LINEAR ALGEBRA (MT–121)

CHAPTER 6

EIGENVALUES AND EIGENVECTORS

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Lecture 21

Eigenvalues and Eigenvectors

- **Eigenvalues** are a fundamental concept in linear algebra with wide-ranging applications in various fields, including
  - Physics
  - Engineering
  - Computer science
  - Economics.
- **Eigenvectors** are the vectors (non-zero) that do not change the direction when any linear transformation is applied.
  - It changes by only a scalar factor.
  - If A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if Ax is a scalar multiple of x. That scalar value is called the **eigenvalue**.

Major Applications

- **Principal Component Analysis (PCA):**
  - Eigenvalues and eigenvectors are used to reduce the dimensionality of data by transforming it into a new coordinate system where the axes correspond to the directions of maximum variance.
- **Differential Equations:**
  - Eigenvalues and eigenvectors play a crucial role in solving systems of ordinary and partial differential equations.
- **Structural Engineering:**
  - They are used in analyzing the stability and natural frequencies of structures like bridges and buildings.
- **Quantum Mechanics:**
  - Eigenvalues represent the possible measurable values (eigenvalues) of physical observables in quantum systems.

Eigenvalues and Eigenvectors

- Given a square matrix **A**, an eigenvalue **λ** and its corresponding eigenvector **x** satisfy the equation:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

- An eigenvector **x** is a non-zero vector that remains in the same direction (up to a scalar multiple) after multiplication by the matrix **A**.
- The corresponding eigenvalue **λ** represents the factor by which the eigenvector is scaled during this transformation.
- There could be infinitely many Eigenvectors corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.

Eigenvalues of a Square Matrix

- Suppose A is an **n x n** square matrix, then **[A – λI]** is called an Eigen or characteristic matrix
- To find eigenvalues and eigenvectors of a matrix A, we solve the Eigen (or characteristic) equation:

$$\det(A - \lambda I) = 0$$

  - Where I is the identity matrix of the same size as A.
  - The solutions to this equation are the eigenvalues λ (eigen roots).
- The sum of the eigenvalues of a matrix equals the sum of its diagonal entries (the **trace** of the matrix).
- The product of the eigenvalues of a matrix equals the **determinant** of the matrix.

Introduction to Eigenvalues

1 An **eigenvector** x lies along the same line as Ax : $\boxed{Ax = \lambda x.}$ The **eigenvalue** is λ.

2 If $Ax = \lambda x$ then $A^2 x = \lambda^2 x$ and $A^{-1} x = \lambda^{-1} x$ and $(A + cI)x = (\lambda + c)x$: the same x.

3 If $Ax = \lambda x$ then $(A - \lambda I)x = \mathbf{0}$ and $A - \lambda I$ is singular and $\boxed{\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = 0.}$ n eigenvalues.

4 Check λ's by $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$ and diagonal sum $a_{11} + a_{22} + \cdots + a_{nn} =$ sum of λ's.

5 Projections have $\lambda = 1$ and 0. Reflections have 1 and -1. Rotations have $e^{i\theta}$ and $e^{-i\theta}$: *complex*!

Finding Eigenvalues and Eigenvectors

Given a 2 x 2 matrix $A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$, find the Eigenvalues and Eigenvectors.

We know that $A - \lambda I$ is singular; so $|A - \lambda I| = 0$

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \text{ so } \begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

Now from the characteristic equation $\lambda^2 + \lambda + 42 = 0$, we find the Eigenvalues as $\lambda_1 = 6, \lambda_2 = -7$

Note that $\lambda_1 + \lambda_2 =$ sum of diagonal of A, and $\lambda_1 \lambda_2 = |A|$

Now since A $x = \lambda\, x$, $\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ ➔ $-6x + 3y = \lambda x$ and $4x + 5y = \lambda y$

When $\lambda = 6$, $x = \begin{bmatrix} \frac{1}{4} \\ 1 \end{bmatrix}$ and when $\lambda = -7$, $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Finding Eigenvalues and Eigenvectors

Given a 3 x 3 matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$, find Evalues and Evectors.

We know that $A - \lambda I$ is singular; so $|A - \lambda I| = 0$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix} = 0$$

So from the characteristic equation $\lambda^3 - 24\lambda^2 + 65\lambda - 42 = 0$, we find the Eigenvalues as $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 21$

