J. Math. Anal. Appl. 458 (2018) 1373–1389

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Ergodic properties of Lévy flights coexisting with subdiffusion and related models

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ARTICLE INFO

Article history:
Received 19 January 2017
Available online 10 October 2017
Submitted by R. Stelzer

Keywords:
Ergodicity
Mixing
Lévy flight
Stable process
Fractional Brownian motion

ABSTRACT

We analyze ergodic properties of two different stationary processes resulting from a combination of Lévy flights and subdiffusion. The first one comes from the Lamperti transformation of subordinated γ-stable process. The second one is a sequence of increments of subordinated Lévy process. We prove that both processes are mixing and ergodic. We also study increments and the Lamperti transformation of time-changed fractional Brownian motion. It appears that these models of anomalous diffusion are also ergodic and mixing. We believe that the derived results will help to verify ergodicity in single particle tracking experiments.

1. Introduction

Ergodicity is one of the key concepts in the theory of dynamical systems and also statistical physics. Suppose we have a dynamical system $(X, \mathcal{A}, \mu, S_t)$. Here X is a phase space, $\mathcal{A}$ is a σ-algebra on X, μ is a probability measure and $S_t : X \to X$, $t \geq 0$, are transformations with the semigroup property: $S_0(x) = x$ and $S_s(S_t(x)) = S_{s+t}(x)$ for $x \in X$, $t, s \geq 0$. It is also assumed that S_t preserve the measure μ, i.e. $\mu(S_t(A)) = \mu(A)$ for all $A \in \mathcal{A}$, $t \geq 0$. All the sets $A \in \mathcal{A}$ satisfying the condition $S_t(A) = A$ for all $t \geq 0$ are called invariant. The system is ergodic if every invariant set $A \in \mathcal{A}$ is trivial, that is $\mu(A) = 0$ or $\mu(A) = 1$. For the ergodic and measure-preserving dynamical system the celebrated Birkhoff's ergodic theorem holds [21]:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(S_t(x))dt = \int_X f(x)\mu(dx)$$

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https://doi.org/10.1016/j.jmaa.2017.10.007
0022-247X/

for all integrable functions. This property is of a great importance in physics. It implies that, for ergodic systems, information contained in one appropriately long trajectory is equal to the information contained in many trajectories.

Another important property closely related to ergodicity is mixing. The dynamical system is mixing if

$$\lim_{t\to\infty} \mu\left(A \cap S_t(B)\right) = \mu(A)\mu(B)$$

for all $A, B \in \mathcal{A}$. In other words the sets A and B are asymptotically independent under the transformation S_t. This is a stronger property than ergodicity [21]. Both of these properties – mixing and ergodicity can be analyzed in the language of stochastic processes. Suppose $Z(t)$, $t \geq 0$ is a real-valued process. In its canonical representation [12] it is defined by a probability measure $\mathbf{P}$ (which corresponds to μ in the language of dynamical systems) on the space of functions – $\mathbb{R}^{\mathbb{R}}$ (corresponds to X). In this space we consider the σ-algebra $\mathcal{B}$ generated by cylinder sets [14] (which corresponds to $\mathcal{A}$). The transformations are now defined as $S_t(f)(y) = f(t+y)$, where f is a function. Thus S_t is a usual shift operator, which describes evolution in time of $Z(t)$. In the context of stochastic processes, the fact that S_t is measure-preserving means that the process $Z(t)$ is stationary, that is $(Z(t_1+s), ..., Z(t_n+s)) \stackrel{d}{=} (Z(t_1), ..., Z(t_n))$ for all $n \in \mathbb{N}$, $0 \leq t_1 \leq ... \leq t_n$, $s \geq 0$. Here $\stackrel{d}{=}$ denotes equality of distributions. In other words all finite-dimensional distributions of $Z(t)$ are invariant to a time shift. Furthermore, a stochastic process $Z(t)$ is called mixing, if for all $m \in \mathbb{N}_0$, $M \in \mathbb{N}$, $0 \leq m \leq M$ and $0 < s_1 < ... < s_M$

$$(Z(s_1), ..., Z(s_m), Z(s_{m+1}+t), ...Z(s_M+t)) \xrightarrow[t\to\infty]{d} (Z(s_1), ..., Z(s_m), R(s_{m+1}), ..., R(s_M)),$$

where $R(t)$ is an independent copy of $Z(t)$.

