Linear Algebra

(MT-121T)

Aftab Alam

Lecture # 24

(Friday, May 10, 2024)

Example

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal, and P^{-1}.

(c) Find A^6 and $f(A)$, where $t^4 - 3t^3 - 6t^2 + 7t + 3$.

(d) Find a "real cube root" of B—that is, a matrix B such that $B^3 = A$ and B has real eigenvalues.

$$\lambda_1 = 1, \lambda_2 = 4$$

$$x_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

$\lambda_1 = 1, \lambda_2 = 4$ $\quad x_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \text{where} \quad P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1366 & 2230 \\ 1365 & 2731 \end{bmatrix}$$

Also, $f(1) = 2$ and $f(4) = -1$. Hence,

$$f(A) = Pf(D)P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$B = P\sqrt[3]{D}P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt[3]{4} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 + \sqrt[3]{4} & -2 + 2\sqrt[3]{4} \\ -1 + \sqrt[3]{4} & 1 + 2\sqrt[3]{4} \end{bmatrix}$$

System of Linear Differential Equations

Solve $\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 0$.

$$\boldsymbol{u}(t) = e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving 2nd Order D. E.

Solve the differential equation $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 9y = 0$

$y = e^{\lambda t}$ $\qquad \dfrac{d^2}{dt^2}\left(e^{\lambda t}\right) - 9\dfrac{d}{dt}\left(e^{\lambda t}\right) = 0$

$\lambda^2 - 9 = 0$

$(\lambda - 3)(\lambda + 3) = 0$

$\lambda_1 = 3, \lambda_2 = -3$

General Solution is $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 e^{3t} + c_2 e^{-3t}$

Specific Solution can be found by applying initial conditions.

Solving 2nd Order D. E. using eigenvalues

Solve the differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 9y = 0$

Let $u = y'$, then $u' = 9y$

$$y'' - 9y = 0$$

$$\begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$u'(t) = A\ u(t)$$

$\lambda_1 = 3, \lambda_2 = -3$

$$u(t) = c_1 e^{3t} x_1 + c_2 e^{-3t} x_2$$

$x_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$u(t) = c_1 e^{3t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

General Solution is $y(t) = c_1 e^{3t} + c_2 e^{-3t}$

Specific Solution can be found by applying initial conditions.

6.3 A Solve $y'' + 4y' + 3y = 0$ by substituting $e^{\lambda t}$ and also by linear algebra.

Solution Substituting $y = e^{\lambda t}$ yields $(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0$. That quadratic factors into $\lambda^2+4\lambda+3 = (\lambda+1)(\lambda+3) = 0$. Therefore $\boldsymbol{\lambda_1 = -1}$ and $\boldsymbol{\lambda_2 = -3}$. The pure solutions are $y_1 = e^{-t}$ and $y_2 = e^{-3t}$. The complete solution $\boldsymbol{y} = c_1y_1 + c_2y_2$ approaches zero.

To use linear algebra we set $\boldsymbol{u} = (y, y')$. Then the vector equation is $\boldsymbol{u}' = A\boldsymbol{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -3y - 4y' \end{aligned} \quad \text{converts to} \quad \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \boldsymbol{u}.$$

This A is a "companion matrix" and its eigenvalues are again -1 and -3:

Same quadratic $$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$$

The eigenvectors of A are $(1, \lambda_1)$ and $(1, \lambda_2)$. Either way, the decay in $y(t)$ comes from e^{-t} and e^{-3t}. With constant coefficients, calculus leads to linear algebra $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Note In linear algebra the serious danger is a shortage of eigenvectors. Our eigenvectors $(1, \lambda_1)$ and $(1, \lambda_2)$ are the same if $\lambda_1 = \lambda_2$. Then we can't diagonalize A. In this case we don't yet have two independent solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$.

