LINEAR ALGEBRA (MT–121)

CHAPTER 4

ORTHOGONALITY

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Lecture 17

FOUR FUNDAMENTALS SUBSPACES

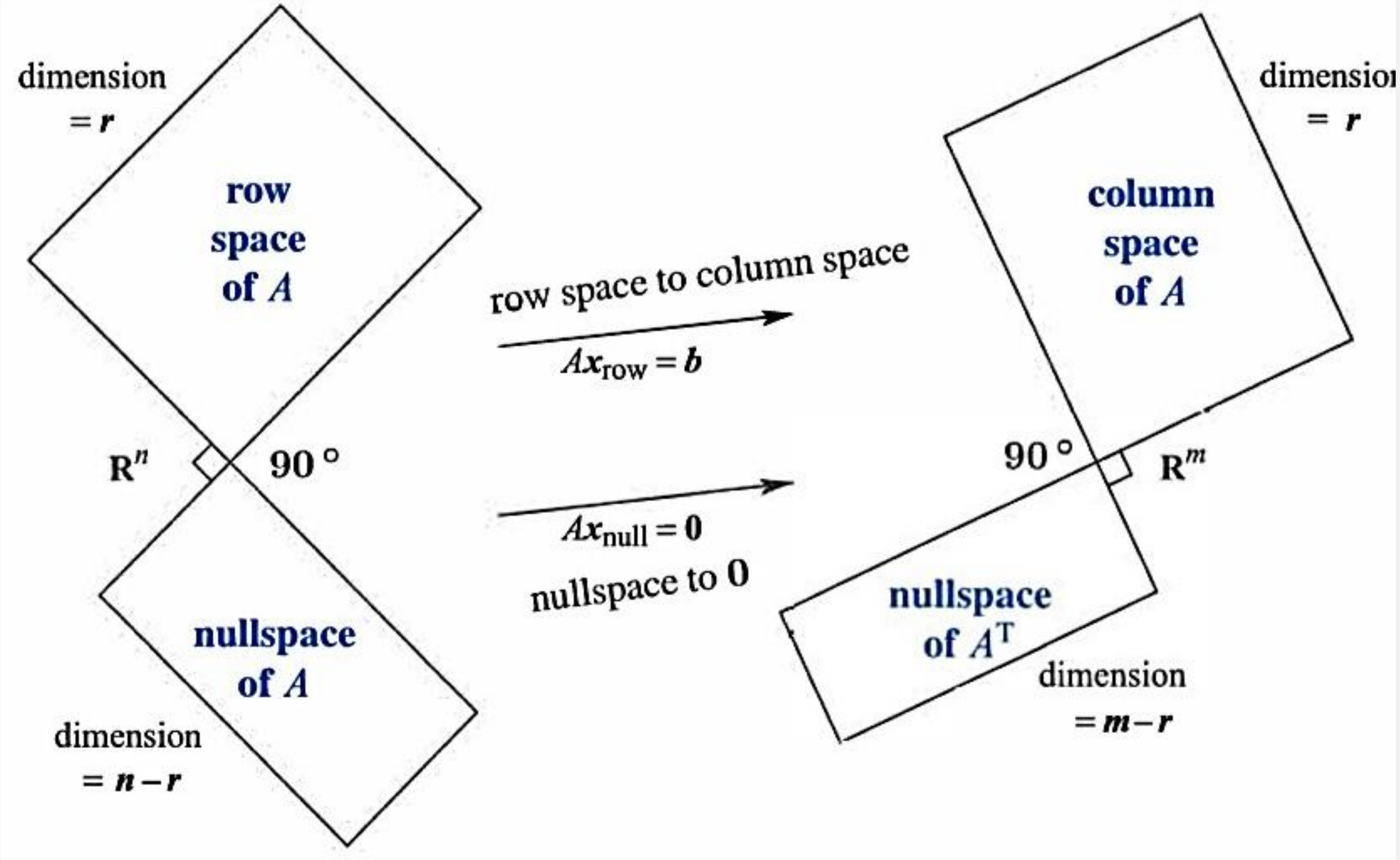
1. The row space is $C(A^{T})$ a subspace of R^{n}.
   Dimension r

2. The column space is $C(A)$ a subspace of R^{m}.
   Dimension r

3. The null space is $N(A)$ a subspace of R^{n}.
   Dimension $n - r$

4. The left null space is $N(A^{T})$ a subspace of R^{m}.
   Dimension $m - r$

TWO PAIRS OF ORTHOGONAL SUBSPACES



$N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in $\mathbf{R}^n$).

