BSSE23058

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Assignment # 4
Linear Algebra.

1. a.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad LU = ?$$

$$E_{21} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$E_{21} A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$E_{31} [E_{21} \times A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = E_{31}^{-1} \times E_{21}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

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b. $A = LDU$

$A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \quad L \quad D \quad U$$

Not possible as Diagonal should be non-zero ~~to~~ not containing zero's.

2. a. $A = \begin{bmatrix} 0 & 0 & 6 & 2 & -4 & -8 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 2 & -3 & 1 & 4 & -7 & 1 \\ 6 & -9 & 0 & 11 & -19 & 3 \end{bmatrix}$

$P_{13} \times A =$

$$\begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 6 & 2 & -4 & -8 \\ 6 & -9 & 0 & 11 & -19 & 3 \end{bmatrix}$$

$R4 \leftarrow R4 + (-3)R1$

$R3 \leftarrow R3 + (-2)R2$

$$\begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -1 & 2 & 0 \end{bmatrix}$$

$R4 \leftarrow R4 + R2$

$$P_{34} \times \begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Rank of A = [

$$\begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = # of pivots of A = 3.

c. Null space of A

N(A) = ?

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 2 & -3 & 1 & 4 & -7 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftarrow \frac{R1}{2}$$

$$\begin{bmatrix} 1 & -3/2 & 1/2 & 2 & -7/2 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R3 \leftarrow \frac{R3}{-4}$$

$$\begin{bmatrix} 1 & -3/2 & 1/2 & 2 & -7/2 & 1 \\ 0 & 0 & 3 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R2 \leftarrow \frac{R2}{3}$$

$$\begin{bmatrix} 1 & -3/2 & 1/2 & 2 & -7/2 & 1 \\ 0 & 0 & 1 & 1/3 & -2/3 & -4/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R1 \leftarrow R1 + \left(\frac{-1}{2}\right)R2$$

$$\begin{bmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 7/6 \\ 0 & 0 & 1 & 1/3 & -2/3 & -4/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R1 = R1 + \left(\frac{7}{6}\right)R3$$

11 C
12 A
13 A
14 B

$$\begin{bmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 \\ 0 & 0 & 1 & +1/3 & -2/3 & -4/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R2 \leftarrow R2 + \left(\tfrac{4}{3}\right) R3$$

$$\begin{array}{c} \begin{bmatrix} 1 & -3/2 & 0 & 11/6 & -19/6 & 0 \\ 0 & 0 & 1 & 1/3 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccccc} P & F & P & F & F & P \end{array} \end{array} \quad \text{RREF form}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad F = \begin{bmatrix} -3/2 & 11/6 & -19/6 \\ 0 & 1/3 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$N(A) = x \begin{bmatrix} -1\times -3/2 \\ 1 \\ -1\times 0 \\ 0 \\ 0 \\ -1\times 0 \end{bmatrix} + y \begin{bmatrix} -1\times 11/6 \\ 0 \\ -1\times 1/3 \\ 1 \\ 0 \\ -1\times 0 \end{bmatrix} + z \begin{bmatrix} -1\times -19/6 \\ 0 \\ -1\times -2/3 \\ 0 \\ 1 \\ -1\times 0 \end{bmatrix}$$

$$N(A) = x \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -11/6 \\ 0 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 19/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Q6 $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$

$Ax = b$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & 2 & 3 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2 + (-1)R1 \\ R3 \leftarrow R3 + (-2)R1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow \frac{R2}{2} \\ R3 \leftarrow \frac{R3}{3} \end{array}$$

$Ux = c.$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 1/2 & -3/2 & 3/2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} R1 \leftarrow R1 + (-3)R3 \\ R2 \leftarrow R2 + \left(\frac{3}{2}\right)R3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 1/2 & 0 & 9/2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad Rx = d.$$

P P F P

$x_1 + 0x_2 + 2x_3 + 0x_4 = -4$

$x_1 + 2x_3 = -4$

$x_1 = -4 - 2x_3$

$0x_1 + 1x_2 + \frac{1}{2}x_3 + 0x_4 = \frac{9}{2}$

$x_2 + \frac{1}{2}x_3 = \frac{9}{2}$

$x_2 = \frac{9}{2} - \frac{1}{2}x_3$

$x_4 = 2$

$$x_p = \begin{bmatrix} -4 - 2x_3 \\ 9/2 - 1/2 x_3 \\ x_3 \\ 2 \end{bmatrix}$$

Let $x_3 = 0$ as free variable.

