Linear Algebra

(MT-121T)

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Lecture # 2

(Friday, February 02, 2024)

Linear System Of Equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Linear System in Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

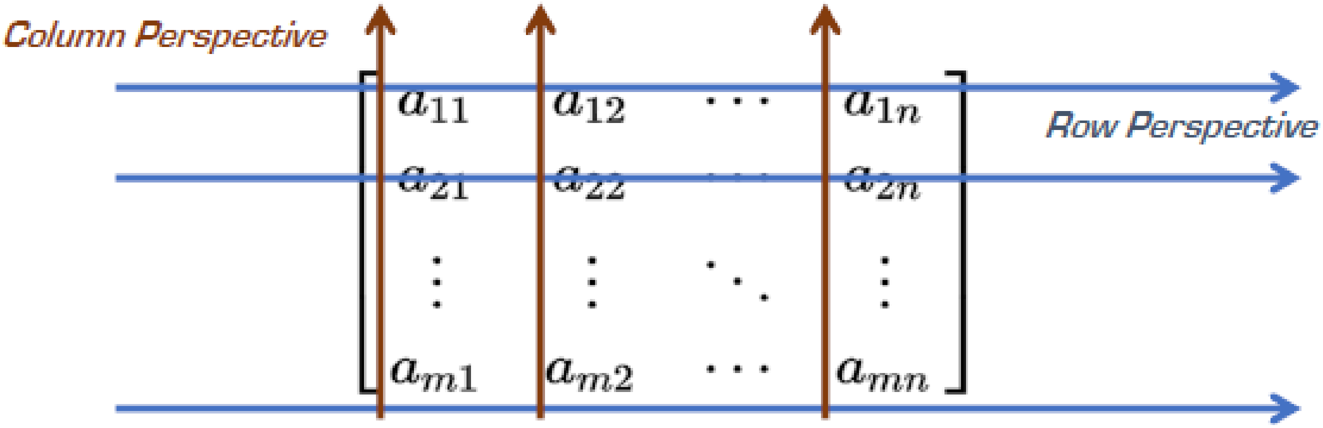
m equations, *n* unknowns

A solution of a linear system is an assignment of values to the variables $x_1, x_2, \ldots, x_n$ such that each of the equations is satisfied.

The set of all possible solutions is called the solution set.

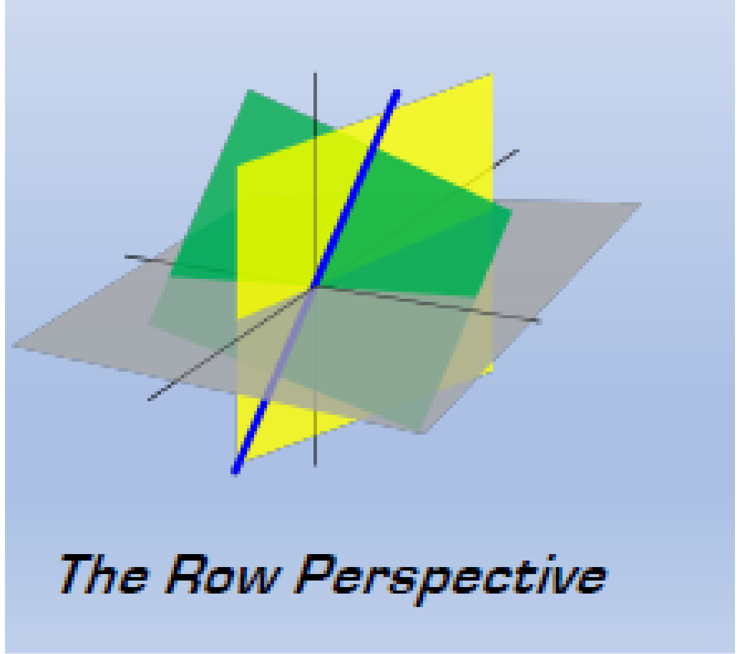
The Two Perspectives

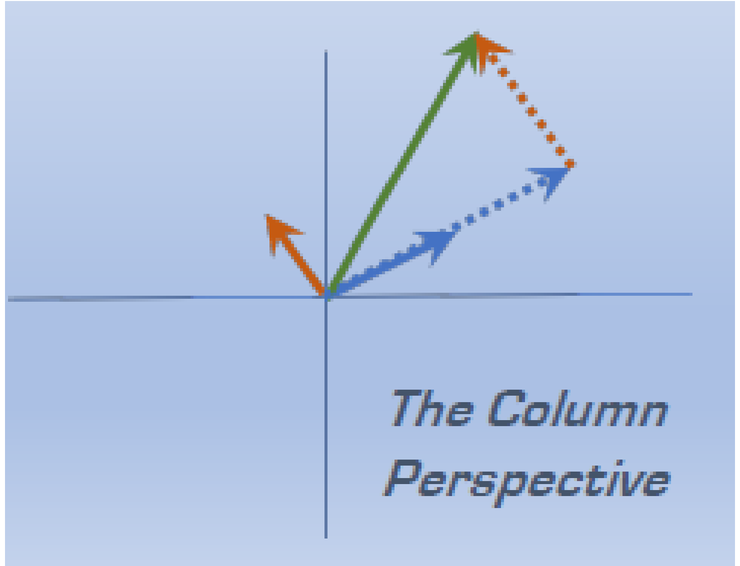
How to interpret matrices?