In recent years the concept of ergodicity breaking has deeply penetrated the field of anomalous diffusion processes and fractional dynamics [3,36,7]. Ergodicity of generalized Langevin equation was studied in [2,20]. A lack of ergodic property in a dynamical system was reported for instance in blinking quantum dots [31]. Khinchin theorem for Lévy flights was proved in [41] (see also [24]). Application of the dynamical functional [8,9] in the verification of ergodicity breaking of experimental data can be found in [19,22,23,27]. In [28] and [25] the authors studied ergodic properties of some classes of anomalous diffusion processes. In [28] it was proved that the stationary fractional Ornstein–Uhlenbeck process $Z(t)$ defined via Lamperti transformation [18,6] of subdiffusive process $B(S_\alpha(t))$, i.e.

$$Z(t) = e^{-\frac{\alpha}{2}t} B(S_\alpha(e^t)),$$

is mixing and ergodic. Here $B(t)$ is a standard Brownian motion. Moreover $S_\alpha(t)$ is an inverse stable subordinator [4] independent of $B(t)$. It is given by

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \quad t \geq 0,$$

where $U_\alpha(t)$ is an α-stable subordinator – strictly increasing Lévy process with the Laplace transform

$$\mathbf{E}e^{-sU_\alpha(t)} = e^{-ts^\alpha}.$$

In the next section we significantly extend this result by replacing Brownian motion $B(t)$ with a γ-strictly stable process $L_\gamma(t)$ with a characteristic function

$$\mathbf{E}\exp\left(ixL_\gamma(t)\right) = \exp\left(t\phi(x)\right).$$

Here

$$\phi(x) = \begin{cases} -\sigma^\gamma |x|^\gamma \left(1 - i\beta \tan\left(\frac{\pi\gamma}{2}\right) \operatorname{sign}(x)\right) & \text{if } \gamma \in (0,2) \setminus \{1\} \\ -\sigma|x| \quad \text{if } \gamma = 1 \end{cases}$$

with $\sigma > 0$ being the scale parameter and $\beta \in [-1, 1]$ being the skewness parameter. The process $L_\gamma(t)$ is also called Lévy flight and is commonly used to model superdiffusive dynamics [34]. The inverse subordinator $S_\alpha(t)$ appears in the limit of continuous-time random walk with heavy tailed waiting times [34,29,26] and introduces the subdiffusive dynamics to the model. Combination of both process $L_\gamma(t)$ and $S(t)$ results in coexistence of Lévy flights and subdiffusion. This phenomenon can arise in physics when a particle performs long flights and is also intermittently trapped in space. Such pattern was observed for instance in an experiment conducted by Solomon, Weeks and Swinney on tracer particles in a two-dimensional rotating flow [39]. Also, the ergodic property plays an important role in the designing of experiments. If the process is ergodic, then instead of observing many trajectories, it is enough to analyze only one sufficiently long trajectory. And vice versa: sometimes obtaining long trajectories is impossible due to technical limitations, for instance in some fluorescent tracking experiments.

Other properties of the subordinated process $L_\gamma(S_\alpha(t))$, including a curious behavior of the first passage time, were extensively studied in [26,15,1]. The process $L_\gamma(S_\alpha(t))$ is α/γ-self-similar. Its probability density function $p(x,t)$ has been used to construct solutions of space-time fractional Cauchy problems of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} p(x,t) = \Delta^{\gamma/2} p(x,t)$$

on bounded domains [10], see also [30,35] for the case of non-zero drift and diffusion coefficients.

Here, we prove that the stationary process obtained via Lamperti transformation of $L_\gamma(S_\alpha(t))$, namely

$$Z(t) = e^{-\frac{\alpha}{\gamma}t} L_\gamma(S_\alpha(e^t)) \tag{1}$$

is mixing and ergodic.

