LINEAR ALGEBRA (MT–121)

CHAPTER 4

ORTHOGONALITY

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Lecture 16

FOUR FUNDAMENTALS SUBSPACES

1. The row space is $C(A^{\mathrm{T}})$ a subspace of R^{n}.
   Dimension r

2. The column space is $C(A)$ a subspace of R^{m}.
   Dimension r

3. The null space is $N(A)$ a subspace of R^{n}.
   Dimension n $- r$

4. The left null space is $N\ (A^{\mathrm{T}})$ a subspace of R^{m}.
   Dimension m $- r$

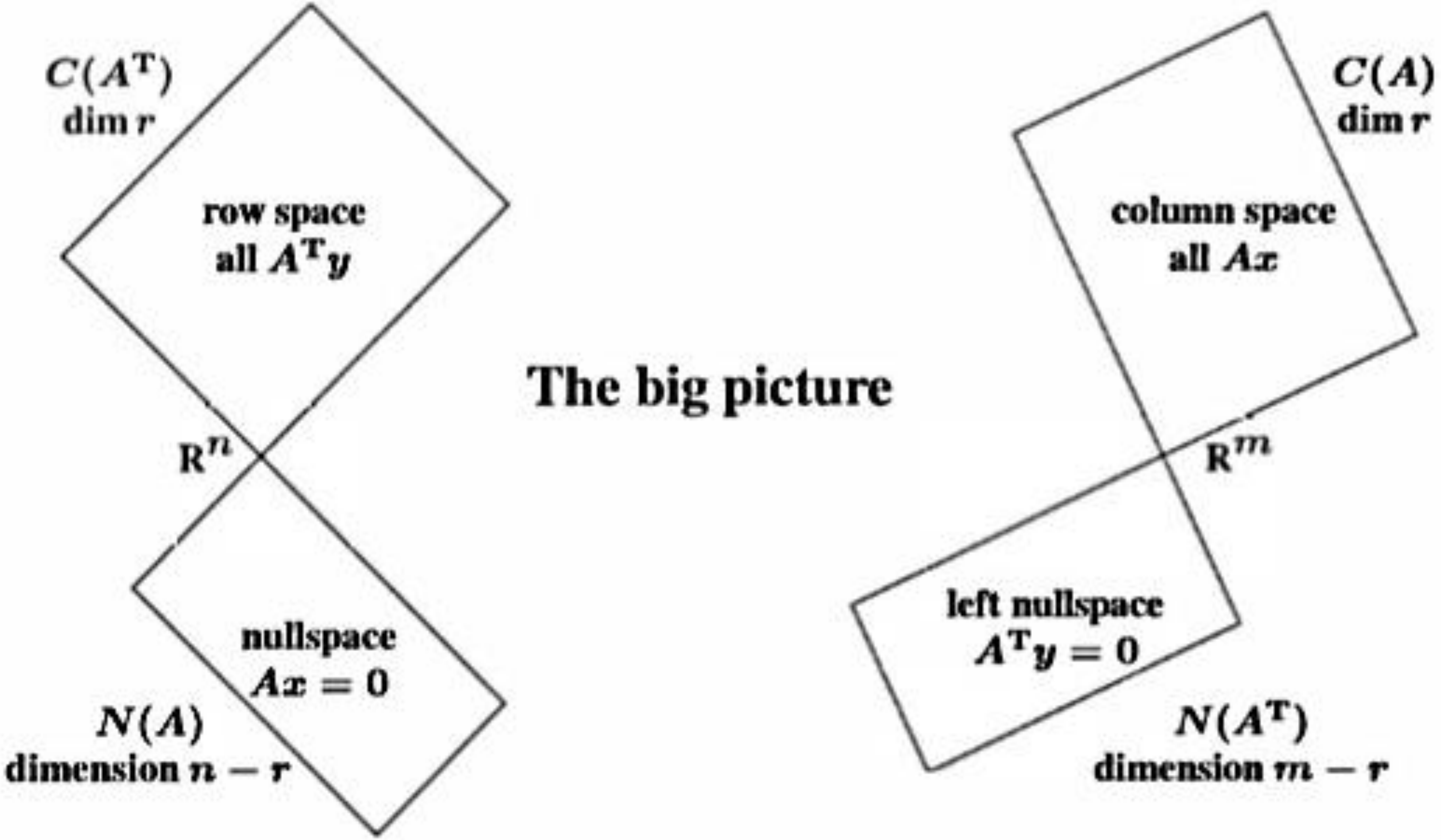


Figure 3.5: The dimensions of the Four Fundamental Subspaces (for R and for A).