Geometry of linear equations:

The Two Perspectives





Taking the Row Perspective

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

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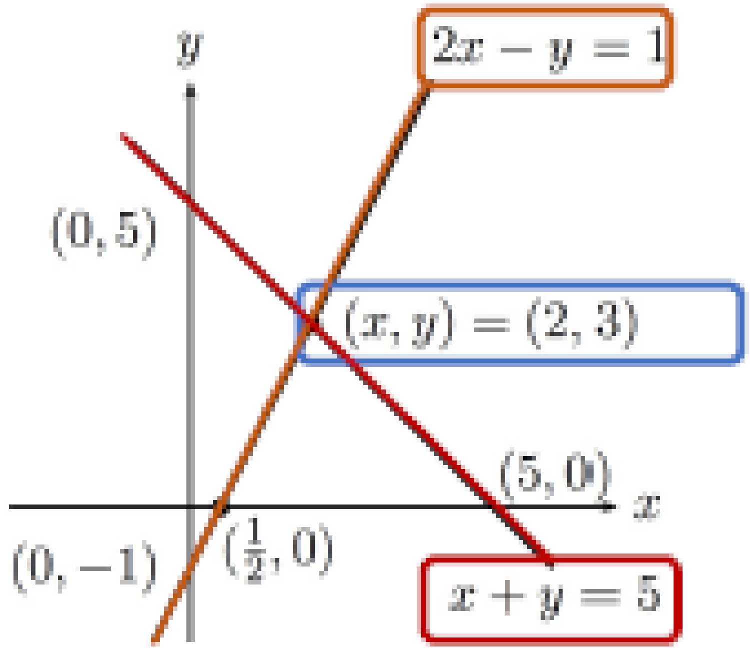


***Row picture*: Intersection of planes**

Plot particular equations and find intersection

Row elements contain coefficients of various variables

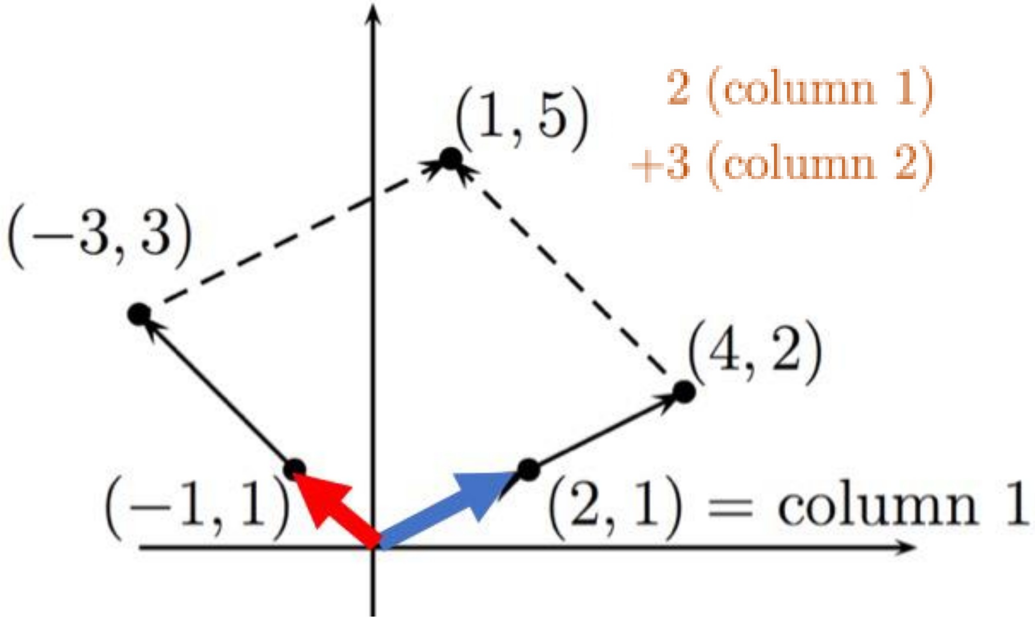
Taking the Column View

$$\begin{bmatrix} \boxed{\begin{matrix}2\\1\end{matrix}} & \boxed{\begin{matrix}-1\\1\end{matrix}} \end{bmatrix}\begin{bmatrix} x\\ y\end{bmatrix} = \boxed{\begin{bmatrix}1\\5\end{bmatrix}}$$

Take x time the first column

Take y time the second column

To obtain the RHS



Column picture: Combination of columns

The Column View

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
$$

We can therefore think of solving a system as mixing of the column vectors to produce the desired vector

