LINEAR ALGEBRA (MT–121)

CHAPTER 6

EIGENVALUES AND EIGENVECTORS

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Eigenvalues and Eigenvectors

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

Eigenvalues provide valuable insights into the behavior and properties of linear transformations represented by matrices, making them a powerful tool in various mathematical and computational contexts.

Eigenvector of an Identity Matrix

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.

Example

Example 1 The matrix A has two eigenvalues $\lambda = 1$ and $\lambda = 1/2$. Look at $\det(A - \lambda I)$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda = 1}$ and $\boldsymbol{\lambda = \frac{1}{2}}$. For those numbers, the matrix $A - \lambda I$ becomes *singular* (zero determinant). The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)\boldsymbol{x}_1 = 0$ is $A\boldsymbol{x}_1 = \boldsymbol{x}_1$ and the first eigenvector is $(\mathbf{.6}, \ \mathbf{.4})$.

$(A - \frac{1}{2}I)\boldsymbol{x}_2 = 0$ is $A\boldsymbol{x}_2 = \frac{1}{2}\boldsymbol{x}_2$ and the second eigenvector is $(\mathbf{1}, \ \mathbf{-1})$:

$$\boldsymbol{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{x}_1 \quad (A\boldsymbol{x} = \boldsymbol{x} \text{ means that } \lambda_1 = 1)$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\,\boldsymbol{x}_2 \text{ so } \lambda_2 = \tfrac{1}{2}).$$

If $\boldsymbol{x}_1$ is multiplied again by A, we still get $\boldsymbol{x}_1$. Every power of A will give $A^n\boldsymbol{x}_1 = \boldsymbol{x}_1$. Multiplying $\boldsymbol{x}_2$ by A gave $\frac{1}{2}\boldsymbol{x}_2$, and if we multiply again we get $(\frac{1}{2})^2$ times $\boldsymbol{x}_2$.

