# Linear Algebra

# (MT-121T)

**Aftab Alam**

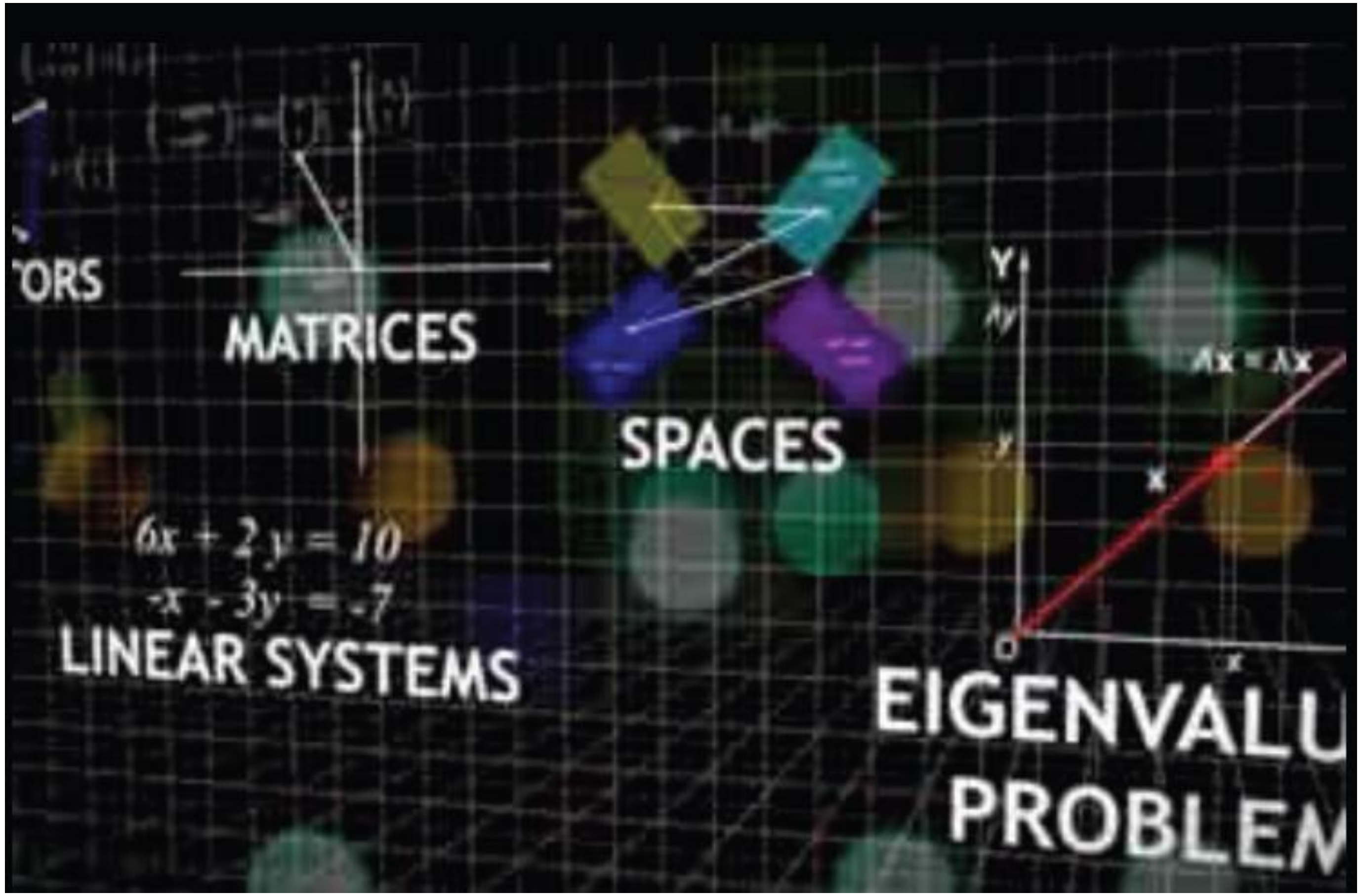
**Lecture # 1**

**(Monday, January 22, 2024)**

# What is Linear Algebra?

- Linear algebra is a branch of mathematics that deals with
  - vector spaces and linear mappings between these spaces.
- LA provides a framework for
  - studying and solving systems of linear equations
- LA represents
  - mathematical concepts and operations using vectors and matrices.

