1 Best Response Dynamics

While the current outcome is not a Pure Nash equilibirum (PNE), we can pick an arbitrary player i and an arbitrary beneficial deviation s'_i for player i and move to outcome $(s'_i, \mathbf{s_{-i}})$.
Recall that the definition of a potential game is one where there exists a function $\Phi : \mathcal{S} \to \mathbb{R}$ where $\mathcal{S}$ is the finite set of strategies with

$$\Phi(s'_i, s_{-i}) - \Phi(s_i, s_{-i}) = c_i(s'_i, s_{-i}) - c_i(s_i, s_{-i})$$

Proposition 1.1. In a finite potential game from any arbitrary outcome, best-response dynamics converge to a PNE.

Proof. In a best-response dynamics approach, every iteration has $\Phi(\mathbf{s^{t+1}}) < \Phi(\mathbf{s^t})$, i.e. the potential decreases. Unless the $\mathbf{s^t}$ is a PNE, our Φ is lower bounded by $\min_{s\in\mathcal{S}} \Phi(s)$ and hence the process must terminate. □

Definition 1.2 ($\epsilon-$Pure Nash Equilibrium)**.** For $\epsilon \in [0, 1]$, and outcome $\mathbf{s}$ is an $\epsilon-$pure NE if for every agent i and deviations $s'_i \in S_i$

$$c_i(s'_i, s_{-i}) \geq (1 - \epsilon)c_i(s_i, s_{-i})$$

An $\epsilon-$best response dynamics is one which permits moves when there is significant improvements (substential lowering of cost or increasing of utility) which is an important factor to for a state to converge to near optimal equilibrium. While a current outcome $\mathbf{s}$ is not an $\epsilon-$PNE, we pick an arbitary player i that has an $\epsilon-$move, i.e. a deviation to s'_i:

$$c_i(s'_i, s_{-i}) < (1 - \epsilon)c_i(\mathbf{s})$$

Lemma 1.3. For $x \in (0, 1)$

$$(1 - x)^{1/x} \leq (e^{-x})^{1/x} = e^{-1}$$

Theorem 1.4 (Fast convergence of $\epsilon-$Best Response Dynamics)**.** Consider an atomic selfish routing game where:

1. All players have the same source s and destination t vertex.
2. Cost function satisfy the "$\alpha-$bound jump condition"

   $$c_e(x) \leq c_e(x + 1) \leq \alpha \cdot c_e(x)$$

   for all edges e.
3. The MaxGain variant of $\epsilon-$BR dynamics is used: in every iteration, amongst all players with an $\epsilon-$move available, the player who can obtain the biggest absolute cost decrease gets to move.

Then an $\epsilon-$PNE is reached in at most

$$\frac{k \cdot \alpha}{\epsilon} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}}$$

iterations, where k is the number of agents, $\mathbf{s^0}$ is the initial state of the system.

Proof. Using lemma 1.5 we pick the agent i with the highest cost to get

$$\begin{aligned}\Phi(\mathbf{s}) - \Phi(s'_i, s_{-i}) = c_i(\mathbf{s}) - c_i(s'_i, s_{-i}) &\geq \frac{\epsilon}{\alpha k} \cdot c_i(\mathbf{s}), \quad \text{by Lemma 1.6} \\ &\geq \frac{\epsilon}{\alpha k} \cdot \Phi(\mathbf{s})\end{aligned}$$

thus we have

$$\left(1 - \frac{\epsilon}{\alpha k}\right) \Phi(\mathbf{s^t}) \geq \Phi(\mathbf{s^{t+1}})$$

thus using Lemma 1.3 we obtain that an $\epsilon-$PNE is reached in $\frac{\alpha k}{\epsilon} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}}$ iterations. □

The two lemmas below are the ones used in the proof.

Lemma 1.5. For all $\mathbf{s} \in \mathcal{S}$ there exists an agent such that

$$c_i(s) \geq \frac{\Phi(\mathbf{s})}{k}$$

Proof. Recall that $\Phi(\mathbf{s}) \leq cost(\mathbf{s})$, then pick the agent that realizes the highest cost, $i = \text{argmax}_i\, c_i(\mathbf{s})$, then

$$c_i(\mathbf{s}) \geq \frac{cost(\mathbf{s})}{k} \geq \frac{\Phi(\mathbf{s})}{k}$$

□

Lemma 1.6. Suppose player i is chosen at outcome s by MaxGain $\epsilon-$best response dynamics and he takes the $\epsilon-$ move s'_i,then

$$c_i(\mathbf{s}) - c_i(s'_i, s_{-i}) \geq \frac{\epsilon}{\alpha} c_j(\mathbf{s}) \tag{1}$$

for any other agent j.

