

Statistics: Homework 4

1. (a) Given p_i and q_i denote the probability of choosing box 1 and 2 respectively if the ball color chosen is i where $i = \{B, W, G\}$, denoting the three different colors. With the given information of the number of different color balls in the different boxes,

$$\begin{aligned}\mathbb{P}(B|1) &= 4/10 & \mathbb{P}(B|2) &= 3/10 & \mathbb{P}(B|3) &= 2/10 \\ \mathbb{P}(W|1) &= 2/10 & \mathbb{P}(W|2) &= 6/10 & \mathbb{P}(W|3) &= 0 \\ \mathbb{P}(G|1) &= 4/10 & \mathbb{P}(G|2) &= 1/10 & \mathbb{P}(G|3) &= 8/10\end{aligned}$$

The risk function is represented by,

$$\begin{aligned}R(\theta, \hat{\theta}_{p,q}) &= \mathbb{E}_\theta(|\theta^2 - \hat{\theta}_{p,q}|^2) \\ &= \mathbb{E}_\theta \left(\sum_{i \in \{B, W, G\}} L(\theta, \hat{\theta}_{p,q}(i)) \mathbb{P}(i|\theta) \right)\end{aligned}$$

where $L(\theta, \hat{\theta}_{p,q}(i)) = L(\theta, 1)p_i + L(\theta, 2)q_i + L(\theta, 3)(1 - p_i - q_i)$. Therefore,

$$\begin{aligned}R(1, \hat{\theta}_{p,q}) &= [q_B + 4(1 - p_B - q_B)] \frac{4}{10} + [q_W + 4(1 - p_W - q_W)] \frac{2}{10} + [q_G + 4(1 - p_G - q_G)] \frac{4}{10} \\ R(2, \hat{\theta}_{p,q}) &= [9p_B + 4q_B + 49(1 - p_B - q_B)] \frac{3}{10} + [9p_W + 4q_W + 49(1 - p_W - q_W)] \frac{6}{10} \\ &\quad + [9p_G + 4q_G + 49(1 - p_G - q_G)] \frac{1}{10}\end{aligned}$$

- (b) Bayes risk is given by

$$r(f, \theta) = \int R(\theta, \hat{\theta}_{p,q}) f(\theta) d\theta$$

but since our scenario is discrete, we instead have

$$\begin{aligned}r(f, \theta) &= \sum_{\theta=1,2} R(\theta, \hat{\theta}_{p,q}) \mathbb{P}(\theta) \\ &= \lambda R(1, \hat{\theta}_{p,q}) + (1 - \lambda) R(2, \hat{\theta}_{p,q})\end{aligned}$$

where $R(1, \hat{\theta}_{p,q})$ and $R(2, \hat{\theta}_{p,q})$ are the values are from (a).

- (c) Given $\lambda = 1/2$, we have

$$r(f, \theta) = \frac{1}{2} \left(R(1, \hat{\theta}_{p,q}) + R(2, \hat{\theta}_{p,q}) \right) = \frac{1}{2} (16.3 - 13.6p_B - 14.7q_B - 24.8p_W - 27.6q_W - 5.6p_G - 5.6q_G)$$

thus to the infimum of Bayes risk is when $q_B = q_W = q_G = 1$.

2.

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library(survey)
library(dplyr)

# read csv file into dataframe car
car <- read.csv('carmpgdat.csv')

# (a) Fitting a multiple linear regression model

# generate a linear model with normally distributed noise with the model MPG~VOL+HP+SP+WT
# covariates <- cbind('VOL','HP','SP','WT')
lm_car <- lm(MPG~VOL+HP+SP+WT, data = car)

# estimated regression function and residual sum of squares
coefficients(lm_car)
# MPG = 192.43775332 - 0.01564501 * VOL + 0.39221231 * HP - 1.29481848 * SP - 1.85980373 * WT

RSS = sum(lm_car$residuals^2)
# RSS = 1027.381

# (b) Mallows's Cp
# Since the AIC criterion is equivalent to Mallows's Cp,

# (i) Forward
base <- lm(MPG~WT, data = car)
step(base, scope = list(upper = lm_car, lower=~1), direction = 'forward', trace = TRUE)

