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Real Analysis: Homework 4

Proof.

(a)

$$\begin{aligned}\mathbb{P}[|X_t - X_s| \geq \epsilon] &= \mathbb{P}[|X_t - X_s|^\alpha \geq \epsilon^\alpha] \\ &\leq \epsilon^{-\alpha}\mathbb{E}[|X_t - X_s|^\alpha], \quad \text{by Markov Inequality} \\ &\leq \epsilon^{-\alpha}|t-s|^{1+\beta}\end{aligned}$$

thus as $s \to t$, we have $\mathbb{P}[|X_t - X_s| \geq \epsilon] \to 0$ which shows that $X_s \to X_t$ in probability as $s \to t$.

(b) We need to show that for $n \geq N(\omega)$,

$$\mathbb{P}\left[\max_{1\leq k\leq 2^n}\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] = 1 \tag{1}$$

so from (a), we get for all $1 \leq k \leq 2^n$,

$$\begin{aligned}\mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] &= 1 - \mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| \geq 2^{-\gamma n}\right] \\ &\geq 1 - (2^{-\gamma n})^{-\alpha}\left|\frac{T}{2^n}\right|^{1+\beta} \\ &= 1 - |T|^{1+\beta}\cdot 2^{-n(1+\beta-\gamma\alpha)} \to 1 \text{ as } n \to \infty\end{aligned}$$

thus we have we result in (1).

(c) We first see that $D = \bigcup_{n=1}^\infty D_n$ where $D_n := \{(k/2^n) \mid k = 0, 1, \ldots, 2^n\}$ is the partition of $[0,1]$ and show, for every $m > N(\omega)$,

$$|X_t - X_s| \leq 2\sum_{j=n+1}^{m} 2^{-\gamma j} \quad \text{for all } t, s \in D_m,\ 0 < t - s < 2^{-N(\omega)} \tag{2}$$

For $m = n+1$, we can only have $t = (k/2^m), s = ((k-1)/2^m)$ and the result follows from (b). Suppose (2) is true for $m = n+1, \ldots, M-1$. We take $s < t$ with $s, t \in D_M$ and consider the numbers $t^1 = \max\{u \in D_{M-1} : u \leq t\}$ and $s^1 = \min\{u \in D_{M-1} : u \geq s\}$ which gives the relationship $s \leq s^1 \leq t^1 \leq t$ and $s - s^1 \leq 2^{-M}, t - t^1 \leq 2^{-M}$. Hence from (1) we have

$$\begin{aligned}|X_{s^1} - X_s| &\leq 2^{-\gamma M} \\ |X_{t^1} - X_t| &\leq 2^{-\gamma M}\end{aligned}$$

and from (2) with $m = M-1$,

$$|X_{t^1} - X_{s^1}| \leq 2\sum_{j=n+1}^{M-1} 2^{-\gamma j}$$

so

$$\begin{aligned}|X_t - X_s| &\leq |X_t - X_{t^1}| + |X_{t^1} - X_{s^1}| + |X_{s^1} - X_s| \\ &\leq 2^{-\gamma M} + 2\sum_{j=n+1}^{M-1} 2^{-\gamma j} + 2^{-\gamma M} = 2\sum_{j=n+1}^{M} 2^{-\gamma j}\end{aligned}$$

which proves (2) for $m = M$. Thus by induction, we have shown

$$|X_t - X_s| \le 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j}, \quad 0 < t - s < 2^{-N(\omega)} \tag{3}$$

To show that X is uniformly continuous with Hölder exponent γ on the dyadic rationals, for any $s, t \in D$, with $0 < t-s < h(\omega) \triangleq 2^{-N(\omega)}$, we choose the $n \ge N(\omega)$ such that $2^{-(n+1)} \le t-s < 2^{-n}$. Using the result from (3),

$$|X_t - X_s| \le 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \le 2^{-\gamma(n+1)} \left(2 \sum_{j=0}^{\infty} 2^{-\gamma j} \right) \le \delta |t-s|^\gamma, \quad 0 < t - s < 2^{-N(\omega)} \tag{4}$$

where $\delta = 2/(1 - 2^{-\gamma})$ and shows it is uniformly continuous.