$N(A^T)$ is the orthogonal complement of the column space $C(A)$ (in $\mathbf{R}^m$).

ORTHOGONAL VECTORS

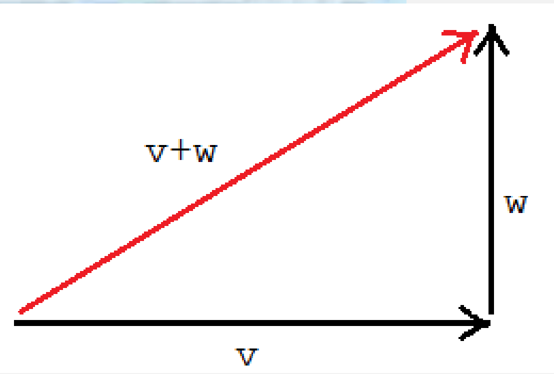
How to come to know that vectors are orthogonal?

Orthogonal vectors $\quad v^{\mathrm{T}}w = 0 \quad$ and $\quad \|v\|^2 + \|w\|^2 = \|v+w\|^2.$

$$v = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad w = \begin{bmatrix}2\\-1\\0\end{bmatrix}, \quad v+w = \begin{bmatrix}3\\1\\3\end{bmatrix}$$

$$||v||^2 = 14, ||w||^2 = 5\ , \ ||v+w||^2 = 19$$

$$vTw = [1\ \ 2\ \ 3]\begin{bmatrix}2\\-1\\0\end{bmatrix} = 2 - 2 + 0 = 0$$



$$v = \begin{bmatrix}1\\3\\5\end{bmatrix}, \quad w = \begin{bmatrix}3\\-1\\2\end{bmatrix}, \quad v+w = \begin{bmatrix}4\\2\\7\end{bmatrix}$$

$$||v||^2 = 35, ||w||^2 = 14\ , \ ||v+w||^2 = 69$$

$$v^Tw = [1\ \ 3\ \ 5]\begin{bmatrix}3\\-1\\2\end{bmatrix} = 3 - 3 + 10 = 10$$

PROJECTIONS

Projecting b onto a with error $e = b - \hat{x}a$

$$a \cdot (b - \hat{x}a) = 0 \quad \text{or} \quad a \cdot b - \hat{x}a \cdot a = 0$$

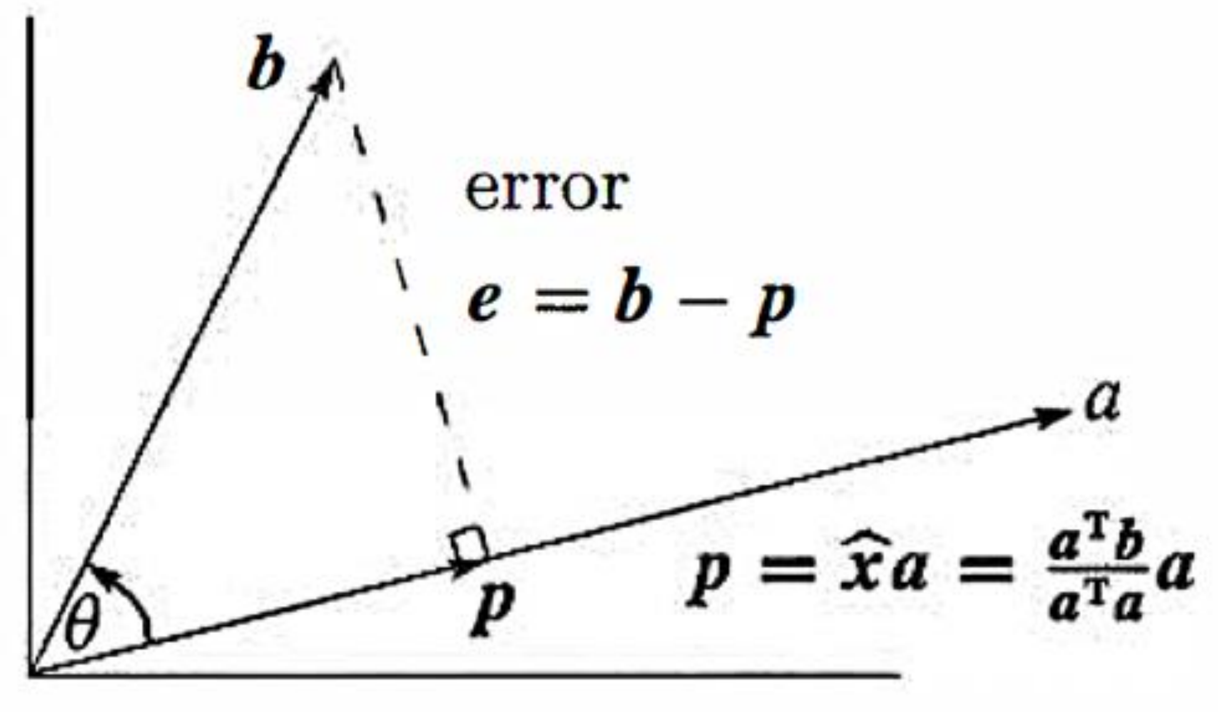
$$\hat{x} = \frac{a \cdot b}{a \cdot a} = \frac{a^{\mathrm{T}} b}{a^{\mathrm{T}} a}.$$