$$x_p = \begin{bmatrix} -4 \\ 9/2 \\ 0 \\ 2 \end{bmatrix}.$$

$Ax = 0$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 2 \\ 1/2 \\ 0 \end{bmatrix}.$$

$$x_n = \begin{array}{c} \begin{bmatrix} 2 \times -1 \\ \frac{1}{2} \times -1 \\ 1 \\ 0 \times -1 \end{bmatrix} \end{array} \begin{array}{c} f \\ p \\ f \\ p \end{array} = \begin{bmatrix} -2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$x_c = x_p + x_n$$

$$= \begin{bmatrix} -4 \\ 9/2 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

Q8

a. v_1, v_2, v_3 — independent

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \leftarrow R1 + (-1) R2$
$R2 \leftarrow R2 + (-1) R3$

$Ax = 0$.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccc} p & p & p \end{array} \end{array}.$$

$Ax = 0$

The vectors v_1, v_2, v_3 are independent as ~~at~~ no free column is there and ~~at~~ no non-zero solution $Ax = 0$.

b. v_1, v_2, v_3, v_4 - dependent vectors

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$

$A x \neq 0$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]$$

$R1 \leftarrow R1 + (-1)R2$

$R2 \leftarrow R2 + (-1)R3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right] \qquad F = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$$

P P P F

3 pivot colums + 1 free colums.

$$x_n = \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix} \end{array} \begin{array}{c} P \\ P \\ P \\ F \end{array}$$

Not all colums are pivots and some null space is non-zero hence the 4 vectors are dependent vectors.

b. $v_1 = w_2 + w_3$, $v_2 = w_1 + w_3$, $v_3 = w_1 + w_2$.

$c_1v_1 + c_2v_2 + c_3v_3 + \cancel{c_4v_4} = 0$

$c_1(w_2+w_3) + c_2(w_1+w_3) + c_3(w_1+w_2) = 0$

$c_1w_2 + c_1w_3 + c_2w_1 + c_2w_3 + c_3w_1 + c_3w_2 = 0$

$w_2(c_1 + c_3) + w_3(c_1 + c_2) + w_1(c_2 + c_3) = 0$

since w_1, w_2, w_3 are linearly independent there linear combination equals to zero.

$c_1 + c_3 = 0$ $\qquad c_1 = -c_3$ $\qquad -c_3 - c_3 = 0$

$c_1 + c_2 = 0$ $\qquad c_2 = -c_1$ $\qquad -2c_3 = 0$

$c_2 + c_3 = 0$ $\qquad c_3 = -c_2$ $\qquad c_3 = 0.$

$c_1 = c_2 = c_3 = 0.$

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Q5 a. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$ $R2 \leftarrow R2 + (-1)R1$
$R3 \leftarrow R3 + (-1)R1$

$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -3 \end{bmatrix}$ $R3 \leftarrow R3 + \left(\frac{-3}{2}\right)R2$

$= \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ Rank = 2.
$\quad\ p \quad p$

Matrix A is not invertible as there are linearly dependent columns. $2(C1) + C2 = C3$ also Rank ≠ is equal to. to rows & columns.

$B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ $R2 \leftarrow R2 + (-1)R1$
$R3 \leftarrow R3 + (-1)R1$

$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & -3 & 1 \end{bmatrix}$ $R3 \leftarrow R3 + \left(\frac{-3}{2}\right)R2$

$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & -7/2 \end{bmatrix}$ Rank = 3
$\quad p \quad p \quad p$

Matrix B is invertible as all rows & columns are independence.

$[B|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$ $R2 \leftarrow R2 + (-1)R1$
$R3 \leftarrow R3 + (-1)R1$

$= \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 3 & -1 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right]$ $R2 \leftarrow \frac{R2}{-2}$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1/2 & -1/2 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{array}\right] \quad R3 \leftarrow R3 + (+3)R2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -7/2 & 1/2 & -3/2 & 1 \end{array}\right] \quad R3 \leftarrow R3 \times \frac{2}{-7}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/7 & 3/7 & -2/7 \end{array}\right] \quad R2 \leftarrow R2 + \left(\frac{+3}{2}\right)R3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/7 & 1/7 & -3/7 \\ 0 & 0 & 1 & -1/7 & 3/7 & -2/7 \end{array}\right] \quad R1 \leftarrow R1 + (1)R3$$

P

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 6/7 & 3/7 & -2/7 \\ 0 & 1 & 0 & 2/7 & 1/7 & -3/7 \\ 0 & 0 & 1 & -1/7 & 3/7 & -2/7 \end{array}\right] \quad R1 \leftarrow R1 + (-1)R2$$

$$B^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4/7 & 2/7 & 1/7 \\ 0 & 1 & 0 & 2/7 & 1/7 & -3/7 \\ 0 & 0 & 1 & -1/7 & 3/7 & -2/7 \end{array}\right]$$

b. Matrix A rank = 2 and it is not invertible. for $Ax=b$ to have many solutions vector b must be in the column space of A. The system will have infinitely many solutions because the null space has dimension 1. $N(A) = n-r = 3-2 = 1$.