Example

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

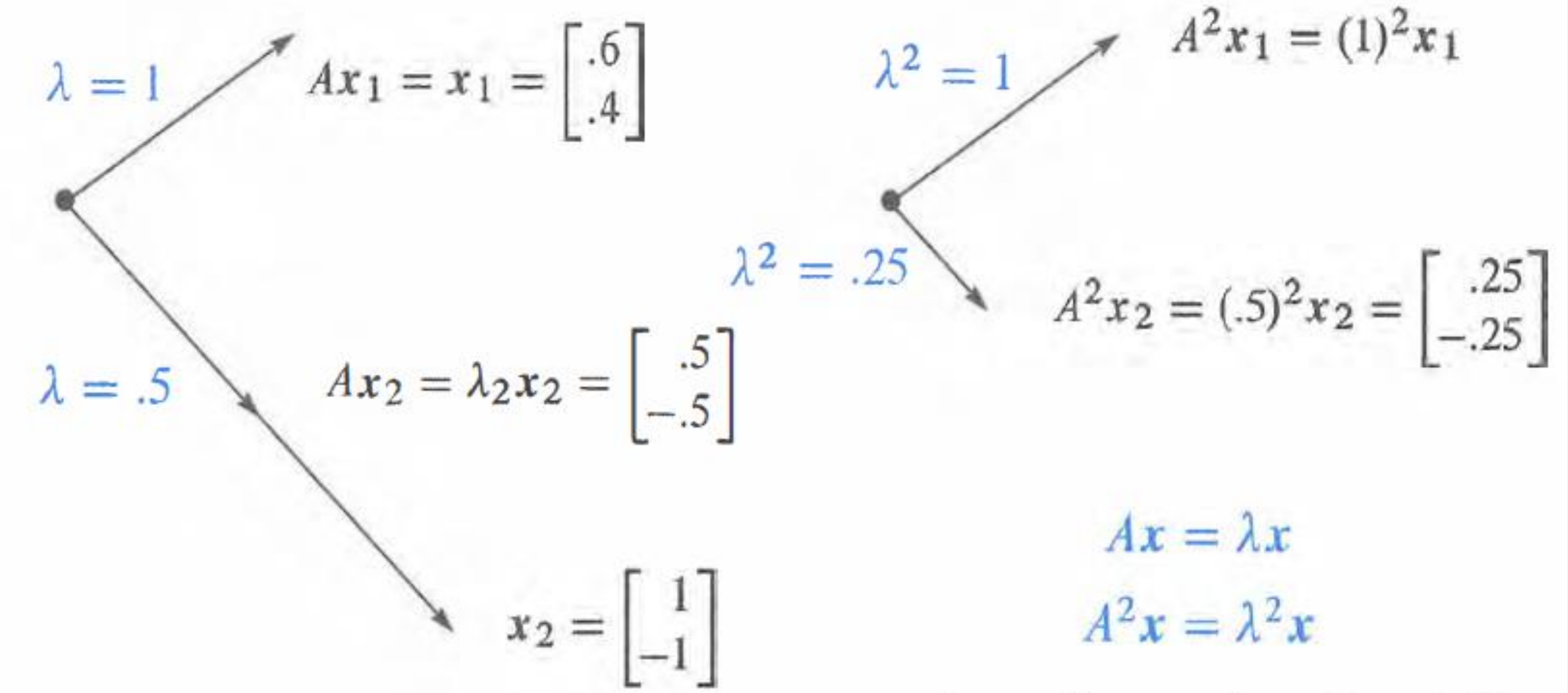

Figure 6.1: The eigenvectors keep their directions. $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$ with $\lambda^2 = 1^2$ and $(.5)^2$.

Use of Eigenvalues to find A^n

Assume a square matrix A as $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ with $\lambda = 1 \text{ and } \frac{1}{2}$

We may find A^2, A^3, ….. as $A^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{bmatrix}$

How to find A^{100}?

It can be found directly using the Eigenvalues of A.

$$A^{100} = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

FACT: If A is a square matrix having λ as its Eigenvalue and n ≥= 0, then λ^n is an Eigenvalue of A^n.

Use of Eigenvalues to find A^n

For $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$, we have $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$, $x_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Note that 1st Column of A is a combination of the two Eigenvectors.

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + (0.2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \mathbf{0.8} \\ \mathbf{0.2} \end{bmatrix} = x_1 + (0.2)x_2$$

Now if we multiply A with 1st column of A to find A^2, it gives $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$, it gives $\begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ (1st column of A^2), which is the same as

$$A\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \left(\frac{1}{2}\right)(0.2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \mathbf{0.7} \\ \mathbf{0.3} \end{bmatrix} = x_1 + \lambda_2(0.2)x_2$$

Now to find the 1st column of A^{100}, we have

$$A^{99}\begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \left(\frac{1}{2}\right)^{99}(0.2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \mathbf{0.7} \\ \mathbf{0.3} \end{bmatrix} = x_1 + \lambda_2^{99}(0.2)x_2$$

Example

The eigenvector x_1 is a "steady state" that doesn't change (because $\lambda_1 = 1$). The eigenvector x_2 is a "decaying mode" that virtually disappears (because $\lambda_2 = .5$). The higher the power of A, the more closely its columns approach the steady state.

This particular A is a ***Markov matrix***. Its largest eigenvalue is $\lambda = 1$. Its eigenvector $x_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach.

Example

Find the eigenvalues and eigenvectors of this symmetric 3 by 3 matrix S:

Symmetric matrix
Singular matrix
Trace $1+2+1=4$

$$S = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of S add to zero, the vector $\boldsymbol{x} = (1,1,1)$ gives $S\boldsymbol{x} = \mathbf{0}$. This is an eigenvector for $\lambda = 0$. To find λ_2 and λ_3 I will compute the 3 by 3 determinant:

$$\det(S-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= (\mathbf{1}-\boldsymbol{\lambda})(-\boldsymbol{\lambda})(\mathbf{3}-\boldsymbol{\lambda}). \end{aligned}$$

Those three factors give $\lambda = 0, 1, 3$. Each eigenvalue corresponds to an eigenvector (or a line of eigenvectors):

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_1 = \mathbf{0}\boldsymbol{x}_1 \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad S\boldsymbol{x}_2 = \mathbf{1}\boldsymbol{x}_2 \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_3 = \mathbf{3}\boldsymbol{x}_3 .$$

I notice again that eigenvectors are perpendicular when S is symmetric. We were lucky to find $\lambda = 0, 1, 3$. For a larger matrix I would use **eig**(A), and never touch determinants.

Diagonalization

- A square matrix A is said to be diagonalizable if it can be expressed in the form A = $X^{-1}\Lambda X$, where X is a matrix whose columns are eigenvectors of A, and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

$$\mathrm{AX} = \mathrm{A}[x_1 \;\; x_2 \dots . \quad xn] = [\lambda x_1 \quad \lambda x_2 \quad \text{------} \quad \lambda xn] = [x_1 \quad x_2 \quad \text{------} \quad x_n]\begin{bmatrix} \lambda_1\, 0 \;\; 0 \;\; \dots \;\; 0 \\ 0 \;\lambda_2 0 \;\; \dots \quad\; 0 \\ \dots \\ 0\; 0\; 0 \;\; \dots . \;\; \lambda_n \end{bmatrix} = X\Lambda$$

- Since $\mathrm{AX} = X\Lambda$,

  $\mathrm{A} = \mathrm{X}\,\Lambda\,\mathrm{X}^{-1}$ (if X is invertible).