- **p** is some multiple of **a**.
  i.e., **p = xa**
- Since **a** is perpendicular to **e**,
  $\mathbf{a^T(b - xa) = 0}$

➔ $\mathbf{xa^Ta = a^Tb}$

➔ $\mathbf{x = a^Tb/a^Ta}$

- What if b is doubled?
  p is doubled.
- What if a is doubled?
  p remains the same



PROJECTIONS

Projecting $\boldsymbol{b}$ onto $\boldsymbol{a}$ with error $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$

$\boldsymbol{a} \cdot (\boldsymbol{b} - \widehat{x}\boldsymbol{a}) = 0 \quad \text{or} \quad \boldsymbol{a} \cdot \boldsymbol{b} - \widehat{x}\boldsymbol{a} \cdot \boldsymbol{a} = 0$

$$\widehat{x} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$

The projection of b onto the line through a is the vector $\boldsymbol{p} = \widehat{x}\boldsymbol{a} = \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}.$

Special case 1: If $\boldsymbol{b} = \boldsymbol{a}$ then $\widehat{x} = 1$. The projection of $\boldsymbol{a}$ onto $\boldsymbol{a}$ is itself. $P\boldsymbol{a} = \boldsymbol{a}$.

Special case 2: If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$. The projection is $\boldsymbol{p} = \boldsymbol{0}$.

