Linear Algebra

(MT-121T)

Aftab Alam

Lecture # 23

(Thursday, May 09, 2024)

Diagonalization

Given the matrix A $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

diagonalize it and give the similarity transformation.

$\lambda = 3, 2, -4 \qquad x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad X^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\mathrm{A} = \mathrm{X}\Lambda\mathrm{X}^{-1}$$

Systems of Differential Equations

1 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ will solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$. Each λ and $\boldsymbol{x}$ give a solution $e^{\lambda t}\boldsymbol{x}$.

2 If $A = X\Lambda X^{-1}$ then $\boxed{\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n.}$

3 A is **stable** and $\boldsymbol{u}(t) \to \boldsymbol{0}$ and $e^{At} \to 0$ when all eigenvalues of A have real part < 0.

4 **Matrix exponential** $e^{At} = I + At + \cdots + (At)^n/n! + \cdots = Xe^{\Lambda t}X^{-1}$ if A is diagonalizable.

5 **Second order equation** / **First order system** $u'' + Bu' + Cu = 0$ is equivalent to $\begin{bmatrix} u \\ u' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -C & -B \end{bmatrix}\begin{bmatrix} u \\ u' \end{bmatrix}$.

Eigenvalues and eigenvectors and $A = X\Lambda X^{-1}$ are perfect for matrix powers A^k. They are also perfect for differential equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$. This section is mostly linear algebra, but to read it you need one fact from calculus: *The derivative of* $e^{\lambda t}$ *is* $\lambda e^{\lambda t}$. The whole point of the section is this: **To convert constant-coefficient differential equations into linear algebra.**

System of Differential Equations

The ordinary equations $\frac{du}{dt} = u$ and $\frac{du}{dt} = \lambda u$ are solved by exponentials:

$$\frac{du}{dt} = u \quad \text{produces} \quad u(t) = Ce^t \qquad \frac{du}{dt} = \lambda u \quad \text{produces} \quad u(t) = Ce^{\lambda t} \tag{1}$$

At time $t = 0$ those solutions include $e^0 = 1$. So they both reduce to $u(0) = C$. This "initial value" tells us the right choice for C. **The solutions that start from the number $\boldsymbol{u(0)}$ at time $\boldsymbol{t = 0}$ are $\boldsymbol{u(t) = u(0)e^t}$ and $\boldsymbol{u(t) = u(0)e^{\lambda t}}$.**

We just solved a 1 by 1 problem. Linear algebra moves to n by n. The unknown is a vector $\boldsymbol{u}$ (now boldface). It starts from the initial vector $\boldsymbol{u}(0)$, which is given. The n equations contain a square matrix A. We expect n exponents $e^{\lambda t}$ in $\boldsymbol{u}(t)$, from n λ's:

System of $\boldsymbol{n}$ equations $\qquad \dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad$ starting from the vector $\boldsymbol{u}(0) = \begin{bmatrix} u_1(0) \\ \cdots \\ u_n(0) \end{bmatrix}$ at $t = 0$.

These differential equations are *linear*. If $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions, so is $C\boldsymbol{u}(t) + D\boldsymbol{v}(t)$.

Summary: System of n Equations

$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}$ starting from the vector u(0) = $\begin{bmatrix} u_1(0) \\ | \\ | \\ u_n(0) \end{bmatrix}$ at t=0.

- These D.Es are linear.
- A is a constant matrix.
- $\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}$ is linear with constant coefficients.

Solving 1st Order D.E.

- To solve a first-order differential equation using eigenvalues and eigenvectors, you need to follow these steps:

1. Write the differential equation in matrix form: **du/dt = Au**, where **A** is a matrix and **u** is a vector.
2. Find the eigenvalues λ and eigenvectors x of the matrix **A**.
3. Diagonalize the matrix **A** by finding the matrix **X**, which is composed of the eigenvectors, and the diagonal matrix $\boldsymbol{\Lambda}$, which contains the eigenvalues.
4. Write the solution as $\boldsymbol{u}(\mathbf{t}) = \mathbf{X}\mathbf{e}^{\Lambda t}\mathbf{X}^{-1}\mathbf{u}(\mathbf{0})$, where **u(0)** is the initial condition.