Diagonalizing a Matrix

1 The columns of $AX = X\Lambda$ are $A\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k$. The eigenvalue matrix Λ is diagonal.

2 $\boldsymbol{n}$ independent eigenvectors in X diagonalize A $\boxed{\boldsymbol{A = X\Lambda X^{-1}} \text{ and } \boldsymbol{\Lambda = X^{-1}AX}}$

3 The eigenvector matrix X also diagonalizes all powers A^k: $\boxed{\boldsymbol{A^k = X\Lambda^k X^{-1}}}$

4 Solve $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ by $\boldsymbol{u}_k = A^k \boldsymbol{u}_0 = X\Lambda^k X^{-1}\boldsymbol{u}_0 = \boxed{\boldsymbol{c_1(\lambda_1)^k x_1 + \cdots + c_n(\lambda_n)^k x_n}}$

5 **No equal eigenvalues** $\Rightarrow$ X is invertible and A can be diagonalized.

Equal eigenvalues $\Rightarrow$ A *might* have too few independent eigenvectors. Then X^{-1} fails.

6 Every matrix $C = B^{-1}AB$ has the **same eigenvalues** as A. These C's are "**similar**" to A.

Diagonalizing a Matrix

Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Put them into the columns of an ***eigenvector matrix*** $\boldsymbol{X}$. Then $X^{-1}AX$ is the ***eigenvalue matrix*** Λ:

Eigenvector matrix X
Eigenvalue matrix Λ

$$X^{-1}AX = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \tag{1}$$

The matrix A is "diagonalized." We use capital lambda for the eigenvalue matrix, because the small λ's (the eigenvalues) are on its diagonal.

Diagonalizing a Matrix

Example 1 This A is triangular so its eigenvalues are on the diagonal: $\lambda = 1$ and $\lambda = 6$.

Eigenvectors go into X $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underset{X^{-1}}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} \mathbf{1} & \mathbf{5} \\ 0 & \mathbf{6} \end{bmatrix}} \underset{X}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{=}{=} \underset{\Lambda}{\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{6} \end{bmatrix}}$$

In other words $A = X\Lambda X^{-1}$. Then watch $A^2 = X\Lambda X^{-1} X\Lambda X^{-1}$. So $\boldsymbol{A^2}$ **is** $\boldsymbol{X\Lambda^2 X^{-1}}$.

A^2 has the same eigenvectors in X and squared eigenvalues in Λ^2.

Diagonalizing a Matrix

Why is $AX = X\Lambda$? A multiplies its eigenvectors, which are the columns of X. The first column of AX is $A\boldsymbol{x}_1$. That is $\lambda_1\boldsymbol{x}_1$. Each column of X is multiplied by its eigenvalue:

A times X
$$AX = A\begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix}.$$

The trick is to split this matrix AX into X times Λ:

X times Λ
$$\begin{bmatrix} \lambda_1\boldsymbol{x}_1 & \cdots & \lambda_n\boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = X\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column $\boldsymbol{x}_1$, as shown. The diagonalization is complete, and we can write $AX = X\Lambda$ in two good ways:

$$AX = X\Lambda \quad \text{is} \quad \boxed{X^{-1}AX = \Lambda} \quad \text{or} \quad \boxed{A = X\Lambda X^{-1}.} \tag{2}$$

$$A^k = (X\Lambda X^{-1})(X\Lambda X^{-1})\ldots(X\Lambda X^{-1}) = X\Lambda^k X^{-1}$$

Remarks

Remark 1 Suppose the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $x_1, \ldots, x_n$ are independent. The eigenvector matrix X will be *invertible*. ***Any matrix that has no repeated eigenvalues can be diagonalized***.

Remark 2 ***We can multiply eigenvectors by any nonzero constants***. $A(cx) = \lambda(cx)$ is still true. In Example 1, we can divide $x = (1, 1)$ by $\sqrt{2}$ to produce a unit vector.

MATLAB and virtually all other codes produce eigenvectors of length $||x|| = 1$.

