

Problem Set #5

Boot Camp 2018

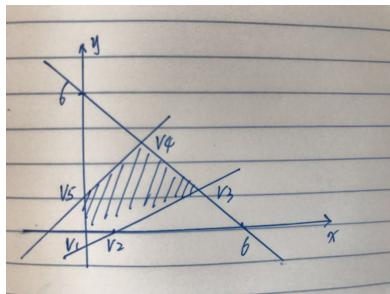
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Exercise 8.1

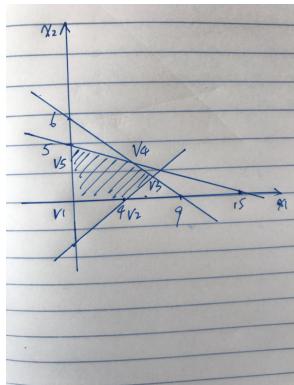
Answer:



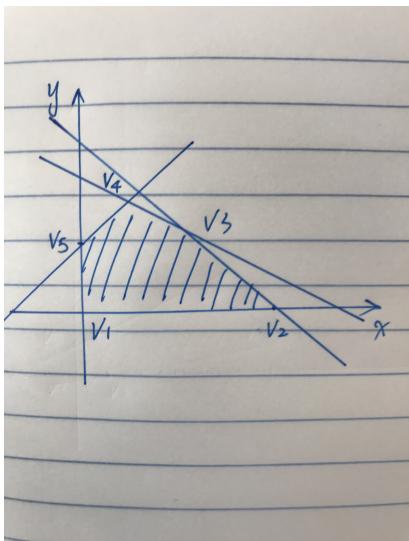
The feasible set is shown above. Observe that there are five vertices. $V_1 = (0, 0)$, $V_2 = (1, 0)$, $V_3 = (\frac{37}{7}, \frac{5}{7})$, $V_4 = (\frac{16}{5}, \frac{14}{5})$, $V_5 = (1, 0)$. Easy calculation yields that $(\frac{37}{7}, \frac{5}{7})$ is an optimizer, and the optimized value is $\frac{165}{7}$.

Exercise 8.2

Answer:



(1) The feasible set is shown above. Observe that there are five vertices. $V_1 = (0, 0)$, $V_2 = (4, 0)$, $V_3 = (6, 2)$, $V_4 = (3, 4)$, $V_5 = (5, 0)$. Easy calculation yields that $(6, 2)$ is an optimizer, and the optimized value is 20.



(2) The feasible set is shown above. Observe that there are five vertices. $V_1 = (0,0)$, $V_2 = (27,0)$, $V_3 = (15,12)$, $V_4 = (5,16)$, $V_5 = (11,0)$. Easy calculation yields that $(15,12)$ is an optimizer, and the optimized value is 132.

Exercise 8.3

Answer:

$$\begin{aligned}
 & \max\{4x + 3y\} \\
 & \text{subject to } 15x + 10y \leq 1800 \\
 & \quad 2x + 2y \leq 300 \\
 & \quad y \leq 200 \\
 & \quad x, y \geq 0.
 \end{aligned}$$

Exercise 8.4

Answer:

$\max\{2x_{AB} + 5x_{BC} + 2x_{CF} + 5x_{AD} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 4x_{DE} + 3x_{EF}\}$
 subject to $x_{AD} + x_{AB} = 10$

$$\begin{aligned}
 x_{BC} + x_{BE} - x_{AB} &= 1 \\
 x_{CF} - x_{BC} &= -2 \\
 x_{DE} - x_{AD} - x_{BD} &= -3 \\
 x_{EF} - x_{BE} - x_{DE} &= 4 \\
 -x_{CF} - x_{EF} - x_{BF} &= -10 \\
 0 \leq x_{AB} &\leq 6 \\
 0 \leq x_{BC} &\leq 6 \\
 0 \leq x_{CF} &\leq 6 \\
 0 \leq x_{AD} &\leq 6 \\
 0 \leq x_{BD} &\leq 6 \\
 0 \leq x_{BE} &\leq 6 \\
 0 \leq x_{BF} &\leq 6 \\
 0 \leq x_{DE} &\leq 6 \\
 0 \leq x_{EF} &\leq 6.
 \end{aligned}$$