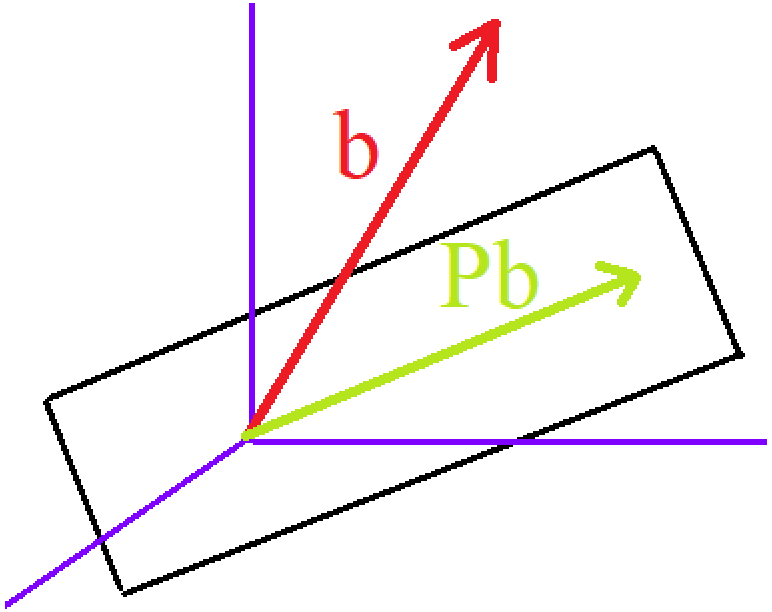
Now we can easily find Eigenvectors as $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/6 \\ 1/6 \\ 1 \end{bmatrix}$

Special Case 1: Singular Matrix

- If **A** matrix is singular, then at least one Eigenvalue $\boldsymbol{\lambda = 0}$.
- In this case, $Ax = 0$.
- the corresponding eigenvectors are the vectors that lie in the null space (also known as **kernel**) of the matrix.
  - In other words, the eigenvectors associated with $\lambda = 0$ are the vectors that, when multiplied by the matrix, result in the zero vector.
  - These eigenvectors represent directions in which the linear transformation represented by the matrix A collapses to zero.

Special Case 2: Projection Matrix

- What are the Eigenvalues for a projection matrix?
- Is **b** an Eigenvector?
  - No, because **Pb** is in a different direction.
- What vectors are Eigenvectors of P?
  - Vectors in the same direction.
  - Any x in the plane.
  - Px = x
    - $\lambda = 1$
    - We have a complete plane of Eigenvalues.
  - Px = 0
    - $\lambda = 0$



Special Case 3: Permutation Matrix

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

So from the characteristic equation $\lambda^2 - 1 = 0,$ we find the Eigenvalues as $\lambda_1 = 1, \lambda_2 = -1$

So the Eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Perpendicular to each other.

Adding a scalar multiple of I to A

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = A + 3I$$

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

So from the characteristic equation $\lambda^2 - 6\lambda + 8 = 0$, we find the Eigenvalues as $\lambda_1 = 2, \lambda_2 = 4$

So the Eigenvectors for λ =2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and for λ=4 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Same as in the previous example.

Since $\mathrm{B} = \mathrm{A} + 3\mathrm{I}$, Eigenvalues are 3 more than previous example.

Special Case 4: Antisymmetric Matrix

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector by 90º.

$|A - \lambda I| = 0$

$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

So from the characteristic equation $\lambda^2 + 1 = 0$, we find the Eigenvalues as $\lambda_1 = i, \lambda_2 = -i$ **(Complex conjugates)**

So the Eigenvectors are $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

<u>Check</u>: $\text{Sum of Eigenvalues} = \lambda_1 + \lambda_2 = i + (-i) = 0$ (Sum of Diagonal of A)

$\text{Sum of Eigenvalues} = \lambda_1 \lambda_2 = (i)(-i) = 1$ (determinant of A)

Special Case 5: Triangular Matrix

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ (Upper Triangular Matrix)

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

So from the characteristic equation $\lambda^2 - 6\lambda + 9 = 0$, we find the Eigenvalues as $\lambda_1 = 3, \lambda_2 = 3$ (Repeated Eigenvalues)

So the Eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

No second independent Eigenvector.
Eigenvectors with distinct Eigenvalues are Linearly independent, but Eigenvectors with same Eigenvalues are not Linearly independent.

Special Case 6: Orthogonal Matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

So from the characteristic equation $\lambda^2 - 1 = 0$, we find the Eigenvalues as $\lambda_1 = 1, \lambda_2 = -1$

Eigenvalues of an Orthogonal Matrix is $|\lambda| = 1$

Special Case 7: Inverse Matrix

If $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, then $\lambda = 2, 4$

Now $A^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{3}{8} \end{bmatrix}$

and $\lambda = \frac{1}{2}, \frac{1}{4}$

If A is a square matrix having λ as its Eigenvalue, then $\lambda^{-1} = \frac{1}{\lambda}$ is an Eigenvalue of A^{-1}.

Special Case 8: Transpose Matrix

If A is a square matrix having λ as its Eigenvalue, then λ is an Eigenvalue of A^T.

Special Case 9: For Matrix powers

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}, \; A^3 = \begin{bmatrix} 0.65 & 0.525 \\ 0.35 & 0.475 \end{bmatrix}$$

$\lambda\ (for\ A) = 1, \frac{1}{2}$ and $\lambda\ (for\ A^2) = 1, \frac{1}{4}$ and $\lambda\ (for\ A^3) = 1, \frac{1}{8}$

If A is a square matrix having λ as its Eigenvalue and n $\geq$= 0, then λ^n is an Eigenvalue of A^n.

Special Case 10: For polynomial of Matrix

If A is a square matrix having λ as its Eigenvalue and p(x) is a polynomial in variable x then λ^n is an Eigenvalue of matrix p(A).