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Stochastic Models: Exercise 2

1. (i)

$$\begin{aligned}\mathbb{P}(S_{n+1}=k) &= \mathbb{P}(S_n=k \mid X_{n+1}=0)\mathbb{P}(X_{n+1}=0)+\mathbb{P}(S_n=k-1 \mid X_{n+1}=1)\mathbb{P}(X_{n+1}=1)\\ &= \mathbb{P}(S_n=k \mid X_{n+1}=0)\cdot q+\mathbb{P}(S_n=k-1 \mid X_{n+1}=1)\cdot p \qquad \text{since } S_n \perp X_{n+1}\\ &= q\,\mathbb{P}(S_n=k)+p\,\mathbb{P}(S_n=k-1)\end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{P}(S_{n+1}=k) &= q\,\mathbb{P}(S_n=k)+p\,\mathbb{P}(S_n=k-1)\\ \sum_{k=0}^{n+1} s^k\mathbb{P}(S_{n+1}=k) &= \sum_{k=0}^{n+1} s^k q\,\mathbb{P}(S_n=k)+\sum_{k=0}^{n+1} s^k p\,\mathbb{P}(S_n=k-1)\\ \sum_{k=0}^{n+1} s^k\mathbb{P}(S_{n+1}=k) &= \sum_{k=0}^{n} s^k q\,\mathbb{P}(S_n=k)+\sum_{k=0}^{n} s^{k+1} p\,\mathbb{P}(S_n=k)\\ &= (q+ps)\sum_{k=0}^{n} s^k\,\mathbb{P}(S_n=k)\end{aligned}$$

let $P_n(s)=\mathbb{E}[s^{S_n}]$, then for all $n \geq 0$ we have the relation,

$$P_{n+1}(s)=(q+ps)P_n(s)$$

and we can inductively deduce that

$$P_n(s)=(q+ps)^n$$

which shows that S_n has a binomial distribution with parameters n, p.

2. The extinction probability π is the smallest fixed point of $P(s)$.

$$\begin{aligned}P(s) &= s\\ as^2+bs+c &= s\\ as^2+(b-1)s+c &= 0\\ s &= \frac{1-b\pm\sqrt{(b-1)^2-4ac}}{2a}\\ s &= \frac{1-b\pm\sqrt{(a+c)^2-4ac}}{2a}\\ s &= \frac{1-b\pm(a-c)}{2a}\end{aligned}$$

hence $s=1$ or $s=c/a$. For extinction to occur, we need the expectation to be greater or equal to 1, i.e. $P'(1)\leq 1$,

$$\begin{aligned}P'(1)=2a+b\leq 1 &\Leftrightarrow a+1-c\leq 1\\ &\Leftrightarrow a\leq c\end{aligned}$$

Extinction occurs when $a \leq c$.

3. (i) We first observe that $\mathbb{P}(Z_n = 0)$ is same as $\mathbb{P}(T \leq n)$ since the latter is the probability that the population becomes extinct before or at the nth generation. We can see that

$$
\begin{aligned}
P(s) = P_1(s) &= q + ps \\
P_2(s) &= q + p(q + ps) = q + pq + p^2 s \\
&\vdots \\
P_n(s) &= q \sum_{i=0}^{n-1} p^i + p^n s
\end{aligned}
$$

we can then solve for the following,

$$
\begin{aligned}
\mathbb{P}(T = n) &= \mathbb{P}(T \leq n) - \mathbb{P}(T \leq n-1) \\
&= \mathbb{P}(Z_n = 0) - \mathbb{P}(Z_{n-1} = 0) \\
&= P_n(0) - P_{n-1}(0) \\
&= q \sum_{k=0}^{n-1} p^k - q \sum_{k=0}^{n-2} p^k = qp^{n-1}
\end{aligned}
$$

(ii) Given the assumption that $Z_0 = i$, define

$$T_i := \inf\{n : Z_n = 0 \mid Z_0 = i\}$$

then $\mathbb{P}(T_i \leq n) = (\mathbb{P}(T \leq n))^i$, then

$$
\begin{aligned}
\mathbb{P}(T_i = n) &= \mathbb{P}(T_i \leq n) - \mathbb{P}(T_i \leq n-1) \\
&= (\mathbb{P}(T \leq n))^i - (\mathbb{P}(T \leq n-1))^i \\
&= \left(q \sum_{k=0}^{n-1} p^k \right)^i - \left(q \sum_{k=0}^{n-2} p^k \right)^i
\end{aligned}
$$

4. (a) We start by finding the pgf of the branching process

$$P(s) = \sum_{k=0}^{\infty} s^k (1-q) q^k = \frac{1-q}{1-sq}$$

where $|s| < 1/q$. The extinction probability is obtained from

$$
\begin{aligned}
s = \frac{1-q}{1-sq} &\Leftrightarrow qs^2 - s + (1-q) = 0 \\
&\Leftrightarrow s = 1 \text{ or } s = \frac{1-q}{q}
\end{aligned}
$$

therefore $\frac{1-q}{q}$ is the extinction probability for $q > 1/2$ since $\frac{1-q}{q} \geq 1$ for $q \leq 1/2$.

(b) For a progenitor Poisson distributed with parameter λ, the population becomes extinct when all the subpopulation of each progenitor becomes extinct which we know from (a) has a probability of $\frac{1-q}{q}$.

$$
\begin{aligned}
\pi \text{ when } Z_0 \text{ is Poisson distributed} &= \sum_{k=0}^{\infty} \mathbb{P}(\text{all subpopulations become extinct} \mid Z_0 = k) \cdot \mathbb{P}(Z_0 = k) \\
&= \sum_{k=0}^{\infty} \left(\frac{1-q}{q} \right)^k \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \cdot e^{\frac{\lambda(1-q)}{q}} = \exp\left(\frac{\lambda}{q} - 2\lambda \right)
\end{aligned}
$$

5. (a) The pgf for $n = 2k$ is

$$g \circ f \circ g \circ \ldots \circ g \circ f(s)$$

where there are k g and f functions in the sequence above. Hence the mean number of individuals in the nth generation is

$$\left.\frac{d(g \circ f \circ g \circ \ldots \circ g \circ f(s))}{ds}\right|_{s=1}$$

(b) The fixed point of the pgf in (a) governs the extinction probability process.

(c) For $i = 1, 2$,

$$P_i(s) = \sum_{k=0}^{\infty} s^k p_i (1 - p_i)^k = \frac{1 - p_i}{1 - sp_i}$$

with $|s| < 1/p_i$.