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Stochastic Models: Exercise 4

1.

$$
\begin{aligned}
m(t) &= \sum_{n=1}^{\infty} F_n(t), \quad \text{where } F_n(t) \text{ is the } n\text{−fold convolution.} \\
&= F(t) + \sum_{n=2}^{\infty} F_n(t), \quad \text{since } F(t) = F_1(t) \\
&= F(t) + \sum_{n=2}^{\infty} F * F_{n-1}(t) \\
&= F(t) + \sum_{n=2}^{\infty} \int_0^t F_{n-1}(t-x)\, dF(x) \\
&= F(t) + \int_0^t \sum_{n=1}^{\infty} F_n(t-x)\, dF(x) \\
&= F(t) + \int_0^t m(t-x)\, dF(x)
\end{aligned}
$$

2. Let $\{N_D(t), t \geq 0\}$ be a given delay renewal process, then

$$
\begin{aligned}
P\left[S_{N_D(t)} \leq s\right] &= \sum_{n=0}^{\infty} P\left[S_n \leq s, S_{n+1} > t\right] \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} P\left[S_n \leq s, S_{n+1} > t\right] \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^{\infty} P\left[S_n \leq s, S_{n+1} > t \mid S_n = y\right] dF_n(y) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t-y)\, d\left(\sum_{n=1}^{\infty} F_n(y)\right) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t-y)\, dm_D(y), \quad \text{since } F_1(y) = G(y)
\end{aligned}
$$

where $m_D(y) = \sum_{n=0}^{\infty} G * F_n(y)$.

3.

$$
\begin{aligned}
P\left[X_{N(t)+1} > x\right] &= \sum_{n=0}^{\infty} P\left[X_{N(t)+1} > x\right] \\
&= \bar{F}(x) + \sum_{n=1}^{\infty} P\left[X_{N(t)+1} > x\right]
\end{aligned}
$$

4.

5. Given the scenario, a new cycle starts each time the policyholder payment rate reverts to r_1.

   (i) Since the claims are made with a Poisson process of rate λ, the interarrival times are exponentially distributed with parameter λ, thus it is not lattice. Hence

$$
P_i = \frac{\mathbf{E}[\text{paying rate } r_i]}{\mathbf{E}[\text{paying rate } r_0] + \mathbf{E}[\text{paying rate } r_1]}
$$

For the given s and letting X denote the interarrival time, we can have either $X > s$ or $X \leq s$, which we shall use to find the expectations.

$$\begin{aligned}
\mathbf{E}[\text{paying rate } r_0] &= \int_s^\infty (x-s)\lambda e^{-\lambda x}\,dx \\
&= \int_s^\infty x\lambda e^{-\lambda x}\,dx - \int_s^\infty s\lambda e^{-\lambda x}\,dx \\
&= \left[-\frac{x}{\lambda}e^{-\lambda x} - \frac{1}{\lambda^2}e^{-\lambda x}\right]_s^\infty + s\left[e^{-\lambda x}\right]_s^\infty = \frac{1}{\lambda}e^{-\lambda x} \\
\mathbf{E}[\text{paying rate } r_1] &= \int_0^s x\lambda e^{-\lambda x}\,dx + \int_s^\infty s\lambda e^{-\lambda x}\,dx \\
&= \lambda\left[-\frac{x}{\lambda}e^{-\lambda x} - \frac{1}{\lambda^2}e^{-\lambda x}\right]_0^s + se^{-\lambda x} = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda x}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{E}[\text{paying at rate } r_0] &= \frac{e^{-\lambda x}/\lambda}{1/\lambda} = e^{-\lambda x} \\
\mathbf{E}[\text{paying at rate } r_0] &= \frac{1/\lambda - e^{-\lambda x}/\lambda}{1/\lambda} = 1 - e^{-\lambda x}
\end{aligned}$$

(ii) The long-run average amount paid per unit time is

$$\begin{aligned}
P_0 r_0 + P_1 r_1 &= r_0 e^{-\lambda x} + r_1(1 - e^{-\lambda x}) \\
&= r_1 + (r_0 - r_1)e^{-\lambda x}
\end{aligned}$$

6. (a)
   (b)
7. (a)
   (b)