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Measure Theoretic Probability: Assignment 1

2.1 Since Ω is finite, then $|\Omega| = n$ for some n. Thus there are 2^n elements in the set of all subsets of Ω denoted by 2^Ω. Thus we have $\varnothing, \Omega \in 2^\Omega$. For any $A \in 2^\Omega$, $A^c \in 2^\Omega$ since A^c is also a subset of Ω. Lastly, there are only finite elements in 2^Ω, thus any finite union of sets in 2^Ω is a subset of Ω thus also in 2^Ω which shows that 2^Ω is a σ-algebra.

2.2 We have $(\mathcal{G}_\alpha)_{\alpha \in A}$ be an arbitrary family of σ-algebras defined on an abstract space Ω. Let $\mathcal{H} = \bigcap_{\alpha \in A} \mathcal{G}_\alpha$, it is a σ-algebra since:

(a) $\varnothing, \Omega \in \mathcal{H}$ since $, \Omega \in \mathcal{G}_\alpha$ for all $\alpha \in A$.

(b) Let $H \in \mathcal{H}$, then $H^c \in \mathcal{H}$ since $H \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$ implies $H^c \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.

(c) Let $H_n \in \mathcal{H}$ for all n, then $\bigcup_n H_n \in \mathcal{H}$ since $\bigcup_n H_n \in \bigcap_{\alpha \in A} \mathcal{G}_\alpha$.

2.3 a) Let $X \in \left(\bigcup_{n=1}^\infty A_n\right)^c$, thus $X \notin A_n$ for any n if and only if $X \in \bigcap_{n=1}^\infty A_n^c$.

b) Let $X \in (\bigcap_{n=1}^\infty A_n)^c$ thus $X \notin A_n$ for all n, so $X \in \bigcup_{n=1}^\infty A_n^c$. For the other containment, let $X \in \bigcup_{n=1}^\infty A_n^c$, then $X \notin A_n$ for some n, thus $X \notin \bigcap_{n=1}^\infty A_n$ and we are done.

2.4 We are given $\mathcal{A}$ to be a σ-algebra and $(A_n)_{n \leq 1}$ be a sequence of events in $\mathcal{A}$. Then

$$\liminf_{n\to\infty} A_n = \bigcup_n^\infty \bigcap_{k\geq n}^\infty A_k = \bigcup_n^\infty \left(\bigcup_{k \geq n}^\infty A_k^c \right)^c$$

since $A_k \in \mathcal{A}$, $A_k^c \in \mathcal{A}$ we have $\bigcup_{k\geq n}^\infty A_k^c$ and its complement to be in $\mathcal{A}$ for all n. Thus $\liminf_{n\to\infty} A_n \in \mathcal{A}$. Next we observe that

$$\limsup_{n\to\infty} A_n = \bigcap_n^\infty \bigcup_{k\geq n}^\infty A_k = \left(\bigcup_n^\infty \left(\bigcup_{k\geq n}^\infty A_k \right)^c \right)^c$$

again $\bigcup_{k\geq n}^\infty A_k$ and its complement are in $\mathcal{A}$ for all n, thus the complement of countably unions of them is also in $\mathcal{A}$. Lastly, let $X \in \liminf_{n\to\infty} A_n$, thus $X \in \bigcap_{k\geq n}^\infty A_k$ for some sufficiently large n and since $\bigcap_{k\geq n}^\infty A_k \subseteq \bigcup_{k\geq n}^\infty A_k$, $X \in \bigcup_{k\geq n}^\infty A_k$ for some sufficiently large n. Thus $X \in \bigcap_n^\infty \bigcup_{k\geq n}^\infty A_k$ which shows $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

2.5 We consider the cases where $X \in \limsup_n A_n \backslash \liminf_n A_n$ and $X \notin \limsup_n A_n \backslash \liminf_n A_n$. For the former, we have $\limsup_n 1_{A_n} - \liminf_n 1_{A_n} = 1 - 0$

2.6 Given $\mathcal{A}$ be a σ-algebra of Ω, $B \in \mathcal{A}$ and $\mathcal{F} = \{A \cap B : A \in \mathcal{A}\}$.

(a) $\varnothing, \Omega \in \mathcal{A}$, thus $\varnothing = \varnothing \cap B = \varnothing$ and $\Omega \cap B = B$ are elements of $\mathcal{F}$.

(b) Let $K \in \mathcal{F}$, then $K = H \cap B$ for some $H \in \mathcal{A}$. Then $K^c = B \backslash K \in \mathcal{F}$ as $K^c = H^c \cap B$.

(c) Let $H_n \in \mathcal{A}$ for all n. Then $H_n \cap B \in \mathcal{F}$ for all n. Thus $\bigcup_n^\infty (H_n \cap B) = (\bigcup_n^\infty H_n) \cap B \in \mathcal{F}$.

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2.13 We shall prove by induction. For $n = 2$, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Suppose it is true for up to $n = k$, i.e.

$$P(\cup_{i=1}^k A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{k+1} P(A_1 \cap A_2 \cap \ldots \cap A_k)$$

thus when $n = k + 1$,

$$P(\cup_{i=1}^{k+1} A_i) =$$

2.14 $P(A \cap B) \leq \min\{P(A), P(B)\} = 1/3$. Then $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = 1/12$.

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2.17 Let $\mathcal{A}$ be the family of all subsets of the infinite set Ω where $\mathcal{A} := \{A \subseteq \Omega : A \text{ or } A^c \text{ is finite.}\}$ We have $\varnothing, \Omega \in \mathcal{A}$ as the null set is finite. For any $A \in \mathcal{A}$, either A or A^c is finite, thus $A^c \in \mathcal{A}$. For $A, B \in \mathcal{A}$, $A \cup B$ is also finite since finite union of finite sets is also finite. Thus $\mathcal{A}$ is an algebra. Countably union does not hold since countably union of finite sets is not necessarily finite.