# Key Concepts in Linear Algebra




- Vectors
- Matrices
- Vector Spaces
- Linear Transformations
- Eigenvalues and Eigenvectors

# Key Concepts in Linear Algebra

## ⁕Vectors:

- Vectors are mathematical objects that represent quantities with both magnitude and direction.
- In linear algebra, vectors can be represented geometrically and algebraically.

## ⁕Matrices:

- Matrices are arrays of numbers, symbols, or expressions arranged in rows and columns.
- They are fundamental to linear algebra and are used to represent linear transformations and systems of linear equations.

# Key Concepts in Linear Algebra

## *Vector Spaces:

- A vector space is a set of vectors equipped with operations of addition and scalar multiplication, satisfying specific properties.
- Common examples include Euclidean spaces and function spaces.

## *Linear Transformations:

- Linear transformations are functions that preserve vector addition and scalar multiplication.
- They are often represented by matrices and play a crucial role in linear algebra.

# Key Concepts in Linear Algebra

## Eigenvalues and Eigenvectors:

- Eigenvalues and eigenvectors are concepts associated with linear transformations and matrices.
- Eigenvalues represent scalars that characterize how the transformation stretches or compresses space
- Eigenvectors are the corresponding non-zero vectors that remain in the same direction after the transformation.

# Applications of Linear Algebra

- LA has wide-ranging applications in various fields such as
  - Physics
  - Computer science
  - Engineering
  - Economics
  - Statistics
- It serves as a fundamental tool for solving problems involving linear relationships and transformations.
- Linear algebra has numerous applications in various fields in the 21st century, playing a critical role in modern technology, science, and engineering.

# Applications of Linear Algebra

## Computer Graphics and Animation:

- Linear algebra is fundamental in computer graphics for rendering images, 3D modeling, and animation.
- Transformations, such as scaling, rotation, and translation, are represented using matrices.

## Machine Learning and Data Science:

- Linear algebra is extensively used in machine learning algorithms, including linear regression, support vector machines, and principal component analysis.
- Matrices and vectors are used to represent and manipulate datasets, features, and parameters.

## Computer Vision:

- Image processing and computer vision applications utilize linear algebra for tasks like image compression, object recognition, and image enhancement.
- Techniques such as convolution and eigenimage analysis rely on linear algebra concepts.

# Applications of Linear Algebra

## ⁕ Network Analysis:

- Linear algebra is employed in the analysis of networks and graphs, including social networks, transportation networks, and communication networks.
- Techniques like adjacency matrices and eigenvalue centrality play a role in understanding network structures.

## ⁕ Quantum Computing:

- Quantum computing, an emerging field, heavily relies on linear algebra for representing quantum states and quantum operations.
- Quantum gates are often represented using unitary matrices.

## ⁕ Economics and Finance:

- Linear algebra is applied in economic modeling, optimization, and financial analysis.
- Portfolio optimization, risk management, and solving systems of linear equations are common tasks in finance.

# Applications of Linear Algebra

## Biomedical Imaging:

- Techniques like MRI (Magnetic Resonance Imaging) and CT (Computed Tomography) use linear algebra for image reconstruction and signal processing.
- Reconstruction algorithms involve solving linear systems of equations.

## Electric Circuits and Signal Processing:

- Electrical engineering relies on linear algebra for circuit analysis.
- Signal processing applications, including filtering and convolution, utilize linear algebraic techniques.

## Robotics:

- In robotics, linear algebra is used for kinematics, dynamics, and control of robotic systems.
- Representations of robot configurations and transformations are expressed using matrices.

