

Real Analysis: Homework 2

1. (a) Let $f(x, y) = \cosh x \cosh y$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned} F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\ F''(h) &= \nabla^2 f(h\vec{v})(\vec{v}, \vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ F'''(h) &= \nabla^3 f(h\vec{v})(\vec{v}, \vec{v}, \vec{v}) = \sum_{i,j,k=1,2} \frac{f(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\ &= (x^3 + 3xy^2)(\sinh hx \cosh hy) + (y^3 + 3x^2y)(\cosh hx \cosh hy) \end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2 \quad F'''(0) = y^3 + 3x^2y$$

Thus the polynomial of third degree that best approximate $f(x, y)$ is $\frac{1}{2}(x^2 + y^2) + \frac{1}{6}(y^3 + 3x^2y)$.

- (b) Let $g(x, y) = \sin(x^2 + y^2)$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$G(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned} G'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\ G''(h) &= \nabla^2 g(h\vec{v})(\vec{v}, \vec{v}) \\ &= x^2(2 \cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\ &\quad - 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\ &\quad + y^2(2 \cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2)) \\ G'''(h) &= \nabla^3 g(h\vec{v})(\vec{v}, \vec{v}, \vec{v}) = \sum_{i,j,k=1,2} \frac{g(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\ &= x^3(-8(hx)^3 \cos((hx)^2 + (hy)^2) - 12(hx) \sin((hx)^2 + (hy)^2)) \\ &\quad y^3(-8(hy)^3 \cos((hx)^2 + (hy)^2) - 12(hy) \sin((hx)^2 + (hy)^2)) \\ &\quad + 3x^2y(-8(hx)^2(hy) \cos((hx)^2 + (hy)^2) - 4(hy) \sin((hx)^2 + (hy)^2)) \\ &\quad + 3xy^2(-8(hx)(hy)^2 \cos((hx)^2 + (hy)^2) - 4(hx) \sin((hx)^2 + (hy)^2)) \end{aligned}$$

and

$$G(0) = 0 \quad G'(0) = 0 \quad G''(0) = 2x^2 + 2y^2 \quad G'''(0) = 0$$

Thus the polynomial of third degree that best approximate $g(x, y)$ is $x^2 + y^2$.

2. (a) We observe that $f(x, y)$ is continuous at all points $(x, y) \neq (0, 0)$ since the denominator is nonzero. Thus we need to show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ to show $f(x, y)$ is everywhere continuous.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} xy \frac{x^2}{x^2 + y^2} - \lim_{(x,y) \rightarrow (0,0)} xy \frac{y^2}{x^2 + y^2}$$

then since,

$$0 \leq \frac{x^2}{x^2 + y^2} \leq 1, \quad -|xy| \leq xy \leq |xy|$$

$$-|xy| \leq xy \frac{x^2}{x^2 + y^2} \leq |xy|$$

and $\lim_{(x,y) \rightarrow (0,0)} \pm|xy| = 0$, which by Squeeze theorem gives us $\lim_{(x,y) \rightarrow (0,0)} xy \frac{x^2}{x^2 + y^2} = 0$. We use the same argument and get the limit to be zero for the second term which shows the desired. We have $\nabla f = (f_x, f_y)$, which are given below,

$$f_x(x, y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}, \quad f_y(x, y) = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2}$$

- (b) The computation gives,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} = \frac{\partial^2 f}{\partial y \partial x}$$

To show it is continuous everywhere we have to show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}$ exists and is well defined along different paths.

$$\text{Along } y = x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} = 0$$

$$\text{Along } y = 2x, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + 36x^6 - 144x^6 - 64x^5}{(x^2 + 4x^2)^3} = -171/125$$

3. We recall the geometric series,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad \text{where } |x| < 1$$

$$\text{substituting } x \text{ with } -x, \quad \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{where } |x| < 1$$

$$\text{substituting } x \text{ with } x^2, \quad \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad \text{where } |x| < 1$$

we can then do integration term wise on the right hand side while integrating $\frac{1}{1+x^2}$,

$$\pi/4 = \tan^{-1}(1) = \int_0^1 \frac{1}{1+t^2} dt = \sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{2k} dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}$$

thus $\pi = 4 \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)$.

4. (a) We will show the result by doing Taylor's expansion at 0, which requires us to compute the higher derivatives of $f(t) = e^{tx - \frac{t^2}{2}}$.

$$\begin{aligned}
 f'(t) &= (x - t)f(t) = P_1(t)f(t) \\
 f''(t) &= [(x - t)^2 - 1]f(t) = P_2(t)f(t) \\
 f'''(t) &= [(x - t)^3 - 3(x - t)]f(t) = P_3(t)f(t) \\
 &\vdots \\
 f^{(n)}(t) &= P_n(t)f(t)
 \end{aligned}$$

then we would now like to show that $P_i(t) = P'_{i-1}(t) + (x - t)P_{i-1}(t)$, $P_0 = 1$. It is clear that it is true for $i = 1$. Suppose it is true for $i = k$, $f^{(k)}(t) = P_k(t)f(t)$, then

$$\begin{aligned}
 f^{(k+1)}(t) &= P'_k(t)f(t) + P_k(t)f'(t) \\
 &= P'_k(t)f(t) + (x - t)P_k(t)f(t)
 \end{aligned}$$

which completes the proof by induction. We know that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, then

$$\begin{aligned}
 H_n(x) &:= \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \\
 e^{tx - \frac{t^2}{2}} &= \sum_{k=0}^{\infty} \frac{(tx - \frac{t^2}{2})^k}{k!} \\
 &= \sum_{k=0}^{\infty} t^k \frac{(-1)^k}{k!} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}
 \end{aligned}$$

(b) (i)

$$H'_n(x =)$$

(c)