Zhangsheng Lai
1002554

Stochastic Models: Exercise 5

1. Let $\{X_n : n \geq 0\}$ be an irreducible Markov chain with period $d \geq 1$ thus for any state i

$$d = \gcd\{n \geq 1 : P[X_n = i \mid X_0 = i] > 0\}$$

Suppose $\{X_{nd} : n \geq 0\}$ is not aperiodic, thus for some integer $k > 1$, we have

$$k = \gcd\{n \geq 1 : P[X_{nd} = i \mid X_0 = i] > 0\}$$

for every state i. This implies that, for all states, the number of transitions needed to return to state i given that it starts from i in $\{X_n : n \geq 0\}$ is of the form lkd where l is a positive integer. This contradicts that $\{X_n : n \geq 0\}$ is a Markov chain with period d since for any integer l, $lkd > d$, thus $k = 1$ as required and $\{X_{nd} : n \geq 0\}$ as aperiodic.

2. Suppose $i \leftrightarrow j$ and let i be positive recurrent, thus $\lim_{n\to\infty} P_{ii}^n > 0$. Let d be the smallest integer such that $P_{ij}^d \neq 0$. Suppose $\lim_{n\to\infty} P_{jj}^n = 0$, then

$$\begin{aligned} 0 = \lim_{n\to\infty} P_{jj}^{n+d} &= \lim_{n\to\infty} \sum_{k=0}^{\infty} P_{jk}^n P_{kj}^d \\ &\geq \lim_{n\to\infty} \sum_{k=0}^{M} P_{jk}^n P_{kj}^d \quad \text{for all } M \\ &= \sum_{k=0}^{M} \pi_k P_{kj}^d \quad \text{for all } M \\ &= \sum_{k=0}^{\infty} \pi_k P_{kj}^d \end{aligned}$$

This implies that for every k, $\pi_k P_{kj}^d = 0$ and in particular $\pi_i P_{ij}^d = 0$. Since $\pi_i > 0$, we will need $P_{ij}^d = 0$, a contradiction.

3. Let $\{X_n : n \geq 0\}$ be an irreducible and aperiodic Markov chain. The chain is doubly stochastic, thus $\sum_i P_{ij} = 1$. For any two states i and j, we have $i \leftrightarrow j$ since the Markov chain is irreducible and together with the aperiodicity, we have $\lim_{n\to\infty} P_{ij}^n = 1/\mu_{jj}$. We can prove by induction that $\sum_{i=0}^{k} P_{ij}^n = 1$. Thus

$$1 = \lim_{n\to\infty} \sum_{i=0}^{k} P_{ij}^n = \sum_{i=0}^{k} \lim_{n\to\infty} P_{ij}^n = (k+1)/\mu_{jj}$$

Thus $\mu_{jj} = k + 1 > 0$ implies all the states are positive recurrent. Thus there exists a unique stationary distribution that is also the limiting distribution, i.e. $\pi_j = 1/\mu_{jj}$. Hence $\pi_j = 1/k + 1$ for all j.

We shall prove the claim that $\sum_{i=0}^{k} P_{ij}^n = 1$. It is easy to see that it holds for $n = 1$. Suppose that it is true for n, then since we have

$$\sum_{i=0}^{k} P_{ij}^{n+1} = \sum_{i=0}^{k} \sum_{l=0}^{k} P_{il}^n P_{lj} = \sum_{l=0}^{k} \left(\sum_{i=0}^{k} P_{il}^n \right) P_{lj}$$

it is also true for $n + 1$, which proves the claim.

4. Let $\{X_n : n \geq 0\}$ be a Markov chain with states $S = \{0, 1, 2, 3, 4\}$ denoting the number of umbrella(s) in the new location after travelling from the previous one, thus $S = \{0, 1, 2, 3, 4\}$.

   (a) We first observe that

$$X_{n+1} := \begin{cases} 4 & X_n = 0 \\ 4 - X_n + 1 & \text{if raining} \\ 4 - X_n & \text{if not raining} \end{cases}$$

with this, the transition matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-p & p \\ 0 & 0 & 1-p & p & 0 \\ 0 & 1-p & p & 0 & 0 \\ 1-p & p & 0 & 0 & 0 \end{pmatrix}$$

The proportion of time that he possibly gets wet is when he is in state 0, and the proportion of time that he gets wet is $p\pi_0$. Solving for $\pi_j = \sum_i \pi_i P_{i,j}$

$$\begin{aligned} \pi_0 &= (1-p)\pi_4 \\ \pi_1 &= (1-p)\pi_3 + p\pi_4 \\ \pi_2 &= (1-p)\pi_2 + p\pi_3 \\ \pi_3 &= (1-p)\pi_1 + p\pi_2 \\ \pi_4 &= \pi_0 + p\pi_1 \end{aligned}$$

which solving for yields π_i are the same for $i = 1, 2, 3, 4$. Thus $(4 + 1 - p)\pi_1 = 1$ and

$$\pi_0 = \frac{1-p}{5-p}, \qquad \pi_i = \frac{1}{5-p} \text{ for } i \neq 0$$

Thus Tom gets wet with probability $\frac{p(1-p)}{5-p}$ in the long run.

