Problem Set #3

Boot Camp 2018

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Exercise 4.2.

Proof:

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

.

Let $\det(\lambda I - D) = 0$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 = 0$$

$\Rightarrow \lambda = 0$

$\therefore$ Algebraic multiplicity is 3. The corresponding eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$Geometric multiplicity is 1.

Exercise 4.4.

Proof:

(i)

Let A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^H = A$ implies that $b = \bar{c}$.

Recall that if $z \in \mathbb{C}$, then $z\bar{z} = |z|^2 \geq 0$

Now, $p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$

$\Delta = (a+d)^2 - 4(ad-bc) = (a-d)^2 + 4b\bar{b} = (a-d)^2 + 4|b|^2 \geq 0$

$\Rightarrow$ Real roots.

(ii)

Suppose λ is an eigenvalue of A where $A^H = -A$. x is the corresponding eigenvector.

Then, $\langle Ax, x\rangle = \langle \lambda x, x\rangle = lam\bar{b}da\langle x, x\rangle$.

Also, $\langle Ax, x\rangle = \langle x, A^H x\rangle = \langle x, -Ax\rangle = -\langle x, \lambda x\rangle = -\lambda\langle x, x\rangle$.

So we have $\bar{\lambda} = -\lambda$

$\Rightarrow$ λ is pure imagery.

Exercise 4.6.

Proof:

Suppose A is an upper triangular matrix.

$$A = \begin{bmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}$$

Then the characteristic polynomial is

$$\det(zI - A) = \begin{vmatrix} z - a_1 & & & * \\ & z - a_2 & & \\ & & \ddots & \\ 0 & & & z - a_n \end{vmatrix} = \Pi_{i=1}^{n}(z - a_i) = 0$$

Note that this polynomial has n zeros, which are $a_1, a_2, \ldots a_n$ respectively.

The case of upper triangular matrix is the same.

Exercise 4.8.

Proof:

(i) Since $V = span(s)$, it suffices to show that the four vectors are linearly independent.

Let $a\sin(x) + b\cos(x) + c\sin(2x) + d\cos(2x) = 0, \quad \forall x \in \mathbb{R}$.

Let $x = 0$: $b + d = 0$

Let $x = \frac{\pi}{2}$: $a - d = 0$

Let $x = \pi$: $-b + d = 0$

Let $x = \frac{\pi}{4}$: $a\sin(\frac{\pi}{4}) + b\cos(\frac{\pi}{4}) + c\sin(\frac{\pi}{2}) + d\cos(\frac{\pi}{2}) = 0$

From the above four conditions we can get $a = 0, b = 0, c = 0, d = 0$.

Since the only case that can let $a\sin(x)+b\cos(x)+c\sin(2x)+d\cos(2x) = 0, \quad \forall x \in \mathbb{R}$. is when $a = b = c = d = 0$,

$\Rightarrow$ They are linearly independent.

(ii)

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

(iii)

$V_1 = \{\sin x, \cos x\}$, $V_2 = \{\sin 2x, \cos 2x\}$

Exercise 4.13.

Proof:

To diagonalize A, we first need to find eigenvalues and eigenvectors.

$p(\lambda) = \lambda^2 - 1.4\lambda + 0.4 = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = \frac{2}{5}$

And $\Sigma_1 = span([2,1]^T), \Sigma_2 = span([1,-1]^T)$

So A is semisimple.

Let

$$p = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

,

then

$$p^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$\Rightarrow D = p^{-1}Ap.$

Exercise 4.15.

Proof:

Since $A \in M_n(\mathbb{F})$ is semisimple, we can diagonalize $A = pDp^{-1}$, where Dd is diagonalized, and $\{\lambda_i\}_1^n$ are the diagonal entries of D.

Now, $f(A) = f(pDp^{-1})$

$= a_0 I + a_1 pDp^{-1} + \cdots + a_n pD^n p^{-1}$

$= p[a_0 I + a_1 D + \cdots + a_n D^n]p^{-1}$

$= pf(D)p^{-1}$

Observe that f(A) and f(D) are similar, so they have the same eigenvalues.

