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Real Analysis: Homework 2

1. (a) Let $f(x,y) = \cosh x \cosh y$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$F'(h) = \langle \nabla f(h\vec{v}), \vec{v}\rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy$$
$$F''(h) = \nabla^2 f(h\vec{v})(\vec{v},\vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$F(0) = 1 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2$$

Thus the polynomial of second degree that best approximate $f(x,y)$ is $1 + \frac{1}{2}(x^2 + y^2)$.

(b) Let $g(x,y) = \sin(x^2 + y^2)$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$G(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned} G'(h) &= \langle \nabla g(h\vec{v}), \vec{v}\rangle = x(2hx\cos((hx)^2 + (hy)^2)) + y(2hy\cos((hx)^2 + (hy)^2)) \\ G''(h) &= \nabla^2 g(h\vec{v})(\vec{v},\vec{v}) \\ &= x^2(2\cos((hx)^2 + (hy)^2) - 4(xh)^2\sin((hx)^2 + (hy)^2)) \\ &- 2xy(4xyh^2\sin((hx)^2 + (hy)^2)) \\ &+ y^2(2\cos((hx)^2 + (hy)^2) - 4(yh)^2\sin((hx)^2 + (hy)^2)) \end{aligned}$$

and

$$G(0) = 0 \quad G'(0) = 0 \quad G''(0) = 2x^2 + 2y^2$$

Thus the polynomial of second degree that best approximate $g(x,y)$ is $x^2 + y^2$.

2. (a) We observe that $f(x,y)$ is continuous at all points $(x,y) \neq (0,0)$ since the denominator is nonzero. Thus we need to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ to show $f(x,y)$ is everywhere continuous.

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} xy\frac{x^2}{x^2 + y^2} - \lim_{(x,y)\to(0,0)} xy\frac{y^2}{x^2 + y^2}$$

then since,

$$0 \leq \frac{x^2}{x^2 + y^2} \leq 1 \ , \qquad -|xy| \leq xy \leq |xy|$$
$$-|xy| \leq xy\frac{x^2}{x^2 + y^2} \leq |xy|$$

and $\lim_{(x,y)\to(0,0)} \pm|xy| = 0$, which by Squeeze theorem gives us $\lim_{(x,y)\to(0,0)} xy\frac{x^2}{x^2+y^2} = 0$. We use the same argument and get the limit to be zero for the second term which shows the desired. We have $\nabla f = (f_x, f_y)$, which are given below,

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, \qquad f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

Again it suffices to show that it is continuous at $(0,0)$. We see that

$$f_x = y\left(1 - \frac{2y^4}{(x^2+y^2)^2}\right), \qquad f_y = -x\left(1 - \frac{2x^4}{(x^2+y^2)^2}\right)$$

and $-1 \leq 1 - \frac{2y^4}{(x^2+y^2)^2}, 1 - \frac{2x^4}{(x^2+y^2)^2} \leq 1$. Thus

$$\begin{aligned}-y \leq & y\left(1 - \frac{2y^4}{(x^2+y^2)^2}\right) \leq y \\ -x \leq & -x\left(1 - \frac{2x^4}{(x^2+y^2)^2}\right) \leq x\end{aligned}$$

which by Squeeze theorem, we get $\lim_{(x,y)\to(0,0)} f_x = 0 = \lim_{(x,y)\to(0,0)} f_y$.

(b) The computation gives,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \frac{\partial^2 f}{\partial y \partial x}$$

To show it is continuous everywhere we have to show that $\lim_{(x,y)\to(0,0)} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$ exists and is well defined along different paths.

$$\begin{aligned}\text{Along } y = x, \quad & \lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = 0 \\ \text{Along } y = 2x, \quad & \lim_{(x,y)\to(0,0)} \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3} = \lim_{(x,y)\to(0,0)} \frac{x^6 + 36x^6 - 144x^6 - 64x^5}{(x^2+4x^2)^3} \\ & = -171/125\end{aligned}$$

3. We recall the geometric series,

$$\begin{aligned}\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \ , \qquad \text{where } |x| < 1 \\ \text{substituting } x \text{ with } -x \ , \frac{1}{1+x} &= \sum_{k=0}^{\infty} (-1)^k x^k \ , \qquad \text{where } |x| < 1 \\ \text{substituting } x \text{ with } x^2 \ , \frac{1}{1+x^2} &= \sum_{k=0}^{\infty} (-1)^k x^{2k} \ , \qquad \text{where } |x| < 1\end{aligned}$$

we can then do integration term wise on the right hand side while integrating $\frac{1}{1+x^2}$,

$$\begin{aligned}\pi/4 = \tan^{-1}(1) = \int_0^1 \frac{1}{1+t^2}\,dt &= \sum_{k=0}^{\infty} (-1)^k \int_0^1 t^{2k}\,dt \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}\end{aligned}$$

thus $\pi = 4\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$.