THE FOUR SUBSPACES OF A

1. A has same row space as R. Same dimension r and same basis.
2. The column space of A has dimension r. The column rank equals the row rank.
3. A has the same null space as R. Same dimension $n - r$ and same basis.
4. (dimension of column space) + (dimension of null space) = dimension of R^n
5. The left null space of A (the null space of A^T) has dimension $m - r$

ORTHOGONALITY

- Orthogonality is a fundamental concept in linear algebra and finds applications in various areas:
  - vector spaces
  - inner product spaces
  - orthogonal bases
  - orthogonal projections
  - least squares approximation

ORTHOGONALITY

- Orthogonality refers to the concept of perpendicularity or independence between vectors.
- Two vectors are orthogonal if their dot product is zero, which geometrically means that they are at right angles to each other in an n-dimensional space.
  - Mathematically, given two vectors **u** and **v** in an n-dimensional real vector space $\mathbf{R}^n$, they are orthogonal if their dot product (also known as inner product or scalar product) is zero:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = 0$$

  - If **u** and **v** are orthogonal, it implies that they are linearly independent, meaning that one cannot be written as a scalar multiple of the other.
  - In geometric terms, if two vectors are orthogonal, they point in different directions and do not lie on the same line.

ORTHOGONAL VECTORS

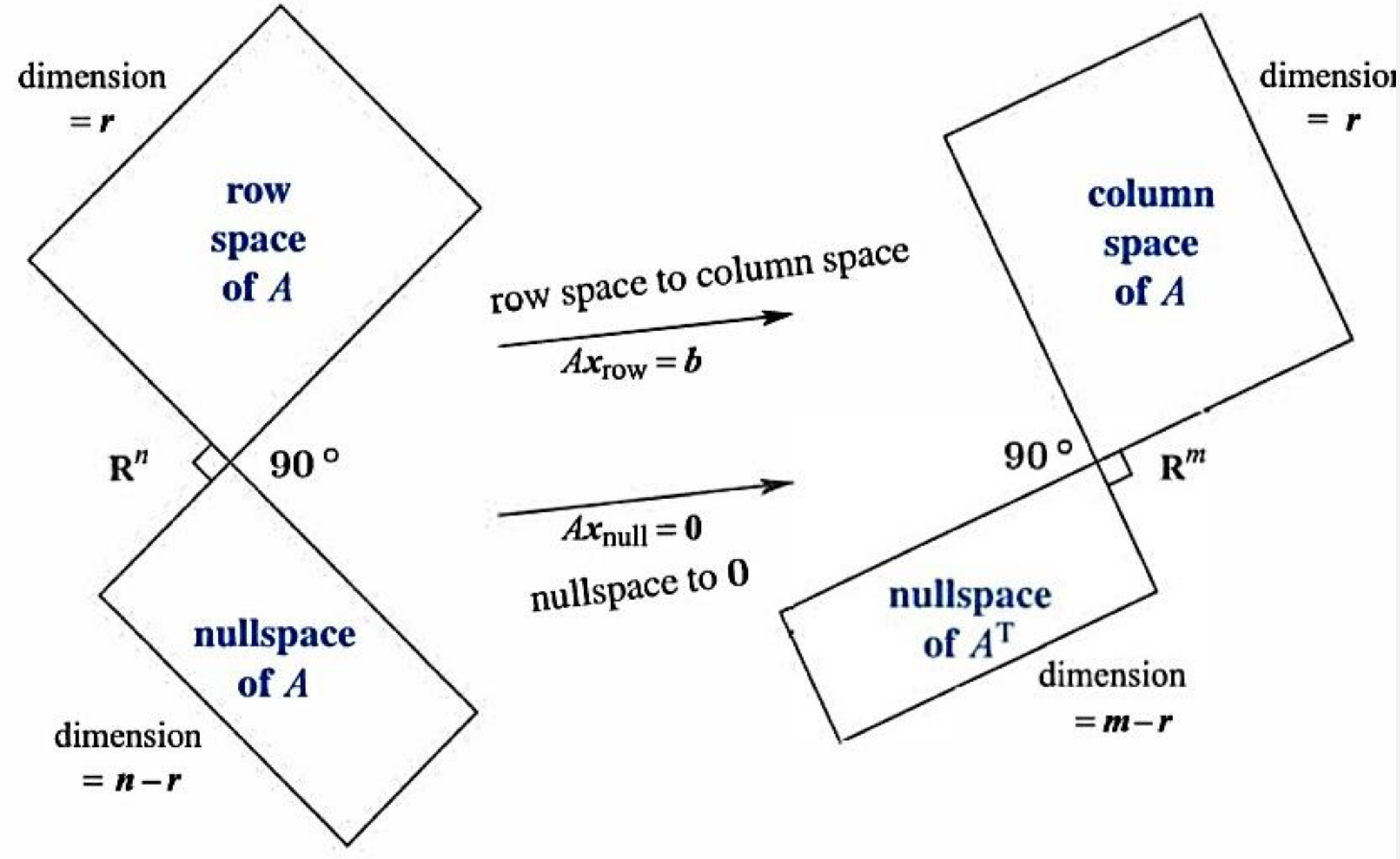
Orthogonal vectors $v^{\mathrm{T}}w = 0$ and $\|v\|^2 + \|w\|^2 = \|v + w\|^2.$

