1 Algorithmic Game Theory

1.1 Nonatomic Selfish Routing

A selfish game occurs in a multicommodity flow network or simply a network. A network is given by a directed graph $G = (V, E)$, with vertex set V and directed edge set E, together with a set $(s_1, t_1,), \dots, (s_k, t_k)$ of source-sink vertex pairs which are called commodities. The $s_i - t_i$ paths of a network are denoted by $\mathcal{P}_i$ and we only consider networks in which $\mathcal{P}_i \neq \varnothing$ for all i, and define $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$. The graph G is allowed to have parallel edges and a vertex can participate in multiple source-sink pairs. The routes chosen by players are described by using flows, which is a nonnegative vector indexed by the set $\mathcal{P}$ of source-sink paths.

Definition 1.1. For a flow f and path $P \in \mathcal{P}_i$, f_P is the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i. A flow is feasible for a vector r if it routes all the traffic: for each $i \in \{1, 2, \dots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

For each edge e of the network has a cost function $c_e : \mathbb{R}^+ \to \mathbb{R}^+$ with the assumptions that the c_e's are nonnegative, continuous and nondecreasing. For any nonatomic selfish routing game, or simply a nonatomic instance, we define it by a triple of the form (G, r, c).

Definition 1.2. The cost of a path P with respect to the flow f is defined as the sum of the costs of the constituent edges: $c_p(f) = \sum_{e \in P} c_e(f_e)$, where $f_e = \sum_{P \in \mathcal{P}_i : e \in P} f_P$ denotes the amount of traffic using the paths that contain the edge e.

Definition 1.3 (Nonatomic equilibrium flow)**.** Let f be a feasible flow for an nonatomic instance (G, r, c). The flow f is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \dots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_p(f) \leq c_{\tilde{P}}(f)$$

Definition 1.4. The social cost of a flow f is

$$SC(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P$$

we can also rewrite it by summing up the cost over all the edges in the network and it turns out that this representation is a useful trick for some proofs.

$$SC(f) = \sum_{e \in E} c_e(f_e) f_e$$

For any instance (G, r, c), a feasible flow is called an *optimal flow* if it minimizes the social cost over all feasible flows.

In examples later, we see that equilibrium flows have social cost higher than optimal flow and although the ideal scenario is for the network to achieve the optimal flow, due to the selfish nature, the network is running on the equilibrium flow. The measure we use to see how bad the equilibrium flow is compared to the optimal flow is the price of anarchy which is defined as follows.

Definition 1.5. The price of anarchy of a nonatomic selfish routing game is the ratio of the cost of an equilibrium flow and that of an optimal flow.

Theorem 1.6 (Existence and uniqueness of equilibrium flows)**.** Let (G, r, c) be a nonatomic instance.

(a) The instance (G, r, c) admits at least one equilibrium flow.

(b) If f and $\tilde{f}$ are equilibrium flows for (G, r, c), then $c_e(f_e) = c_e(\tilde{f}_e)$ for every edge e.

Proof. to be done later ☐

Definition 1.7. Let $c_e^*(x) = (x \cdot c_e(x))'$ denote the *marginal cost function* for the edge e.

Proposition 1.8 (Characterization of optimal flows)**.** Let (G, r, c) be a nonatomic instance such that for every edge e, the function $x \cdot c_e(x)$ is convex and continuously differentiable. Let c_e^* denote the marginal cost function of the edge e. Then f^* is an optimal flow (G, r, c) iff for every commodity $i \ in \{1, 2, \dots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_p^* > 0$

$$c_p^*(f^*) \leq c_{\tilde{p}}^*(f^*)$$

Corollary 1.9 (Equivalence of equilibrium and optimal flows)**.** Let (G, r, c) be a nonatomic instance such that for every edge e, the function $x \cdot c_e(x)$ is convex and continuously differentiable. Let $c_e^*(x)$ denote the marginal cost function of the edge e. Then f^* is an optimal flow for (G, r, c) iff it is an equilibrium flow for (G, r, c^*).

The consequence above says that there is a one to one correspondence between an optimal flow for (G, r, c) and the equilibrium flow for (G, r, c^*). In real life scenarios, the equilibrium flow is easily achieved as every agent selfishly tries to maximize their utility or minimize their cost, but the optimal equilibrium is not so obvious as some agents might have to be "sacrificed" in order to get a minimal social cost. Therefore, to find the optimal flow for (G, r, c), we could do it by finding the equilibrium flow of (G, r, c^*)

Definition 1.10 (Pigou-like networks)**.** A Pigou-like network is one with the traffic being routed, $r > 0$ and consists of two nodes and two links: one with cost function $c(x)$ and the other with a constant cost fixed at $c(r)$, i.e. cost function of the upper cost function $c(x)$ evaluated at r.

