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Real Analysis: Homework 3

1. A function which is in $\mathcal{C}^1(\mathbb{R})$ but not in $\mathcal{C}^2(\mathbb{R})$ means a function that has continuous first derivative but its second derivative is not continuous. Consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f'(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad f''(x) := \begin{cases} 2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   we see that the first derivate of $f(x)$ is continuous but the second derivate is not continuous at $x = 0$.

2. Given $f : [0, 1] \to \mathbb{R}$ continuous with $\int_0^1 f(x)x^n \, dx = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. By MVT for integrals, for every $n \in \mathbb{Z}_{\geq 0}$ there exists $c_n \in (0, 1)$ such that

$$0 = \int_0^1 f(x)x^n \, dx = f(c_n) \int_0^1 x^n \, dx = \frac{f(c_n)}{n+1}$$

3.

4. (a)

   (b)