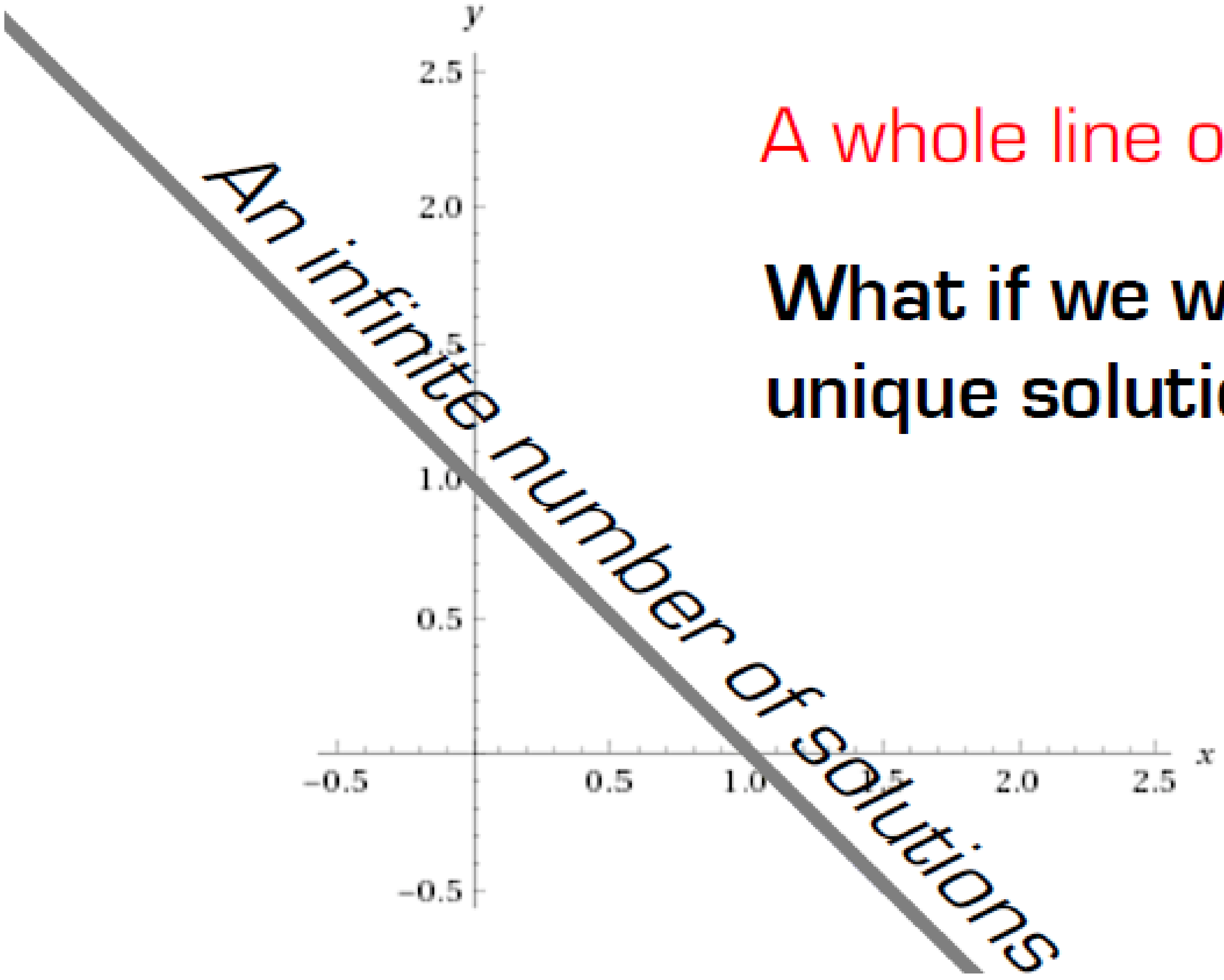
$$
x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

Solution set with m equations, n unknowns

In terms of the solution set, there are three possibilities:

1. The system has a single unique solution.
2. The system has no solution (typically $m>n$).
3. The system has infinitely many solutions (typically $m<n$).

How do we solve x + y = 1?



A whole line of solutions

What if we want a unique solution?

