Types of convergence

Let $(X_n, 1 \le n \le \infty)$ be a sequence of random variables defined on a common space $(\Omega, \mathcal{F}, \mathbb{P})$.

We say X_n converges almost surely to X_∞ and write $X_n \xrightarrow{a.s.} X_\infty$

if $\mathbb{P}(\lim_{n\to\infty} X_n = X_\infty) = \mathbb{P}(\{\omega : \lim_{n\to\infty} X_n(\omega) = X_\infty(\omega)\}) = 1.$

We say X_n converges in probability to X_∞ and write $X_n \xrightarrow{P} X_\infty$ if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X_\infty| > \varepsilon) = 0$$

we say X_n converges in distribution to X_∞ and write $X_n \xrightarrow{d} X_\infty$

if $\lim_{n\to\infty} \mathbb{P}(X_n \le x) = \mathbb{P}(X_\infty \le x)$ for all x with $\mathbb{P}(X_\infty = x) = 0$

i.e. if $F_{X_n}(x) \to F_{X_\infty}(x)$ whenever F_{X_∞} is continuous at x.

Later we say X_n converges in distribution if for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X_\infty)]$.

But for now $\mathbb{E}$ not defined!

NB With $U_n = \sum_{k=1}^{n} \frac{1}{2^k} R_k = 0.R_1R_2 \ldots R_n$, we have $U_n \xrightarrow{a.s.} U$ and $|U_n - U| \le 2^{-n}$

so $U_n \xrightarrow{a.s.} U$ and $U_n \xrightarrow{P} U$. However, $\mathbb{P}(U_n \in \mathbb{Q}) = 1$, $\mathbb{P}(U \in \mathbb{Q}) = 0$, So a.s. conv/conv. in prob. does not imply $\mathbb{P}(X_n \in A) \to \mathbb{P}(X_\infty \in A)$ for all $A \in \mathcal{B}(\mathbb{R})$. More care is needed.

Example Let $(B_n, n\geq 1)$ be independent with $B_n = \text{Bernoulli}(\frac{1}{n})$.

Then $B_n \xrightarrow{P} 0$ because for all $\varepsilon>0$, $\mathbb{P}(|B_n - 0|>\varepsilon) = \mathbb{P}(B_n=1) = \frac{1}{n} \to 0$

However,

$$\mathbb{P}(\lim_{n\to\infty} B_n = 0) = \mathbb{P}(\overbrace{B_n=1 \text{ finitely often}}^{B_n=0 \text{ eventually}}) = 1 - \mathbb{P}(B_n=1 \text{ i.o.}) = 0$$

$\sum_{n\geq 1} \mathbb{P}(B_n=1) = \sum_{n\geq 1}\frac{1}{n} = \infty$ so $\mathbb{P}(B_n=1 \text{ i.o.}) = 1$ by the second Borel-Cantelli Lemma.

Therefore $X_n \xrightarrow{P} X_\infty \not\Rightarrow X_n \xrightarrow{a.s.} X_\infty$.

Proposition If $X_n \xrightarrow{a.s.} X_\infty$ then $X_n \xrightarrow{P} X_\infty$

Proof Fix $\varepsilon>0$. Note that $\{\lim_{n\to\infty} X_n = X_\infty\} \subseteq \{\limsup_{n\to\infty}|X_n - X_\infty| \leq \varepsilon\}$

and $\{\sup_{m\geq n}|X_m - X_\infty|\leq\varepsilon\} \uparrow \{\limsup_{n\to\infty}|X_n - X_\infty|\leq\varepsilon\}$

so
$$\mathbb{P}(\lim_{n\to\infty} X_n = X_\infty) \leq \mathbb{P}(\limsup_{n\to\infty}|X_n - X_\infty|\leq\varepsilon)$$
$$= \lim_{n\to\infty}\mathbb{P}(\sup_{m\geq n}|X_m - X_\infty|\leq\varepsilon) \leq \lim_{n\to\infty}\mathbb{P}(|X_n - X_\infty|\leq\varepsilon).$$

If $X_n \xrightarrow{a.s.} X_\infty$ then LHS $=1$ so $\lim_{n\to\infty}\mathbb{P}(|X_n-X_\infty|\leq\varepsilon)=1$ so $X_n \xrightarrow{P} X_\infty$ $\square$

Proposition If $X_n \xrightarrow{p} X_\infty$ then there exists a subsequence n_k s.t. $X_{n_k} \xrightarrow{a.s.} X_\infty$

Proof Suppose $X_n \xrightarrow{p} X_\infty$.

For each $k \in \mathbb{N}$ let n_k be large enough that $\mathbb{P}(|X_{n_k} - X_\infty| > \frac{1}{k}) < \frac{1}{2^k}$.