Also note that $f(D)$ is also diagonal, so each entry along the diagonal is $f(D)_{ii} = a_0 + a_1 d_{ii} + \cdots + a_n d_{ii}^n = f(d_{ii})$, where $D = [d_{ij}]_{ij}$

Hence, the eigenvalues of $f(D)$ are just its diagonals, which are:

$\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}$

Exercise 4.16.

Proof:

(i)

$$A = pD^np_{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\therefore lim_{n\to\infty} A^n = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Let $B = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, then it follows immediately from the definition of limit.

(ii)

The choice of norm does not affect the answer.

(iii)

Let $f(x) = 3+5x+x^3$, then the eigenvalues of $f(A)$ are $f(\lambda_1) = f(1) = 9$, $f(\lambda_2) = f(0.4) = 5.064$.

Exercise 4.18.

Proof:

Take $\vec{y}$ an eigenvector corresponding to λ. Then,

$x^TAy = x^T\lambda y = (\lambda x^T y)$

$\Rightarrow x^TA = \lambda x^T$

Exercise 4.20.

Proof:

Let $B = U^HAU$, then,

$B^H = U^HA^HU = U^HAU = B$, since $A^H = A$

Exercise 4.28.

Proof:

(i)

$p(\vec{x}) = \frac{\langle x, Ax\rangle}{||x||^2}$

Observe that the denominator is always a real number.

Hence to show that $p(\vec{x}) \in \mathbb{R}, it suffices to show that \langle x, Ax\rangle \in \mathbb{R}$.

Now $\langle x, Ax\rangle = \langle A^H, x\rangle = -\langle Ax, x\rangle$

Since by definition, $\langle x, Ax\rangle = \langle A\bar{x}, x\rangle$, we have $\langle Ax, x\rangle = \langle A\bar{x}, x\rangle \in \mathbb{R}$

This implies

$$\langle x, Ax\rangle \in \mathbb{R}$$

$\Rightarrow p(x) \in \mathbb{R}$

(ii)

If $A^H = -A$, then

$\langle x, Ax\rangle = \langle A^H, x\rangle = -\langle Ax, x\rangle$

Also, $\langle x, Ax\rangle = \langle A\bar{x}, x\rangle$

$\therefore \langle A\bar{x}, x\rangle = -\langle Ax, x\rangle$

This implies $\langle x, Ax\rangle = \langle A\bar{x}, x\rangle \in \mathbb{C} \setminus \mathbb{R} \cup \{0\}$

Hence $p(\vec{x}) = \frac{\langle x, Ax\rangle}{||x||^2}$ is pure imaginary number.

Exercise 4.25.

Proof:

(i)

Since $A \in M_n(\mathbb{C})$ is a normal matrix, its eigenspace $\{x_1, x_2, \ldots, x_n\}$ spans $\mathbb{C}^n$.

Observe that $\forall j = 1, 2, \ldots, n$,

$(x_1x_1^H + x_2x_2^H + \cdots + x_nx_n^H)x_j = x_1x_1^Hx_j + x_2x_2^Hx_j + \cdots + x_nx_n^Hx_j = x_j$

This holds for any j.

Since $\{x_1, x_2, \ldots, x_n\}$ spans $\mathbb{C}^n$, $\forall \vec{v} = \mathbb{C}^n$, $\vec{v} = \sum a_i\vec{x}_i$

Let $B = x_1x_1^H + x_2x_2^H + \cdots + x_nx_n^H$, then $B\vec{v} = \sum a_iB\vec{x}_i = \sum a_i\vec{x}_i = \vec{v}$.

Let $\vec{v} = \vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ respective, then we get

$Be_1 = e_1, Be_2 = e_2, \ldots, Be_n = e_n$

Hence $B = I$

(ii)

Since A is a normal matrix and $\{x_1, x_2, \ldots, x_n\}$ forms an orthonormal eigenbasis, A admits a diagonalization.

$A = pDp{-}1 = pDp^H$, where

$$p = [x_1, x_2, \ldots, x_n] \quad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$p^{-1} = p^H = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

, since p is an orthonormal matrix.

Hence,

$$A = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^H \\ x_2^H \\ \vdots \\ x_n^H \end{bmatrix} = \sum \lambda_i x_i x_i^H$$

Exercise 4.27.

