Linear Algebra

(MT-121T)

AFTAB ALAM

LECTURE # 3

(THURSDAY, FEBRUARY 15, 2024)

Quiz 1 Solution

Given vectors

$$\mathbf{u} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(a) Calculate the dot products $\mathbf{u}.\mathbf{v}, \mathbf{u}.\mathbf{w}, \mathbf{u}.(\mathbf{v}+\mathbf{w})$ and $\mathbf{w}.\mathbf{v}$.

(b) Compute the lengths $\|v\|$ and $\|w\|$.

(c) Find the unit vectors of v and w.

(d) Find the $\cos\theta$ between vectors v and w.

Answers:

a).

u·v=-6x4+8x3=0

u·w=-6x1+8x2=10

u·(v+w)=-6x5+8x5=10

w·v=1x4+2x3=10

b) .

$\|\mathbf{u}\| = \sqrt{6^2+8^2} = 10$

$\|\mathbf{v}\| = \sqrt{4^2+32} = 5$

$\|\mathbf{w}\| = \sqrt{1^2+2^2} = \sqrt{5}$

|u.v|=0

|v.w|=10

$$\hat{v} = \frac{v}{\|v\|} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$\widehat{w} = \frac{w}{\|w\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\cos\theta = \frac{v.w}{\|v\|.\|w\|} = \frac{(5)(2)}{5\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Problem 1

Given vectors

$$\mathbf{u} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

a). Check the Schwarz inequalities

$|\boldsymbol{u}.\boldsymbol{v}| \leq ||\boldsymbol{u}||\ ||\boldsymbol{v}||$ and

$|\boldsymbol{v}.\boldsymbol{w}| \leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||$

b). Choose vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ that make $\mathbf{0^o}$, $\mathbf{90^o}$ and $\mathbf{180^o}$ angles with $\boldsymbol{w}$.

a).

$0 \leq 50$

$10 \leq 5\sqrt{5}$

b). Vector **a** that makes a 0° angle with **a** = λ**w**, where λ is any scalar. Let's choose λ = 2, so **a** =(2,4).

Let's find a vector **b** such that **b**·**w** = 0, **b**·**w** = $(x, y)\cdot(1, 2)$ = $x+2y$ = 0. We can choose x = −2 and y = 1, so **b** =(−2, 1).

Vector **c** that makes a 180 ° angle with **w**
c = −λ**w**, where λ is any scalar.
Let's choose λ = 1, so **c** = (−1,−2).

Problem 2

Find the angle θ between the pairs of vectors,

a) $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ *and* $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

$$\mathbf{v} \cdot \mathbf{w} = (1)(1) + (\sqrt{3})(0) = 1$$

Next, let's calculate the magnitudes of $\mathbf{v}$ and $\mathbf{w}$:

$$|\mathbf{v}| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$|\mathbf{w}| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$$

Now, let's plug these values into the dot product formula to find $\cos\theta$:

$$1 = (2)(1)\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \arccos\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ$$

Problem 2

Find the angle θ between the pairs of vectors,

b) $v = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ *and* $w = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

$$\mathbf{v} \cdot \mathbf{w} = (2)(2) + (2)(-1) + (-1)(2) = 4 - 2 - 2 = 0$$

Next, let's calculate the magnitudes of $\mathbf{v}$ and $\mathbf{w}$:

$$|\mathbf{v}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$|\mathbf{w}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

Now, let's plug these values into the dot product formula to find $\cos\theta$:

$$0 = (3)(3)\cos\theta$$

$$\cos\theta = \frac{0}{9} = 0$$

$$\theta = \arccos(0)$$

$$\theta = 90^\circ$$

Problem 2

Find the angle θ between the pairs of vectors,

c) $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ *and* $w = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

$$|\mathbf{v}| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$|\mathbf{w}| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Now, let's plug these values into the dot product formula to find $\cos\theta$:

$$2 = (2)(2)\cos\theta$$

$$\cos\theta = \frac{2}{4} = \frac{1}{2}$$

Finally, we find θ by taking the arccosine of $\frac{1}{2}$:

$$\theta = \arccos\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ$$

Problem 2

Find the angle θ between the pairs of vectors,

$$d)\ v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \ and \ \ w = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$\mathbf{v} \cdot \mathbf{w} = (3)(-1) + (1)(-2) = -3 - 2 = -5$$

