Linear Algebra

(MT-121T)

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Lecture # 25

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Symmetric Matrices

1 A symmetric matrix S has n **real eigenvalues** λ_i and n **orthonormal eigenvectors** $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$.

2 Every real symmetric S can be diagonalized: $\boxed{S = Q\Lambda Q^{-1} = \boldsymbol{Q\Lambda Q^{\mathrm{T}}}}$

3 The number of positive eigenvalues of S equals the number of positive pivots.

4 Antisymmetric matrices $A = -A^{\mathrm{T}}$ have *imaginary* λ's and *orthonormal* (*complex*) $\boldsymbol{q}$'s.

5 Section 9.2 explains why the test $S = S^{\mathrm{T}}$ becomes $\boldsymbol{S} = \overline{\boldsymbol{S}}^{\mathrm{T}}$ for *complex matrices*.

$S = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \overline{S}^{\mathrm{T}}$ has real $\lambda = 1, -1$. $\quad A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\overline{A}^{\mathrm{T}}$ has $\lambda = i, -i$.

1. A symmetric matrix has only *real eigenvalues*.

2. The *eigenvectors* can be chosen *orthonormal*.

All Symmetric Matrices are Diagonalizable

Principal Axis Theorem

(Spectral Theorem) Every symmetric matrix has the factorization $S = Q\Lambda Q^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in the columns of Q:

Symmetric diagonalization $\quad S = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}} \quad \text{with} \quad Q^{-1} = Q^{\mathrm{T}}. \quad (1)$

Example

Example 2 The eigenvectors of a 2 by 2 symmetric matrix have a special form:

$$\textbf{Not widely known} \quad S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \quad \text{has} \quad \boldsymbol{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}.$$

$$\boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{x}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0.$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $\boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{x}_2 = 0$. Eagle eyes might notice the special case $S = I$, when b and $\lambda_1 - a$ and $\lambda_2 - c$ and $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are all zero. Then $\lambda_1 = \lambda_2 = 1$ is repeated. But of course $S = I$ has perpendicular eigenvectors.

Symmetric Matrices and Q

Symmetric matrices S have orthogonal eigenvector matrices Q. Look at this again:

Symmetry $\quad S = X\Lambda X^{-1}$ becomes $S = Q\Lambda Q^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = I$.

This says that every 2 by 2 symmetric matrix is (**rotation**)(**stretch**)(**rotate back**)

$$S = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} q_1^{\mathrm{T}} \\ q_2^{\mathrm{T}} \end{bmatrix}. \qquad (5)$$

Columns q_1 and q_2 multiply rows $\lambda_1 q_1^{\mathrm{T}}$ and $\lambda_2 q_2^{\mathrm{T}}$ to produce $S = \lambda_1 q_1 q_1^{\mathrm{T}} + \lambda_2 q_2 q_2^{\mathrm{T}}$.

Every symmetric matrix $\quad S = Q\Lambda Q^{\mathrm{T}} = \lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}}$

S has correct eigenvectors
Those q's are orthonormal $\quad Sq_i = (\lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}})\, q_i = \lambda_i q_i$

Eigenvalues versus Pivots

Example 4 This symmetric matrix has one positive eigenvalue and one positive pivot:

Matching signs $S = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has pivots 1 and -8 eigenvalues 4 and -2.

The signs of the pivots match the signs of the eigenvalues, one plus and one minus. This could be false when the matrix is not symmetric:

Opposite signs $B = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$ has pivots 1 and 2 eigenvalues -1 and -2.

Worked Example

6.4 A What matrix A has eigenvalues $\lambda = 1, -1$ and eigenvectors $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$ and $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$? Which of these properties can be predicted in advance?

$$A = A^{\mathrm{T}} \qquad A^2 = I \qquad \det A = -1 \qquad \text{pivot are} + \text{and} - \qquad A^{-1} = A$$

Solution All those properties can be predicted! With real eigenvalues $1, -1$ and orthonormal $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, the matrix $A = Q\Lambda Q^{\mathrm{T}}$ must be symmetric. The eigenvalues 1 and -1 tell us that $A^2 = I$ (since $\lambda^2 = 1$) and $A^{-1} = A$ (same thing) and $\det A = -1$. The two pivots must be positive and negative like the eigenvalues, since A is symmetric.

