Department of Computer and Software Engineering
Mid-Term Examination
(Spring 2024)

Course Name	Discrete Structures	Course Code	SE103T
Time Allowed	120 mins	Total Marks	30
Student Name		Student Roll No	

CLOs:
1. Analyze mathematical arguments using propositional logic and rules of inference.
2. Apply set operations build sequences and compute summations.
3. Solve various computing problem using combinatorics, graphs and trees.

Instructions:

1. This Mid-term will assess your CLO-1 and CLO-2 as per OBE.
2. Use of pencils is not allowed to answer the questions.
3. Total time is the maximum time allowed to solve the paper.
4. It is a closed book and closed notes exam.
5. All questions are required to be solved with detailed explanation and reasoning to get full marks.
6. Anybody found cheating or helping any other in cheating will get their paper cancelled.

Marks Distribution								
Q1/4 CLO-1	Q2/6 CLO-1	Q3/5 CLO-1	Q4/5 CLO-1	Q5/2 CLO-1	Q6/2 CLO-1	Q7/2 CLO-1	Q8/4 CLO-2	Total/30

Instructor Name: Abdul Basit **Signature:** ______________________

Q1: Let p and q be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of the following compound propositions as an English sentence. [0.5 × 8]

a) $\neg p$

b) $p \lor q$

c) $\neg p \land q$

d) $q \rightarrow p$

e) $\neg q \rightarrow \neg p$

f) $\neg p \rightarrow \neg q$

g) $p \leftrightarrow q$

h) $\neg q \lor (\neg p \land q)$

Solution:

10. **a)** The election is not decided.

b) The election is decided, or the votes have been counted.

c) The election is not decided, and the votes have been counted.

d) If the votes have been counted, then the election is decided.

e) If the votes have not been counted, then the election is not decided.

f) If the election is not decided, then the votes have not been counted.

g) The election is decided if and only if the votes have been counted.

h) Either the votes have not been counted, or else the election is not decided and the votes have been counted. Note that we were able to incorporate the parentheses by using the words *either* and *else*.

Q2: Construct a truth table for each of the following compound propositions. [1 × 6]

a) $\mathbf{p \oplus p}$

b) $\mathbf{p \oplus \neg p}$

c) $\mathbf{p \oplus \neg q}$

d) $\mathbf{\neg p \oplus \neg q}$

e) $\mathbf{(p \oplus q) \vee (p \oplus \neg q)}$

f) $\mathbf{(p \oplus q) \wedge (p \oplus \neg q)}$

Solution:

34. For parts **(a)** and **(b)** we have the following table (column two for part **(a)**, column four for part **(b)**).

p	$p \oplus p$	$\neg p$	$p \oplus \neg p$
T	F	F	T
F	F	T	T

For parts **(c)** and **(d)** we have the following table (columns five and six).

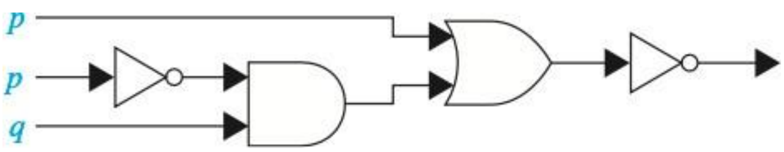
p	q	$\neg p$	$\neg q$	$p \oplus \neg q$	$\neg p \oplus \neg q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

For parts **(e)** and **(f)** we have the following table (columns five and six). This time we have omitted the column explicitly showing the negation of q. Note that the first is a tautology and the second is a contradiction (see definitions in Section 1.3).

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \vee (p \oplus \neg q)$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	T	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	T	T	F

Q3: [1 + 2 + 2]

a) **Find the output of the following combinatorial circuit.**

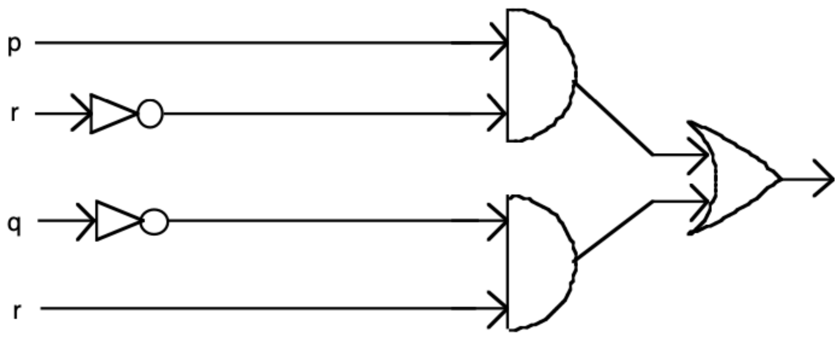


b) **Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output (p ∧ ¬ r) ∨ (¬ q ∧ r) from input bits p, q, and r.**

Solution:

40. **a)** Each of p and q is negated and fed to the OR gate. Therefore the output is $(\neg p) \vee (\neg q)$.
b) $\neg(p \vee ((\neg p) \wedge q))$

42. We have the inputs come in from the left, in some cases passing through an inverter to form their negations. Certain pairs of them enter AND gates, and the outputs of these enter the final OR gate.



c) Show that $[(p \lor q) \land (p \rightarrow r) \land (q \rightarrow r)] \rightarrow r$ is a tautology using truth table.