$Ax = b$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 1 & -1 & 1 & b_2 \\ 1 & -2 & 0 & b_3 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2 + (-1)R1 \\ R3 \leftarrow R3 + (-1)R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & -2 & -2 & b_2-b_1 \\ 0 & -3 & -3 & b_3-b_1 \end{array}\right] \quad R3 \leftarrow R3 + \left(-\tfrac{3}{2}\right)R2.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & -2 & -2 & b_2-b_1 \\ 0 & 0 & 0 & b_3-\frac{5}{2}b_1-\frac{3}{2}b_2 \end{array}\right]$$

p p

$b_3 - \frac{5}{2}b_1 - \frac{3}{2}b_2 = 0.$

$b_3 = \frac{5}{2}b_1 + \frac{3}{2}b_2$ } this should be the condition that cannot be violated to get multiple solutions.

eg:- $b = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$

c. $Bx = b$ can not have many solutions as Rank = 3 which means null space of $Bx = 0$ that contains only zero vector. Therefore $Bx=b$ has only one unique solution which is the ~~non~~-zero null space.

Q4

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

p p p

$x_3 = b_3$ | $x_3 = 0$

$x_2 + x_3 = b_2$ | $x_2 = 1$

$x_1 + x_2 + x_3 = b_1$ | $x_1 = 1$

Consider $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

All the colums of this upper triangular matrix are pivots. Rank = 3 that is equal to the no. of rows & columns also diagonals element are non-zero. This is a independent Matrix. all colums & rows are independenet. Hence the system has a solution.

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

P P

Rank = 2 since only two columns are linearly independent and 1 is a free column. This system does not have a unique solution ~~or~~ 1 because free columns can be set leading to multiple solutions $Ax=b$.

3. a.

$$x+2y+z=b$$
$$2x+y+2z=2$$
$$3x+3y+az=3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 2 & 1 & 2 & 2 \\ 3 & 3 & a & 3 \end{array}\right] \quad \begin{array}{l} R2 \leftarrow R2+(-2)R1 \\ R3 \leftarrow R3+(-3)R1. \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & -3 & 0 & 2-b \\ 0 & -3 & a-3 & 3-3b \end{array}\right] \quad R3 \leftarrow R3+R2.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & -3 & 0 & 2-b \\ 0 & 0 & a-3 & 5-4b \end{array}\right] \quad R2 \leftarrow \frac{R2}{-3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b \\ 0 & 1 & 0 & 2-b/3 \\ 0 & 0 & a-3 & 5-4b \end{array}\right] \quad R1 \leftarrow R1+(-2)R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{b+4}{3} \\ 0 & 1 & 0 & \frac{2-b}{-3} \\ 0 & 0 & a-3 & 5-4b \end{array}\right]$$

P P P.

$$x+z = \frac{b+4}{3}$$
$$y = \frac{2-b}{-3}$$
$$(a-3)z = 5-4b$$
$$z = \frac{5-4b}{a-3}.$$

For system to have a unique solution, the matrix should have three pivots = Rank = 3, no free variables. Thats why z and y cannot be zero. In order for them not to be zero. $z = \frac{5-4b}{a-3}$ $a-3 \neq 0$ & $a \neq 3$ and $b \neq \frac{3}{2}$

for y to be not zero, if z & y are ~~not~~ zero that will

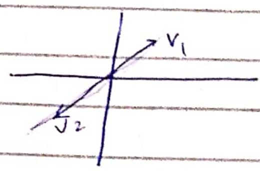
b. result in free columns, not resulting in unique solution for the system.

b. We have no solution when $a = 3$ and $b \neq \frac{3}{2}$.
if $a - 3 = 0$ if $\frac{2-b}{-3} = 0$.

~~In this case only one pivot would be there and rest zero leading to inconsistency~~
There will be three columns but no valid solution.

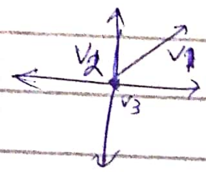
c. When $a = 3$ & $b = \frac{3}{2}$ the system will have infinite number of solutions, as there will be 2 free columns leading to infinite no. of solutions.

Q7 a. $V_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $V_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$



Both these vectors are ~~depen~~ linearly dependent on each other $V_1 = -1 \cdot V_2$.
Hence there subspace is that they span a line in R^3.

b. $V_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$



These vectors (V_1, V_2) are linearly independent, they span a plane in R^3. ~~The Null vector is dependent.~~

c. Whole Number Components Vectors would mean with number 0, 1, 2, -1, -2. but if I multiply with a fraction or decimal or add it with that decimal or fraction, it does not come in the given subspace and it clearly does not fulfil the rules of subspace. that set of vectors span a space if there linear combination fill the space.
Hence whole numbe vectors does not form a vector subspace of $\mathbb{R}^3$

d. Vectors with positive components would mean only positive numbers in the vectors. but if I multiply it with a negative scalar number or add or subtract with a negative number; It does not come in the given subspace and it clearly does not fulfill the rules of subspace that states that set of vectors span a space if there linear combination fill the space. Hence vectors with positive components does not form a vector subspace of $\mathbb{R}^3$.