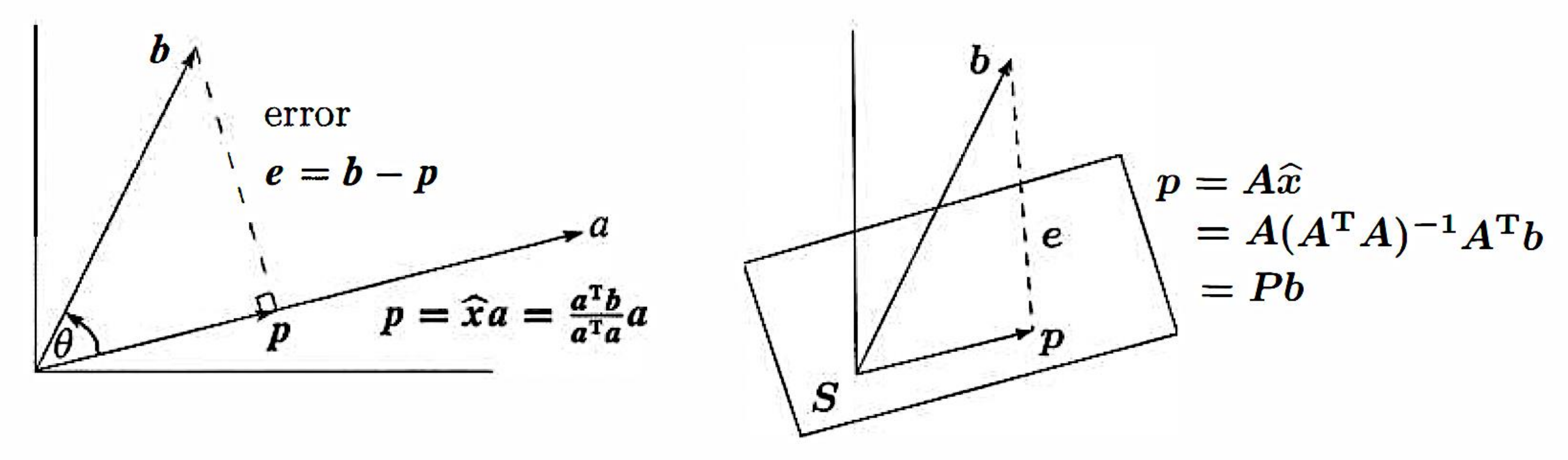


Figure: The projection p of b onto a line and onto S = Column Space of A

Projection matrix P $p = a\hat{x} = a\dfrac{a^Tb}{a^Ta} = Pb$ ***when the matrix is*** $P = \dfrac{aa^T}{a^Ta}$.

PROPERTIES OF THE PROJECTION MATRIX

- $p = Pb$
  - where P is a matrix and b is a vector.
- Column Space of the matrix is
  - $C(P)$ = line through **a**
- Rank of P?
  - is 1.
  - Column times a row.
- Is the matrix symmetric?
  - Yes because $P^T = P$
- What happens if do the projection twice?
  - We get the same point.
  - $P^2 = P$

PROPERTIES OF THE PROJECTION MATRIX

1 The projection of a vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is the closest point $\boldsymbol{p} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} \,/\, \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$.

2 The error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$: Right triangle $\boldsymbol{b}\,\boldsymbol{p}\,\boldsymbol{e}$ has $||\boldsymbol{p}||^2 + ||\boldsymbol{e}||^2 = ||\boldsymbol{b}||^2$.

3 The **projection** of $\boldsymbol{b}$ onto a subspace $\boldsymbol{S}$ is the closest vector $\boldsymbol{p}$ in $\boldsymbol{S}$; $\boldsymbol{b} - \boldsymbol{p}$ is orthogonal to $\boldsymbol{S}$.

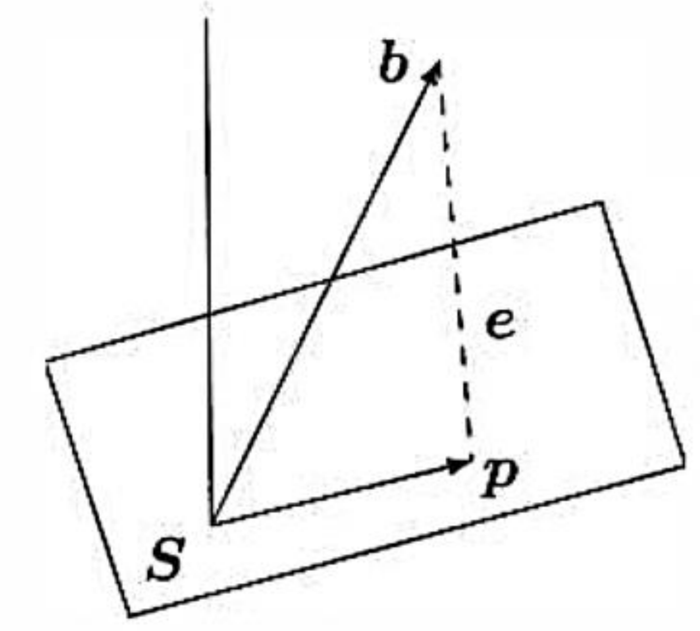
4 $A^{\mathrm{T}}A$ is invertible (and symmetric) only if A has independent columns : $\boldsymbol{N}(A^{\mathrm{T}}A) = \boldsymbol{N}(A)$.

5 Then the projection of $\boldsymbol{b}$ onto the column space of A is the vector $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

6 The **projection matrix** onto $C(A)$ is $\boxed{P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.}$ It has $\boldsymbol{p} = P\boldsymbol{b}$ and $P^2 = P = P^{\mathrm{T}}$.

