BSSE 23058
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Section A.

"Assigment # 2.
Calculus and Analytical Geometry"

Q1

a. $V(t) = \frac{t+1}{t+4}$

Quotient rule by definition $f'\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{h\,v(x+h)\,v(x)}$

$V'(t) = \lim_{h\to 0} \frac{(t+h+1)(t+4) - (t+1)(t+h+4)}{h(t+h+4)(t+4)}$

$= \lim_{h\to 0} \frac{t^2+4t+ht+4h+t+4-(t^2+th+4t+t+h+4)}{h(t^2+4t+ht+4h+4t+16)}$

$= \lim_{h\to 0} \frac{t^2+5t+ht+4h+4-t^2-5t-th-h-4}{h(t^2+8t+8h+ht+16)}$

$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}(t^2+8t+8h+ht+16)}$

$= \lim_{h\to 0} \frac{3}{t^2+8t+8(0)+(0)t+16}$

$V'(t) = \frac{3}{t^2+8t+16}$

b. $f(x) = \sqrt{1-9x}$.

$f(x) = (1-9x)^{1/2}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(1-9(x+h))^{1/2} - (1-9x)^{1/2}}{h}$

$= \lim_{h\to 0} \frac{(1-9x-9h)^{1/2} - (1-9x)^{1/2}}{h} \times \frac{(1-9x-9h)^{1/2} + \sqrt{1-9x}}{\sqrt{1-9x-9h} + \sqrt{1-9x}}$

$= \lim_{h\to 0} \frac{(\cancel{1}-\cancel{9x}-9h)-(\cancel{1}-\cancel{9x})}{h(\sqrt{1-9x-9h}+\sqrt{1-9x})}$

$$= \lim_{h\to 0} \frac{-9h}{h(\sqrt{1-9x-9h}+\sqrt{1-9x})}$$

$$= \lim_{h\to 0} \frac{-9}{\sqrt{1-9x-0}+\sqrt{1-9x}}$$

$$= \frac{-9}{\sqrt{1-9x}+\sqrt{1-9x}}$$

$$= \frac{-9}{2\sqrt{1-9x}}$$

dc. $g(x) = x^3 - 2x^2 + x - 1$

$$g'(x) = \lim_{h\to 0} \frac{((x+h)^3 - 2(x+h)^2 + (x+h) - 1) - (x^3 - 2x^2 + x - 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3h^2x + h^3 - \cancel{2x^2} - 4hx - 2h^2 + \cancel{x} + h - \cancel{1} - \cancel{x^3} + \cancel{2x^2} - \cancel{x} + \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3h^2x + h^3 - 4hx - 2h^2 + h}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3hx + h^2 - 4x - 2h + 1$$

$$= \lim_{h\to 0} 3x^2 + 0 + 0 - 4x - 0 + 1$$

$$g'(x) = 3x^2 - 4x + 1$$

d. $R(z) = \frac{5}{z} \Rightarrow 5z^{-1}$

$$R'(z) = \lim_{h\to 0} \frac{5(z+h)^{-1} - 5z^{-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{5}{z+h} - \frac{5}{z}}{h} \Rightarrow \lim_{h\to 0} \frac{\cancel{5z} - \cancel{5z} - 5h}{z^2 + zh}$$

$$= \lim_{h\to 0} \frac{-5h}{hz^2+h^2z}$$

$$= \lim_{h\to 0} \frac{-5(\cancel{h})}{\cancel{h}(z^2+hz)}$$

$$= \lim_{h\to 0} \frac{-5}{z^2+0z}$$

$$R'(x) = \frac{-5}{z^2}$$

Q2

a. $f(x) = x^3 g(x)$; $g(-7) = 2$; $g'(-7) = 9$; $f'(-7) = ?$

$u = x^3$ $\quad u' = 3x^2$ $\quad u'(-7) = 3(-7)^2 = 147$

$v = g(x) =$ $\quad v' = g'(x)$ $\quad v'(-7) = 9$

$v = g(-7) = 2$ $\quad u(-7) = -343$

$f'(x) = u'v + v'u$

$f'(-7) = 147 \times 2 + 9 \times -343$

$= 294 - 3087$

$= -2793$

b. $f(x) = \dfrac{\sqrt{x} + 2x}{7x - 4x^2}$

$$f'(x) = \frac{\left(\frac{1}{2}(x)^{-1/2} + 2\right)(7x - 4x^2) - (\sqrt{x} + 2x)(7 - 8x)}{(7x - 4x^2)^2}$$

