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Algorithmic Game Theory: HW 2

1. Let G be a cost-minimization game which has the function Φ such that

$$C_i(s_i', s_{-i}) < C_i(s)$$
$$\Phi(s_i', s_{-i}) < \Phi(s)$$

As the game is finite, we can use Φ and best-response dynamics such that the cost decreases with every beneficial deviation made by a particular agent until no more beneficial deviation exists for every agent and call this strategy s^*. Then this s^* is a pure Nash since for any agent i and any deviation s_i', we cannot have

$$C_i(s_i', s_{-i}) < C_i(s^*)$$

since it implies $\Phi(s_i', s_{-i}) < \Phi(s^*)$ and contradicts $\Phi(s^*) < \Phi(s)$ for all pure strategies $s \neq s^*$. Thus this game has at least on PNE.

2. (a) Consider the utility maximizing game below starting with the the initial outcome (A_1, B_1), from which best-response dynamics cycles forever, avoiding the pure Nash of (A_3, B_2).

		P1		
		A_1	A_2	A_3
P2	B_1	$1,4$	$2,3$	$0,0$
	B_2	$0,0$	$0,0$	$5,5$
	B_3	$4,1$	$3,2$	$0,0$

(b) Consider the cost minimization game below, where for player P1, weights are decreased whenever A_1 or A_2 is played. Thus the weights are concentrated on A_3 in the long run and hence the time-averaged history of joint play would point to playing A_3 with close to 1 probability. Similar argument goes for P2 and thus average history of joint play will point to A_3, B_3 being the PNE.

		P1		
		A_1	A_2	A_3
P2	B_1	$1,1$	$1,1$	$0,0$
	B_2	$1,1$	$1,1$	$0,0$
	B_3	$0,0$	$0,0$	$0,0$

3. For a fixed t' such that i is the smallest integer such that $t' \leq 2^i$. Then $\epsilon = \sqrt{\frac{\ln n}{t'}} \geq \sqrt{\frac{\ln n}{2^i}}$ and the regret is at most $2\sqrt{2^i \ln n}$ up till time t, i.e.

$$\sum_{t=1}^{t'} \nu^t \leq OPT + 2\sqrt{2^i \ln n}$$

Let $kt' \geq T$, then

$$\sum_{t=1}^{T} \nu^t \leq \sum_{t=1}^{kt'} \nu^t \leq OPT + k \cdot 2\sqrt{2^i \ln n} = OPT + k\sqrt{2^i}(2\sqrt{\ln n}) \leq OPT + 2\sqrt{T \ln n}$$

The last inequality holds since

$$\sqrt{T} \geq k\sqrt{2^i} \implies T \geq k^2 2^i$$

4. Let $f_\epsilon(x) = (1-\epsilon)^x$ and $g_\epsilon(x) = 1 + \epsilon x$, then

$$\begin{aligned}
&f_\epsilon(0) = 1 = g_\epsilon(0) \\
&f_\epsilon(1) = 1 - \epsilon = g_\epsilon(1) \\
&\left.\begin{aligned} f'_\epsilon(x) &= (1-\epsilon)^x \ln(1-\epsilon) \\ g'_\epsilon(x) &= \epsilon \end{aligned}\right\} f'_\epsilon(0) = \ln(1-\epsilon) < 0 = g'_\epsilon(0)
\end{aligned}$$

also f_ϵ is a convex function as $f''_\epsilon(x) = (1-\epsilon)^x \left[\ln(1-\epsilon)\right]^2 > 0$ for $\epsilon \in (0, 1/2]$. This this proves $f_\epsilon(x) \leq g_\epsilon(x)$ since the initial gradient of f_ϵ is smaller then g_ϵ

5.

6. (a) Let $\hat{x}, \hat{y}$ be a mixed Nash equilibrium then,

$$\hat{x}^T A\hat{y} \geq x^T A\hat{y} \quad \text{for all mixed distributions } x \tag{1}$$
$$\hat{x}^T A\hat{y} \leq \hat{x}^T Ay \quad \text{for all mixed distributions } y \tag{2}$$

if and only if,

$$\hat{x} \in \arg\max_x \left(x^T A\hat{y}\right) \subseteq \arg\max_x \left(\min_y x^T Ay\right)$$
$$\hat{y} \in \arg\min_y \left(\hat{x}^T Ay\right) \subseteq \arg\min_x \left(\max_x x^T Ay\right)$$

(b) Let x_1, y_1 and x_2, y_2 be the given mixed Nash equilibria of a two-player zero-sum game. Thus by the above result, for $i = 1, 2$

$$x_i \in \arg\max_x \left(\min_y x^T Ay\right)$$
$$y_{3-i} \in \arg\min_x \left(\max_x x^T Ay\right)$$

thus

$$x_i^T Ay_{3-i} \geq x^T Ay_{3-i} \quad \text{for all mixed distributions } x$$
$$x_i^T Ay_{3-i} \leq x_i^T Ay \quad \text{for all mixed distributions } y$$

7. (a)

(b)