$y = 1 - x$

$x=0;\ y=1$

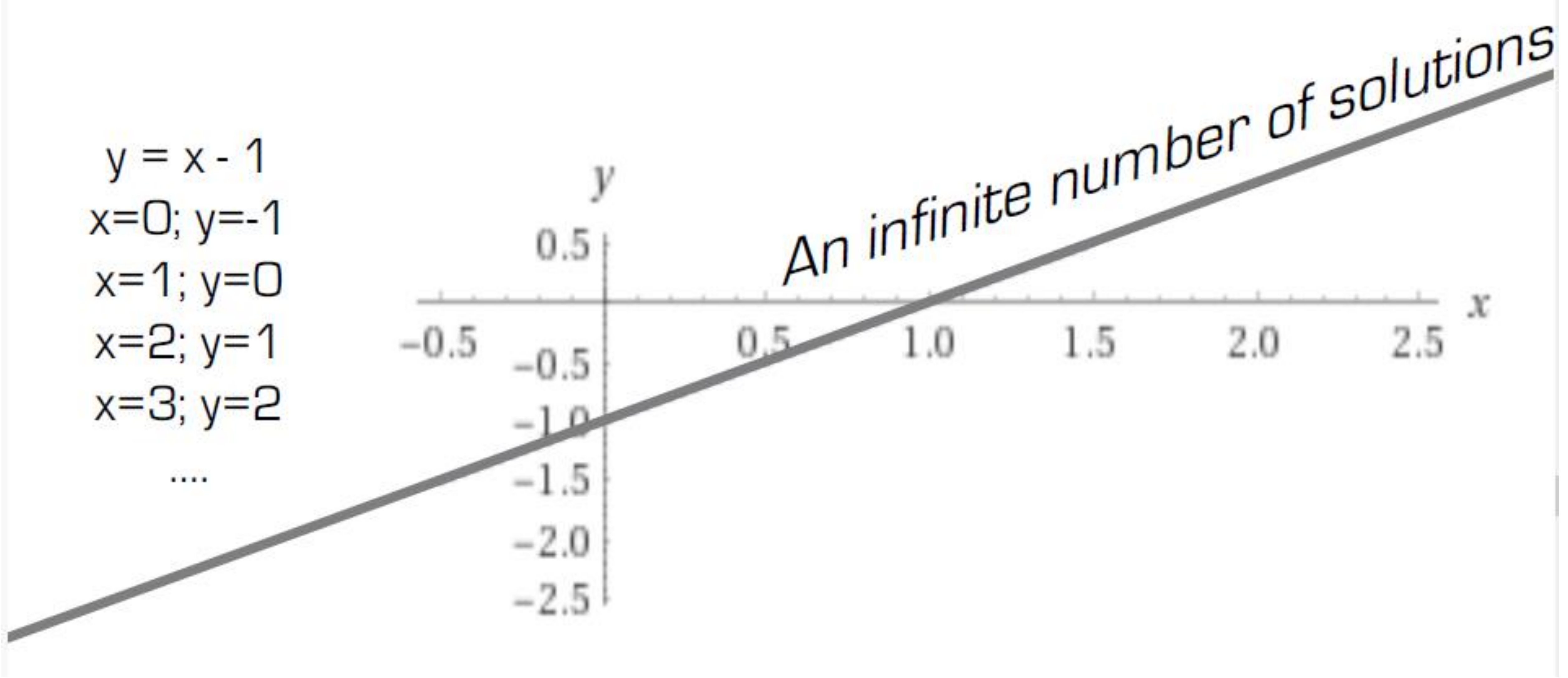
$x=1;\ y=0$

$x=2;\ y=-1$

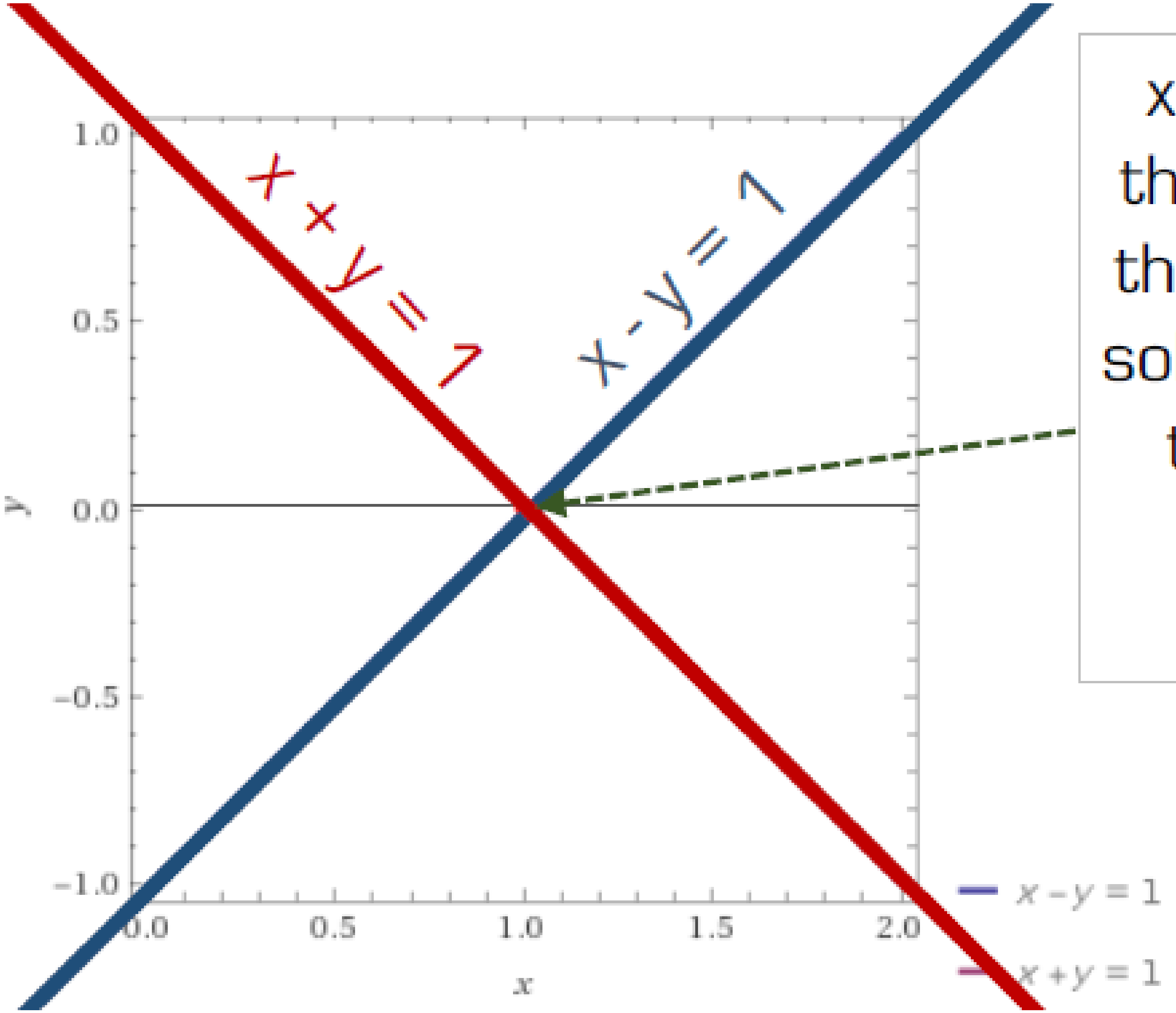
$x=3;\ y=-2$

....

How do we solve x - y = 1?



How do we solve a system of equations?
$x - y = 1$ and $x + y = 1$



x = 1 and y = 0 is the **unique** solution that simultaneously solves the system of these equations

How do we write these equations
x - y = 1 and x + y = 1 in the form of a matrix?

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Inconsistent Equations

