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Real Analysis: Homework 3

1. A function which is in $\mathcal{C}^1(\mathbb{R})$ but not in $\mathcal{C}^2(\mathbb{R})$ means a function that has continuous first derivative but its second derivative is not continuous. Consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   we see that the first derivate of $f(x)$ is continuous but the second derivate is not continuous at $x = 0$.

2.

3. Let ϕ be λ-Hölder bi-continuous then for $v_1, v_2, u_1, u_2 \in T$, we have

$$\sup_{v \in [0,T]} |\phi(u_2, v) - \phi(u_1, v)| \leq C_u |u_2 - u_1|^\lambda$$
$$\sup_{u \in [0,T]} |\phi(u, v_2) - \phi(u, v_1)| \leq C_v |v_2 - v_1|^\lambda$$

   then we also observe that

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq |\phi(u_1, v_1) - \phi(u_1, v_2)| + |\phi(u_2, v_2) - \phi(u_2, v_1)|$$
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq |\phi(u_1, v_1) - \phi(u_2, v_1)| + |\phi(u_2, v_2) - \phi(u_1, v_2)|$$

   which gives us

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq 2C_v |v_2 - v_1|^\lambda$$
$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| \leq 2C_u |u_2 - u_1|^\lambda$$

   multiplying them together, we have

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)|^2 \leq 4C_v C_u |v_2 - v_1|^\lambda |u_2 - u_1|^\lambda$$

   squaring both sides, we have shown that all λ-Hölder bi-continuous are strongly $\lambda/2$-Hölder bi-continuous.

4. (a)

   (b)