DEFINITION Two subspaces V and W of a vector space are ***orthogonal*** if every vector v in V is perpendicular to every vector w in W:

Orthogonal subspaces $v^{\mathrm{T}}w = 0$ *for all v in V and all w in W.*

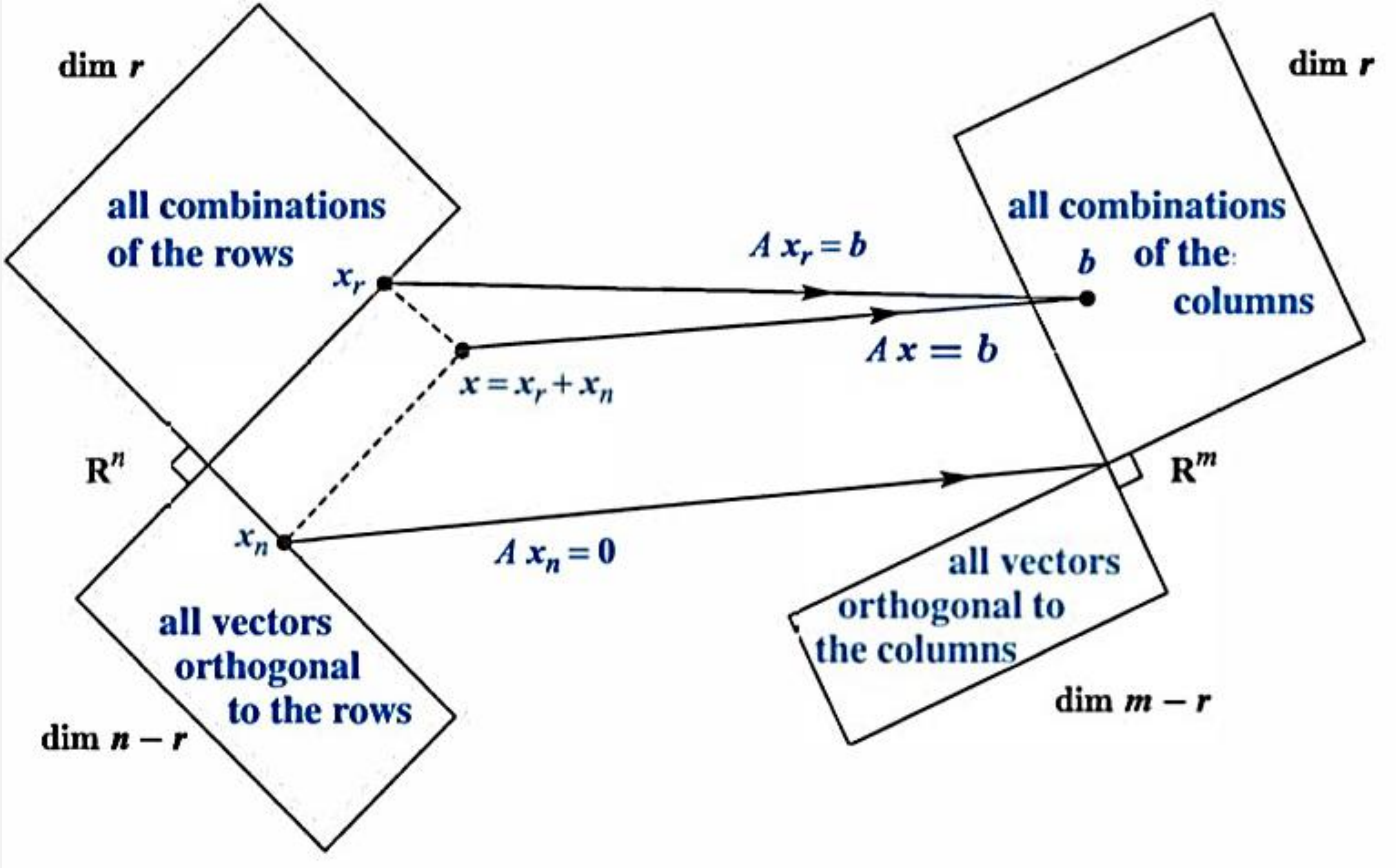
Every vector x in the nullspace is perpendicular to every row of A, because $Ax = \mathbf{0}$. ***The nullspace $N(A)$ and the row space $C(A^{\mathrm{T}})$ are orthogonal subspaces of*** $\mathbf{R}^n$.