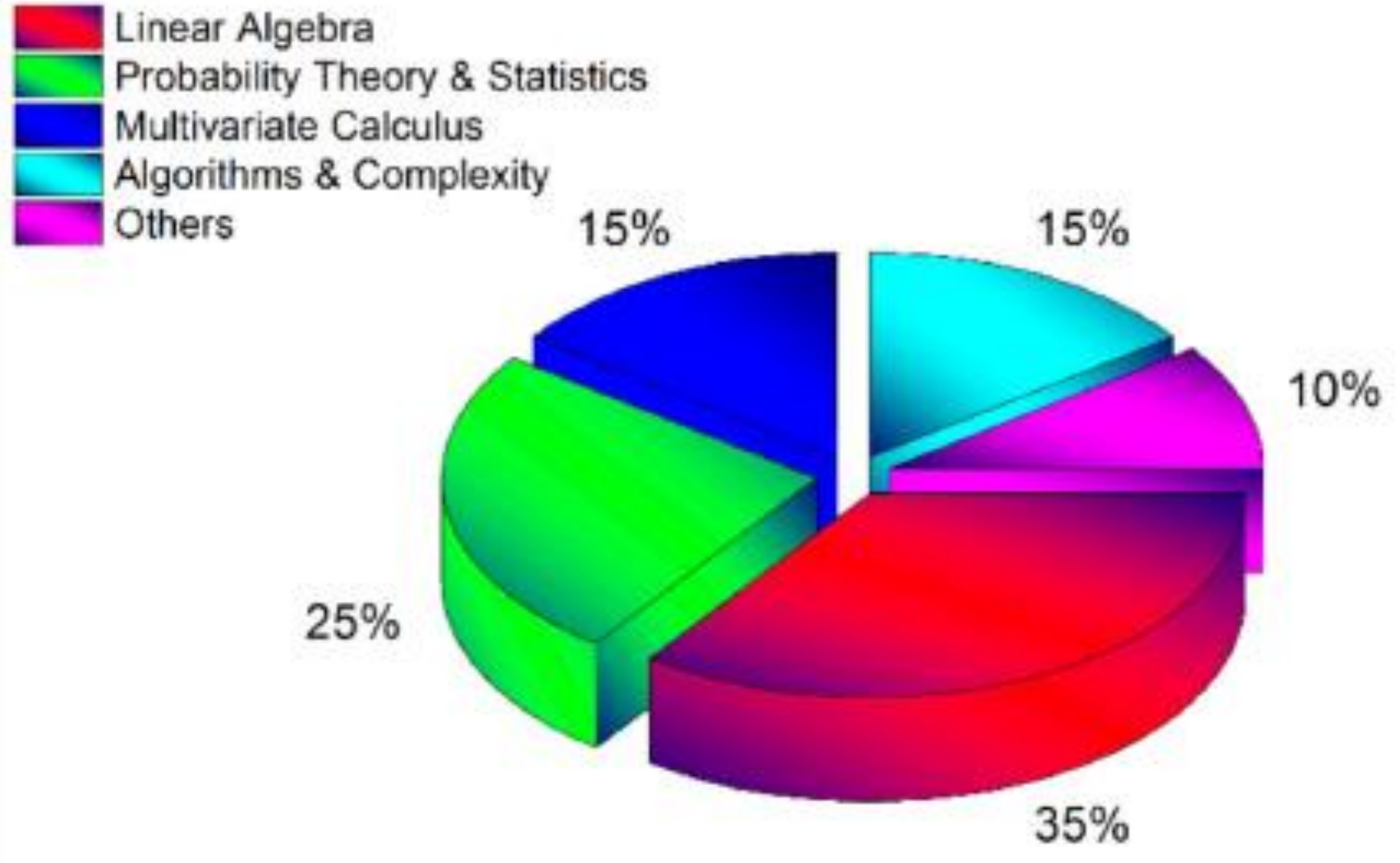
# Applications of Linear Algebra

## * Physics and Engineering:

- Linear algebra is foundational in physics for describing physical systems, especially those with multiple dimensions.
- Quantum mechanics, fluid dynamics, and electromagnetism often involve linear algebraic methods.

