Linear Algebra

(MT-121T)

Aftab Alam

Lecture # 5

(Thursday, February 22, 2024)

Steps to Follow in Elimination

1. A linear system ($A\boldsymbol{x} = \boldsymbol{b}$) becomes **upper triangular** ($U\boldsymbol{x} = \boldsymbol{c}$) after elimination.
2. We **subtract** ℓ_{ij} times equation j from equation i, to make the (i, j) entry zero.
3. The **multiplier** is $\ell_{ij} = \dfrac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. **Pivots** can not be zero!
4. When zero is in the pivot position, **exchange rows** if there is a nonzero below it.
5. The upper triangular $U\boldsymbol{x} = \boldsymbol{c}$ is solved by **back substitution** (starting at the bottom).

When **breakdown** is permanent, $A\boldsymbol{x} = \boldsymbol{b}$ has no solution or infinitely many.

Steps of Elimination

1 The first step multiplies the equations $A\boldsymbol{x} = \boldsymbol{b}$ by a matrix E_{21} to produce $E_{21}A\boldsymbol{x} = E_{21}\boldsymbol{b}$.

2 That matrix $E_{21}A$ has a zero in row 2, column 1 because x_1 is eliminated from equation 2.

3 E_{21} is the **identity matrix** (diagonal of 1's) minus the multiplier a_{21}/a_{11} in row 2, column 1.

4 Matrix-matrix multiplication is n matrix-vector multiplications: $\boldsymbol{EA} = [\boldsymbol{Ea_1} \ \ldots \ \boldsymbol{Ea_n}]$.

5 We must also multiply $E\boldsymbol{b}$! So E is multiplying the **augmented matrix** $[A\,\boldsymbol{b}] = [\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_n\,\boldsymbol{b}]$.

6 Elimination multiplies $A\boldsymbol{x} = \boldsymbol{b}$ by $E_{21}, E_{31}, \ldots, E_{n1}$, then $E_{32}, E_{42}, \ldots, E_{n2}$, and onward.

7 The **row exchange matrix** is not E_{ij} but P_{ij}. To find P_{ij}, exchange rows i and j of I.

Problem 1

a) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ *gives* $\begin{bmatrix} y \\ x \end{bmatrix}$.

b) What 2 by 2 matrix R rotates every vector by 90°? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ *gives* $\begin{bmatrix} y \\ -x \end{bmatrix}$.

c) What 2 by 2 matrix R^2 rotates every vector by 180°? R^2 times $\begin{bmatrix} x \\ y \end{bmatrix}$ *gives* $\begin{bmatrix} -x \\ -y \end{bmatrix}$.

Problem 2

a) What 2 by 2 matrix E subtracts the first component from the second component?

$$E\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

b) What 3 by 3 matrix E subtracts the first component from the second component?

$$E\begin{bmatrix}3\\5\\7\end{bmatrix} = \begin{bmatrix}3\\2\\7\end{bmatrix}$$

Problem 3

a) What 3 by 3 matrix E multiples (x, y, z) to give $(x, y, z + x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z - x)$? If you multiply $(3, 4, 5)$ by E then multiply by E^{-1}, the two results are (___________) and (____________).

b) What 2 by 2 matrix P_1 projects by the vector (x, y) onto the axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5,7)$ by P_1 and then multiply by P_2, you get (___________) and (____________).

Elimination Using Matrices

$$2x + 4y - 2z = 2$$
$$4x + 9y - 3z = 8$$
$$-2x - 3y + 7z = 10$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Augmented Matrix => [A b] = $\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$

$$R_2 - 2R_1 = E_{21}\,[A \mid b]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$R_3 + R_1 = E_{31}(E_{21}\,[A \mid b])$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

$$R_3 - R_2 = E_{32}(E_{31}(E_{21}\,[A \mid b]))$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Back-substitution:

$$z = 2$$

$$y + z = 4 \ \rightarrow y = 2$$

$$2x + 4y - 2z = 2 \ \rightarrow x = 1 - 2y + z$$
$$x = 1 \ - 2(2) + 2$$
$$x = -1$$

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (-\mathbf{1}, \mathbf{2}, \mathbf{2})$$

Follow Up Question 1

a) Which three matrices $E_{21}, E_{31}\ and\ E_{32}$ put A into triangular form U?

Multiply those E's to get on matrix M that does elimination: $MA = U$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

b) Include $b = (1,0,0)$ as fourth column to produce [A b]. Carry out the

elimination steps on this augmented matrix to solve $Ax = b$.

Follow Up Question 2

This 4 x 4 matrix will need elimination matrices E_{21} and E_{32} and E_{31}. What are those matrices?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Follow Up Question 3

a) E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix M = $P_{23}E_{21}$ does both steps at once?

b) P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 and row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M's are the same but the E's are different.

Follow Up Question 4

a) What 3 by 3 matrix E_{13} will add row 3 to row 1?

b) What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1?

c) What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

Quiz 2

Write down the augmented matrix [A b] with an extra column:

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

Apply E_{21} and then P_{32} to reach a triangular system. Solve by back substitution. What combined matrix $P_{32}E_{21}$ will do both steps at once?