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Stochastic Models: Exercise 3

1. (i) The setup can be modelled by disjoint Poisson process with s being the 24-hour time,

$$\lambda(s) := \begin{cases} 5t-35, & 8 \le s < 11 \\ 20, & 11 \le s < 13 \\ -2t+46, & 13 \le s < 17 \end{cases}$$

we see that the first and last are nonhomogeneous Poisson process and the middle one is a homogeneous. A Poisson process can be used to model the number of customers that arrive at different time intervals as it is a counting process.

(ii)

$$\begin{aligned} \mathbb{P}(N(9.5)-N(8.5)=0) &= e^{-(m(9.5)-m(8.5))}\frac{m(9.5)-m(8.5)^0}{0!} \\ &= e^{-10} \end{aligned}$$

(iii)

$$\begin{aligned} \mathbb{E}(\text{number of arrivals from 8:30AM-9:30AM}) &= \sum_{k=0}^{\infty} k \cdot e^{-(m(9.5)-m(8.5))}\frac{(m(9.5)-m(8.5))^k}{k!} \\ &= 10\sum_{k=1}^{\infty} e^{-10}\frac{10^{k-1}}{(k-1)!} \\ &= 10 \end{aligned}$$

2.

$$\begin{aligned} \mathbb{P}(N(I_1)=k_1,\dots,N(I_n)=k_n \mid N(U)=k) &= \frac{\mathbb{P}(N(I_1)=k_1,\dots,N(I_n)=k_n, N(u)=k)}{\mathbb{P}(N(U)=k)} \\ &= \left(\prod_{i=1}^{n} e^{-\lambda c_i}\frac{(\lambda c_i)^{k_i}}{k_i!}\right) \Bigg/ e^{-\lambda c}\frac{(\lambda c)^k}{k!} \\ &= \frac{k!}{k_1!k_2!\dots k_n!}\left(\frac{c_1}{c}\right)^{k_1}\left(\frac{c_2}{c}\right)^{k_2}\dots\left(\frac{c_n}{c}\right)^{k_n} \end{aligned}$$

3. Let N_i denote the number of families with number of member of size i migrating to Batan Island over a t week period and let such an event be called a type-i event for $i = 1,2,3,4$. Hence $N_i(t)$ is a Poisson process and $\mathbb{E}(N_i(t)) = \lambda t p_i = 10p_i$. Let $M(t) = \sum_i iN_i(t)$ denote the number of individuals migrating during a t-week period.

$$\begin{aligned} \mathbb{E}(M(t)) &= \sum_i i\mathbb{E}(N_i(t)) \\ &= (1+4)\frac{10}{6} + (2+3)\frac{10}{3} \\ &= 25 \end{aligned}$$

To find variance, we first find $\mathbb{E}(N_i(t)^2)$

$$\begin{aligned} \mathbb{E}(N_i(t)^2) &= \sum_{n=0}^{\infty} n^2 e^{-\lambda t p_i}\frac{(\lambda t p_i)^n}{n!} \\ &= \sum_{n=2}^{\infty} e^{-\lambda t p_i}\frac{(\lambda t p_i)^{n-2}}{(n-2)!} + \sum_{n=1}^{\infty} e^{-\lambda t p_i}\frac{(\lambda t p_i)^{n-1}}{(n-1)!} \\ &= \lambda t p_i + (\lambda t p_i)^2 \end{aligned}$$

and so $Var(N_i(t)) = \lambda t p_i$.

$$
\begin{aligned}
Var(M(t)) &= \sum_i i^2 Var(N_i(t)) \\
&= (1^2 + 4^2)\frac{10}{6} + (2^2 + 3^2)\frac{10}{3} = \frac{215}{3}
\end{aligned}
$$

4. Suppose that the mini toys are collected at times chosen according to a homogeneous Poisson process with unit rate 1 per order of a meal. Thinning the Poisson process, for each toy i, we have $\{N_i(t), t \in \mathbb{Z}_{\geq 0}\}$ to be a Poisson process with rate $\lambda p_i = p_i$ and independent of each other. Let X_i denote the first interarrival time of the Poisson process $\{N_i(t), t \geq 0\}$ and we known that $\mathbb{P}(X_i \leq t) = e^{-tp_i}$. The number of meals to be ordered such is all the mini toys are collected is $X = \max_{1 \leq i \leq m} X_i$.

$$
\mathbb{P}(X < t) = \prod_{i=1}^{m} \mathbb{P}(X_i \leq t)
$$

thus

$$
\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty \mathbb{P}(X > t)\, dt \\
&= \int_0^\infty \prod_{i=1}^{m} \mathbb{P}(X_i > t)\, dt \\
&= \int_0^\infty \prod_{i=1}^{m} (1 - e^{-tp_i})\, dt
\end{aligned}
$$

Since there is a one to one correspondence between the number of order of meals and the time, $\mathbb{E}(X) = \mathbb{E}(N)$.