Example

Solve $\frac{d\boldsymbol{u}}{dt} = \boldsymbol{A}\boldsymbol{u}$ if $\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ starting from the vector $\mathrm{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$u_1(t) = e_1^{\lambda_1 t} x_1, \quad u_2(t) = e_2^{\lambda_2 t} x_2$

$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u} \rightarrow \frac{\boldsymbol{d}}{\boldsymbol{dt}} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \rightarrow \frac{d\boldsymbol{y}}{dt} = z, \ \frac{d\boldsymbol{z}}{dt} = \boldsymbol{y}$

From $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \lambda_1 = 1, \lambda_2 = -1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, u(\mathrm{t}) = \mathrm{C}u_1(\mathrm{t}) + \mathrm{D}u_2(\mathrm{t}) = \mathrm{C}e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u(0) = \mathrm{C} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ → C = 3, D = 1

So $\boldsymbol{u(t) = 3e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

du/dt = Au

The same three steps that solved $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$ now solve $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

1. Write $\boldsymbol{u}(0)$ as a **combination** $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$ **of the eigenvectors of** A.

2. Multiply each eigenvector $\boldsymbol{x}_i$ by **its growth factor** $e^{\lambda_i t}$.

3. The solution is the same combination of those pure solutions $e^{\lambda t}\boldsymbol{x}$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \cdots + c_n e^{\lambda_n t}\boldsymbol{x}_n. \tag{6}$$

Example 2

Example 2 Solve $du/dt = Au$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A:

Typical example
Equation for u
Initial condition $u(0)$

$$\frac{du}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{u} \quad \text{starting from} \quad \boldsymbol{u}(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}$$

The eigenvectors are $\boldsymbol{x}_1 = (1,0,0)$ and $\boldsymbol{x}_2 = (1,1,0)$ and $\boldsymbol{x}_3 = (1,1,1)$.

$$\boldsymbol{Xc} = \boldsymbol{u}(\mathbf{0})$$

Step 1 The vector $\boldsymbol{u}(0) = (9,7,4)$ is $\mathbf{2}\boldsymbol{x}_1 + \mathbf{3}\boldsymbol{x}_2 + \mathbf{4}\boldsymbol{x}_3$. Thus $(c_1, c_2, c_3) = (\mathbf{2}, \mathbf{3}, \mathbf{4})$.

Step 2 The factors $e^{\lambda t}$ give exponential solutions $e^t\boldsymbol{x}_1$ and $e^{2t}\boldsymbol{x}_2$ and $e^{3t}\boldsymbol{x}_3$.

Step 3 The combination that starts from $\boldsymbol{u}(0)$ is $\boldsymbol{u}(t) = \mathbf{2}e^t\boldsymbol{x}_1 + \mathbf{3}e^{2t}\boldsymbol{x}_2 + \mathbf{4}e^{3t}\boldsymbol{x}_3$.

Example

Solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \boldsymbol{u}$ starting from $\boldsymbol{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 0$.

$$\boldsymbol{u}(t) = e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Second Order Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (that is *Newton's Law*). The force includes the damping $-by'$ and the elastic force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k.

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative of y brings down a factor λ. We want $y = e^{\lambda t}$ to solve the equation:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \text{becomes} \quad (m\lambda^2 + b\lambda + k)\,e^{\lambda t} = 0. \tag{8}$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2. Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1y_1 + c_2y_2$ give the complete solution unless $\lambda_1 = \lambda_2$.

Second Order Differential Equations

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a *vector equation for* y *and* y': first derivative only. Suppose the mass is $m = 1$. Two equations for $\boldsymbol{u} = (y,\ y')$ give $d\boldsymbol{u}/dt = A\boldsymbol{u}$:

$$\begin{aligned} \boldsymbol{dy/dt} &= \boldsymbol{y'} \\ \boldsymbol{dy'/dt} &= \boldsymbol{-ky - by'} \end{aligned} \quad \text{converts to} \quad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \boldsymbol{-k} & \boldsymbol{-b} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}. \qquad (9)$$

The first equation $dy/dt = y'$ is trivial (but true). The second is equation (8) connecting y'' to y' and y. Together they connect $\boldsymbol{u}'$ to $\boldsymbol{u}$. So we solve $\boldsymbol{u}' = A\boldsymbol{u}$ by eigenvalues of A:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \qquad \boldsymbol{u}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

Exponential of a Matrix

We want to write the solution $\boldsymbol{u}(t)$ ***in a new form*** $e^{At}\boldsymbol{u}(0)$. First we have to say what e^{At} means, with a matrix in the exponent. To define e^{At} for matrices, we copy e^x for numbers.

