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Stochastic Models: Exercise 1

1. Let X_k denote the number of meals ordered to collect the kth different type of mini toy given that $k-1$ different toys have been collected. Then each X_k is a geometric distribution with $p = \frac{N-(k-1)}{N}$ and
$$E[X_k] = \frac{N}{N-k+1}$$
The expected number of meals ones need to order before collecting a complete set of at least one toy is
$$\begin{aligned} E\left[\sum_{k=1}^{N} X_k\right] &= \sum_{i=1}^{N} \frac{N}{N-k+1} \\ &= N \sum_{k=1}^{N} \frac{1}{k} \end{aligned}$$

2. First observe that
$$\sum_{k=1}^{\infty} \mathbb{P}(X \geq k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{P}(X \geq k) = \lim_{n\to\infty} \sum_{k=0}^{n} \mathbb{P}(X > k) = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$$
We shall now show that it equals $E[X]$ by first defining
$$I_n := \begin{cases} 1 & \text{if } X \geq n \text{ for } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$
then $\mathbb{P}(X \geq n) = E[I_n]$.
$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) &= \sum_{k=1}^{\infty} E[I_k] \\ &= E\left[\sum_{k=1}^{\infty} I_k\right], \text{ by Fubini's Theorem} \\ &= E[X] \end{aligned}$$
since the infinite sum of the indicator random variables is equals to X.
$$\begin{aligned} \int_0^{\infty} \mathbb{P}(X > x)\, dx &= \int_0^{\infty} \int_x^{\infty} f_X(t)\, dt\, dx \\ &= \int_0^{\infty} \int_0^{t} f_X(t)\, dx\, dt \\ &= \int_0^{\infty} t f_X(t)\, dt \\ &= \int_{-\infty}^{\infty} t f_X(t)\, dt \\ &= E[X] \end{aligned}$$
Using the earlier result,
$$\begin{aligned} E[X^n] &= \int_0^{\infty} \mathbb{P}(X^n > x)\, dx \\ &= \int_0^{\infty} \mathbb{P}(X > x^{1/n})\, dx, \text{only consider positive root of } x^{1/n} \text{ since } X \text{ is non-negative} \end{aligned}$$
we can do a change of variables, by letting $y = x^n$
$$dx = ny^{n-1}dy$$
$$\int_0^{\infty} \mathbb{P}(X > x^{1/n})\, dx = \int_0^{\infty} ny^{n-1}\mathbb{P}(X > y)\, dy$$
and we are done.

3.

(a) Let $Y = F(X)$ and $0 \leq Y \leq 1$ since F_X is a cdf.

$$\begin{aligned} F_Y(y) = \mathbb{P}(Y \leq y) &= \mathbb{P}(F(X) \leq y) \\ &= \mathbb{P}(X \leq F^{-1}(y)), \text{ since } F \text{ is non-decreasing} \\ &= F(F^{-1}(y)) = y \end{aligned}$$

since $F_Y(y) = y$, Y is the uniform distribution as the cdf of U over $(0, 1)$, $F_U(x) = x$.

(b) Let $W = F^{-1}(U)$, then

$$\begin{aligned} F_W(w) = \mathbb{P}(W \leq w) &= \mathbb{P}(F^{-1}(U) \leq w) \\ &= \mathbb{P}(U \leq F(w)), \text{ since } F_X \text{ is non-decreasing} \\ &= F(w) \end{aligned}$$

since $F_W(w) = F(w)$ for all w, $F^{-1}(U)$ has the same distribution as F.

4.

$$E[X^2] = \int_0^a x^2 \, dF(x) \, dx \leq \int_0^a ax \, dF(x) = aE[X], \text{ since } 0 \leq x \leq a.$$

then

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &\leq aE[X] - E[X]^2 \\ &= a^2 \left(\frac{E[X]}{a} - \left(\frac{E[X]}{a} \right)^2 \right) \\ &= a^2 \left[\alpha(1 - \alpha) \right], \text{ where } \alpha = \frac{E[X]}{a} \end{aligned}$$

it suffices to show that $\alpha(1-\alpha) \leq 1/4$ and we are done which is true since we know that the function $y = x - x^2$ attains a maximum value of $y = 1/4$ at $x = 1/2$. Therefore,

$$Var[X] \leq \frac{a}{4}$$