5. (i) (a) No. This can be show as follows, where $m(t) = \int_0^t \alpha(u)\, du$

$$
\begin{aligned}
\mathbb{P}(E_1 > t) &= \mathbb{P}(N(t) = 0) \\
&= e^{-m(t)} \\
&= exp\left(-\int_0^t \alpha(u)\, du\right) \\
\mathbb{P}(E_2 > t \mid E_1 = s) &= \mathbb{P}(N(t+s) - N(s) = 0 \mid E_1 = s) \\
&= \mathbb{P}(N(t+s) - N(s) = 0), \quad \text{by independent increments} \\
&= e^{-(m(t+s) - m(s))} \\
&= exp\left(-\int_t^{t+s} \alpha(u)\, du\right)
\end{aligned}
$$

(b) From (a), we have $\mathbb{P}(E_1 \leq t) = 1 - exp\left(-\int_0^t \alpha(u)\, du\right)$

(ii) (a) For each busstop i, let A_{ij} represent the number of passengers that board at stop i and alight at stop j. We claim that $A_{ij} \sim Pois(\lambda_i p_{ij})$ for $j = i+1, i+2 \ldots, n$. Since $D_j = \sum_{i=1}^{j-1} A_{ij}$, we have $D_j \sim Pois(\tilde{\lambda}_j)$, where $\tilde{\lambda}_j = \sum_{i=1}^{j-1} \lambda_i p_{ij}$ as the sum of finitely many Poisson random variable is Poisson. To show the claim, let $X \sim Pois(\lambda)$ and suppose the Poisson random variable can be classified into n distinct types with type i occurring with probability p_i, $\sum_{i=1}^{n} p_i = 1$, also let N_i denote the number of occurrence of type i thus $X = \sum_{i=1}^{n} N_i$. Now

$$
\begin{aligned}
\mathbb{P}(N_1 = k_1, \ldots, N_n = k_n) &= \mathbb{P}(N_1 = k_1, \ldots, N_n = k_n \mid X = k) \cdot \mathbb{P}(X = k) \\
&= \mathbb{P}(N_1 = k_1, \ldots, N_n = k_n \mid X = k) \cdot e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \binom{k}{k_1, k_2, \ldots, k_n} \cdot e^{-\lambda}\frac{\lambda^k}{k!} \\
&= \prod_{i=1}^{n} e^{-\lambda p_i}\frac{\lambda^{k_i}}{k_i!}
\end{aligned}
$$

(b) It follows from (a) that $\mathbb{E}(D_j) = \sum_{i=1}^{j-1} \lambda_i p_{ij}$

6. (a) Let $N_1(t)$ and $N_2(t)$ be the type-I and type-II events where Irma Pince finds a misplaced book and fails to find a misplaced book respectively. Hence, the N_is are independent Poisson process with rate λp_i where $i = 1, 2$. This the misplacements found by Irma Pince follows a homogeneous Poisson process. For $t = 100$, $\mathbb{E}(N_1(100)) = 90\lambda$.

   (b) For each shelf i, we can the classify the event of find a misplaced book in the shelf as a type-I event and not finding a misplaced book as a type-II event. Then the $N_1(t)$ of shelf i is a Poisson process with rate λp_i. Define $N(t) = N_1(t) + N_2(t) + N_3(t)$ and we claim that it is a Poisson process with rate of process $\lambda(p_1 + p_2 + p_3)$. Hence the desired probability is,

$$\mathbb{P}(N(3) = 5) = e^{-\lambda(p_1+p_2+p_3)} \frac{(\lambda(p_1 + p_2 + p_3))^5}{5!}$$

   Here, we shall proof the claim that the sum of two independent Poisson process is a Poisson process. Let $\{N(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ be two independent Poisson process with rate λ_1 and λ_2 respectively. We shall show that $\{N(t) + M(t), t \geq 0\}$ is also a Poisson process by showing the four conditions.

   (i) $N(0) + M(0) = 0$

   (ii) The independent and stationary increments are inherited from $N(t), M(t)$.

   (iii)

$$\begin{aligned}
\mathbb{P}(N(h) + M(h) = 1) &= \mathbb{P}(N(h) = 1, M(h) = 0) + \mathbb{P}(N(h) = 0, M(h) = 1) \\
&= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (\lambda_2 h + o(h))(1 - \lambda_1 h + o(h)), \quad \text{since } N \perp M \\
&= (\lambda_1 + \lambda_2)h + o(h)
\end{aligned}$$

   (iv) Lastly,

$$\begin{aligned}
\mathbb{P}(N(t) + M(t) \geq 2) &= 1 - (\mathbb{P}(N(t) = 0, M(t) = 0) + \mathbb{P}(N(t) = 1, M(t) = 0) + \mathbb{P}(N(t) = 0, M(t) = 1)) \\
&= 1 - (1 - \lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) \\
&- (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) - (1 - \lambda_1 h + o(h))(\lambda_2 h + o(h)) \\
&= o(h)
\end{aligned}$$

   which proves our earlier claim.