

Real Analysis: Homework 2

1. (a) Let $f(x, y) = \cosh x \cosh y$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned} F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\ F''(h) &= \nabla^2 f(h\vec{v})(\vec{v}, \vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

and

$$F(0) = 1 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2$$

Thus the polynomial of second degree that best approximate $f(x, y)$ is $1 + \frac{1}{2}(x^2 + y^2)$.

- (b) Let $g(x, y) = \sin(x^2 + y^2)$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$G(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned} G'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\ G''(h) &= \nabla^2 g(h\vec{v})(\vec{v}, \vec{v}) \\ &= x^2(2 \cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\ &\quad - 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\ &\quad + y^2(2 \cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2)) \end{aligned}$$

and

$$G(0) = 0 \quad G'(0) = 0 \quad G''(0) = 2x^2 + 2y^2$$

Thus the polynomial of second degree that best approximate $g(x, y)$ is $x^2 + y^2$.

2. (a) We observe that $f(x, y)$ is continuous at all points $(x, y) \neq (0, 0)$ since the denominator is nonzero. Thus we need to show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ to show $f(x, y)$ is everywhere continuous.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} xy \frac{x^2}{x^2 + y^2} - \lim_{(x,y) \rightarrow (0,0)} xy \frac{y^2}{x^2 + y^2}$$

then since,

$$\begin{aligned} 0 \leq \frac{x^2}{x^2 + y^2} \leq 1, \quad -|xy| \leq xy \leq |xy| \\ -|xy| \leq xy \frac{x^2}{x^2 + y^2} \leq |xy| \end{aligned}$$

and $\lim_{(x,y) \rightarrow (0,0)} \pm |xy| = 0$, which by Squeeze theorem gives us $\lim_{(x,y) \rightarrow (0,0)} xy \frac{x^2}{x^2+y^2} = 0$. We use the same argument and get the limit to be zero for the second term which shows the desired. We have $\nabla f = (f_x, f_y)$, which are given below,

$$f_x(x, y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}, \quad f_y(x, y) = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2}$$

Again it suffices to show that it is continuous at $(0, 0)$. We see that

$$f_x = y \left(1 - \frac{2y^4}{(x^2 + y^2)^2} \right), \quad f_y = -x \left(1 - \frac{2x^4}{(x^2 + y^2)^2} \right)$$

and $-1 \leq 1 - \frac{2y^4}{(x^2+y^2)^2}, 1 - \frac{2x^4}{(x^2+y^2)^2} \leq 1$. Thus

$$\begin{aligned} -y &\leq y \left(1 - \frac{2y^4}{(x^2 + y^2)^2} \right) \leq y \\ -x &\leq -x \left(1 - \frac{2x^4}{(x^2 + y^2)^2} \right) \leq x \end{aligned}$$

which by Squeeze theorem, we get $\lim_{(x,y) \rightarrow (0,0)} f_x = 0 = \lim_{(x,y) \rightarrow (0,0)} f_y$.

(b) The computation gives,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} = \frac{\partial^2 f}{\partial y \partial x}$$

To show it is continuous everywhere we have to show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}$ exists and is well defined along different paths.

$$\text{Along } y = x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} = 0$$

$$\begin{aligned} \text{Along } y = 2x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 36x^6 - 144x^6 - 64x^5}{(x^2 + 4x^2)^3} \\ &= -171/125 \end{aligned}$$

3. We recall the geometric series,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{where } |x| < 1$$

$$\text{substituting } x \text{ with } -x, \quad \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{where } |x| < 1$$

$$\text{substituting } x \text{ with } x^2, \quad \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad \text{where } |x| < 1$$

we can then do integration term wise on the right hand side while integrating $\frac{1}{1+x^2}$,

$$\begin{aligned} \pi/4 = \tan^{-1}(1) &= \int_0^1 \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{2k} dt \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} \end{aligned}$$

$$\text{thus } \pi = 4 \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right).$$

4. (a) Let $f(t) = e^{-\frac{(x-t)^2}{2}} = e^{-x^2/2} \cdot e^{xt-t^2/2}$. Then $f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$, where

$$\begin{aligned} f^{(k)}(0) &= \left. \frac{d^k}{dt^k} e^{-\frac{(x-t)^2}{2}} \right|_{t=0} \\ &= (-1)^k \left. \frac{d^k}{du^k} e^{-\frac{u^2}{2}} \right|_{u=x} \quad \text{letting } u = x - t, \text{ so } \frac{d}{du} = -\frac{d}{dt} \\ &= (-1)^k \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \end{aligned}$$

then

$$\begin{aligned} e^{-x^2/2} \cdot e^{xt-t^2/2} &= \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \\ e^{xt-t^2/2} &= e^{x^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \\ &= \sum_{k=0}^{\infty} t^k H_k(x) \end{aligned}$$

It converges because it is a Taylor series of an exponential which has radius of convergence, $R < \infty$.

(b) (i)

$$\begin{aligned} \frac{d}{dx} H_n(x) &= x H_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{x^2}{2}} \\ &= x H_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (-x e^{-\frac{x^2}{2}}) \\ &= x H_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \left[-x \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} - n \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}} \right], \quad \text{by General Leibniz Rule} \\ &= \frac{(-1)^{n-1}}{(n-1)!} e^{\frac{x^2}{2}} \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}} = H_{n-1}(x) \end{aligned}$$

(ii)

(iii) Let $y = -x$, then

$$\begin{aligned} H_n(y) &= \frac{(-1)^n}{n!} e^{\frac{y^2}{2}} \frac{d^n}{dy^n} e^{-\frac{y^2}{2}} \\ &= \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} (-1)^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad \text{since } \frac{d}{dy} = -\frac{d}{dx} \\ &= (-1)^n H_n(x) \end{aligned}$$

(c) (i)

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{sx-\frac{s^2}{2}} e^{tx-\frac{t^2}{2}} e^{-\frac{x^2}{2}} dx = e^{st} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-(s+t))^2}{2}} dx = e^{st}$$

(ii) Applying the partial derivatives $\left. \frac{\partial^{n+m}}{\partial s^n \partial t^m} \right|_{s,t=0}$ to (i), we get

$$\begin{aligned} 0 &= \left. \frac{\partial^{n+m}}{\partial s^n \partial t^m} \right|_{s,t=0} e^{st} = \left. \frac{\partial^{n+m}}{\partial s^n \partial t^m} \right|_{s,t=0} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{sx-\frac{s^2}{2}} e^{tx-\frac{t^2}{2}} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left. \frac{\partial^n}{\partial s^n} \right|_{s=0} e^{sx-\frac{s^2}{2}} \left. \frac{\partial^m}{\partial t^m} \right|_{t=0} e^{tx-\frac{t^2}{2}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

and since from 4(a),

$$\left. \frac{\partial^n}{\partial s^n} \right|_{s=0} e^{sx - \frac{s^2}{2}} = H_n(x)$$

we are done.