$$= \frac{\left(\frac{1}{2\sqrt{x}} + 2\right)(7x - 4x^2) - (\sqrt{x} + 2x)(7 - 8x)}{(7x - 4x^2)^2}$$

$$= \frac{\frac{7\sqrt{x}}{2} - 2\sqrt{x^3} + \cancel{14x} - 8x^2 - 7\sqrt{x} + 8\sqrt{x^3} - \cancel{14x} + 16x}{(7x - 4x^2)^2}$$

$$= \frac{-\frac{7\sqrt{x}}{2} + 16\sqrt{x^3} - 8x^2 + 16x}{(7x - 4x^2)^2}$$

c. $y = (1 + \sqrt{3x^3})(x^{-3} + 2\sqrt[3]{x})$

$$y' = \left(1 + \frac{1}{2} \times 3 \times 3x^2 \frac{1}{\sqrt{3x^3}}\right)(x^{-3} + 2\sqrt[3]{x}) + \left(-3x^{-4} + 2 \times \frac{1}{3}(x)^{-\frac{2}{3}}\right)(1 + \sqrt{3x^3})$$

$$y' = \left(1 + \frac{9}{2\sqrt{3x^3}}\right)(x^{-3} + 2\sqrt[3]{x}) + \left(-3x^{-4} + \frac{2}{3\sqrt[3]{x^2}}\right)(1 + \sqrt{3x^3})$$

Q3

a. $Z(v) = \frac{v + \tan(v)}{1 + \csc(v)}$

$$Z'(v) = \frac{(1 + \sec^2 v)(1 + \operatorname{cosec} v) - (v + \tan(v))(1 - \operatorname{cosec} v \cot v)}{(1 + \operatorname{cosec} v)^2}$$

b. $f(w) = \tan(w)\sec(w)$

$$f'(w) = (\tan(w))'(\sec w) + (\sec w)'(\tan(w))$$

$$= (\sec^2 w)(\sec w) + (\tan w \sec w)\tan w$$

$$= \sec^3 w + \tan^2 w \sec w$$

$$= \sec w(\sec^2 w + \tan^2 w)$$

$$= \sec w(\cancel{\sec^2 w} \; 1 + \tan^2 w + \tan^2 w)$$

$$= \sec w(1 + 2\tan^2 w)$$

c. $f(t) = \frac{1 + 5t}{\ln(t)}$

$$f'(t) = \frac{(5t+1)'(\ln t) - (\ln t)'(1 + 5t)}{[\ln(t)]^2}$$

$$= \frac{(5)(\ln t) - \left(\frac{1}{t}\right)(1+5t)}{(\ln(t))^2}$$

$$= \frac{5\ln(t) - \frac{1+5t}{t}}{(\ln(t))^2}$$

d. d. $h(y) = \frac{y}{1-e^y}$

$$h'(y) = \frac{(1)(1+e^y) - (y)(-e^y)}{(1-e^y)^2}.$$

$$= \frac{1+e^y + ye^{+y}}{(1+e^y)^2}.$$

e. $R(w) = 3^w \log(w)$.

$$R'(w) = (3^w)' \log w + (\log w)' \, 3^w.$$

$$= 3^w \ln(w) \log w + \frac{1}{w \ln 10} 3^w$$

$$= 3^w \ln(w) \log w + \frac{3^w}{w \ln(10)}.$$

i. $f(x) = \tan(x) + 9\cos(x)$ at $x = \pi$ find tangent to line.

$$f'(x) = \sec^2 x + 9(-\sin x)$$

$$= \frac{1}{\cos^2 x} - 9\sin x$$

$$f'(\pi) = \frac{1}{\cos^2(\pi)} - 9\sin(\pi)$$

$$= \frac{1}{(-1)^2} - 0 = 1$$

$f(\pi) = \tan(\pi) + 9\cos(\pi)$
$= -9$ $(\pi, -9)$
$y - (-9) = 1(x - \pi)$
$y = x - \pi - 9.$

f. $h(x) = \dfrac{\sin^{-1}(x)}{1+x}$

$h'(x) = \dfrac{\frac{1}{\sqrt{1-x^2}} \times (1+x) - \sin^{-1}(x)(1)}{(1+x)^2}$

$= \dfrac{\frac{1+x}{\sqrt{1-x^2}} - \sin^{-1}(x)}{(1+x)^2}$

g. $g(t) = \operatorname{cosec}^{-1}(t) - 4\cot^{-1}(t)$

$g'(t) = \cancel{\frac{1}{x\sqrt{x^2-1}}} \quad \frac{1}{t\sqrt{t^2-1}} - 4\left(-\frac{1}{1+t^2}\right)$