The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$. When you change x to a square matrix At, this series defines the matrix exponential e^{At}:

Matrix exponential e^{At}	$\boxed{e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots}$	(14)
Its t derivative is Ae^{At}	$A + A^2t + \frac{1}{2}A^3t^2 + \cdots = \boxed{Ae^{At}}$	
Its eigenvalues are $e^{\lambda t}$	$(I + At + \frac{1}{2}(At)^2 + \cdots)\boldsymbol{x} = \boxed{(1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \cdots)\,\boldsymbol{x}}$	

The number that divides $(At)^n$ is "n factorial". This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after $1, 2, 6$ are $4! = 24$ and $5! = 120$. They grow quickly. The series always converges and its derivative is always Ae^{At}. Therefore $e^{At}\boldsymbol{u}(0)$ solves the differential equation with one quick formula—*even if there is a shortage of eigenvectors.*

Exponential of a Matrix

Use the series $$e^{At} = I + X\Lambda X^{-1}t + \tfrac{1}{2}(X\Lambda X^{-1}t)(X\Lambda X^{-1}t) + \cdots$$

Factor out X and X^{-1} $$= X\left[I + \Lambda t + \tfrac{1}{2}(\Lambda t)^2 + \cdots\right]X^{-1} \qquad (15)$$

e^{At} is diagonalized! $$\boxed{e^{At} = X\, e^{\Lambda t}\, X^{-1}.}$$

e^{At} has the same eigenvector matrix X as A. Then Λ is a diagonal matrix and so is $e^{\Lambda t}$. The numbers $e^{\lambda_i t}$ are on the diagonal. Multiply $Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0)$ to recognize $\boldsymbol{u}(t)$:

$$e^{At}\boldsymbol{u}(0) = Xe^{\Lambda t}X^{-1}\boldsymbol{u}(0) = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \qquad (16)$$

This solution $e^{At}\boldsymbol{u}(0)$ is the same answer that came in equation (6) from three steps

Repeated Roots

Example 4 When you substitute $y = e^{\lambda t}$ into $y'' - 2y' + y = 0$, you get an equation with **repeated roots**: $\lambda^2 - 2\lambda + 1 = 0$ is $(\lambda - 1)^2 = 0$ with $\boldsymbol{\lambda = 1, 1}$. A differential equations course would propose e^t and te^t as two independent solutions. Here we discover why.

Linear algebra reduces $y'' - 2y' + y = 0$ to a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 2y' - y \end{bmatrix} \quad \text{is} \quad \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\boldsymbol{u}. \tag{18}$$

A has a **repeated eigenvalue** $\boldsymbol{\lambda = 1, 1}$ (with trace $= 2$ and $\det A = 1$). The only eigenvectors are multiples of $\boldsymbol{x} = (1, 1)$. *Diagonalization is not possible*, A has only one line of eigenvectors. So we compute e^{At} from its definition as a series:

$$\textbf{Short series} \qquad \boldsymbol{e^{At}} = e^{It}e^{(A-I)t} = e^t\,[I + (A - I)t]. \tag{19}$$

That "infinite" series for $e^{(A-I)t}$ ended quickly because $(A - I)^2$ is the zero matrix! You can see $\boldsymbol{te^t}$ in equation (19). The first component of $\boldsymbol{e^{At}}\,\boldsymbol{u}(0)$ is our answer $y(t)$:

$$\begin{bmatrix} y \\ y' \end{bmatrix} = e^t\left[I + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t\right]\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \qquad y(t) = e^t y(0) - \boldsymbol{te^t} y(0) + \boldsymbol{te^t} y'(0).$$