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Algebraic Geometry: Homework 1

1. R be a ring and S a multiplicative subset of R with $1 \in S$ and $0 \notin S$

   (i) It is reflexive, since for any $t \in S$, $t(rs - rs) = 0$, thus $(r, s) \sim (r, s)$. Suppose $(r, s) \sim (r', s')$, so there exist $t \in S$ such that $t(rs' - r's) = 0$ which also means $t(r's - rs') = 0$ and we have symmetry. Lastly, let $(r, s) \sim (t, u)$ and $(t, u) \sim (v, w)$, then there exist $a, b \in S$ such that

   $$a(ru - ts) = 0$$
   $$b(tw - vu) = 0$$

   Then $abwru - abwts = 0$, $bastw - basvu = 0$ and summing them gives $abu(rw - vs) = 0$ with $abu \in S$ which shows transitivity.

   (ii) Same as part (i).

   (iii) Define the action of R_s on M_s, $\phi : R_s \times M_s \to M_s$, $((r, s), (m, s')) \mapsto (rm, ss')$. We first show that the action is well-defined. Let $(r_1, s_1) \sim (r_2, s_2) \in R_s$ and $(m_1, t_1) \sim (m_1, t_1) \in M_s$. Then $\phi((r_i, s_i), (m_i, t_i)) = (r_i m_i, s_i t_i)$ for $i = 1, 2$. Since $a(r_1 s_2 - r_2 s_1) = 0$ and $b(m_1 t_2 - m_2 t_1) = 0$ with some $a, b \in S$,

   $$ab(r_1 s_2 m_1 t_2 - r_2 s_1 m_1 t_2) = 0$$
   $$ab(r_2 s_1 m_1 t_2 - r_2 s_1 m_2 t_1) = 0$$

   thus we have $ab(r_1 m_1 s_2 t_2 - r_2 m_2 s_1 t_1)$, so $(r_1 m_1, s_1 t_1) \sim (r_2 m_2, s_2 t_2)$. M_s is a R_s-module, since for $m_i \in M$, $r_i \in R$ and $s_i, t_i \in S$,

   •$(r, s)((m_1, t_1) + (m_2, t_2)) = (r, s)((m_1 t_2 + m_2 t_1, t_1 t_2)) = (rm_1 t_2 + rm_2 t_1, st_1 t_2) = (rm_1, st_1) + (rm_2, st_2)$
   •$((r_1, s_1) + (r_2, s_2))(m, t) = (mr_1 s_2 + mr_2 s_1, s_1 s_2 t) = (r_1 m, s_1 t) + (r_2 m, s_2 t)$
   •$((r_1, s_1)(r_2, s_2))(m, t) = (r_1 r_2 m, s_1 s_2 t) = (r_1, s_1)(r_2 m, s_2 t) = ((r_1, s_1)(r_2, s_2))(m, t)$
   •$(1_R, 1_S)(m, t) = (m, t)$

2. Given morphisms of R-modules, $\phi : M \to N$ and $\psi : N \to P$, it is an *exact sequence* if the image of ϕ is equal to the kernel of ψ in N.

   (i) If $M = 0_M$, then $\phi(0_M) = \{0_N\} = \ker(\psi)$, thus ψ is injective.

   (ii) If $P = 0_P$, then $\phi(M) = \ker(\psi) = N$, thus ϕ is surjective.

   (iii) For a prime ideal $\mathfrak{p}$ with $S = R - \mathfrak{p}$, and $R_S = R_\mathfrak{p}$, $M_S = M_\mathfrak{p}$. Let $p(r, s) \in \mathfrak{p}R_\mathfrak{p}$ with $p \in \mathfrak{p}$ and $(r, s) \in R_\mathfrak{p}$. It is an ideal since, $p(r, s)(r', s') = p(rr', ss') \in \mathfrak{p}R_\mathfrak{p}$.
   $\mathfrak{p}$ does not contain any unit of R, else $\mathfrak{p} = R$ and $S = \varnothing$. Thus for $(p, s) \in \mathfrak{p}R_\mathfrak{p}$, it is not a unit in $R_\mathfrak{p}$ and thus it is a proper ideal. It is maximal since for any $(r, s) \in R_\mathfrak{p} - \mathfrak{p}R_\mathfrak{p}$, it is a unit of $R_\mathfrak{p}$ since $r \in S$ and its inverse is (s, r). Thus $\mathfrak{p}R_\mathfrak{p} + ((r, s)) = R_\mathfrak{p}$ for any $(r, s) \in R_\mathfrak{p} - \mathfrak{p}R_\mathfrak{p}$

   (iv) Given the natural maps $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ and $\psi_\mathfrak{p} : N_\mathfrak{p} \to P_\mathfrak{p}$ given by $(m, s) \mapsto (\phi(m), s)$ and $(n, s) \mapsto (\psi(n), s)$. Let $(m_1, s_1) \sim (m_2, s_2)$ thus $t(m_1 s_2 - m_2 s_1) = 0_M$ for some $t \in S$. Then $\phi(t(m_1 s_2 - m_2 s_1)) = t(\phi(m_1)s_2 - \phi(m_2)s_1) = \phi(0_M) = 0_N$. Thus $(\phi(m_1), s_1) \sim (\phi(m_2), s_2)$. Similar argument for $\psi_\mathfrak{p}$.

   (v) Let $(\phi(m), s) \in \phi_\mathfrak{p}(M_\mathfrak{p})$, then $(\psi_\mathfrak{p} \circ \phi_\mathfrak{p}(m), s)$

3. $R_\mathfrak{p}$

4.

