1 Best Response Dynamics

While the current outcome is not a Pure Nash equilibirum (PNE), we can pick an arbitrary player i and an arbitrary beneficial deviation s_i' for player i and move to outcome $(s_i', \mathbf{s_{-i}})$.
Recall that the definition of a potential game is one where there exists a function $\Phi : \mathcal{S} \to \mathbb{R}$ where $\mathcal{S}$ is the finite set of strategies with

$$\Phi(s_i', s_{-i}) - \Phi(s_i, s_{-i}) = c_i(s_i', s_{-i}) - c_i(s_i, s_{-i})$$

Proposition 1.1. In a finite potential game from any arbitrary outcome, best-response dynamics converge to a PNE.

Proof. In a best-response dynamics approach, every iteration has $\Phi(\mathbf{s^{t+1}}) < \Phi(\mathbf{s^t})$, i.e. the potential decreases. Unless the $\mathbf{s^t}$ is a PNE, our Φ is lower bounded by $\min_{s \in \mathcal{S}} \Phi(s)$ and hence the process must terminate. □

Definition 1.2 ($\epsilon-$Pure Nash Equilibrium)**.** For $\epsilon \in [0, 1]$, and outcome $\mathbf{s}$ is an $\epsilon-$pure NE if for every agent i and deviations $s_i' \in S_i$

$$c_i(s_i', s_{-i}) \geq (1 - \epsilon) c_i(s_i, s_{-i})$$

An *epsilon*−best response dynamics is one which permits moves when there is significant improvements (substential lowering of cost or increasing of utility) which is an important factor to for a state to converge to near optimal equilibrium. While a current outcome $\mathbf{s}$ is not an $\epsilon-$PNE, we pick an arbitary player i that has an $\epsilon-$move, i.e. a deviation to s_i':

$$c_i(s_i', s_{-i}) < (1 - \epsilon) c_i(\mathbf{s})$$

Theorem 1.3 (Fast convergence of $\epsilon-$Best Response Dynamics)**.** Consider an atomic selfish routing game where:

1. All players have the same source s and destination t vertex.
2. Cost function satisfy the "$\alpha-$bound jump condition"

   $$c_e(x) \leq c_e(x + 1) \leq \alpha \cdot c_e(x)$$

   for all edges e.
3. The MaxGain variant of $\epsilon-$BR dynamics is used: in every iteration, amongst all players with an $\epsilon-$move available, the player who can obtain the biggest absolute cost decrease gets to move.

Then an $\epsilon-$PNE is reached in at most

$$\frac{k \cdot \alpha}{\epsilon} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}}$$

iterations, where k is the number of agents, $\mathbf{s^0}$ is the initial state of the system.

Lemma 1.4. For $x \in (0,1)$

$$(1-x)^{1/x} \leq (e^{-x})^{1/x} = e^{-1}$$

Theorem 1.5. Consider a $(\lambda,\mu)-$cost minimization game with a positive potential function Φ such that $\Phi(\mathbf{s}) \leq cost(\mathbf{s})$ for every outcome $\mathbf{s}$. Let $\mathbf{s^0}, \mathbf{s^1}, \ldots, \mathbf{s^T}$ be a sequence generated by MaxGain best response dynamics, $\mathbf{s}^*$ a minimum cost outcome and $1 > \gamma > 0$ is a parameter, Then for all but

$$\frac{k}{\gamma(1-\mu)} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}} \tag{1}$$

outcomes $\mathbf{s}^t$ satisfy

$$cost(\mathbf{s^t}) \leq \left(\frac{\lambda}{(1-\mu)(1-\gamma)}\right) \cdot cost(\mathbf{s}^*) \tag{2}$$

Proof.

$$\begin{aligned}
cost(\mathbf{s^t}) &\leq \sum_i c_i(\mathbf{s^t}) \\
&= \sum_i \left[c_i(s_i^*, s_{-i}^t) + \delta_i(\mathbf{s^t})\right], \quad \delta_i(\mathbf{s^t}) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) \\
&\leq \lambda \cdot cost(\mathbf{s}^*) + \mu \cdot cost(\mathbf{s^t}) + \sum_i \delta_i(\mathbf{s^t}) \\
cost(\mathbf{s^t}) &\leq \frac{\lambda}{1-\mu} \cdot cost(\mathbf{s}^*) + \frac{1}{1-\mu} \cdot \sum_i \delta_i(\mathbf{s^t}) \qquad (3)
\end{aligned}$$

we shall let $\Delta(\mathbf{s^t}) = \sum_i \delta_i(\mathbf{s^t})$ in the remaining parts of the proof. We shall now define a state $\mathbf{s^t}$ to be bad if it does not satisfy (2) and by (3), when $\mathbf{s^t}$ is bad we get

$$\Delta(\mathbf{s^t}) \geq \gamma(1-\mu) \cdot cost(\mathbf{s^t})$$

By the MaxGain definition and the inequality relating the potential function and cost,

$$\max_i \delta_i(\mathbf{s^t}) \geq \frac{\Delta(\mathbf{s^t})}{k} \geq \frac{\gamma(1-\mu)}{k} \cdot cost(\mathbf{s^t}) \geq \frac{\gamma(1-\mu)}{k} \cdot \Phi(\mathbf{s^t})$$

and we get what we desire as

$$\Phi(\mathbf{s^t}) - \Phi(s_i^*, s_{-i}^t) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) = \delta_i(\mathbf{s^t})$$

and hence

$$\left(1 - \frac{\gamma(1-\mu)}{k}\right) \Phi(\mathbf{s^t}) \geq \Phi(\mathbf{s^{t+1}}) \tag{4}$$

whenever $\mathbf{s^t}$ is a bad state. The equation in (4) says that for every MaxGain best response dynamics, if the state is bad, the new state $\mathbf{s^{t+1}}$ is smaller than the previous state $\mathbf{s^t}$ by a factor of $1 - \frac{\gamma(1-\mu)}{k}$. By Lemma 1.4, the potential decreases by a factor of e for every $\frac{k}{\gamma(1-\mu)}$ bad states encountered. Thus solving

$$e^{-n}\Phi(\mathbf{s^0}) \geq \Phi_{min}$$

shows (1). ☐