Literature Review: Real-Time Dynamic Pricing for Multiproduct Models with Time-Dependent Customer Arrival Rates

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1 Introduction

In the single product dynamic pricing model introduced in class, we looked at a monopolist seller who sells a single indivisible product which has finite amount of units with no replenishment over a continuous time horizon $[0, T]$. The unit price of the product, π_t, is decided by the seller at each point of time $t \in [0, T]$. Besides, customers' product valuations follow a distribution over $\mathbb{R}_+$. However, in practice, sellers often have a wide range of products with different features to cater to customers' varying preferences and purchasing powers.

Thus we look at this multiproduct model [3] which has a long supply lead time and short selling season with substitutes available when replenishment is not allowed. For example, if a customer' favorite product is not available and stock won't be replenished, then very likely the customer would like to find a substitute. Thus the retailer can control the prices of corresponding products upon each point of time to maximize his or her revenue. Instead of considering customers' product valuation as some random variable, this paper explicitly constructs a model that sees customer purchase probability as a function of quality and price. Also, a time dependent customer arrival rate is modeled by a nonhomogeneous Poisson process and the data of this paper is forecasted correspondingly. Lastly, this paper assumes that the products are equally dissimiliar, which means the addition or removal of one affects the choice for all other products with the same portion.

2 The Multinomial Logit (MNL) Model

2.1 Summary of the MNL model

The MNL model is one of the most commonly used models to capture customers' preferences. It describes preferences dynamically as prices vary. A customer makes choices from a range of products to maximise his or her utility, which is represented by the features of products, such as price and quality. The utility of product i is defined by the logit demand function $v^i(r^i) = \exp((q^i - r^i)/\mu)$, which is a positive function defined by quality q^i and price r^i. And μ is a constant representing the stochastic preference of the choice process. Hence, the customer's expected demand probability of product i is defined as

$$P^i(r) = \frac{v^i(r^i)}{v^0 + \sum_{j=1}^n v^j(r^j)}, \quad i = 1, \dots n \tag{1}$$

and v^0 denotes the utility of not making any purchase.

Customer's arrivals are assumed to follow a nonhomogenous Poisson process with a time-dependent rate $\lambda(t)$. In a small time interval δt, the probability of one arrival is $\lambda(t)\delta t$. The prices of the products at time t is $r_t = (r_t^1, \dots, r_t^n) \in \mathbb{R}^n$ is related to the current inventory level. If certain product i is sold out before the end of the selling season, as we do not allow replenishments, its price is set to $r_t^i = \infty$ for all t occuring after $c_t^i = 0$. If all the products are sold upon time $t < T$, then the selling season ends at t. All unsold products are salvaged at $t = 0$.

Let $\mathbf{r} = \{r_t, t \in [T, 0]\} = (r^1, \dots, r^n)$ denote a pricing policy for the entire season, where r^i denotes the prices of product i over the selling season. We denote the probabilty upon a customer's arrival at time t to choose product $i \in \mathbf{n} = \{1, \dots, n\}$ by $P_t^i(r_t)$ while $P_t^0(r_t)$ denotes the no-purchase probability. Using (1) with the no-purchase utility $v^0 = \exp(u_0/\mu)$, we have the demand probability of product i and of no-purchase

$$P_t^i(r_t) = \frac{\exp((q^i - r_t^i)/\mu)}{\sum_{i \in \mathbf{n}} \exp((q_i - r_t^i)/\mu) + \exp(u_0/\mu)} \tag{2}$$

$$P_t^0(r_t) = \frac{\exp(u_0/\mu)}{\sum_{i \in \mathbf{n}} \exp((q_i - r_t^i)/\mu) + \exp(u_0/\mu)} \tag{3}$$

and by construction, $P_t^0(r_t) + \sum_{i \in \mathbf{n}} P_t^i(r_t) = 1$.

Hence, the revenue rate at time t is

$$\Phi_t(r_t) := \lambda(t) \sum_{i=1}^n r_t^i P_t^i(r_t) \tag{4}$$

By setting price r_t to take different values, the retailer controls the probability of customers' buying so the revenue rate is optimized. From (2) and (3) we obtain $P_t^i(r_t)/P_t^0 = \exp((q^i - r_t^i - u_0)/\mu)$. Furthermore, we have

$$r_t^i = q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0, \quad i = 1, \ldots, n \tag{5}$$

which is a mapping from the sales probability space to the price space. In (2) and (3) we have a mapping in the opposite direction. However, these two mappings are not inverses of each other as the mapping from the price space to the sales probability space is not injective and hence cannot be a bijection. This can be easily observed for a given set of pricing $r_t = (r_t^1, \ldots, r_t^n)$, choosing $r_t' = (r_t^1 \pm \ln k, \ldots, r_t^n \pm \ln k)$ for some positive real constant k will produce the same demand probability of product. Thus the vector of prices obtained from a given sales probability has to be chosen from a set of allowable prices [2].