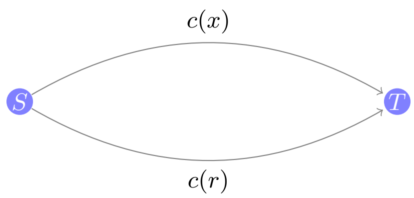

Figure 1: Pigou-like network with traffic r

Definition 1.11 (Pigou bound)**.** Let $\mathcal{C}$ be a nonempty set of cost functions. The *Pigou bound* $\alpha(\mathcal{C})$ for $\mathcal{C}$ is

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x)c(r)}$$

Theorem 1.12 (Tightness of the Pigou bound)**.** Let $\mathcal{C}$ be the set of cost functions and $\alpha(\mathcal{C})$ the Pigou bound for $\mathcal{C}$. If (G, r, c) is a nonatomic instance with cost functions in $\mathcal{C}$, then the price of anarchy of (G, r, c) is at most $\alpha(\mathcal{C})$.

Proof. the idea is to instantiate each edge as a pigou-like network. □

1.2 Network Overprovisioning

Let us now consider selfish routing networks with cost function

$$c_e(x) := \begin{cases} \frac{1}{R_e - x} & \text{if } x < R_e \\ +\infty & \text{if } x \geq R_e \end{cases}$$

Definition 1.13. A network is $\beta-$overprovisioned, meaning

$$f_e \leq (1 - \beta) R_e$$

for all edges $e \in E$ at equilibrium where R_e is the capacity of edge e.

The worst case price of anarchy in $\beta-$overprovisioned networks is at most

$$PoA = \frac{1}{2}\left(1 + \sqrt{\frac{1}{\beta}}\right)$$

from the above result, we can observe that

$$\begin{aligned} \beta \to 1, &\quad PoA \to 1 \\ \beta \to 0, &\quad PoA \to \infty \end{aligned}$$

Recall that for nonlinear Pigou's example we have the price of anarchy to be unbounded. Hence to

Theorem 1.14. If f is an equilibrium flow for (G, r, c) and f^* is a feasible for $(G, 2r, c)$, then $SC(f) \leq SC(f^*)$.

Proof. By the definition of equilibrium flow, we have

$$\sum_{e \in E} c_e(f) f_e = \sum_{P \in \mathcal{P}} c_P(f) f_P = r \cdot L$$

where L is the common cost of every path the equilibrium flow. Fixing the cost of the paths at the equilibrium flow but changing the flow to the optimal flow instead, we get

$$\sum_{e\in E} c_e(f)f_e^* = \sum_{P\in\mathcal{P}} c_P(f)f_P^* \geq 2r \cdot L$$

To complete the proof, it suffices to show that:

$$\sum_{e\in E} c_e(f^*)f_e^* \geq \sum_{e\in E} c_e(f)f_e^* - \sum_{e\in E} c_e(f)f_e$$

We will show the above by showing that for each edge $e \in E$

$$f_e c_e(f_e) \geq f_e^*(c_e(f_e) - c_e(f_e^*))$$

which says that the error made in using the cost of the edges at the equilibrium flow instead of the optimal flow is at most the cost at the equilibrium flow. By considering cases:

(i) $f_e \leq f_e^*$ which will give a negative right term as c_e is a nondecreasing function thus the inequality is clearly true.

(ii) $f_e > f_e^*$ add diagram here.

□

1.3 Location Games

1.4 Positive Externalities

A *network cost-sharing game* takes place in a graph $G = (V, E)$ which can be directed and undirected and each edge E carries a fixed cost $\gamma_e \geq 0$. There are k players and each player i has a source vertex $s_i \in V$ and a sink vertex $t_i \in V$ and its strategy set is the $s_i - t_i$ paths of the graph. Outcomes of the game are path vectors $(P_1, \ldots, P_k)$, with the semantics that the subnetwork $(V, \cup_{i=1}^k P_i)$ gets formed.
The γ_e is thought of as a fixed cost for building edge e, i.e. laying down high-speed internet fiber to a neighbourhood and this cost is independent of the number of users using that edge. The cost incurred by a player is the sum of all the cost of the edges used by the player, but instead of incurring higher cost for a highly utilized edge, the cost is lower when more people are using that edge, hence its name. Let ℓ_e denote the load of an edge e and for $\ell_e > 0$, the players that utilize edge e are jointly responsible for the fixed cost of the edge, γ_e. Here we will assume that the cost of an edge are split equally amongst the players. The cost of a player i is thus given by

$$C_i(\mathbf{P}) = \sum_{e\in P_i} \frac{\gamma_e}{\ell_e}$$