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Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that $f(y,z) = 5y^2 + z^2 - 3zy - 3y \geq 0$ for every $y, z \in \{0, 1, 2, \ldots\}$. We shall use $\mathbb{Z}_{>0}$ to denote the set $\{0, 1, 2, \ldots\}$ subsequently. We can rewrite $f(y,z)$ to get

$$f(y,z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \tag{1}$$

which we will show that the $f(y,z)$ in this form is nonnegative. All that is left to show is that $\frac{11}{4}y^2 - 3y \geq 0$ for all $y \in \mathbb{Z}_{>0}$ but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (1) holds for all $z \in \mathbb{Z}_{>0}$ when $y = 1$. Solving for the inequality below,

$$\begin{aligned} f(1,z) = \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 &\Leftrightarrow (z-1)(z-2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that $f(1,z) < 0$ for $z \in (1,2)$ and hence positive for all $z \in \mathbb{Z}_{>0}$ and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let $\mathcal{P}_i$ denote the set of paths from an origin s_i to a sink t_i with $\mathcal{P}_i \neq \emptyset$.

   Definition (flow)**.** For a *flow* f and path $P \in \mathcal{P}_i$, f_P is the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i. A flow is feasible for a vector $r = (r_1, \ldots, r_k)$ if it routes all the traffic: for each $i \in \{1, 2, \ldots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

   Definition (Nonatomic equilibrium flow)**.** Let f be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow f is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \ldots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_p(f) \leq c_{\tilde{P}}(f)$$

   where $c_p(f)$ denotes the cost of travelling on path P for flow f.

   (ii) Let $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$. Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_p(f_p) \cdot f_p = \sum_{e \in E} c_e(f_e) \cdot f_e$$

   where E is the set of directed edges on the graph G.

   (iii) Let L_i be the common cost of all paths in use between s_i and t_i at equilibrium.

$$\text{SC(NE)} = \sum_{i=1}^k \sum_{p \in \mathcal{P}_i} f_P c_P(f) = \sum_{i=1}^k r_i L_i \tag{2}$$

On the other hand, for each source-sink pair $s_i - t_i$

$$\sum_{P \in \mathcal{P}_i} f_P^* c_P(f) \geq r_i L_i$$

$$\Rightarrow \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P^* c_P(f) \geq \sum_{i=1}^{k} r_i L_i \tag{3}$$

Now, substract (2) from (3) and express the sum as summing over edges instead of over paths, we have

$$\sum_{e \in \mathcal{E}} (f_e^* - f_e) c_e(f_e) \geq 0 \tag{4}$$

Apply the Pigou bound for each edge with $c \leftarrow c_e, r \leftarrow f_e$ and $x \leftarrow f_e^*$.

$$\begin{aligned}
\alpha(C) &\geq \frac{rc(r)}{xc(x) + (r-x)c(r)} \\
\Rightarrow \alpha(C) &\geq \frac{f_e c_e(f_e)}{f_e^* c_e(f_e^*) + (f_e - f_e^*) c_e(f_e)} \\
\sum_{e \in \mathcal{E}} f_e^* c_e(f_e^*) &\geq \sum_{e \in \mathcal{E}} \frac{1}{\alpha(C)} f_e c_e(f_e) + (f_e^* - f_e) c_e(f_e) \\
\text{SW(OPT)} &\geq \frac{1}{\alpha(C)} \text{SW(NE)} \\
\alpha(C) &\geq \text{PoA}
\end{aligned}$$

(iv) We start by showing that

$$\begin{aligned}
\inf_x \left\{ \left(\frac{ax+b}{ar+b} - 1 \right) \right\} &= \inf_x \left\{ x \left(\frac{a(x-r)}{ar+b} \right) \right\} \\
&= \frac{a}{ar+b} \inf_x \left\{ x^2 - rx \right\} \\
&= -\frac{r^2}{4} \cdot \frac{a}{ar+b}
\end{aligned}$$

with that we can begin out proof.

$$\begin{aligned}
\sup_{c \in \mathcal{C}} \sup_{x,r} \frac{rc(r)}{xc(x) - (r-x)c(r)} &= \sup_{c \in \mathcal{C}} \sup_{x,r} \frac{r}{r + x \left(\frac{c(x)}{c(r)} - 1 \right)}, \text{ since } c(r) > 0 \\
&= \sup_{a,b \geq 0} \sup_{x,r} \frac{r}{r + x \left(\frac{ax+b}{ar+b} - 1 \right)} \\
&= \sup_{a,b \geq 0} \sup_{r} \frac{r}{r - \frac{r^2}{4} \frac{a}{ar+b}} \\
&= \sup_{a,b \geq 0} \sup_{r} \frac{1}{1 - \frac{ar}{4(ar+b)}} \\
&= \frac{1}{1 - 1/4}
\end{aligned}$$

the last equality follows as the supremum of $\frac{ar}{4(ar+b)}$ occurs when $b = 0$.

