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Statistics: Homework 1

1.19 (a) Let X_1, X_2 and X_3 denote the computer owners that use Macintosh, Windows and Linux respectively and let V denote the event that the user's system is infected with the virus. We want to find $\mathbb{P}(X_2|V)$

$$\begin{aligned}\mathbb{P}(X_2|V) &= \frac{\mathbb{P}(V|X_2)\mathbb{P}(X_2)}{\sum_{i=1}^{3}\mathbb{P}(V|X_i)\mathbb{P}(X_i)}\\ &= \frac{(.82)(.5)}{(.65)(.3)+(.82)(.5)+(.5)(.2)}\\ &= 0.581560284\end{aligned}$$

(b) $\mathbb{P}(V) = (.65)(.3) + (.82)(.5) + (.5)(.2) = .705$

(c) Let A and B denote the event that the second person has a system that was also infected by a virus and the second person is known to have the same computer system as the first person. We observe that A and B are independent events as the probability of getting a virus on your computer system is the same regardless of whether the second person has the same computer system as the first person. Thus $\mathbb{P}(A|B) = \mathbb{P}(A) = \mathbb{P}(V) = .705$

2.4 (a)

$$F_X(x) := \begin{cases} \frac{1}{4}x & 0 < x < 1 \\ \frac{3}{8}x - \frac{7}{8} & 3 < x < 5 \\ 1 & x > 5 \end{cases}$$

(b)

$$\begin{aligned}\mathbb{P}(Y \le y) &= \mathbb{P}(1/X \le y)\\ &= \mathbb{P}(X \ge 1/y)\\ &= 1 - \mathbb{P}(X \le 1/y)\end{aligned}$$

From (a):

$$F_Y(y) := \begin{cases} \frac{15}{8} - \frac{3}{8y} & 1/5 < y < 1/3 \\ \frac{3}{4} & 1/3 \le y \le 1 \\ 1 - \frac{1}{4y} & y > 1 \end{cases}$$

$$f_Y(y) := \begin{cases} \frac{3}{8y^2} & 1/5 < y < 1/3 \\ \frac{1}{4y^2} & y > 1 \\ 0 & \text{otherwise} \end{cases}$$

Alternatively, we can do it in this way:

2.11 (a) We see that $\mathbb{P}(X = 1) = p = \mathbb{P}(Y = 0)$. Since the state space contains $\{H, T\}$, we have $1 - \mathbb{P}(X = 1, Y = 0) = 1 - p = \mathbb{P}(X = 0, Y = 1)$. But since

$$\mathbb{P}(X = 1)\mathbb{P}(Y = 0) = p^2 \neq p = \mathbb{P}(X = 1, Y = 0)$$

X and Y are dependent.

(b) By total law of probability,

$$\begin{aligned}
\mathbb{P}(X=x) &= \sum_{n=x}^{\infty} \mathbb{P}(X=x|N=n)\cdot\mathbb{P}(N=n) \\
&= \sum_{n=x}^{\infty} \binom{n}{x} p^x(1-p)^{n-x}\cdot e^{-\lambda}\frac{\lambda^n}{n!} \\
&= e^{-\lambda}\frac{(\lambda p)^x}{x!}\sum_{n=x}^{\infty}\frac{[\lambda(1-p)]^{n-x}}{(n-x)!} \\
&= e^{-\lambda p}\frac{(\lambda p)^x}{x!}
\end{aligned}$$

in a similar fashion, we have

$$\mathbb{P}(Y=y) = e^{-\lambda(1-p)}\frac{(\lambda(1-p))^y}{y!}$$

for the joint distribution of X and Y,

$$\begin{aligned}
\mathbb{P}(X=x, Y=n-x) &= \mathbb{P}(X=x, Y=n-x|N=n)\cdot\mathbb{P}(N=n) \\
&= \binom{n}{x}p^x(1-p)^{n-x}\cdot e^{-\lambda}\frac{\lambda^n}{n!}
\end{aligned}$$

now

$$\begin{aligned}
\mathbb{P}(X=x)\cdot\mathbb{P}(Y=y) &= e^{-\lambda p}\frac{(\lambda p)^x}{x!}\cdot e^{-\lambda(1-p)}\frac{(\lambda(1-p))^y}{y!} \\
&= \binom{x+y}{x}p^x(1-p)^y e^{-\lambda}\frac{\lambda^{x+y}}{(x+y)!}
\end{aligned}$$

which shows that X and Y are independent.