Second Order Differential Equations

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a *vector equation for y and y'*: first derivative only. Suppose the mass is $m = 1$. Two equations for $\boldsymbol{u} = (y,\ y')$ give $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} \boldsymbol{dy/dt} &= \boldsymbol{y'} \\ \boldsymbol{dy'/dt} &= \boldsymbol{-ky - by'} \end{aligned} \qquad \text{converts to} \qquad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{-k} & \boldsymbol{-b} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \tag{9}$$

The first equation $dy/dt = y'$ is trivial (but true). The second is equation (8) connecting y'' to y' and y. Together they connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve $\boldsymbol{u}' = A\boldsymbol{u}$ by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{bmatrix} \qquad \text{has determinant} \qquad \lambda^2 + b\lambda + k = 0.$$

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Exponential of a Matrix

We want to write the solution $\boldsymbol{u}(t)$ *in a new form* $e^{At}\boldsymbol{u}(0)$. First we have to say what e^{At} means, with a matrix in the exponent. To define e^{At} for matrices, we copy e^x for numbers.

The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. When you change x to a square matrix At, this series defines the matrix exponential e^{At}:

Matrix exponential e^{At}	$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots$	(14)
Its t derivative is Ae^{At}	$A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$	
Its eigenvalues are $e^{\lambda t}$	$(I + At + \frac{1}{2}(At)^2 + \cdots)\boldsymbol{x} = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots)\,\boldsymbol{x}$	

The number that divides $(At)^n$ is "n factorial". This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after $1, 2, 6$ are $4! = 24$ and $5! = 120$. They grow quickly. The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}\boldsymbol{u}(0)$ solves the differential equation with one quick formula—*even if there is a shortage of eigenvectors.*

Exponential of a Matrix

Use the series $$e^{At} = I + X\Lambda X^{-1}t + \tfrac{1}{2}(X\Lambda X^{-1}t)(X\Lambda X^{-1}t) + \cdots$$

Factor out X and X^{-1} $$= X\left[I + \Lambda t + \tfrac{1}{2}(\Lambda t)^2 + \cdots\right]X^{-1} \tag{15}$$

e^{At} **is diagonalized!** $$\boxed{e^{At} = X\,e^{\Lambda t}\,X^{-1}.}$$

e^{At} has the same eigenvector matrix X as A. Then Λ is a diagonal matrix and so is $e^{\Lambda t}$. The numbers $e^{\lambda_i t}$ are on the diagonal. Multiply $Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0)$ to recognize $\boldsymbol{u}(t)$:

$$e^{At}\boldsymbol{u}(0) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \tag{16}$$

This solution $e^{At}\boldsymbol{u}(0)$ is the same answer that came in equation (6) from three steps

Exponential of a Matrix

The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

1. Write $\boldsymbol{u}(0)$ as a **combination** $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ **of the eigenvectors of** A.
2. Multiply each eigenvector $\boldsymbol{x}_i$ by **its growth factor** $e^{\lambda_i t}$.
3. The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n. \tag{6}$$

Repeated Roots

Example 4 When you substitute $y = e^{\lambda t}$ into $y'' - 2y' + y = 0$, you get an equation with **repeated roots**: $\lambda^2 - 2\lambda + 1 = 0$ is $(\lambda - 1)^2 = 0$ with $\boldsymbol{\lambda = 1, 1}$. A differential equations course would propose e^t and te^t as two independent solutions. Here we discover why.

Linear algebra reduces $y'' - 2y' + y = 0$ to a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' - y \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \boldsymbol{u}. \tag{18}$$

A has a **repeated eigenvalue** $\boldsymbol{\lambda = 1, 1}$ (with trace $= 2$ and $\det A = 1$). The only eigenvectors are multiples of $\boldsymbol{x} = (1, 1)$. *Diagonalization is not possible*, A has only one line of eigenvectors. So we compute e^{At} from its definition as a series:

$$\textbf{Short series} \qquad \boldsymbol{e^{At}} = e^{It}e^{(A-I)t} = e^t\,[I + (A - I)t]. \tag{19}$$

That "infinite" series for $e^{(A-I)t}$ ended quickly because $(A - I)^2$ is the zero matrix! You can see $\boldsymbol{te^t}$ in equation (19). The first component of $\boldsymbol{e^{At}}\,\boldsymbol{u}(0)$ is our answer $y(t)$:

$$\begin{bmatrix} y \\ y' \end{bmatrix} = e^t \left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right] \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \qquad y(t) = e^t y(0) - \boldsymbol{te^t} y(0) + \boldsymbol{te^t} y'(0).$$

Exponential of a Matrix

Solve the differential equation $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ for the general solution.