$= \dfrac{1}{t\sqrt{t^2-1}} + \dfrac{4}{1+t^2}$

h. $g(z) = \dfrac{z+1}{\tanh(z)}$

$g'(z) = \dfrac{(1)\tanh(z) - (z+1)\operatorname{sech}^2(z)}{\tanh(z)}$

$= \dfrac{\tanh(z) - z\operatorname{sech}^2 z}{\tanh(z)} - \operatorname{sech}^2 z$

Q4

a. $f(x) = \left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$

$f'(x) = -1\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2} \times \left(\sqrt[3]{12x} + \sin^2(3x)\right)'$

$\cancel{= -1\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2} \times \left(\frac{1}{3}(12x)^{-\frac{2}{3}} \times 12^{4} + 3\times 2\sin(3x)\cos(3x)\right)}$

$= \dfrac{-\left(\sqrt[3]{12} \times \frac{1}{3} x^{-\frac{2}{3}} + 3\times 2\sin(3x)\cos(3x)\right)}{\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{+2}}$

$= \dfrac{-\frac{\sqrt[3]{12}}{3x^{\frac{2}{3}}} - 6\sin(3x)\cos(3x)}{\left(\sqrt[3]{12x} + \sin^2(3x)\right)^2}$

b. $g(x) = \left[\ln(x^2+1) - \tan^{-1}(6x)\right]^{10}$

$g'(x) = 10\left(\ln(x^2+1) - \tan^{-1}(6x)\right)^9 \times \left(\frac{1}{x^2+1} \times 2x - \frac{1}{1+(6x)^2} \times 6\right)$

$= 10\left(\ln(x^2+1) - \tan^{-1}(6x)\right)^9 \left(\frac{2x}{x^2+1} - \frac{6}{1+36x^2}\right).$

c. $k(x) = \dfrac{1+e^{-2x}}{x+\tan(12x)}$

$k'(x) = \dfrac{(-2e^{-2x})(x+\tan(12x)) - (1+\sec^2(12x)\times 12)(1+e^{-2x})}{(x+\tan(12x))^2}$

$= \dfrac{(-2e^{-2x})(x+\tan(12x)) - (1+12\sec^2(12x))(1+e^{-2x})}{(x+\tan(12x))^2}.$

a. $\frac{x}{y^3} = 1$ b. $x^2 + y^3 = 4$ c. $x^2 + y^2 = 2$

Q5

1.

a. $y = (x)^{1/3}$

$y' = \frac{1}{3} x^{-\frac{2}{3}}$

$y' = \frac{1}{3\sqrt[3]{x^2}}$

b. $y^3 = 4 - x^2$

$y = (4 - x^2)^{1/3}$

$y' = \frac{1}{3}(4 - x^2)^{-\frac{2}{3}} \times -2x$

$y' = \frac{-2x}{3\sqrt[3]{(4-x^2)^2}}$

c. $y^2 = 2 - x^2$

$y = (2 - x^2)^{1/2}$

$y' = \frac{1}{\cancel{2}}(2 - x^2)^{-\frac{1}{2}} \times -\cancel{2}x$

$y = \frac{-x}{\sqrt{2-x^2}}$

2.

a. $x = y^3$

$\frac{dx}{dx} = 3y^2 \frac{dy}{dx}$

$\frac{1}{3y^2} = \frac{dy}{dx}$

b. $x^2 + y^3 = 4$

$2x + (3y^2)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{3y^2}$

c. $x^2 + y^2 = 2$

$2x + 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y}$

3. No, the derivative of a & b are not same.

6

a. $g(w) = e^{w^3 - 2w^2 - 7w}$ on $\left[-\frac{1}{2}, \frac{5}{2}\right]$

$g'(w) = 0$

$e^{w^3 - 2w^2 - 7w} \times (3w^2 - 4w - 7) = 0$

$\cancel{\left(3 \times \left(\frac{-1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) - 7\right) e^{\left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right)^2 - 7\left(-\frac{1}{2}\right)} = 0}$

$(3w^2 - 4w - 7) \cdot e^{w^3 - 2w^2 - 7w} = 0$

$3w^2 - 4w - 7 = 0$

$3w^2 - 7w + 3w - 7 = 0$

$w(w3 - 7) + 1(3w - 7) = 0$

$(w+1)(3w+7) = 0$

$w = -1 \quad w = -\frac{7}{3}$

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$$g(-\tfrac{1}{2}) = e^{(-\frac{1}{2})^3 - 2(-\frac{1}{2})^2 - 7(-\frac{1}{2})}$$

$$= e^{\frac{23}{8}}$$ (Absolute maximum)

$$g(\tfrac{5}{2}) = e^{(\frac{5}{2})^3 - 2(\frac{5}{2})^2 - 7(\frac{5}{2})}$$

$$= e^{\frac{-115}{8}}.$$ (Absolute minimum).