In [25] ergodic properties of a different process were studied. Let $U_\eta(t)$ be an arbitrary subordinator with the Laplace transform

$$\mathbf{E} e^{-sU_\eta(t)} = e^{-t\eta(s)}.$$

The Laplace exponent $\eta(s)$ can be represented as [4]

$$\eta(s) = \lambda s + \int_{(0,\infty)} (1 - e^{sx})\, \nu(dx),$$

where λ is a drift parameter and $\nu(dx)$ is the Lévy measure. The measure ν is supported on a positive half line and satisfies the condition $\int_{(0,\infty)} \min(1,x)\, \nu(dx) < \infty$. For simplicity we take $\lambda = 0$. The inverse process is defined as

$$S_\eta(t) = \inf\{\tau > 0 : U_\eta(\tau) > t\}, \quad t \geq 0.$$

To ensure that trajectories of $S_\eta(t)$ are almost surely (a.s.) continuous we impose an additional condition on ν, namely $\nu(0,\infty) = \infty$. This causes that sample paths of U_η are a.s. strictly increasing and as a result we have the continuity of $S_\eta(t)$ (a.s.). Instead of taking Lamperti transformation we can analyze stationary increments of the process. However, $S_\eta(t)$ does not have stationary increments – we have to modify it. The generalized subordinator $\tilde{U}_\eta(t)$ is given by

$$\tilde{U}_\eta(t) = U_\eta(t) + U_0,$$

where U_0 is a random variable independent of $U_\eta(t)$ with distribution

$$\mathbf{P}(U_0 \leq x) = \frac{1}{\mathbf{E}U_\eta(1)} \int_0^x \nu\left([y, \infty)\right) dy.$$

It is necessary here to assume that $\mathbf{E}U_\eta(1) < \infty$. The generalized (delayed) inverse subordinator

$$\tilde{S}_\eta(t) = \inf\{\tau > 0 : \tilde{U}_\eta(\tau) > t\}, \quad t \geq 0$$

has now stationary increments (Proposition 4 in [17]). Note that each trajectory of $\tilde{S}_\eta(t)$ starts with a flat period of length U_0, which makes the increments of $\tilde{S}_\eta(t)$ stationary. This is in perfect analogy to the property of delayed renewal processes in the general renewal theory (see [17] and references therein). The inverse subordinator corresponds to a renewal process which can also be delayed in a similar manner to have stationary increments. Since $\tilde{S}_\eta(t)$ has stationary increments, also the process $X(t) = B(\tilde{S}_\eta(t))$ has stationary increments (we assume here that $B(t)$ and $\tilde{S}_\eta(t)$ are independent). Consequently, the sequence of increments

$$Y(n) = X(n+1) - X(n), \quad n \in \mathbb{N}$$

is a stationary process with discrete time. In [25] the authors proved that $Y(n)$ is ergodic and mixing. The third section of this article generalizes this result. Brownian motion $B(t)$ is replaced with an arbitrary Lévy process $L_\psi(t)$. The characteristic function of $L_\psi(t)$ is [38]

$$\mathbf{E}e^{ixL_\psi(t)} = e^{t\psi(x)}.$$

The characteristic exponent ψ is given by the Lévy–Khintchine formula

$$\psi(x) = ibx - \frac{1}{2}a^2x^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{ixy} - 1 - ixy\mathbf{1}_{|y|<1}\right) \mu(dx).$$

Here b is a drift parameter, a^2 is the variance of the Gaussian part of L_ψ and μ is the Lévy measure satisfying $\int_{\mathbb{R}\setminus\{0\}} \min\left(1, x^2\right) \mu(dx) < \infty$. We prove that the process

$$Y(n) = X(n+1) - X(n), \quad n \in \mathbb{N}, \tag{2}$$

where $X(t) = L_\psi(\tilde{S}_\eta(t))$, with $L_\psi(t)$ and $\tilde{S}_\eta(t)$ being independent, is mixing and ergodic. The last two sections are devoted to the time-changed fractional Brownian motion and its ergodic properties.

2. Lamperti transformation of the subordinated γ-stable process

In this section we prove that the Lamperti transformation of the process $L_\gamma(S_\eta(t))$ is ergodic and mixing. Before we state and prove the theorem let us begin with the following lemma.

