Lemma 0.1. For $x \in (0,1)$

$$(1-x)^{1/x} \leq (e^{-x})^{1/x} = e^{-1}$$

Theorem 0.2. Consider a $(\lambda, \mu)-$cost minimization game with a positive potential function Φ such that $\Phi(\mathbf{s}) \leq cost(\mathbf{s})$ for every outcome $\mathbf{s}$. Let $\mathbf{s^0}, \mathbf{s^1}, \ldots, \mathbf{s^T}$ be a sequence generated by MaxGain best response dynamics, $\mathbf{s}^*$ a minimum cost outcome and $1 > \gamma > 0$ is a parameter, Then for all but

$$\frac{k}{\gamma(1-\mu)} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}} \tag{1}$$

outcomes $\mathbf{s}^t$ satisfy

$$cost(\mathbf{s^t}) \leq \left(\frac{\lambda}{(1-\mu)(1-\gamma)} \right) \cdot cost(\mathbf{s}^*) \tag{2}$$

Proof.

$$\begin{aligned}
cost(\mathbf{s^t}) &\leq \sum_i c_i(\mathbf{s^t}) \\
&= \sum_i \left[c_i(s_i^*, s_{-i}^t) + \delta_i(\mathbf{s^t}) \right], \quad \delta_i(\mathbf{s^t}) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) \\
&\leq \lambda \cdot cost(\mathbf{s}^*) + \mu \cdot cost(\mathbf{s^t}) + \sum_i \delta_i(\mathbf{s^t}) \\
cost(\mathbf{s^t}) &\leq \frac{\lambda}{1-\mu} \cdot cost(\mathbf{s}^*) + \frac{1}{1-\mu} \cdot \sum_i \delta_i(\mathbf{s^t})
\end{aligned} \tag{3}$$

we shall let $\Delta(\mathbf{s^t}) = \sum_i \delta_i(\mathbf{s^t})$ in the remaining parts of the proof. We shall now define a state $\mathbf{s^t}$ to be bad if it does not satisfy (2) and by (3), when $\mathbf{s^t}$ is bad we get

$$\Delta(\mathbf{s^t}) \geq \gamma(1-\mu) \cdot cost(\mathbf{s^t})$$

By the MaxGain definition and the inequality relating the potential function and cost,

$$\max_i \delta_i(\mathbf{s^t}) \geq \frac{\Delta(\mathbf{s^t})}{k} \geq \frac{\gamma(1-\mu)}{k} \cdot cost(\mathbf{s^t}) \geq \frac{\gamma(1-\mu)}{k} \cdot \Phi(\mathbf{s^t})$$

and we get what we desire as

$$\Phi(\mathbf{s^t}) - \Phi(s_i^*, s_{-i}^t) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) = \delta_i(\mathbf{s^t})$$

and hence

$$\left(1 - \frac{\gamma(1-\mu)}{k}\right) \Phi(\mathbf{s^t}) \geq \Phi(\mathbf{s^{t+1}}) \tag{4}$$

whenever $\mathbf{s^t}$ is a bad state. The equation in (4) says that for every MaxGain best response dynamics, if the state is bad, the new state $\mathbf{s^{t+1}}$ is smaller than the previous state $\mathbf{s^t}$ by a factor of $1 - \frac{\gamma(1-\mu)}{k}$. By Lemma 0.1, the potential decreases by a factor of e for every $\frac{k}{\gamma(1-\mu)}$ bad states encountered. Thus solving

$$e^{-n}\Phi(\mathbf{s^0}) \geq \Phi_{min}$$

shows (1). □