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Problems to Ponder

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space with $A_n \uparrow A$ and $B_n \downarrow B$ where $A_n, B_n \in \mathcal{F}$ for $n \geq 1$. WLOG, consider $A_n \uparrow A$, then since $A = \bigcup_n A_n$, $A \in \mathcal{F}$. By considering the complements of $B_n^c = H_n$, $H_n \uparrow H = \bigcup_n B_n^c \in \mathcal{F}$ implies $\bigcap_n B_n \in \mathcal{F}$. Use countably additive property to prove $\lim_{n\to\infty} \mathbb{P}(A_n)$.

2. We are given Ω to be infinite, $\mathcal{F} = \{A \subset \Omega : A \text{ is finite or } A^c \text{ is finite}\}$ and $\mu : \mathcal{F} \to [0, \infty)$ by $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A^c is finite. We shall show that $\mathcal{F}$ is an algebra,

   (a) $\varnothing, \Omega \in \mathcal{F}$ since $\varnothing = \Omega^c$ is finite.

   (b) For $A \in \mathcal{F}$ either A or A^c is finite, thus $A^c \in \mathcal{F}$.

   (c) For $A, B \in \mathcal{F}$, if both are finite, $A \cup B \in \mathcal{F}$. If both A^c, B^c are finite, $(A \cup B)^c = A^c \cap B^c \in \mathcal{F}$. If A, B^c is finite, $(A \cup B)^c \subset B^c$, thus $A \cup B$ is finite.

   Therefore, $\mathcal{F}$ is an algebra. To show that μ is finitely additive on $\mathcal{F}$, we have to show that for $A_k \in \mathcal{F}$, $k \geq 1$ and pairwise disjoint, $\mu(\cup_{k=1}^n A_k) = \sum_{i=1}^n \mu(A_k)$. This trivially holds when the A_k's are finite. When we have A_k infinite, we can have at most one since if A_k, A_j are infinite, since they are disjoint, $A_j \subset A_k^c$ which is a contradiction. Thus for any collection of disjoint A_k's there is at most one of them which is infinite and thus it satisfies the finite additivity property.

3. Let μ be the Lebesgue probability measure on the σ-algebra $\mathcal{F}$ of $[0, 1]$ and μ_* the outer measure on all subsets of $[0, 1]$ defined to be $\mu_*(A) = \inf\{\sum_{k=1}^\infty |I_k|;\text{ each } I_k \text{ is an interval and } \{I_k\} \text{ is a countable cover for } A\}$. We also define $\mathcal{B}$ to be the Borel σ-algebra on $[0, 1]$ and N is called a null set if $\mu_*(N) = 0$ where $N \subset [0, 1]$. We want to show that $\overline{\mathcal{B}} := \{B \cup N : B \in \mathcal{B}\} = \mathcal{F}$. Take $B \cup N \in \overline{\mathcal{B}}$ for some $B \in \mathcal{B}$, then for any $E \subseteq [0, 1]$

$$\begin{aligned}
\mu_*((B \cup N) \cap E) + \mu_*((B \cup N)^c \cap E) &= \mu_*((B \cap E) \cup (N \cap E))) + \mu_*((B^c \cap N^c) \cap E) \\
&= \mu_*(B \cap E) + \mu_*(N \cap E) + \mu_*(B^c \cap E) - \mu_*(N \cap E) \\
&= \mu_*(E)
\end{aligned}$$

   thus $\overline{\mathcal{B}} \subseteq \mathcal{F}$. For the converse, take $A \in \mathcal{F}$, then we want to show $A = B \cup N$ for some $B \in \mathcal{B}$.