Zhangsheng Lai
1002554

Real Analysis: Homework 4

Proof.

(a)

$$\begin{aligned}\mathbb{P}[|X_t - X_s| \geq \epsilon] &= \mathbb{P}[|X_t - X_s|^\alpha \geq \epsilon^\alpha] \\ &\leq \epsilon^{-\alpha}\mathbb{E}[|X_t - X_s|^\alpha], \quad \text{by Markov Inequality} \\ &\leq \epsilon^{-\alpha}|t-s|^{1+\beta}\end{aligned}$$

thus as $s \to t$, we have $\mathbb{P}[|X_t - X_s| \geq \epsilon] \to 0$ which shows that $X_s \to X_t$ in probability as $s \to t$.

(b) We need to show that for $n \geq N(\omega)$,

$$\mathbb{P}\left[\max_{1\leq k\leq 2^n}\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] = 1 \tag{1}$$

so from (a), we get for all $1 \leq k \leq 2^n$,

$$\begin{aligned}\mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| < 2^{-\gamma n}\right] &= 1 - \mathbb{P}\left[\left|X_{\frac{kT}{2^n}} - X_{\frac{(k-1)T}{2^n}}\right| \geq 2^{-\gamma n}\right] \\ &\geq 1 - (2^{-\gamma n})^{-\alpha}\left|\frac{T}{2^n}\right|^{1+\beta} \\ &= 1 - |T|^{1+\beta}\cdot 2^{-n(1+\beta-\gamma\alpha)} \to 1 \text{ as } n \to \infty\end{aligned}$$

thus we have we result in (1).

(c) We first see that $D = \bigcup_{n=1}^\infty D_n$ where $D_n := \{(k/2^n) \mid k = 0, 1, \ldots, 2^n\}$ is the partition of $[0,1]$ and show, for every $m > N(\omega)$,

$$|X_t - X_s| \leq 2\sum_{j=n+1}^{m} 2^{-\gamma j} \quad \text{for all } t, s \in D_m,\ 0 < t-s < 2^{-N(\omega)} \tag{2}$$

For $m = n+1$, we can only have $t = (k/2^m), s = ((k-1)/2^m)$ and the result follows from (b). Suppose (2) is true for $m = n+1, \ldots, M-1$. We take $s < t$ with $s, t \in D_M$ and consider the numbers $t^1 = \max\{u \in D_{M-1} : u \leq t\}$ and $s^1 = \min\{u \in D_{M-1} : u \geq s\}$ which gives the relationship $s \leq s^1 \leq t^1 \leq t$ and $s - s^1 \leq 2^{-M}, t - t^1 \leq 2^{-M}$ (need to explain this later). Hence from (1) we have

$$|X_{s^1} - X_s| \leq 2^{-\gamma M}$$
$$|X_{t^1} - X_t| \leq 2^{-\gamma M}$$

and from (2) with $m = M-1$,

$$|X_{t^1} - X_{s^1}| \leq 2\sum_{j=n+1}^{M-1} 2^{-\gamma j}$$

so

$$
\begin{aligned}
|X_t - X_s| &\le |X_t - X_{t^1}| + |X_{t^1} - X_{s^1}| + |X_{s^1} - X_s| \\
&\le 2^{-\gamma M} + 2\sum_{j=n+1}^{M-1} 2^{-\gamma j} + 2^{-\gamma M} = 2\sum_{j=n+1}^{M} 2^{-\gamma j}
\end{aligned}
$$

which proves (2) for $m = M$. Thus by induction, we have shown

$$
|X_t - X_s| \le 2\sum_{j=n+1}^{\infty} 2^{-\gamma j}, \quad 0 < t - s < 2^{-N(\omega)} \tag{3}
$$

To show that X is uniformly continuous with Hölder exponent γ on the dyadic rationals, for any $s, t \in D$, with $0 < t-s < h(\omega) \triangleq 2^{-N(\omega)}$, we choose the $n \ge N(\omega)$ such that $2^{-(n+1)} \le t-s < 2^{-n}$. Using the result from (3),

$$
|X_t - X_s| \le 2\sum_{j=n+1}^{\infty} 2^{-\gamma j} \le 2^{-\gamma(n+1)}\left(2\sum_{j=0}^{\infty} 2^{-\gamma j}\right) \le \delta |t-s|^{\gamma}, \quad 0 < t - s < 2^{-N(\omega)} \tag{4}
$$

where $\delta = 2/(1-2^{-\gamma})$ and shows it is uniformly continuous.

(d) Set $\widetilde{X}$ to be equal to X on the dyadic rationals. For $t \in [0,T]\backslash D$, we choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, the uniform continuity and Cauchy criterion implies that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t and we set $\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. Thus $\widetilde{X}$ is continuous and satisfies (4), since for $0 < t - s < 2^{-N(\omega)}$, for any $\epsilon > 0$, where $t_n \to t, s_n \to s$, we can choose a sufficient large n such that

$$
\begin{gathered}
|\widetilde{X}_t(\omega) - X_{t_n}(\omega)| < \epsilon/2 \\
|X_{s_n}(\omega) - \widetilde{X}_s(\omega)| < \epsilon/2 \\
|X_{t_n}(\omega) - X_{s_n}(\omega)| \le \delta |t-s|^{\gamma}
\end{gathered}
$$

then we have

$$
\begin{aligned}
|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)| &= |\widetilde{X}_t(\omega) - X_{t_n}(\omega) + X_{t_n}(\omega) - X_{s_n}(\omega) + X_{s_n}(\omega) - \widetilde{X}_s(\omega)| \\
&= |\widetilde{X}_t(\omega) - X_{t_n}(\omega)| + |X_{t_n}(\omega) - X_{s_n}(\omega)| + |X_{s_n}(\omega) - \widetilde{X}_s(\omega)| \le \delta|t-s|^{\gamma} + \epsilon
\end{aligned}
$$

and since ϵ is arbitrary, it shows

$$
\mathbb{P}\left[\omega; \sup_{0<t-s<h(\omega)} \left|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)\right| \le \delta|t-s|^{\gamma}\right] = 1
$$

To see that $\widetilde{X}$ is a modification of X, we observe that $\widetilde{X}_t = X_t$ almost surely for $t \in D$ and for $t \in [0,T]\backslash D$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability as $X_{s_n} \to \widetilde{X}_t$ almost surely, so $\widetilde{X}_t = X_t$ almost surely.

(e)

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx &= \frac{1}{\sqrt{2\pi}}\left[-x^{n-1}e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -(n-1)x^{n-2}e^{-\frac{x^2}{2}}\,dx \\
&= (n-1)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{n-2}e^{-\frac{x^2}{2}}\,dx \\
&= (n-1)(n-3)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{n-4}e^{-\frac{x^2}{2}}\,dx \\
\text{In general, } &= (n-1)(n-3)\ldots(n-(2k-1))\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{n-2k}e^{-\frac{x^2}{2}}\,dx
\end{aligned}
$$

Thus we have

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}}\,dx := \begin{cases} 0, & n \text{ is odd} \\ (n-1)!!, & n \text{ is even} \end{cases}
$$

where $n!!$ is the double factorial, the product of all numbers from 1 to n that have the same parity as n.

$\square$