3. (i) Let Φ be the potential function of a potential game and c_i denote the cost function of the agents for $i \in \{1, 2, \dots, k\}$. To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of s_i and solely dependent on s_{-i}. If we consider two alternative distinct strategies for agent i, $s_i', s_i'' \neq s_i$

$$\begin{aligned} c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i', s_{-i}) - \Phi(s_i', s_{-i})] \\ c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i'', s_{-i}) - \Phi(s_i'', s_{-i})] \end{aligned}$$

hence we can choose $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$, where $-$ represents any choice of strategy of agent i.

(ii) Let Φ_1 and Φ_2 be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \tag{5}$$
$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \tag{6}$$

where $D_i^k(s_{-i})$ denotes the dummy term for $k = 1, 2$. Taking (5)$-$(6),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at s_{-i} and any strategy of agent i, s_i.

(iii) First, we prove that for any potential game, the equality holds. Let Φ be a potential function. By definition,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) = \phi(s_i^2, s_{-i}^1) - \phi(s^1)$$

Furthermore, from part (i) we have

$$c_i(s^2) = \phi(s^2) + D_i(s_{-i}^2)$$

and the same results apply for agent j. So

$$\begin{aligned} \text{LHS} &= c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) \\ &= \phi(s_i^2, s_{-i}^1) - \phi(s^1) - c_j(s_i^2, s_{-i}^1) + \phi(s^2) + D_j(s_{-j}^2) \\ &= \phi(s^2) - \phi(s^1) + D_j(s_{-j}^2) - D_j(s_i^2, s_{-i}^1) \end{aligned}$$

Now, observe that $D_j(s_{-j}^2) = D_j(s_i^2, s_{-i}^1)$ because in both cases i plays s_i^2 while the rest of the players (different from i and j) play the same in s^1 and s^2 anyways. Therefore

$$\text{LHS} = \phi(s^2) - \phi(s^1)$$

In the same manner we prove that RHS $= \phi(s^2) - \phi(s^1)$. Therefore, for any potential game, LHS $=$ RHS.

Now we need to prove the reverse, i.e. if the identity holds for every pair s^1, s^2 then the potential function exists. We want to show that for every pair i, j,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) = \phi(s_i^2, s_{-i}^1) - \phi(s^1)$$

and so on. Our idea is to use the observation that (s_i^2, s_{-i}^1) and (s_j^1, s_{-j}^2) are the same outcome, and manipulate terms so that the relationship $D_j(s_{-j}^2) = D_j(s_i^2, s_{-i}^1)$ appears, which links us back to the first part.

4. (a) Let $\tilde{f}$ be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player $i \in \{1, 2, \ldots, n\}$, any two parallel links P_i, P_j where $1 \leq i < j \leq k$,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of $\tilde{f}$ on P_i equals the flow of f on P_j, ($\tilde{f}_{P_i} = f_{P_j}$) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

since the above inequality is true for any two parallel links, we sum it over the n parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^{n} \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^{n} \sum_{m=1}^{f_{P_i}} c_{P_j}(m) = \Phi(f)$$

where Φ is the potential function.

(b) Consider the triangular network of internet service providers (ISP) that we discussed in class (Figure 1). There are 3 ISPs 1, 2, 3 serving a common market located at 4. Suppose the direct links from the ISPs to the user has high cost, and the links between the ISPs are cheap. Now, if 1 is using the direct link 1-4, then 3 will have an incentive to piggy-back ride and use 3-1-4, abandoning the link 3-4. Then 2 has two choices: either to use his own direct link 2-4, or to use a 3-hop link 2-3-1-4. The latter is not desirable (as it will cause a lot of delay), so he will bear the high cost and run the link 2-4. When that's the case, 1 will have the incentive to piggy-back ride, abandoning 1-4 and using 1-2-4 instead. But this will cause 3 to choose the direct link 3-4 instead of piggy-back ride through 1. This is contrary to his previous strategy. The cycle repeats. Effectively, there are no Nash equilibria in this game.