Lemma 1. *The stationary process $W(t) = e^{-\alpha t}S_\alpha(e^t)$ is mixing.*

Proof. In the proof we use the method from [28]. To prove the mixing property it suffices to show the following convergence of multidimensional characteristic functions (see Eq. 5.13 in [32])

$$\begin{aligned}&\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k W\left(s_k\right)+i\sum_{k=m+1}^{M} z_k W\left(s_k+t\right)\right)\\&\xrightarrow{t\to\infty}\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k W\left(s_k\right)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k W(s_k)\right)\end{aligned}\tag{3}$$

for every $m \in \mathbb{N}_0, M \in \mathbb{N}$, $0 \le m \le M$ and $z_1, ..., z_M \in \mathbb{R}$, $0 < s_1 < ... < s_M$. For technical reasons it is more convenient to prove a more general formula

$$\begin{aligned}&\mathbf{E}\exp\left(\sum_{k=1}^{m} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}-x\right)+\sum_{k=m+1}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha\left(e^{s_k+t}-x\right)\right)\\&\xrightarrow{t\to\infty}\mathbf{E}\exp\left(\sum_{k=1}^{m} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}-x\right)\right)\mathbf{E}\exp\left(\sum_{k=m+1}^{M} a_k e^{-\alpha s_k} S_\alpha(e^{s_k})\right)\end{aligned}\tag{4}$$

for every $x \in [0, e^{s_1}]$, $m \in \mathbb{N}_0, M \in \mathbb{N}$, $0 \le m \le M$, $a_k = ib_k$, $b_k \in \mathbb{R}$, $k = 1, ..., M$, $0 < s_1 < ... < s_M$. Notice that when $x = 0$ we recover Eq. (3). The proof of Eq. (4) is by induction on M.

Step I – $M = 1$. There are two cases – $m = 0$ and $m = 1$. In the former one we use the formula for Laplace transform of the inverse subordinator $\mathbf{E}\exp\left(zS_\alpha(t)\right) = E_\alpha(zt^\alpha)$, where $E_\alpha(x)$ is the Mittag-Leffler function [37]:

$$E_\alpha(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}.$$

Then the left hand side of our Eq. (4) reads

$$\begin{aligned}&\mathbf{E}\exp\left(a_1 e^{-\alpha(s_1+t)} S_\alpha\left(e^{s_1+t}-x\right)\right)=E_\alpha\left(a_1 e^{-\alpha(s_1+t)}\left(e^{s_1+t}-x\right)^\alpha\right)\\&\xrightarrow{t\to\infty} E_\alpha\left(a_1\right)=\mathbf{E}\exp\left(a_1 e^{-\alpha s_1} S_\alpha\left(e^{s_1}\right)\right),\end{aligned}$$

hence Eq. (4) holds in this case. When $m = 1$ Eq. (4) is trivially fulfilled.

Step II – assume that Eq. (4) holds for $M - 1$. We will prove it for M. Throughout the proof the following formula for finite dimensional distribution of $S_\alpha(t)$ will be used repeatedly [13]:

$$\begin{aligned}&\mathbf{E}\exp\left(\sum_{k=1}^{n} z_k S_\alpha(t_k)\right)=\mathbf{E}\exp\left(\sum_{k=2}^{n} z_k S_\alpha(t_k)\right)\\&+\frac{z_1}{\sum_{k=1}^{n} z_k}\int_0^{t_1}\mathbf{E}\exp\left(\sum_{k=2}^{n} z_k S_\alpha(t_k-y)\right)d_y E_\alpha\left(y^\alpha\sum_{k=1}^{n} z_k\right),\end{aligned}\tag{5}$$

for $z_k \in \mathbf{C}$, $n \ge 2$ and $0 \le t_1 \le ... \le t_n$. In [13] this is given only for $z_k \in \mathbb{R}$, but the proof of the formula remains valid if we generalize it to complex coefficients. We will analyze here also two cases – $m = 0$ and $m \ne 0$. For the first one we can transform the left hand side of Eq. (4) using the above recurrence formula into two summands:

$$\mathbf{E}\exp\left(\sum_{k=1}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha\left(e^{s_k+t}-x\right)\right) = \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha\left(e^{s_k+t}-x\right)\right)$$
$$+\frac{z_1 e^{-\alpha s_1}}{\sum_{k=1}^{M} a_k e^{-\alpha s_k}} \int\limits_0^{e^{s_1+t}-x} \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^{s_k+t}-x-y)\right) d_y E_\alpha\left(y^\alpha \sum_{k=1}^{M} a_k e^{-\alpha(s_k+t)}\right). \tag{6}$$