# Why Linear Algebra is so important in future?

* Linear Algebra is the mathematics of AI and ML.




Importance of Maths Topics Needed For Machine Learning

# Commonly used notations of Linear Algebra

## ⁕ Vectors:

- A vector **v** can be represented using column notation:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ . \\ . \\ v_n \end{bmatrix}$$

- Row notation is also used for vectors:

$$v = [v_1, v_2, v_2, .. ,v_n]$$

# Commonly used notations of Linear Algebra

## ⁎ Matrices:

- A matrix $A$ can be represented using square or rectangular brackets:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- $m \times n$ matrix with elements $a_{ij}$ can be written as

$$[a_{ij}]_{m \times n}$$

# Commonly used notations of Linear Algebra

## ⁎ Vector and Matrix Operations:

- Vector addition: $\mathbf{u} + \mathbf{v}$
- Scalar multiplication: $c\,\mathbf{v}$
- Matrix multiplication: $A\,B$
- Transposition: $\mathbf{v}^T$ or $A^T$ (transpose)

# Commonly used notations of Linear Algebra

## ⁕ Linear Transformations:

- Linear transformation $T$ acting on vector $\mathbf{v}$: $T(\mathbf{v})$
- Matrix representation of a linear transformation: $T(\mathbf{v}) = A\mathbf{v}$

## ⁕ Systems of Linear Equations:

- System of linear equations $Ax = b$, where
  - $A$ is a matrix,
  - $x$ is a column vector of variables,
  - $b$ is a column vector on the right side.

## ⁕ Eigenvalues and Eigenvectors:

- Eigenvalue equation: $A\mathbf{v} = \lambda\mathbf{v}$, where
  - $\lambda$ is the eigenvalue
  - $\mathbf{v}$ is the corresponding eigenvector.

# Commonly used notations of Linear Algebra

## * Linear Independence and Span:

- Linearly independent vectors:

$\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ are linearly independent.

- Span of vectors:

$\mathrm{span}(\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\})$.

## * Inner Product:

- Inner product of vectors:

$\langle \mathbf{u}, \mathbf{v} \rangle$ or $\mathbf{u}\cdot\mathbf{v}$

## * Projection:

- Projection of vector **v** onto vector **u**:

$\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$

# Operations performed on Vectors

- **Vector Addition:**
  - **Description:** Combining two vectors to obtain a new vector.
  - **Notation: u + v**
  - **Result:** A vector with components obtained by adding corresponding components of **u** and **v**
- **Scalar Multiplication:**
  - **Description:** Multiplying a vector by a scalar (a single numerical value).
  - **Notation:** c**v or v** · c
  - **Result:** A vector with components obtained by multiplying each component of **v** by the scalar c
- **Vector Subtraction:**
  - **Description:** Subtracting one vector from another.
  - **Notation: u – v**
  - **Result:** A vector with components obtained by subtracting corresponding components of **v** from **u**

# Operations performed on Vectors

## ⁕ Dot Product (Scalar Product):

- **Description:** Multiplying corresponding components of two vectors and summing the results.
- **Notation:** $\mathbf{u} \cdot \mathbf{v}$ **or** $\langle \mathbf{u}, \mathbf{v} \rangle$
- **Result:** A scalar (single numerical value).

## ⁕ Cross Product (Vector Product):

- **Description:** Producing a new vector that is perpendicular to the plane formed by the original vectors.
- **Notation:** $\mathbf{u} \times \mathbf{v}$
- **Result:** A vector

## ⁕ Scalar Triple Product:

- **Description:** Combining the dot product and cross product of three vectors.
- **Notation:** $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ **or** $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
- **Result:** A scalar

# Operations performed on Vectors

## ⁕ Vector Triple Product:

- **Description:** Combining the cross product and cross product of three vectors.
- **Notation:** $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$
- **Result:** A vector

## ⁕ Magnitude (Norm):

- **Description:** Calculating the length or magnitude of a vector.
- **Notation:** $\|v\|$
- **Result:** A scalar