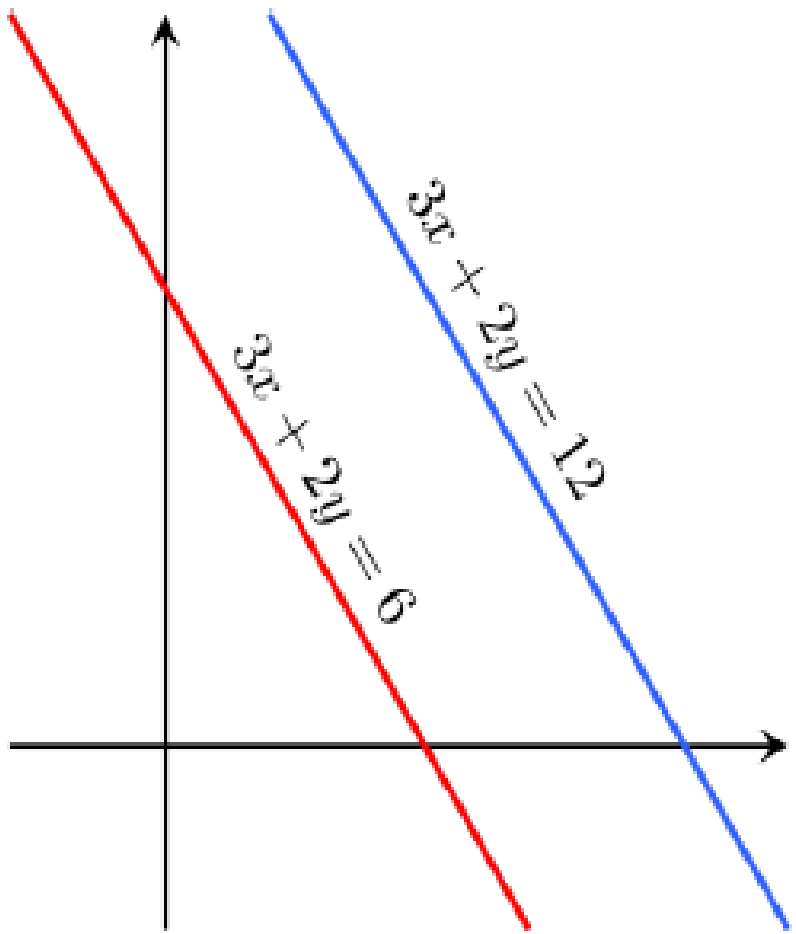
- When we have an equal number of variables and equations but still can't solve.
- What makes these equations inconsistent?

$$x + 2y + z = 3$$
$$3x - 2y - 4z = 4$$
$$-2x - 4y - 2z = 5$$

-2R1 and R3 are inconsistent

Consistency and Inconsistency

- These equations are inconsistent.
  - In fact, by subtracting the first equation from the second one and multiplying both sides of the result by 1/6, we get 0 = 1.
  - The graphs of these equations on the xy-plane are a pair of parallel lines.