# Start: AIC=378.69
# MPG ~ 1
#
# Df Sum of Sq  RSS    AIC
# + WT    1    6641.5 1466.0 240.45
# + HP    1    5058.0 3049.4 300.51
# + SP    1    3842.6 4264.9 328.02
# + VOL   1    1101.6 7005.9 368.72
# <none>             8107.5 378.69
#
# Step: AIC=240.45
# MPG ~ WT
#
# Df Sum of Sq  RSS    AIC
# + SP    1    82.981 1383.0 237.68
# + HP    1    35.380 1430.6 240.45
# <none>             1466.0 240.45
# + VOL   1     3.883 1462.1 242.24
#
# Step: AIC=237.68
# MPG ~ WT + SP
#
# Df Sum of Sq  RSS    AIC
# + HP    1    349.37 1033.7 215.80
# + VOL   1     45.97 1337.0 236.90
# <none>             1383.0 237.68
#
# Step: AIC=215.8
# MPG ~ WT + SP + HP
#
# Df Sum of Sq  RSS    AIC
# <none>             1033.7 215.8
# + VOL   1     6.2685 1027.4 217.3
#
# Call:
# lm(formula = MPG ~ WT + SP + HP, data = car)
#
# Coefficients:

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# (Intercept)      WT      SP      HP
# 194.1296      -1.9221      -1.3200      0.4052

# (ii) Backward
step(lm_car,direction = 'backward', trace = TRUE)

# Start: AIC=217.3
# MPG ~ VOL + HP + SP + WT
#
# Df Sum of Sq  RSS   AIC
# - VOL  1      6.27 1033.7 215.80
# <none>                  1027.4 217.30
# - HP   1     309.67 1337.0 236.90
# - SP   1     373.36 1400.7 240.72
# - WT   1    1013.76 2041.2 271.59
#
# Step: AIC=215.8
# MPG ~ HP + SP + WT
#
# Df Sum of Sq  RSS   AIC
# <none>                  1033.7 215.80
# - HP   1     349.37 1383.0 237.68
# - SP   1     396.97 1430.6 240.45
# - WT   1    1322.87 2356.5 281.37
#
# Call:
# lm(formula = MPG ~ HP + SP + WT, data = car)
#
# Coefficients:
# (Intercept)      HP      SP      WT
# 194.1296      0.4052     -1.3200     -1.9221
#
# Both forward and backward stepwise approach give the same submodel to be selected.
# The selected submodel is MPG ~ HP + SP + WT with the regression formula:
# MPG = 194.1296 + HP * 0.4052 - SP * 1.3200 - WT * 1.9221

## For the forward stepwise approach, the base model is important
## as it will change the outcome. For example for this, if we were
## to start with VOL instead of any of the other covariates, we
## will end up with also the VOL covariate. By not starting with with
## the VOL covariate we will end up with a model without VOL which
## corresponds to the backward stepwise approach and also the
## Zheng-Loh model selection. As for the backward stepwise approach
## we do not have such a problem as we start the model with all the
## covariates and reduce it down by computing the AIC of the model
## with different covariate missing then remove the covariate that
## gives the smallest AIC when removed.

# (c) Zheng-Loh Model Selection
# Wald test for the covariates

regTermTest(lm_car, 'VOL', null=NULL,df=Inf, method = "Wald")
regTermTest(lm_car, 'HP', null=NULL,df=Inf, method = "Wald")
regTermTest(lm_car, 'SP', null=NULL,df=Inf, method = "Wald")
regTermTest(lm_car, 'WT', null=NULL,df=Inf, method = "Wald")

# Wald test for VOL
# in lm(formula = MPG ~ VOL + HP + SP + WT, data = car)
# Chisq = 0.4698075 on 1 df: p= 0.49308
#
# Wald test for HP
# in lm(formula = MPG ~ VOL + HP + SP + WT, data = car)

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# Chisq = 23.20929 on 1 df: p= 1.4529e-06
#
# Wald test for SP
# in lm(formula = MPG ~ VOL + HP + SP + WT, data = car)
# Chisq = 27.98266 on 1 df: p= 1.2241e-07
#
# Wald test for WT
# in lm(formula = MPG ~ VOL + HP + SP + WT, data = car)
# Chisq = 75.97941 on 1 df: p= < 2.22e-16
#
# Arranging in descending order we have:
# WT > SP > HP > VOL
# We can do this as the Chisq test statistic is just the square
# of the test statistic of the Wald test.
#
# Let lm_j be the linear model with the jth largest Wald test statistic

n <- nrow(car)

lm_1 <- lm(MPG ~ WT, data = car)
jhat_1 = sum(lm_1$residuals^2) + (1 * RSS/(n-4) * log(n))
print (jhat_1)
# jhat_1 = 1524.045

lm_2 <- lm(MPG ~ WT + SP, data = car)
jhat_2 = sum(lm_2$residuals^2) + (2 * RSS/(n-4) * log(n))
print (jhat_2)
# jhat_2 = 1499.108

lm_3 <- lm(MPG ~ WT + SP + HP, data = car)
jhat_3 = sum(lm_3$residuals^2) + (3 * RSS/(n-4) * log(n))
print (jhat_3)
# jhat_3 = 1207.78

jhat_4 = RSS + (4 * RSS/(n-4) * log(n))
print (jhat_4)
# jhat_4 = 1259.555

# Thus the Zheng-Loh model selection method will select WT, SP and HP
# as the covariates for predicting the MPG. Comparing it to (b), we see
# that the Zheng-Loh model selection method gives similar outcome to using
# Mallows's Cp model forward and backward stepwise to selecting a model.