Q9 a.

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix} \quad \begin{array}{l} R2 \leftarrow R2 + \left(-\frac{3}{2}\right)R1 \\ R3 \leftarrow R3 + \left(-\frac{1}{2}\right)R1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 13/2 & 7 & -5/2 \\ 0 & 9/2 & 2 & 13/2 \end{bmatrix} \quad R3 \leftarrow R3 + \left(-9/13\right)R2$$

$$\begin{array}{c} \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 13/2 & 7 & -5/2 \\ 0 & 0 & -37/13 & 107/13 \end{bmatrix} \\ \begin{array}{cccc} P & P & P & \end{array} \end{array} \quad \text{Rank} = 3$$

C1, C2, C3 are linearly independent column

b. Basis of $C(A) = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \right\}$

Dimension of $C(A) = 3$

c. Basis of $N(A) = \left\{ \begin{bmatrix} -\frac{69}{37} \\ -\frac{101}{37} \\ 107/37 \\ 1 \end{bmatrix} \right\}$

$$\text{RREF} = \begin{array}{c} \begin{array}{cccc} r & r & r & F \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{69}{37} & 0 \\ 0 & 1 & 0 & \frac{101}{37} & 0 \\ 0 & 0 & 1 & -\frac{107}{37} & 0 \end{array}\right] \end{array}$$

Dimension of $N(A) = n - r$

$= 4 - 3$

$= 1$

d. Basis of $C(A^T) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \\ 7 \end{bmatrix} \right\}$

Dimension of $C(A^T) = r = 3$

$$A^T = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 5 & 4 \\ 0 & 7 & 2 \\ 1 & -1 & 7 \end{bmatrix} \begin{matrix} R2 \leftarrow R2 + \left(\frac{1}{2}\right)R1 \\ \\ R4 \leftarrow R4 + \left(-\frac{1}{2}\right)R1 \end{matrix} \sim \begin{bmatrix} 2 & 3 & 1 \\ 0 & 13/2 & 9/2 \\ 0 & 7 & 2 \\ 0 & -5/2 & 13/2 \end{bmatrix}$$

$R3 \leftarrow R3 + \left(-\frac{14}{13}\right)R2$

$R4 \leftarrow R4 + \left(\frac{5}{13}\right)R2$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 13/2 & 9/2 \\ 0 & 0 & -37/13 \\ 0 & 0 & 107/13 \end{bmatrix} R4 \leftarrow R4 + \left(\frac{107}{37}\right)R3 \quad \begin{bmatrix} 2 & 3 & 1 \\ 0 & 13/2 & 9/2 \\ 0 & 0 & -37/13 \\ 0 & 0 & 0 \end{bmatrix}$$

P P P. Rank = 3

e. Basis of left Null space $N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Dimension of $N(A^T) = m - r = 3 - 3$

$= 0.$

RREF, $Ax = 0$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 13/2 & 9/2 \\ 0 & 0 & -37/13 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

P P P

Q10

a. $v_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \\ -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}$ $v_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ -4 & 2 & 3 \\ -3 & -2 & 2 \end{bmatrix}$ $R2 \leftarrow R2 + (-1)R1$
$R3 \leftarrow R3 + (4)R1$
$R4 \leftarrow R4 + (3)R1$

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \\ 0 & 4 & 8 \end{bmatrix}$ $R3 \leftarrow R3 + (5)R2$
$R4 \leftarrow R4 + (2)R2$

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ $R4 \leftarrow R4 + \left(\frac{1}{2}\right)R3$
$R2 \leftarrow \frac{R2}{-2}$ $R3 \leftarrow \frac{R3}{-4}$

$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R2 \leftarrow R2 + \left(-\frac{3}{2}\right)R3$
$R1 \leftarrow R1 + (-2)R3$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R1 \leftarrow R1 + (-2)R2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
$\quad P \quad P \quad P$

Basis = $\left\{ \begin{bmatrix} 1 \\ 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right\}$

11 C
12 A
13 A
14 B

b. $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix}$ $v_4 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad R2 \leftarrow R2 + (-1)R1$$

$$P_{32} \times \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad R4 \leftarrow R4 + (-1)R2$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{Rank} = 4 \sim$$

P P P P

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 3 \end{bmatrix} \right\}.$$