

## Statistics: Homework 1

- 1.19 (a) Let  $X_1, X_2$  and  $X_3$  denote the computer owners that use Macintosh, Windows and Linux respectively and let  $V$  denote the event that the user's system is infected with the virus. We want to find  $\mathbb{P}(X_2|V)$

$$\begin{aligned}\mathbb{P}(X_2|V) &= \frac{\mathbb{P}(V|X_2)\mathbb{P}(X_2)}{\sum_{i=1}^3 \mathbb{P}(V|X_i)\mathbb{P}(X_i)} \\ &= \frac{(.82)(.5)}{(.65)(.3) + (.82)(.5) + (.5)(.2)} \\ &= 0.581560284\end{aligned}$$

(b)

2.4 (a)

$$F_X(x) := \begin{cases} \frac{1}{4}x & 0 < x < 1 \\ \frac{3}{8}x - \frac{7}{8} & 3 < x < 5 \\ 1 & x > 5 \end{cases}$$

(b)

$$\begin{aligned}\mathbb{P}(Y \leq y) &= \mathbb{P}(1/X \leq y) \\ &= \mathbb{P}(X \geq 1/y) \\ &= 1 - \mathbb{P}(X \leq 1/y)\end{aligned}$$

From (a):

$$\begin{aligned}F_Y(y) &:= \begin{cases} \frac{15}{8} - \frac{3}{8y} & 1/5 < y < 1/3 \\ 1 - \frac{1}{4y} & y \geq 1 \\ 0 & \text{otherwise} \end{cases} \\ f_Y(y) &:= \begin{cases} \frac{3}{8y^2} & 1/5 < y < 1/3 \\ \frac{1}{4y^2} & y \geq 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Alternatively, we can do it in this way:

- 2.11 (a) We see that  $\mathbb{P}(X = 1) = p = \mathbb{P}(Y = 0)$ . Since the state space contains  $\{H, T\}$ , we have  $1 - \mathbb{P}(X = 1, Y = 0) = 1 - p = \mathbb{P}(X = 0, Y = 1)$ . But since

$$\mathbb{P}(X = 1)\mathbb{P}(Y = 0) = p^2 \neq p = \mathbb{P}(X = 1, Y = 0)$$

$X$  and  $Y$  are dependent.

(b) By total law of probability,

$$\begin{aligned}\mathbb{P}(X = x) &= \sum_{n=x}^{\infty} \mathbb{P}(X = x|N = n) \cdot \mathbb{P}(N = n) \\ &= \sum_{n=x}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} \frac{(\lambda p)^x}{x!} \sum_{n=x}^{\infty} \frac{[\lambda(1-p)]^{n-x}}{(n-x)!} \\ &= e^{-\lambda p} \frac{(\lambda p)^x}{x!}\end{aligned}$$

in a similar fashion, we have

$$\mathbb{P}(Y = y) = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!}$$

for the joint distribution of  $X$  and  $Y$ ,

$$\begin{aligned} \mathbb{P}(X = x, Y = n - x) &= \mathbb{P}(X = x, Y = n - x | N = n) \cdot \mathbb{P}(N = n) \\ &= \binom{n}{x} p^x (1-p)^{n-x} \cdot e^{-\lambda} \frac{\lambda^n}{n!} \end{aligned}$$

now

$$\begin{aligned} \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y) &= e^{-\lambda p} \frac{(\lambda p)^x}{x!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1-p))^y}{y!} \\ &= \binom{x+y}{x} p^x (1-p)^y e^{-\lambda} \frac{\lambda^{x+y}}{(x+y)!} \end{aligned}$$

which shows that  $X$  and  $Y$  are independent.

3.4 Let  $Y_i$  denote the jump of the particle at the  $i$ th unit. Then  $X_n = \sum_{i=1}^n Y_i$ . The  $Y_i$ 's are iid, with  $\mathbb{E}(Y_i) = 1 - 2p$  and  $\mathbb{V}(Y_i) = 1 - (1 - 2p)^2 = 4p(1-p)$  for  $i = 1, 2, \dots, n$ .

$$\begin{aligned} \mathbb{E}(X_n) &= \sum_{i=1}^n \mathbb{E}(Y_i) = n(1 - 2p) \\ \mathbb{V}(X_n) &= \sum_{i=1}^n \mathbb{V}(Y_i) = n \cdot 4p(1-p) \end{aligned}$$

4.3 Using Chebyshev's and Hoeffding's inequality we have

$$\begin{aligned} \mathbb{P}(|\bar{X}_n - p| > \epsilon) &\leq \frac{1}{4n\epsilon^2} \\ \mathbb{P}(|\bar{X}_n - p| > \epsilon) &\leq 2e^{-2n\epsilon^2} \end{aligned}$$

The inequality  $(1+x)^r \leq e^{rx}$  for  $r > 0, x > 0$ , thus for  $r = 1$

$$\begin{aligned} x &< 1 + x \leq e^x \\ 1/x &> e^{-x} \\ \frac{1}{2n\epsilon^2} &> e^{-2n\epsilon^2} \\ \frac{1}{n\epsilon^2} &> 2e^{-2n\epsilon^2} \end{aligned}$$

5.7 We first note that

$$\mathbb{V}\left(n^{-1} \sum_{i=1}^n X_i^2 - p\right) = n^{-2} \sum_{i=1}^n \mathbb{V}(X_i^2) = \frac{p(1-p)}{n}$$

thus for any given  $\epsilon > 0$ ,

$$\mathbb{P}\left(\left|n^{-1} \sum_{i=1}^n X_i^2 - p\right| > \epsilon\right) \leq \frac{\mathbb{V}(n^{-1} \sum_{i=1}^n X_i^2 - p)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

which proves the convergence in probability. We now prove its convergence in quadratic mean:

$$\mathbb{E}\left(\left[n^{-1} \sum_{i=1}^n X_i^2 - p\right]^2\right) = \mathbb{E}\left(n^{-2} \left[\sum_{i=1}^n X_i^2\right]^2\right) - 2\mathbb{E}\left(p/n \sum_{i=1}^n X_i^2\right) + \mathbb{E}(p^2) \quad (1)$$

simplifying the first term, we get

$$\begin{aligned}
\mathbb{E} \left( n^{-2} \left[ \sum_{i=1}^n X_i^2 \right]^2 \right) &= n^{-2} \left[ \mathbb{E} \left( \sum_{i=1}^n X_i^4 \right) + 2 \mathbb{E} \left( \sum_{i \neq j} X_i^2 X_j^2 \right) \right] \\
&= n^{-2} \left[ \mathbb{E} \left( \sum_{i=1}^n X_i^4 \right) + 2 \sum_{i \neq j} \mathbb{E} (X_i^2) \mathbb{E} (X_j^2) \right] \\
&= n^{-2} \left[ np + 2 \binom{n}{2} p^2 \right] \\
&= \frac{p}{n} + \frac{n-1}{n} p^2
\end{aligned}$$

the simplification of the last 2 terms in (1) gives  $-p^2$ . Thus

$$\mathbb{E} \left( \left[ n^{-1} \sum_{i=1}^n X_i^2 - p \right]^2 \right) = \frac{p - p^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

which concludes the prove of convergence in quadratic mean.