Exercise 8.5

Answer:

(1)

$$\begin{aligned}
 \xi &= 3x_1 + x_2 \\
 w_1 &= 15 - x_1 - 3x_2 \\
 w_2 &= 18 - 2x_1 - 3x_2 \\
 w_3 &= 4 - x_1 + x_2
 \end{aligned}$$

Observe that we can increase x_1 to 9.

$$\begin{aligned}
 \xi &= 12 + 4x_2 - 3w_3 \\
 w_1 &= 11 - 3x_2 - w_3 \\
 w_2 &= 10 - 5x_2 + 2w_3 \\
 x_1 &= 4 + x_2 - w_3
 \end{aligned}$$

Observe that we can increase x_2 to 2.

$$\begin{aligned}
 \xi &= 20 - 0.8w_2 - 1.4w_3 \\
 w_1 &= 5 + 0.6w_2 - 2.2w_3 \\
 x_2 &= 2 - 0.2w_2 + 0.4w_3 \\
 x_1 &= 6 + 0.2w_2 - 0.6w_3
 \end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 20 when $x_1 = 6, x_2 = 2$.

(2)

$$\begin{aligned}\xi &= 4x + 6y \\ w_1 &= 27 - x - y \\ w_2 &= 27 - x - y \\ w_3 &= 90 - 2x - 5y\end{aligned}$$

Observe that we can increase x to 27.

$$\begin{aligned}\xi &= 108 - 4w_2 + 2y \\ w_1 &= 38 - w_2 - 2y \\ x &= 27 - w_2 - y \\ w_3 &= 36 + 2w_2 - 3y\end{aligned}$$

Observe that we can increase y to 12.

$$\begin{aligned}\xi &= 132 - \frac{8}{3}w_2 + \frac{2}{3}w_3 \\ w_1 &= 38 - \frac{7}{3}w_2 - \frac{2}{3}w_3 \\ x &= 15 - \frac{5}{3}w_2 - \frac{1}{3}w_3 \\ y &= 12 + \frac{2}{3}w_2 + \frac{1}{3}w_3\end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 132 when $x = 15, y = 12$.

Exercise 8.6

Answer:

Observe that our problem

$$\begin{aligned}&\max\{4x + 3y\} \\ \text{subject to } &15x + 10y \leq 1800 \\ &2x + 2y \leq 300 \\ &y \leq 200 \\ &x, y \geq 0\end{aligned}$$

is equivalent to

$$\begin{aligned}&\max\{4x + 3y\} \\ \text{subject to } &3x + 2y \leq 360 \\ &x + y \leq 150 \\ &y \leq 200 \\ &x, y \geq 0.\end{aligned}$$

$$\begin{aligned}
\xi &= 4x + 3y \\
w_1 &= 360 - 3x - 2y \\
w_2 &= 150 - x - y \\
w_3 &= 200 - 7
\end{aligned}$$

Observe that we can increase y to 150.

$$\begin{aligned}
\xi &= 450 + x - 3w_2 \\
w_1 &= 60 - x + 2w_2 \\
y &= 150 - x - w_2 \\
w_3 &= 50 + x + w_2
\end{aligned}$$

Observe that we can increase x to 60.

$$\begin{aligned}
\xi &= 510 - w_1 - w_2 \\
x &= 60 - w_1 + 2w_2 \\
y &= 90 + w_1 - 3w_2 \\
w_3 &= 110 - w_1 + 3w_2
\end{aligned}$$

There is no more term to increase and we conclude that the optimal value is 510 when $x = 60, y = 90$. So the company should produce 60 toy soldiers and 90 toy dolls.