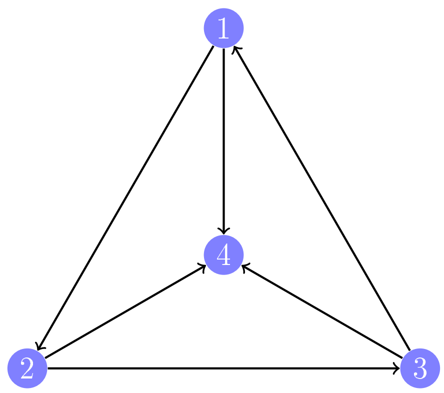
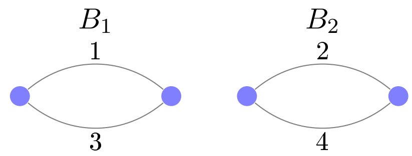


Figure 1: Example network with non-parallel links

5. (a) The congestion game to consider is shown below with $m = n = 2$ where n and m are the number of strategies for P1 and P2 respectively,. A strategy of P1 is the upper or lower edges of the networks; we shall let A_1 be the upper edges. A strategy of P2 is the choice of a disjoint network, here the left network is P2's strategy B_1. For $1 \leq i \leq n$ and $1 \leq j \leq m$, $e_{i,j}$ is the ith edge of the jth disjoint network.



P2

P1		B_1	B_2
	A_1	$1, 1$	$2, 2$
	A_2	$3, 3$	$4, 4$

Hence the tuple of bijective functions from the strategies of the network to the strategies in the partnership game for the congestion game shown above is:

$$f_1(A_i, B_j) = \{e_{i,j}\}_{1 \le j \le m}, \quad \text{the } i\text{th edge in every disjoint network}$$
$$f_2(A_i, B_j) = \{e_{i,j}\}_{1 \le i \le n}, \quad \text{all the edges in the } j\text{th disjoint network}$$

which says that every strategy in the congestion game is a subset of edges in the family of disjoint networks. The cost function c_k^1 for the partnership game G^1 is shown in the table above and the cost function for the congestion game $c_k^2(\{e_{p,j}\}_{1 \le j \le m}, \{e_{i,q}\}_{1 \le i \le n})$ is the cost of the unique edge found from the intersection $\{e_{p,j}\}_{1 \le j \le m} \cap \{e_{i,q}\}_{1 \le i \le n}$. Note that although we used fixed numbers as the cost in the 2 player 2 strategies example we illustrated above, we can see them as just placeholders and it works well for any partnership game of that form even with different cost. We can generalize this from 2 players to N players in a similar fashion.

(b) Still working on it.

(c) We know that the cost function of a potential game can be decomposed into the sum of a potential function and a dummy function. We claim that:

i. in a partnership game, the potential function is $\phi(s) = c_i(s) \ \forall i$. Therefore, the cost function of a partnership game which is also a potential game, can be decomposed into a sum of a potential function and a zero function as its dummy function

$$c_i(s) = \phi(s) + \mathbf{0}$$

ii. in a dummy game, the potential function is $\phi(s) = 0$. The cost function of a dummy game which is also a potential game can be decomposed into a sum of itself as the dummy function and a potential function which is the zero function.

$$c_i(s) = \mathbf{0} + D_i(s_{-i})$$

Now, if we have a potential game such that

$$c_i(s) = \phi(s) + D_i(s_{-i})$$

then we can effectively map the partnership game with potential function $\phi(s)$ into a congestion game G_1 and map the dummy game with dummy function $D_i(s_{-i})$ into another congestion game G_2. What we did is effectively map a potential game into a congestion game with two separate networks G_1 and G_2, where each player has to play an action in G_1 and an action in G_2 simultaneously (isomorphism is preserved under addition). Thus every potential game is isomorphic to a congestion game.

