 1 > 0$ implies all the states are positive recurrent. Thus there exists a unique stationary distribution that is also the limiting distribution, i.e. $\pi_j = 1/\mu_{jj}$. Hence $\pi_j = 1/k + 1$ for all j.

We shall prove the claim that $\sum_{i=0}^{k} P_{ij}^n = 1$. It is easy to see that it holds for $n = 1$. Suppose that it is true for n, then since we have

$$\sum_{i=0}^{k} P_{ij}^{n+1} = \sum_{i=0}^{k}\sum_{l=0}^{k} P_{il}^n P_{lj} = \sum_{l=0}^{k}\left(\sum_{i=0}^{k} P_{il}^n\right) P_{lj}$$

it is also true for $n + 1$, which proves the claim.

4. Let $\{X_n : n \geq 0\}$ be a Markov chain with states $S = \{0, 1, 2, 3, 4\}$ denoting the number of umbrella(s) in his home. The transition matrix $\mathbb{P}$ is such that $P_{i,i-1} = p = 1 - P_{i,i}$ for $i \neq 0$ and $P_{0,0} = 1$.

   (a)

   (b)

5. (a) Let $\{X_n : n \geq 0\}$ be a Markov chain with states of the form $(i, k - i)$ for $i = 0, 1, \ldots, k$. We shall denote state $(i, k - i)$ by i. Then the transition matrix is given by

$$P_{0,0} = 3(1/2)^2 = P_{k,k} \quad P_{0,1} = (1/2)^2 = P_{k,k-1}$$
$$P_{i,i} = 2(1/2)^2 \quad P_{i,i+1} = (1/2)^2 = P_{i,i-1} \quad \text{for } i \neq 0, k$$

(b) We first note that the Markov chain is irreducible as $i \leftrightarrow j$ for any states i and j. It is aperiodic as $P[X_1 = i \mid X_0 = i] > 0$ for all states. The transition matrix is also doubly stochastic as

$$\begin{aligned} \text{when } i \neq 0, k \quad & \sum_i P_{i,j} = P_{i-1,i} + P_{i,i} + P_{i+1,i} = 1 \\ \text{when } i = 0 \quad & \sum_i P_{i,j} = P_{0,0} + P_{1,0} = 1 \\ \text{when } i = k \quad & \sum_i P_{i,j} = P_{k-1,k} + P_{k,k} = 1 \end{aligned}$$

Hence by question 3, we have $\pi_0 = \pi_k = 1/k$. Thus the proportion of time where there is only shoes at one door is $2/k$ and since he choose to depart the front or back door with equal chance, he runs barefooted $1/k$ of the time.

6. (a)

   (b)