TWO PAIRS OF ORTHOGONAL SUBSPACES



$N(A)$ is the orthogonal complement of the row space $C(A^{\mathrm{T}})$ (in $\mathbf{R}^n$).

$N(A^{\mathrm{T}})$ is the orthogonal complement of the column space $C(A)$ (in $\mathbf{R}^m$).



The true action of A on $x = x_p + x_n$. Row space vector x_r on column space, null space vector x_n to zero.

EXAMPLE

Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2(0) + 1(1) + (-1)(1) = 0$$
$$\mathbf{v}_2 \cdot \mathbf{v}_3 = 0(1) + 1(-1) + (1)(1) = 0$$
$$\mathbf{v}_1 \cdot \mathbf{v}_3 = 2(1) + 1(-1) + (-1)(1) = 0$$

So the vectors are mutually perpendicular

The vectors are orthogonal, and hence linearly independent. Since any three linearly independent vectors in R^3 form a basis for R^3 (by fundamental theorem of Invertible matrices, it follows that the given vectors is an orthogonal basis for R^3.

EXAMPLE

Construct an orthonormal basis for $\mathbb{R}^3$ from the vectors

Since we already know that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are an orthogonal basis, we normalize them to get

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$\mathbf{q}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{q}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Then $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

EXAMPLES: REQUIREMENTS FOR A BASIS

Q : Find a basis for each of these subspaces of R^4:

2. All vectors whose components add to zero.

Answer: To find a basis for the subspace of R^4 consisting of all vectors whose components add to zero, we can start by considering the vectors of the form

(x, y, z, w) where (x + y + z + w = 0).

One possible approach is to use three vectors as a basis, ensuring that they are linearly independent and span the subspace. We can choose vectors that are not collinear and sum to zero. Here's one way to do it:

1. Choose two linearly independent vectors, for example, (1, -1, 0, 0) and (0, 1, -1, 0)

2. The third vector can be obtained by taking the negative sum of the first two vectors, i.e., -(1, -1, 0, 0) - (0, 1, -1, 0) = (-1, 0, 1, 0)

We can verify that these vectors satisfy the required conditions: