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Statistics: Homework 4

1. (a) Given p_i and q_i denote the probability of choosing box 1 and 2 respectively if the ball color chosen is i where $i = \{B, W, G\}$, denoting the three different colors. With the given information of the number of different color balls in the different boxes,

$$\begin{array}{lll} \mathbb{P}(B|1) = 4/10 & \mathbb{P}(B|2) = 2/10 & \mathbb{P}(B|3) = 4/10 \\ \mathbb{P}(W|1) = 3/10 & \mathbb{P}(W|2) = 6/10 & \mathbb{P}(W|3) = 1/10 \\ \mathbb{P}(G|1) = 2/10 & \mathbb{P}(G|2) = 0 & \mathbb{P}(G|3) = 8/10 \end{array}$$

The risk function is represented by,

$$\begin{aligned} R(\theta, \hat{\theta}_{p,q}) &= \mathbb{E}_\theta(|\theta^2 - \hat{\theta}_{p,q}|^2) \\ &= \mathbb{E}_\theta \left(\sum_{i \in \{B,W,G\}} L(\theta, \hat{\theta}_{p,q}(i)) \mathbb{P}(i|\theta) \right) \end{aligned}$$

where $L(\theta, \hat{\theta}_{p,q}(i)) = L(\theta, 1)p_i + L(\theta, 2)q_i + L(\theta, 3)(1 - p_i - q_i)$.Therefore,

$$\begin{aligned} R(1, \hat{\theta}_{p,q}) &= [q_B + 4(1 - p_B - q_B)]\frac{4}{10} + [q_W + 4(1 - p_W - q_W)]\frac{3}{10} + [q_G + 4(1 - p_G - q_G)]\frac{2}{10} \\ R(2, \hat{\theta}_{p,q}) &= [9p_B + 4q_B + 49(1 - p_B - q_B)]\frac{2}{10} + [9p_W + 4q_W + 49(1 - p_W - q_W)]\frac{6}{10} \end{aligned}$$

(b) Bayes risk is given by

$$r(f, \theta) = \int R(\theta, \hat{\theta}_{p,q}) f(\theta)\, d\theta$$

but since our scenario is discrete, we instead have

$$\begin{aligned} r(f, \theta) &= \sum_{\theta=1,2} R(\theta, \hat{\theta}_{p,q}) \mathbb{P}(\theta) \\ &= \lambda R(1, \hat{\theta}_{p,q}) + (1 - \lambda) R(2, \hat{\theta}_{p,q}) \end{aligned}$$

where $R(1, \hat{\theta}_{p,q})$ and $R(2, \hat{\theta}_{p,q})$ are the values are from (a).

(c) Given $\lambda = 1/2$, we have

$$r(f, \theta) = \frac{1}{2}\left(R(1, \hat{\theta}_{p,q}) + R(2, \hat{\theta}_{p,q}) \right) = \frac{1}{20}(428 - 96p_B - 102q_B - 252p_W - 279q_W - 8p_G - 6q_G)$$

thus to the infimum of Bayes risk is when $q_B = q_W = p_G = 1$.