Last class

Proposition Let F be any CDF. Then there exists a random variable $X: \Omega \to \mathbb{R}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \mathrm{Leb}_{[0,1]})$ such that X has CDF $F_X = F$.

Step 1 $F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x \ge 1 \end{cases}$. Let $U = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} \frac{1}{2^k} R_k}_{= 0.R_1R_2\ldots R_k}$.

Then $F_U = F$.

NB: The f^n $X: \Omega \to \mathbb{R}$ $X(\omega) = \omega$ also has

$$\mathbb{P}(X \le x) = \mathbb{P}(\{\omega : X(\omega) \le x\}) = \mathbb{P}(\{\omega : \omega \le x\}) = \mathrm{Leb}[0,x] = \begin{cases} 0, & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

which a) seems much easier and b) is actually equivalent.
The advantage of having done things "the hard way" is that it generalizes more easily when we want to construct multiple independent random variables.

Proposition Let F be any CDF. Then there exists a random variable $X:\Omega\to\mathbb{R}$ on probability space $(\Omega,\mathcal{F},\mathbb{P}) = ([0,1], \mathcal{B}[0,1], \mathrm{Leb}_{[0,1]})$ such that X has CDF $F_X = F$.

Proof **Step 1**

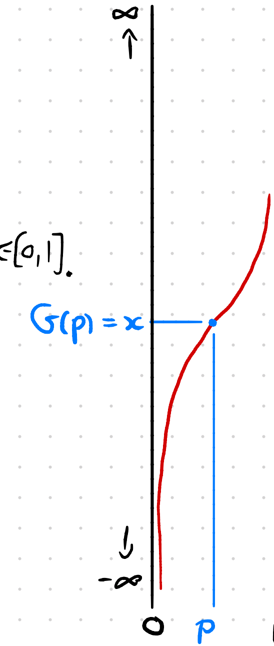
F is Uniform$[0,1]$ CDF, $F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$. Set $U = \sum_{k\ge 1} 2^{-k} R_k$.

U is $\uparrow$ limit of random variables so a random variable.

Then for any $n \ge 1$, $0 \le m < 2^n$, if $\frac{m}{2^n} = 0.b_1 b_2 \dots b_n$ then

$\mathbb{P}(U \in (m/2^n, (m+1)/2^n)) = \mathbb{P}(R_1 = b_1, \dots, R_n = b_n) = \frac{1}{2^n}$

So $\mathbb{P}(U \le \frac{m}{2^n}) = \frac{m}{2^n}$. Since this holds $\forall n \ge 1$, $0 \le m < 2^n$ it follows that $\mathbb{P}(U \le x) = x$ $\forall x \in [0,1]$.



Step 2 Fix any CDF $F:\mathbb{R}\to[0,1]$, and let $G:[0,1]\to\mathbb{R}$ be defined by

$G(p) = \inf\{x : F(x) \ge p\}$. (if F is invertible then $G = F^{-1}$)

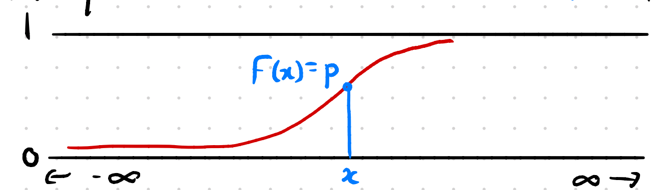
Exercise G is Borel.

Claim For $q \in [0,1]$, $r \in \mathbb{R}$, have $q \le F(r) \iff G(q) \le r$

Proof $\boxed{\text{If } q \le F(r)}$ then $\{x : F(x) \ge q\} \subset [r, \infty)$ so $G(q) = \inf\{x : F(x) \ge q\} \le r$

$\boxed{\text{If } F(r) < q}$ then by right-continuity $\exists\, s > r$ s.t. $F(s) < q$.

Then $\{x : F(x) \ge q\} \subset (s, \infty)$ so $G(q) \ge s > r$ $\square$



Now let $X = G(U)$. Then X is a r.v. since G is Borel, &

$\mathbb{P}(X \le r) = \mathbb{P}(G(U) \le r) = \mathbb{P}(U \le F(r)) = F(r)$ $\square$

Theorem Fix any seq. $(F_n, n \geq 1)$ of CDFs.
Then there exists a seq. of independent r.v.s $(X_n, n \geq 1)$ such that X_n has CDF F_n.

Proof Let's split the r.v.s $(R_n, n \geq 1)$ into infinitely many independent, infinite bunches.
E.g. List primes as $(p_1, p_2, p_3, \ldots) = (2, 3, 5, \ldots)$, set $Q_{j,k} := R_{p_j^k}$
so that $(Q_{i,k}, k \geq 1)$ is disjoint from $(Q_{j,k}, k \geq 1)$ for $i \neq j$.

Then let $U_i = \sum_{k \geq 1} \frac{1}{2^k} Q_{i,k}$. The r.v.s $(U_i, i \geq 1)$ are each Uniform$[0,1]$
and are independent since $\sigma(U_i) \subseteq \sigma(Q_{i,k}, k \geq 1) = \mathcal{F}_i$,
and the σ-fields $(\mathcal{F}_i, i \geq 1)$ are independent.

Now set $X_n = G_n(U_n)$, where $G_n(p) = \inf\{x : F_n(x) \geq p\}$.