Then

$$\mathbb{P}\left(\lim_{k\to\infty} X_{n_k} \neq X_\infty\right) = \mathbb{P}\left(\exists m \in \mathbb{N} : \limsup_{k\to\infty} |X_{n_k} - X_\infty| > \frac{1}{m}\right)$$

$$\leq \sum_{m\in\mathbb{N}} \mathbb{P}\left(|X_{n_k} - X_\infty| > \frac{1}{m} \text{ i.o.}\right) = \circledast$$

However,

$$\sum_{k\geq 1} \mathbb{P}\left(|X_{n_k} - X_\infty| > \frac{1}{m}\right) \leq m + \sum_{k>m} \frac{1}{2^k} = m + \frac{1}{2^m} < \infty$$

so $\mathbb{P}(|X_{n_k} - X_\infty| > \frac{1}{m} \text{ i.o.}) = 0$ by first Borel-Cantelli lemma. So $\circledast = 0$. $\square$

Proposition If $X_n \xrightarrow{p} X_\infty$ then $X_n \xrightarrow{d} X_\infty$

Lemma: For any random variable X, the set $\{x \in \mathbb{R} : \mathbb{P}(X = x) > 0\}$ is countable.

Proof: For each x with $\mathbb{P}(X = x) > 0$ the interval $(\mathbb{P}(X<x), \mathbb{P}(X \leq x))$ is non-empty, so we may choose $q(x) \in \mathbb{Q} \cap (\mathbb{P}(X<x), \mathbb{P}(X\leq x))$. These intervals are disjoint so the values $q(x)$ are all distinct; so this gives an injective map from $\{x \in \mathbb{R} : \mathbb{P}(X=x) > 0\}$ to $\mathbb{Q}$. $\square$

Proof of Proposition

Fix $x \in \mathbb{R}$ with $\mathbb{P}(X_\infty = x) = 0$. Then $\mathbb{P}(X_\infty \le x) = \mathbb{P}(X_\infty < x) = \lim_{\delta \to 0} \mathbb{P}(X_\infty < x - \delta)$

and $\mathbb{P}(X_\infty \le x) = \lim_{\delta \to 0} \mathbb{P}(X_\infty \le x + \delta)$

So for all $\varepsilon > 0$ there is $\delta > 0$ s.t. $\mathbb{P}(X_\infty \le x - \delta), \mathbb{P}(X_\infty \le x + \delta) \in \mathbb{P}(X_\infty \le x) \pm \varepsilon$

Therefore,

$$\limsup_{n \to \infty} \mathbb{P}(X_n \le x) \le \limsup_{n \to \infty} \Big(\mathbb{P}(X_\infty \le x + \delta) + \mathbb{P}(|X_n - X_\infty| > \delta)\Big) = \mathbb{P}(X_\infty \le x + \delta) \le \mathbb{P}(X_\infty \le x) + \varepsilon$$

and

$$\liminf_{n \to \infty} \mathbb{P}(X_n \le x) \ge \liminf_{n \to \infty} \Big(\mathbb{P}(X_\infty \le x - \delta) - \mathbb{P}(|X_n - X_\infty| > \delta)\Big) = \mathbb{P}(X_\infty \le x - \delta) \ge \mathbb{P}(X_\infty \le x) - \varepsilon.$$

$\square$

Definition Fix random variables $(X_i, i \in I)$ with $X_i : \Omega_i \to \mathbb{R}$
some probability spaces $((\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i \in I)$

A **coupling** of $(X_i, i \in I)$ is a sequence $(Y_i, i \in I)$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $F_{X_i} = F_{Y_i}$ for all $i \in I$.

Example $X_1 \sim \mathrm{Unif}\{1, \dots, 6\}$, $X_2 = \mathrm{Unif}\{1, \dots, 6\}$

Could take $(\Omega, \mathcal{F}, \mathbb{P}) = ([6], 2^{[6]}, \text{Unif. measure})$, $X_1(\omega) = \omega$, $X_2(\omega) = 7 - \omega$. Or, $(\Omega, \mathcal{F}, \mathbb{P}) = ([6]^2, 2^{[6]^2}, \text{Unif. Measure on } [6]^2)$, $X_1(a,b) = a$, $X_2(a,b) = b$. (Independent coupling) or...

