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Statistics: Homework 1

1.19 Let X_1, X_2 and X_3 denote the computer owners that use Macintosh, Windows and Linux respectively and let V denote the event that the user's system is infected with the virus. We want to find $\mathbb{P}(X_2|V)$

$$\begin{aligned}\mathbb{P}(X_2|V) &= \frac{\mathbb{P}(V|X_2)\mathbb{P}(X_2)}{\sum_{i=1}^{3}\mathbb{P}(V|X_i)\mathbb{P}(X_i)}\\ &= \frac{(.82)(.5)}{(.65)(.3)+(.82)(.5)+(.5)(.2)}\\ &= 0.581560284\end{aligned}$$

2.4 (a)

$$F_X(x) := \begin{cases}\frac{1}{4}x & 0<x<1\\ \frac{3}{8}x-\frac{7}{8} & 3<x<5\\ 0 & \text{otherwise}\end{cases}$$

(b)

$$\begin{aligned}\mathbb{P}(Y\le y) &= \mathbb{P}(1/X\le y)\\ &= \mathbb{P}(X\ge 1/y)\\ &= 1-\mathbb{P}(X\le 1/y)\end{aligned}$$

From (a):

$$F_Y(y) := \begin{cases}\frac{15}{8}-\frac{3}{8y} & 1/5<y<1/3\\ 1-\frac{1}{4y} & y\ge 1\\ 0 & \text{otherwise}\end{cases}$$

$$f_Y(y) := \begin{cases}\frac{3}{8y^2} & 1/5<y<1/3\\ \frac{1}{4y^2} & y\ge 1\\ 0 & \text{otherwise}\end{cases}$$

2.11 (a) We see that $\mathbb{P}(X=1)=p=\mathbb{P}(Y=0)$. Since the state space contains $\{H,T\}$, we have $1-\mathbb{P}(X=1,Y=0)=1-p=\mathbb{P}(X=0,Y=1)$. But since

$$\mathbb{P}(X=1)\mathbb{P}(Y=0)=p^2\ne p=\mathbb{P}(X=1,Y=0)$$

X and Y are dependent.

(b) By total law of probability,

$$\begin{aligned}\mathbb{P}(X=x) &= \sum_{n=x}^{\infty}\mathbb{P}(X=x|N=n)\cdot\mathbb{P}(N=n)\\ &= \sum_{n=x}^{\infty}\binom{n}{x}p^x(1-p)^{n-x}\cdot e^{-\lambda}\frac{\lambda^n}{n!}\\ &= e^{-\lambda}\frac{(\lambda p)^x}{x!}\sum_{n=x}^{\infty}\frac{[\lambda(1-p)]^{n-x}}{(n-x)!}\\ &= e^{-\lambda p}\frac{(\lambda p)^x}{x!}\end{aligned}$$

in a similar fashion, we have

$$\mathbb{P}(Y=y)=e^{-\lambda(1-p)}\frac{(\lambda(1-p))^y}{y!}$$

for the joint distribution of X and Y,

$$\begin{aligned}\mathbb{P}(X = x, Y = n - x) &= \mathbb{P}(X = x, Y = n - x | N = n) \cdot \mathbb{P}(N = n) \\ &= \binom{n}{x} p^x (1 - p)^{n-x} \cdot e^{-\lambda} \frac{\lambda^n}{n!}\end{aligned}$$

now

$$\begin{aligned}\mathbb{P}(X = x) \cdot \mathbb{P}(Y = y) &= e^{-\lambda p} \frac{(\lambda p)^x}{x!} \cdot e^{-\lambda(1-p)} \frac{(\lambda(1 - p))^y}{y!} \\ &= \binom{x + y}{x} p^x (1 - p)^y e^{-\lambda} \frac{\lambda^{x+y}}{(x + y)!}\end{aligned}$$

which shows that X and Y are independent.

3.4 Let Y_i denote the jump of the particle at the ith unit. Then $X_n = \sum_{i=1}^n Y_i$. The Y_i's are iid, with $\mathbb{E}(Y_i) = 1-2p$ and $\mathbb{V}(Y_i) = 1 - (1 - 2p)^2 = 4p(1 - p)$ for $i = 1, 2, \ldots, n$.

$$\begin{aligned}\mathbb{E}(X_n) &= \sum_{i=1}^n \mathbb{E}(Y_i) = n(1 - 2p) \\ \mathbb{V}(X_n) &= \sum_{i=1}^n \mathbb{V}(Y_i) = n \cdot 4p(1 - p)\end{aligned}$$

4.3 Using Chebyshev's and Hoeffding's inequality we have

$$\begin{aligned}\mathbb{P}(|\overline{X}_n - p| > \epsilon) &\leq \frac{1}{4n\epsilon^2} \\ \mathbb{P}(|\overline{X}_n - p| > \epsilon) &\leq 2e^{-2n\epsilon^2}\end{aligned}$$

The inequality $(1 + x)^r \leq e^{rx}$ for $r > 0, x > 0$, thus for $r = 1$

$$\begin{aligned}x < 1 + x &\leq e^x \\ 1/x &> e^{-x} \\ \frac{1}{2n\epsilon^2} &> e^{-2n\epsilon^2} \\ \frac{1}{n\epsilon^2} &> 2e^{-2n\epsilon^2}\end{aligned}$$

5.7 We first note that

$$\mathbb{V}\left(n^{-1} \sum_{i=1}^n X_i^2 - p\right) = n^{-2} \sum_{i=1}^n \mathbb{V}(X_i^2) = \frac{p(1 - p)}{n}$$

thus for any given $\epsilon > 0$,

$$\mathbb{P}\left(|n^{-1} \sum_{i=1}^n X_i^2 - p| > \epsilon\right) \leq \frac{\mathbb{V}(n^{-1} \sum_{i=1}^n X_i^2 - p)}{\epsilon^2} = \frac{p(1 - p)}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

which proves the convergence in probability. We now prove its convergence in quadratic mean:

$$\mathbb{E}\left(\left[n^{-1} \sum_{i=1}^n X_i^2 - p\right]^2\right) = \mathbb{E}\left(n^{-2} \left[\sum_{i=1}^n X_i^2\right]^2\right) - 2\,\mathbb{E}\left(p/n \sum_{i=1}^n X_i^2\right) + \mathbb{E}\left(p^2\right) \tag{1}$$

simplifying the first term, we get

$$\begin{aligned}\mathbb{E}\left(n^{-2} \left[\sum_{i=1}^n X_i^2\right]^2\right) &= n^{-2} \left[\mathbb{E}\left(\sum_{i=1}^n X_i^4\right) + 2\,\mathbb{E}\left(\sum_{i \neq j} X_i^2 X_j^2\right)\right] \\ &= n^{-2} \left[\mathbb{E}\left(\sum_{i=1}^n X_i^4\right) + 2 \sum_{i \neq j} \mathbb{E}\left(X_i^2\right) \mathbb{E}\left(X_j^2\right)\right] \\ &= n^{-2} \left[np + 2\binom{n}{2} p^2\right] \\ &= \frac{p}{n} + \frac{n - 1}{n} p^2\end{aligned}$$

the simplification of the last 2 terms in (1) gives $-p^2$. Thus

$$\mathbb{E}\left(\left[n^{-1}\sum_{i=1}^{n} X_i^2 - p\right]^2\right) = \frac{p-p^2}{n} \to 0 \text{ as } n \to \infty$$

which concludes the prove of convergence in quadratic mean.