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Algebraic Geometry: Homework 1

1. R be a ring and S a multiplicative subset of R with $1 \in S$ and $0 \notin S$

   (i) It is reflexive, since for any $t \in S$, $t(rs - rs) = 0$, thus $(r, s) \sim (r, s)$. Suppose $(r, s) \sim (r', s')$, so there exist $t \in S$ such that $t(rs' - r's) = 0$ which also means $t(r's - rs') = 0$ and we have symmetry. Lastly, let $(r, s) \sim (t, u)$ and $(t, u) \sim (v, w)$, then there exist $a, b \in S$ such that

   $$a(ru - ts) = 0$$
   $$b(tw - vu) = 0$$

   Then $abwru - abwts = 0$, $bastw - basvu = 0$ and summing them gives $abu(rw - vs) = 0$ with $abu \in S$ which shows transitivity.

   (ii) Same as part (i).

   (iii) Define the action of R_s on M_s, $\phi : R_s \times M_s \to M_s$, $((r, s), (m, s')) \mapsto (rm, ss')$. We first show that the action is well-defined. Let $(r_1, s_1) \sim (r_2, s_2) \in R_s$ and $(m_1, t_1) \sim (m_1, t_1) \in M_s$. Then $\phi((r_i, s_i), (m_i, t_i)) = (r_i m_i, s_i t_i)$ for $i = 1, 2$. Since $a(r_1 s_2 - r_2 s_1) = 0$ and $b(m_1 t_2 - m_2 t_1) = 0$ with some $a, b \in S$,

   $$ab(r_1 s_2 m_1 t_2 - r_2 s_1 m_1 t_2) = 0$$
   $$ab(r_2 s_1 m_1 t_2 - r_2 s_1 m_2 t_1) = 0$$

   thus we have $ab(r_1 m_1 s_2 t_2 - r_2 m_2 s_1 t_1)$, so $(r_1 m_1, s_1 t_1) \sim (r_2 m_2, s_2 t_2)$. M_s is a R_s-module, since for $m_i \in M$, $r_i \in R$ and $s_i, t_i \in S$,

   •$(r, s)((m_1, t_1) + (m_2, t_2)) = (r, s)((m_1 t_2 + m_2 t_1, t_1 t_2)) = (rm_1 t_2 + rm_2 t_1, st_1 t_2) = (rm_1, st_1) + (rm_2, st_2)$
   •$((r_1, s_1) + (r_2, s_2))(m, t) = (mr_1 s_2 + mr_2 s_1, s_1 s_2 t) = (r_1 m, s_1 t) + (r_2 m, s_2 t)$
   •$((r_1, s_1)(r_2, s_2))(m, t) = (r_1 r_2 m, s_1 s_2 t) = (r_1, s_1)(r_2 m, s_2 t) = ((r_1, s_1)(r_2, s_2))(m, t)$
   •$(1_R, 1_S)(m, t) = (m, t)$

2. Given morphisms of R-modules, $\phi : M \to N$ and $\psi : N \to P$, it is an *exact sequence* if the image of ϕ is equal to the kernel of ψ in N.

   (i) If $M = 0_M$, then $\phi(0_M) = \{0_N\} = \ker(\psi)$, thus ψ is injective.

   (ii) If $P = 0_P$, then $\phi(M) = \ker(\psi) = N$, thus ϕ is surjective.

   (iii) For a prime ideal $\mathfrak{p}$ with $S = R - \mathfrak{p}$, and $R_S = R_\mathfrak{p}$, $M_S = M_\mathfrak{p}$. Let $p(r, s) \in \mathfrak{p}R_\mathfrak{p}$ with $p \in \mathfrak{p}$ and $(r, s) \in R_\mathfrak{p}$. It is an ideal since, $p(r, s)(r', s') = p(rr', ss') \in \mathfrak{p}R_\mathfrak{p}$.
   $\mathfrak{p}$ does not contain any unit of R, else $\mathfrak{p} = R$ and $S = \varnothing$. Thus for $(p, s) \in \mathfrak{p}R_\mathfrak{p}$, it is not a unit in $R_\mathfrak{p}$ and thus it is a proper ideal. It is maximal since for any $(r, s) \in R_\mathfrak{p} - \mathfrak{p}R_\mathfrak{p}$, it is a unit of $R_\mathfrak{p}$ since $r \in S$ and its inverse is (s, r). Thus $\mathfrak{p}R_\mathfrak{p} + ((r, s)) = R_\mathfrak{p}$ for any $(r, s) \in R_\mathfrak{p} - \mathfrak{p}R_\mathfrak{p}$

   (iv) Given the natural maps $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ and $\psi_\mathfrak{p} : N_\mathfrak{p} \to P_\mathfrak{p}$ given by $(m, s) \mapsto (\phi(m), s)$ and $(n, s) \mapsto (\psi(n), s)$. Let $(m_1, s_1) \sim (m_2, s_2)$ thus $t(m_1 s_2 - m_2 s_1) = 0_M$ for some $t \in S$. Then $\phi(t(m_1 s_2 - m_2 s_1)) = t(\phi(m_1) s_2 - \phi(m_2) s_1) = \phi(0_M) = 0_N$. Thus $(\phi(m_1), s_1) \sim (\phi(m_2), s_2)$. Similar argument for $\psi_\mathfrak{p}$.

   (v) Let $(\phi(m), s) \in \phi_\mathfrak{p}(M_\mathfrak{p})$, then $(\psi_\mathfrak{p} \circ \phi_\mathfrak{p}(m), s)$

3. (i)

4. (i)

5. (i) Let $S \subseteq k[\underline{x}]$, then $V(S) = \{p \in A^n | F(p) = 0,\ \forall p \in S\}$. Thus $I(V(S)) = \{F \in k[\underline{x}] | F(\underline{a}) = 0,\ \forall \underline{a} \in V(S)\}$.
   $I(V(S)) \supseteq S$

   (ii)

6. (i)

7. (i)