Problem Set #1

Boot Camp 2018

Name: Zunda Xu

Email: zunda@uchicago.edu

Exercise 1.3.

Proof:

(1) $\mathcal{G}_1$ is not algebra. Since for $\forall\ A \in \mathcal{G}_1$, A is open, then A^c is closed in R. Thus $A^c \notin \mathcal{G}_1$. Hence $\mathcal{G}_1$ is not algebra.

(2) $\mathcal{G}_2$ is algebra. First we check $\emptyset$, since interval $(a, a] = \emptyset$ and $(a, a] \in \mathcal{G}_2$, thus $\emptyset \in \mathcal{G}_2$. Then we check its closeness. the complements of the intervals $(a, b], (-\infty, b]$ and (a, ∞) are the intervals $(-\infty, a] \cup (b, \infty), (b, \infty)$ and $(-\infty, a]$ respectively, and obviously they all belong to $\mathcal{G}_2$, thus for $\forall\ A \in \mathcal{G}_2$, A^c is also a finite union of intervals of the form $(a, b], (-\infty, b]$ and (a, ∞), which means $A^c \in \mathcal{G}_2$, similarly, for $\forall A_1, A_2, \ldots, A_n \in \mathcal{G}_2$, $\cup_{n=1}^{N} A_n \in \mathcal{G}_2$ too. Hence, $\mathcal{G}_2$ is algebra. However, $\mathcal{G}_2$ is not σ-algebra, since we can not determine whether $\forall\ A \in \mathcal{G}_2$ is a countable union or not.

(3) $\mathcal{G}_3$ is not only algebra, but also σ-algebra. Same to the proof in (2), we can easily know $\mathcal{G}_3$ is algebra. Also for $\forall A_1, A_2, \dots \in \mathcal{G}_3$, they are all countable unions of $(a, b], (-\infty, b]$ and (a, ∞), thus $\cup_{n=1}^{\infty} A_n$ is a countable union of $(a, b], (-\infty, b]$ and $(a, \infty) \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{G}_3$. Therefore, $\mathcal{G}_3$ is also closed under countable unions $\Rightarrow \mathcal{G}_3$ is σ-algebra.

Exercise 1.7.

Proof:

(1) First, we will show the left part. Since $\forall$ σ-algebra $\mathcal{A}$, $\emptyset \in \mathcal{A}$. Also, $X = (\emptyset)^c$, according to the definition of σ-algebra, $\forall$ σ-algebra $\mathcal{A}$, $X \in \mathcal{A}$. Hence, $\{\emptyset, X\} \subset \mathcal{A}$.

(2) Second, we will show the right part. Since $P(X) = \{A : A \subset X\}$, also for $\forall$

σ-algebra $\mathcal{A}$, $\forall A \in \mathcal{A}$, $A \subset X \Rightarrow A \in P(X)$. Hence, $\mathcal{A} \subset P(X)$.

In all, we have shown that $\{\emptyset, X\} \subset \mathcal{A} \subset P(X)$.

Exercise 1.10.

Proof:

Let $\mathcal{F} = \cap_\alpha \mathcal{S}_\alpha$, then if $A \in \mathcal{F} \Rightarrow A \in \mathcal{S}_\alpha$ for $\forall \alpha$. Since $\mathcal{S}_\alpha$ is σ-algebra for $\forall \alpha$, we have $A^c \in \mathcal{S}_\alpha$. Similarly, if $An \in \mathcal{F}$ for $\forall n = 1, 2, \ldots$, then $An \in \mathcal{S}_\alpha$ for every $n, \alpha \Rightarrow \cup_{n=1}^\infty A_n \in \mathcal{S}_\alpha$ for every $\alpha \Rightarrow \cup_{n=1}^\infty A_n \in \mathcal{F}$. Hence, $\cap_\alpha \mathcal{S}_\alpha$ is also σ-algebra.

Exercise 1.17.

Proof:

(1) Since $B = (B \setminus A) \cup (A \cap B)$, $A \subset B \Rightarrow A \cap B = A$, $B = (B \setminus A) \cup A$. Also $(B \setminus A) \cup A = \emptyset$, so B is the union of the two disjoint sets $(B \setminus A)$ and A. Since μ is a non-negative additive function, we have $\mu(B) = \mu(B \setminus A) + \mu(A)$. Since $\mu(B \setminus A) \geqslant 0$ $\Rightarrow \mu(B) \geqslant \mu(A)$

(2) Define a new sequence $(F_n)_{n \in \mathbb{N}}$ in S by: $F_n = \cup_{k=1}^n A_n$, then for $\forall n \in \mathbb{N}, F_n \subset F_{n+1}$, also $\cup_{n=1}^\infty F_n = \cup_{n=1}^\infty A_n$. Thus, $\mu(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(A_1 \cup A_2 \cdots \cup A_n) \leqslant \lim_{n\to\infty} \sum_{k=1}^n \mu(A_k) = \sum_{k \in \mathbb{N}} \mu(A_k)$.

Exercise 1.18.

