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Real Analysis: Homework 1

1. (a) $\mathbb{R}$ is second-countable by considering the countable basis

$$\mathcal{B} := \{(r-\epsilon, r+\epsilon) | r \in \mathbb{Q}, \text{ for any arbitrary } \epsilon > 0\}$$

We now claim that $\mathcal{B}^n = \{U_1 \times \ldots \times U_n) | \text{ each } U_i \in \mathcal{B} \text{ for } i = 1, \ldots, n\}$ is a countable basis for $\mathbb{R}^n$. It is clear that $\mathcal{B}^n$ is countable as the Cartesian product of countable sets is still countable. To show $\mathcal{B}^n$ is a basis for $\mathbb{R}^n$:

(1) Pick $x = (x_1 \ldots, x_n) \in \mathbb{R}^n$ and consider the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $(x_1 \ldots, x_n) \mapsto x_i$. Thus for each $\pi_i(x) = x_i$ we can find $B_i \in \mathcal{B}$ such that $x_i \in B_i$. Thus $B_1 \times \ldots \times B_n$ is the basis element in $\mathcal{B}^n$ containing x.

(2) Let x belong to the intersection of two basis elements $U = B_1 \times \ldots \times B_n, U' = B'_1 \times \ldots \times B'_n$. Using the projection map, $\pi_i(U) = B_i, \pi_i(U') = B'_i$ and thus there is a basis element $A_i \subseteq B_i \cap B'_i$ for some $A_i \in \mathcal{B}$. Thus $A = A_i \times \ldots \times A_n$ is the basis element in $\mathcal{B}^n$ such that $A \subseteq U \cap U'$.

Thus we have shown that $\mathcal{B}^n$ is a countable basis for $\mathbb{R}^n$.

(b) Let U be an open set of $\mathbb{R}$. Then since $\mathbb{R}$ is second-countable, there exists a countable basis $\mathcal{B}$, for $\mathbb{R}$. Thus for each $x \in U$ we can find $B \in \mathcal{B}$ with $x \in B$. Suppose $\{B_\alpha\}_{\alpha \in A}$ is a countable set with $B_\alpha \in \mathcal{B}$ such that each $x \in U$ is in some B_α. Still need some work...

2. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Let (X, τ'_X) be a finer topology than (X, τ_X) then $\tau'_X \supseteq \tau_X$. Thus for any $U \in \tau_Y$, $f^{-1}(U) \in \tau_X \subseteq \tau'_X$. Thus $f^{-1}(U) \in \tau'_X$ and $f : (X, \tau'_X) \to (Y, \tau_Y)$ remains continuous. Let (Y, τ'_Y) is a topology coarser than (Y, τ_Y) and so $\tau_Y \supseteq \tau'_Y$. Hence for $U \in \tau'_Y \subseteq \tau_Y$, we have $f^{-1}(U) \in \tau_X$. Thus $f : (X, \tau_X) \to (Y, \tau'_Y)$ remains continuous.

3. We shall show that $\int_{\mathbb{R}} e^{-x^2/2}\, dx = \sqrt{2\pi}$. Start by considering its square,

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2/2}\, dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2/2}\, dx\right)\left(\int_{\mathbb{R}} e^{-y^2/2}\, dy\right) \\
&= \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dydx \\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2/2} r\, drd\theta \quad \text{by polar change of coordinates.} \\
&= 2\pi \int_0^\infty e^{-r^2/2} r\, dr \\
&= 2\pi \int_0^\infty \frac{1}{2} e^{-s/2}\, ds \quad \text{change of coordinates, } s = r^2 \\
&= \pi \Big[-2e^{-s/2} \Big]_0^\infty = 2\pi
\end{aligned}$$

which shows what is required and hence $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2}\, dx = 1$.

4. Let $x_n > 0$ for all n and $x_n \to a$ with $a > 0$. Then

$$\log\left((x_1 x_2^2 \ldots x_n^n)^{\frac{1}{n^2}}\right) = \sum_{i=1}^n \frac{i}{n^2} \log x_i$$

5.