Linear Algebra

(MT-121T)

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Lecture # 6

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$A = LU$

Forward from A ***to*** U **:** $\quad E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$

Back from U ***to*** A **:** $\quad E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$

The second line is our factorization $LU = A$. Instead of E_{21}^{-1} we write L.

$\boldsymbol{A = LU}$	***This is elimination without row exchanges***. The upper triangular U has the pivots on its diagonal. The lower triangular L has all 1's on its diagonal. ***The multipliers*** ℓ_{ij} ***are below the diagonal of L***.

Example 1

Example 1 Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. The last step subtracts $\frac{2}{3}$ times row 2 from row 3. The lower triangular L has $\ell_{21} = \frac{1}{2}$ and $\ell_{32} = \frac{2}{3}$. Multiplying LU produces A:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU.$$

The $(3, 1)$ multiplier is zero because the $(3, 1)$ entry in A is zero. No operation needed.

Elimination = Factorization: $A = LU$

$(\boldsymbol{E_{32}E_{31}E_{21}})\boldsymbol{A} = \boldsymbol{U}$ **becomes** $\boldsymbol{A} = (\boldsymbol{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}})\ \boldsymbol{U}$ **which is** $\boldsymbol{A} = \boldsymbol{LU}$

- The factors L and U are triangular matrices.
- The factorization that comes from elimination is $A = LU$.
- $A = LU$ is elimination without row exchanges.
- The upper triangle U has the pivots on its diagonal.
- The lower triangular L has all 1's on its diagonal.
- The multipliers (from the inverse of elimination matrix) are below the diagonal of L.

Elimination = Factorization: $A = LU$

1 Each elimination step E_{ij} is inverted by L_{ij}. Off the main diagonal change $-\ell_{ij}$ to $+\ell_{ij}$.

2 The whole forward elimination process (with no row exchanges) is inverted by $\boldsymbol{L}$:

$$\boldsymbol{L} = (L_{21}L_{31}\ldots L_{n1})(L_{32}\ldots L_{n2})(L_{43}\ldots L_{n3})\ldots(L_{n\,n-1}).$$

3 That product matrix L is still lower triangular. **Every multiplier ℓ_{ij} is in row i, column j.**

4 The original A is recovered from U by $\boldsymbol{A} = \boldsymbol{LU}$ = (lower triangular)(upper triangular).

5 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ reaches $U\boldsymbol{x} = \boldsymbol{c}$. Then back-substitution solves $U\boldsymbol{x} = \boldsymbol{c}$.

6 Solving a triangular system takes $n^2/2$ multiply-subtracts. Elimination to find U takes $n^3/3$.

Problem 1

What matrix E puts A into triangular form $EA = U$? Multiply by

E^{-1} $(which\ is\ equal\ to\ L)$ to factor A into LU:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Problem 2

What three elimination matrices

E_{21}, E_{31} *and* E_{32} put A into its upper triangular form

$\boldsymbol{E_{32}E_{31}E_{21}A = U}$? Multiply by $\boldsymbol{E_{32}^{-1}}, \boldsymbol{E_{31}^{-1}}$ and $\boldsymbol{E_{21}^{-1}}$ to

factor A into L times U.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Note that $(E_{32}E_{31}E_{21})^{-1}$ is not the same as $(E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})$

Example 3

Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x = b$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = c$:

$$\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \end{aligned} \quad \rightarrow \quad \begin{aligned} x + y &= 5 \\ y &= 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

The matrix for that step is L = ________.

Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get $Ux = c$.

$$x = (3, 2)$$

LDU Decomposition

- How to decompose a matrix A into LDU form?
- To split a matrix A into its LDU factorization, where
  - L is a lower triangular matrix,
  - D is a diagonal matrix, and
  - U is an upper triangular matrix, we typically perform the following steps:

1. Perform Gaussian Elimination with Partial Pivoting (LU Decomposition)
2. Extract D from U
3. Normalize the Diagonal of L

LDU Decomposition

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

LDU Decomposition

- Perform LDU decomposition on $B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}$

- $B = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Inverse Matrices

The matrix A is *invertible* if there exists a matrix A^{-1} that inverts A

$$AA^{-1} = A^{-1}A = I \text{ (two sided inverse)}$$

For Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Inverse of a Product:

If A and B are invertible, then so is AB. The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}$$

Example

If E subtracts 5 times row 1 from row 2, then E^{-1} adds 5 times row 1 to row 2.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose F subtracts 4 times row 2 from row 3, then F^{-1} adds it back.

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \qquad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \qquad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix}$$

$$E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

- In elimination order F follows E. In reverse order, E^{-1} follows F^{-1}.
- $E^{-1}F^{-1}$ is quick. The multipliers 5, 4 fall into place below the diagonal of 1’s