Exercise 8.7

Answer:

(1) Observe that the origin is not in the feasible set, so we need an auxillary problem to find an initial vertex, which is

$$\begin{aligned}
&\max\{-x_0\} \\
\text{subject to } &-4x_1 - 2x_2 - x_0 \leq -8 \\
&-2x_1 + 3x_2 - x_0 \leq 6 \\
&x_1 + x_0 \leq 3 \\
&x_0, x_1, x_2 \geq 0
\end{aligned}$$

We use simplex method to solve it:

$$\begin{aligned}
\xi &= -x_0 \\
w_1 &= -8 + 4x_1 + 2x_2 + x_0 \\
w_2 &= 6 + 2x_1 - 3x_2 + x_0 \\
w_3 &= 3 - x_1 - x_0
\end{aligned}$$

We start with $x_0 = 8$.

$$\begin{aligned}\xi &= -8 - w_1 + 4x_1 + 2x_2 \\ x_0 &= 8 + w_1 - 4x_1 - 2x_2 \\ w_2 &= 14 + w_1 - 2x_1 - 5x_2 \\ w_3 &= -5 - w_1 + 3x_1 + 2x_2\end{aligned}$$

Observe that we can increase x_1 to 2.

$$\begin{aligned}\xi &= -x_0 \\ x_1 &= 2 + 0.25w_1 - 0.5x_2 - 0.25x_0 \\ w_2 &= 10 + 0.5w_1 + x_2 + 0.5x_0 \\ w_3 &= 1 - 0.25w_1 + 0.5x_2 - 0.75x_0\end{aligned}$$

Hence we can see that a feasible vertex is $x_1 = 2, x_2 = 0$. We start from the original problem and let x_1 be 2.

$$\begin{aligned}\xi &= 2 + 1.5x_2 + 0.5w_1 \\ x_1 &= 2 - 0.5x_2 + 0.5w_1 \\ w_2 &= 10 + w_1 - 4x_2 \\ w_3 &= 1 + 0.5x_2 - 0.5w_1\end{aligned}$$

Observe that we can increase w_1 to 2.

$$\begin{aligned}\xi &= 3 + 2x_2 - w_3 \\ x_1 &= 3 - w_3 \\ w_2 &= 12 - 3x_2 - 2w_3 \\ w_1 &= 2 + x_2 - 2w_3\end{aligned}$$

Observe that we can increase x_2 to 4.

$$\begin{aligned}\xi &= 11 - \frac{2}{3}w_2 - \frac{7}{3}w_3 \\ x_1 &= 3 - w_3 \\ x_2 &= 4 - \frac{1}{3}w_2 - \frac{2}{3}w_3 \\ w_1 &= 6 - \frac{1}{3}w_2 - \frac{8}{3}w_3\end{aligned}$$

There is no more term to increase. So the optimal value is 11 when $x_1 = 3, x_2 = 4$.

(2) Write down the auxillary problem and we can see that x_0 can never be 0. So this problem is infeasible.

(3) We write down the table.

$$\begin{aligned}\xi &= -3x_1 + x_2 \\ w_1 &= 4 - x_2 \\ w_2 &= 6 + 2x_1 - 3x_2\end{aligned}$$

Observe that we can increase x_2 to 2.

$$\begin{aligned}\xi &= 2 - \frac{7}{3}x_1 - \frac{1}{3}w_2 \\ w_1 &= 2 - \frac{2}{3}x_1 + \frac{1}{3}w_2 \\ x_2 &= 2 + \frac{2}{3}x_1 - \frac{1}{3}w_2\end{aligned}$$

There is no more term to increase. So the optimal value is 2 when $x_1 = 0, x_2 = 2$.

Exercise 8.8

Answer:

$$\max -x_1 - x_2 - x_3 \text{ s.t. } x_1, x_2, x_3 \geq 0.$$

Exercise 8.9

Answer:

$$\max x_1 + x_2 + x_3 \text{ s.t. } x_1, x_2, x_3 \geq 0.$$

Exercise 8.10

Answer:

$$\max x_1 + x_2 + x_3 \text{ s.t. } x_1, x_2 \geq 0, x_3 \geq 3 \text{ and } x_3 \leq 2.$$

Exercise 8.11

$$\max x_1 + x_2 + x_3 \text{ s.t. } , x_1 + x_2 + x_3 \geq 1, \text{ and } 0 \leq x_1, x_2, x_3 \leq 2.$$

The auxillary problem is

$$\max -x_0 \text{ s.t. } -x_1 - x_2 - x_3 - x_0 \leq -1 \text{ and } x_0, x_1, x_2, x_3 \geq 0.$$

Exercise 8.12

Answer:

By Bland's rule, among possible leaving and entering variables, we should always choose the one with the smallest index.

$$\begin{aligned}\xi &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ x_5 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ x_6 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ x_7 &= 1 - x_1\end{aligned}$$