Proof:

(1) If $A = \emptyset$, then $A \cap B = \emptyset \Rightarrow \mu(A \cap B) = \mu(\emptyset) = 0 \Rightarrow \lambda(A) = \lambda(\emptyset) = \mu(A \cap B) = 0$. Hence, $\lambda(\emptyset) = 0$.

(2) $\lambda(\cup_{n=1}^\infty A_n) = \mu((\cup_{n=1}^\infty A_n) \cap B) = \mu(\cup_{n=1}^\infty (A_n \cap B)) = \sum_{n=1}^\infty \mu(A_n \cap B) = \sum_{n=1}^\infty \lambda(A_n)$.

Hence, $\lambda(A)$ is also a also a measure.

Exercise 1.20.

Proof:

Since $\mu(Ai)$ is an decreasing sequence, so the limit exists. Set $C_i = A_1 \setminus A_i$ $\Rightarrow \mu(A_i) + \mu(C_i) = \mu(A_1)$, since $\mu(C_i) \leqslant \mu(A_1) < \infty \Rightarrow \mu(A_i) = \mu(A_1) - \mu(C_i)$, Moreover, the sets C_i form an increasing sequence of sets $\Rightarrow \lim_i \mu(C_i) = \mu(\cup_i C_i)$. Furthermore, $\cup_i C_i = \cup_i(A_1 \setminus A_i) = A_1 \setminus (\cap_i A_i) \Rightarrow \mu(\cap_i A_i) = \mu(A_1) - \mu(\cup_i C_i) = \mu(A_1) - \lim_i \mu(C_i) = \lim_i(\mu(A_1) - \mu(C_i)) = \lim_i \mu(A_i)$.

Exercise 2.10.

Proof:

To proof the statement, we will shown that the $\geqslant$ can be replaced by $\leqslant$ in the Theorem 2.8. Since $B = (B \cap E) \cup (B \cap E^c)$ and μ^* is an outer-measure $\Rightarrow \mu^*$ is countably sub-additive $\Rightarrow \mu^*((B \cap E) \cup (B \cap E^c)) \leqslant \mu^*(B \cap E) + \mu^*(B \cap E^c)$ $\Rightarrow \mu^*(B) \leqslant \mu^*(B \cap E) + \mu^*(B \cap E^c)$. From the Theorem 2.8, we obtain $\mu^*(B) \geqslant \mu^*(B \cap E) + \mu^*(B \cap E^c)$, hence we get the statement $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

Exercise 2.14.

Proof:

Define $\mathcal{O} = \{A : A \text{ is open}, A \subset \mathbb{R}\}$, ν is a premeasure on $\mathbb{R}$, denote μ^* as the outer measure generated by ν. Let $\sigma(\mathcal{O})$ be the σ-algebra generated by $\mathcal{O}$ and $\mathcal{M}$ denote the σ-algebra from the Caratheodory construction. By Theorem 2.12, we obtain $\sigma(\mathcal{O}) \subset \mathcal{M}$, since $\sigma(\mathcal{O})$ is the σ-algebra generated by $\mathcal{O}$, which is the Borel-algebra. Hence, we have shown $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.

Exercise 3.1.

Proof:

Let $a \in \mathbb{R}$, then $\{a\} \subset [a - \epsilon/2, a + \epsilon/2]$ for $\forall \epsilon > 0$. Define the Lebesgue Measure λ^*, then we have $\lambda^*(a) \leqslant \lambda^*([a - \epsilon/2, a + \epsilon/2]) = \epsilon$ for $\forall \epsilon > 0 \Rightarrow \lambda^*(a) = 0$, for $\forall a \in \mathbb{R}$. Let $A = \cup_{n=1}^{\infty}\{a_n\}$, then A is a countable set, then $\lambda^*(A) \leqslant \sum_{n=1}^{\infty} \lambda^*(a_n) = 0$.

Exercise 3.4.

Proof:

(1). First, let set $A = \{x \in X : f(x) < a\}, \quad \forall a \in \mathbf{R}$. $\because$ $\mathbb{M}$ is σ-algebra, $\therefore A^C = \{x \in X : f(x) \geqslant a\} \in \mathbb{M}, \quad \forall a \in \mathbf{R}$, and the definition still holds.

(2). Then, we show that set $\{x \in X : f(x) > a\} \in \mathbb{M}$. Define $\{a_n = a + \frac{1}{n}\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, then $\lim_{n\to\infty} a_n = a$. By the proof above, we know that $A_n = \{x \in X : f(x) \geqslant a_n\} \in \mathbb{M} \quad \forall a_n \in \mathbb{R}$. Thus $\bigcup_{n=1}^{\infty} A_n \in \mathbb{M} \Rightarrow \bigcup_{n=1}^{\infty}\{x \in X : f(x) \geqslant a\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > \lim_{n\to\infty} a_n\} \Rightarrow \{x \in X, f(x) > a\} \in \mathbb{M}$

(3). Thus, by the same logic in 1), $\{x \in X, f(x) > a\}$ $\{x \in X, f(x) \leqslant a\}$. Thus we have the sets composed of all four operators belonging to $\mathbb{M}$.

