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Real Analysis: Homework 5

1. (a)

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}F(\epsilon,t) &= \int_0^\infty \frac{\partial}{\partial \epsilon} e^{-\epsilon x}\frac{\sin xt}{x}\,dx \\
&= -\int_0^\infty e^{-\epsilon x}\sin xt\,dx \\
&= -\left\{\left[\sin xt \cdot -\frac{1}{\epsilon}e^{-\epsilon x}\right]_0^\infty - \int_0^\infty -\frac{1}{\epsilon}e^{-\epsilon x} t\cos xt\,dx\right\} \\
&= -\frac{t}{\epsilon}\int_0^\infty e^{-\epsilon x}\cos xt\,dx \\
&= -\frac{t}{\epsilon}\left\{\left[\cos xt \cdot -\frac{1}{\epsilon}e^{-\epsilon x}\right]_0^\infty - \int_0^\infty -\frac{1}{\epsilon}e^{-\epsilon x}\cdot -t\sin xt\,dx\right\} \\
&= \frac{t}{\epsilon^2} + \frac{t^2}{\epsilon^2}\int_0^\infty e^{-\epsilon x}\sin xt\,dx
\end{aligned}$$

with some algebraic manipulation we obtain

$$\int_0^\infty e^{-\epsilon x}\sin xt\,dx = -\frac{t}{t^2+\epsilon^2}$$

(b) We observe that

$$\left|e^{-\epsilon x}\frac{\sin xt}{x}\right| \le \frac{e^{-\epsilon x}}{x}$$

thus

$$\sup_{x,t\in\mathbb{R}}\left|e^{-\epsilon x}\frac{\sin xt}{x} - 0\right| \to 0 \text{ as } \epsilon\to\infty$$

Hence,

$$\lim_{\epsilon\to\infty} F(\epsilon,t) = \int_0^\infty \lim_{\epsilon\to\infty} e^{-\epsilon x}\frac{\sin xt}{x}\,dx = 0$$

(c) As $e^{-\epsilon x}\frac{\sin xt}{x}$ is nonnegative, converges uniformly to $\frac{\sin xt}{x}$ as $\epsilon\to 0$, by Monotone convergence theorem, we have

$$\begin{aligned}
\lim_{\epsilon\to 0} F(\epsilon,t) &= \int_0^\infty \frac{\sin xt}{x}\,dx \\
&= \int_0^\infty\int_0^\infty e^{-xy}\sin xt\,dy\,dx \\
&= \int_0^\infty\left(\int_0^\infty e^{-xy}\sin xt\,dx\right)dy \qquad (1)
\end{aligned}$$

we work on the inner integral first,

$$\begin{aligned}\int_0^\infty e^{-xy}\sin xt\,dx &= \left[\sin xt - \frac{1}{y}e^{-xy}\right]_0^\infty - \int_0^\infty -\frac{1}{y}e^{-xy}t\cos xt\,dx \\ &= \left[-\frac{1}{y}e^{-xy}\sin xt - \frac{t}{y^2}e^{-xy}\cos xt\right]_0^\infty - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx \\ &= \left[\frac{-ye^{-xy}\sin xt - te^{-xy}\cos xt}{y^2}\right]_0^\infty - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx \\ &= \frac{t}{y^2} - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx\end{aligned}$$

thus

$$\int_0^\infty e^{-xy}\sin xt\,dx = \frac{t}{t^2+y^2}$$

which then we apply it to (1) to get

$$\begin{aligned}\lim_{\epsilon\to 0} F(\epsilon,t) &= \int_0^\infty \frac{t}{t^2+y^2}\,dy \\ &= \left[\tan^{-1}\frac{y}{t}\right]_0^\infty = \frac{\pi}{2}\mathrm{sgn}(t)\end{aligned}$$

2.

3. (a)

(b)

(c)

(d)