From the induction assumption the first summand converges to

$$\mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}\right)\right).$$

For the second summand of Eq. (6) a substitution $y = ue^t$ in the integral yields

$$\int\limits_0^{e^{s_1+t}-x} \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^{s_k+t}-x-y)\right) d_y E_\alpha\left(y^\alpha \sum_{k=1}^{M} a_k e^{-\alpha(s_k+t)}\right)$$
$$= \int\limits_0^{e^{s_1}-xe^{-t}} \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^t(e^{s_k}-u)-x)\right) d_u E_\alpha\left(u^\alpha \sum_{k=1}^{M} a_k e^{-\alpha s_k}\right).$$

Let us denote $\tau_k = \log\left(e^{s_k}-u\right)$ and $y_k = a_k e^{\alpha(\tau_k - s_k)}$. For all $u \in (0, e^{s_1})$ there exists $T > 0$ such that for $t > T$ we have from the induction assumption

$$\mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^t(e^{s_k}-u)-x)\right) = \mathbf{E}\exp\left(\sum_{k=2}^{M} y_k e^{-\alpha(\tau_k+t)} S_\alpha(e^{\tau_k+t}-x)\right)$$
$$\xrightarrow{t\to\infty} \mathbf{E}\exp\left(\sum_{k=2}^{M} y_k e^{-\alpha\tau_k} S_\alpha(e^{\tau_k})\right) = \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha s_k} S_\alpha(e^{s_k}-u)\right).$$

Now since $e^{s_1} - xe^{-t} \le e^{s_1}$ and

$$\left\|\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^t(e^{s_k}-u)-x)\right)\right\| < 1$$

we may apply the dominated convergence theorem

$$\int\limits_0^{e^{s_1}-xe^{-t}} \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^t(e^{s_k}-u)-x)\right) d_u E_\alpha\left(u^\alpha \sum_{k=1}^{M} a_k e^{-\alpha s_k}\right)$$
$$\xrightarrow{t\to\infty} \int\limits_0^{e^{s_1}} \mathbf{E}\exp\left(\sum_{k=2}^{M} a_k e^{-\alpha s_k} S_\alpha(e^{s_k}-u)\right) d_u E_\alpha\left(u^\alpha \sum_{k=1}^{M} a_k e^{-\alpha s_k}\right).$$

This combined with Eq. (5) ends the proof of the case $m = 0$. Let us move to the second possible case $m \neq 0$. Once again from Eq. (5) the left hand side of Eq. (4) equals

$$
\begin{aligned}
&\mathbf{E}\exp\left(\sum_{k=2}^{m} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}-x\right)+\sum_{k=m+1}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha\left(e^{s_k+t}-x\right)\right)\\
&\quad+\frac{a_1 e^{-\alpha s_1}}{\sum_{k=1}^{m} a_k e^{-\alpha s_k}+\sum_{k=m+1}^{M} a_k e^{-\alpha(s_k+t)}}\\
&\quad\times\int_0^{e^{s_1}-x}\mathbf{E}\exp\left(\sum_{k=2}^{m} a_k e^{-\alpha s_k} S_\alpha(e^{s_k}-x-y)+\sum_{k=m+1}^{M} a_k e^{-\alpha(s_k+t)} S_\alpha(e^{s_k+t}-x-y)\right)\\
&\qquad\qquad d_y E_\alpha\left(y^\alpha\left(\sum_{k=1}^{m} a_k e^{-\alpha s_k}+\sum_{k=m+1}^{M} a_k e^{-\alpha(s_k+t)}\right)\right).
\end{aligned}
\tag{7}
$$

From the induction assumption the first summand converges to

$$
\mathbf{E}\exp\left(\sum_{k=2}^{m} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}-x\right)\right)\mathbf{E}\exp\left(\sum_{k=m+1}^{M} a_k e^{-\alpha s_k} S_\alpha\left(e^{s_k}\right)\right).
$$