b. $f(y) = \sin\left(\frac{y}{3}\right) + \frac{2y}{9}$ on $[-10, 15]$

$$f'(y) = \cos\left(\frac{y}{3}\right) \times \frac{1}{3} + \frac{2}{9}$$

$$9 \times 0 - 2 = 3\cos\left(\frac{y}{3}\right)$$

$$-\frac{2}{3} = \cos\left(\frac{y}{3}\right).$$

$$\cos^{-1}\left(-\frac{2}{3}\right) \times 3 = y.$$

$$7 = y.$$

$$f(-10) = \sin\left(-\frac{10}{3}\right) + \frac{2(10)}{9}.$$

$$= -2.0$$ (Absolute Minimum).

$$f(15) = \sin\left(\frac{15}{3}\right) + \frac{2(15)}{9}$$

$$= 2.4$$ (Absolute. Maximum)

. $y = \ln(x^2 + 4x + 14)$ on $[4, -2]$

$$y' = \frac{2x + 4}{x^2 + 4x + 14}.$$

substitute $x = 4.$

$$y = \ln(4^2 + 4(4) + 14).$$

$$= 3.8$$ (Absolute Maximum).

substitute, $x = -2$

$y = \ln((-2)^2 + 4(-2) + 14)$

$= 2.3$ (Absolute Minimum)

Q7

a. $y = \lim_{x\to\infty} [e^x + x]^{\frac{1}{x}}$ $\Rightarrow$ $\ln y = g(x) \ln f(x)$.

in this case $\frac{1}{x} = g(x)$

$e^x + x = f(x)$

$y = \lim_{x\to\infty} [e^x + x]^{\frac{1}{x}}$.

$\ln y = \ln [e^x + x]^{1/x}$

$\ln y = \frac{1}{x} \ln [e^x + x]$. Apply rule. $\frac{f(x)'}{g(x)'}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x} \ln [e^x + n]$.

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{e^x + x} \times (e^x + 1)}{1}$.

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\cancel{e^\infty} + 1}{\cancel{e^\infty} + 1}$

$\lim_{x\to\infty} \ln y = 1$.

$y = e^1$

b. $\lim_{t\to\infty} \frac{\ln(3t)}{t^2}$ $\Rightarrow$ $\lim_{x\to a} \frac{f'(x)}{g'(x)}$

$\lim_{t\to\infty} \frac{\frac{1}{3t} \times 3}{2t}$

$\lim_{t\to\infty} \frac{\cancel{3}}{3t} \div 2t$

$$\lim_{t\to\infty} \frac{1}{2t^2}$$

$$\lim_{t\to\infty} \frac{1}{2(\infty)^2} = 0.$$

c. $\lim_{x\to 1^+} \left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right]$ using rule $fg = \frac{f}{1/g}$.

$f(x) = \tan\left(\frac{\pi}{2}x\right)$. $\quad g(x) = (x-1)$.

$$\lim_{x\to 1^+} \frac{\tan\left(\frac{\pi}{2}x\right)}{\frac{1}{(x-1)}} = \frac{\sec^2\left(\frac{\pi}{2}x\right)\times\frac{\pi}{2}}{\frac{-1}{(x-1)^2}}.$$

$$\lim_{x\to 1^+} \frac{\frac{\pi}{2}\left(\frac{1}{\cos^2\left(\frac{\pi}{2}\right)}\right)}{\frac{-1}{(1-1)^2}}$$

$$\frac{\frac{\pi}{2}\times\frac{1}{0}}{\frac{-1}{0}} = -\infty$$

Q8

a. $e^{0.1}$ linear approximation.

$$L(x) = f(a) + f'(a)(x-a).$$

$f(x) = e^x$ $\quad L(0.1) = f(0) + f'(0)(0.1-0)$

$f(0) = e^0 = 1$ $\quad = 1 + 1(0.1)$

$f'(x) = e^x$ $\quad = 1 + 0.1$

$f'(0) = e^0 = 1$ $\quad L(0.1) = 1.1$

$e^{0.1} = 1.105 = 1.1.$

b. $f(t) = \cos(2t)$ at $t = \frac{1}{2}$.

$f'(t) = -2\sin 2t$.

