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Real Analysis: Homework 2

1. (a) Let $f(x, y) = \cosh x \cosh y$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$F(h) := f(\vec{x} + h\vec{v}) = f(h\vec{v}) = \cosh hx \cosh hy$$

then

$$\begin{aligned} F'(h) &= \langle \nabla f(h\vec{v}), \vec{v} \rangle = x \sinh hx \cosh hy + y \cosh hx \sinh hy \\ F''(h) &= \nabla^2 f(h\vec{v})(\vec{v}, \vec{v}) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cosh hx \cosh hy & \sinh hx \sinh hy \\ \sinh hx \sinh hy & \cosh hx \cosh hy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = x^2 + y^2$$

Thus the polynomial of second degree that best approximate $f(x, y)$ is $\frac{1}{2}(x^2 + y^2)$.

(b) Let $g(x, y) = \sin(x^2 + y^2)$, with $\vec{x} = (0, 0)$, $\vec{v} = (x, y)$,

$$F(h) := g(\vec{x} + h\vec{v}) = g(h\vec{v}) = \sin((hx)^2 + (hy)^2)$$

then

$$\begin{aligned} F'(h) &= \langle \nabla g(h\vec{v}), \vec{v} \rangle = x(2hx \cos((hx)^2 + (hy)^2)) + y(2hy \cos((hx)^2 + (hy)^2)) \\ F''(h) &= \nabla^2 g(h\vec{v})(\vec{v}, \vec{v}) \\ &= x^2(2\cos((hx)^2 + (hy)^2) - 4(xh)^2 \sin((hx)^2 + (hy)^2)) \\ &- 2xy(4xyh^2 \sin((hx)^2 + (hy)^2)) \\ &+ y^2(2\cos((hx)^2 + (hy)^2) - 4(yh)^2 \sin((hx)^2 + (hy)^2)) \end{aligned}$$

and

$$F(0) = 0 \quad F'(0) = 0 \quad F''(0) = 2x^2 + 2y^2$$

Thus the polynomial of second degree that best approximate $g(x, y)$ is $x^2 + y^2$.

2. (a)

(b)

$$\frac{\partial^2 f}{\partial x \partial y}$$

3.

4.