

## Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that  $f(y, z) = 5y^2 + z^2 - 3zy - 3y \geq 0$  for every  $y, z \in \{0, 1, 2, \dots\}$ . We shall use  $\mathbb{Z}_{>0}$  to denote the set  $\{0, 1, 2, \dots\}$  subsequently. We can rewrite  $f(y, z)$  to get

$$f(y, z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \quad (1)$$

which we will show that the  $f(y, z)$  in this form is nonnegative. All that is left to show is that  $\frac{11}{4}y^2 - 3y \geq 0$  for all  $y \in \mathbb{Z}_{>0}$  but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (1) holds for all  $z \in \mathbb{Z}_{>0}$  when  $y = 1$ . Solving for the inequality below,

$$\begin{aligned} f(1, z) &= \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 \Leftrightarrow (z-1)(z-2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that  $f(1, z) < 0$  for  $z \in (1, 2)$  and hence positive for all  $z \in \mathbb{Z}_{>0}$  and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let  $\mathcal{P}_i$  denote the set of paths from an origin  $s_i$  to a sink  $t_i$  with  $\mathcal{P}_i \neq \emptyset$ .

**Definition** (flow). For a flow  $f$  and path  $P \in \mathcal{P}_i$ ,  $f_P$  is the amount of traffic of commodity  $i$  that chooses the path  $P$  to travel from  $s_i$  to  $t_i$ . A flow is feasible for a vector  $r = (r_1, \dots, r_k)$  if it routes all the traffic: for each  $i \in \{1, 2, \dots, k\}$ ,  $\sum_{P \in \mathcal{P}_i} f_P = r_i$ .

**Definition** (Nonatomic equilibrium flow). Let  $f$  be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow  $f$  is an *equilibrium flow* if, for every commodity  $i \in \{1, 2, \dots, k\}$  and every pair  $P, \tilde{P} \in \mathcal{P}_i$  of  $s_i - t_i$  paths with  $f_P > 0$ ,

$$c_P(f) \leq c_{\tilde{P}}(f)$$

where  $c_P(f)$  denotes the cost of travelling on path  $P$  for flow  $f$ .

- (ii) Let  $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$ . Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_P(f_P) \cdot f_P = \sum_{e \in E} c_e(f_e) \cdot f_e$$

where  $E$  is the set of directed edges on the graph  $G$ .

- (iii)

(iv) We start by showing that

$$\begin{aligned}\inf_x \left\{ \left( \frac{ax+b}{ar+b} - 1 \right) \right\} &= \inf_x \left\{ x \left( \frac{a(x-r)}{ar+b} \right) \right\} \\ &= \frac{a}{ar+b} \inf_x \{x^2 - rx\} \\ &= -\frac{r^2}{4} \cdot \frac{a}{ar+b}\end{aligned}$$

with that we can begin our proof.

$$\begin{aligned}\sup_{c \in \mathcal{C}} \sup_{x,r} \frac{rc(r)}{xc(x) - (r-x)c(r)} &= \sup_{c \in \mathcal{C}} \sup_{x,r} \frac{r}{r+x \left( \frac{c(x)}{c(r)} - 1 \right)}, \text{ since } c(r) > 0 \\ &= \sup_{a,b \geq 0} \sup_{x,r} \frac{r}{r+x \left( \frac{ax+b}{ar+b} - 1 \right)} \\ &= \sup_{a,b \geq 0} \sup_r \frac{r}{r - \frac{r^2}{4} \frac{a}{ar+b}} \\ &= \sup_{a,b \geq 0} \sup_r \frac{1}{1 - \frac{ar}{4(ar+b)}} \\ &= \frac{1}{1 - 1/4}\end{aligned}$$

the last equality follows as the supremum of  $\frac{ar}{4(ar+b)}$  occurs when  $b = 0$ .

