Zhangsheng Lai
1002554

Real Analysis: Homework 2

1. (a) Let $f(x,y) = \cosh x \cosh y$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned}
F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\
F''(h) &= \nabla^2 f(h\vec{v})(\vec{v}, \vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
F'''(h) &= \nabla^3 f(h\vec{v})(\vec{v}, \vec{v}, \vec{v}) = \sum_{i,j,k=1,2} \frac{f(h\vec{v})}{\partial e_i \partial e_j \partial e_k} v_i v_j v_k \\
&= (x^3 + 3xy^2)(\sinh hx \cosh hy) + (y^3 + 3x^2 y)(\cosh hx \cosh hy)
\end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2 \quad F'''(0) =$$

Thus the polynomial of second degree that best approximate $f(x,y)$ is $\frac{1}{2}(x^2 + y^2)$.

(b) Let $g(x,y) = \sin(x^2 + y^2)$, with $\vec{x} = (0,0)$, $\vec{v} = (x,y)$,

$$F(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned}
F'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\
F''(h) &= \nabla^2 g(h\vec{v})(\vec{v}, \vec{v}) \\
&= x^2(2\cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\
&- 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\
&+ y^2(2\cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2))
\end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = 2x^2 + 2y^2$$

Thus the polynomial of second degree that best approximate $g(x,y)$ is $x^2 + y^2$.

2. (a)

(b)

$$\frac{\partial^2 f}{\partial x \partial y}$$

3. We recall the geometric series,

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \ , \qquad \text{where } |x|<1$$

$$\text{substituting } x \text{ with } -x \ , \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \ , \qquad \text{where } |x|<1$$

$$\text{substituting } x \text{ with } x^2 \ , \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \ , \qquad \text{where } |x|<1$$

we can then do integration term wise on the right hand side while integrating $\frac{1}{1+x^2}$,

$$\begin{aligned}\pi/4 = \tan^{-1}(1) = \int_0^1 \frac{1}{1+t^2}\,dt &= \sum_{k=0}^{\infty}(-1)^k \int_0^1 t^{2k}\,dt \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\end{aligned}$$

thus $\pi = 4\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$.

4. (a) We know that $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$, then

$$\begin{aligned}H_n(x) &:= \frac{(-1)^n}{n!}e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}} \\ e^{tx-\frac{t^2}{2}} &= \sum_{k=0}^{\infty}\frac{(tx-\frac{t^2}{2})^k}{k!} \\ &= \sum_{k=0}^{\infty} t^k \frac{(-1)^k}{k!}e^{\frac{x^2}{2}}\frac{d^k}{dx^k}e^{-\frac{x^2}{2}}\end{aligned}$$

(b) (i)

$$H_n'(x=)$$

(c)