Thus, by choosing from the set of allowable prices, we have a one-to-one mapping from the price space to the sales probability space. The revenue rate as in (4) can be rewritten as

$$\Phi_t(P_t) = \lambda(t) \sum_{i=1}^{n} \left[q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0\right] P_t^i \tag{6}$$

where $P_t = (P_t^1, \ldots, P_t^i)$. This is a very commonly used technique in this field to find the maximum revenue. As in the price space, revenue-price function is not quasi-concave which makes it difficult to observe the optimal solution.

2.2 Optimal control of the MNL model

Let N_t^i be the Poisson counting process for the number of products sold up to time t. If a customer arrives and chooses the product i at time t then $dN_t^i = 1$, otherwise $dN_t^i = 0$. $N_t = (N_t^1, \ldots, N_t^n)$ is defined to be a multivariate stochastic process controlled by the price vector r_t through the probability P_t. Given an initial inventory c_T, the expected revenue of the price scheme $\mathbf{r}$ for the entire selling season is

$$\mathbf{w}_T^{\mathbf{r}}(c_T) = \left(E_{\mathbf{r}} \left[\int_0^T r_t^i \, dN_t^i \right] \right)_{i=1,\ldots,n} \tag{7}$$

Using the relationship between pricing and sales probability, the paper we studies rewrites the expected reveue in terms of corresponding probability scheme $\mathbf{P} = \{P_t, t \in [T, 0)\}$ to get

$$\mathbf{w}_T^{\mathbf{P}}(c_T) = \left[\int_0^T \lambda(t) \sum_{i=1}^{n} P_t^i [q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0] \, dt \right] = \int_0^T \Phi_t(P_t) \, dt \tag{8}$$

By writing the expected revenue in terms of sales probabilities instead of pricing, it reduces the space from $\mathbb{R}_+^n$ to $[0,1]^n$. This allows the search of the optimal control and value to be much more managable (to remove or to justify it better) as we are searching for a solution from a probability simplex. The retailer searchs for a pricing policy $\hat{\mathbf{r}}$ that maximizes the expected revenue over the selling season, which is denoted as

$$\hat{\mathbf{w}}_T^{\mathbf{P}}(c_T) = \max_{P_t \in [0,1)^n, t\in[T,0)} \int_0^T \lambda(t) \sum_{i=1}^n P_t^i \left(q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0\right) dt \tag{9}$$

subject to $dc_t^i = -dN_t^i$, which simply means one reduce in the inventory is one increase in sales, or a unit change in the Poisson counting process. This formulation is observed to be a optimal control problem; our utility is $\mathbf{w}_T^{\mathbf{P}}(c_T)$, with scalar utility rate function $\Phi_t(\cdot)$, the system state vector is c_t and the control vector we are trying to find is P_t for some time t. If we used the expected revenue formulation in (6) instead, the control vector we are trying to find will be r_t. Now let

$$\mathbf{w}_t^{\mathbf{P}}(c_t) = \int_0^t \lambda(s) \sum_{i=1}^n P_s^i \left(q^i - u_0 - \mu \ln P_s^i + \mu \ln P_s^0\right) ds \tag{10}$$

and

$$\hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) = \max_{P_s \in [0,1)^n, s\in[t,0)} \mathbf{w}_t^{\mathbf{P}}(c_t) \tag{11}$$

The authors then solve the optimal control problem by applying the Hamilton-Jacobi-Bellman (HJB) sufficient conditions. The HJB sufficient conditions help us to find optimality by saying that the solution to the equation,

$$\frac{\partial \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)}{\partial t} = \max_{P_t \in [0,1)^n, t\in[T,0)} \sum_{i=1}^n \lambda(t) P_t^i \left(q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)\right) \tag{12}$$

will give us the optimal revenue rate, where $\Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) = \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) - \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t - e^i)$, is the marginal value of product i. And e^i is an n-dimensional vector with the i-th component being 1 and the rest zeros. Also $\hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)$ satisfies the boundary and initial conditions

$$\hat{\mathbf{w}}_t^{\mathbf{P}}(0) = 0 \quad \forall t \tag{13}$$
$$\hat{\mathbf{w}}_0^{\mathbf{P}}(c_0) = 0 \quad \forall c_0 \tag{14}$$

The boundary and initial conditions state that if there is zero inventory then at any time the optimal revenue rate is zero; and if the selling season has ended regardless of the amount of inventory left, the optimal revenue rate will also be zero. The HJB conditions are equivalent to maximizing the revenue rate $\Phi_t(P_t)$ at each t.