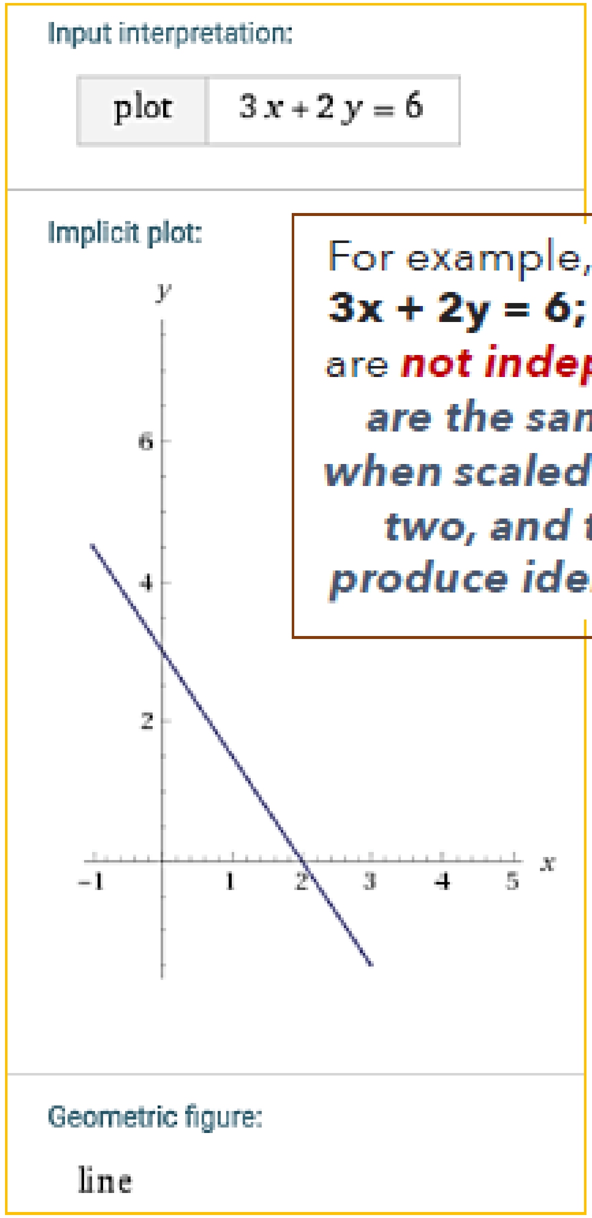
Inconsistency: a more intricate example

- It is possible for three linear equations to be inconsistent, even though any two of them are consistent together.
- Adding the first two equations together gives $3x + 2y = 2$, which can be subtracted from the third equation to yield $0 = 1$

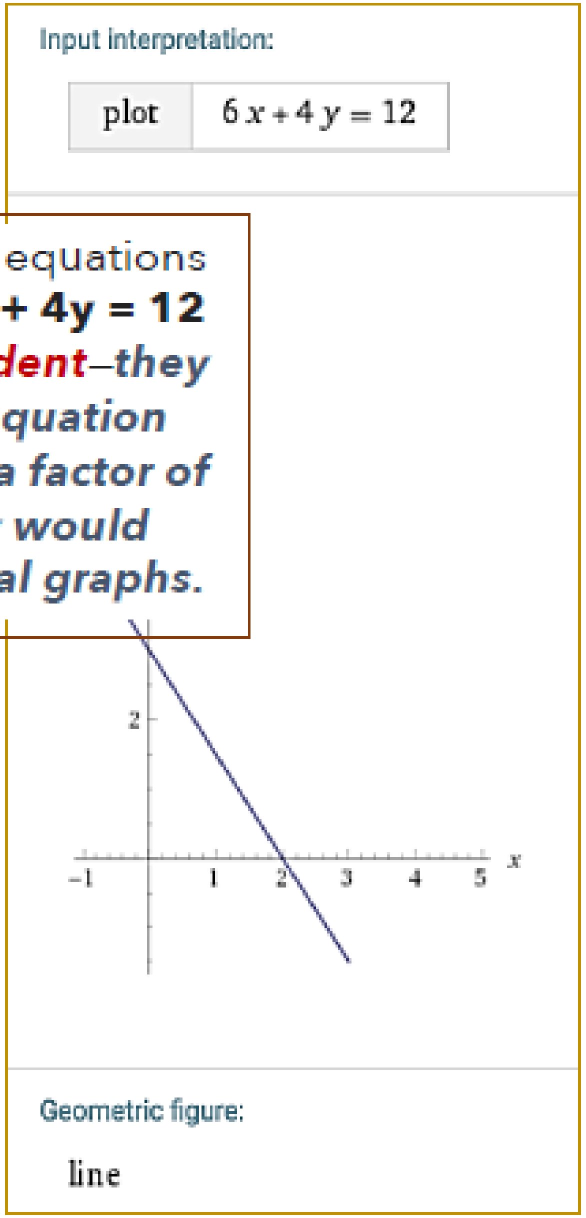
$$
\begin{aligned}
x + y &= 1 \\
2x + y &= 1 \\
3x + 2y &= 3
\end{aligned}
$$