Next, let's calculate the magnitudes of $\mathbf{v}$ and $\mathbf{w}$:

$$|\mathbf{v}| = \sqrt{3^2 + 1^2} = \sqrt{9 + 1} = \sqrt{10}$$

$$|\mathbf{w}| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$$

Now, let's plug these values into the dot product formula to find $\cos\theta$:

$$-5 = \sqrt{10} \cdot \sqrt{5} \cdot \cos\theta$$

$$\cos\theta = \frac{-5}{\sqrt{10} \cdot \sqrt{5}} = \frac{-5}{\sqrt{50}}$$

Problem 3

The parallelogram with sides $\boldsymbol{v}$ = $(\mathbf{4}, \mathbf{2})$ and $\boldsymbol{w}$ = $(-\mathbf{1}, \mathbf{2})$ is a rectangle.
Check the Pythagoras formula $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$ which is for right triangles only,

$(\boldsymbol{lenght\ of\ v})^2 + (\boldsymbol{lenght\ of\ w})^2 = (\boldsymbol{lenght\ of\ v + w})^2$

$$\|\mathbf{d}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

Given $\mathbf{v} = (4, 2)$ and $\mathbf{w} = (-1, 2)$:

$$\|\mathbf{v}\|^2 = 4^2 + 2^2 = 16 + 4 = 20$$

$$\|\mathbf{w}\|^2 = (-1)^2 + 2^2 = 1 + 4 = 5$$

So,

$$\|\mathbf{d}\|^2 = 20 + 5 = 25$$

Taking the square root of both sides gives us the length of $\mathbf{d}$:

$$\|\mathbf{d}\| = \sqrt{25} = 5$$

The length of the diagonal $\mathbf{d}$ of the rectangle is 5 units.

Thus, the parallelogram with sides $\mathbf{v} = (4, 2)$ and $\mathbf{w} = (-1, 2)$ is indeed a rectangle, and the Pythagorean theorem holds for its diagonal.

Problem 4

Find nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that are perpendicular to $(\mathbf{1}, \mathbf{0}, \mathbf{1})$ and to each other.

Let's denote $\boldsymbol{v}$ = (a, b, c) and $\boldsymbol{w}$ = (d, e, f).

We need to satisfy the following conditions:

- $\boldsymbol{v}$ and $\boldsymbol{w}$ must be perpendicular to (1, 0, 1).
  - The dot product of $\boldsymbol{v}$ and (1, 0, 1) should be zero: [a + 0 + c = 0 → a = -c]
  - The dot product of $\boldsymbol{w}$ and (1, 0, 1) should be zero: [d + 0 + f = 0 → d = -f]
- $\boldsymbol{v}$ and $\boldsymbol{w}$ must be perpendicular to each other. $\boldsymbol{v}.\,\boldsymbol{w} = 0$ [ad + be + cf = 0]

Let's (c = 1) and (d = 1), then

a = -1 and f = -1 (since a =-c and d = -f).

For simplicity, let's set (b = 2).

Therefore, one possible solution is: [$\boldsymbol{v}$ = (-1, 2, 1)] [$\boldsymbol{w}$ = (1, 1, -1)]