Proof:

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

By definition, $\forall x, \quad x^H A x > 0$

Now, let

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_i^{-1}$$

then

$$e_1^H A e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = a_{11} > 0$$

Similarly, let $x = e_2, e_3, \dots, e_n$

we have $a_{22} > 0, a_{33} > 0, \dots, a_{nn} > 0$

Here all diagonal elements are positive and real.

Exercise 4.28.

Proof:

First we introduce the following lemmas used in the proof.

- Lemma 1: The diagonals of a positive semi-definite matrix are greater than or equal to zero. (Proof similar to exercise 4.27)

- Lemma 2: $tr(AB) = tr(BA)$ (Proof can be found in Problem Set 2)

- Lemma 3: If $A \in M_n(\mathbb{F})$ is a positive semi-definite matrix, $D \in M_n(\mathbb{F})$ is a diagonal matrix with non-negative diagonals, then $0 \leq tr(AD) \leq tr(A)tr(D)$.

  Proof. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

  ,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

  ,

  then $tr(AD) = \sum_{i=1}^{n} a_{ii} d_i \geq 0$, since $a_{ii} \geq 0$ and $d_i \geq 0$ for $\forall i$

  $tr(A)tr(D) = (\sum_{i=1}^{n} a_{ii})(\sum_{i=1}^{n} d_i) = \sum_{i=1}^{n} a_{ii} d_i + \sum_{i \neq j} a_{ii} d_j \geq \sum_{i=1}^{n} a_{ii} d_i$

  $\Rightarrow tr(A)tr(D) \geq tr(AD) \geq 0$ □

Now since B is a positive semi-definite matrix, it admits a diagonalization s.t. $B = PDP^{-1} = PDP^{H}$, where

$$P = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is an orthonormal eigenbasis,

$$D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

is diagonal matrix with $d_i \geq 0 \quad \forall i$.

Then $tr(AB) = tr(APDP^H) = tr(P^H APD) \leq tr(P^H AP) tr(D)$

$= tr(APP^H) tr(D) = tr(A) tr(D) = tr(A) tr(B)$.

Meanwhile, $||AB||_F^2 = \text{ tr}(AA^H BB^H) \leq \text{ tr}(AA^H) \text{ tr}(BB^H) = ||A||_F ||B||_F^2$,

which makes $|| \cdot ||_F$ a matrix norm.

Exercise 4.31.

Proof:

(i)

Suppose A has rank r, then $A^H A$ is positive definite and has r distinct eigenvalues.

Let $s = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be an orthonormal eigenspace of $A^H A$, and $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$ be the corresponding eigenvectors, where $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$.

Since s spans $\mathbb{F}^n$, $\forall \vec{x} \in \mathbb{F}^n$, we have

$\vec{x} = \sum_{i=1}^n c_i \vec{v_i}$, $c_I \in]\mathbb{F}, \forall i$, and

$||x||_2 = \sqrt{(\sum c_i v_i^T)(\sum c_i v_i)} = \sqrt{(\sum c_i^2)}$

Hence if $||x||_2 = 1, then \sum_{i=1}^n c_i^2 = 1$

Now, observe that $\|Ax\|_2^2 = \langle Ax, Ax \rangle = (Ax)^H Ax = x^H A^H Ax$

$= (\sum_{i=1}^n c_i \vec{v_i}^H)(A^H A)(\sum_{i=1}^n c_i \vec{v_i})$

$= (\sum_{i=1}^n c_i \vec{v_i}^H)(\sum_{i=1}^n c_i A^H A \vec{v_i}^H)$

$= (\sum_{i=1}^n c_i \vec{v_i}^H)(\sum_{i=1}^n c_i \sigma^2 \vec{v_i}^H) = \sum c_i^2 \sigma^2$, where $s = \{v_1, v_2, \ldots, v_n\}$

Note that when $\sigma c_i^2 = 1$, and $\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_n^2$,

$\sum c_i^2 \sigma_i^2 \leq \sigma_1^2$

Hence, $\|A\|_2^2 = \sup_{\|x\|_2 = 1} \|Ax\|_2^2 = \sigma_1^2$

$\Rightarrow \|A\|_2 = \sigma_1$

(ii)

Since $A = U \Sigma V^H$

$A^{-1} = (U \Sigma V^H)^{-1} = (V^H)^{-1} \Sigma^{-1} (U)^{-1} = V \Sigma^{-1} U^H$

