

Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that $f(y, z) = 5y^2 + z^2 - 3zy - 3y \geq 0$ for every $y, z \in \{0, 1, 2, \dots\}$. We shall use $\mathbb{Z}_{>0}$ to denote the set $\{0, 1, 2, \dots\}$ subsequently. We can rewrite $f(y, z)$ to get

$$f(y, z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \quad (1)$$

which we will show that the $f(y, z)$ in this form is nonnegative. All that is left to show is that $\frac{11}{4}y^2 - 3y \geq 0$ for all $y \in \mathbb{Z}_{>0}$ but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (1) holds for all $z \in \mathbb{Z}_{>0}$ when $y = 1$. Solving for the inequality below,

$$\begin{aligned} f(1, z) &= \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 \Leftrightarrow (z - 1)(z - 2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that $f(1, z) < 0$ for $z \in (1, 2)$ and hence positive for all $z \in \mathbb{Z}_{>0}$ and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let \mathcal{P}_i denote the set of paths from an origin s_i to a sink t_i with $\mathcal{P}_i \neq \emptyset$.

Definition (flow). For a flow f and path $P \in \mathcal{P}_i$, f_P is the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i . A flow is feasible for a vector $r = (r_1, \dots, r_k)$ if it routes all the traffic: for each $i \in \{1, 2, \dots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

Definition (Nonatomic equilibrium flow). Let f be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow f is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \dots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_P(f) \leq c_{\tilde{P}}(f)$$

where $c_P(f)$ denotes the cost of travelling on path P for flow f .

- (ii) Let $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$. Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_P(f_P) \cdot f_P = \sum_{e \in E} c_e(f_e) \cdot f_e$$

where E is the set of directed edges on the graph G .

(iii)

(iv)

3. (i) Let Φ be the potential function of a potential game and c_i denote the cost function of the agents for $i \in \{1, 2, \dots, k\}$. To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of s_i and solely dependent on s_{-i} . If we consider two alternative distinct strategies for agent i , $s'_i, s''_i \neq s_i$

$$c_i(s_i, s_{-i}) = \Phi(s_i, s_{-i}) + [c_i(s'_i, s_{-i}) - \Phi(s'_i, s_{-i})]$$

$$c_i(s_i, s_{-i}) = \Phi(s_i, s_{-i}) + [c_i(s''_i, s_{-i}) - \Phi(s''_i, s_{-i})]$$

hence we can choose $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$, where $-$ represents any choice of strategy of agent i .

- (ii) Let Φ_1 and Φ_2 be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \quad (2)$$

$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \quad (3)$$

where $D_i^k(s_{-i})$ denotes the dummy term for $k = 1, 2$. Taking (2)–(3),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at s_{-i} and any strategy of agent i , s_i .

- (iii) (\Rightarrow): Let Φ be a potential function for a finite game. We want to show:

$$c_i(s_i^2, s_{-i}^1) - c_i(s_i^1) + c_j(s_j^2) - c_j(s_j^1, s_{-j}^1) = c_j(s_j^2, s_{-j}^1) - c_j(s_j^1) + c_i(s_i^2) - c_i(s_i^2, s_{-j}^1)$$

consider the left hand side,

$$c_i(s_i^2, s_{-i}^1) - c_i(s_i^1) + c_j(s_j^2) - c_j(s_j^1, s_{-j}^1) = \Phi(s_i^2, s_{-i}^1) - c_i(s_i^1) + c_j(s_j^2) - c_j(s_j^1, s_{-j}^1)$$

(\Leftarrow):

4. (a) Let \tilde{f} be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player $i \in \{1, 2, \dots, n\}$, any two parallel links P_i, P_j where $1 \leq i < j \leq k$,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of \tilde{f} on P_i equals the flow of f on P_j , ($\tilde{f}_{P_i} = f_{P_j}$) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

since the above inequality is true for any two parallel links, we sum it over the n parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^n \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^n \sum_{m=1}^{f_{P_i}} c_{P_i}(m) = \Phi(f)$$

where Φ is the potential function.

(b)

5. (a) Let G^1 and G^2 be a partnership game and congestion game respectively and suppose that they have the same number of players k . For the bijection f_i to exist from the strategies A_i of player i in G^1 to the strategies B_i of player i in G^2 , we require $|A_i| = |B_i|$ for all $i \in \{1, \dots, k\}$.