Zhangsheng Lai
1002554

Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that $f(y,z) = 5y^2 + z^2 - 3zy - 3y \geq 0$ for every $y, z \in \{0, 1, 2, \ldots\}$. We shall use $\mathbb{Z}_{>0}$ to denote the set $\{0, 1, 2, \ldots\}$ subsequently. We can rewrite $f(y,z)$ to get

$$f(y,z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \tag{1}$$

which we will show that the $f(y,z)$ in this form is nonnegative. All that is left to show is that $\frac{11}{4}y^2 - 3y \geq 0$ for all $y \in \mathbb{Z}_{>0}$ but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (**??**) holds for all $z \in \mathbb{Z}_{>0}$ when $y = 1$. Solving for the inequality below,

$$\begin{aligned} f(1,z) = \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 &\Leftrightarrow (z-1)(z-2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that $f(1,z) < 0$ for $z \in (1,2)$ and hence positive for all $z \in \mathbb{Z}_{>0}$ and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let $\mathcal{P}_i$ denote the set of paths from an origin s_i to a sink t_i with $\mathcal{P}_i \neq \emptyset$.

   Definition (flow)**.** For a *flow* f and path $P \in \mathcal{P}_i$, f_P is the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i. A flow is feasible for a vector $r = (r_1, \ldots, r_k)$ if it routes all the traffic: for each $i \in \{1, 2, \ldots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

   Definition (Nonatomic equilibrium flow)**.** Let f be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow f is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \ldots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_p(f) \leq c_{\tilde{P}}(f)$$

   where $c_p(f)$ denotes the cost of travelling on path P for flow f.

   (ii) Let $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$. Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_p(f_p) \cdot f_p = \sum_{e \in E} c_e(f_e) \cdot f_e$$

   where E is the set of directed edges on the graph G.

   (iii)

(iv) We start by showing that

$$\begin{aligned}
\inf_x \left\{ \left(\frac{ax+b}{ar+b} - 1 \right) \right\} &= \inf_x \left\{ x \left(\frac{a(x-r)}{ar+b} \right) \right\} \\
&= \frac{a}{ar+b} \inf_x \left\{ x^2 - rx \right\} \\
&= -\frac{r^2}{4} \cdot \frac{a}{ar+b}
\end{aligned}$$

with that we can begin out proof.

$$\begin{aligned}
\sup_{c \in \mathcal{C}} \sup_{x,r} \frac{rc(r)}{xc(x) - (r-x)c(r)} &= \sup_{c \in \mathcal{C}} \sup_{x,r} \frac{r}{r + x\left(\frac{c(x)}{c(r)} - 1\right)}, \text{ since } c(r) > 0 \\
&= \sup_{a,b \geq 0} \sup_{x,r} \frac{r}{r + x\left(\frac{ax+b}{ar+b} - 1\right)} \\
&= \sup_{a,b \geq 0} \sup_{r} \frac{r}{r - \frac{r^2}{4}\frac{a}{ar+b}} \\
&= \sup_{a,b \geq 0} \sup_{r} \frac{1}{1 - \frac{ar}{4(ar+b)}} \\
&= \frac{1}{1 - 1/4}
\end{aligned}$$

the last equality follows as the supremum of $\frac{ar}{4(ar+b)}$ occurs when $b = 0$.

3. (i) Let Φ be the potential function of a potential game and c_i denote the cost function of the agents for $i \in \{1, 2, \ldots, k\}$. To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of s_i and solely dependent on s_{-i}. If we consider two alternative distinct strategies for agent i, $s_i', s_i'' \neq s_i$

$$\begin{aligned}
c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i', s_{-i}) - \Phi(s_i', s_{-i})] \\
c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i'', s_{-i}) - \Phi(s_i'', s_{-i})]
\end{aligned}$$

hence we can choose $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$, where $-$ represents any choice of strategy of agent i.

(ii) Let Φ_1 and Φ_2 be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \tag{2}$$
$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \tag{3}$$

where $D_i^k(s_{-i})$ denotes the dummy term for $k = 1, 2$. Taking (**??**)$-$(**??**),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at s_{-i} and any strategy of agent i, s_i.

(iii) ($\Rightarrow$): Let Φ be a potential function for a finite game. We want to show:

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = c_j(s_j^2, s_{-j}^1) - c_j(s^1) + c_i(s^2) - c_i(s_j^2, s_{-j}^1)$$

consider the left hand side,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = \Phi(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1)$$

($\Leftarrow$):

4. (a) Let $\tilde{f}$ be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player $i \in \{1, 2, \ldots, n\}$, any two parallel links P_i, P_j where $1 \leq i < j \leq k$,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of $\tilde{f}$ on P_i equals the flow of f on P_j, ($\tilde{f}_{P_i} = f_{P_j}$) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

since the above inequality is true for any two parallel links, we sum it over the n parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^{n} \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^{n} \sum_{m=1}^{f_{P_i}} c_{P_j}(m) = \Phi(f)$$

where Φ is the potential function.

(b)

5. (a) Let G^1 be a congestion game.