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Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that $f(y,z) = 5y^2 + z^2 - 3zy - 3y \geq 0$ for every $y, z \in \{0, 1, 2, \ldots\}$. We shall use $\mathbb{Z}_{>0}$ to denote the set $\{0, 1, 2, \ldots\}$ subsequently. We can rewrite $f(y,z)$ to get

$$f(y,z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \tag{1}$$

which we will show that the $f(y,z)$ in this form is nonnegative. All that is left to show is that $\frac{11}{4}y^2 - 3y \geq 0$ for all $y \in \mathbb{Z}_{>0}$ but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (1) holds for all $z \in \mathbb{Z}_{>0}$ when $y = 1$. Solving for the inequality below,

$$\begin{aligned} f(1,z) = \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 &\Leftrightarrow (z-1)(z-2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that $f(1,z) < 0$ for $z \in (1,2)$ and hence positive for all $z \in \mathbb{Z}_{>0}$ and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let $\mathcal{P}_i$ denote the set of paths from an origin s_i to a sink t_i with $\mathcal{P}_i \neq \emptyset$.

   Definition (flow)**.** For a *flow* f and path $P \in \mathcal{P}_i$, f_P is the amount of traffic of commodity i that chooses the path P to travel from s_i to t_i. A flow is feasible for a vector $r = (r_1, \ldots, r_k)$ if it routes all the traffic: for each $i \in \{1, 2, \ldots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

   Definition (Nonatomic equilibrium flow)**.** Let f be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow f is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \ldots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_p(f) \leq c_{\tilde{P}}(f)$$

   where $c_p(f)$ denotes the cost of travelling on path P for flow f.

   (ii) Let $\mathcal{P} = \bigcup_{i=1}^k \mathcal{P}_i$. Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_p(f_p) \cdot f_p = \sum_{e \in E} c_e(f_e) \cdot f_e$$

   where E is the set of directed edges on the graph G.

   (iii)

(iv) We start by showing that

$$\begin{aligned}
\inf_x \left\{ \left(\frac{ax+b}{ar+b} - 1 \right) \right\} &= \inf_x \left\{ x \left(\frac{a(x-r)}{ar+b} \right) \right\} \\
&= \frac{a}{ar+b} \inf_x \left\{ x^2 - rx \right\} \\
&= -\frac{r^2}{4} \cdot \frac{a}{ar+b}
\end{aligned}$$

with that we can begin out proof.

$$\begin{aligned}
\sup_{c\in\mathcal{C}} \sup_{x,r} \frac{rc(r)}{xc(x)-(r-x)c(r)} &= \sup_{c\in\mathcal{C}} \sup_{x,r} \frac{r}{r+x\left(\frac{c(x)}{c(r)}-1\right)}, \text{ since } c(r)>0 \\
&= \sup_{a,b\geq 0} \sup_{x,r} \frac{r}{r+x\left(\frac{ax+b}{ar+b}-1\right)} \\
&= \sup_{a,b\geq 0} \sup_{r} \frac{r}{r-\frac{r^2}{4}\frac{a}{ar+b}} \\
&= \sup_{a,b\geq 0} \sup_{r} \frac{1}{1-\frac{ar}{4(ar+b)}} \\
&= \frac{1}{1-1/4}
\end{aligned}$$

the last equality follows as the supremum of $\frac{ar}{4(ar+b)}$ occurs when $b=0$.

3. (i) Let Φ be the potential function of a potential game and c_i denote the cost function of the agents for $i \in \{1, 2, \dots, k\}$. To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of s_i and solely dependent on s_{-i}. If we consider two alternative distinct strategies for agent i, $s_i', s_i'' \neq s_i$

$$\begin{aligned}
c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i', s_{-i}) - \Phi(s_i', s_{-i})] \\
c_i(s_i, s_{-i}) &= \Phi(s_i, s_{-i}) + [c_i(s_i'', s_{-i}) - \Phi(s_i'', s_{-i})]
\end{aligned}$$

hence we can choose $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$, where $-$ represents any choice of strategy of agent i.

(ii) Let Φ_1 and Φ_2 be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \tag{2}$$
$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \tag{3}$$

where $D_i^k(s_{-i})$ denotes the dummy term for $k = 1, 2$. Taking (2)$-$(3),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at s_{-i} and any strategy of agent i, s_i.