Then X_n has CDF F_n (as in prop), and $(X_n, n \geq 1) = (G_n(U_n), n \geq 1)$ are independent since $(U_n, n \geq 1)$ are independent. $\square$

Exercise If $(Y_i, i \in I)$ are mutually independent, $(I_n, n \geq 1)$ partitions I and for each n, $g_n : \mathbb{R}^{I_n} \to \mathbb{R}$ is $(\sigma(Y_i, i \in I_n) / \mathcal{B}(\mathbb{R}))$-measurable, then with $X_n = g_n(Y_i, i \in I_n)$, the r.v.s $(X_n, n \geq 1)$ are independent.

Kolmogorov's 0-1 Law

Let $(X_n\ n\in\mathbb{N})$ be a countable collection of independent random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Write $\mathcal{T} := \bigcap_{M\subset\mathbb{N} \text{ finite}} \sigma(X_n,\ n\in\mathbb{N}\setminus M)$ for the tail σ-field.

Then for all $E\in\mathcal{T}$, $\mathbb{P}(E)\in\{0,1\}$.

Proof Fix $E\in\mathcal{T}$. For any $n\in\mathbb{N}$ and $F\in\sigma(X_n)$, since $\mathcal{T}\subset\sigma(X_m,\ m\in\mathbb{N}\setminus\{n\})$, the events E and F are independent.

Let $\mathcal{G}=\sigma(X_n, n\in\mathbb{N}) = \sigma\left(\bigcup_{n\in\mathbb{N}}\sigma(X_n)\right)$. Then E is indep. of all events in $\bigcup_{n\in\mathbb{N}}\sigma(X_n)$

so E is indep. of $\mathcal{G}$: For all $F\in\mathcal{G}$, $\mathbb{P}(E\cap F)=\mathbb{P}(E)\mathbb{P}(F)$.

However, $\mathcal{T}\subset\mathcal{G}$ so $E\in\mathcal{G}$ so $\mathbb{P}(E\cap E)=\mathbb{P}(E)\cdot\mathbb{P}(E)$; $\mathbb{P}(E)=\mathbb{P}(E^2)$ so $\mathbb{P}(E)\in\{0,1\}$ $\square$

Example Law of large numbers. Let $(X_n, n\geq 1)$ be independent r.v.s, $\mathcal{T}$ = Tail σ-field, let $S_n=\sum_{i=1}^{n}X_i$.

Write $S^+ = \limsup_{n\to\infty} S_n/n$, let $S^- = \liminf_{n\to\infty} S_n/n$.

For all $x\in\mathbb{R}$, $\{S^+\geq x\}\in\mathcal{T}$, so $\mathbb{P}(S^+\geq x)\in\{0,1\}$.

Letting $x^+ := \sup\{x : \mathbb{P}(S^+\geq x)=1\}$ then for $y>x^+$, $\mathbb{P}(S^+\geq y)<1$ so $\mathbb{P}(S^+\geq y)=0$.

Thus $\mathbb{P}(S^+=x^+)=1$; likewise $\mathbb{P}(S^-=x^-)=1$.

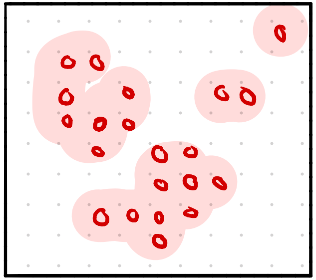
Also $\mathbb{P}(\lim_{n\to\infty}\frac{S_n}{n} \text{ exists})\in\{0,1\}$. [Because event is tail or because prob $=\begin{cases}1 & \text{if } x^+=x^- \\ 0 & \text{if } x^+\neq x^-\end{cases}$]

Example Percolation

Let $\mathbb{Z}^d$ be the d-dim. integer lattice, let $B=(B_v, v\in\mathbb{Z}^d)$ be indep. Bernoulli(p) random variables, let $\mathcal{T}$ = Tail σ-field of B. Write $\mathbb{Z}^d(B)=\{v\in\mathbb{Z}^d : B_v=1\}$.

For $x,y\in\mathbb{Z}^d$ say x is connected to y in $\mathbb{Z}^d(B)$, and write $x\overset{B}{\longleftrightarrow}y$, if there is a nearest-neighbour path from x to y containing only elements of $\mathbb{Z}^d(B)$.

Define $\mathcal{C}(x):=\{y\in\mathbb{Z}^d : x\overset{B}{\longleftrightarrow}y\}$. **Note** If $y\in\mathcal{C}(x)$ then $\mathcal{C}(x)=\mathcal{C}(y)$.



Let $E=\{\exists x\in\mathbb{Z}^d : |\mathcal{C}(x)|=\infty\}=\{\mathbb{Z}^d(B)$ contains an ∞ connected component$\}$

This is a tail event since an ∞ connected component can not be created or destroyed by finitely many sites. [Exercise: prove this carefully!]

Thus $\theta(p):=\mathbb{P}(E_\infty)\in\{0,1\}$.

Define $p_c=p_c(\mathbb{Z}^d):=\sup\{p : \theta(p)=0\}$. Then $\theta(p)=1$ for all $p>p_c$.

But unlike with $\limsup S_n/n$, this doesn't imply $\theta(p_c)=1$

Conjecture $\theta(p_c)=0$: at the "critical" value there is no ∞ connected component.