Solution:

p	q	r	$(p \lor q) \land (p \rightarrow r) \land (p \rightarrow r)$	$[(p \lor q) \land (p \rightarrow r) \land (p \rightarrow r)] \rightarrow r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Q4: Let C(x) be the statement "x has a cat," let D(x) be the statement "x has a dog," and let F(x) be the statement "x has a ferret." Express each of these statements in terms of C(x), D(x), F(x), quantifiers, and logical connectives. Let the domain consist of all students in your class. [1 × 5]

a) A student in your class has a cat, a dog, and a ferret.

b) **All students in your class have a cat, a dog, or a ferret.**

c) **Some student in your class has a cat and a ferret, but not a dog.**

d) **No student in your class has a cat, a dog, and a ferret.**

e) **For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.**

Solution:

10. a) We assume that this means that one student has all three animals: $\exists x(C(x) \wedge D(x) \wedge F(x))$.

b) $\forall x(C(x) \vee D(x) \vee F(x))$ c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

d) This is the negation of part (a): $\neg\exists x(C(x) \wedge D(x) \wedge F(x))$.

e) Here the owners of these pets can be different: $(\exists x\, C(x)) \wedge (\exists x\, D(x)) \wedge (\exists x\, F(x))$. There is no harm in using the same dummy variable, but this could also be written, for example, as $(\exists x\, C(x)) \wedge (\exists y\, D(y)) \wedge (\exists z\, F(z))$.

Q5: [1 + 1]

a) **Identify the error or errors in this argument that supposedly shows that if ∀x(P(x) ∨ Q(x)) is true then ∀xP(x) ∨ ∀xQ(x) is true.**

1.	**∀x(P (x) ∨ Q(x))**	**Premise**
2.	**P(c) ∨ Q(c)**	**Universal instantiation from (1)**
3.	**P(c)**	**Simplification from (2)**
4.	**∀xP(x)**	**Universal generalization from (3)**
5.	**Q(c)**	**Simplification from (2)**
6.	**∀xQ(x)**	**Universal generalization from (5)**
7.	**∀x(P (x) ∨ ∀xQ(x))**	**Conjunction from (4) and (6)**

Solution:

24. Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

b) **Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy."**

Solution:

30. Let a be "Allen is a good boy"; let h be "Hillary is a good girl"; let d be "David is happy." Then our assumptions are $\neg a \vee h$ and $a \vee d$. Using resolution gives us $h \vee d$, as desired.

Q6: Prove that if n is an integer and 3n + 2 is even, then n is even using: [1 + 1]

a) **a proof by contraposition.**

b) **a proof by contradiction.**

Solution:

18. a) We must prove the contrapositive: If n is odd, then $3n+2$ is odd. Assume that n is odd. Then we can write $n = 2k+1$ for some integer k. Then $3n+2 = 3(2k+1)+2 = 6k+5 = 2(3k+2)+1$. Thus $3n+2$ is two times some integer plus 1, so it is odd.

b) Suppose that $3n+2$ is even and that n is odd. Since $3n+2$ is even, so is $3n$. If we add subtract an odd number from an even number, we get an odd number, so $3n - n = 2n$ is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

Q7: [1 + 1]

a) **Use a proof by cases to show that min(a, min(b, c)) = min(min(a, b), c) whenever a, b, and c are real numbers.**

Solution:

4. There are three main cases, depending on which of the three numbers is smallest. If a is smallest (or tied for smallest), then clearly $a \leq \min(b, c)$, and so the left-hand side equals a. On the other hand, for the right-hand side we have $\min(a, c) = a$ as well. In the second case, b is smallest (or tied for smallest). The same reasoning shows us that the right-hand side equals b; and the left-hand side is $\min(a, b) = b$ as well. In the final case, in which c is smallest (or tied for smallest), the left-hand side is $\min(a, c) = c$, whereas the right-hand side is clearly also c. Since one of the three has to be smallest we have taken care of all the cases.

b) **Prove using the notion of without loss of generality that 5x + 5y is an odd integer when x and y are integers of opposite parity.**

Solution:

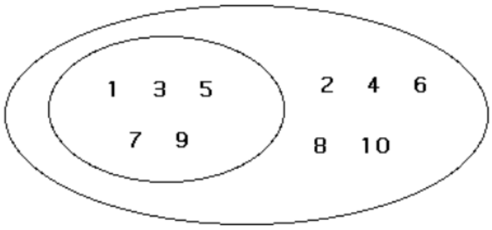
6. Because x and y are of opposite parities, we can assume, without loss of generality, that x is even and y is odd. This tells us that $x = 2m$ for some integer m and $y = 2n + 1$ for some integer n. Then $5x + 5y = 5(2m) + 5(2n + 1) = 10m + 10n + 1 = 10(m + n) + 1 = 2 \cdot 5(m + n) + 1$, which satisfies the definition of being an odd number.

Q8: [1 + 1 + 1 + 1]

a) **Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.**

Solution:

12. The numbers 1, 3, 5, 7, and 9 form a subset of the set of all ten positive integers under discussion, as shown here.



b) **Find A^3 when A = {0, a}.**

Solution:

34. Recall that A^3 consists of all the ordered triples (x, y, z) of elements of A.

a) $\{(a, a, a)\}$ **b)** $\{(0,0,0), (0,0,a), (0,a,0), (0,a,a), (a,0,0), (a,0,a), (a,a,0), (a,a,a)\}$

c) **Show that if A and B are finite sets, then A ∪ B is a finite set.**

Solution:

44. A finite set is a set with k elements for some natural number k. Suppose that A has n elements and B has m elements. Then the number of elements in $A \cup B$ is at most $n + m$ (it might be less because $A \cap B$ might be nonempty). Therefore by definition, $A \cup B$ is finite.

d) Find the domain and range of the function that assigns to each positive integer its largest decimal digit.

Solution:

b) Since the largest decimal digit of a strictly positive integer cannot be 0, we have domain $\mathbf{Z}^+$ and range $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

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