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Statistics: Homework 1

1.19 (a) Let X_1, X_2 and X_3 denote the computer owners that use Macintosh, Windows and Linux respectively and let V denote the event that the user's system is infected with the virus. We want to find $\mathbb{P}(X_2|V)$

$$
\begin{aligned}
\mathbb{P}(X_2|V) &= \frac{\mathbb{P}(V|X_2)\mathbb{P}(X_2)}{\sum_{i=1}^{3} \mathbb{P}(V|X_i)\mathbb{P}(X_i)} \\
&= \frac{(.82)(.5)}{(.65)(.3) + (.82)(.5) + (.5)(.2)} \\
&= 0.581560284
\end{aligned}
$$

(b) $\mathbb{P}(V) = (.65)(.3) + (.82)(.5) + (.5)(.2) = .705$

(c) Let A and B denote the event that the second person has a system that was also infected by a virus and the second person is known to have the same computer system as the first person. We observe that A and B are independent events as the probability of getting a virus on your computer system is the same regardless of whether the second person has the same computer system as the first person. Thus $\mathbb{P}(A|B) = \mathbb{P}(A) = \mathbb{P}(V) = .705$

2.4 (a)

$$
F_X(x) := \begin{cases}
\frac{1}{4}x & 0 < x < 1 \\
\frac{1}{4} & 1 \le x \le 3 \\
\frac{3}{8}x - \frac{7}{8} & 3 < x < 5 \\
1 & x > 5
\end{cases}
$$

(b)

$$
\begin{aligned}
\mathbb{P}(Y \le y) &= \mathbb{P}(1/X \le y) \\
&= \mathbb{P}(X \ge 1/y) \\
&= 1 - \mathbb{P}(X \le 1/y)
\end{aligned}
$$

From (a):

$$
F_Y(y) := \begin{cases}
\frac{15}{8} - \frac{3}{8y} & 1/5 < y < 1/3 \\
\frac{3}{4} & 1/3 \le y \le 1 \\
1 - \frac{1}{4y} & y > 1
\end{cases}
$$

$$
f_Y(y) := \begin{cases}
\frac{3}{8y^2} & 1/5 < y < 1/3 \\
\frac{1}{4y^2} & y > 1 \\
0 & \text{otherwise}
\end{cases}
$$

2.11 (a) We see that $\mathbb{P}(X = 1) = p = \mathbb{P}(Y = 0)$. Since the state space contains $\{H, T\}$, we have $1 - \mathbb{P}(X = 1, Y = 0) = 1 - p = \mathbb{P}(X = 0, Y = 1)$. But since

$$
\mathbb{P}(X = 1)\mathbb{P}(Y = 0) = p^2 \ne p = \mathbb{P}(X = 1, Y = 0)
$$

X and Y are dependent.

(b) By total law of probability,

$$\mathbb{P}(X = x) = \sum_{n=x}^{\infty} \mathbb{P}(X = x | N = n) \cdot \mathbb{P}(N = n)$$

$$= \sum_{n=x}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \frac{(\lambda p)^x}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!}$$

$$= e^{-\lambda p} \frac{(\lambda p)^x}{x!}$$

in a similar fashion, we have

$$\mathbb{P}(Y = y) = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!}$$

for the joint distribution of X and Y,

$$\mathbb{P}(X = x, Y = n - x) = \mathbb{P}(X = x, Y = n - x | N = n) \cdot \mathbb{P}(N = n)$$

$$= \binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

now

$$\mathbb{P}(X = x) \cdot \mathbb{P}(Y = y) = e^{-\lambda p} \frac{(\lambda p)^x}{x!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!}$$

$$= \binom{x+y}{x} p^x (1-p)^y e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!} = \mathbb{P}(X = x, Y = y)$$

which shows that X and Y are independent.

3.4 Let Y_i denote the jump of the particle at the ith unit. Then $X_n = \sum_{i=1}^{n} Y_i$. The Y_i's are iid, with $\mathbb{E}(Y_i) = 1 - 2p$ and $\mathbb{V}(Y_i) = 1 - (1 - 2p)^2 = 4p(1 - p)$ for $i = 1, 2, \ldots, n$.

$$\mathbb{E}(X_n) = \sum_{i=1}^{n} \mathbb{E}(Y_i) = n(1 - 2p)$$

$$\mathbb{V}(X_n) = \sum_{i=1}^{n} \mathbb{V}(Y_i) = n \cdot 4p(1 - p)$$

4.3 Using Chebyshev's and Hoeffding's inequality we have

$$\mathbb{P}(|\overline{X}_n - p| > \epsilon) \leq \frac{1}{4n\epsilon^2}$$

$$\mathbb{P}(|\overline{X}_n - p| > \epsilon) \leq 2e^{-2n\epsilon^2}$$

The inequality $(1 + x)^r \leq e^{rx}$ for $r > 0, x > 0$, thus for $r = 1$

$$x < 1 + x \leq e^x$$

$$1/x > e^{-x}$$

$$\frac{1}{2n\epsilon^2} > e^{-2n\epsilon^2}$$

$$\frac{1}{n\epsilon^2} > 2e^{-2n\epsilon^2}$$

5.7 We first note that

$$\mathbb{V}\left(n^{-1} \sum_{i=1}^{n} X_i^2 - p\right) = n^{-2} \sum_{i=1}^{n} \mathbb{V}(X_i^2) = \frac{p(1-p)}{n}$$

thus for any given $\epsilon > 0$,

$$\mathbb{P}\left(|n^{-1}\sum_{i=1}^{n} X_i^2 - p| > \epsilon\right) \leq \frac{\mathbb{V}(n^{-1}\sum_{i=1}^{n} X_i^2 - p)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

which proves the convergence in probability. We now prove its convergence in quadratic mean:

$$\mathbb{E}\left(\left[n^{-1}\sum_{i=1}^{n} X_i^2 - p\right]^2\right) = \mathbb{E}\left(n^{-2}\left[\sum_{i=1}^{n} X_i^2\right]^2\right) - 2\,\mathbb{E}\left(p/n\sum_{i=1}^{n} X_i^2\right) + \mathbb{E}\left(p^2\right) \tag{1}$$

simplifying the first term, we get

$$\mathbb{E}\left(n^{-2}\left[\sum_{i=1}^{n} X_i^2\right]^2\right) = n^{-2}\left[\mathbb{E}\left(\sum_{i=1}^{n} X_i^4\right) + 2\,\mathbb{E}\left(\sum_{i\neq j} X_i^2 X_j^2\right)\right]$$

$$= n^{-2}\left[\mathbb{E}\left(\sum_{i=1}^{n} X_i^4\right) + 2\sum_{i\neq j}\mathbb{E}\left(X_i^2\right)\mathbb{E}\left(X_j^2\right)\right]$$

$$= n^{-2}\left[np + 2\binom{n}{2}p^2\right]$$

$$= \frac{p}{n} + \frac{n-1}{n}p^2$$

the simplification of the last 2 terms in (1) gives $-p^2$. Thus

$$\mathbb{E}\left(\left[n^{-1}\sum_{i=1}^{n} X_i^2 - p\right]^2\right) = \frac{p-p^2}{n} \to 0 \text{ as } n \to \infty$$

which concludes the prove of convergence in quadratic mean.