

Algorithmic Game Theory: HW 2

1. Let G be a cost-minimization game which has the function Φ such that

$$\begin{aligned} C_i(s'_i, s_{-i}) &< C_i(s) \\ \Phi(s'_i, s_{-i}) &< \Phi(s) \end{aligned}$$

As the game is finite, we can use Φ and best-response dynamics such that the cost decreases with every beneficial deviation made by a particular agent until no more beneficial deviation exists for every agent and call this strategy s^* . Then this s^* is a pure Nash since for any agent i and any deviation s'_i , we cannot have

$$C_i(s'_i, s_{-i}) < C_i(s^*)$$

since it implies $\Phi(s'_i, s_{-i}) < \Phi(s^*)$ and contradicts $\Phi(s^*) < \Phi(s)$ for all pure strategies $s \neq s^*$. Thus this game has at least one PNE.

2. (a) Consider the utility maximizing game below starting with the the initial outcome (A_1, B_1) , from which best-response dynamics cycles forever, avoiding the pure Nash of (A_3, B_2) .

		P1		
		A_1	A_2	A_3
P2	B_1	1, 4	2, 3	0, 0
	B_2	0, 0	0, 0	5, 5
	B_3	4, 1	3, 2	0, 0

- (b) Consider the cost minimization game below, where for player P1, weights are decreased whenever A_1 or A_2 is played. Thus the weights are concentrated on A_3 in the long run and hence the time-averaged history of joint play would point to playing A_3 with close to 1 probability. Similar argument goes for P2 and thus average history of joint play will point to A_3, B_3 being the PNE.

		P1		
		A_1	A_2	A_3
P2	B_1	1, 1	1, 1	0, 0
	B_2	1, 1	1, 1	0, 0
	B_3	0, 0	0, 0	0, 0

3. For a fixed t' such that i is the smallest integer such that $t' \leq 2^i$. Then $\epsilon = \sqrt{\frac{\ln n}{t'}} \geq \sqrt{\frac{\ln n}{2^i}}$ and the regret is at most $2\sqrt{2^i \ln n}$ up till time t , i.e.

$$\sum_{t=1}^{t'} \nu^t \leq OPT + 2\sqrt{2^i \ln n}$$

Let $kt' \geq T$ for some integer k , then

$$\sum_{t=1}^T \nu^t \leq \sum_{t=1}^{kt'} \nu^t \leq OPT + k \cdot 2\sqrt{2^i \ln n} = OPT + (2\sqrt{k^2 \cdot 2^i \ln n}) \leq OPT + 2\sqrt{k}\sqrt{T \ln n}$$

and hence the expected regret is still $O(\sqrt{\ln n/T})$ with respect to every fixed action.

4. Let $f_\epsilon(x) = (1 - \epsilon)^x$ and $g_\epsilon(x) = 1 + \epsilon x$, then

$$\left. \begin{aligned} f_\epsilon(0) &= 1 = g_\epsilon(0) \\ f_\epsilon(1) &= 1 - \epsilon = g_\epsilon(1) \\ f'_\epsilon(x) &= (1 - \epsilon)^x \ln(1 - \epsilon) \\ g'_\epsilon(x) &= \epsilon \end{aligned} \right\} f'_\epsilon(0) = \ln(1 - \epsilon) < 0 = g'_\epsilon(0)$$

also f_ϵ is a convex function as $f''_\epsilon(x) = (1 - \epsilon)^x [\ln(1 - \epsilon)]^2 > 0$ for $\epsilon \in (0, 1/2]$. This this proves $f_\epsilon(x) \leq g_\epsilon(x)$ since the initial gradient of f_ϵ is smaller than g_ϵ

5.

6. (a) Let \hat{x}, \hat{y} be a mixed Nash equilibrium then,

$$\hat{x}^T A \hat{y} \geq x^T A \hat{y} \quad \text{for all mixed distributions } x \quad (1)$$

$$\hat{x}^T A \hat{y} \leq \hat{x}^T A y \quad \text{for all mixed distributions } y \quad (2)$$

if and only if,

$$\begin{aligned} \hat{x} &\in \arg \max_x (x^T A \hat{y}) \subseteq \arg \max_x \left(\min_y x^T A y \right) \\ \hat{y} &\in \arg \min_y (\hat{x}^T A y) \subseteq \arg \min_y \left(\max_x x^T A y \right) \end{aligned}$$

(b) Let x_1, y_1 and x_2, y_2 be the given mixed Nash equilibria of a two-player zero-sum game. Thus by the above result, for $i = 1, 2$

$$\begin{aligned} x_i &\in \arg \max_x \left(\min_y x^T A y \right) \\ y_{3-i} &\in \arg \min_y \left(\max_x x^T A y \right) \end{aligned}$$

thus

$$\begin{aligned} x_i^T A y_{3-i} &\geq x^T A y_{3-i} \quad \text{for all mixed distributions } x \\ x_i^T A y_{3-i} &\leq x_i^T A y \quad \text{for all mixed distributions } y \end{aligned}$$

which by definition tells us that (x_i, y_{3-i}) is a mixed Nash equilibrium for $i = 1, 2$.

7. (a) Let (A, B) be a given general bimatrix game with player one and player two having n and m different strategies to choose from their strategy set denoted by \mathbf{x} and \mathbf{y} respectively, thus A and B are $n \times m$ and $m \times n$ matrices respectively. The expected payoffs of player one is then $\mathbf{x}^T A \mathbf{y}$ and the expected payoffs of player two is $\mathbf{y}^T B \mathbf{x}$.

(b)