The induction assumption together with the dominated convergence theorem also yield that the limit of the second summand of Eq. (7) when $t\to\infty$ is

$$
\begin{aligned}
&\mathbf{E}\exp\left(\sum_{k=m+1}^{M} a_k e^{-\alpha s_k} S_\alpha(e^{s_k})\right)\frac{a_1 e^{-\alpha s_1}}{\sum_{k=1}^{m} a_k e^{-\alpha s_k}}\\
&\quad\times\int_0^{e^{s_1}-x}\mathbf{E}\exp\left(\sum_{k=2}^{m} a_k e^{-\alpha s_k} S_\alpha(e^{s_k}-x-y)\right) d_y E_\alpha\left(y^\alpha\sum_{k=1}^{m} a_k e^{-\alpha s_k}\right).
\end{aligned}
$$

Therefore by Eq. (5) we obtain Eq. (4) which ends the proof. □

Since mixing is stronger than ergodicity, we obtain

Corollary 1. *The stationary process $W(t)=e^{-\alpha t}S_\alpha(t)$ is ergodic.*

We now move to the main theorem of this section.

Theorem 1. *The process $Z(t)$ defined by Eq. (1) is mixing.*

Proof. Similarly as in Lemma 1, in order to prove that $Z(t)$ is mixing it is sufficient to prove that (see Eq. 5.13 in [32])

$$
\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k Z(s_k)+i\sum_{k=m+1}^{M} z_k Z(s_k+t)\right)\xrightarrow{t\to\infty}\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k Z(s_k)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k Z(s_k)\right),
$$

for each choice of $m,M\in\mathbb{N}$, $1\le m\le M$, $0<s_1<\ldots<s_M$ and $z_1,\ldots,z_M\in\mathbb{R}$. The characteristic function of a finite-dimensional distribution of Lévy process $L_\gamma(t)$ is given by:

$$
\mathbf{E}\exp\left(i\sum_{k=1}^{M} a_k L_\gamma(t_k)\right)=\exp\left(\sum_{k=1}^{M} t_k\left(\phi\left(\sum_{j=k}^{M} a_j\right)-\phi\left(\sum_{j=k+1}^{M} a_j\right)\right)\right),
$$

where $M \in \mathbb{N}$, $0 \leq t_1 \leq ... \leq t_M$ and $a_1, ..., a_M \in \mathbb{R}$. To simplify the notation let us put for $k \in \{1, ..., M\}$:

$$a_k = \begin{cases} z_k e^{-\frac{\alpha}{\gamma} s_k} & \text{if } k \in \{1, ..., m\} \\ z_k e^{-\frac{\alpha}{\gamma}(s_k+t)} & \text{if } k \in \{m+1, ..., M\} \end{cases}, \quad c_k = \phi\left(\sum_{j=k}^{M} a_j\right) - \phi\left(\sum_{j=k+1}^{M} a_j\right).$$

Since $L_\gamma(t)$ and $S_\alpha(t)$ are independent we get

$$\begin{aligned} &\mathbf{E} \exp\left(i \sum_{k=1}^{m} z_k e^{-\frac{\alpha}{\gamma} s_k} L_\gamma(S_\alpha(e^{s_k})) + i \sum_{k=m+1}^{M} z_k e^{-\frac{\alpha}{\gamma}(s_k+t)} L_\gamma(S_\alpha(e^{s_k+t}))\right) \\ &\quad = \mathbf{E} \exp\left(i \sum_{k=1}^{m} c_k S_\alpha(e^{s_k}) + i \sum_{k=m+1}^{M} c_k S_\alpha(e^{s_k+t})\right). \end{aligned}$$

Now set

$$\begin{aligned} \tilde{a_k} &= z_k e^{-\frac{\alpha}{\gamma} s_k} \quad k = 1, ..., M \\ \tilde{c_k} &= \begin{cases} \phi\left(\sum_{j=k}^{m} \tilde{a_j}\right) - \phi\left(\sum_{j=k+1}^{m} \tilde{a_j}\right) & \text{if } k \in 1, ..., m \\ \phi\left(\sum_{j=k}^{M} \tilde{a_j}\right) - \phi\left(\sum_{j=k+1}^{M} \tilde{a_j}\right) & \text{if } k \in m+1, ..., M. \end{cases} \end{aligned}$$