Find the 1st Column of e^{At}

$$y''' + 2y'' - y' - 2y = 0$$

Transform this equation into 1st order D.E. $\quad y''' = -2y'' + y' + 2y$

$$\begin{bmatrix} y''' \\ y'' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y'' \\ y' \\ y \end{bmatrix}$$

$$u(t) = c_1 e^{t} x_1 + c_2 e^{-t} x_2 + c_3 e^{-2t} x_3$$

General Solution is $y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t}$

$$u'(t) = A\ u(t)$$

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$$

$$X = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}, \quad e^{\Lambda t} = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} \frac{1}{6} \cdots\cdots \\ -\frac{1}{2} \cdots\cdots \\ \frac{1}{3} \cdots\cdots \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$e^{At} = X e^{\Lambda t} X^{-1}$$

Symmetric Matrices

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q\Lambda Q^{\mathrm{T}}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

4 Antisymmetric matrices $A = -A^{\mathrm{T}}$ have *imaginary* λ's and *orthonormal* (*complex*) $\boldsymbol{q}$'s.

5 Section 9.2 explains why the test $S = S^{\mathrm{T}}$ becomes $\boldsymbol{S} = \overline{\boldsymbol{S}}^{\mathrm{T}}$ for *complex matrices*.

$S = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \overline{S}^{\mathrm{T}}$ has real $\lambda = 1, -1$. $\quad A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\overline{A}^{\mathrm{T}}$ has $\lambda = i, -i$.

1. **A symmetric matrix has only *real eigenvalues*.**
2. **The *eigenvectors* can be chosen *orthonormal*.**

Symmetric Matrices

Example 1 Find the λ's and $\boldsymbol{x}$'s when $S = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $S - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$.

Solution The determinant of $S - \lambda I$ is $\lambda^2 - 5\lambda$. The eigenvalues are 0 and 5 (*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue because S is singular, and $\lambda = 5$ matches the *trace* down the diagonal of S: $0 + 5$ agrees with $1 + 4$.

Two eigenvectors are $(\mathbf{2}, -\mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$—orthogonal but not yet orthonormal. The eigenvector for $\lambda = 0$ is in the *nullspace* of A. The eigenvector for $\lambda = 5$ is in the *column space*. We ask ourselves, why are the nullspace and column space perpendicular? The Fundamental Theorem says that the nullspace is perpendicular to the ***row space***—not the column space. But our matrix is *symmetric*! Its row and column spaces are the same. Its eigenvectors $(2, -1)$ and $(1, 2)$ must be (and are) perpendicular.

These eigenvectors have length $\sqrt{5}$. Divide them by $\sqrt{5}$ to get unit vectors. Put those unit eigenvectors into the columns of Q. Then $Q^{-1}SQ$ is Λ and $Q^{-1} = Q^{\mathrm{T}}$:

$$Q^{-1}SQ = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda.$$

Principal Axis Theorem

(Spectral Theorem) Every symmetric matrix has the factorization $S = Q\Lambda Q^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q:

Symmetric diagonalization $\quad S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}. \quad (1)$

Example

Example 2 The eigenvectors of a 2 by 2 symmetric matrix have a special form:

$$\textbf{Not widely known} \quad S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{has} \quad \boldsymbol{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}.$$

$$\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 = 0$. Eagle eyes might notice the special case $S = I$, when b and $\lambda_1 - a$ and $\lambda_2 - c$ and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are all zero. Then $\lambda_1 = \lambda_2 = 1$ is repeated. But of course $S = I$ has perpendicular eigenvectors.

