Sep. 4 Lecture 3

§1.3 Construction of Lebesgue measure on $\mathbb{R}$

We are looking for a measure on $\mathbb{R}$ s.t.

① the measure of an interval I is just the length of I, denoted by $\ell(I)$

(no matter I is open / closed / half-open-half-closed / finite / infinite)

② the measure is translation invariant, $\mu(A) = \mu(A+x)$ $\forall x \in \mathbb{R}$, $\forall$ "measurable" A

Step 1 Define outer measure

Definition For any $A \subseteq \mathbb{R}$, define

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n,\ I_n\text{'s are open intervals} \right\}$$

m^* is called an outer measure on $\mathbb{R}$

Proposition m^* has the following properties

(1) $\forall A \in \mathbb{R}$, $m^*(A) \geq 0$, and $m^*(\phi) = 0$.

(2) (monotonicity) if $A \subseteq B \subseteq \mathbb{R}$. then $m^*(A) \le m^*(B)$

(3) (countable subadditivity) if $\{A_n : n \ge 1\}$ is a sequence of subsets of $\mathbb{R}$.

then $m^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} m^*(A_n)$. (*)

Proof: Let $\{A_n : n \ge 1\}$ be a sequence of subsets of $\mathbb{R}$. If $m^*(A_n) = \infty$ for some $n \ge 1$, then (*) holds trivially. Next, assume $m^*(A_n) < \infty$ $\forall n \ge 1$.

Fix arbitrary $\varepsilon > 0$. For each A_n, take a sequence of open intervals $\{I_{n,i} : i \ge 1\}$ s.t. $A_n \subseteq \bigcup_{i=1}^{\infty} I_{n,i}$. and $\sum_{i=1}^{\infty} \ell(I_{n,i}) \le m^*(A_n) + \varepsilon/2^n$.

Therefore, $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} I_{n,i}$ and hence

$$m^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{n,i}) \le \sum_{n=1}^{\infty} (m^*(A_n) + \varepsilon/2^n) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon$$

Since ε is arbitrary, we have proven (*) $\square$.

(More generally, any set function on $2^{\mathbb{R}}$ that satisfies (1)(2)(3) is considered to be an outer measure on $\mathbb{R}$.)

(4) If $I \subseteq \mathbb{R}$ is an interval, then $m^*(I) = \ell(I)$

Proof: We first prove the statement for closed finite interval $I = [a, b]$ $(a, b \in \mathbb{R}, a \le b)$

For any $\varepsilon > 0$, set $I_1 := (a - \varepsilon/2, b + \varepsilon/2)$. Then, $I \subseteq I_1$ and hence

$$m^*(I) \le \ell(I_1) = b - a + \varepsilon = \ell(I) + \varepsilon.$$

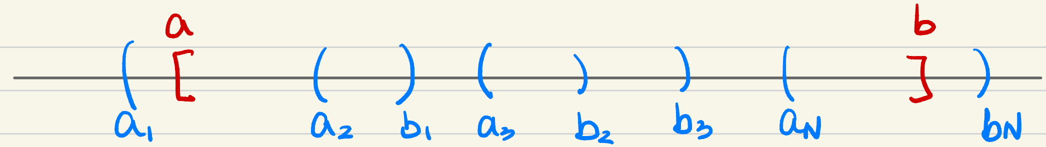
Since $\varepsilon > 0$ is arbitrary. $m^*(I) \le \ell(I)$.

To prove the reverse inequality, we need to show that $\forall \{I_n : n \ge 1\}$ open intervals s.t. $I \subseteq \bigcup_{n=1}^{\infty} I_n$, $\ell(I) \le \sum_{n=1}^{\infty} \ell(I_n)$

Let $\{I_n : n \ge 1\}$ be one such covering. W.L.O.G., assume $\ell(I_n) < \infty$ $\forall n \ge 1$. (otherwise the concerned relation holds trivially). Assume $I_n = (a_n, b_n)$ $\forall n \ge 1$.

Since $I = [a, b]$ is compact, it can be covered by finitely many I_n's.

We may further extract a finite collection, say, $\{I_1, I_2, \cdots, I_N\}$.



such that $I \subseteq \bigcup_{n=1}^{N} I_n$ with $I_n \cap I \neq \phi \quad \forall\, 1 \le n \le N$,

and $a_1 < a$, $b < b_N$. $\quad a_n < b_{n-1} \quad \forall\, 2 \le n \le N$.

Then, $$\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) = b_1 - a_1 + \sum_{n=2}^{N} (b_n - a_n)$$
$$\geq b_1 - a_1 + \sum_{n=2}^{N} (b_n - b_{n-1})$$
$$= b_N - a_1 \geq b - a = \ell(I)$$

So, we have proven $m^*(I) = \ell(I)$ for I being a closed finite interval.

