

Literature Review: Real-Time Dynamic Pricing for Multiproduct Models with Time-Dependent Customer Arrival Rates

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1 Introduction

In the single product dynamic pricing model that we were introduced in class, we looked at a monopolist seller which finite units x_0 of a single indivisible product with no replenishment over a finite and continuous horizon $[0, T)$. The unit price π_t is decided by the seller at each point of time $t \in [0, T)$ and customers product valuations follow a distribution over \mathbb{R}_+ . However, sellers often have a wide range of products that the customers can choose from, with the products having similar functionalities; catering to customers of varying purchasing power.

Thus we look to a multiproduct model [2] where the retailer who encounters a long supply leadtime and short selling season with substitute products when replenishment products are unavailable. For example, when shopping, if the item of your choice is not available and its stock won't be replenish, then you will have to find a substitute product. Thus the retailer can control the price of the products with time so as to maximize his or her revenue. Instead of considering the customers product valuation as a general random variable, we explicitly construct a model that sees customer purchase probability as a function of the quality and the price. Also, a time dependent customer arrival rate is modeled by a nonhomogeneous Poisson process (add that it is forecasted in the model). Lastly, the paper assumes that the products are equally dissimilar; the addition or removal of one product affects the choice of all other products in the same portion.

2 The Multinomial Logit Model

2.1 Summary of the MNL model

The MNL model is a multiproduct dynamic pricing problem and assumes the products are nominal. It describes dynamic consumer choice preferences over substitute products as prices are varied. The customer makes choices from a range of products to maximise his or her utility. The utility of the i product is defined by the logit demand function $v^i(r^i) = \exp((q^i - r^i)/\mu)$ which is a positive function of the quality q^i , the price r^i of product i for $i = 1, \dots, n$ and μ is the constant representing the stochastic preference of the choice process. The customer expected demand probability of product i is defined as

$$P^i(r) = \frac{v^i(r^i)}{v^0 + \sum_{j=1}^n v^j(r^j)}, \quad i = 1, \dots, n \quad (1)$$

and v^0 denotes the utility of not making any purchase.

The customer's arrivals are assumed to follow a nonhomogenous Poisson process with a time-dependent rate $\lambda(t)$ and in a small time interval δt , the probability of one arrival is $\lambda(t)\delta t$. The price of the products at time t is $r_t = (r_t^1, \dots, r_t^n) \in \mathbb{R}^n$ is decided according to the current inventory level. If some product i is sold out before the end of the selling season, as we do not allow replenishments, the price is set to $r_t^i = \infty$ for all t occurring after $c_t^i = 0$. If all the products are sold at $t < T$, then the selling season ends at t . All unsold products are salvaged at $t = 0$.

Let $\mathbf{r} = \{r_t, t \in [T, 0]\} = (r^1, \dots, r^n)$ denote a pricing policy for the entire season, where r^i denotes the trend of the price of product i over the selling season. We denote the probability for a customer to arrive at time t to choose product $i \in \mathbf{n} = \{1, \dots, n\}$ by $P_t^i(r_t)$ and $P_t^0(r_t)$ denotes the no-purchase probability. Using (1) with the no-purchase utility $v^0 = \exp(u_0/\mu)$, we have the demand of product i and of no-purchase

$$P_t^i(r_t) = \frac{\exp((q^i - r_t^i)/\mu)}{\sum_{i \in \mathbf{n}} \exp((q_i - r_t^i)/\mu) + \exp(u_0/\mu)} \quad (2)$$

$$P_t^0(r_t) = \frac{\exp(u_0/\mu)}{\sum_{i \in \mathbf{n}} \exp((q_i - r_t^i)/\mu) + \exp(u_0/\mu)} \quad (3)$$

and by construction, $P_t^0(r_t) + \sum_{i \in \mathbf{n}} P_t^i(r_t) = 1$.

The revenue rate at time t is

$$\Phi_t(r_t) := \lambda(t) \sum_{i=1}^n r_t^i P_t^i(r_t) \quad (4)$$

By setting the price r_t , the retailer controls the probability of customer choice process so the revenue rate is optimized. From (2) and (3) we obtain $P_t^i(r_t)/P_t^0 = \exp((q^i - r_t^i - u_0)/\mu)$.

$$r_t^i = q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0, \quad i = 1, \dots, n \quad (5)$$

which is a mapping from the sales probability space to the price space, and in (2) and (3) we have a mapping in the opposite direction. However, these two mappings are not inverses of each other as the mapping from the price space to the sales probability space is not injective and hence cannot be a bijection. This can be easily observed for given pricing $r_t = (r_t^1, \dots, r_t^n)$, choosing $r_t' = (r_t^1 \pm \ln k, \dots, r_t^n \pm \ln k)$ for some positive real constant k will produce the same demand probability of product. Thus the vector of prices obtained from a given sales probability has to be chosen from a set of allowable prices [1].