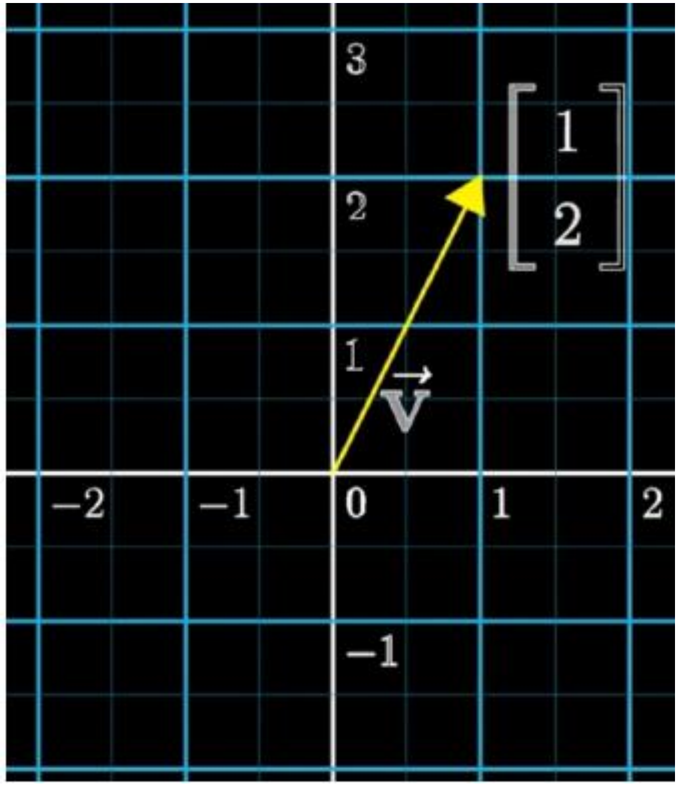
# What is a Vector?

$v \in \mathbf{R^n}$ is an n-tuple of real numbers

**Examples:**

$$v = (v_1, v_2) \in (\boldsymbol{R}, \boldsymbol{R}) \equiv \boldsymbol{R}^2$$

$$v = (v_1, v_2, v_3) \in (\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}) \equiv \boldsymbol{R}^3$$




- When you see ≡ between two mathematical expressions, it means that the expressions are not only equal in value but are also identical or represent the same mathematical object.
- The term "tuple" is used to describe a finite, ordered collection of elements. The letter "n" indicates the number of elements in the tuple, making it an n-tuple.
- In linear algebra, an n-tuple refers to an ordered list or sequence of n elements, where each element can be a number, variable, or any other mathematical object.

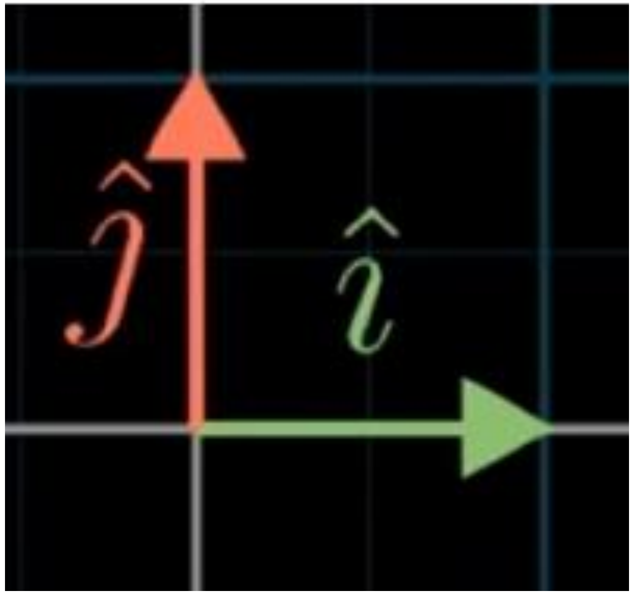
# Vectors

$$v = (v_1, v_2, v_3) \in (\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}) \equiv \boldsymbol{R}^3$$

- In the context of linear algebra, n-tuples are commonly used to represent vectors. A vector with n components is essentially an n-tuple. For instance, a **3-dimensional vector v** can be represented $(v_1, v_2, v_3)$, where $v_1, v_2$ and $v_3$ are the components of the vector.