Next, assume I is a finite interval with end points $a < b$

$\forall\ 0 < \varepsilon < \frac{1}{2}(b-a)$, $\quad [a+\varepsilon, b-\varepsilon] \subseteq I \subseteq [a-\varepsilon, b+\varepsilon]$

By (monotonicity) and the case above, $\quad b-a-2\varepsilon \le m^*(I) \le b-a+2\varepsilon$

Since ε can be arbitrarily small, $m^*(I) = b-a = \ell(I)$.

Finally, if I is an infinite interval, then $\forall M>0$, $\exists$ closed finite interval I_M s.t. $I_M \subseteq I$ and $\ell(I_M) \geq M$.

It follows again from (monotonicity) that $m^*(I) \geq M$

Since M is arbitrarily large, $m^*(I) = \infty = \ell(I)$ $\square$.

Corollary of (4): $\forall x \in \mathbb{R}$, $m^*(\{x\}) = 0$,

and $\forall C \subseteq \mathbb{R}$ at most countable, $m^*(C) = 0$.

(5) m^* is translation invariant, i.e. $\forall A \subseteq \mathbb{R}$ $\forall x \in \mathbb{R}$, $m^*(A) = m^*(A+x)$

(Proof: It's sufficient to notice that $\{I_n : n \geq 1\}$ is an open-interval covering of A if and only if $\{I_n + x : n \geq 1\}$ is an open-interval covering of $A+x$. Furthermore, $\ell(I_n) = \ell(I_n + x)$ $\forall n \geq 1$. $\square$)

(6) $\forall A \subseteq \mathbb{R}$. $m^*(A) = \inf\{m^*(B) : B \text{ open and } A \subseteq B\}$

Proof: By (monotonicity) $m^*(A) \leq m^*(B)$ $\forall$ B open and $A \subseteq B$

So, $m^*(A) \leq$ RHS. To show the reverse, W.L.O.G., assume $m^*(A) < \infty$.

$\forall \varepsilon > 0$, $\exists \{I_n : n \geq 1\}$ open-interval covering of A s.t.

$$\sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \varepsilon.$$

Set $B := \bigcup_{n=1}^{\infty} I_n$. Then, B is open, $A \subseteq B$ and $m^*(B) \overset{\text{countable subadd.}}{\leq} \sum_{n=1}^{\infty} m^*(I_n) \overset{(4)}{=} \sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \varepsilon$.

Since $\varepsilon > 0$ is arbitrarily small, we have $m^*(A) \geq$ RHS. □

(7) If $A = A_1 \cup A_2 \subseteq \mathbb{R}$ with $d(A_1, A_2) > 0$, then $m^*(A) = m^*(A_1) + m^*(A_2)$

Proof: We already know $m^*(A) \leq m^*(A_1) + m^*(A_2)$ (finite subaddtivity)

We only need to prove the reverse inequality. Assume that $d(A_1, A_2) = \delta > 0$.

Again, W.L.O.G., assume $m^*(A) < \infty$. $\forall \varepsilon > 0$, take $\{I_n : n \geq 1\}$ to be an open-interval covering of A s.t. $\sum_{n=1}^{\infty} \ell(I_n) \le m^*(A) + \varepsilon$.

$\forall n \ge 1$, since I_n is a finite open interval, it is possible to further divide I_n into $\{I_{n,i} : 1 \le i \le k_n\}$ s.t. $I_{n,i}$'s are open intervals s.t. $I_n \subseteq \bigcup_{i=1}^{k_n} I_{n,i}$

$\ell(I_{n,i}) < \delta \;\; \forall\, 1 \le i \le k_n$ and $\sum_{i=1}^{k_n} \ell(I_{n,i}) \le \ell(I_n) + \varepsilon/2^n$.

We rename $\{I_{n,i} : 1 \le i \le k_n,\ n \ge 1\}$ as $\{J_m : m \ge 1\}$ which is again an open-interval covering of A and

$$\sum_{m=1}^{\infty} \ell(J_m) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \ell(I_{n,i}) \le \sum_{n=1}^{\infty} \ell(I_n) + \varepsilon \le m^*(A) + 2\varepsilon.$$

Since $\forall m \ge 1$, $\ell(J_m) < \delta$, J_m can intersect at most one A_1, A_2.

Set $M_1 := \{m \ge 1 : J_m \cap A_1 \ne \phi\}$. $M_2 := \{m \ge 1 : J_m \cap A_2 \ne \phi\}$

Then, $M_1 \cap M_2 = \phi$ and $\{J_m : m \in M_p\}$ is an open-interval covering of A_p. $p = 1, 2$

Therefore, $m^*(A_1) + m^*(A_2) \le \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \le \sum_{m=1}^{\infty} \ell(J_m) \le m^*(A) + 2\varepsilon$. $\square$