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Real Analysis: Homework 1

1. (a) $\mathbb{R}$ is second-countable by considering the countable basis

$$\mathcal{B} := \{(a,b) | a < b, a, b \in \mathbb{Q}\}$$

We now claim that $\mathcal{B}^n = \{U_1 \times \ldots \times U_n)|$ each $U_i \in \mathcal{B}$ for $i = 1, \ldots, n\}$ is a countable basis for $\mathbb{R}^n$. It is clear that $\mathcal{B}^n$ is countable as the Cartesian product of countable sets is still countable. To show $\mathcal{B}^n$ is a basis for $\mathbb{R}^n$:

(1) Pick $x = (x_1 \ldots, x_n) \in \mathbb{R}^n$ and consider the projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $(x_1 \ldots, x_n) \mapsto x_i$. Thus for each $\pi_i(x) = x_i$ we can find $B_i \in \mathcal{B}$ such that $x_i \in B_i$. Thus $B_1 \times \ldots \times B_n$ is the basis element in $\mathcal{B}^n$ containing x.

(2) Let x belong to the intersection of two basis elements $U = B_1 \times \ldots \times B_n, U' = B_1' \times \ldots \times B_n'$. Using the projection map, $\pi_i(U) = B_i, \pi_i(U') = B_i'$ and thus there is a basis element $A_i \subseteq B_i \cap B_i'$ for some $A_i \in \mathcal{B}$. Thus $A = A_i \times \ldots \times A_n$ is the basis element in $\mathcal{B}^n$ such that $A \subseteq U \cap U'$.

Thus we have shown that $\mathcal{B}^n$ is a countable basis for $\mathbb{R}^n$.

(b) Let U be an open set of $\mathbb{R}$. If U is a union of countably many open sets we can simply pick the disjoint open intervals from that union and we are done. Suppose U is an uncountable union of open sets, and without loss of generality assume that they are disjoint, $U = \sqcup_{\alpha \in A} V_\alpha$ for uncountable A, then since $\mathbb{R}$ is second-countable, there exists a countable basis $\mathcal{B}$ for $\mathbb{R}$. Thus for each $x \in V_\alpha$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq V_\alpha$. Consider the set

$$\left\{ \bigcup_{x \in V_\alpha} B_x \text{ for all } \alpha \in A \right\}$$

we claim that this set is at most countable and is a disjoint set of open intervals. Every element of the set is open since it is a union of $B \in \mathcal{B}$. It is disjoint since by construction $V_\alpha = \bigcup_{x \in V_\alpha} B_x$. Lastly, suppose that the set above is uncountable, then since there are countably many $B \in \mathcal{B}$, $\left(\bigcup_{x \in V_{\alpha_1}} B_x\right) \cap \left(\bigcup_{x \in V_{\alpha_2}} B_x\right) = V_{\alpha_1} \cap V_{\alpha_2} \neq \varnothing$ for some distinct α_1, α_2 which contradicts the earlier assumption.

2. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous function. Let (X, τ_X') be a finer topology than (X, τ_X) then $\tau_X' \supseteq \tau_X$. Thus for any $U \in \tau_Y$, $f^{-1}(U) \in \tau_X \subseteq \tau_X'$. Thus $f^{-1}(U) \in \tau_X'$ and $f : (X, \tau_X') \to (Y, \tau_Y)$ remains continuous. Let (Y, τ_Y') is a topology coarser than (Y, τ_Y) and so $\tau_Y \supseteq \tau_Y'$. Hence for $U \in \tau_Y' \subseteq \tau_Y$, we have $f^{-1}(U) \in \tau_X$. Thus $f : (X, \tau_X) \to (Y, \tau_Y')$ remains continuous.

3. We shall show that $\int_{\mathbb{R}} e^{-x^2/2}\,dx = \sqrt{2\pi}$. Start by considering its square,

$$\begin{aligned}
\left(\int_{\mathbb{R}} e^{-x^2/2}\,dx\right)^2 &= \left(\int_{\mathbb{R}} e^{-x^2/2}\,dx\right)\left(\int_{\mathbb{R}} e^{-y^2/2}\,dy\right) \\
&= \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dydx \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2} r\,drd\theta \quad \text{by polar change of coordinates.} \\
&= 2\pi\int_0^{\infty} e^{-r^2/2} r\,dr \\
&= 2\pi\int_0^{\infty} \frac{1}{2} e^{-s/2}\,ds \quad \text{change of coordinates, } s = r^2 \\
&= \pi\Big[-2e^{-s/2}\Big]_0^{\infty} = 2\pi
\end{aligned}$$

which shows what is required and hence $\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-x^2/2}\,dx = 1$.

4. Let $x_n > 0$ for all n and $x_n \to a$ with $a > 0$. Since $f(x) = \log x$ is a continuous function, it suffices to show

$$\frac{1}{n^2}\sum_{k=1}^{n} k\log x_k \to \frac{1}{2}\log a$$

Given $\epsilon > 0$, there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$, $\log x_n \in B(\log a, \epsilon)$, an epsilon ball centered at $\log a$. Consider the sequence $s_n := \frac{1}{n^2}\sum_{i=1}^{n} i\log x_i$. Then for some positive integer k, there exists N_1 such that

$$s_n = \frac{1}{n^2}\sum_{i=1}^{k} i\log x_i + \frac{1}{n^2}\sum_{i=k+1}^{n} i\log x_i \to \frac{1}{n^2}\sum_{i=k+1}^{n} i\log x_i$$

for $n \geq N_1$. Let $s = \limsup_{n\to\infty} s_n$, for any $\epsilon > 0$, then there exists N_2 such that we have $s_n < s + \epsilon$ for $n \geq N_2$. Taking $N = \max\{N_1, N_2\}$,

$$\frac{1}{n^2}\sum_{i=k+1}^{n} i\log x_i < s + \epsilon$$

5. We first observe that

$$\int_0^1 f(x)\,dx = \sum_{j=0}^{n-1}\int_{j/n}^{j+1/n} f(x)\,dx$$

thus

$$
\begin{aligned}
\left|\sum_{j=0}^{n-1}\frac{f\left(j/n\right)}{n}-\int_0^1 f(x)\,dx\right| &= \left|\sum_{j=0}^{n-1}\int_{j/n}^{(j+1)/n} f\left(j/n\right)-f(x)\,dx\right| \\
&= \left|\sum_{j=0}^{n-1}\int_{j/n}^{(j+1)/n} (j/n-x)\cdot\frac{f\left(j/n\right)-f(x)}{j/n-x}\,dx\right| \\
&\leq M\left|\sum_{j=0}^{n-1}\int_{j/n}^{(j+1)/n}(j/n-x)\,dx\right| \text{ by MVT for integrals and then MVT} \\
&= M\left|\sum_{j=0}^{n-1}-\frac{1}{2}\left(\frac{j}{n}-\frac{j+1}{n}\right)^2\right| \\
&= \frac{M}{2n}
\end{aligned}
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