6. (a) We consider two networks A, B that resemble $n-$gons where n is even, i.e. there are n edges and n vertices for each network as shown in Figure 2. The strategy of each player i is

$$S_i = \{\{a_i, b_i\}, \{a_{i+1}, b_{i-1}, b_{i+1}\}\} = \{s_i^1, s_i^2\}$$

where a_i, b_i denotes edges in A and B respectively and the cost function of each edge is simply $c_e(x) = x$. We claim that when all the player were to play s_i^1 it is a Nash equilibrium and we have the optimal value of potential which has value $2n$, since every edge is inhabit by a single player. Any player i that deviates from playing s_i^1 will increase the potential by $2+2+2-1$ (hence the claim it is a Nash) since it will share an edge with player $i+1$ in A and players $i-1, i+1$ in B and hence increase the potential. If all the players were to play s_i^2, we see that the potential will be $n+n+2n$ where the first n is incurred from A and the next two terms are from B as we sum up the cost from 1 to the load of the edge. It is easy to see that everyone playing s_i^2 is a Nash; for any player that deviates from s_i^2, will have an increased cost of $+1$ coming from A, an increased cost of $+2$ from B and hence this completes the proof.

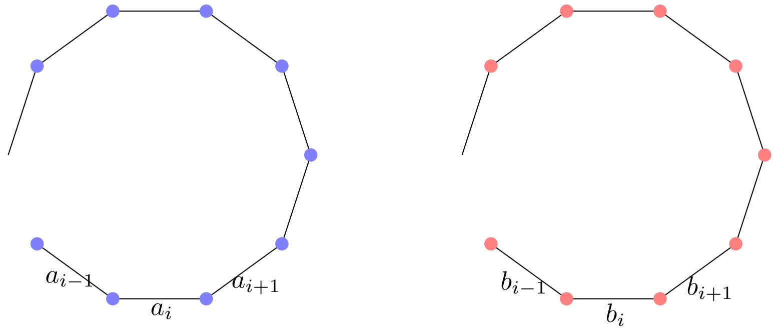


Figure 2: Example with price of potential anarchy equals 2.

(b) First, we understand that this question is concerning the atomic case, because for the non-atomic case, all Nash equilibria have the same potential value, as we proved in class. The first obvious bound is that

$$\phi(s) \leq SW(s)$$
$$\phi(s^*) \leq SW(s^*)$$
$$\Rightarrow PoPA = \frac{\phi(s)}{\phi(s^*)} \leq PoA = \frac{SW(s)}{SW(s^*)} \leq \frac{5}{2}$$

Idea for proving a tight bound: apply the same 4-step strategy as for a normal PoA bound, but the tricky part is that the potential function is not a sum over all agents like the social welfare. Below are what we have got so far:

$$\phi(s) = \sum_{e \in \mathcal{E}} \sum_{i=1}^{l_e} c_e(i)$$

$$\phi(s^*) = \sum_{e \in \mathcal{E}} \sum_{i=1}^{l_e^*} c_e(i)$$

Now we create a hybrid term

$$\phi(s_i^*, s_{-i}) - \phi(s) = \sum_{e \in s_i^* \setminus s_i} c_e(f_e + 1) - \sum_{e \in s_i \setminus s_i^*} c_e(f_e) \geq 0$$

To go from $\phi(s)$ to $\phi(s^*)$ we let agents deviate one by one

$$\begin{aligned}
\phi(s) &\leq \phi(s_1^*, s_{-1}) + \sum_{e \in s_1 \setminus s_1^*} c_e(f_e) - \sum_{e \in s_1^* \setminus s_1} c_e(f_e + 1) \\
&\leq \phi(s_1^*, s_2^*, s_{3..n}) + \sum_{e \in s_1 \setminus s_1^*} c_e(f_e) - \sum_{e \in s_1^* \setminus s_1} c_e(f_e + 1) + \sum_{e \in s_2 \setminus s_2^*} c_e(f_e) - \sum_{e \in s_2^* \setminus s_2} c_e(f_e + 1) \\
&\leq ... \\
&\leq \phi(s^*) + \sum_{i=1}^{n} \sum_{e \in s_i \setminus s_i^*} c_e(f_e) - \sum_{i=1}^{n} \sum_{e \in s_i^* \setminus s_i} c_e(f_e + 1)
\end{aligned}$$

Now if we can manipulate the double sums so that we can create $\phi(s)$ and $\phi(s^*)$ then we are done. We are trying to break the $c_e(f_e)$ term into two components, one involves $\phi(s)$ and one involves a dummy term.