WHY PROJECTION?

- Ax = b may have no solution.
  - More equations than unknowns
  - Can't be solved
  - Ax is in the column space and b is probably not.
  - Solve the closest problem that can be solved.
  - Instead of solving Ax = b, solve $A\hat{x}$ = P (projection of b onto column space).



EXAMPLE

(a). Find the projection matrix P onto a line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{a^Ta}\, aa^T = \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

$$P = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$ Projection Matrix

EXAMPLE

b) Find projection vectors for given vectors b, c and d. Also find error vector e in each case.

$$b = (1,1,1); \qquad c = (-1,4,8)\,; \qquad d = (2,-3,4)$$

$$p = Pb = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$p = \begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix}\;;\; e = b - p = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}5/9\\10/9\\10/9\end{bmatrix} = \begin{bmatrix}4/9\\-1/9\\-1/9\end{bmatrix}$$

(2)

$$\mathrm{þ}_c = Pc = 1/9 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix} = 1/9 \begin{bmatrix} -1 + 2(4) + 2(8) \\ 2(-1) + 4(4) + 4(8) \\ 2(-1) + 4(4) + 4(8) \end{bmatrix} = \begin{bmatrix} 23/9 \\ 46/9 \\ 46/9 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{c} - \mathrm{þ}_c = \begin{bmatrix} -1 \\ 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 23/9 \\ 46/9 \\ 46/9 \end{bmatrix} = \begin{bmatrix} -32/9 \\ -10/9 \\ -26/9 \end{bmatrix}$$

(3)

$$\mathrm{þ}_d = P_d = 1/9 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 1/9 \begin{bmatrix} 2 + 2(-3) + 2(4) \\ 2(2) + 4(-3) + 4(4) \\ 2(2) + 4(-3) + 4(4) \end{bmatrix} = \begin{bmatrix} 4/9 \\ 8/9 \\ 8/9 \end{bmatrix}$$

$$\mathbf{e} = \mathbf{d} - \mathrm{þ}_d = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4/9 \\ 8/9 \\ 8/9 \end{bmatrix} = \begin{bmatrix} 14/9 \\ -35/9 \\ 28/9 \end{bmatrix}$$

PROJECTION ONTO A SUBSPACE

The combination $\boldsymbol{p} = \widehat{x}_1\boldsymbol{a}_1 + \cdots + \widehat{x}_n\boldsymbol{a}_n = A\widehat{\boldsymbol{x}}$ that is closest to $\boldsymbol{b}$ comes from $\widehat{\boldsymbol{x}}$:

Find $\widehat{\boldsymbol{x}}$ $(n \times 1)$ $\qquad \boxed{A^{\mathrm{T}}(\boldsymbol{b} - A\widehat{\boldsymbol{x}}) = \mathbf{0} \quad \text{or} \quad A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}.}$ $\qquad$ (5)

This symmetric matrix $A^{\mathrm{T}}A$ is n by n. It is invertible if the $\boldsymbol{a}$'s are independent. The solution is $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$. The ***projection*** of $\boldsymbol{b}$ onto the subspace is $\boldsymbol{p}$:

Find $\boldsymbol{p}$ $(m \times 1)$ $\qquad \boxed{\boldsymbol{p} = A\widehat{\boldsymbol{x}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}.}$ $\qquad$ (6)

The next formula picks out the ***projection matrix*** that is multiplying $\boldsymbol{b}$ in (6):

Find P $(m \times m)$ $\qquad \boxed{P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}.}$ $\qquad$ (7)

Compare with projection onto a line, when A has only one column: $A^{\mathrm{T}}A$ is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

For $n = 1$ $\qquad \boxed{\widehat{x} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}}$ **and** $\boxed{\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}}$ **and** $\boxed{P = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.}$

EXAMPLE

If A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$, find $\hat{x}, p$ *and* P.

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A\hat{x} = b$$

$$A^T A\hat{x} = A^T b$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & 6 \\ 0 & 2 & -6 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$p = Pb = A(A^T A)^{-1} A^T b$$

$$P = A(A^T A)^{-1} A^T$$

$$(A^T A)^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}^{-1}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix} \text{ (Projection Matrix)}$$