$f'(\frac{1}{2}) = \frac{1}{2}. -2\sin(2 \times \frac{1}{2})$

$= -2\sin(1)$

$f(\frac{1}{2}) = \cos(1)$

$=$

$L(t) = \cos(1) - 2\sin(1)(t - \frac{1}{2})$

$L(1) = \cos(1) - 2\sin(1)(1 - \frac{1}{2})$

$= -0.30$

$f(1) = \cos(2 \times 1)$

$= -0.42$.

Approx. value of $\cos(2) = -0.30$

Exact value of $\cos(2) = -0.42$.

$L(9) = \cos(1) - 2\sin(1)(9 - \frac{1}{2})$

$= -13.8$.

$f(9) = \cos(2 \times 9)$

$= \cos(18)$

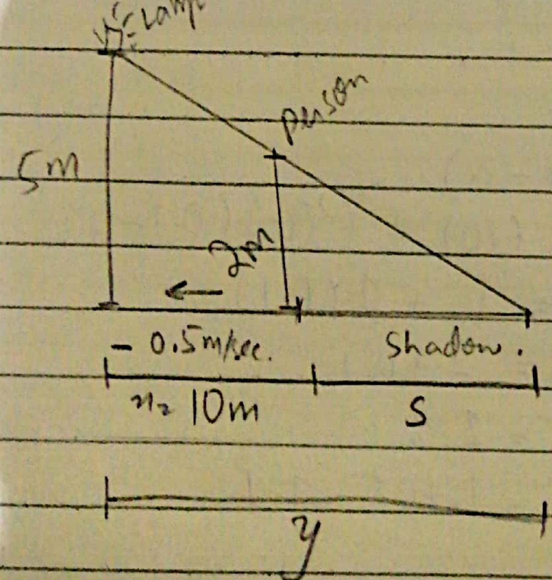
$= 0.66$.

Approx. value of $\cos(18) = -13.8$

Exact value of $\cos(18) = 0.66$.

Q9

a.



By using similar triangle rules

$\frac{2}{5} = \frac{S}{x + S}$.

$S = \frac{2x}{3}$.

$y = x + S$

$= x + \frac{2x}{3}$

$= \frac{3x + 2x}{3}$

$y = \frac{5x}{3}$.

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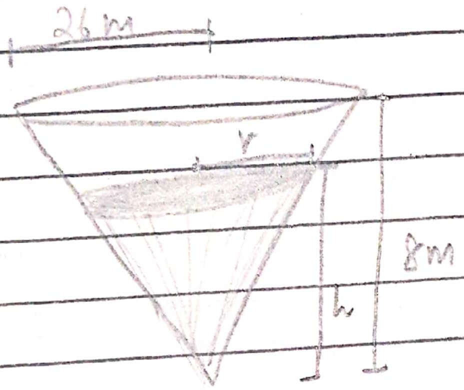
$$\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt} \qquad \frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

$$= \frac{2}{3} \times -0.5 \qquad = \frac{5}{3} \times -0.5$$

$$= \frac{-1}{3} \qquad = \frac{-5}{6}$$

a. The tip of the shadow is moving towards the person.
b. The tip of the shadow is moving towards the wall

b.



$\frac{dh}{dt} = ?$ $\quad \frac{dV}{dt} = 12\,m^3/s$

$V = \frac{1}{3}\pi r^2 h.$

$\cancel{\frac{dV}{dt} = \frac{1}{3}\pi (10)^2 \frac{dh}{dt}}$

Using Similar triangle.

$$\frac{r}{h} = \frac{\cancel{26}^{13}}{\cancel{8}_4}$$

$$r = \frac{13}{4} h.$$

$$V = \frac{1}{3}\pi\left(\frac{13h}{4}\right)^2 h$$

$$= \frac{169}{48}\pi h^3.$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$12 = \frac{169}{48}\pi \times 3 \times h^2 \times \frac{dh}{dt}$$

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$$12 = \frac{169}{16}\pi h^2 \frac{dh}{dt}$$

$$12 = \frac{169}{16}\pi \times \left(\frac{40}{13}\right)^2 \frac{dh}{dt}$$

$$\frac{32448}{270400\pi} = \frac{dh}{dt}$$

$$\frac{3}{25\times\frac{22}{7}\pi} = \frac{dh}{dt}$$

$$\frac{21}{550} = \frac{dh}{dt}$$

$\Rightarrow$ when $r=10$

$$h = \frac{4}{13}r$$

$$= \frac{4}{13}\times 10$$

$$= \frac{40}{13}$$