- n-tuples are versatile and find applications not only in linear algebra but also in various mathematical disciplines and computer science. They provide a concise and ordered way to represent collections of elements with a specific number and order.

# Orthogonal Basis



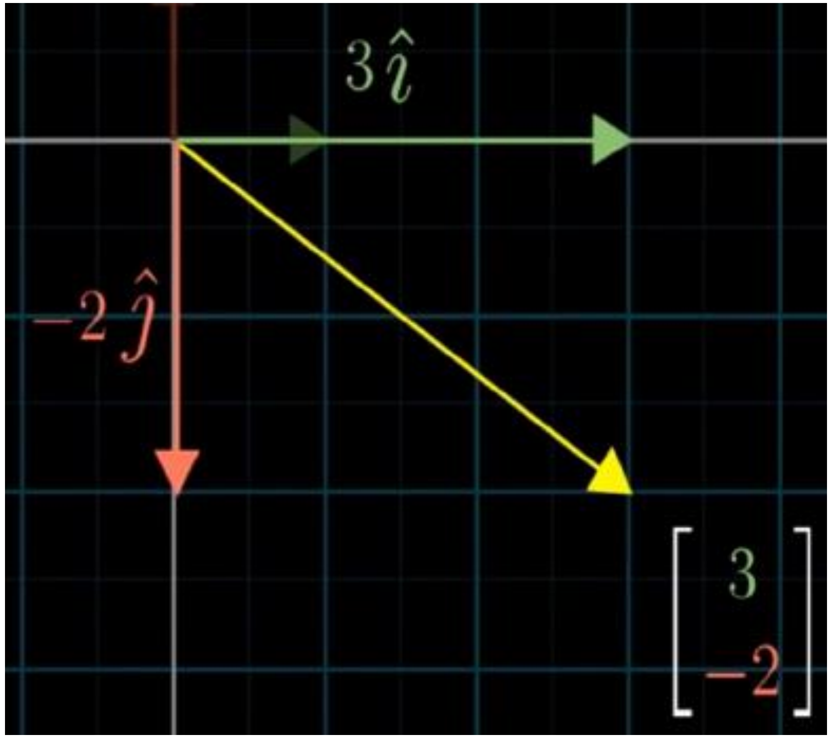

- Vectors are easier to understand when they are described in terms of orthogonal basis.

- In LA, an **orthogonal basis** is a set of vectors that are mutually perpendicular to each other (dot product is ZERO) and have unit length (each vector in the basis has a length of 1).

- **Mutual Orthogonality:** $v_i \cdot v_j = 0$ for $i \neq j$
- **Unit Length:** $\|v_i\| = 1$

