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# Real Analysis: Homework 2

1. (a) Let $f(x,y) = \cosh x \cosh y$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned}
F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\
F''(h) &= \nabla^2 f(h\vec{v})(\vec{v},\vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
F'''(h) &= \nabla^3 f(h\vec{v})(\vec{v},\vec{v},\vec{v}) = \sum_{i,j,k=1,2} \frac{f(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\
&= (x^3 + 3xy^2)(\sinh hx \cosh hy) + (y^3 + 3x^2 y)(\cosh hx \cosh hy)
\end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2 \quad F'''(0) = y^3 + 3x^2 y$$

Thus the polynomial of third degree that best approximate $f(x,y)$ is $\frac{1}{2}(x^2 + y^2) + \frac{1}{6}(y^3 + 3x^2 y)$.

(b) Let $g(x,y) = \sin(x^2 + y^2)$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$G(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned}
G'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\
G''(h) &= \nabla^2 g(h\vec{v})(\vec{v},\vec{v}) \\
&= x^2(2\cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\
&- 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\
&+ y^2(2\cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2)) \\
G'''(h) &= \nabla^3 g(h\vec{v})(\vec{v},\vec{v},\vec{v}) = \sum_{i,j,k=1,2} \frac{g(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\
&= x^3(-8(hx)^3 \cos((hx)^2 + (hy)^2) - 12(hx) \sin((hx)^2 + (hy)^2)) \\
& y^3(-8(hy)^3 \cos((hx)^2 + (hy)^2) - 12(hy) \sin((hx)^2 + (hy)^2)) \\
&+ 3x^2 y(-8(hx)^2(hy) \cos((hx)^2 + (hy)^2) - 4(hy) \sin((hx)^2 + (hy)^2)) \\
&+ 3xy^2(-8(hx)(hy)^2 \cos((hx)^2 + (hy)^2) - 4(hx) \sin((hx)^2 + (hy)^2))
\end{aligned}$$

and

$$G(0) = 0 \quad G'(0) = 0 \quad G''(0) = 2x^2 + 2y^2 \quad G'''(0) = 0$$

Thus the polynomial of third degree that best approximate $g(x,y)$ is $x^2 + y^2$.

2. (a) We observe that $f(x,y)$ is continuous at all points $(x,y) \neq (0,0)$ since the denominator is nonzero. Thus we need to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ to show $f(x,y)$ is everywhere continuous.

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} xy \cdot \lim_{(x,y)\to(0,0)} \left( \frac{1}{1+(y/x)^2} + \frac{1}{1+(x/y)^2} \right)$$

Let $z = x/y$, then $h(z) = 1/(1+z^2)$ is a continuous function for all $x, y \in \mathbb{R}$. Thus if $\lim_{(x,y)\to(0,0)} z = z_0$, $\lim_{(x,y)\to(0,0)} h(z) = h(z_0)$. Using L' Hopital's Rule, we get $z_0 = 1$ and thus the limit of the second term above is 2. Since the limit of the first term above is 0, we show that as $(x,y) \to (0,0)$, $f(x,y) \to 0$. Thus $f(x,y)$ is everywhere continuous.

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, \qquad f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

(b) The computation gives,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \frac{\partial^2 f}{\partial y \partial x}$$

To show it is continuous everywhere we have to show that $\lim_{(x,y)\to(0,0)} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$ exists and is well defined along different paths.

$$\begin{aligned}
\text{Along } y = x, \quad &\lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = 0 \\
\text{Along } y = 2x, \quad &\lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \lim_{(x,y)\to(0,0)} \frac{x^6 + 36x^6 - 144x^6 - 64x^5}{(x^2+4x^2)^3} \\
&= -171/125
\end{aligned}$$

3. We recall the geometric series,

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \ , \qquad \text{where } |x| < 1 \\
\text{substituting } x \text{ with } -x \ , \frac{1}{1+x} &= \sum_{k=0}^{\infty} (-1)^k x^k \ , \qquad \text{where } |x| < 1 \\
\text{substituting } x \text{ with } x^2 \ , \frac{1}{1+x^2} &= \sum_{k=0}^{\infty} (-1)^k x^{2k} \ , \qquad \text{where } |x| < 1
\end{aligned}$$

we can then do integration term wise on the right hand side while integrating $\frac{1}{1+x^2}$,

$$\begin{aligned}
\pi/4 = \tan^{-1}(1) = \int_0^1 \frac{1}{1+t^2}\, dt &= \sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{2k}\, dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}
\end{aligned}$$

thus $\pi = 4\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$.

4. (a) We will show the result by doing Taylor's expansion at 0, which requires us to compute the higher derivatives of $f(t) = e^{tx-\frac{t^2}{2}}$.

$$\begin{aligned}
f'(t) &= (x-t)f(t) = P_1(t)f(t) \\
f''(t) &= \left[(x-t)^2 - 1\right] f(t) = P_2(t)f(t) \\
f'''(t) &= \left[(x-t)^3 - 3(x-t)\right] f(t) = P_3(t)f(t) \\
&\vdots \\
f^{(n)}(t) &= P_n(t)f(t)
\end{aligned}$$

then we would now like to show that $P_i(t) = P'_{i-1}(t) + (x-t)P_{i-1}(t)$, $P_0 = 1$. It is clear that it is true for $i = 1$. Suppose it is true for $i = k$, $f^{(k)}(t) = P_k(t)f(t)$, then

$$\begin{aligned}
f^{(k+1)}(t) &= P'_k(t)f(t) + P_k(t)f'(t) \\
&= P'_k(t)f(t) + (x-t)P_k(t)f(t)
\end{aligned}$$

which completes the proof by induction. We know that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, then

$$\begin{aligned}
H_n(x) &:= \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \\
e^{tx-\frac{t^2}{2}} &= \sum_{k=0}^{\infty} \frac{(tx - \frac{t^2}{2})^k}{k!} \\
&= \sum_{k=0}^{\infty} t^k \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}
\end{aligned}$$

(b) (i)

$$H'_n(x =)$$

(c)