$\Rightarrow$ This is still an SVD of A^{-1}

And

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \ddots \\ & \frac{1}{\sigma_n} & \end{bmatrix}$$

i.e. The singular values of $A^{-1} are\ \frac{1}{\sigma_1} \leq \cdots \leq \frac{1}{\sigma_n}$

By (i), $\|A^{-1}\|_2$ is the largest singular value of A^{-1}, i.e. $\frac{1}{\sigma_n}$

(iii)

Since $A = U\Sigma V^H$

$A^H = (V^H)^H \Sigma^H U^H = V\Sigma^H U = V\Sigma U$

$\Rightarrow A^H$ and A has the same singular values.

So $\left\|A^H\right\|_2^2 = \|A\|_2^2 = \sigma_1^2$

(A^T) is just A^H restricted on $\mathbb{R}$.

So $\left\|A^T\right\|_2^2 = \left\|A^H\right\|_2^2$

By the previous argument, we know that $A^H A$ has an orthonormal eigenbasis $\{v_1, v_2, \ldots, v_n\}$,

and $\forall \|x\|_2 = 1$,

$\left\|A^H Ax\right\|_2 = \left\|A^H A \sum c_i v_i\right\|_2 = \sqrt{(\sum c_i \sigma_i^2 v_i^T)(\sum c_i \sigma_i^2 v_i)} = \sqrt{\sum c_i \sigma_i^4} \leq \sigma_1^2$

Hence $\left\|A^H A\right\|_2 = \sup_{\|x\|=1} \left\|A^H Ax\right\| = \sigma_1^2$

It follows that $\left\|A^H A\right\|_2 = \|A\|_2^2 = \left\|A^H\right\|_2^2 = \left\|A^T\right\|_2^2 = \sigma_1^2$

(iv)

Lemma: Let Q be an orthonormal matrix, then $||AQ||_2 = ||A||_2$.

Proof. Let $S_1 = \{\ ||AQ\vec{x}||, ||x||_2 = 1\ \}$, $S_2 = \{\ ||Ax||, ||x||_2 = 1$ Since Q is orthonormal, so Q is also invertible.

$\forall s_1 \in S_1, \exists x, \|x\| = 1, \quad s.t. \|AQx\|_2 = s_1$

Now, let $y = Qx$, it follows that $\|Qx\| = \|y\|_2 = 1$

Since orthonormal matrix preserves length, $\|Ay\|_2 = \|AQx\|_2 = s_1 \in S_2$

i.e. $S_1 \subset S_2$

$\forall s_2 \in S_2, \exists x, \|x\|_2 = 1 \quad s.t. \|Ax\|_2 = s_2$

Now, let $y = Q^{-1}x$, then $\|y\|_2 = \|Q^{-1}x\|_2 = 1$

Hence $\|AQy\|_2 = \|AQQ^{-1}x\|_2 = \|Ax\|_2 = s_2 \in S_1$

i.e. $S_2 \subset S_1$

$\therefore \|AQ\|_2 = \sup S_1 = \sup S_2 = \|A\|_2$ $\square$

Now $\|UAV\|_2 = \|UA\|_2$ by lemma since V is an orthonormal matrix.

$\|UA\|_2 = \sup_{\|x\|_2=1} \sqrt{(UAx)^H(UAx)} = \sup_{\|x\|_2=1} \sqrt{x^H A^H U^H U A x}$

$= \sup_{\|x\|_2=1} \sqrt{\langle Ax, Ax \rangle} = \|A\|_2$

Hence, $\|UAV\|_2 = \|UA\|_2 = \|A\|_2$

Exercise 4.32.