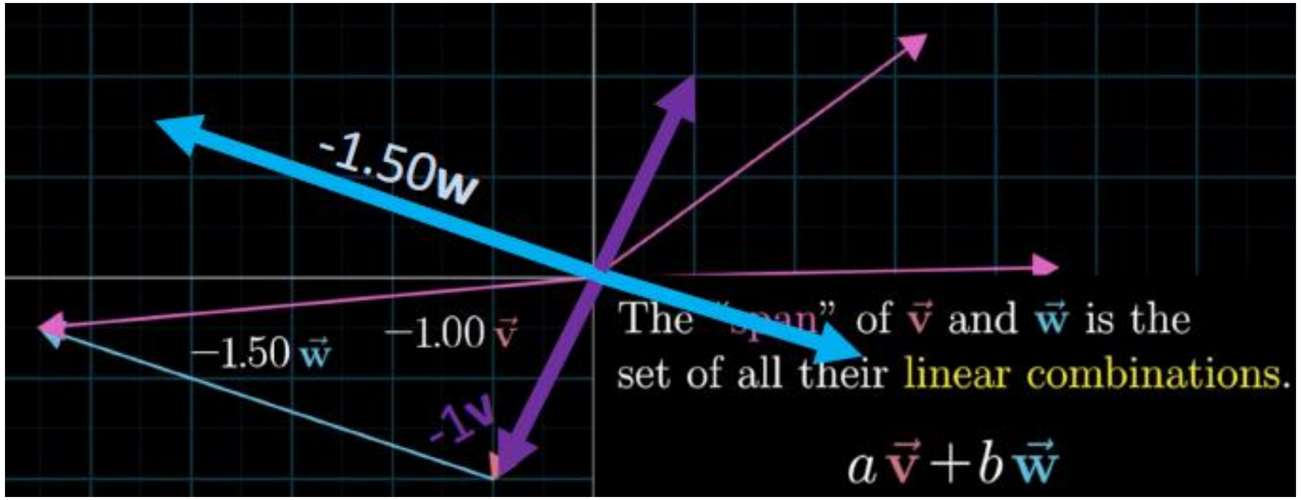
# Linear Combination

- A linear combination of vectors is a combination of those vectors where each vector is multiplied by a scalar and then added together.
- Let's say you have vectors $(v_1, v_{2,\ldots} \ldots v_n)$ and scalars $(c_1, c_{2,\ldots} \ldots c_n)$. The linear combination of these vectors is given by $(c_1 v_1 + c_2 v_{2,\ldots} \ldots + c_n v_n)$.



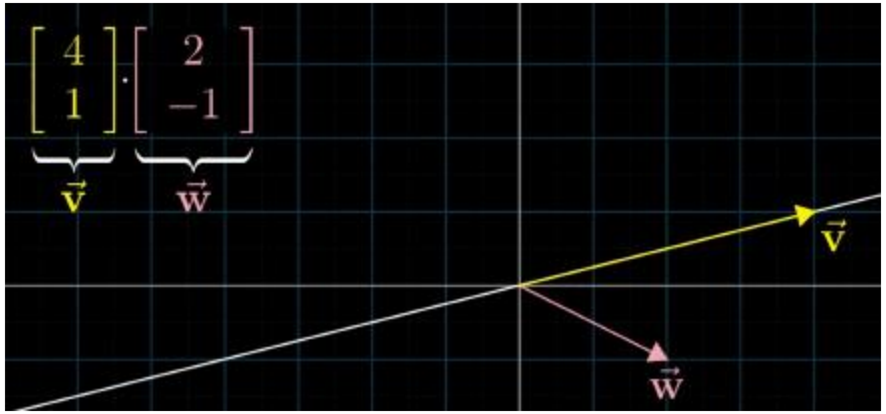
# Linear Combination and Subspace

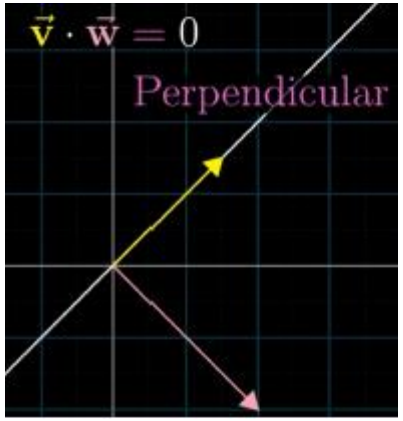
- A subspace is a subset of a vector space that is itself a vector space. It means that a subspace is closed under vector addition and scalar multiplication. In other words, if you take any vectors **u** and **v** from the subspace and any scalar c, then c**u** and **u+v** must also belong to the subspace.
- Linear combinations provide a way to generate subspaces, and subspaces are sets of vectors that are closed under linear combinations. Understanding these concepts is crucial in linear algebra for studying vector spaces and their properties.
- Using linear combinations of basis vectors, we can find the subspace subsuming all the linear combinations of our vectors.