Symmetric Matrices and Q

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

Symmetry $\quad S = X\Lambda X^{-1}$ becomes $S = Q\Lambda Q^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$.

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix}. \tag{5}$$

Columns q_1 and q_2 multiply rows $\lambda_1 q_1^{\mathrm{T}}$ and $\lambda_2 q_2^{\mathrm{T}}$ to produce $S = \lambda_1 q_1 q_1^{\mathrm{T}} + \lambda_2 q_2 q_2^{\mathrm{T}}$.

Every symmetric matrix $\quad S = Q\Lambda Q^{\mathrm{T}} = \lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}}$

S has correct eigenvectors
Those q's are orthonormal $\quad Sq_i = (\lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}})\, q_i = \lambda_i q_i$

Complex Eigenvalues for Real Matrices

For any real matrix, $S\boldsymbol{x} = \lambda \boldsymbol{x}$ gives $S\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}$. For a symmetric matrix, λ and $\boldsymbol{x}$ turn out to be real. Those two equations become the same. But a *non*symmetric matrix can easily produce λ and $\boldsymbol{x}$ that are complex. Then $A\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}$ is true but different from $A\boldsymbol{x} = \lambda \boldsymbol{x}$. We get another complex eigenvalue (which is $\overline{\lambda}$) and a new eigenvector (which is $\overline{\boldsymbol{x}}$):

For real matrices, complex λ's and $\boldsymbol{x}$'s come in "conjugate pairs."

$$\lambda = a + ib$$
$$\overline{\lambda} = a - ib$$

If $\quad A\boldsymbol{x} = \lambda \boldsymbol{x} \quad$ ***then*** $\quad A\overline{\boldsymbol{x}} = \overline{\lambda}\,\overline{\boldsymbol{x}}.$

Complex Eigenvalues for Real Matrices

Example 3 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Those eigenvalues are conjugate to each other. They are λ and $\overline{\lambda}$. The eigenvectors must be x and $\overline{x}$, because A is real:

$$\begin{aligned} &\textbf{This is } \lambda x \qquad A x = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &\textbf{This is } \overline{\lambda}\,\overline{x} \qquad A\overline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta) \begin{bmatrix} 1 \\ i \end{bmatrix}. \end{aligned} \tag{9}$$

Those eigenvectors $(1, -i)$ and $(1, i)$ are complex conjugates because A is real.

For this rotation matrix the absolute value is $|\lambda| = 1$, because $\cos^2\theta + \sin^2\theta = 1$. ***This fact*** $|\lambda| = 1$ ***holds for the eigenvalues of every orthogonal matrix*** Q.

Eigenvalues versus Pivots

Example 4 This symmetric matrix has one positive eigenvalue and one positive pivot:

Matching signs $S = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has pivots 1 and -8 eigenvalues 4 and -2.

The signs of the pivots match the signs of the eigenvalues, one plus and one minus. This could be false when the matrix is not symmetric:

Opposite signs $B = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$ has pivots 1 and 2 eigenvalues -1 and -2.

Symmetric: $S^{\mathrm{T}} = S = Q\Lambda Q^{\mathrm{T}}$	real eigenvalues	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{S}^{\mathrm{T}} = S$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $S^{\mathrm{T}} = S$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $A = BCB^{-1}$	$\lambda(A) = \lambda(C)$	B times eigenvector of C
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation : cosine-sine	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: $P_{i,i+1} = 1, P_{n1} = 1$	$\lambda_k = e^{2\pi ik/n} =$ roots of 1	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Circulant: $c_0 I + c_1 P + \cdots$	$\lambda_k = c_0 + c_1\, e^{2\pi ik/n} + \cdots$	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = X\Lambda X^{-1}$	diagonal of Λ	columns of X are independent
Schur: $A = QTQ^{-1}$	diagonal of triangular T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $A = BJB^{-1}$	diagonal of J	each block gives 1 eigenvector
SVD: $A = U\Sigma V^{\mathrm{T}}$	r singular values in Σ	eigenvectors of $A^{\mathrm{T}}A$, AA^{T} in V, U