Zhangsheng Lai
1002554

Real Analysis: Homework 5

1. (a)

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}F(\epsilon,t) &= \int_0^\infty \frac{\partial}{\partial \epsilon} e^{-\epsilon x}\frac{\sin xt}{x}\,dx \\
&= -\int_0^\infty e^{-\epsilon x}\sin xt\,dx \\
&= -\left\{\left[\sin xt \cdot -\frac{1}{\epsilon}e^{-\epsilon x}\right]_0^\infty - \int_0^\infty -\frac{1}{\epsilon}e^{-\epsilon x} t\cos xt\,dx\right\} \\
&= -\frac{t}{\epsilon}\int_0^\infty e^{-\epsilon x}\cos xt\,dx \\
&= -\frac{t}{\epsilon}\left\{\left[\cos xt \cdot -\frac{1}{\epsilon}e^{-\epsilon x}\right]_0^\infty - \int_0^\infty -\frac{1}{\epsilon}e^{-\epsilon x}\cdot -t\sin xt\,dx\right\} \\
&= \frac{t}{\epsilon^2} + \frac{t^2}{\epsilon^2}\int_0^\infty e^{-\epsilon x}\sin xt\,dx
\end{aligned}$$

with some algebraic manipulation we obtain

$$\int_0^\infty e^{-\epsilon x}\sin xt\,dx = -\frac{t}{t^2+\epsilon^2}$$

(b) We observe that

$$\left|e^{-\epsilon x}\frac{\sin xt}{x}\right| \leq \frac{e^{-\epsilon x}}{x}$$

thus

$$\sup_{x,t\in\mathbb{R}}\left|e^{-\epsilon x}\frac{\sin xt}{x} - 0\right| \to 0 \text{ as } \epsilon \to \infty$$

Hence,

$$\lim_{\epsilon\to\infty} F(\epsilon,t) = \int_0^\infty \lim_{\epsilon\to\infty} e^{-\epsilon x}\frac{\sin xt}{x}\,dx = 0$$

(c) As $e^{-\epsilon x}\frac{\sin xt}{x}$ is nonnegative, converges uniformly to $\frac{\sin xt}{x}$ as $\epsilon \to 0$, by Monotone convergence theorem, we have

$$\begin{aligned}
\lim_{\epsilon\to 0} F(\epsilon,t) &= \int_0^\infty \frac{\sin xt}{x}\,dx \\
&= \int_0^\infty \int_0^\infty e^{-xy}\sin xt\,dy\,dx \\
&= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin xt\,dx\right)\,dy \qquad (1)
\end{aligned}$$

we work on the inner integral first,

$$\begin{aligned}
\int_0^\infty e^{-xy}\sin xt\,dx &= \left[\sin xt - \frac{1}{y}e^{-xy}\right]_0^\infty - \int_0^\infty -\frac{1}{y}e^{-xy}t\cos xt\,dx \\
&= \left[-\frac{1}{y}e^{-xy}\sin xt - \frac{t}{y^2}e^{-xy}\cos xt\right]_0^\infty - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx \\
&= \left[\frac{-ye^{-xy}\sin xt - te^{-xy}\cos xt}{y^2}\right]_0^\infty - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx \\
&= \frac{t}{y^2} - \frac{t^2}{y^2}\int_0^\infty e^{-xy}\sin xt\,dx
\end{aligned}$$

thus

$$\int_0^\infty e^{-xy}\sin xt\,dx = \frac{t}{t^2+y^2}$$

which then we apply it to (1) to get

$$\begin{aligned}
\lim_{\epsilon\to 0} F(\epsilon, t) &= \int_0^\infty \frac{t}{t^2+y^2}\,dy \\
&= \left[\tan^{-1}\frac{y}{t}\right]_0^\infty = \frac{\pi}{2}\operatorname{sgn}(t)
\end{aligned}$$

2.

