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Real Analysis: Homework 1

1. (a) $\mathbb{R}$ is second-countable by considering the countable basis

$$\mathcal{B} := \{(r-\epsilon, r+\epsilon) | r \in \mathbb{Q}, \text{ for any arbitrary } \epsilon > 0\}$$

We now claim that $\mathcal{B}^n = \{U_1 \times \ldots \times U_n) | \text{ each } U_i \in \mathcal{B} \text{ for } i = 1, \ldots, n\}$ is a countable basis for $\mathbb{R}^n$. It is clear that $\mathcal{B}^n$ is countable as the Cartesian product of countable sets is still countable. To show $\mathcal{B}^n$ is a basis for $\mathbb{R}^n$:

(1) Pick $x = (x_1 \ldots, x_n) \in \mathbb{R}^n$ and consider the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $(x_1 \ldots, x_n) \mapsto x_i$. Thus for each $\pi_i(x) = x_i$ we can find $B_i \in \mathcal{B}$ such that $x_i \in B_i$. Thus $B_1 \times \ldots \times B_n$ is the basis element in $\mathcal{B}^n$ containing x.

(2) Let x belong to the intersection of two basis elements $U = B_1 \times \ldots \times B_n, U' = B_1' \times \ldots \times B_n'$. Using the projection map, $\pi_i(U) = B_i, \pi_i(U') = B_i'$ and thus there is a basis element $A_i \subseteq B_i \cap B_i'$ for some $A_i \in \mathcal{B}$. Thus $A = A_i \times \ldots \times A_n$ is the basis element in $\mathcal{B}^n$ such that $A \subseteq U \cap U'$.

Thus we have shown that $\mathcal{B}^n$ is a countable basis for $\mathbb{R}^n$.

(b) Let U be an open set of $\mathbb{R}$. If U is a union of countably many open sets we can simply pick the disjoint open intervals from that union and we are done. Suppose U is an uncountable union of open sets, and without loss of generality assume that they are disjoint, $U = \sqcup_{\alpha \in A} V_\alpha$ for uncountable A, then since $\mathbb{R}$ is second-countable, there exists a countable basis $\mathcal{B}$ for $\mathbb{R}$. For each $x \in V_\alpha$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V_\alpha$. Taking the $\bigcup_{x \in V_\alpha} B_x$ for all $\alpha \in A$, we claim that this set is at most countable and is a disjoint set of open intervals. It is open since it is a union of $B \in \mathcal{B}$. For $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \neq \alpha_2$, $\bigcup_{x \in V_{\alpha_i}} B_x \subseteq V_{\alpha_i}$, thus $\left(\bigcup_{x \in V_{\alpha_1}} B_x\right) \cap \left(\bigcup_{x \in V_{\alpha_2}} B_x\right)$ are disjoint since $V_\alpha \cap V_{\alpha'} = \varnothing$.

2. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Let (X, τ_X') be a finer topology than (X, τ_X) then $\tau_X' \supseteq \tau_X$. Thus for any $U \in \tau_Y$, $f^{-1}(U) \in \tau_X \subseteq \tau_X'$. Thus $f^{-1}(U) \in \tau_X'$ and $f : (X, \tau_X') \to (Y, \tau_Y)$ remains continuous. Let (Y, τ_Y') is a topology coarser than (Y, τ_Y) and so $\tau_Y \supseteq \tau_Y'$. Hence for $U \in \tau_Y' \subseteq \tau_Y$, we have $f^{-1}(U) \in \tau_X$. Thus $f : (X, \tau_X) \to (Y, \tau_Y')$ remains continuous.

3.

4. Let $x_n > 0$ for all n and $x_n \to a$ with $a > 0$. Then

$$\log\left((x_1 x_2^2 \ldots x_n^n)^{\frac{1}{n^2}}\right) = \sum_{i=1}^{n} \frac{i}{n^2} \log x_i$$

5. By Mean Value Theorem, there is some $c \in (0,1)$ such that

$$\int_0^1 f(x)\, dx = f'(c)$$