Independence

- The equations of a linear system are independent if none of the equations can be derived algebraically from the others.
- When the equations are independent, each equation contains new information about the variables, and removing any of the equations increases the size of the solution set.



For example, the equations
3x + 2y = 6; 6x + 4y = 12
are ***not independent***—*they are the same equation when scaled by a factor of two, and they would produce identical graphs.*



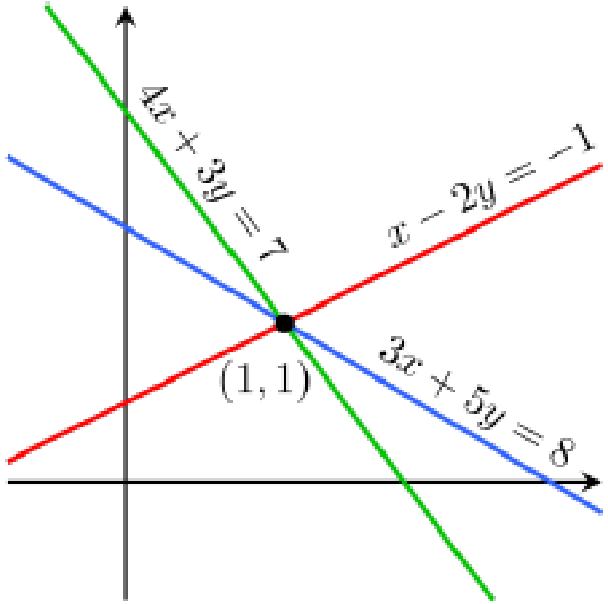
More complicated example of dependence

$$
\begin{aligned}
x - 2y &= -1 \\
3x + 5y &= 8 \\
4x + 3y &= 7
\end{aligned}
$$

With two unknowns and three equations, the system should be overdetermined and unsolvable.

But (1,1) solves this system

But in this case, the system is solvable since we have effectively two equations (i.e., the rank = 2)



The third equation is the sum of the other two.

Indeed, any one of these equations can be derived from the other two, and any one of the equations can be removed without affecting the solution set.