The matrix will be a reflection. Vectors in the direction of $\boldsymbol{x}_1$ are unchanged by A (since $\lambda = 1$). Vectors in the perpendicular direction are reversed (since $\lambda = -1$). The reflection $A = Q\Lambda Q^{\mathrm{T}}$ is across the "θ-line". Write c for $\cos\theta$ and s for $\sin\theta$:

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & 2cs \\ 2cs & s^2 - c^2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Notice that $\boldsymbol{x} = (1, 0)$ goes to $A\boldsymbol{x} = (\cos 2\theta, \sin 2\theta)$ on the 2θ-line. And $(\cos 2\theta, \sin 2\theta)$ goes back across the θ-line to $\boldsymbol{x} = (1, 0)$.

Complex Eigenvalues for Real Matrices

For any real matrix, $S\,x = \lambda\,x$ gives $S\,\overline{x} = \overline{\lambda}\,\overline{x}$. For a symmetric matrix, λ and x turn out to be real. Those two equations become the same. But a *non*symmetric matrix can easily produce λ and x that are complex. Then $A\,\overline{x} = \overline{\lambda}\,\overline{x}$ is true but different from $A\,x = \lambda\,x$. We get another complex eigenvalue (which is $\overline{\lambda}$) and a new eigenvector (which is $\overline{x}$):

For real matrices, complex λ's and x's come in "conjugate pairs."

$$\lambda = a + ib \qquad \overline{\lambda} = a - ib$$

If $\quad A\,x = \lambda\,x \quad$ ***then*** $\quad A\overline{x} = \overline{\lambda}\,\overline{x}$.

Complex Eigenvalues for Real Matrices

Example 3 $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has $\lambda_1 = \cos\theta + i\sin\theta$ and $\lambda_2 = \cos\theta - i\sin\theta$.

Those eigenvalues are conjugate to each other. They are λ and $\overline{\lambda}$. The eigenvectors must be x and $\overline{x}$, because A is real:

$$\begin{aligned}
&\textbf{This is } \lambda x \qquad && Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&\textbf{This is } \overline{\lambda}\,\overline{x} \qquad && A\overline{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta) \begin{bmatrix} 1 \\ i \end{bmatrix}.
\end{aligned} \tag{9}$$

Those eigenvectors $(1, -i)$ and $(1, i)$ are complex conjugates because A is real.

For this rotation matrix the absolute value is $|\lambda| = 1$, because $\cos^2\theta + \sin^2\theta = 1$. ***This fact*** $|\lambda| = 1$ ***holds for the eigenvalues of every orthogonal matrix Q.***

Positive Definite Matrices

1 **Symmetric** S : all eigenvalues $> 0 \Leftrightarrow$ all pivots $> 0 \Leftrightarrow$ all upper left determinants > 0.

2 The matrix S is then **positive definite**. The energy test is $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$ for all vectors $\boldsymbol{x} \neq \mathbf{0}$.

3 One more test for positive definiteness : $S = A^{\mathrm{T}} A$ with independent columns in A.

4 **Positive semidefinite** S allows $\lambda = 0$, pivot $= 0$, determinant $= 0$, energy $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 0$.

5 The equation $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ gives an ellipse in $\mathbf{R}^n$ when S is symmetric positive definite.

Definition *S is positive definite if $x^{\mathrm{T}} S x > 0$ for every nonzero vector x:*

2 by 2 $$\boxed{x^{\mathrm{T}} S x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \boxed{ax^2 + 2bxy + cy^2 > 0.}$$

Positive Definite Matrices

A 2×2 real symmetric matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ is positive definite if and only if the diagonal entries a and d are positive and the determinant $|A| = ad - bc = ad - b^2$ is positive.

Consider the following symmetric matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

A is not positive definite, because $|A| = 4 - 9 = -5$ is negative. B is not positive definite, because the diagonal entry -3 is negative. However, C is positive definite, because the diagonal entries 1 and 5 are positive, and the determinant $|C| = 5 - 4 = 1$ is also positive.

Positive Definite Matrices

When a symmetric matrix S has one of these five properties, it has them all:

1. All ***n pivots*** of S are positive.
2. All ***n upper left determinants*** are positive.
3. All ***n eigenvalues*** of S are positive.
4. $x^{\mathrm{T}} S x$ is positive except at $x = \mathbf{0}$. This is the ***energy-based*** definition.
5. S equals $A^{\mathrm{T}} A$ for a matrix A with ***independent columns***.