4. (a) Let $f(t) = e^{-\frac{(x-t)^2}{2}} = e^{-x^2/2} \cdot e^{xt-\frac{t^2}{2}}$. Then $f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$, where

$$\begin{aligned} f^{(k)}(0) &= \left.\frac{d^k}{dt^k} e^{-\frac{(x-t)^2)}{2}}\right|_{t=0} \\ &= (-1)^k \left.\frac{d^k}{du^k} e^{-\frac{u^2}{2}}\right|_{u=x} \qquad \text{letting } u = x-t, \text{ so } \frac{d}{du} = -\frac{d}{dt} \\ &= (-1)^k \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \end{aligned}$$

then

$$\begin{aligned} e^{-x^2/2} \cdot e^{xt-\frac{t^2}{2}} &= \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \\ e^{xt-\frac{t^2}{2}} &= e^{x^2/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}} \\ &= \sum_{k=0}^{\infty} t^k H_k(x) \end{aligned}$$

It converges because it is a Taylor series of an exponential which has radius of convergence, $R < \infty$.

(b) (i)

$$\begin{aligned} H_n'(x) &= xH_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{x^2}{2}} \\ &= xH_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (-xe^{-\frac{x^2}{2}}) \\ &= xH_n(x) + \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \left[-x \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} - n \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}} \right], \text{ by General Leibniz Rule} \\ &= \frac{(-1)^{n-1}}{(n-1)!} e^{\frac{x^2}{2}} \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}} = H_{n-1}(x) \end{aligned}$$

(ii)

$$\begin{aligned} (n+1)H_n(x) &= (n+1) \frac{(-1)^{n+1}}{(n+1)!} e^{\frac{x^2}{2}} \frac{d^{n+1}}{dx^{n+1}} e^{-\frac{x^2}{2}} \\ &= \frac{(-1)^{n+1}}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} - xe^{-\frac{x^2}{2}} \\ &= \frac{(-1)^{n+1}}{n!} e^{\frac{x^2}{2}} \left[-x \frac{d^n}{dx^n} e^{\frac{-x^2}{2}} - n \frac{d^{n-1}}{dx^{n-1}} e^{\frac{-x^2}{2}} \right] \\ &= xH_n(x) - H_{n-1}(x) \end{aligned}$$

(iii) Let $y = -x$, then

$$\begin{aligned} H_n(y) &= \frac{(-1)^n}{n!} e^{\frac{y^2}{2}} \frac{d^n}{dy^n} e^{-\frac{y^2}{2}} \\ &= \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} (-1)^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \qquad \text{since } \frac{d}{dy} = -\frac{d}{dx} \\ &= (-1)^n H_n(x) \end{aligned}$$

(c) (i)

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{sx-\frac{s^2}{2}}\, e^{tx-\frac{t^2}{2}}\, e^{-\frac{x^2}{2}}\, dx = e^{st}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{(x-(s+t))^2}{2}}\, dx = e^{st}$$

(ii) Applying the partial derivatives $\left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s,t=0}$ to (i), we get

$$\begin{aligned} 0 = \left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s,t=0} e^{st} &= \left.\frac{\partial^{n+m}}{\partial s^n \partial t^m}\right|_{s,t=0} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{sx-\frac{s^2}{2}} e^{tx-\frac{t^2}{2}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \left.\frac{\partial^n}{\partial s^n}\right|_{s=0} e^{sx-\frac{s^2}{2}}\ \left.\frac{\partial^m}{\partial t^m}\right|_{t=0} e^{tx-\frac{t^2}{2}} e^{-\frac{x^2}{2}}\, dx \end{aligned}$$

and since from 4(a),

$$\left.\frac{\partial^n}{\partial s^n}\right|_{s=0} e^{sx-\frac{s^2}{2}} = H_n(x)$$

we are done.