5. (i) Let S be any subset of $k[x_1, \ldots, x_n]$. For every $\mathbf{a} \in V(S)$, $F(\mathbf{a}) = 0$ for every $F \in S$. Thus $F \in I(V(S))$ for every $F \in S$ and $S \subseteq I(V(S))$.

   (ii) $V(I(V(S))) = V(S)$

6. (i) Let $X \subseteq A^n(k)$, then $I(X) := \{F \in k[x_1, \ldots, x_n] | F(\mathbf{a}) = 0 \text{ for all } \mathbf{a} \in X\}$. For each $F^n \in I(X)$ where $n > 0$ integer, suppose $F \notin I(X)$, i.e. there exists $\mathbf{a} \in X$ such that $F(\mathbf{a}) \neq 0$. But $(F(\mathbf{a}))^n = F^n(\mathbf{a}) = 0$ which implies $F(\mathbf{a}) \in k$ is a zero divisor, a contradiction since fields do not have zero divisors. Thus $F^n \in I(X)$ and shows $I(X) \supseteq \mathrm{Rad}(I(X))$. The other containment is obvious since for $F \in I(X), F \in \mathrm{Rad}(I(X))$ by choosing $n = 1$.

(ii) Let $X \subseteq A^n(k)$, then for any $F \in I(X)$, $F(\mathbf{a}) = 0$ for all $\mathbf{a} \in X$, thus $\mathbf{a} \in V(I(X))$, thus $V(I(X)) \supseteq X$.

(iii) Since $I(X) = I$ is a radical ideal, by Hilbert's Nullstellensatz, $I(V(I)) = \text{Rad}(I) = I$.

7. (i) Let J be an ideal of R and $\pi(J) := \{j + I | j \in J\}$. Thus for $r + I \in R/I, j + I \in \pi(J)$, $(j + I)(r + I) = jr + I$ and $(r + I)(j + I) = rj + I$ are both in $\pi(J)$ since $rj, jr \in J$. It is easy to see that $\pi(J)$ is closed under addition, and $\pi(J)$ is an ideal of R/I.

(ii) Let J' be an ideal of R/I and $\pi^{-1}(J') := \{j \in R | j + I \in J'\}$. Let $j, j' \in \pi^{-1}(J')$, then $j - j' \in \pi^{-1}(J')$ as $(j - j') + I \in J'$. Also for $r \in R, j \in \pi^{-1}(J')$, $rj, jr \in \pi^{-1}(J')$ as $rj + I, jr + I \in J'$. Thus $\pi^{-1}(J')$ is an ideal in R. $J \supseteq I$ as $0_{R/I} \in J'$ and thus $\pi^{-1}(0_{R/I}) \supseteq I$.

(iii) To show the bijection, we have to show that $\pi \circ \pi^{-1} = 1_{R/I}$ and $\pi^{-1} \circ \pi = 1_R$. For $J' \subseteq R/I$ ideal, $\pi^{-1}(J') := \{j \in R | j + I \in J'\}$, thus $\pi(j) = j + I \in J'$ for $j \in \pi^{-1}(J')$ and so $\pi \circ \pi^{-1}(J') = J'$. Now let $J \subseteq R$ ideal, $\pi(J) := \{j + I | j \in J\}$ and so $\pi^{-1}(j + I) = j \in J$ so $\pi^{-1} \circ \pi = 1_R$.

Since we have a one to one correspondence between {Ideals $J \supseteq I$} and {Ideals $J' \subseteq R/I$}, if ...

(iv) Let J' be a radical ideal, i.e. $J' = \text{Rad}(J') := \{r + I \in R/I | r^n + I \in J'$ for some integer $n > 0\}$. Then $J = \pi^{-1}(J') := \{j \in R | j + I \in J'\}$ ideal. Take $j^n \in J$ for some integer $n > 0$ and $\pi(j^n) = j^n + I \in J'$, which also implies $j + I \in J'$ since J' is a radical ideal. Thus $j = \pi^{-1}(j + I) \in J$, i.e. J is a radical ideal. Conversely, let $J \subseteq R$ radical ideal. Then for $j^n + I \in J'$, $\pi^{-1}(j^n + I) = j^n \in J$ and so $j \in J$. Hence, $\pi(j) = j + I \in J'$ which proves J' is a radical ideal.

Let J' be a prime ideal, then for $ab + I \in J'$, either $a + I$ or $b + I$ is in J'. Let $cd \in \pi^{-1}(J')$ ideal, then $\pi(cd) = cd + I \in J'$, thus $c + I$ or $d + I$ is in J'. Thus $\pi^{-1}(c + I) = c \in J$ or $\pi^{-1}(d + I) = d \in J$, thus $\pi^{-1}(J')$ is also a prime ideal. Conversely, let $J \subseteq R$ prime ideal. Take $ab + I \in J'$, then $ab = \pi^{-1}(ab + I) \in J$. Thus $a \in J$ or $b \in J$ and we have $a + I \in J'$ or $b + I \in J'$. Thus J' is a prime ideal.

The proof for maximal ideals follows from the result from (iii). Suppose J' is maximal, thus for $J' \subseteq K' \subseteq R/I$, $K' = J'$ or $K' = R/I$. So if $J = \pi^{-1}(J')$ is not maximal, there exists a $K \neq J, R$ such that $J \subseteq K \subseteq R$, then $\pi(K)$ is an ideal in R/I that will contradict the maximality of J'. The converse direction follows the same argument.