$$\begin{aligned}
|x^3 - a - bx - cx^2|^2 &= x^6 - 2cx^5 + (c^2-2b)x^4 \\
&\quad + (2bc-2a)x^3 + (2ac+b^2)x^2 + 2abx + a^2 \\
\frac{1}{2}\int_{-1}^1 |x^3 - a - bx - cx^2|^2\,dx &= \frac{1}{2}\Bigg[\frac{1}{7}x^7 - \frac{1}{3}cx^6 + \frac{1}{5}(c^2-2b)x^5 + \frac{1}{4}(2bc-2a)x^4 \\
&\quad + \frac{1}{3}(2ac+b^2)x^3 + abx^2 + a^2x\Bigg]_{-1}^1 \\
&= \frac{1}{5}c^2 + \frac{2}{3}2ac + 2a^2 + \frac{1}{7} + \frac{1}{3}b^2 - \frac{2}{5}b
\end{aligned}$$

we see that it is minimum when $a = 0 = c$ and when $b = 3/5$. Thus

$$\min_{a,b,c}\frac{1}{2}\int_{-1}^1 |x^3 - a - bx - cx^2|^2\,dx = 4/175$$

3. (a) Let $n \neq m$,

$$\begin{aligned}
\frac{1}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx &= \frac{1}{2L}\int_0^L \cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\,dx \\
&= \frac{1}{2L}\left[-\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L} + \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\right]_0^L = 0 \\
\frac{1}{L}\int_0^L \sin^2\frac{n\pi x}{L}\,dx &= \frac{1}{L}\int_0^L 1 - \frac{1}{2}\cos\frac{2n\pi x}{L}\,dx \\
&= \frac{1}{L}\left[x + \frac{L}{4n\pi}\sin\frac{2n\pi x}{L}\right]_0^L = 1
\end{aligned}$$

Thus $\left\{\sin\frac{n\pi x}{L}\right\}, n \in \mathbb{N}$ is orthonormal. It is a basis as they are linearly independent. Suppose not, then there exists $\sin\frac{a\pi x}{L} a \in \mathbb{N}$ such that it can be expressed as a linear combination of $\left\{\sin\frac{n\pi x}{L}\right\}, n \in \mathbb{N}, a \neq n$,

$$\sin\frac{a\pi x}{L} = \sum_{n\in\mathbb{N}, n\neq a} c_n \sin\frac{n\pi x}{L}$$

but then by the result proven above we have

$$1 = \frac{1}{L}\int_0^L \sin^2\frac{a\pi x}{L}\,dx = \sum_{n\in\mathbb{N}, n\neq a} c_n \frac{1}{L}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{a\pi x}{L}\,dx = 0$$

which is a contradiction, thus $\left\{\sin\frac{n\pi x}{L}\right\}, n \in \mathbb{N}$ is orthonormal. (still need to show it is a basis?)

(b) The heat equation with Dirichlet boundary conditions is given by

$$\frac{\partial u}{\partial t} = k\frac{\partial u}{\partial x^2} \tag{2}$$
$$u(0,t) = u(L,t) = 0 \tag{3}$$
$$u(x,0) = f(x) \tag{4}$$

we will show (3) and (4) first,

$$\begin{aligned}
u(0,t) &= \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt}\sin\frac{n\pi 0}{L} = 0 \\
u(L,t) &= \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt}\sin n\pi = 0 \\
u(x,0) &= \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 k0}\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)
\end{aligned}$$

to show (2), we assume that term-by-term differentiation of the infinite series exists, then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} e^{-(\frac{n\pi}{L})^2 kt}\cos\frac{n\pi x}{L} \\
\frac{\partial u}{\partial x^2} &= \sum_{n=1}^{\infty} A_n \cdot -\left(\frac{n\pi}{L}\right)^2 e^{-(\frac{n\pi}{L})^2 kt}\sin\frac{n\pi x}{L} = \frac{\partial u}{\partial t}
\end{aligned}$$

(c)

(d)