Remark 3 The eigenvectors in X come in the same order as the eigenvalues in Λ. To reverse the order in Λ, put the eigenvector $(1, 1)$ before $(1, 0)$ in X:

New order $6, 1$ $$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda_{\textbf{new}}$$

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices cannot be diagonalized*. Here are two examples:

Not diagonalizable $$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Remarks

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Example 2 **Powers of A** The Markov matrix $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ in the last section had $\lambda_1 = 1$ and $\lambda_2 = .5$. Here is $A = X\Lambda X^{-1}$ with those eigenvalues in the diagonal Λ:

$$\textbf{Markov example} \qquad \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = X\Lambda X^{-1}.$$

The eigenvectors $(.6, .4)$ and $(1, -1)$ are in the columns of X. They are also the eigenvectors of A^2. Watch how A^2 has the same X, and ***the eigenvalue matrix of*** $\boldsymbol{A^2}$ ***is*** $\boldsymbol{\Lambda^2}$:

$$\textbf{Same } \boldsymbol{X} \textbf{ for } \boldsymbol{A^2} \qquad \boxed{A^2} = X\Lambda X^{-1} X\Lambda X^{-1} = \boxed{X\Lambda^2 X^{-1}}. \tag{4}$$

Just keep going, and you see why the high powers A^k approach a "steady state":

$$\textbf{Powers of } \boldsymbol{A} \qquad A^k = X\Lambda^k X^{-1} = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix}.$$

As k gets larger, $(.5)^k$ gets smaller. In the limit it disappears completely. That limit is A^∞:

$$\textbf{Limit } \boldsymbol{k \to \infty} \qquad A^\infty = \begin{bmatrix} .6 & 1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ .4 & -.6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Question *When does* $A^k \to$ *zero matrix*? **Answer** *All* $|\lambda| < 1$.

Suppose the eigenvalue matrix Λ is fixed. As we change the eigenvector matrix X, we get a whole family of different matrices $A = X\Lambda X^{-1}$—*all with the same eigenvalues in* Λ. All those matrices A (with the same Λ) are called **similar**.

All the matrices $A = BCB^{-1}$ are "similar." They all share the eigenvalues of C.

Fibonacci Numbers

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ***comes from*** $F_{k+2} = F_{k+1} + F_k.$

Let $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$. The rule $\begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix}$ is $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k.$

Every step multiplies by $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. After 100 steps we reach $\boldsymbol{u}_{100} = A^{100}\boldsymbol{u}_0$:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \ldots, \quad \boldsymbol{u}_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$ leads to $\det(A - \lambda I) = \lambda^2 - \lambda - 1$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \approx -.618$$

Fibonacci Numbers

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \Lambda^{100} = \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \lambda_2^{100} \end{bmatrix}$$

$$u_{100} = Au_{99} = \mathrm{A}(\mathrm{A}u_{98}) = \mathrm{A}^{100}u_o$$

$$\mathrm{A}^{100} = \mathrm{X}\,\Lambda^{100}\,X^{-1}$$

$$\mathrm{A}^{100} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \approx 0 \end{bmatrix} \left(\frac{1}{\lambda_1 - \lambda_2}\right) \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} = \left(\frac{1}{\lambda_1 - \lambda_2}\right) \begin{bmatrix} \lambda_1^{101} & -\lambda_1^{101}\lambda_2 \\ \lambda_1^{100} & -\lambda_1^{100}\lambda_2 \end{bmatrix}$$

Fibonacci Numbers

$$u_{100} = A^{100} u_o$$

but $u_{100} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$

So $\begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix} = \left(\frac{1}{\lambda_1 - \lambda_2}\right) \begin{bmatrix} \lambda_1^{101} & -\lambda_1^{101}\lambda_2 \\ \lambda_1^{100} & -\lambda_1^{100}\lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix} = \left(\frac{1}{\lambda_1 - \lambda_2}\right) \begin{bmatrix} \lambda_1^{101} \\ \lambda_1^{100} \end{bmatrix}$$

$$\boldsymbol{F_{100} = \left(\frac{1}{\lambda_1 - \lambda_2}\right) \lambda_1^{100} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{100}}$$

Fibonacci Numbers

$$u_{100} = \frac{(\lambda_1)^{100} x_1 - (\lambda_2)^{100} x_2}{\lambda_1 - \lambda_2}$$

We want F_{100} = second component of u_{100}. The second components of x_1 and x_2 are 1. The difference between $\lambda_1 = (1+\sqrt{5})/2$ and $\lambda_2 = (1-\sqrt{5})/2$ is $\sqrt{5}$. And $\lambda_2^{100} \approx 0$.

$$\text{100th Fibonacci number} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{100}. \quad (10)$$

Every F_k is a whole number. The ratio F_{101}/F_{100} must be very close to the limiting ratio $(1+\sqrt{5})/2$. The Greeks called this number the "*golden mean*". For some reason a rectangle with sides 1.618 and 1 looks especially graceful.