Positive Definite Matrices

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

eigenvalues $1, 1, 4$

determinants 2 and 3 and 4

pivots 2 and $3/2$ and $4/3$

$S - I$ will be *semidefinite*: eigenvalues $0, 0, 3$

$S - 2I$ is *indefinite* because $\lambda = -1, -1, 2$

Example

Test these symmetric matrices S and T for positive definiteness:

$$S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution The pivots of S are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. The eigenvalues of S are $2-\sqrt{2}$ and 2 and $2+\sqrt{2}$, all positive. That completes tests **1**, **2**, and **3**. Any one test is decisive!

Example (contd...)

I have three candidates A_1, A_2, A_3 to suggest for $S = A^{\mathrm{T}}A$. They all show that S is positive definite. A_1 is a first difference matrix, 4 by 3, to produce $-1, 2, -1$ in S:

$$S = A_1^{\mathrm{T}} A_1 \qquad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

The three columns of A_1 are independent. Therefore S is positive definite.

Example (contd...)

A_2 comes from $S = LDL^{\mathrm{T}}$ (the symmetric version of $S = LU$). Elimination gives the pivots $2, \frac{3}{2}, \frac{4}{3}$ in D and the multipliers $-\frac{1}{2}, 0, -\frac{2}{3}$ in L. **Just put $\boldsymbol{A_2 = L\sqrt{D}}$.**

$$LDL^{\mathrm{T}} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}} = A_2^{\mathrm{T}} A_2.$$

A_2 is the Cholesky factor of S

This triangular choice of A has square roots (not so beautiful). It is the "Cholesky factor" of S and the MATLAB command is $A = \mathrm{chol}(S)$. In applications, the rectangular A_1 is how we build S and this Cholesky A_2 is how we break it apart.

Example (contd...)

Eigenvalues give the symmetric choice $A_3 = Q\sqrt{\Lambda}Q^{\mathrm{T}}$. This is also successful with $A_3^{\mathrm{T}} A_3 = Q\Lambda Q^{\mathrm{T}} = S$. All tests show that the $-1, 2, -1$ matrix S is positive definite.

To see that the energy $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}$ is positive, we can write it as a sum of squares. The three choices A_1, A_2, A_3 give three different ways to split up $\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x}$:

$$\boldsymbol{x}^{\mathrm{T}} S\boldsymbol{x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \qquad \textbf{Rewrite with squares}$$

$$||A_1\boldsymbol{x}||^2 = x_1^2 + \left(x_2 - x_1\right)^2 + \left(x_3 - x_2\right)^2 + x_3^2 \qquad \textbf{Using differences in } \boldsymbol{A_1}$$

$$||A_2\boldsymbol{x}||^2 = 2\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{2}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{3}\,x_3^2 \qquad \textbf{Using } \boldsymbol{S = LDL^{\mathrm{T}}}$$

$$||A_3\boldsymbol{x}||^2 = \lambda_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{x})^2 + \lambda_3(\boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{x})^2 \qquad \textbf{Using } \boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$$

Example (contd…)

$$T = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

Now turn to T (top of this page). The $(1,3)$ and $(3,1)$ entries move away from 0 to b. This b must not be too large! *The determinant test is easiest*. The 1 by 1 determinant is 2, the 2 by 2 determinant T is still 3. The 3 by 3 determinant involves b:

Test on T $\qquad \det T = 4 + 2b - 2b^2 = (1+b)(4-2b)$ must be positive.

At $b = -1$ and $b = 2$ we get $\det T = 0$. *Between $b = -1$ and $b = 2$ this matrix T is positive definite*. The corner entry $b = 0$ in the matrix S was safely between -1 and 2.

Minimum of a Function

First derivatives are zero $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \mathbf{0}$ at the minimum point.

Next comes the linear algebra version of the usual calculus test $d^2 f/dx^2 > 0$:

Second derivative matrix S is positive definite $S = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}$

Here $\boldsymbol{F_{xy}} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \boldsymbol{F_{yx}}$ is a 'mixed" second derivative.

Practice Problem

For which a and c is this matrix positive definite? For which a and c is it positive semidefinite (this includes definite)?

$$S = \begin{bmatrix} a & a & a \\ a & a+c & a-c \\ a & a-c & a+c \end{bmatrix}$$

All 5 tests are possible.