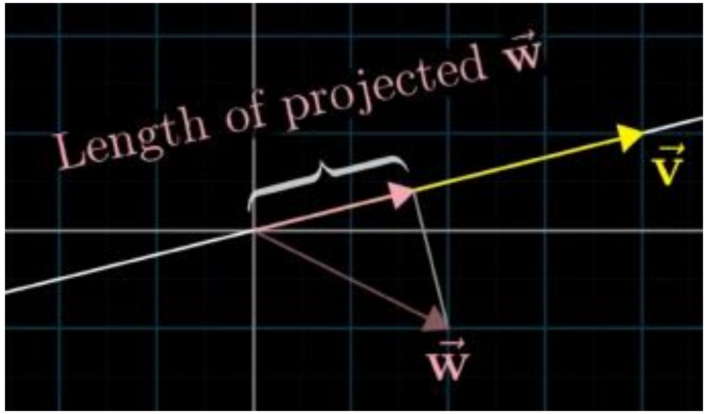


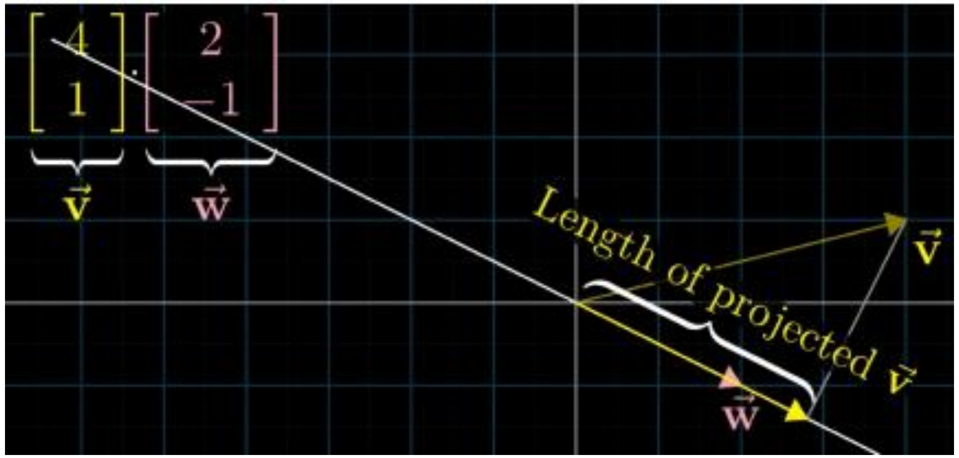
# Projecting a vector onto a subspace

- What about a vector that is outside this space? How can we approximate that vector through a vector in this subspace?
- Answer: Project that vector onto the subspace.



# Projecting a vector onto a subspace

- Projecting a vector onto a subspace involves finding the component of the vector that lies within that subspace. The projection provides a way to represent the original vector as a combination of vectors in the subspace.
  - Let's consider a vector v and a subspace W with basis vectors $\{w_1, w_2, \ldots, w_n\}$. The projection of v onto the subspace W, denoted as $\text{proj}_W(v)$ can be calculated using the following formula:

$$\mathbf{proj_W(v)} = \frac{v \cdot w_1}{\|w_1\|^2} w_1 + \frac{v \cdot w_2}{\|w_2\|^2} w_2 + \ldots. + \frac{v \cdot w_n}{\|w_n\|^2} w_n$$

  - Here $\mathbf{v} \cdot \mathbf{w}_i$ represents the dot product of **v** and the i-th basis vector $\mathbf{w}_i$
  - $\| \mathbf{w}_i \|$ is the magnitude (length) of the i-th basis vector.
  - The fraction $\frac{v \cdot w_i}{\|w_i\|^2} w_i$ represents the scalar projection of **v** onto $\mathbf{w}_i$.
- The projection ensures that the resulting vector lies within the subspace *W*. Geometrically, it represents the closest point in the subspace to the original vector **v**.

# Signal and Systems Perspective

- Linear Algebra is useful both for depicting data (input/output signal) compactly and for characterizing system action (the linear transformation performed on input data to produce the output data).