Thus, by choosing from the set of allowable prices, we have a one-to-one mapping from the price space to the sales probability space. The revenue rate as in (4) can be rewritten as

$$\Phi_t(P_t) = \lambda(t) \sum_{i=1}^n [q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0] P_t^i \quad (6)$$

where $P_t = (P_t^1, \dots, P_t^n)$

2.2 Optimal control of the MNL model

Let N_t^i be the Poisson counting process of the number of products sold up to time t . If a customer arrives and chooses the product i at time t then $dN_t^i = 1$, otherwise $dN_t^i = 0$. $N_t = (N_t^1, \dots, N_t^n)$ is defined to be a multivariate stochastic process controlled by the price vector r_t through the probability P_t . Given an initial inventory c_T , the expected revenue of the price scheme \mathbf{r} for the entire selling season is

$$\mathbf{w}_T^{\mathbf{r}}(c_T) = \left(E_{\mathbf{r}} \left[\int_0^T r_t^i dN_t^i \right] \right)_{i=1, \dots, n} \quad (7)$$

Using the relationship between the pricing and the sales probability, the paper rewrites the expected revenue in terms of its corresponding probability scheme $\mathbf{P} = \{P_t, t \in [T, 0]\}$ to get

$$\mathbf{w}_T^{\mathbf{P}}(c_T) = \left[\int_0^T \lambda(t) \sum_{i=1}^n P_t^i [q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0] dt \right] = \int_0^T \Phi_t(P_t) dt \quad (8)$$

By writing the expected revenue in terms of sales probabilities instead of pricing, it reduces the space from \mathbb{R}_+^n to $[0, 1]^n$. This allows the search of the optimal control and

value to be much more managable (to remove or to justify it better) as we are searching for a solution from a probability simplex. The retailer searches for a pricing policy $\hat{\mathbf{r}}$ that maximizes the expected revenue over the selling season, which is denoted as

$$\hat{\mathbf{w}}_T^{\mathbf{P}}(c_T) = \max_{P_t \in [0,1]^n, t \in [T,0]} \int_0^T \lambda(t) \sum_{i=1}^n P_t^i (q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0) dt \quad (9)$$

such that $dc_t^i = -dN_t^i$, which simply means any change in the inventory levels of product i is the negative of the change in the Poisson counting process of the number of products i sold up till time i . This formulation is observed to be a optimal control problem; our utility is $\mathbf{w}_T^{\mathbf{P}}(c_T)$, with scalar utility rate function $\Phi_t(\cdot)$, the system state vector is c_t and the control vector we are trying to find is P_t for some time t . If we used the expected revenue formulation in (6) instead, the control vector we are trying to find will be r_t . Now let

$$\mathbf{w}_t^{\mathbf{P}}(c_t) = \int_0^t \lambda(s) \sum_{i=1}^n P_s^i (q^i - u_0 - \mu \ln P_s^i + \mu \ln P_s^0) ds \quad (10)$$

and

$$\hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) = \max_{P_s \in [0,1]^n, s \in [t,0]} \mathbf{w}_t^{\mathbf{P}}(c_t) \quad (11)$$

The authors then solve the optimal control problem by applying the Hamilton-Jacobi-Bellman (HJB) sufficient conditions. The HJB provide sufficient conditions for optimality, which says that the solution for the equation,

$$\frac{\partial \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)}{\partial t} = \max_{P_t \in [0,1]^n, t \in [T,0]} \sum_{i=1}^n \lambda(t) P_t^i (q^i - u_0 - \mu \ln P_t^i + \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)) \quad (12)$$

gives us the optimal revenue rate, where $\Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) = \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) - \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t - e^i)$, is the marginal value of product i , with e^i being a n -dimensional vector with the i -th component being 1 and the rest zeros, and $\hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)$ satisfying the boundary and initial conditions

$$\hat{\mathbf{w}}_t^{\mathbf{P}}(0) = 0 \quad \forall t \quad (13)$$

$$\hat{\mathbf{w}}_0^{\mathbf{P}}(c_0) = 0 \quad \forall c_0 \quad (14)$$

The boundary and initial conditions states that if there is zero inventory then at any time the optimal revenue rate is zero, and if the selling season has ended regardless of the amount of inventory left, the optimal revenue rate will also be zero. The HJB conditions are equivalent to maximizing the revenue rate $\Phi_t(P_t)$ at each t .

