Emmanouil Leontaris 1001984
Nguyen Tan Thai Hung 1001986
Zhangsheng Lai 1002554

Algorithmic Game Theory: HW 2

Question 1

Let G be a cost-minimization game which admits the function Φ such that, for every outcome s, every player i, and every possible deviation s'_i

$$C_i(s'_i, s_{-i}) < C_i(s) \implies \Phi(s'_i, s_{-i}) < \Phi(s) \tag{1}$$

As the game is finite, a global minimum exists for Φ (not necessarily unique). Let s^* denote an outcome at which Φ achieves its global minimum. If follows that s^* has to be a pure Nash equilibrium, because if s^* is not a PNE, then there exists an agent i who can deviate from s_i to s'_i and hence $C_i(s'_i, s_{-i}) < C_i(s)$. This implies, following (1), that $\Phi(s'_i, s_{-i}) < \Phi(s)$, which contradicts the fact that s is the global minimum. Therefore, a PNE always exist (and it corresponds to a global minimum of Φ).

Question 2

(a) Consider the utility maximizing game below starting with the the initial outcome (A_1, B_1), from which best-response dynamics cycles forever, avoiding the pure Nash of (A_3, B_2).

		P1		
		A_1	A_2	A_3
P2	B_1	$1,4$	$2,3$	$0,0$
	B_2	$0,0$	$0,0$	$5,5$
	B_3	$4,1$	$3,2$	$0,0$

(b) Consider the game at a traffic light where the equilibrium for both players is a correlated equilibrium where they play the strategy (stop,go) and (go,stop) with equal probabilities. Thus by using no-regret dynamics, it will converge to a correlated equilibrium which is not a mixed Nash.

		P1	
		stop	go
P2	stop	$0,0$	$0,1$
	go	$1,0$	$-5,-5$

Question 3

For a fixed t' such that i is the smallest integer such that $t' \leq 2^i$. Then $\epsilon = \sqrt{\frac{\ln n}{t'}} \geq \sqrt{\frac{\ln n}{2^i}}$ and the regret is at most $2\sqrt{2^i \ln n}$ up till time t, i.e.

$$\sum_{t=1}^{t'} \nu^t \leq OPT + 2\sqrt{2^i \ln n}$$

Let $kt' \geq T$ for some integer k, then

$$\sum_{t=1}^{T} \nu^t \leq \sum_{t=1}^{kt'} \nu^t \leq OPT + k \cdot 2\sqrt{2^i \ln n} = OPT + (2\sqrt{k^2 \cdot 2^i \ln n}) \leq OPT + 2\sqrt{k}\sqrt{T \ln n}$$

and hence the expected regret is still $O(\sqrt{\ln n / T})$ with respect to every fixed action.

Question 4

Let $f_\epsilon(x) = (1-\epsilon)^x$ and $g_\epsilon(x) = 1 - \epsilon x$, then

$$\begin{aligned} &f_\epsilon(0) = 1 = g_\epsilon(0) \\ &f_\epsilon(1) = 1 - \epsilon = g_\epsilon(1) \\ &\left.\begin{aligned} f'_\epsilon(x) &= (1-\epsilon)^x \ln(1-\epsilon) \\ g'_\epsilon(x) &= -\epsilon \end{aligned}\right\} f'_\epsilon(0) = \ln(1-\epsilon) < -\epsilon = g'_\epsilon(0) \end{aligned}$$

The last inequality holds since

$$\ln(1-\epsilon) = -\epsilon - \frac{\epsilon^2}{2!(1-\xi)^2} \quad \text{by Taylor's theorem in Lagrange form}$$

for some $\xi \in (0, \epsilon)$. Also f_ϵ is a convex function as $f''_\epsilon(x) = (1-\epsilon)^x \left[\ln(1-\epsilon)\right]^2 > 0$ for $\epsilon \in (0, 1/2]$ on the interval $[0, 1]$. This this proves $f_\epsilon(x) \leq g_\epsilon(x)$.
Now let $\hat{f}_\epsilon(x) = (1+\epsilon)^x$, then by Taylor's expansion

$$(1+\epsilon)^x = 1 + \epsilon x + \frac{x(x-1)(1-\xi)^{x-2}}{2!}\epsilon^2$$

for some $\xi \in (0, \epsilon)$. We observe that $\frac{x(x-1)(1-\xi)^{x-2}}{2!}\epsilon^2 \leq 0$ since $x \in [0, 1]$ we have proved that $(1+\epsilon)^x \leq 1 + \epsilon x$.