Proof. For the case when $j = i$, when $\alpha = 1$, it is exactly the definition of the $\epsilon-$move. Now consider when $i \neq j$ with j having an $\epsilon-$move, by MaxGain dynamics and $\alpha = 1$,

$$c_i(\mathbf{s}) - c_i(s'_i, s_{-i}) \geq c_j(\mathbf{s}) - c_j(s'_j, s_{-j}) > \epsilon \cdot c_j(\mathbf{s})$$

the proof is completed by with the case where j does not have an $\epsilon-$move, which we consider $-$ since s'_i is such a great deviation for player i, why isn't it good for player j? That is

$$c_i(s'_i, s_{-i}) < (1-\epsilon) c_i(\mathbf{s})$$

while

$$c_j(s'_i, s_{-j}) \geq (1-\epsilon) c_j(\mathbf{s})$$

and here we used the condition that the agents have the same source and sink vertex, i.e. they have the same set of strategies. An observation made here is that (s'_i, s_{-i}) and (s'_i, s_{-j}) have at least $k-1$ strategies in common (not that s'_i is played by agent i in the former and agent j in the latter.) □

Theorem 1.7. Consider a $(\lambda, \mu)-$cost minimization game with a positive potential function Φ such that $\Phi(\mathbf{s}) \leq cost(\mathbf{s})$ for every outcome $\mathbf{s}$. Let $\mathbf{s^0}, \mathbf{s^1}, \ldots, \mathbf{s^T}$ be a sequence generated by MaxGain best response dynamics, $\mathbf{s}^*$ a minimum cost outcome and $1 > \gamma > 0$ is a parameter, Then for all but

$$\frac{k}{\gamma(1-\mu)} \log \frac{\Phi(\mathbf{s^0})}{\Phi_{min}} \tag{2}$$

outcomes $\mathbf{s}^t$ satisfy

$$cost(\mathbf{s^t}) \leq \left(\frac{\lambda}{(1-\mu)(1-\gamma)} \right) \cdot cost(\mathbf{s}^*) \tag{3}$$

Proof.

$$\begin{aligned}
cost(\mathbf{s^t}) &\leq \sum_i c_i(\mathbf{s^t}) \\
&= \sum_i \left[c_i(s_i^*, s_{-i}^t) + \delta_i(\mathbf{s^t}) \right], \quad \delta_i(\mathbf{s^t}) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) \\
&\leq \lambda \cdot cost(\mathbf{s}^*) + \mu \cdot cost(\mathbf{s^t}) + \sum_i \delta_i(\mathbf{s^t}) \\
cost(\mathbf{s^t}) &\leq \frac{\lambda}{1-\mu} \cdot cost(\mathbf{s}^*) + \frac{1}{1-\mu} \cdot \sum_i \delta_i(\mathbf{s^t}) \qquad (4)
\end{aligned}$$

we shall let $\Delta(\mathbf{s^t}) = \sum_i \delta_i(\mathbf{s^t})$ in the remaining parts of the proof. We shall now define a state $\mathbf{s^t}$ to be bad if it does not satisfy (3) and by (4), when $\mathbf{s^t}$ is bad we get

$$\Delta(\mathbf{s^t}) \geq \gamma(1-\mu) \cdot cost(\mathbf{s^t})$$

By the MaxGain definition and the inequality relating the potential function and cost,

$$\max_i \delta_i(\mathbf{s^t}) \geq \frac{\Delta(\mathbf{s^t})}{k} \geq \frac{\gamma(1-\mu)}{k} \cdot cost(\mathbf{s^t}) \geq \frac{\gamma(1-\mu)}{k} \cdot \Phi(\mathbf{s^t})$$

and we get what we desire as

$$\Phi(\mathbf{s^t}) - \Phi(s_i^*, s_{-i}^t) = c_i(\mathbf{s^t}) - c_i(s_i^*, s_{-i}^t) = \delta_i(\mathbf{s^t})$$

and hence

$$\left(1 - \frac{\gamma(1-\mu)}{k}\right) \Phi(\mathbf{s^t}) \geq \Phi(\mathbf{s^{t+1}}) \tag{5}$$

whenever $\mathbf{s^t}$ is a bad state. The equation in (5) says that for every MaxGain best response dynamics, if the state is bad, the new state $\mathbf{s^{t+1}}$ is smaller than the previous state $\mathbf{s^t}$ by a factor of $1 - \frac{\gamma(1-\mu)}{k}$. By Lemma 1.3, the potential decreases by a factor of e for every $\frac{k}{\gamma(1-\mu)}$ bad states encountered. Thus solving

$$e^{-n}\Phi(\mathbf{s^0}) \geq \Phi_{min}$$

shows (2). □