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Real Analysis: Homework 4

Proof.

(a)

$$\begin{aligned}\mathbb{P}[|X_t - X_s| \geq \epsilon] &= \mathbb{P}[|X_t - X_s|^\alpha \geq \epsilon^\alpha] \\ &\leq \epsilon^{-\alpha}\mathbb{E}[|X_t - X_s|^\alpha], \quad \text{by Markov Inequality} \\ &\leq \epsilon^{-\alpha}|t-s|^{1+\beta}\end{aligned}$$

thus as $s \to t$, we have $\mathbb{P}[|X_t - X_s| \geq \epsilon] \to 0$ which shows that $X_s \to X_t$ in probability as $s \to t$.

(b) We need to show that

$$\mathbb{P}\left[\max_{1\leq k\leq 2^n}\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] = 1$$

so from (a), we get for all $1 \leq k \leq 2^n$,

$$\begin{aligned}\mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] &= 1 - \mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| \geq 2^{-\gamma n}\right] \\ &\geq 1 - (2^{-\gamma n})^{-\alpha}\left|\frac{T}{2^n}\right|^{1+\beta} \\ &= 1 - |T|^{1+\beta}\cdot 2^{-n(1+\beta-\gamma\alpha)} \to 1 \text{ as } n \to \infty\end{aligned}$$

which shows the desired.

(c) For all $t, s \in D$ such that $|t-s| < 2^{-N(\omega)}$, we need to show that

$$\mathbb{P}\left[|X_t - X_s| \leq 2\sum_{j=n+1}^{\infty} 2^{-\gamma j}\right] = 1$$

From (a),

$$\mathbb{P}\left[|X_t - X_s| > 2\sum_{j=n+1}^{\infty} 2^{-\gamma j}\right] \leq \left(2\sum_{j=n+1}^{\infty} 2^{-\gamma j}\right)^{-\alpha}|t-s|^{1+\beta}$$

(d)

(e)

□