Quetion 5

Consider the online decision-making setting where every time step t the adversary chooses a payoff vector $\pi^t : A \to [0, 1]$ where the time-averaged regret is defined as $\frac{1}{T} \max_{a \in A} \sum_{t=1}^{T} \pi^t(a) - \frac{1}{T} \sum_{t=1}^{T} \pi^t(a^t)$. Let $\Gamma^t = \sum_{a \in A} w^t(a)$ and define $OPT = \sum_{t=1}^{T} c^t(a^*)$ as the cumulative cost for the best fixed action a^*. Then

$$\begin{aligned} \Gamma^T &\leq w^T(a^*) \\ &= w^1(a^*) \prod_{t=1}^{T} (1+\epsilon)^{\pi^t(a^*)} = (1+\epsilon)^{OPT} \end{aligned}$$

Te expectec cost of the MW algorithm at time t is

$$\sum_{a\in A} p^t(a).\pi^t(a) = \sum_{a\in A} \frac{w^t(a)}{\Gamma^t}\pi^t(a)$$

We can rewrite Γ^{t+1} in terms of Γ^t in the following manner

$$\begin{aligned}
\Gamma^{t+1} &= \sum_{a\in A} w^{t+1}(a) \\
&= \sum_{a\in A} w^t(a)\cdot(1+\epsilon)^{\pi^t(a)} \\
&\le \sum_{a\in A} w^t(a)\cdot(1+\epsilon\pi^t(a)) \quad \text{by Q} \\
&\le \Gamma^t \sum_{a\in A} p^t(a)\cdot(1+\epsilon\pi^t(a)) \\
&\le \Gamma^t \sum_{a\in A} p^t(a) + p^t(a)\epsilon\pi^t(a) \\
&\le \Gamma^t(1+\epsilon\nu^t) \quad \text{where } \nu^t \text{ is the expected utility at time } t.
\end{aligned}$$

Combining the results obtained from before,

$$\begin{aligned}
(1+\epsilon)^{OPT} \le \Gamma^T \le \Gamma^1 \prod_{t=1}^{T}(1+\epsilon\nu^t) \\
OPT\ln(1+\epsilon) \le \ln n + \sum_{t=1}^{T}\ln(1+\epsilon\nu^t) \\
OPT(\epsilon-\epsilon^2) \le OPT\ln(1+\epsilon) \le \ln n + \sum_{t=1}^{T}\ln(1+\epsilon\nu^t) \le \ln n + \sum_{t=1}^{T}\epsilon\nu^t \\
OPT(\epsilon-\epsilon^2) \le \ln n + \sum_{t=1}^{T}\epsilon\nu^t \\
OPT \le (\ln n)/\epsilon + \sum_{t=1}^{T}\nu^t + \epsilon \\
OPT \le (\ln n)/\epsilon + \epsilon T + \sum_{t=1}^{T}\nu^t
\end{aligned}$$

Hence equalizing the two error terms, we get $\epsilon = \sqrt{\ln n/T}$, similar to the case of the cost vector. Thus it has regret $O(\sqrt{\ln n/T})$

Question 6

(a) Let $\hat{x}, \hat{y}$ be a mixed Nash equilibrium then,

$$\hat{x}^T A\hat{y} \ge x^T A\hat{y} \quad \text{for all mixed distributions } x \tag{2}$$

$$\hat{x}^T A\hat{y} \le \hat{x}^T Ay \quad \text{for all mixed distributions } y \tag{3}$$

if and only if,

$$\hat{x} \in \arg\max_x \left(x^T A \hat{y}\right) \subseteq \arg\max_x \left(\min_y x^T A y\right)$$
$$\hat{y} \in \arg\min_y \left(\hat{x}^T A y\right) \subseteq \arg\min_x \left(\max_x x^T A y\right)$$