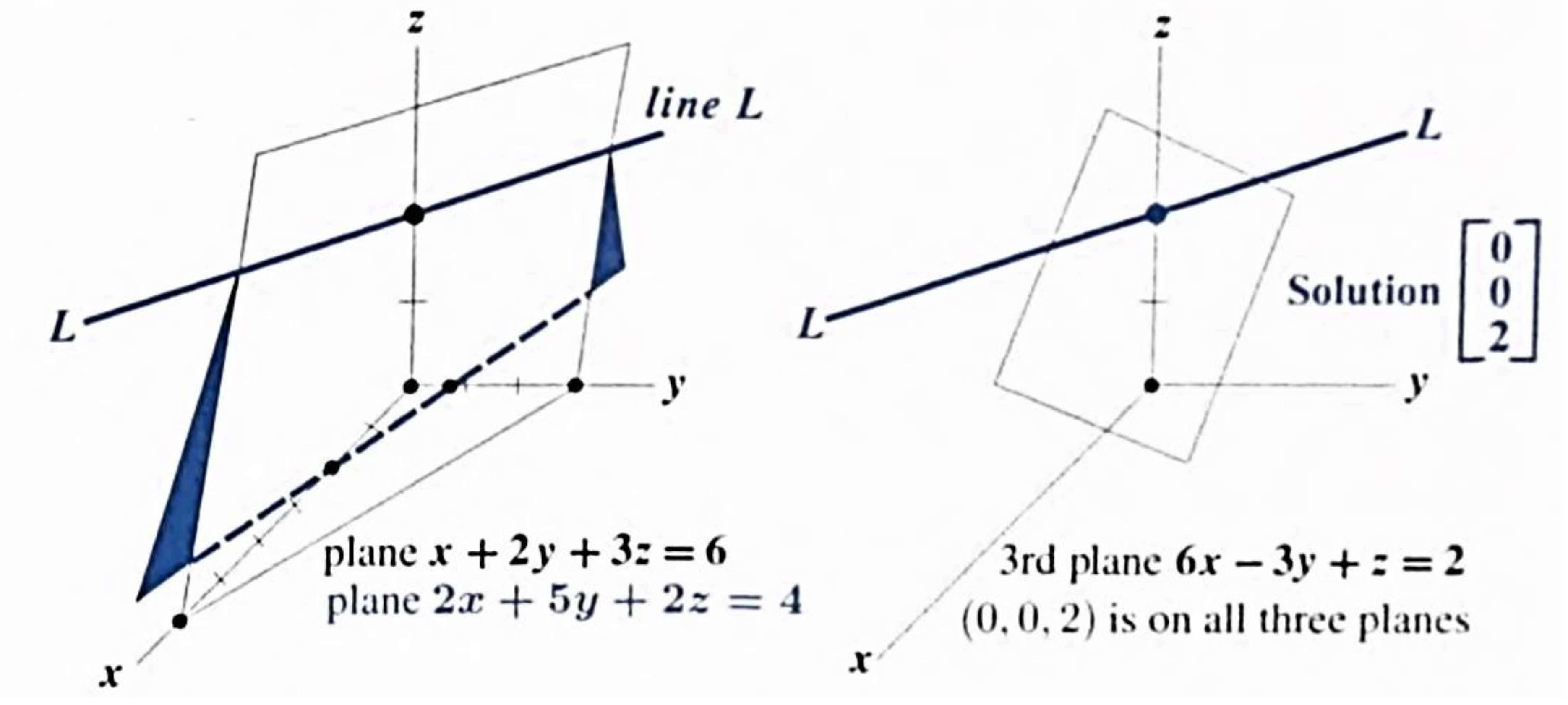
These vectors are nonzero, perpendicular to (1, 0, 1), and perpendicular to each other.

Solving a Linear System

$x + 2y + 3z = 6$

$2x + 5y + 2z = 4$

$6x - 3y + z = 2$



: *Row picture*: Two planes meet at a line L. Three planes meet at a point.

Column Picture

Column picture: Combine the columns with weights $(x, y, z) = (0, 0, 2)$.