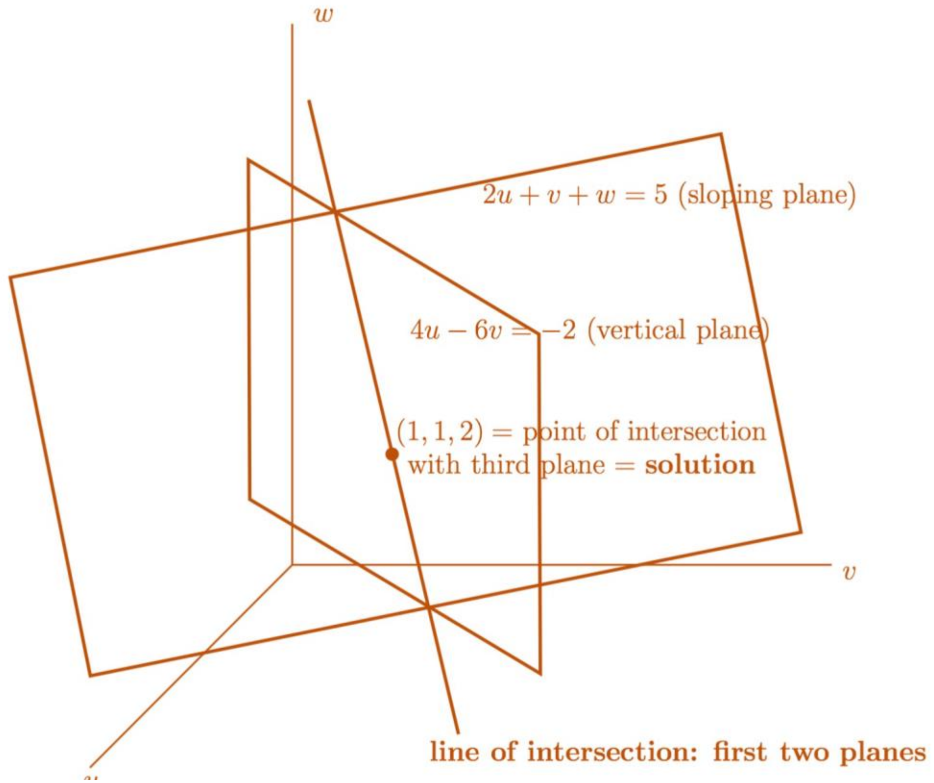
3 Equations and 3 unknowns

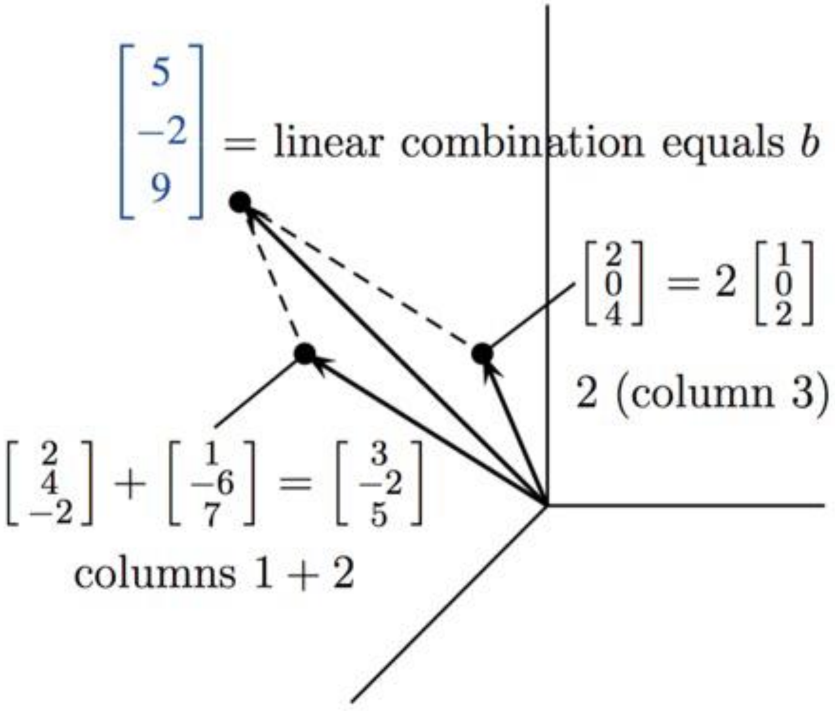
Three planes

$$\begin{aligned} 2u &+ v &+ w &= 5 \\ 4u &- 6v & &= -2 \\ -2u &+ 7v &+ 2w &= 9. \end{aligned}$$

Column form

$$u\begin{bmatrix}2\\4\\-2\end{bmatrix} + v\begin{bmatrix}1\\-6\\7\end{bmatrix} + w\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix} = b.$$





(b) Add columns $1 + 2 + (3 + 3)$

The solution is ***u, v, w = (1, 1, 2)***

Lengths and Dot Products

1 The "dot product" of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{w} = (1)(4) + (2)(5) = 4 + 10 = \mathbf{14}$.

2 $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$ are perpendicular because $\boldsymbol{v} \cdot \boldsymbol{w}$ is zero:
$(1)(4) + (3)(-4) + (2)(4) = \mathbf{0}$.

3 The length squared of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ is $\boldsymbol{v} \cdot \boldsymbol{v} = 1 + 9 + 4 = 14$. **The length is** $||\boldsymbol{v}|| = \sqrt{\mathbf{14}}$.

4 Then $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{||\boldsymbol{v}||} = \dfrac{\boldsymbol{v}}{\sqrt{14}} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ has length $||\boldsymbol{u}|| = \mathbf{1}$. Check $\dfrac{1}{14} + \dfrac{9}{14} + \dfrac{4}{14} = 1$.

5 The angle θ between $\boldsymbol{v}$ and $\boldsymbol{w}$ has $\cos\theta = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}|| \; ||\boldsymbol{w}||}$.

6 The angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $\cos\theta = \dfrac{1}{(1)(\sqrt{2})}$. That angle is $\theta = 45°$.

7 All angles have $|\cos\theta| \le 1$. So all vectors have $\boxed{|\boldsymbol{v} \cdot \boldsymbol{w}| \le ||\boldsymbol{v}|| \; ||\boldsymbol{w}||.}$

Inequalities

COSINE FORMULA If v and w are nonzero vectors then $\dfrac{v \cdot w}{\|v\|\,\|w\|} = \cos\theta$.

SCHWARZ INEQUALITY $|v \cdot w| \leq \|v\|\,\|w\|$

TRIANGLE INEQUALITY $\|v + w\| \leq \|v\| + \|w\|$

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

By the Schwarz inequality, $v \cdot w = 4$ is less than $\|v\|\,\|w\| = 5$. By the triangle inequality, side $3 = \|v + w\|$ is less than side 1 + side 2. For $v + w = (3, 3)$ the three sides are $\sqrt{18} < \sqrt{5} + \sqrt{5}$. Square this triangle inequality to get $18 < 20$.