Proposition (Skorohod representation theorem)

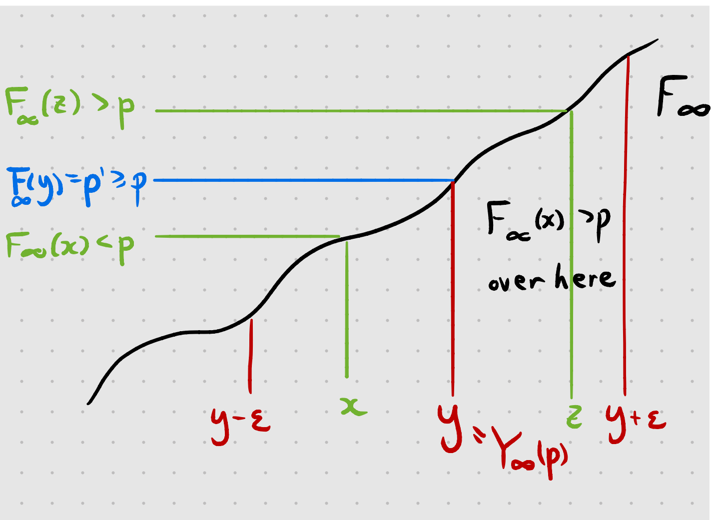
If $X_n \xrightarrow{d} X_\infty$ then there exists a coupling $(Y_n, 1 \le n \le \infty)$ of $(X_n, 1 \le n \le \infty)$ such that $Y_n \xrightarrow{a.s.} Y_\infty$.

Proof Our coupling lives on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \mathrm{Leb}_{[0,1]})$.

Let $U : \Omega \to \mathbb{R}$, $U(x) = x$, so U is Uniform$[0,1]$. Then, for $1 \le n \le \infty$, writing $F_n = F_{X_n}$ let $Y_n : \Omega \to \mathbb{R}$, $Y_n(p) = \inf\{x : F_n(x) \ge p\}$. Then $Y_n = Y_n(U)$ has $F_{Y_n} = F_n$

It remains to prove that $\mathbb{P}(\lim_{n\to\infty} Y_n = Y_\infty) = 1$.

Note: For all $1 \le n \le \infty$, $Y_n(p)$ is increasing so has at most countably many discontinuity points.



Fix $p \in [0,1]$ s.t. Y_∞ continuous at p

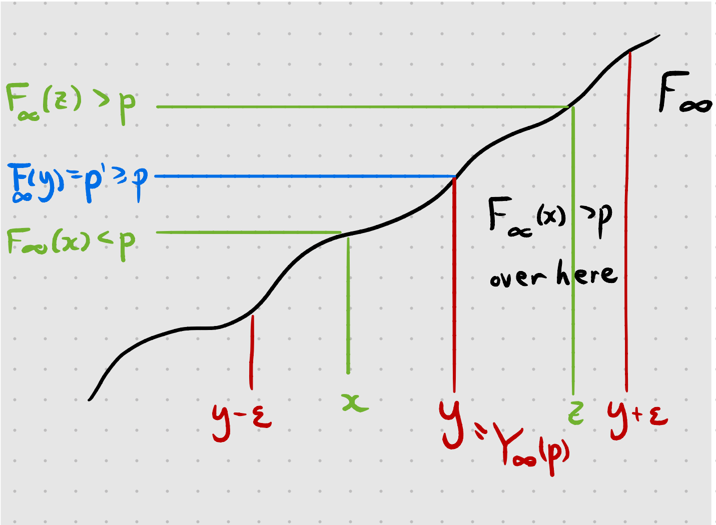
Write $y = Y_\infty(p) = \inf\{x \in \mathbb{R} : F_\infty(x) \ge p\}$

Then $F_\infty(x) < p$ for $x < y$, $F_\infty(y) \ge p$, and

and $F_\infty(z) > p$ for $z > y$

(if $F_\infty(z) = p$ for some $z > y$ then for all $q > p$, $Y_\infty(q) = \inf\{x : F_\infty(x) \ge q\} \ge z$, contradicting that Y_∞ cts at p)

Now fix $\varepsilon > 0$, and choose $x < y < z$ with x, z continuity points of F_∞ and with $z - x < \varepsilon$.



Then $F_\infty(x) < p$ and $F_\infty(z) > p$.

Since z is a continuity point of F_∞ and $X_n \xrightarrow{d} X_\infty$, it follows that $F_n(z) \to F_\infty(z)$ so $F_n(z) > p$ for all n sufficiently large.

Thus for n large $Y_n(p) \leq z < y+\varepsilon$.

Also, $F_\infty(x) < p$ and x is a continuity point of F_∞, so $F_n(x) < p$ for n large, so for n large,

$$Y_n(p) > x > y - \varepsilon.$$

Thus

$$y - \varepsilon < \liminf_{n\to\infty} Y_n(p) \leq \limsup_{n\to\infty} Y_n(p) \leq y+\varepsilon,$$

so $\lim_{n\to\infty} Y_n(p) = y$.

Since p was an arbitrary continuity point of Y_n, it follows that $Y_n \xrightarrow{a.e.} Y$. $\square$