(iii) ($\Rightarrow$): Let Φ be a potential function for a finite game. We want to show:

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = c_j(s_j^2, s_{-j}^1) - c_j(s^1) + c_i(s^2) - c_i(s_j^2, s_{-j}^1)$$

consider the left hand side,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = \Phi(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1)$$

($\Leftarrow$):

4. (a) Let $\tilde{f}$ be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player $i \in \{1, 2, \ldots, n\}$, any two parallel links P_i, P_j where $1 \leq i < j \leq k$,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of $\tilde{f}$ on P_i equals the flow of f on P_j, ($\tilde{f}_{P_i} = f_{P_j}$) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

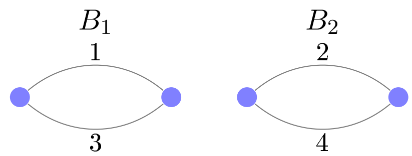
since the above inequality is true for any two parallel links, we sum it over the n parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^{n} \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^{n} \sum_{m=1}^{f_{P_i}} c_{P_j}(m) = \Phi(f)$$

where Φ is the potential function.

(b)

5. (a) The approach to find construct the bijection from the strategies from a partnership game G^1 to a congestion game G^2 will be to first find one for a 2 player game where the player have 2 strategies respectively.



		P2	
		B_1	B_2
P1	A_1	$1, 1$	$2, 2$
	A_2	$3, 3$	$4, 4$

The congestion game to consider is shown above and a strategy of P1 is the upper or lower edges of the networks; we shall let A_1 be the upper edges. A strategy of P2 is the choice of a disjoint network, here the left network is P2's strategy B_1. For $1 \leq i \leq n$ and $1 \leq j \leq m$, where n and m are the number of strategies for P1 and P2 respectively, e_{ij} is the ith edge of the jth disjoint network. Hence the tuple of bijective functions from the strategies of the network to the strategies in the partnership game for the congestion game shown above is:

$$\begin{aligned} f_1(A_i, B_j) &= \{e_{i,j}\}_{1 \leq j \leq m}, \quad \text{the } i\text{th edge in every disjoint network} \\ f_2(A_i, B_j) &= \{e_{i,j}\}_{1 \leq i \leq n}, \quad \text{all the edges in the } j\text{th disjoint network} \end{aligned}$$

which says that every strategy in the congestion game is a subset of edges in the family of disjoint networks. The cost function c_k^1 for the partnership game G^1 is shown in the table above and the cost function for the congestion game $c_k^2(\{e_{pj}\}_{1\leq j\leq m}, \{e_{iq}\}_{1\leq i\leq n})$ is the cost of the unique edge found from the intersection $\{e_{pj}\}_{1\leq j\leq m} \cap \{e_{iq}\}_{1\leq i\leq n}$. Note that although we used fixed numbers as the cost in the 2 player 2 strategies example we illustrated above, we can see them as just placeholders and it works well for any partnership game of that form even with different cost. We can generalize this from 2 players to N players in a similar fashion.

(b)

(c)

6. (a) We consider two networks A, B that resemble $n-$gons where n is even, i.e. there are n edges and n vertices for each network as shown in Figure 1. The strategy of each player i is

$$S_i = \{\{a_i, b_i\}, \{a_{i+1}, b_{i-1}, b_{i+1}\}\} = \{s_i^1, s_i^2\}$$

where a_i, b_i denotes edges in A and B respectively and the cost function of each edge is simply $c_e(x) = x$. We claim that when all the player were to play s_i^1 it is a Nash equilibrium and we have the optimal value of potential which has value $2n$, since every edge is inhabit by a single player. Any player i that deviates from playing s_i^1 will increase the potential by 2+2+2-1 (hence the claim it is a Nash) since it will share an edge with player $i+1$ in A and players $i-1, i+1$ in B and hence increase the potential. If all the players were to play s_i^2, we see that the potential will be $n + n + 2n$ where the first n is incurred from A and the next two terms are from B as we sum up the cost from 1 to the load of the edge. It is easy to see that everyone playing s_i^2 is a Nash; for any player that deviates from s_i^2, will have an increased cost of $+1$ coming from A, an increased cost of $+2$ from B and hence this completes the proof.

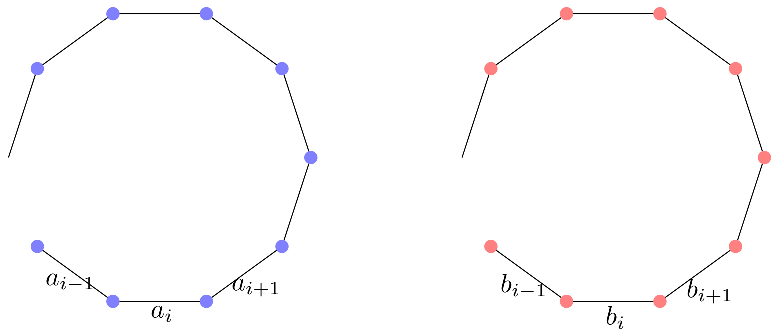


Figure 1: Example with price of potential anarchy equals 2.

(b)