2.3 Optimal Dynamic Pricing Policy

To find the optimal dynamic pricing policy, we just need to find the maximizer P_t^* of the HJB equation in (12). Define

$$\Psi_t^{\mathbf{P}}(c_t) = \sum_{i=1}^{n} \lambda(t) P_t^i \left[q^i - u_0 = \mu \ln P_t^i + \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) \right]$$

and the fact that $\Psi_t^{\mathbf{P}}(c_t)$ is concave is derived from showing that its Hessian is negative and thus it is semi-negative definite. Thus it has a global maximum over P_t which can be found by setting below formula to zero:

$$\frac{\partial \Psi_t^{\mathbf{P}}(c_t)}{\partial P_t^i} = \lambda(t) \left[q^i - u_0 + \mu \ln P_t^i - \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) - \mu \right] \tag{15}$$

as we have $\sum_{i=1}^n P_t^i + P_t^0 = 1$, it suffices to find P_t^i for $i > 0$.

In the paper, they solved for the maximizer to be of the form

$$(P_t^i)^* = \frac{\mu}{m_t(c_t)} \exp\left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - m_t(c_t)}{\mu} \right) \tag{16}$$

where $i \in \mathbb{R}(c_t)$, the set of products with non-negative inventory and

$$m_t(c_t) = \frac{\mu}{1 - \sum_{i=1}^n (P_t^i)^*} = \frac{\mu}{P_t^{0*}} \tag{17}$$

By (16), (17) and the probability simplex constraint, we obtain

$$\left[\frac{m_t(c_t)}{\mu} - 1 \right] \exp\left[\frac{m_t(c_t) + u_0}{\mu} \right] = \sum_{i=1}^{n} \exp\left[\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)}{\mu} \right] \tag{18}$$

where $m_t(c_t)$ is the unique solution. This uniqueness comes from the concave property of $\Psi_t^{\mathbf{P}}(c_t)$ over all probability simplex P_t.

Using the results that we have derived, the optimal price scheme is

$$(r_t^i)^* = \Delta \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) + m_t(c_t) \tag{19}$$

and updating the HJB optimality sufficiency conditions,

$$\frac{\partial}{\partial t} \hat{\mathbf{w}}_t^{\mathbf{P}^*} = \lambda(t)(m_t(c_t) - \mu) \tag{20}$$

By integrating both sides with respect to t from 0 to T, the maximum expected revenue at the beginning of the selling season in terms of $m_t(c_t)$ is

$$\hat{\mathbf{w}}_t^{\mathbf{P}^*} = \int_0^T \lambda(t) \left[m_t(c_t) - \mu\right] dt \tag{21}$$

From the discussion above, we see that the solution to the optimal pricing policy depends on the term $m_t(c_t)$. However, $m_t(c_t)$ is found both in the base and exponent of (16) and thus there is no closed form solution for $m_t(c_t)$ and hence $(r_t^i)^*$ is also intractable.

2.3.1 Our observations

In our observations, contrary to (16), we arrived at a different result, shown as below

$$\begin{aligned}
(P_t^i)^* &= \frac{\mu}{m_t(c_t)} \exp\left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - \mu - m_t(c_t)(P_t^i)^*}{\mu}\right) \\
&= P_t^{0*} \exp\left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - \mu - m_t(c_t)(P_t^i)^*}{\mu}\right)
\end{aligned} \tag{22}$$

However, $\Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)$ relies on P_t^{0*} and $(P_t^i)^*$, thus we do not have a closed form solution for (22). Updating the equation in (18), we have the equation below.

$$\left[\frac{m_t(c_t)}{\mu} - 1\right] \exp\left[1 + \frac{u_0}{\mu}\right] = \sum_{i=1}^{n} \exp\left[\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)}{\mu}\right] \tag{23}$$

We observe that by letting

[1]

3 Dynamic Programming For Optimal Dynamic Pricing

4 Discussions and Conclusions

This is where we can add our comments and our inputs, how the model can be further improved or how we can find estimates for the solution. Closing conclusions, futher areas that can be explored and research opportunities.

References

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