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3.

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library(dplyr)
library(magrittr)

# Reading data with separator tab
raw_riasec <- read.csv('RIASEC.csv', sep = '\t')

# (a) CLEANING UP
# List of realistic traits
realistic_trait <- c('R1', 'R2', 'R3', 'R4', 'R5', 'R6', 'R7', 'R8')

# Extracting out the realistic traits
raw_realistic <- raw_riasec[, realistic_trait]

# Removing rows with -1 from the dataframe
# realistic <- raw_realistic[raw_realistic != -1, ]

realistic <- raw_realistic %>%
  filter(R1 > 0 & R2 > 0 & R3 > 0 & R4 > 0 & R5 > 0 & R6 > 0 & R7 > 0 & R8 > 0)

# (b) MODEL SELECTION

# Computing the score for the R trait
realistic <- realistic %>%
  rowwise() %>%
  mutate(Rscore = mean(c(R1, R2, R3, R4, R5, R6, R7, R8)))

# Generating the training and validation set
tr_realistic <- realistic[1:6500,]
val_realistic <- realistic[-(1:6500),]

# Building the linear model
lm_riasec <- lm(Rscore~R1, data = tr_realistic)
RSS_tr = sum(lm_riasec$residuals^2)
print(RSS_tr)
# RSS_tr = 2902.039

# estimated regression function and residual sum of squares
print(lm_riasec$coefficients)
# R1 = 1.0181052 + 0.4235993 * R1

# (c) VALIDATION
reg.fn <- function(x) 1.0181052 + 0.4235993 * x

val_realistic %<>%
  mutate(pred_Rscore = reg.fn(R1),
         residuals = reg.fn(R1) - Rscore )

avg_RSS_tr = mean(lm_riasec$residuals^2)
avg_RSS_val = mean(val_realistic$residuals^2)

print (avg_RSS_tr) # 0.4464676
print (avg_RSS_val) # 0.5201091

# The residual sum of squares for the validation set using the regression function
# is larger than the residual sum of square for the training set but are of the
# same order. Thus the model generalizes well.

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4. (a)

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# (a) Using Monte Carlo method with N samples

N <- 100000
I <- function(u){X = rgamma(1.5,2.3,n = N); return(sum(1<X & X<2)/N)}
I_hat <- I

# I_hat = 0.17552

# Estimated standard error by resampling Monte Carlo 10000 times.
se_mc <- sqrt(var(sapply(1:N,I)))

# se_mc = 0.001204692

# Estimated standard error using 10000 bootstrap samples.
B <- 10000
base_sample <- rgamma(1.5,1/2.3,n = N)
I_bootstrap <- function(u){X = sample(base_sample, N, replace = TRUE);return(sum(1<X & X<2)/N)}
se_bootstrap <- sqrt(var(sapply(1:B,I_bootstrap)))

# se_bootstrap = 0.001286245
```

(b) We see that we can rewrite I as the following,

$$I = \int_1^2 f(x|1.5, 2.3) dx = \int_0^\infty h(x)f(x|1.5, 2.3) dx$$

where $h(x) = \mathbb{1}(1 \leq x \leq 2)$, an indicator function that is 1 when $1 \leq x \leq 2$ and 0 otherwise. Thus we can use the following estimator \hat{I} for I where $X_i \sim \text{Gamma}(1.5, 2.3)$,

$$\hat{I} = \frac{1}{N} \sum_{i=1}^N h(X_i)$$

Hence by viewing $h(X) \sim \text{Bernoulli}(p)$, where $p = \mathbb{P}(1 \leq X \leq 2)$ for $X \sim \text{Gamma}(1.5, 2.3)$ the standard error is given by

$$\text{Var}(\hat{I}) = \frac{p(1-p)}{N}$$

where we sourced for the value of p from the website thus getting

$$\mathbb{P}(1 \leq X \leq 2) = \int_0^2 f(x|1.5, 2.3) dx - \int_0^1 f(x|1.5, 2.3) dx = 0.97325 - 0.79645$$

Evaluating the value of the standard error we get $\text{se} = 0.0012064069$