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Real Analysis: Homework 3

1. A function which is in $\mathcal{C}^1(\mathbb{R})$ but not in $\mathcal{C}^2(\mathbb{R})$ means a function that has continuous first derivative but its second derivative is not continuous. Consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

we see that the first derivate of $f(x)$ is continuous but the second derivate is not continuous at $x = 0$.

2. By Stone-Weierstrass Theorem, for the given function $f(x)$, there exists a sequence of polynomials $P_n(x)$ that converges uniformly to $f(x)$, i.e. $\sup_{x\in[0,1]} |f(x) - P_n(x)| < 1/n$ as $n \to 0$. Now, we consider the inner product $\langle f(x), g(x)\rangle := \int_0^1 f(x)g(x)\,dx$, and if we managed to show that $\langle f(x), f(x)\rangle = 0$, we are done. To show that,

$$\begin{aligned} \left|\int_0^1 f(x)^2\,dx\right| &= \left|\int_0^1 f(x)^2\,dx - \int_0^1 f(x)P_n(x)\,dx\right|, \text{ since } \int_0^1 f(x)x^n\,dx = 0 \text{ for all } n. \\ &\leq \int_0^1 |f(x)|\,|f(x) - P_n(x)|\,dx \\ &\leq \int_0^1 |f(x)|\,dx \cdot \sup_{x\in[0,1]} |f(x) - P_n(x)| \\ &\leq \frac{1}{n}\int_0^1 |f(x)|\,dx \text{ for all } n. \end{aligned}$$

Thus we need $\langle f(x), f(x)\rangle = 0$, which completes the proof.

3. Let ϕ be λ-Hölder bi-continuous then for $v_1, v_2, u_1, u_2 \in T$, we have

$$\begin{aligned} \sup_{v\in[0,T]} |\phi(u_2, v) - \phi(u_1, v)| &\leq C_u|u_2 - u_1|^\lambda \\ \sup_{u\in[0,T]} |\phi(u, v_2) - \phi(u, v_1)| &\leq C_v|v_2 - v_1|^\lambda \end{aligned}$$

then we also observe that

$$\begin{aligned} |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_1, v_2)| + |\phi(u_2, v_2) - \phi(u_2, v_1)| \\ |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_2, v_1)| + |\phi(u_2, v_2) - \phi(u_1, v_2)| \end{aligned}$$

which gives us

$$\begin{aligned} |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_v|v_2 - v_1|^\lambda \\ |\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_u|u_2 - u_1|^\lambda \end{aligned}$$

multiplying them together, we have

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)|^2 \leq 4C_vC_u|v_2 - v_1|^\lambda|u_2 - u_1|^\lambda$$

squaring both sides, we have shown that all λ-Hölder bi-continuous are strongly $\lambda/2$-Hölder bi-continuous.

4. (a) For $0 \leq a < b \leq T$ with $0 \leq \alpha < 1/4$, we first make the following observations:

$$(r_1 - b)^{3/4} \leq (r_1 - a)^{3/4} \tag{1}$$
$$\frac{1}{(r_1 - a)^{\alpha+1/4}} \leq \frac{1}{(b - a)^{\alpha+1/4}} \tag{2}$$

then

$$\begin{aligned}
\int_b^T \frac{1}{(r_1 - b)^{\alpha}(r_1 - a)^{\alpha+1}}\, dr_1 &= \int_b^T \frac{(r_1 - b)^{-\alpha-3/4+3/4}}{(r_1 - a)^{\alpha+1}}\, dr_1 \\
&\leq \int_b^T \frac{(r_1 - b)^{-\alpha-3/4}(r_1 - a)^{3/4}}{(r_1 - a)^{\alpha+1}}\, dr_1 \\
&= \int_b^T \frac{(r_1 - b)^{-\alpha-3/4}}{(r_1 - a)^{\alpha+1/4}}\, dr_1 \\
&= \frac{1}{(b - a)^{\alpha+1/4}} \int_b^T (r_1 - b)^{-\alpha-3/4}\, dr_1 \\
&= \frac{1}{(b - a)^{\alpha+1/4}} \left[\frac{(r_1 - b)^{1/4-\alpha}}{1/4 - \alpha}\right]_b^T \\
&= 4\frac{(T - b)^{1/4-\alpha}}{(b - a)^{\alpha+1/4}}
\end{aligned}$$

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$$\int_b^T \frac{1}{(r_1 - b)^{\alpha}(r_1 - a)^{\alpha+1}}\, dr_1 \leq \frac{(T - b)^{1/4-\alpha}}{(b - a)^{\alpha+1/4}}$$

(b) Given $\psi(u, v) := \mathbb{1}_{[0,v)}(u)\tilde{\psi}(u, v)$ we can understand it as

$$\psi(u, v) := \begin{cases} \tilde{\psi}(u, v) & u < v \\ 0 & u \geq v \end{cases}$$

then to do the double integral of $f(u, v)^2$ over $R = [0, T] \times [0, T]$ is equivalent to integrating over the region $\{(u, v) \in R \mid u < v\}$. Thus

$$\int_0^T \int_0^T f(u, v)^2\, du\, dv = \int_0^T \int_0^v f(u, v)^2\, du\, dv$$