(b) From (a), we can generalise it to for r umbrellas:

$$\pi_0 = \frac{1-p}{r+1-p}, \qquad \pi_i = \frac{1}{r+1-p} \text{ for } i \neq 0$$

which then we have to solve for r when $p = 0.6$,

$$\frac{p(1-p)}{r+1-p} < 0.01$$

which yields $23.6 < r$, thus Tom should have 24 umbrellas.

5. (a) Let $\{X_n : n \geq 0\}$ be a Markov chain with states of the form $(i, k-i)$ for $i = 0, 1, \dots, k$. We shall denote state $(i, k-i)$ by i. Then the transition matrix is given by

$$P_{0,0} = 3(1/2)^2 = P_{k,k} \quad P_{0,1} = (1/2)^2 = P_{k,k-1}$$
$$P_{i,i} = 2(1/2)^2 \quad P_{i,i+1} = (1/2)^2 = P_{i,i-1} \quad \text{for } i \neq 0, k$$

(b) We first note that the Markov chain is irreducible as $i \leftrightarrow j$ for any states i and j. It is aperiodic as $P[X_1 = i \mid X_0 = i] > 0$ for all states. The transition matrix is also doubly stochastic as

$$\begin{aligned} \text{when } i \neq 0, k \quad & \sum_i P_{i,j} = P_{i-1,i} + P_{i,i} + P_{i+1,i} = 1 \\ \text{when } i = 0 \quad & \sum_i P_{i,j} = P_{0,0} + P_{1,0} = 1 \\ \text{when } i = k \quad & \sum_i P_{i,j} = P_{k-1,k} + P_{k,k} = 1 \end{aligned}$$

Hence by question 3, we have $\pi_0 = \pi_k = 1/k + 1$. Thus the proportion of time where there is only shoes at one door is $2/k + 1$ and since he choose to depart the front or back door with equal chance, he runs barefooted $1/k + 1$ of the time.

6. (a) We first note that with X_n being the number of rolls in the warehouse at the beginning of the nth day, we have $X_{n+1} = X_n - 1 + k$, where k is the number of rolls delivered by the local distributor in the evening. Thus, the states are $S = \{0, 1, 2, \dots\}$ and the transition probabilities that make up the transition matrix $\mathbb{P}$ is

$$P_{i,j} := \begin{cases} a_{j-i+1} & \text{if } j \geq i-1,\ i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

when $i = 0$, we have $P_{0,j} = a_j$.

(b) We observe that this Markov chain is irreducible and aperiodic, thus all the states are either positive or null recurrent. The stationary distribution need to satisfy

$$\pi_j = \sum \pi_i P_{i,j} = \pi_0 P_{0,j} + \pi_1 P_{1,j} + \pi_2 P_{2,j} + \ldots + \pi_{j+1} P_{j+1,j}$$

Trying out for the first few values:

$$\begin{aligned}
\pi_0 &= \pi_0 a_0 + \pi_1 a_0 \\
\pi_1 &= \pi_0 a_1 + \pi_1 a_1 + \pi_2 a_0 \\
\pi_2 &= \pi_0 a_2 + \pi_1 a_2 + \pi_2 a_1 + \pi_3 a_0
\end{aligned}$$

we can generalise it to

$$\begin{aligned}
\pi_{n+1} &= q\pi_{n-1} + (1-q)\pi_{n+1} \implies \pi_{n+1} = \frac{1}{1-q}(\pi_n - q\pi_{n-1}) \\
\pi_1 &= \frac{q}{1-q}\pi_0 \\
\pi_2 &= \frac{1}{1-q}\left(\frac{q}{1-q}\pi_0 - q\pi_0\right) = \left(\frac{q}{1-q}\right)^2 \pi_0 \\
\pi_3 &= \frac{1}{1-q}\left(\left(\frac{q}{1-q}\right)^2 \pi_0 - \frac{q^2}{1-q}\pi_0\right) = \left(\frac{q}{1-q}\right)^3 \pi_0 \\
&\vdots \\
\pi_n &= \left(\frac{q}{1-q}\right)^n \pi_0
\end{aligned}$$

For it to be a stationary distribution we need:

$$\pi_0 \sum_{i=0}^{\infty} \left(\frac{q}{1-q}\right)^i = 1$$

to converge which happens when $|q/1-q| < 1$.