(d) Set $\widetilde{X}$ to be equal to X on the dyadic rationals. For $t \in [0,T] \backslash D$, we choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, the uniform continuity and Cauchy criterion implies that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t and we set $\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. Thus $\widetilde{X}$ is continuous and satisfies (4), since for $0 < t - s < 2^{-N(\omega)}$, for any $\epsilon > 0$, where $t_n \to t, s_n \to s$, we can choose a sufficient large n such that

$$\begin{aligned} |\widetilde{X}_t(\omega) - X_{t_n}(\omega)| &< \epsilon/2 \\ |X_{s_n}(\omega) - \widetilde{X}_s(\omega)| &< \epsilon/2 \\ |X_{t_n}(\omega) - X_{s_n}(\omega)| &\le \delta |t-s|^\gamma \end{aligned}$$

then we have

$$\begin{aligned} |\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)| &= |\widetilde{X}_t(\omega) - X_{t_n}(\omega) + X_{t_n}(\omega) - X_{s_n}(\omega) + X_{s_n}(\omega) - \widetilde{X}_s(\omega)| \\ &= |\widetilde{X}_t(\omega) - X_{t_n}(\omega)| + |X_{t_n}(\omega) - X_{s_n}(\omega)| + |X_{s_n}(\omega) - \widetilde{X}_s(\omega)| \le \delta |t-s|^\gamma + \epsilon \end{aligned}$$

and since ϵ is arbitrary, it shows

$$\mathbb{P}\left[\omega; \sup_{0<t-s<h(\omega)} \left| \widetilde{X}_t(\omega) - \widetilde{X}_s(\omega) \right| \le \delta |t-s|^\gamma \right] = 1$$

To see that $\widetilde{X}$ is a modification of X, we observe that $\widetilde{X}_t = X_t$ almost surely for $t \in D$ and for $t \in [0,T] \backslash D$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability as $X_{s_n} \to \widetilde{X}_t$ almost surely, so $\widetilde{X}_t = X_t$ almost surely.

(e)

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\, dx &= \frac{1}{\sqrt{2\pi}} \left[-x^{n-1} e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -(n-1) x^{n-2} e^{-\frac{x^2}{2}}\, dx \\ &= (n-1) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n-2} e^{-\frac{x^2}{2}}\, dx \\ &= (n-1)(n-3) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n-4} e^{-\frac{x^2}{2}}\, dx \\ \text{In general, } &= (n-1)(n-3)\dots(n-(2k-1)) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{n-2k} e^{-\frac{x^2}{2}}\, dx \end{aligned}$$

Thus we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx := \begin{cases} 0, & n \text{ is odd} \\ (n-1)!!, & n \text{ is even} \end{cases}$$

where $n!!$ is the double factorial, the product of all numbers from 1 to n that have the same parity as n.

To show that Brownian motion is almost surely locally α-Hölder continuous for all $\alpha < 1/2$, we start with a Brownian motion $\{W_t\}_{t\in[0,T]}$. Then $W_t - W_s$ and $\sqrt{t-s}\,W_1$ are both normally distributed with mean 0 and variance $t-s$, for $0 \le s < t \le T$ and for each $n = 2, 3, \ldots$

$$\mathbb{E}\left[|W_t - W_s|^{2n}\right] = |t-s|^n \mathbb{E}\left[|W_1|^{2n}\right] = C_n|t-s|^n$$

where $C_n = \mathbb{E}\left[|W_1|^{2n}\right] = \frac{(2n)!}{2^n\,n!} < \infty$ from earlier result in (e). Thus by the main theorem proved above, we can find a continuous modification $\{\widetilde{W}_t\}_{t\in[0,T]}$ of $\{W_t\}_{t\in[0,T]}$ and also it shows that $\{\widetilde{W}_t\}_{t\in[0,T]}$ is almost surely continuous α-Hölder continuous for all $0 < \alpha < \frac{\beta}{\alpha} = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}$ as desired. Thus Brownian motion is almost surely α-Hölder continuous for all $\alpha < 1/2$.

$\square$