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# Algorithmic Game Theory: HW 1

1. To show the desired inequality, it suffices to show that $f(y,z) = 5y^2 + z^2 - 3zy - 3y \geq 0$ for every $y, z \in \{0, 1, 2, \ldots\}$. We shall use $\mathbb{Z}_{>0}$ to denote the set $\{0, 1, 2, \ldots\}$ subsequently. We can rewrite $f(y,z)$ to get

$$f(y,z) = \left(\frac{3}{2}y - z\right)^2 + \frac{11}{4}y^2 - 3y \tag{1}$$

which we will show that the $f(y,z)$ in this form is nonnegative. All that is left to show is that $\frac{11}{4}y^2 - 3y \geq 0$ for all $y \in \mathbb{Z}_{>0}$ but solving for the inequality, we have

$$\frac{11}{4}y^2 - 3y \geq 0 \Leftrightarrow y \geq \frac{12}{11} \text{ or } y = 0$$

meaning we are left to prove that (1) holds for all $z \in \mathbb{Z}_{>0}$ when $y = 1$. Solving for the inequality below,

$$\begin{aligned} f(1,z) = \left(z - \frac{3}{2}\right)^2 - \frac{1}{4} < 0 &\Leftrightarrow (z-1)(z-2) < 0 \\ &\Leftrightarrow 1 < z < 2 \end{aligned}$$

which says that $f(1,z) < 0$ for $z \in (1,2)$ and hence positive for all $z \in \mathbb{Z}_{>0}$ and we are done.

2. (i) In an nonatomic congestion games with multicommodity networks, let $\mathcal{P}_i$ denote the set of paths from an origin $s_i$ to a sink $t_i$ with $\mathcal{P}_i \neq \emptyset$.

   **Definition** (flow)**.** For a *flow* $f$ and path $P \in \mathcal{P}_i$, $f_P$ is the amount of traffic of commodity $i$ that chooses the path $P$ to travel from $s_i$ to $t_i$. A flow is feasible for a vector $r = (r_1, \ldots, r_k)$ if it routes all the traffic: for each $i \in \{1, 2, \ldots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = r_i$.

   **Definition** (Nonatomic equilibrium flow)**.** Let $f$ be a feasible flow for an nonatomic congestion games with multicommodity networks. The flow $f$ is an *equilibrium flow* if, for every commodity $i \in \{1, 2, \ldots, k\}$ and every pair $P, \tilde{P} \in \mathcal{P}_i$ of $s_i - t_i$ paths with $f_P > 0$,

$$c_p(f) \leq c_{\tilde{P}}(f)$$

   where $c_p(f)$ denotes the cost of travelling on path $P$ for flow $f$.

   (ii) Let $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$. Then the total cost of a multicommodity network is

$$\sum_{P \in \mathcal{P}} c_p(f_p) \cdot f_p = \sum_{e \in E} c_e(f_e) \cdot f_e$$

   where $E$ is the set of directed edges on the graph $G$.

   (iii)

   (iv)

3. (i) Let $\Phi$ be the potential function of a potential game and $c_i$ denote the cost function of the agents for $i \in \{1, 2, \ldots, k\}$. To prove the required, it suffices to show that

$$c_i(s_i, s_{-i}) - \Phi(s_i, s_{-i})$$

is independent of the choice of $s_i$ and solely dependent on $s_{-i}$. If we consider two alternative distinct strategies for agent $i$, $s_i', s_i'' \neq s_i$

$$c_i(s_i, s_{-i}) = \Phi(s_i, s_{-i}) + [c_i(s_i', s_{-i}) - \Phi(s_i', s_{-i})]$$
$$c_i(s_i, s_{-i}) = \Phi(s_i, s_{-i}) + [c_i(s_i'', s_{-i}) - \Phi(s_i'', s_{-i})]$$

hence we can choose $D_i(s_{-i}) = c_i(-, s_{-i}) - \Phi(-, s_{-i})$, where $-$ represents any choice of strategy of agent $i$.

(ii) Let $\Phi_1$ and $\Phi_2$ be two potential functions of a game. From 3 (i) we have

$$c_i(s_i, s_{-i}) = \Phi_1(s_i, s_{-i}) + D_i^1(s_{-i}) \tag{2}$$
$$c_i(s_i, s_{-i}) = \Phi_2(s_i, s_{-i}) + D_i^2(s_{-i}) \tag{3}$$

where $D_i^k(s_{-i})$ denotes the dummy term for $k = 1, 2$. Taking (2)−(3),

$$\Phi_1(s_i, s_{-i}) - \Phi_2(s_i, s_{-i}) = D_i^1(s_{-i}) - D_i^2(s_{-i})$$

we have shown that two distinct potential functions differ by a constant, more precisely the difference of their dummy term evaluated at $s_{-i}$ and any strategy of agent $i$, $s_i$.

(iii) ($\Rightarrow$): Let $\Phi$ be a potential function for a finite game. We want to show:

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = c_j(s_j^2, s_{-j}^1) - c_j(s^1) + c_i(s^2) - c_i(s_j^2, s_{-j}^1)$$

consider the left hand side,

$$c_i(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1) = \Phi(s_i^2, s_{-i}^1) - c_i(s^1) + c_j(s^2) - c_j(s_i^2, s_{-i}^1)$$

($\Leftarrow$):

4. (a) Let $\tilde{f}$ be an equilibrium flow for an atomic selfish routing network of parallel links. Then for every player $i \in \{1, 2, \ldots, n\}$, any two parallel links $P_i, P_j$ where $1 \leq i < j \leq k$,

$$c_{P_i}(\tilde{f}) \leq c_{P_j}(f)$$

here, the flow of $\tilde{f}$ on $P_i$ equals the flow of $f$ on $P_j$, ($\tilde{f}_{P_i} = f_{P_j}$) i.e. any player routing their commodity to any path will have a cost equal to or larger than the equilibrium flow.

$$\sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{m=1}^{f_{P_j}} c_{P_j}(m)$$

since the above inequality is true for any two parallel links, we sum it over the $n$ parallel links and we are done.

$$\Phi(\tilde{f}) = \sum_{i=1}^{n} \sum_{m=1}^{\tilde{f}_{P_i}} c_{P_i}(m) \leq \sum_{i=1}^{n} \sum_{m=1}^{f_{P_i}} c_{P_j}(m) = \Phi(f)$$

where $\Phi$ is the potential function.

(b)

5. (a) Let $G^1$ and $G^2$ be a partnership game and congestion game respectively and suppose that they have the same number of players $k$. For the bijection $f_i$ to exists from the strategies $A_i$ of player $i$ in $G^1$ to the strategies $B_i$ of player $i$ in $G^2$, we require $|A_i| = |B_i|$ for all $i \in \{1, \ldots, k\}$.