(b) Let x_1, y_1 and x_2, y_2 be the given mixed Nash equilibria of a two-player zero-sum game. Thus by the above result, for $i = 1, 2$

$$x_i \in \arg\max_x \left(\min_y x^T A y\right)$$
$$y_{3-i} \in \arg\min_x \left(\max_x x^T A y\right)$$

thus

$$x_i^T A y_{3-i} \geq x^T A y_{3-i} \quad \text{for all mixed distributions } x$$
$$x_i^T A y_{3-i} \leq x_i^T A y \quad \text{for all mixed distributions } y$$

which by definition tells us that (x_i, y_{3-i}) is a mixed Nash equilibrium for $i = 1, 2$.

Quetion 7

(a) Let G be a general bimatrix game. The mixed strategy of player 1 is a vector x of length m. The mixed strategy of player 2 is a vector y of length n. The payoff matrices for the two players are A and B, each of size $m \times n$. The expected payoff for player 1 is $x^\mathsf{T} A y$ and the expected payoff for player 2 is $y^\mathsf{T} B^\mathsf{T} x$.

We form a symmetric bimatrix game by concatenating the strategies and the payoff matrices of both players together. Let G' be a new bimatrix game formed from G with the following construction:

- The payoff matrix for player 1 is

$$K = \begin{bmatrix} 0 & A \\ B^\mathsf{T} & 0 \end{bmatrix}$$

  and the payoff matrix for player 2 is K^T.
- The strategy set of each player is the concatenation of the strategy set of both players (player 1 first). Thus a mixed strategy of each player has the form $z = \begin{bmatrix} x \\ y \end{bmatrix}$, where x and y are constructed from a mixed strategy of players 1 and 2 in G by dividing each coordinate by 2. This is to ensure that z is still a probability distribution.

It follows that G' is a symmetric bimatrix game. Let (z, z) be a symmetric mixed Nash equilibrium of this game. We aim to prove that this equilibrium can be translated back to a mixed Nash equilibrium of the original game by taking the first m coordinates, multiplied by 2, to be the mixed strategy of player 1, and the next n coordinates to be the mixed strategy of player 2. Mathematically, we know that z maximizes the payoffs $z^\mathsf{T} K z$ of each player; i.e. the optimal expected payoff for each player is

$$\begin{bmatrix} x^\mathsf{T} & y^\mathsf{T} \end{bmatrix} \begin{bmatrix} 0 & A \\ B^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^\mathsf{T} A y + y^\mathsf{T} B^\mathsf{T} x$$

and we aim to prove that x and y are the mixed Nash strategies of the original game.

Proof. Let i be a coordinate such that $x_i > 0$. Because z is an MNE, strategy i must have the highest expected payoff for player 1

$$(Kz)_i \geq (Kz)_j \quad \forall j$$

but

$$Kz = \begin{bmatrix} 0 & A \\ B^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ay \\ B^\top x \end{bmatrix}$$

So the i^{th} entry that is the largest in Kz is also the largest in Ay. This implies that x is the mixed Nash strategy for player 1 in the original game. With similar argument, we also conclude that y is the mixed Nash strategy for player 2. Thus (x, y) is an MNE of the original game. $\square$

Note: if we concatenate the payoff matrices in this way

$$K = \begin{bmatrix} A & 0 \\ 0 & B^\top \end{bmatrix}$$

then the new strategy of player 1 is $\begin{bmatrix} x \\ y \end{bmatrix}$ and the strategy of player 2 is $\begin{bmatrix} y \\ x \end{bmatrix}$. This cannot lead to a symmetric Nash equilibrium of the form $\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right)$ which is desirable. Therefore, we adjust this a bit to have the appropriate form mentioned above.

(b) We aim to prove that if (z^1, z^2) is an MNE of G' then (z^1, z^1) is also an MNE. Or we can somehow use the result from question 6 where if (x1, y1) and (x2, y2) are both MNE then so are (x1, y2) and (x2, y1).