Proof:

(i)

We need the following lemmas:

- lemma 1: if $A, B \in M_n(\mathbb{F}), then tr(AB) = tr(BA)$
- lemma 2: $||A||_p^2 = tr(A^T A)$

Proof. let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{1m} & \dots & a_{mn} \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

Observe that $(A^T A)_{nm} = a_{1n}^2 + a_{2n}^2 + \dots + a_{mn}^2$

$\Rightarrow tr(A^T A) = ||A||_p^2$ $\square$

Now, $||UAV||_1^2 = tr((UAV)^T(UAV)) = tr(V^T A^T U^T UAV) = tr(V^T A^T AV) = tr(VV^T A^A) = tr(A^T A) = ||A||_1^2$

$\Rightarrow ||UAV||_2 = ||A||_2$

(ii)

Observe that $A = U\Sigma V^T$, with U and V^T orthonormal and

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$$

Now, $\|A\|_p^2 = tr(A^T A) = tr((U\Sigma V)^T(U\Sigma V)) = tr(U\Sigma^T U^T U\Sigma V^T) = tr(U\Sigma^2 V^T)$

$= tr(\Sigma^2) = \sum_{i=1}^r \sigma_i^2$

Hence $\|A\|_p = \sqrt{\sum_{i=1}^r \sigma_i^2}$

Exercise 4.33.

Proof:

Note that the $Y^H Ax$ will be a field element. Consider it as a linear map $Y^H Ax : \mathbb{F} \to \mathbb{F}$, then the spectral norm of this map is:

$||Y^H Ax||_2 = \sup_{f \in \mathbb{F}} \frac{||(Y^H Ax)f||_2}{||f||_2} = |Y^H Ax|$

, where the first norm is spectral norm and the norm in fraction is the standard 2-norm.

Exercise 4.36.

Proof:

One example can be

$$A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$$

then

$$A^T A = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

$\det(A^T A - \lambda I) = \lambda^2 - 50\lambda + 400 = 0$

$\lambda_1 = 40, \lambda_2 = 10$

Thus its singular value are $s_1 = \sqrt{40}, s_2 = \sqrt{10}$

To calculate its eigenvalues,

$\det(A - \lambda I) = \lambda^2 - 9\lambda + 20 = 0$

Thus its eigenvalues are $\lambda_1 = 4, \lambda_2 = 5$, which are different from its singular values.

Exercise 4.38.

Proof:

(i) Suppose $U\Sigma V^H$ is an SVD of A, then $A^\dagger = V\Sigma^{-1}U^H$

$AA^\dagger A = (U\Sigma V^H)(V\Sigma^{-1}U^H)(U\Sigma V^H) = U\Sigma V^H = A$

(ii)

$A^\dagger AA^\dagger = (V\Sigma^{-1}U^H)(U\Sigma V^H)(V\Sigma^{-1}U^H) = V\Sigma^{-1}U^H = A^\dagger$

(iii)

$(AA^\dagger)^H = ((U\Sigma V^H)(V\Sigma^{-1}U^H))^H = U\Sigma^{-1}V^H V\Sigma U^H = UU^H = AA^\dagger$

(iv)

$(A^\dagger A)^H = ((V\Sigma^{-1}U^H)(U\Sigma V^H))^H = V\Sigma U^H U\Sigma^{-1}V^H = VV^H = A^\dagger A$

(v)

By prop (iii) $\Rightarrow AA^\dagger$ is hermitian.

Also by prop (i), $AA^\dagger AA^\dagger = AA^\dagger \Rightarrow AA^\dagger$ is idempotent.

Next we will check whether $\mathcal{R}(AA^\dagger) = \mathcal{R}(A)$.

It is trivially $\mathcal{R}(AA^\dagger) \subset \mathcal{R}(A)$, and by prop(i) $\Rightarrow \mathcal{R}(A) \subset \mathcal{R}(AA^\dagger)$

$\Rightarrow \mathcal{R}(AA^\dagger) = \mathcal{R}(A)$

(vi)

By prop (iv) $A^\dagger A$ is hermitian.

Also by prop (ii) $A^\dagger AA^\dagger A = A^\dagger A \Rightarrow A^\dagger A$ is idempotent

Next we will check whether $\mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$

By prop (iv), $AA^\dagger = (AA^\dagger)^H = A^H(A^\dagger)^H \implies \mathcal{R}(A^\dagger A) \subset \mathcal{R}(A^H)$

Then we take the hermitian of both sides of prop (i),

we have $(A^\dagger A)^H A^H = (A^\dagger A)^H A^H A^H \Rightarrow \mathcal{R}(A^H) \subset \mathcal{R}(A^\dagger A)$

$\Rightarrow \mathcal{R}(A^\dagger A) = \mathcal{R}(A^H)$