

## Real Analysis: Homework 3

1. A function which is in  $\mathcal{C}^1(\mathbb{R})$  but not in  $\mathcal{C}^2(\mathbb{R})$  means a function that has continuous first derivative but its second derivative is not continuous. Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

we see that the first derivate of  $f(x)$  is continuous but the second derivate is not continuous at  $x = 0$ .

2. By Stone-Weierstrass Theorem, for the given function  $f(x)$ , there exists a sequence of polynomials  $P_n(x)$  that converges uniformly to  $f(x)$ , i.e.  $\sup_{x \in [0,1]} |f(x) - P_n(x)| < 1/n$  as  $n \rightarrow \infty$ . Now, we consider the inner product  $\langle f(x), g(x) \rangle := \int_0^1 f(x)g(x) dx$ , and if we managed to show that  $\langle f(x), f(x) \rangle = 0$ , we are done. To show that,

$$\begin{aligned} \left| \int_0^1 f(x)^2 dx \right| &= \left| \int_0^1 f(x)^2 dx - \int_0^1 f(x)P_n(x) dx \right|, \text{ since } \int_0^1 f(x)x^n dx = 0 \text{ for all } n. \\ &\leq \int_0^1 |f(x)| |f(x) - P_n(x)| dx \\ &\leq \int_0^1 |f(x)| dx \cdot \sup_{x \in [0,1]} |f(x) - P_n(x)| \\ &\leq \frac{1}{n} \int_0^1 |f(x)| dx \text{ for all } n. \end{aligned}$$

Thus we need  $\langle f(x), f(x) \rangle = 0$ , which completes the proof.

3. Let  $\phi$  be  $\lambda$ -Hölder bi-continuous then for  $v_1, v_2, u_1, u_2 \in T$ , we have

$$\begin{aligned} \sup_{v \in [0,T]} |\phi(u_2, v) - \phi(u_1, v)| &\leq C_u |u_2 - u_1|^\lambda \\ \sup_{u \in [0,T]} |\phi(u, v_2) - \phi(u, v_1)| &\leq C_v |v_2 - v_1|^\lambda \end{aligned}$$

then we also observe that

$$\begin{aligned} |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_1, v_2)| + |\phi(u_2, v_2) - \phi(u_2, v_1)| \\ |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_2, v_1)| + |\phi(u_2, v_2) - \phi(u_1, v_2)| \end{aligned}$$

which gives us

$$\begin{aligned} |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_v |v_2 - v_1|^\lambda \\ |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_u |u_2 - u_1|^\lambda \end{aligned}$$

multiplying them together, we have

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)|^2 \leq 4C_v C_u |v_2 - v_1|^\lambda |u_2 - u_1|^\lambda$$

squaring both sides, we have shown that all  $\lambda$ -Hölder bi-continuous are strongly  $\lambda/2$ -Hölder bi-continuous.

4. (a) For  $0 \leq a < b \leq T$  with  $0 \leq \alpha < 1/4$  and  $b \leq r_1 \leq T$ , we first make the following observations:

$$(r_1 - b)^{3/4} \leq (r_1 - a)^{3/4} \quad (1)$$

$$\frac{1}{(r_1 - a)^{\alpha+1/4}} \leq \frac{1}{(b - a)^{\alpha+1/4}} \quad (2)$$

then

$$\begin{aligned} \int_b^T \frac{1}{(r_1 - b)^\alpha (r_1 - a)^{\alpha+1}} dr_1 &= \int_b^T \frac{(r_1 - b)^{-\alpha-3/4+3/4}}{(r_1 - a)^{\alpha+1}} dr_1 \\ &\leq \int_b^T \frac{(r_1 - b)^{-\alpha-3/4} (r_1 - a)^{3/4}}{(r_1 - a)^{\alpha+1}} dr_1 \\ &= \int_b^T \frac{(r_1 - b)^{-\alpha-3/4}}{(r_1 - a)^{\alpha+1/4}} dr_1 \\ &\leq \frac{1}{(b - a)^{\alpha+1/4}} \int_b^T (r_1 - b)^{-\alpha-3/4} dr_1 \\ &= \frac{1}{(b - a)^{\alpha+1/4}} \left[ \frac{(r_1 - b)^{1/4-\alpha}}{1/4 - \alpha} \right]_b^T \\ &= C \frac{(T - b)^{1/4-\alpha}}{(b - a)^{\alpha+1/4}} \text{ for some constant } C. \end{aligned}$$

- (b) Given  $\psi(u, v) := \mathbb{1}_{[0, v)}(u) \tilde{\psi}(u, v)$  we can understand it as

$$\psi(u, v) := \begin{cases} \tilde{\psi}(u, v) & u < v \\ 0 & u \geq v \end{cases}$$

then to do the double integral of  $f(u, v)^2$  over  $R = [0, T] \times [0, T]$  is equivalent to integrating over the region  $\{(u, v) \in R \mid u < v\}$ . Thus

$$\int_0^T \int_0^T f(u, v)^2 du dv = \int_0^T \int_0^v \left| \int_u^T \int_v^T \frac{\tilde{\psi}(u, v) - \tilde{\psi}(u, r_2) - \tilde{\psi}(r_1, v) + \tilde{\psi}(r_1, r_2)}{(r_1 - u)^{1+\alpha} (r_2 - v)^{1+\alpha}} dr_2 dr_1 \right|^2 du dv$$

We will now massage the term inside the first double integral,

$$\begin{aligned} &\left| \int_u^T \int_v^T \frac{\tilde{\psi}(u, v) - \tilde{\psi}(u, r_2) - \tilde{\psi}(r_1, v) + \tilde{\psi}(r_1, r_2)}{(r_1 - u)^{1+\alpha} (r_2 - v)^{1+\alpha}} dr_2 dr_1 \right| \\ &\leq \int_u^T \int_v^T \left| \frac{\tilde{\psi}(u, v) - \tilde{\psi}(u, r_2) - \tilde{\psi}(r_1, v) + \tilde{\psi}(r_1, r_2)}{(r_1 - u)^{1+\alpha} (r_2 - v)^{1+\alpha}} \right| dr_2 dr_1 \\ &\leq \int_u^T \int_v^T \frac{|\tilde{\psi}(u, v) - \tilde{\psi}(u, r_2) - \tilde{\psi}(r_1, v) + \tilde{\psi}(r_1, r_2)|}{|(r_1 - u)^{1+\alpha} (r_2 - v)^{1+\alpha}|} dr_2 dr_1 \\ &\leq \int_u^T \int_v^T \frac{|(r_1 - u)^\lambda (r_2 - v)^\lambda|}{|(r_1 - u)^{1+\alpha} (r_2 - v)^{1+\alpha}|} dr_2 dr_1, \quad \text{by strongly } \lambda\text{-H\"older bi-continuous property} \end{aligned}$$

but I'm stuck.