3.4 Let Y_i denote the jump of the particle at the ith unit. Then $X_n = \sum_{i=1}^n Y_i$. The Y_i's are iid, with $\mathbb{E}(Y_i) = 1-2p$ and $\mathbb{V}(Y_i) = 1-(1-2p)^2 = 4p(1-p)$ for $i = 1, 2, \ldots, n$.

$$\begin{aligned}
\mathbb{E}(X_n) &= \sum_{i=1}^n \mathbb{E}(Y_i) = n(1-2p) \\
\mathbb{V}(X_n) &= \sum_{i=1}^n \mathbb{V}(Y_i) = n\cdot 4p(1-p)
\end{aligned}$$

4.3 Using Chebyshev's and Hoeffding's inequality we have

$$\begin{aligned}
\mathbb{P}(|\overline{X}_n - p| > \epsilon) &\le \frac{1}{4n\epsilon^2} \\
\mathbb{P}(|\overline{X}_n - p| > \epsilon) &\le 2e^{-2n\epsilon^2}
\end{aligned}$$

The inequality $(1+x)^r \le e^{rx}$ for $r > 0, x > 0$, thus for $r = 1$

$$\begin{aligned}
x < 1 + x &\le e^x \\
1/x &> e^{-x} \\
\frac{1}{2n\epsilon^2} &> e^{-2n\epsilon^2} \\
\frac{1}{n\epsilon^2} &> 2e^{-2n\epsilon^2}
\end{aligned}$$

5.7 We first note that

$$\mathbb{V}\left(n^{-1}\sum_{i=1}^n X_i^2 - p\right) = n^{-2}\sum_{i=1}^n \mathbb{V}(X_i^2) = \frac{p(1-p)}{n}$$

thus for any given $\epsilon > 0$,

$$\mathbb{P}\left(|n^{-1}\sum_{i=1}^{n} X_i^2 - p| > \epsilon\right) \leq \frac{\mathbb{V}(n^{-1}\sum_{i=1}^{n} X_i^2 - p)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

which proves the convergence in probability. We now prove its convergence in quadratic mean:

$$\mathbb{E}\left(\left[n^{-1}\sum_{i=1}^{n} X_i^2 - p\right]^2\right) = \mathbb{E}\left(n^{-2}\left[\sum_{i=1}^{n} X_i^2\right]^2\right) - 2\,\mathbb{E}\left(p/n\sum_{i=1}^{n} X_i^2\right) + \mathbb{E}\left(p^2\right) \tag{1}$$

simplifying the first term, we get

$$\begin{aligned}
\mathbb{E}\left(n^{-2}\left[\sum_{i=1}^{n} X_i^2\right]^2\right) &= n^{-2}\left[\mathbb{E}\left(\sum_{i=1}^{n} X_i^4\right) + 2\,\mathbb{E}\left(\sum_{i \neq j} X_i^2 X_j^2\right)\right] \\
&= n^{-2}\left[\mathbb{E}\left(\sum_{i=1}^{n} X_i^4\right) + 2\sum_{i \neq j} \mathbb{E}\left(X_i^2\right)\,\mathbb{E}\left(X_j^2\right)\right] \\
&= n^{-2}\left[np + 2\binom{n}{2}p^2\right] \\
&= \frac{p}{n} + \frac{n-1}{n}p^2
\end{aligned}$$

the simplification of the last 2 terms in (1) gives $-p^2$. Thus

$$\mathbb{E}\left(\left[n^{-1}\sum_{i=1}^{n} X_i^2 - p\right]^2\right) = \frac{p - p^2}{n} \to 0 \text{ as } n \to \infty$$

which concludes the prove of convergence in quadratic mean.