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Machine Learning: Homework 4

1. For (a) and (b), as D is not part of the requirement, we assume D to be a variable that is not connected to any of the other variables when represented as a DAG.

   (a) True. Consider B to be that event of flipping a coin with probability of heads to be p, A to be the event that it mirrors the outcome of the coin flip with probability 1 and C to be the event that it mirrors the opposite of the outcome of the coin flip with probability 1. There is no direct dependency between A and C, but we see that

$$\mathbb{P}(A=a|B=b) := \begin{cases} 1, & a=H, b=H \\ 1, & a=T, b=T \\ 0, & \text{otherwise} \end{cases}$$
$$\mathbb{P}(C=c|B=b) := \begin{cases} 1, & c=H, b=T \\ 1, & c=T, b=H \\ 0, & \text{otherwise} \end{cases}$$

   and observe that

$$\mathbb{P}(A) = \sum_{B\in\{H,T\}} \mathbb{P}(B)\mathbb{P}(A|B) \qquad \mathbb{P}(C) = \sum_{B\in\{H,T\}} \mathbb{P}(B)\mathbb{P}(C|B)$$
$$\mathbb{P}(A,C) = \sum_{B\in\{H,T\}} \mathbb{P}(B)\mathbb{P}(A|B)\mathbb{P}(C|B)$$

   Since $\mathbb{P}(A=H)\mathbb{P}(C=H) = p(1-p)$ and $\mathbb{P}(A=H, C=H) = 0$, it is shown that A and C are not independent, since $p \neq 0$. Alternatively, this can be explained by the common causes effect.

   (b) False. Let B be the event that the weather is sunny, A be the event of getting heads when flipping a fair coin and C be the event that is True when A is heads. Clearly both A and C is independent of B, however,

$$1 = \mathbb{P}(C=True|A=Heads) \neq \mathbb{P}(C=True) = 0.5$$
$$0 = \mathbb{P}(C=True|A=Tails) \neq \mathbb{P}(C=True) = 0.5$$

   which shows that A and C are not independent of each other.

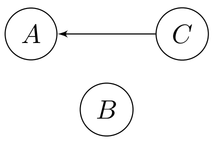
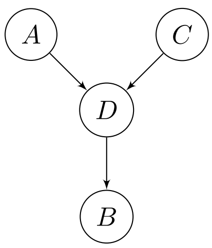


Figure 1: Model of 1(b)

   (c) False. Consider the following model below, we see that it fulfills the requirement that A is independent of B



   given D and C is independent of B given D by chaining. However, the explaining away effect tells us that A and C are dependent of each other.

2. (a) The parameters associated with the HMM are the transmission probabilities, $a_{i,j} = p(y_{next} = j|y = i)$ for $i, j \in \{\text{START}, X, Y, Z, \text{STOP}\}$ and the emission probabilities, $b_j(o) = p(x = o|y = j)$ where $o \in \{a, b, c\}$ and $j \in \{X, Y, Z\}$.

$a_{i,j} : i\backslash j$	X	Y	Z	STOP
START	1/2	0	1/2	0
X	2/5	2/5	1/5	
Y	1/5	0	1/5	3/5
Z	2/5	3/5	0	0

$b_j(0)$:$b_u(o)$ $u\backslash o$	a	b	c
X	2/5	1/5	2/5
Y	2/5	2/5	1/5
Z	1/5	3/5	1/5

(b) To solve for the probability $p(x_1 = a, x_2 = b)$, we use the fact that

$$p(y_0, \ldots, y_3, x_1, x_2) = \prod_{i=0}^{3} a_{y_i, y_{i-1}} \prod_{i=1}^{2} b_{y_i}(x_i).$$

$$\text{and} \quad p(x_1 = a, x_2 = b) = \sum_{\substack{y_0=\text{START}, \\ y_1, y_2 \in \{X,Y,Z\}, \\ y_3=\text{STOP}}} p(y_0, y_1, y_2, y_3, x_1 = a, x_2 = b) \tag{1}$$

The only nonzero term in the summation of (1) is when $y_1 = Z, y_2 = X$ which gives a probability of 0.00648. Thus $p(x_1 = a, x_2 = b) = 0.00648$.

(c) We see that X_1, X_6 has no parents, $X_2, X_3, X_4, X_5, X_7, X_8, X_{10}, X_{11}$ has one parent each and X_9 has 3 parents, thus the number of free parameters is $(2 \times 1) + (8 \times 2) + (1 \times 8) = 26$

(d) We see from the probability tables that X_3 is independent of X_2 and X_{10} is independent of X_9. Thus X_1 is independent of X_3 and X_{11} is independent of X_9. Thus

$$\begin{aligned} P(X_5 = 2|X_3 = 1, X_{11} = 2, X_1 = 1) = P(X_5 = 2|X_3 = 1) &= \sum_{v \in \{1,2\}} P(X_4 = v|X_3) P(X_5|X_4) \\ &= (0.1 \times 0.5) + (0.9 \times 0.4) = 0.41 \end{aligned}$$