Aug. 30

Lecture 2

§1.1 $\sigma$-algebras (continued) Given a space $X$, $\mathcal{F}$ a collection of subsets of $X$ is a $\sigma$-algebra if (1) $X \in \mathcal{F}$ (2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(3) $\{A_n : n \geq 1\} \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Important example on $\mathbb{R}$: The Borel $\sigma$-algebra on $\mathbb{R}$ (of subsets of $\mathbb{R}$) denoted by $\mathcal{B}_{\mathbb{R}}$, is defined as: $\mathcal{B}_{\mathbb{R}} = \sigma(\{\text{open subsets of } \mathbb{R}\})$

If $A \in \mathcal{B}_{\mathbb{R}}$, $A$ is called a Borel set.

<u>Proposition</u>

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,b) : a,b \in \mathbb{R},\ a<b\})$$

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,b] : a,b \in \mathbb{R},\ a<b\})$$

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$$\mathcal{B}_{\mathbb{R}} = \sigma(\{[a,b] : a,b \in \mathbb{R}\ a<b\})$$

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(-\infty,a) : a \in \mathbb{R}\}) = \sigma(\{(-\infty,a] : a \in \mathbb{R}\})$$

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,\infty) : a \in \mathbb{R}\}) = \sigma(\{[a,\infty) : a \in \mathbb{R}\})$$

The generator of $\mathcal{B}_{\mathbb{R}}$ is not unique

$\mathcal{B}_{\mathbb{R}}$ is generated by open/ left-open-right-closed/ left-closed-right-open/ closed intervals, or infinite open/closed intervals.

On the other hand, all the intervals, union / intersection of interval are Borel sets.

All the singletons are Borel sets. i.e. $\forall x \in \mathbb{R}$, $\{x\} \in \mathcal{B}_{\mathbb{R}}$ $\left( \{x\} = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n}) \right)$

All finite and countable sets are Borel sets, e.g. $\mathbb{Q} \in \mathcal{B}_{\mathbb{R}}$

## §1.2. measures

Given a space $X$ and a $\sigma$-algebra $\mathcal{F}$ of subsets of $X$, $(X, \mathcal{F})$ is called a measurable space, e.g., $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. We are now ready to define "measures"

Definition: Given $(X, \mathcal{F})$ a measurable space, $\mu: \mathcal{F} \to [0, \infty]$ is a non-negative set function defined on $\mathcal{F}$. Then, $\mu$ is called a measure if

(i) $\mu(\phi) = 0$ and (ii) if $\{A_n : n \geq 1\} \subseteq \mathcal{F}$ is a sequence of pairwise disjoint sets.

then $\mu\left( \bigsqcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$ (countable additivity)

In this case, $(X, \mathcal{F}, \mu)$ is called a measure space

We say $\mu$ is finite, if $\mu(X) < \infty$; $\mu$ is $\sigma$-finite, if $\exists \{A_n : n \geq 1\} \subseteq \mathcal{F}$ s.t. $X = \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ $\forall n \geq 1$; $\mu$ is a probability measure (and hence $(X, \mathcal{F}, \mu)$ is a probability space) if $\mu(X) = 1$.

Example of measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

$\rightarrow$ cardinality

1. $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$. $\forall A \in \mathcal{B}_{\mathbb{R}}$. $\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$

(Convention: $\phi$ is considered as a finite set with $|\phi| = 0$)

$\mu$ is the counting measure on $\mathbb{R}$

2. Let $x_0 \in \mathbb{R}$. $\mu$ is s.t. $\forall A \in \mathcal{B}_{\mathbb{R}}$. $\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise.} \end{cases}$

$\mu$ is the point mass measure at $x_0$ ($\mu$ is a probability measure)

<u>Theorem</u> (Properties of measure) Let $(X, \mathcal{F}, \mu)$ is a measure space.

1. (finite additivity) If $A_1, \cdots, A_N \in \mathcal{F}$ are pairwise disjoint sets, then $\mu(\bigcup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n)$.

2. (Monotonicity) Given $A, B \in \mathcal{F}$. if $A \subseteq B$, then $\mu(A) \le \mu(B)$.

In the proof, DON'T take "$\mu(B) - \mu(A)$". because both quantities may be $\infty$.

Instead, write $B = A \sqcup (B \setminus A)$ union of disjoint sets $\Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A) \ge \mu(A)$.

3. (countable / finite subadditivity) Given $\{A_n : n \ge 1\} \subseteq \mathcal{F}$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n)$$

(Proof: Define $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \ge 2$

$\Rightarrow \{B_n : n \ge 1\} \subseteq \mathcal{F}$ pairwise disjoint and $\bigsqcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) \underset{\text{countable add.}}{=} \sum_{n=1}^{\infty} \mu(B_n) \underset{\text{monotonicity}}{\le} \sum_{n=1}^{\infty} \mu(A_n). \ \square \ )$$

Similarly. $\forall N \ge 1$. $\mu\left(\bigcup_{n=1}^{N} A_n\right) \le \sum_{n=1}^{N} \mu(A_n)$

Remark If $N \in \mathcal{F}$ is s.t. $\mu(N) = 0$, then $N$ is called a ($\mu$-) null set.

If $\{N_n : n \ge 1\} \subseteq \mathcal{F}$ is a sequence of ($\mu$-) null sets, then $\bigcup_{n=1}^{\infty} N_n$ is also a null set

4. (continuity from below) Given $\{A_n : n\geq 1\} \subseteq \mathcal{F}$ s.t. $A_n \subseteq A_{n+1}$ $\forall n\geq 1$ ("$A_n \uparrow$")

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) \qquad \left(\text{view } \bigcup_{n=1}^{\infty} A_n \text{ as } \text{"}\lim_{n\to\infty} A_n\text{"}\right)$$

Proof: Define $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ $\forall n\geq 2$.

$\Rightarrow \{B_n : n\geq 1\} \subseteq \mathcal{F}$ pairwise disjoint and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

finite add. $A_n = \bigcup_{i=1}^{n} B_i$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{+\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) \overset{\uparrow}{=} \lim_{n\to\infty} \mu(A_n)$$

5 (continuity from above) Given $\{A_n : n\geq 1\} \subseteq \mathcal{F}$ s.t $A_n \supseteq A_{n+1}$ $\forall n\geq 1$ ("$A_n \downarrow$")

if $\mu(A_1) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$

Proof: Set $B_n = A_1 \setminus A_n$ $\forall n\geq 1$. Then, $\{B_n : n\geq 1\} \subseteq \mathcal{F}$ is s.t. $B_n \uparrow$ and

(continuity from below)

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right) \Rightarrow \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n) \quad (*)$$

Since $\mu(A_1) < \infty$ and $A_n \downarrow$, $\mu(A_n) \leq \mu(A_1) < \infty$ (all the $A_n$'s have finite measure)

$\mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$. Similarly, $\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$

So, (*) $\Rightarrow$ $\mu(A_1) - \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) \Rightarrow \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$. $\square$

<u>Remark</u>: The hypothesis "$\mu(A_1) < \infty$" in (continuity from above) is necessary.

Example: $\mu$ is the counting measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ (i.e. $\forall A \in \mathcal{B}_{\mathbb{R}}$ $\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$)

$E_n = \{n, n+1, n+2, \cdots\}$ $\forall n \geq 1$. Then, $\mu(E_n) = \infty$ $\forall n \geq 1$. but $\mu(\bigcap_{n=1}^{\infty} E_n) = \mu(\phi) = 0$.