Zhangsheng Lai
1002554

Real Analysis: Homework 2

1. (a) Let $f(x,y) = \cosh x \cosh y$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned} F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\ F''(h) &= \nabla^2 f(h\vec{v})(\vec{v},\vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ F'''(h) &= \nabla^3 f(h\vec{v})(\vec{v},\vec{v},\vec{v}) = \sum_{i,j,k=1,2} \frac{f(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\ &= (x^3 + 3xy^2)(\sinh hx \cosh hy) + (y^3 + 3x^2 y)(\cosh hx \cosh hy) \end{aligned}$$

and

$$F(0) = 0 \quad F'(0) {=}\ 0 \quad F''(0) = x^2 + y^2$$

Thus the polynomial of second degree that best approximate $f(x,y)$ is $\frac{1}{2}(x^2+y^2)$.

(b) Let $g(x,y) = \sin(x^2+y^2)$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned} F'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\ F''(h) &= \nabla^2 g(h\vec{v})(\vec{v},\vec{v}) \\ &= x^2(2\cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\ &\quad - 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\ &\quad + y^2(2\cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2)) \end{aligned}$$

and

$$F(0) = 0 \quad F'(0) {=}\ 0 \quad F''(0) = 2x^2 + 2y^2$$

Thus the polynomial of second degree that best approximate $g(x,y)$ is x^2+y^2.

2. (a)

(b)

$$\frac{\partial^2 f}{\partial x \partial y}$$

3.

4. (a)

(b) (i)

$$H_n'(x =)$$

(c)