Zhangsheng Lai
1002554

Stochastic Models: Exercise 3

1. (i)

   (ii)

$$\begin{aligned}\mathbb{P}(N(9.5)-N(8.5)=0) &= e^{-(m(9.5)-m(8.5))}\frac{m(9.5)-m(8.5)^0}{0!}\\ &= e^{-10}\end{aligned}$$

   (iii)

$$\begin{aligned}\mathbb{E}(\text{number of arrivals from 8:30AM-9:30AM}) &= \sum_{k=0}^{\infty} k\cdot e^{-(m(9.5)-m(8.5))}\frac{(m(9.5)-m(8.5))^k}{k!}\\ &= 10\sum_{k=1}^{\infty} e^{-10}\frac{10^{(k-1)}}{(k-1)!}\\ &= 10\end{aligned}$$

2.

$$\begin{aligned}\mathbb{P}(N(I_1)=k_1,\dots,N(I_n)=k_n \mid N(U)=k) &= \frac{\mathbb{P}(N(I_1)=k_1,\dots,N(I_n)=k_n, N(u)=k)}{\mathbb{P}(N(U)=k)}\\ &= \left(\prod_{i=1}^{n} e^{-\lambda c_i}\frac{(\lambda c_i)^{k_i}}{k_i!}\right)\Bigg/\, e^{-\lambda c}\frac{(\lambda c)^k}{k!}\\ &= \frac{k!}{k_1!k_2!\dots k_n!}\left(\frac{c_1}{c}\right)^{k_1}\left(\frac{c_2}{c}\right)^{k_2}\dots\left(\frac{c_n}{c}\right)^{k_n}\end{aligned}$$

3. Let N_i denote the number of families with number of member of size i migrating to Batan Island over a t week period and let such an event be called a type-i event for $i=1,2,3,4$. Hence $N_i(t)$ is a Poisson process and $\mathbb{E}(N_i(t))=\lambda t p_i = 10p_i$. Let $M(t)=\sum_i iN_i(t)$ denote the number of individuals migrating during a t-week period.

$$\begin{aligned}\mathbb{E}(M(t)) &= \sum_i i\mathbb{E}(N_i(t))\\ &= (1+4)\frac{10}{6}+(2+3)\frac{10}{3}\\ &= 25\end{aligned}$$

   To find variance, we first find $\mathbb{E}(N_i(t)^2)$

$$\begin{aligned}\mathbb{E}(N_i(t)^2) &= \sum_{n=0}^{\infty} n^2 e^{-\lambda t p_i}\frac{(\lambda t p_i)^n}{n!}\\ &= \sum_{n=2}^{\infty} e^{-\lambda t p_i}\frac{(\lambda t p_i)^{n-2}}{(n-2)!}+\sum_{n=1}^{\infty} e^{-\lambda t p_i}\frac{(\lambda t p_i)^{n-1}}{(n-1)!}\\ &= \lambda t p_i + (\lambda t p_i)^2\end{aligned}$$

   and so $Var(N_i(t))=\lambda t p_i$.

$$\begin{aligned}Var(M(t)) &= \sum_i i^2 Var(N_i(t))\\ &= (1^2+4^2)\frac{10}{6}+(2^2+3^2)\frac{10}{3}=\frac{215}{3}\end{aligned}$$

4.

5. (i) (a) No. This can be show as follows, where $m(t) = \int_0^t \alpha(u)\,du$

$$
\begin{aligned}
\mathbb{P}(E_1 > t) &= \mathbb{P}(N(t) = 0) \\
&= e^{-m(t)} \\
&= exp\left(-\int_0^t \alpha(u)\,du\right) \\
\mathbb{P}(E_2 > t \mid E_1 = s) &= \mathbb{P}(N(t+s) - N(s) = 0 \mid E_1 = s) \\
&= \mathbb{P}(N(t+s) - N(s) = 0), \quad \text{by independent increments} \\
&= e^{-(m(t+s)-m(s))} \\
&= exp\left(-\int_t^{t+s} \alpha(u)\,du\right)
\end{aligned}
$$

(b) From (a), we have $\mathbb{P}(E_1 \leq t) = 1 - exp\left(-\int_0^t \alpha(u)\,du\right)$

(ii)

6. (a) Let $N_1(t)$ and $N_2(t)$ be the type-I and type-II events where Irma Pince finds a misplaced book and fails to find a misplaced book respectively. Hence, the N_is are independent Poisson process with rate λp_i where $i = 1, 2$. This the misplacements found by Irma Pince follows a homogeneous Poisson process. For $t = 100$, $\mathbb{E}(N_1(100)) = 90\lambda$.

(b) For each shelf i, we can the classify the event of find a misplaced book in the shelf as a type-I event and not finding a misplaced book as a type-II event. Then the $N_1(t)$ of shelf i is a Poisson process with rate λp_i. Define $N(t) = N_1(t) + N_2(t) + N_3(t)$ and we claim that it is a Poisson process with rate of process $\lambda(p_1 + p_2 + p_3)$. Hence the desired probability is,

$$
\mathbb{P}(N(3) = 5) = e^{-\lambda(p_1+p_2+p_3)} \frac{(\lambda(p_1 + p_2 + p_3))^5}{5!}
$$

Here, we shall proof the claim that the sum of two independent Poisson process is a Poisson process. Let $\{N(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ be two independent Poisson process with rate λ_1 and λ_2 respectively. We shall show that $\{N(t) + M(t), t \geq 0\}$ is also a Poisson process by showing the conditions.

(i) N(0)+M(0)=0

(ii) The independent and stationary increments are inherited from $N(t), M(t)$.

(iii)

$$
\begin{aligned}
\mathbb{P}(N(h) + M(h) = 1) &= \mathbb{P}(N(h) = 1, M(h) = 0) + \mathbb{P}(N(h) = 0, M(h) = 1) \\
&= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (\lambda_2 h + o(h))(1 - \lambda_1 h + o(h)), \quad \text{since } NM \\
&= (\lambda_1 + \lambda_2)h + o(h)
\end{aligned}
$$