Equality of Norms in $\mathbb{R}^n$

Real Analysis Presentation

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Definition

A norm on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ with the following properties:
For all $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$

1. $\|x\| \geq 0$, $\|x\| = 0$ iff $x = \mathbf{0}$
2. $\|ax\| = |a|\|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

1-norm

For $x \in \mathbb{R}^n$,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

It is easy to see that it satisfy properties 1 and 2, so we shall show the triangle of inequality property

$$\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1$$

Definition

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ are equivalent if there exists two constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b$$

Theorem (Extreme Value Theorem)

Let $f : X \to \mathbb{R}$ be continuous. If X is compact[1]*, then there exists points c and d such that*

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in X$.

[1] closed and bounded by Heine-Borel theorem when in $X = \mathbb{R}^n$

Steps

1. Transitivity property of norm equivalence
2. Consideration of only $\|x\|_1 = 1$
3. Continuity of any norm $\|\cdot\|_a$ under the $1-$norm
4. Maximum and minimum of $\|\cdot\|_a$ on the $1-$norm unit sphere

Transitivity property of norm equivalence

Suppose we have two norms $\|\cdot\|_a, \|\cdot\|_{a'}$ that are equivalent to constants $c_1, c_2 > 0$ and $c_1', c_2' > 0$ respectively:

$$
\begin{aligned}
c_1\|x\|_1 \leq &\|x\|_a \leq c_2\|x\|_1 \\
c_1'\|x\|_1 \leq &\|x\|_{a'} \leq c_2'\|x\|_1
\end{aligned}
$$

then

$$
\frac{c_1'}{c_2}\|x\|_a \leq c_1'\|x\|_1 \leq \|x\|_{a'} \leq c_2'\|x\|_1 \leq \frac{c_2'}{c_1}\|x\|_a
$$

which shows that if every norm in $\mathbb{R}^n$ is equivalent to the 1-norm, it implies that all the norms in $\mathbb{R}^n$ are equivalent.

Transitivity property of norm equivalence

Consideration of only $\|x\|_1 = 1$

We want to show that for every norm $\|\cdot\|_a$ in $\mathbb{R}^n$,

$$c_1\|x\|_1 \leq \|x\|_a \leq c_2\|x\|_1$$

this is trivially true for $x = 0$ and thus for $x \neq 0$, we can normalize x such that it is unit length in the 1-norm, by scaling x be a multiple of $1/\|x\|_1$, i.e.

$$c_1 = c_1\|u\|_1 \leq \|u\|_a \leq c_2\|u\|_1 = c_2 \tag{1}$$

where $u = x/\|x\|_1$. This means if we can show that (1) holds for the set $\{x \in \mathbb{R}^n | \|x\|_1 = 1\}$, it is same as showing for all $x \in \mathbb{R}^n$.

Consideration of only $\|x\|_1 = 1$

Continuity of any norm $\| \cdot \|_a$ under the $1-$norm

Showing that every norm $\| \cdot \|_a$ is continuous under the 1- norm allows us to apply the generalised extreme value theorem from earlier.

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Continuity of any norm $\|\cdot\|_a$ under the 1—norm

Maximum and minimum of $\| \cdot \|_a$ on the $1-$norm unit sphere

Here we proof the claim that $S := \{x \in \mathbb{R}^n | \|x\|_1 = 1\}$ is a compact set.

Proof.

Let y be any limit point of S, then given any $\delta > 0$, we have an open ball of radius δ centered at y, denoted by $B_\delta(y)$[2] such that $B_\delta(y) \cap S \neq \varnothing$. Pick $z \in B_\delta(y) \cap S$, then

$$\delta > \|y - z\|_1 > |\|y\|_1 - \|z\|_1| = |\|y\|_1 - 1|$$

since δ is arbitrary, we have $\|y\|_1 = 1$ and thus we have shown S is closed since its limit points are all in S. S is bounded by construction of the set.

□

[2] $B_\delta(y) := \{x \mid \|y - x\|_1 < \delta\}$