Sep. 11

Lecture 5

Theorem. $\forall a, b \in \mathbb{R}$, $a<b$. $(a,b) \in \mathcal{M}$ and $m((a,b)) = b-a$.

Proof: Given $B \subseteq \mathbb{R}$, it's sufficient to prove that $m^*(B) \geq m^*(B \cap (a,b)) + m^*(B \cap (a,b)^c)$

W.L.O.G., assume $m^*(B) < \infty$. $\forall \varepsilon > 0$, choose an open-interval covering $\{I_n : n \geq 1\}$ of B s.t.

$\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(B) + \varepsilon$. For each $n \geq 1$, set $J_n := I_n \cap (a,b)$, $K_n := I_n \cap (-\infty, a]$, $L_n := I_n \cap [b, \infty)$

Then, J_n, K_n, L_n are almost disjoint intervals. So, $\ell(I_n) = \ell(J_n) + \ell(K_n) + \ell(L_n)$

Therefore, $m^*(B) \geq \sum_{n=1}^{\infty} \ell(I_n) - \varepsilon = \sum_{n=1}^{\infty} \ell(J_n) + \sum_{n=1}^{\infty} \left(\ell(K_n) + \ell(L_n)\right) - \varepsilon$

$d(K_n, L_n) > 0$

$$= \sum_{n=1}^{\infty} m^*(I_n \cap (a,b)) + \sum_{n=1}^{\infty} m^*(K_n \cup L_n) - \varepsilon$$

$\overset{\|}{I_n \cap (a,b)^c}$

(countable subadd.) $\geq m^*\left(\bigcup_{n=1}^{\infty} I_n \cap (a,b)\right) + m^*\left(\bigcup_{n=1}^{\infty} I_n \cap (a,b)^c\right) - \varepsilon$

(monotonicity) $\geq m^*(B \cap (a,b)) + m^*(B \cap (a,b)^c) - \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we conclude that $m^*(B) = m^*(B \cap (a,b)) + m^*(B \cap (a,b)^c)$ $\square$

Important corollary: $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}$. i.e. all Borel sets are Lebesgue measurable

§1.4. Properties of the Lebesgue measure.

<u>Remark</u>: The construction of the Lebesgue measure presented here is a particular example of the general result known as Carathéodory's Extension Theorem (not covered in this class)

<u>Proposition</u> The Lebesgue measure m has the following regularity properties

(1). $\forall A \in \mathcal{M}$. $\varepsilon > 0$, $\exists$ open set $G \subseteq \mathbb{R}$ s.t. $A \subseteq G$ and $m(G \setminus A) \leq \varepsilon$

(2). $\forall A \in \mathcal{M}$. $\varepsilon > 0$, $\exists$ closed set $F \subseteq \mathbb{R}$ s.t $F \subseteq A$ and $m(A \setminus F) \leq \varepsilon$

(3) $\forall A \in \mathcal{M}$. $m(A) = \inf\{m(G) : G \text{ open and } A \subseteq G\}$ i.e., m is outer regular.

(4) $\forall A \in \mathcal{M}$. $m(A) = \sup\{m(K) : K \text{ compact and } K \subseteq A\}$ i.e., m is inner regular.

(5). $\forall A \in \mathcal{M}$ with $m(A) < \infty$, $\forall \varepsilon > 0$. $\exists$ compact set $K \subseteq A$ s.t. $m(A \setminus K) < \varepsilon$.

(6) $\forall A \in \mathcal{M}$ with $m(A) < \infty$. $\forall \varepsilon > 0$, $\exists$ a finite union of (open) intervals $B = \bigcup_{n=1}^{N} I_n$
s.t. $m(A \triangle B) = m(A \setminus B) + m(B \setminus A) \leq \varepsilon$.

Proof: (1)(2)(4) are assignment problems. (3) follows from the same property of m^*, i.e.
$m^*(A) = \inf\{m^*(G) : G \text{ open and } A \subseteq G\}$

(5): Given $A \in \mathcal{M}$ with $m(A) < \infty$ and $\varepsilon > 0$, by (3), one can choose closed $F \subseteq A$ s.t. $m(A \setminus F) \le \varepsilon/2$.

For every $n \ge 1$, set $F_n := F \cap [-n, n]$. Then, F_n's are compact and $F_n \uparrow$ with $\bigcup_{n=1}^{\infty} F_n = F$.

Hence $A \setminus F_n \downarrow$ and $\bigcap_{n=1}^{\infty} (A \setminus F_n) = A \setminus F$. Since $m(A) < \infty$, the continuity of m from above implies that $\lim_{n\to\infty} m(A \setminus F_n) = m(A \setminus F) \le \varepsilon/2 \Rightarrow \exists N$ large enough s.t. $m(A \setminus F_N) \le \varepsilon$.

(6): Given $A \in \mathcal{M}$ with $m(A) < \infty$ and $\varepsilon > 0$, choose $\{I_n : n \ge 1\}$ open intervals s.t.

$A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $m(A) \ge \sum_{n=1}^{\infty} \ell(I_n) - \varepsilon/2$. Since $\sum_{n=1}^{\infty} \ell(I_n)$ is a convergent series, $\exists N \ge 1$ s.t. $\sum_{n=N+1}^{\infty} \ell(I_n) \le \varepsilon/2$. Set $B = \bigcup_{n=1}^{N} I_n$.

