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Real Analysis: Homework 3

1. A function which is in $\mathcal{C}^1(\mathbb{R})$ but not in $\mathcal{C}^2(\mathbb{R})$ means a function that has continuous first derivative but its second derivative is not continuous. Consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

we see that the first derivate of $f(x)$ is continuous but the second derivate is not continuous at $x = 0$.

2. By Stone-Weierstrass Theorem, for the given function $f(x)$, there exists a sequence of polynomials $P_n(x)$ that converges uniformly to $f(x)$, i.e. $\sup_{x\in[0,1]} |f(x) - P_n(x)| < 1/n$ as $n \to 0$. Now, we consider the inner product $\langle f(x), g(x)\rangle := \int_0^1 f(x)g(x)\, dx$, and if we managed to show that $\langle f(x), f(x)\rangle = 0$, we are done. To show that,

$$\begin{aligned}
\left|\int_0^1 f(x)^2\, dx\right| &= \left|\int_0^1 f(x)^2\, dx - \int_0^1 f(x)P_n(x)\, dx\right|, \text{ since } \int_0^1 f(x)x^n\, dx = 0 \text{ for all } n. \\
&\leq \int_0^1 |f(x)|\, |f(x) - P_n(x)|\ dx \\
&\leq \int_0^1 |f(x)|\ dx \cdot \sup_{x\in[0,1]} |f(x) - P_n(x)| \\
&\leq \frac{1}{n}\int_0^1 |f(x)|\ dx \text{ for all } n.
\end{aligned}$$

Thus we need $\langle f(x), f(x)\rangle = 0$, which completes the proof.

3. Let ϕ be λ-Hölder bi-continuous then for $v_1, v_2, u_1, u_2 \in T$, we have

$$\begin{aligned}
\sup_{v\in[0,T]} |\phi(u_2, v) - \phi(u_1, v)| &\leq C_u|u_2 - u_1|^\lambda \\
\sup_{u\in[0,T]} |\phi(u, v_2) - \phi(u, v_1)| &\leq C_v|v_2 - v_1|^\lambda
\end{aligned}$$

then we also observe that

$$\begin{aligned}
|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_1, v_2)| + |\phi(u_2, v_2) - \phi(u_2, v_1)| \\
|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq |\phi(u_1, v_1) - \phi(u_2, v_1)| + |\phi(u_2, v_2) - \phi(u_1, v_2)|
\end{aligned}$$

which gives us

$$\begin{aligned}
|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_v|v_2 - v_1|^\lambda \\
|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)| &\leq 2C_u|u_2 - u_1|^\lambda
\end{aligned}$$

multiplying them together, we have

$$|\phi(u_1, v_1) - \phi(u_1, v_2) - \phi(u_2, v_1) + \phi(u_2, v_2)|^2 \leq 4C_vC_u|v_2 - v_1|^\lambda|u_2 - u_1|^\lambda$$

squaring both sides, we have shown that all λ-Hölder bi-continuous are strongly $\lambda/2$-Hölder bi-continuous.

4. (a) For $0 \leq a < b \leq T$ with $0 \leq \alpha < 1/4$ and $b \leq r_1 \leq T$, we first make the following observations:

$$(r_1 - b)^{3/4} \leq (r_1 - a)^{3/4} \tag{1}$$

$$\frac{1}{(r_1 - a)^{\alpha+1/4}} \leq \frac{1}{(b-a)^{\alpha+1/4}} \tag{2}$$

then

$$\begin{aligned}
\int_b^T \frac{1}{(r_1-b)^\alpha (r_1-a)^{\alpha+1}}\, dr_1 &= \int_b^T \frac{(r_1-b)^{-\alpha-3/4+3/4}}{(r_1-a)^{\alpha+1}}\, dr_1 \\
&\leq \int_b^T \frac{(r_1-b)^{-\alpha-3/4}(r_1-a)^{3/4}}{(r_1-a)^{\alpha+1}}\, dr_1 \\
&= \int_b^T \frac{(r_1-b)^{-\alpha-3/4}}{(r_1-a)^{\alpha+1/4}}\, dr_1 \\
&\leq \frac{1}{(b-a)^{\alpha+1/4}} \int_b^T (r_1-b)^{-\alpha-3/4}\, dr_1 \\
&= \frac{1}{(b-a)^{\alpha+1/4}} \left[\frac{(r_1-b)^{1/4-\alpha}}{1/4-\alpha}\right]_b^T \\
&= 4\frac{(T-b)^{1/4-\alpha}}{(b-a)^{\alpha+1/4}}
\end{aligned}$$

(b) Given $\psi(u,v) := \mathbb{1}_{[0,v)}(u)\tilde{\psi}(u,v)$ we can understand it as

$$\psi(u,v) := \begin{cases} \tilde{\psi}(u,v) & u < v \\ 0 & u \geq v \end{cases}$$

then to do the double integral of $f(u,v)^2$ over $R = [0,T] \times [0,T]$ is equivalent to integrating over the region $\{(u,v) \in R \mid u < v\}$. Thus

$$\int_0^T \int_0^T f(u,v)^2\, du\, dv = \int_0^T \int_0^v \left| \int_u^T \int_v^T \frac{\tilde{\psi}(u,v) - \tilde{\psi}(u,r_2) - \tilde{\psi}(r_1,v) + \tilde{\psi}(r_1,r_2)}{(r_1-u)^{1+\alpha}(r_2-v)^{1+\alpha}} dr_2\, dr_1 \right|^2 du\, dv$$

We will now massage the term inside the first double integral,

$$\begin{aligned}
&\left| \int_u^T \int_v^T \frac{\tilde{\psi}(u,v) - \tilde{\psi}(u,r_2) - \tilde{\psi}(r_1,v) + \tilde{\psi}(r_1,r_2)}{(r_1-u)^{1+\alpha}(r_2-v)^{1+\alpha}} dr_2\, dr_1 \right| \\
&\leq \int_u^T \int_v^T \left| \frac{\tilde{\psi}(u,v) - \tilde{\psi}(u,r_2) - \tilde{\psi}(r_1,v) + \tilde{\psi}(r_1,r_2)}{(r_1-u)^{1+\alpha}(r_2-v)^{1+\alpha}} \right| dr_2\, dr_1 \\
&\leq \int_u^T \int_v^T \frac{\left|\tilde{\psi}(u,v) - \tilde{\psi}(u,r_2) - \tilde{\psi}(r_1,v) + \tilde{\psi}(r_1,r_2)\right|}{\left|(r_1-u)^{1+\alpha}(r_2-v)^{1+\alpha}\right|} dr_2\, dr_1 \\
&\leq \int_u^T \int_v^T \frac{\left|(r_1-u)^{\lambda}(r_2-v)^{\lambda}\right|}{\left|(r_1-u)^{1+\alpha}(r_2-v)^{1+\alpha}\right|} dr_2\, dr_1, \qquad \text{by strongly } \lambda\text{-Hölder bi-continuous property}
\end{aligned}$$