Exercise 3.7.

Proof:

(1). For case of $f + g$: Let $F(x, y) = x + y$, then $f + g = F(f, g)$ and $f + g$ is a continuous function. $\Rightarrow$ By property 4, we show that $f + g$ is measurable.

(2). For case of $f \cdot g$: Let $F(x, y) = xy$, then $fg = F(f, g)$ and fg is a continuous function. $\Rightarrow$ By property 4, we show that $f \cdot g$ is measurable.

(3). Let $f = \sup_{n\in\mathbb{N}} f_n(x), \quad g = \sup_{n\in\mathbb{N}} g_n(x)$. Also, let $\{K_n \mid_{n\in\mathbb{N}}\} = \{\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}\}$ $\Rightarrow sup_{n\in\mathbb{N}}K_n(x) = \max(sup_{n\in\mathbb{N}}f_n(x), sup_{n\in\mathbb{N}}g_n(x)) = \max(f, g) \Rightarrow \forall n,\ K_n(x)$ is measurable. $\Rightarrow \{K_n(x)\}_{n\in\mathbb{N}}$ is measurable $\Rightarrow$ By property (2) we show that $\max(f, g)$ is measurable.

(4). Similar to the above proof, change sup to inf, then $\inf_{n\in\mathbb{N}} K_n(x) = min(f, g)$, $\Rightarrow min(f, g)$ is measurable.

(5). $|f| = \max(f, -f)$ by proof (3) $\Rightarrow$ we know that $|f|$ is measurable.

Exercise 3.14.

Proof:

$\forall \epsilon > 0$, we constrict intervals and simple function as the proof in note. $\exists N_1 \in \mathbb{N}$, s.t. $\frac{1}{2^{N_1}} < \epsilon$, $\exists N_2 \in \mathbb{N}$, s.t. $f(x) < N_2$. Let $N = \max\{N_1, N_2\}$, for $n > N$,

$\forall x \in X, x \in E_i^n$ for $0 \le i \le N, i \in \mathbf{N} \Rightarrow f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$ and $s_n(x) = \frac{i-1}{2^n} \Rightarrow$ $|f(x) - s_n(x)| < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon \Rightarrow$ the convergence in (1) is uniform.

Exercise 4.13.

Proof:

Since $0 \leqslant ||f|| < M$ on $E \in \mathbb{M}$, and $\mu(E) < \infty$, by proposition 4.5, $0 \leqslant \int_E ||f|| d\mu \leqslant M\mu(E) < \infty \Rightarrow f \in \mathbb{L}^1(\mu, E)$.

Exercise 4.14.

Proof:

We will prove by contradiction. Without loss of generality, we just show the condition that $f = \infty$. Suppose $\exists A \subset E$ with positive measure μ, s.t. $f = \infty$ somewhere on A. Then, $\infty = \int_A f d\mu \leqslant \int_E f d\mu \leqslant \int_E ||f|| d\mu \Rightarrow f \notin \mathbb{L}^1(\mu, E)$, which contradicts with $f \in \mathbb{L}^1(\mu, E)$.

Exercise 4.15.

Proof:

Let $S(f) = \{s : 0 \leqslant s \leqslant f$, s measurable and simple$\}$. $f < g \Rightarrow f^+ < g^+$ and $f^- > g^- \Rightarrow S(f^+) \subset S(g^+) \Rightarrow \int_E f^+ d\mu \leqslant \int_E g^+ d\mu$. Similarly, $\Rightarrow S(g^-) \subset S(f^-) \Rightarrow$ $\int_E g^- d\mu \leqslant \int_E f^- d\mu \Rightarrow \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leqslant \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu$. Hence $\int_E f d\mu \leqslant \int_E g d\mu$.

Exercise 4.16.

Proof:

Take an arbitrary simple function $s(x) = \sum_1^N c_i \chi_i E_i$, where E_i is measurable. Then since $A \subset E \Rightarrow A \cap E_i \subset E \cap E_i \quad \forall i \Rightarrow \mu(A \cap E_i) \le \mu(E \cap E_i) \quad \forall i \Rightarrow \int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \le \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu \Rightarrow \int_A ||f|| d\mu \le \int_E ||f|| d\mu < \infty$ $\Rightarrow f \in \mathbb{L}^1(\mu, A)$.

Exercise 4.21.

Proof:

Let $\lambda(\cdot)$ be a measure on μ, and $A = (A \backslash B) \cup (A \cap B)$. $\because B \subset A, \Rightarrow A = (A \backslash B) \cup B$ $\Rightarrow \lambda(A) = \lambda((A \setminus B) \cup B) = \lambda(A \setminus B) + \lambda(B) \Rightarrow \int_A f d\mu = \int_{A \backslash B} f d\mu + \int_B f d\mu$. $\because \int_{A \backslash B} f d\mu = 0 \Rightarrow \int_A f d\mu = \int_B f d\mu \Rightarrow \int_A f d\mu \leqslant \int_B f d\mu$