2.3 Optimal Dynamic Pricing Policy

To find the optimal dynamic pricing policy, we just need to find the maximizer P_t^* of the HJB equation in (12). Define

$$\Psi_t^{\mathbf{P}}(c_t) = \sum_{i=1}^n \lambda(t) P_t^i [q^i - u_0 = \mu \ln P_t^i + \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t)]$$

and the fact that $\Psi_t^{\mathbf{P}}(c_t)$ is concave is derived from showing that its Hessian is negative and thus it is semi-negative definite. Thus it has a global maximum over P_t which can be found by solving,

$$\frac{\partial \Psi_t^{\mathbf{P}}(c_t)}{\partial P_t^i} = \lambda(t) [q^i - u_0 + \mu \ln P_t^i - \mu \ln P_t^0 - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}}(c_t) - \mu] \quad (15)$$

as we have $\sum_{i=1}^n P_t^i + P_t^0 = 1$, it suffices to find P_t^i for $i > 0$.

In the paper, they solved for the maximizer to be of the form

$$(P_t^i)^* = \frac{\mu}{m_t(c_t)} \exp \left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - m_t(c_t)}{\mu} \right) \quad (16)$$

where $i \in \mathbb{R}(c_t)$, the set of products with non-negative inventory and

$$m_t(c_t) = \frac{\mu}{1 - \sum_{i=1}^n (P_t^i)^*} = \frac{\mu}{P_t^{0*}} \quad (17)$$

By (16), (17) and the probability simplex constrain, we obtain

$$\left[\frac{m_t(c_t)}{\mu} - 1 \right] \exp \left[\frac{m_t(c_t) + u_0}{\mu} \right] = \sum_{i=1}^n \exp \left[\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)}{\mu} \right] \quad (18)$$

where $m_t(c_t)$ is the unique solution. This uniqueness follows from the concave property of $\Psi_t^{\mathbf{P}}(c_t)$ over all probability simplex P_t .

Using the results that we have derived above, the optimal price scheme is

$$(r_t^i)^* = \Delta \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) + m_t(c_t) \quad (19)$$

and updating the HJB optimality sufficiency conditions,

$$\frac{\partial}{\partial t} \hat{\mathbf{w}}_t^{\mathbf{P}^*} = \lambda(t)(m_t(c_t) - \mu) \quad (20)$$

By integrating both sides with respect to t from 0 to T , the maximum expected revenue at the beginning of the selling season in terms of $m_t(c_t)$ is

$$\hat{\mathbf{w}}_t^{\mathbf{P}^*} = \int_0^T \lambda(t) [m_t(c_t) - \mu] dt \quad (21)$$

From the discussion above, we see that the solution to the optimal pricing policy depends on the term $m_t(c_t)$. However, $m_t(c_t)$ is found both in the base and exponent of (16) and thus there is no closed form solution for $m_t(c_t)$ and hence also $(r_t^i)^*$

2.3.1 Our observations

In our observations, contrary to (16), we arrived at a different result

$$\begin{aligned} (P_t^i)^* &= \frac{\mu}{m_t(c_t)} \exp \left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - \mu - m_t(c_t)(P_t^i)^*}{\mu} \right) \\ &= P_t^{0*} \exp \left(\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t) - u_0 - \mu - m_t(c_t)(P_t^i)^*}{\mu} \right) \end{aligned} \quad (22)$$

However, $\Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)$ relies on P_t^{0*} and $(P_t^i)^*$ s, thus we do not have a closed form solution for (22). Updating the equation in (18), we have the equation below.

$$\left[\frac{m_t(c_t)}{\mu} - 1 \right] \exp \left[1 + \frac{u_0}{\mu} \right] = \sum_{i=1}^n \exp \left[\frac{q^i - \Delta^i \hat{\mathbf{w}}_t^{\mathbf{P}^*}(c_t)}{\mu} \right] \quad (23)$$

3 Dynamic Programming For Optimal Dynamic Pricing

4 Discussions and Conclusions

This is where we can add our comments and our inputs, how the model can be further improved or how we can find estimates for the solution. Closing conclusions, further areas that can be explored and research opportunities.

References

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- [2] J. S. Li and S. Chen. Real-time dynamic pricing for multiproduct models with time-dependent customer arrival rates. *Proceedings of the American Control Conference*, pages 2196–2201, 2009.