Then,
$$m(A \triangle B) = m\Big(A \setminus \bigcup_{n=1}^{N} I_n\Big) + m\Big(\bigcup_{n=1}^{N} I_n \setminus A\Big)$$
$$\le \sum_{n=N+1}^{\infty} \ell(I_n) + m\Big(\bigcup_{n=1}^{\infty} I_n \setminus A\Big) \le \varepsilon/2 + \sum_{n=1}^{\infty} \ell(I_n) - m(A) \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Proposition The Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$ is complete, i.e., given any $A \subseteq \mathbb{R}$, if $\exists B \in \mathcal{M}$ s.t. $A \subseteq B$ and $m(B) = 0$, then $A \in \mathcal{M}$ and $m(A) = 0$.

(any subset of a null set is a null set) equivalently, $\forall F \subseteq \mathbb{R}$. if $\exists E, G \in \mathcal{M}$ s.t. $E \subseteq F \subseteq G$ and $m(G \setminus E) = 0$,

then $F \in \mathcal{M}$ and $m(F) = m(E) = m(G)$.

Proof: Assume $A \subseteq \mathbb{R}$ is a subset of $B \in \mathcal{M}$ with $m(B) = 0$. To show $A \in \mathcal{M}$.

it suffices to show that $\forall E \subseteq \mathbb{R}$. $m^*(E) \geq m^*(E \cap A) + m^*(E \cap A^c)$

Since $B \in \mathcal{M}$ and $A \subseteq B$, $m^*(E \cap A^c) = m^*(E \cap A^c \cap B) + m^*(E \cap A^c \cap B^c)$

$$\leq m^*(B) + m^*(E \cap B^c) = 0 + m^*(E \cap B^c).$$

Obviously. $m^*(E \cap A) \leq m^*(E \cap B)$ ($m^*(E \cap B)$ is also 0, but we won't need it.)

$$m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E \cap B) + m^*(E \cap B^c) = m^*(E) \qquad \square.$$

Proposition (*) Up to rescaling, m is the unique (non-trivial) measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ (consider restriction of m on $\mathcal{B}_{\mathbb{R}}$) that is finite on compact sets and translation invariant. i.e. if μ is another such measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then $\mu = c \cdot m$ for some $c > 0$ ($c = \mu((0,1))$).

$\forall A \in \mathcal{B}_{\mathbb{R}}$ $\mu(A) = c\,m(A)$

The proof of the uniqueness of the Lebesgue measure makes use of the following general theorem (a corollary of Dynkin's π-d Theorem)

Theorem: Given a space X, $\mathcal{C}$ is called a π-system (of subsets of X) if $\mathcal{C}$ is a collection of subsets of X s.t. $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ (closed under finite intersection)

Assume μ_1 and μ_2 are two finite measures on $\sigma(\mathcal{C})$.

If $\mu_1(X) = \mu_2(X)$ and $\mu_1(A) = \mu_2(A)$ $\forall A \in \mathcal{C}$

then $\mu_1 = \mu_2$ on $\sigma(\mathcal{C})$. i.e., $\forall B \in \sigma(\mathcal{C})$. $\mu_1(B) = \mu_2(B)$.

In other words, two finite measures are identitical, if they have the same total mass and they match on a generating π-system.

Back to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, $\mathcal{C} := \{\phi\} \cup \{(a,b) : a, b \in \mathbb{R}, a<b\}$ is a generating π-system of $\mathcal{B}_{\mathbb{R}}$ (NOT the unique one)

$\forall n \geq 1$, $\mathcal{B}_{[-n,n]} := \sigma(\mathcal{C})_{\cap [-n,n]}$ is the Borel σ-algebra of subsets of $[-n,n]$,

and $\mathcal{C}_{\cap[-n,n]} := \{A \cap [-n,n] : A \in \mathcal{C}\}$ is a generating π-system of $\mathcal{B}_{[-n,n]}$.

With the theorem above, we first prove a related uniqueness result for the Lebesgue measure.

Proposition: If μ is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ s.t. $\mu(I) = \ell(I)$ for any interval $I \subseteq \mathbb{R}$, then $\mu = m$ (μ must be the Lebesgue measure).

Proof: $\forall n \geq 1$, consider μ and m restricted on $\mathcal{B}_{[-n,n]}$, where both of them are finite measures. and $\mu([-n,n]) = m([-n,n]) = 2n$. We have that $\mathcal{C}_{\cap[-n,n]}$ as defined above is a generating π-system of $\mathcal{B}_{[-n,n]}$. Since $\mu(\phi) = m(\phi) = 0$ and $\forall\, a, b \in \mathbb{R}$, $a<b$. $\mu((a,b) \cap [-n,n]) = m((a,b) \cap [-n,n]) = \ell((a,b) \cap [-n,n])$

That is, $\mu = m$ on $\mathcal{C}_{\cap[-n,n]}$. By the above theorem, $\mu = m$ on $\mathcal{B}_{[-n,n]}$.

For any $A \in \mathcal{B}_{\mathbb{R}}$, by the continuity from below.

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap [-n,n]) = \lim_{n\to\infty} m(A \cap [-n,n]) = m(A) \qquad \square$$

Now we prove the general uniqueness proposition (*)