We choose to increase x_1 to zero.

$$\begin{aligned}\xi &= -27x_2 + x_3 - 44x_4 - 20x_5 \\ x_1 &= 3x_2 + x_3 - 2x_4 - 2x_5 \\ x_6 &= 4x_2 + 2x_3 - 8x_4 + x_5 \\ x_7 &= 1 - 3x_2 - x_3 + 2x_4 + 2x_5\end{aligned}$$

We choose to increase x_3 to 1.

$$\begin{aligned}\xi &= 1 - 30x_2 - 42x_4 - 18x_5 - x_7 \\ x_1 &= 1 - x_7 \\ x_6 &= 2 - 2x_2 - 4x_4 + 5x_5 - 2x_7 \\ x_3 &= 1 - 3x_2 + 2x_4 + 2x_5 - x_7\end{aligned}$$

Now we conclude that the maximum is 1. This is obtained when $\mathbf{x} = (1, 0, 1, 0)$.

Exercise 8.15

Proof.

$$\begin{aligned}\mathbf{c}^T \mathbf{x} &= \mathbf{x}^T \mathbf{c} \\ &\leq \mathbf{x}^T (A^T \mathbf{y}) \\ &= (A\mathbf{x})^T \mathbf{y} \\ &\leq \mathbf{b}^T \mathbf{y}\end{aligned}$$

□

Exercise 8.17

Proof. Our primal problem is

$$\max \mathbf{c}^T \mathbf{x} \text{ s.t. } A\mathbf{x} \preceq \mathbf{b} \text{ and } \mathbf{x} \succeq \mathbf{0}.$$

The corresponding dual problem is

$$\min \mathbf{b}^T \mathbf{y} \text{ s.t. } A^T \mathbf{y} \succeq \mathbf{c} \text{ and } \mathbf{y} \succeq \mathbf{0},$$

which is equivalent to

$$\max -\mathbf{b}^T \mathbf{y} \text{ s.t. } -A^T \mathbf{y} \preceq -\mathbf{c} \text{ and } \mathbf{y} \succeq \mathbf{0}.$$

Now take this as the primal problem, and observe that the dual problem is

$$\min -\mathbf{c}^T \mathbf{z} \text{ s.t. } -A\mathbf{z} \succeq -\mathbf{b} \text{ and } \mathbf{z} \succeq \mathbf{0},$$

which is equivalent to

$$\max \mathbf{c}^T \mathbf{z} \text{ s.t. } A\mathbf{z} \preceq \mathbf{b} \text{ and } \mathbf{z} \succeq \mathbf{0}.$$

This is the same as the original primal problem. □

Exercise 8.18

Answer:

We first solve for the primal problem.

$$\begin{aligned}\xi &= x_1 + x_2 \\ w_1 &= 3 - 2x_1 - x_2 \\ w_2 &= 5 - x_1 - 3x_2 \\ w_3 &= 4 - 2x_1 - 3x_2\end{aligned}$$

Observe that we can increase x_1 to 1.5.

$$\begin{aligned}\xi &= 1.5 + 0.5x_2 - 0.5x_1 \\ x_1 &= 1.5 - 0.5x_2 - 0.5w_1 \\ w_2 &= 3.5 - 2.5x_2 + 0.5w_1 \\ w_3 &= 1 - 2x_2 + w_1\end{aligned}$$

Observe that we can increase x_2 to 0.5.

$$\begin{aligned}\xi &= 1.75 - 0.25w_1 - 0.25w_3 \\ x_1 &= 1.25 - 0.75w_1 - 0.25w_3 \\ w_2 &= 2.25 - 0.75w_1 + 1.75w_3 \\ x_2 &= 0.5 + 0.5w_1 - 0.5w_3\end{aligned}$$

We conclude that the maximum value is 1.75. Now, the dual problem is

$$\min 3y_1 + 5y_2 + 4y_3 \text{ s.t. } 2y_1 + y_2 + 2y_3 \geq 1, y_1 + 3y_2 + 3y_3 \geq 1, \text{ and } y_1, y_2, y_3 \geq 0.$$

Using the same technique we can see that the minimum value is attained at $(0.25, 0, 0.25, 0)$, and the minimum is 1.75.