

Stochastic Models: Exercise 4

1.

$$\begin{aligned}
 m(t) &= \sum_{n=1}^{\infty} F_n(t), \quad \text{where } F_n(t) \text{ is the } n\text{-fold convolution.} \\
 &= F(t) + \sum_{n=2}^{\infty} F_n(t), \quad \text{since } F(t) = F_1(t) \\
 &= F(t) + \sum_{n=2}^{\infty} F * F_{n-1}(t) \\
 &= F(t) + \sum_{n=2}^{\infty} \int_0^t F_{n-1}(t-x) dF(x) \\
 &= F(t) + \int_0^t \sum_{n=1}^{\infty} F_n(t-x) dF(x) \\
 &= F(t) + \int_0^t m(t-x) dF(x)
 \end{aligned}$$

2. Let $\{N_D(t), t \geq 0\}$ be a given delay renewal process, then

$$\begin{aligned}
 P[S_{N_D(t)} \leq s] &= \sum_{n=0}^{\infty} P[S_n \leq s, S_{n+1} > t] \\
 &= \bar{F}(t) + \sum_{n=1}^{\infty} P[S_n \leq s, S_{n+1} > t] \\
 &= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^{\infty} P[S_n \leq s, S_{n+1} > t \mid S_n = y] dF_n(y) \\
 &= \bar{F}(t) + \int_0^s \bar{F}(t-y) d\left(\sum_{n=1}^{\infty} F_n(y)\right) \\
 &= \bar{F}(t) + \int_0^s \bar{F}(t-y) dm_D(y), \quad \text{since } F_1(y) = G(y)
 \end{aligned}$$

where $m_D(y) = \sum_{n=0}^{\infty} G * F_n(y)$.

3.

$$\begin{aligned}
 P[X_{N(t)+1} > x] &= \sum_{n=0}^{\infty} P[X_{N(t)+1} > x] \\
 &= \bar{F}(x) + \sum_{n=1}^{\infty} P[X_{N(t)+1} > x]
 \end{aligned}$$

4.

5. Given the scenario, a new cycle starts each time the policyholder payment rate reverts to r_1 .

- (i) Since the claims are made with a Poisson process of rate λ , the interarrival times are exponentially distributed with parameter λ , thus it is not lattice. Hence

$$P_i = \frac{\mathbf{E}[\text{paying rate } r_i]}{\mathbf{E}[\text{paying rate } r_0] + \mathbf{E}[\text{paying rate } r_1]}$$

For the given s and letting X denote the interarrival time, we can have either $X > s$ or $X \leq s$, which we shall use to find the expectations.

$$\begin{aligned}
\mathbf{E}[\text{paying rate } r_0] &= \int_s^\infty (x-s)\lambda e^{-\lambda x} dx \\
&= \int_s^\infty x\lambda e^{-\lambda x} dx - \int_s^\infty s\lambda e^{-\lambda x} dx \\
&= \left[-\frac{x}{\lambda} e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_s^\infty + s [e^{-\lambda x}]_s^\infty = \frac{1}{\lambda} e^{-\lambda s} \\
\mathbf{E}[\text{paying rate } r_1] &= \int_0^s x\lambda e^{-\lambda x} dx + \int_s^\infty s\lambda e^{-\lambda x} dx \\
&= \lambda \left[-\frac{x}{\lambda} e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^s + s e^{-\lambda s} = \frac{1}{\lambda} - \frac{1}{\lambda} e^{-\lambda s}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{E}[\text{paying at rate } r_0] &= \frac{e^{-\lambda s}/\lambda}{1/\lambda} = e^{-\lambda s} \\
\mathbf{E}[\text{paying at rate } r_1] &= \frac{1/\lambda - e^{-\lambda s}/\lambda}{1/\lambda} = 1 - e^{-\lambda s}
\end{aligned}$$

(ii) The long-run average amount paid per unit time is

$$\begin{aligned}
P_0 r_0 + P_1 r_1 &= r_0 e^{-\lambda s} + r_1 (1 - e^{-\lambda s}) \\
&= r_1 + (r_0 - r_1) e^{-\lambda s}
\end{aligned}$$

6. (a) To find the rate of occurrence of d -events, we want to find $\lim_{t \rightarrow \infty} N(t)/t$ which we know to be $1/\mu$, where μ is the interarrival time of a d -event. Since we are given that the events occur according to a Poisson process, the interarrival times are exponentially distributed with parameter λ and a d -event occurs when the interarrival time is less than d .

(b)

7. (a)

(b)