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Real Analysis: Homework 3

1. A function which is in $\mathcal{C}^1(\mathbb{R})$ but not in $\mathcal{C}^2(\mathbb{R})$ means a function that has continuous first derivative but its second derivative is not continuous. Consider $f : \mathbb{R} \to \mathbb{R}$,
$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
we see that the first derivate of $f(x)$ is continuous but the second derivate is not continuous at $x = 0$.

2.

3. Let ϕ be λ-Hölder bi-continuous then for $v_1, v_2, u_1, u_2 \in T$, we have
$$\sup_{v\in[0,T]} |\phi(u_2, v) - \phi(u_1, v)| \leq C_u|u_2 - u_1|^\lambda$$
$$\sup_{u\in[0,T]} |\phi(u, v_2) - \phi(u, v_1)| \leq C_v|v_2 - v_1|^\lambda$$
then we also observe that
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq |\phi(u_1, v_1) - \phi(u_1, v_2)| + |\phi(u_2, v_2) - \phi(u_2, v_1)|$$
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq |\phi(u_1, v_1) - \phi(u_2, v_1)| + |\phi(u_2, v_2) - \phi(u_1, v_2)|$$
which gives us
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq 2C_v|v_2 - v_1|^\lambda$$
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq 2C_u|u_2 - u_1|^\lambda$$
multiplying them together, we have
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)|^2 \leq 4C_vC_u|v_2 - v_1|^\lambda|u_2 - u_1|^\lambda$$
squaring both sides, we have shown that all λ-Hölder bi-continuous are strongly $\lambda/2$-Hölder bi-continuous.

4. (a) For $0 \leq a < b \leq T$, WTS
$$\int_b^T \frac{1}{(r_1 - b)^\alpha (r_1 - a)^{\alpha+1}}\, dr_1 \leq \frac{(T - b)^{1/4-\alpha}}{(b - a)^{\alpha+1/4}}$$
(b) Given $\psi(u, v) := \mathbb{1}_{[0,v)}(u)\tilde{\psi}(u, v)$ we can understand it as
$$\psi(u, v) := \begin{cases} \tilde{\psi}(u, v) & u < v \\ 0 & u \geq v \end{cases}$$
then to do the double integral of $f(u, v)^2$ over $R = [0, T] \times [0, T]$ is equivalent to integrating over the region $\{(u, v) \in R \mid u < v\}$. Thus
$$\int_0^T \int_0^T f(u, v)^2\, du\, dv = \int_0^T \int_0^v f(u, v)^2\, du\, dv$$