3. (i) Let  $\Phi$  be the potential function of a potential game and  $c_i$  denote the cost function of the agents for  $i \in \{1, 2, \dots, k\}$ . To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of  $s_i$  and solely dependent on  $s_{-i}$ . If we consider two alternative distinct strategies for agent  $i$ ,  $s'_i, s''_i \neq s_i$

$$\begin{aligned}c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s'_i, s_{-i}) - \Phi(s'_i, s_{-i})] \\ c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s''_i, s_{-i}) - \Phi(s''_i, s_{-i})]\end{aligned}$$

hence we can choose  $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$ , where  $-$  represents any choice of strategy of agent  $i$ .

- (ii) Let  $\Phi_1$  and  $\Phi_2$  be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \tag{2}$$

$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \tag{3}$$

where  $D_i^k(s_{-i})$  denotes the dummy term for  $k = 1, 2$ . Taking (2)–(3),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at  $s_{-i}$  and any strategy of agent  $i$ ,  $s_i$ .

(iii) ( $\Rightarrow$ ): Let  $\Phi$  be a potential function for a finite game. We want to show:

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = c_j(s_j^2, s_{-j}^1) - c_j(s^1) + c_i(s^2) - c_i(s_j^2, s_{-j}^1)$$

consider the left hand side,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = \Phi(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1)$$

( $\Leftarrow$ ):

4. (a) Let  $\tilde{f}$  be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player  $i \in \{1, 2, \dots, n\}$ , any two parallel links  $P_i, P_j$  where  $1 \leq i < j \leq k$ ,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of  $\tilde{f}$  on  $P_i$  equals the flow of  $f$  on  $P_j$ , ( $\tilde{f}_{P_i} = f_{P_j}$ ) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

since the above inequality is true for any two parallel links, we sum it over the  $n$  parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^n \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^n \sum_{m=1}^{f_{P_i}} c_{P_i}(m) = \Phi(f)$$

where  $\Phi$  is the potential function.

(b)

5. (a) Let  $G^1$  be a congestion game.
6. (a) We consider two networks  $A, B$  that resemble  $n$ -gons where  $n$  is even, i.e. there are  $n$  edges and  $n$  vertices for each network as shown in Figure 1. The strategy of each player  $i$  is

$$S_i = \{\{a_i, b_i\}, \{a_{i+1}, b_{i-1}, b_{i+1}\}\} = \{s_i^1, s_i^2\}$$

where  $a_i, b_i$  denotes edges in  $A$  and  $B$  respectively and the cost function of each edge is simply  $c_e(x) = x$ . We claim that when all the player were to play  $s_i^1$  it is a Nash equilibrium and we have the optimal value of potential which has value  $2n$ , since every edge is inhabit by a single player. Any player  $i$  that deviates from playing  $s_i^1$  will increase the potential by  $2+2+2-1$  (hence the claim it is a Nash) since it will share an edge with player  $i+1$  in  $A$  and players  $i-1, i+1$  in  $B$  and hence increase the potential. If all the players were to play  $s_i^2$ , we see that the potential will be  $n + n + 2n$  where the first  $n$  is incurred from  $A$  and the next two terms are from  $B$  as we sum up the cost from 1 to the load of the edge. It is easy to see that everyone playing  $s_i^2$  is a Nash; for any player that deviates from  $s_i^2$ , will have an increased cost of  $+1$  coming from  $A$ , an increased cost of  $+2$  from  $B$  and hence this completes the proof.

(b)

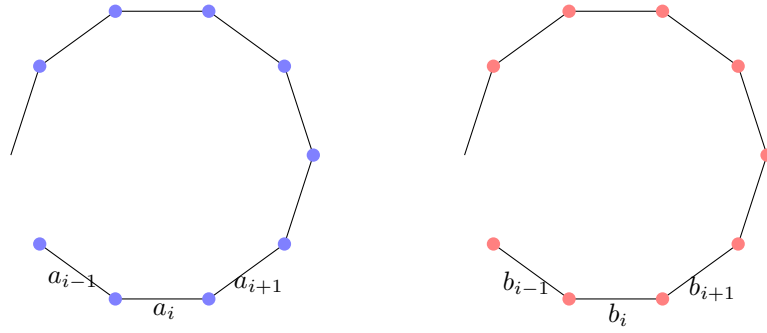


Figure 1: Example with price of potential anarchy equals 2.