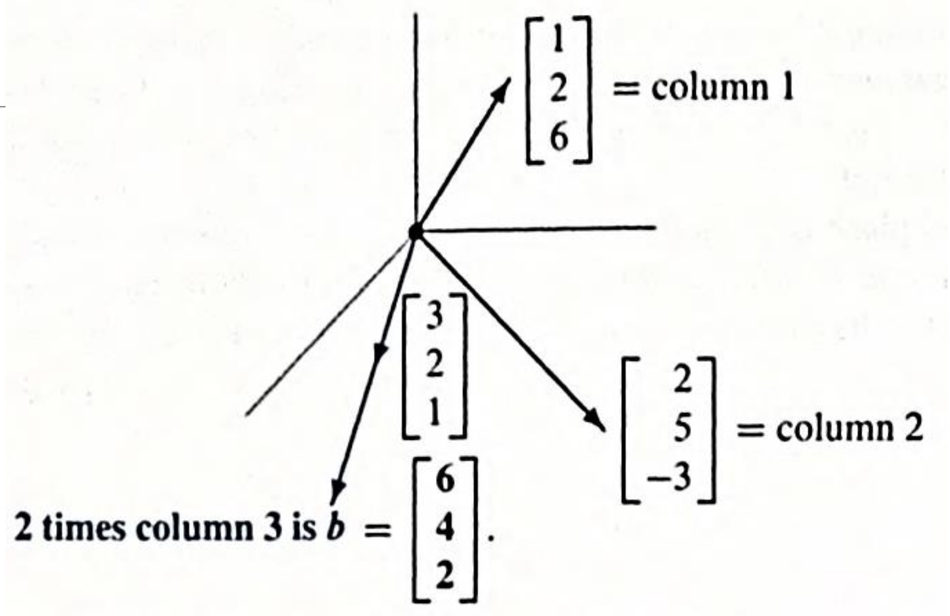
- The column picture combines three columns to produce b = (6,4,2)

- The column picture starts with the vector form of the equations Ax = b

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix} = b$$

Correct Combination x, y, z = (0, 0, 2)

$$0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix} = b$$



Example 1

Describe the column picture of these three equations, and solve by careful inspection of the columns.

$x + 3y + 3z = -3$

$2x + 2y + 2z = -2$

$3x + 5y + 6z = -5$

$$x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}3\\2\\5\end{bmatrix} + z\begin{bmatrix}3\\2\\6\end{bmatrix} = \begin{bmatrix}-3\\-2\\-5\end{bmatrix}$$

By careful inspection, we find that (x, y, z) = (0, -1, 0)

Example 2 (a)

The first of these equations plus the second equals the third:

$x + y + z = 2$

$x + 2y + z = 3$

$2x + 3y + 2z = 5$

a) The first two planes meet along a line. The third plane contains that line, because if *x, y, z* satisfy the first two equations then they also ___________.

satisfy the equation of the third plane.

Example 2 (b)

The first of these equations plus the second equals the third:

$x + y + z = 2$

$x + 2y + z = 3$

$2x + 3y + 2z = 5$

b) The equations have infinitely many solutions (the whole line **L**). Find three solutions on L.

Substituting $\boldsymbol{x} = \mathbf{0}$ into these equations, we get: $y + z = 2$ and $2y + z = 3$ ➔ **y = 1, z = 1**

Substituting $\boldsymbol{x} = \mathbf{1}$ into these equations, we get: $1 + y + z = 2$ and $1 + 2y + z = 3$ ➔ **y = 1, z = 0**

Substituting $\boldsymbol{x} = \mathbf{-1}$ into these equations, we get: $-1 + y + z = 2$ and $-1 + 2y + z = 3$ ➔ **y = 2, z = 1**

So the three solutions are (0,1,1), (1, 1, 0) and (−1, 2, 1).

Example 2 (c)

3rd Equation slightly changed.

- $x + y + z = 2$
- $x + 2y + z = 3$
- $2x + 3y + 2z = 9$

Now the three equations have no solution, why not?

The given system of equations has no solution because the third equation is inconsistent with the first two equations.

Example 2 (d)

- $x + y + z = 2$
- $x + 2y + z = 3$
- $2x + 3y + 2z = 5$

In the given system of equations, the columns are (1,1,2) and (1,2,3) and (1,1,2). This is a singular matrix because the third column is _______________.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

Swap the columns and rows, then the 3rd column will be a linear combination of the first and second columns.