

## Real Analysis: Homework 1

1. (a)  $\mathbb{R}$  is second-countable by considering the countable basis

$$\mathcal{B} := \{(r - \epsilon, r + \epsilon) \mid r \in \mathbb{Q}, \text{ for any arbitrary } \epsilon > 0\}$$

We now claim that  $\mathcal{B}^n = \{U_1 \times \dots \times U_n \mid \text{each } U_i \in \mathcal{B} \text{ for } i = 1, \dots, n\}$  is a countable basis for  $\mathbb{R}^n$ . It is clear that  $\mathcal{B}^n$  is countable as the Cartesian product of countable sets is still countable. To show  $\mathcal{B}^n$  is a basis for  $\mathbb{R}^n$ :

- (1) Pick  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and consider the projection map  $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}, (x_1, \dots, x_n) \mapsto x_i$ . Thus for each  $\pi_i(x) = x_i$  we can find  $B_i \in \mathcal{B}$  such that  $x_i \in B_i$ . Thus  $B_1 \times \dots \times B_n$  is the basis element in  $\mathcal{B}^n$  containing  $x$ .
- (2) Let  $x$  belong to the intersection of two basis elements  $U = B_1 \times \dots \times B_n, U' = B'_1 \times \dots \times B'_n$ . Using the projection map,  $\pi_i(U) = B_i, \pi_i(U') = B'_i$  and thus there is a basis element  $A_i \subseteq B_i \cap B'_i$  for some  $A_i \in \mathcal{B}$ . Thus  $A = A_1 \times \dots \times A_n$  is the basis element in  $\mathcal{B}^n$  such that  $A \subseteq U \cap U'$ .

Thus we have shown that  $\mathcal{B}^n$  is a countable basis for  $\mathbb{R}^n$ .

- (b) Let  $U$  be an open set of  $\mathbb{R}$ . If  $U$  is a union of countably many open sets we can simply pick the disjoint open intervals from that union and we are done. Suppose  $U$  is an uncountable union of open sets, and without loss of generality assume that they are disjoint,  $U = \sqcup_{\alpha \in A} V_\alpha$  for uncountable  $A$ , then since  $\mathbb{R}$  is second-countable, there exists a countable basis  $\mathcal{B}$  for  $\mathbb{R}$ . For each  $x \in V_\alpha$ , there exists  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq V_\alpha$ . Taking the  $\bigcup_{x \in V_\alpha} B_x$  for all  $\alpha \in A$ , we claim that this set is at most countable and is a disjoint set of open intervals. It is open since it is a union of  $B \in \mathcal{B}$ . For  $\alpha_1, \alpha_2 \in A$  with  $\alpha_1 \neq \alpha_2$ ,  $\bigcup_{x \in V_{\alpha_1}} B_x \subseteq V_{\alpha_1}$ , thus  $\left(\bigcup_{x \in V_{\alpha_1}} B_x\right) \cap \left(\bigcup_{x \in V_{\alpha_2}} B_x\right) = \emptyset$ .

2. Let  $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$  be a continuous function. Let  $(X, \tau'_X)$  be a finer topology than  $(X, \tau_X)$  then  $\tau'_X \supseteq \tau_X$ . Thus for any  $U \in \tau_Y$ ,  $f^{-1}(U) \in \tau_X \subseteq \tau'_X$ . Thus  $f^{-1}(U) \in \tau'_X$  and  $f : (X, \tau'_X) \rightarrow (Y, \tau_Y)$  remains continuous. Let  $(Y, \tau'_Y)$  is a topology coarser than  $(Y, \tau_Y)$  and so  $\tau_Y \supseteq \tau'_Y$ . Hence for  $U \in \tau'_Y \subseteq \tau_Y$ , we have  $f^{-1}(U) \in \tau_X$ . Thus  $f : (X, \tau_X) \rightarrow (Y, \tau'_Y)$  remains continuous.

3. We shall show that  $\int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}$ . Start by considering its square,

$$\begin{aligned} \left(\int_{\mathbb{R}} e^{-x^2/2} dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2/2} dx\right) \left(\int_{\mathbb{R}} e^{-y^2/2} dy\right) \\ &= \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} dy dx \\ &= \int_0^{2\pi} \int_0^\infty e^{-r^2/2} r dr d\theta \quad \text{by polar change of coordinates.} \\ &= 2\pi \int_0^\infty e^{-r^2/2} r dr \\ &= 2\pi \int_0^\infty \frac{1}{2} e^{-s/2} ds \quad \text{change of coordinates, } s = r^2 \\ &= \pi \left[ -2e^{-s/2} \right]_0^\infty = 2\pi \end{aligned}$$

which shows what is required and hence  $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} dx = 1$ .

4. Let  $x_n > 0$  for all  $n$  and  $x_n \rightarrow a$  with  $a > 0$ . Then

$$\log \left( (x_1 x_2^2 \dots x_n^n)^{\frac{1}{n^2}} \right) = \sum_{i=1}^n \frac{i}{n^2} \log x_i$$

5. By Mean Value Theorem, there is some  $c \in (0, 1)$  such that

$$\int_0^1 f(x) dx = f'(c)$$