The energy $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x}$ equals

$a\,(x_1 + x_2 + x_3)^2 + c\,(x_2 - x_3)^2.$

Symmetric: $S^{\mathrm{T}} = S = Q\Lambda Q^{\mathrm{T}}$	real eigenvalues	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{S}^{\mathrm{T}} = S$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $S^{\mathrm{T}} = S$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $A = BCB^{-1}$	$\lambda(A) = \lambda(C)$	B times eigenvector of C
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation : cosine-sine	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: $P_{i,i+1} = 1, P_{n1} = 1$	$\lambda_k = e^{2\pi ik/n} =$ roots of 1	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Circulant: $c_0 I + c_1 P + \cdots$	$\lambda_k = c_0 + c_1\, e^{2\pi ik/n} + \cdots$	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = X\Lambda X^{-1}$	diagonal of Λ	columns of X are independent
Schur: $A = QTQ^{-1}$	diagonal of triangular T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $A = BJB^{-1}$	diagonal of J	each block gives 1 eigenvector
SVD: $A = U\Sigma V^{\mathrm{T}}$	r singular values in Σ	eigenvectors of $A^{\mathrm{T}}A$, AA^{T} in V, U

Singular Value Decomposition

1 An image is a large matrix of grayscale values, one for each pixel and color.

2 When nearby pixels are correlated (not random) the image can be compressed.

3 The SVD separates any matrix A into rank one pieces $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = (\textbf{column})(\textbf{row})$.

4 The columns and rows are eigenvectors of symmetric matrices AA^{T} and $A^{\mathrm{T}}A$.

Singular Value Decomposition

- SVD enables us to factor any m x n matrix A into the product of three matrices, $A = U\,\Sigma\,VT$ where
  - U is an orthogonal m x m matrix
  - V is an orthogonal n x n matrix.
  - Σ is an m x n matrix with a diagonal upper left submatrix D of positive entries decreasing in magnitude, the remaining entries being zeros.

$$\Sigma = \begin{bmatrix} D & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}, \text{ where } D = \begin{bmatrix} \sigma_1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \sigma_r \end{bmatrix} \text{ with } \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0,$$

for $r \leq m, n$

Examples of Σ

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$\sigma_1 = 9, \sigma_2 = 3,$

$r = 2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = 6, \sigma_2 = 2,$

$r = 2$

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = 7, \sigma_2 = 5, \sigma_3 = 2,$

$r = 3$

$$\begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = 8, \sigma_2 = 7, \sigma_3 = 1$

$r = 3$

Example 1

Find a singular value decomposition of the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

We want to find the matrices U, Σ, V such that $A = U\Sigma V^t$. It is useful to keep track of the sizes of the various matrices. We get

$$\begin{array}{cccc} A & = U & \Sigma & V^t \\ m \times n & m \times m & m \times n & n \times n \end{array} \qquad \begin{array}{cccc} A & = U & \Sigma & V^t \\ 2 \times 3 & 2 \times 2 & 2 \times 3 & 3 \times 3 \end{array}$$

(general $m \times n$ matrix A) (our 2×3 matrix A)

Example 1 (contd...)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Finding V: The columns of V will be eigenvectors of A^tA. We get

$$A^tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

The eigenvalues of A^tA are computed and found to be $\lambda = 5, 1, 0$ (in descending order of magnitude).

$$\underset{(\lambda = 5)}{\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}}, \quad \underset{(\lambda = 1)}{\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \quad \underset{(\lambda = 0)}{\mathbf{v}_3 = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}}$$

$$V = \begin{bmatrix} 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$

Finding the Singular Values: The *singular values* of A are the positive square roots of the eigenvalues of A^tA. The singular values are

$$\sigma_1 = \sqrt{5}, \sigma_2 = 1, \sigma_3 = 0$$

Example 1 (contd...)

Finding Σ: Σ is to be a 2×3 matrix with upper left block being a diagonal matrix D with diagonal elements $\sigma_1 = \sqrt{5}, \sigma_2 = 1$.

$$D = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Finding U: U is a 2×2 matrix. The column vectors of U are selected to be the vectors $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These vectors are orthonormal. U is the orthogonal matrix

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$A = U\Sigma V^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ -2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix}$$