From Lemma 1 the process $e^{-\alpha t} S_\alpha(e^t)$ is mixing and

$$\begin{aligned} &\mathbf{W}_t = \left(e^{-s_1\alpha} S_\alpha(e^{s_1}), ..., e^{-s_m\alpha} S_\alpha(e^{s_m}), e^{-(s_{m+1}+t)\alpha} S_\alpha(e^{s_{m+1}+t}), ..., e^{-(s_M+t)\alpha} S_\alpha(e^{s_M+t})\right) \\ &\xrightarrow[t\to\infty]{d} \left(e^{-s_1\alpha} S_\alpha(e^{s_1}), ..., e^{-s_m\alpha} S_\alpha(e^{s_m}), e^{-s_{m+1}\alpha} R_\alpha(e^{s_{m+1}}), ..., e^{-s_M\alpha} R_\alpha(e^{s_M})\right) = \mathbf{W}, \end{aligned}$$

where $R_\alpha(t)$ is an independent copy of $S_\alpha(t)$. Therefore a linear combination of the components of the random vector $\mathbf{W}_t$ also converges to the corresponding linear combination of $\mathbf{W}$

$$\sum_{k=1}^{m} \tilde{c_k} S_\alpha(e^{s_k}) + \sum_{k=m+1}^{M} \tilde{c_k} e^{-\alpha t} S_\alpha(e^{s_k+t}) \xrightarrow[t\to\infty]{d} \sum_{k=1}^{m} \tilde{c_k} S_\alpha(e^{s_k}) + \sum_{k=m+1}^{M} \tilde{c_k} R_\alpha(e^{s_k}). \tag{8}$$

Notice that $\tilde{c_k} = e^{\alpha t} c_k$ for $k \in \{m+1, ...M\}$, therefore

$$\sum_{k=m+1}^{M} c_k S_\alpha(e^{s_k+t}) = \sum_{k=m+1}^{M} \tilde{c_k} e^{-\alpha t} S_\alpha(e^{s_k+t}). \tag{9}$$

Furthermore for $k \in 1, ..., m$ we have $\tilde{a_k} = a_k$ and thus

$$c_k - \tilde{c_k} = \phi\left(\sum_{j=k}^{M} a_j\right) - \phi\left(\sum_{j=k+1}^{M} a_j\right) - \left(\phi\left(\sum_{j=k}^{m} a_j\right) - \phi\left(\sum_{j=k+1}^{m} a_j\right)\right) \xrightarrow{t\to\infty} 0.$$

This, in turn, implies that

$$\sum_{k=1}^{m} (c_k - \tilde{c_k}) S_\alpha(e^{s_k}) \xrightarrow[t\to\infty]{a.s.} 0. \tag{10}$$

After combining Eqs. (8), (9) and (10) we get

$$\sum_{k=1}^{m} c_k S_\alpha(e^{s_k}) + \sum_{k=m+1}^{M} c_k S_\alpha(e^{s_k+t}) \xrightarrow[t\to\infty]{d} \sum_{k=1}^{m} \tilde{c_k} S_\alpha(e^{s_k}) + \sum_{k=m+1}^{M} \tilde{c_k} R_\alpha(e^{s_k}).$$

Notice that

$$\left\| \exp\left[i\sum_{k=1}^{m} c_k S_\alpha(e^{s_k}) + i\sum_{k=m+1}^{M} c_k S_\alpha(e^{s_k+t}) \right] \right\| < 1$$

almost surely and the convergence in distribution together with independence of $S_\alpha(t)$ and $R_\alpha(t)$ implies that

$$\begin{aligned}
&\mathbf{E}\exp\left(i\sum_{k=1}^{m} c_k S_\alpha(e^{s_k}) + i\sum_{k=m+1}^{M} c_k S_\alpha(e^{s_k+t}) \right) \xrightarrow{t\to\infty} \mathbf{E}\exp\left(i\sum_{k=1}^{m} \tilde{c_k} S_\alpha(e^{s_k}) + i\sum_{k=m+1}^{M} \tilde{c_k} R_\alpha(e^{s_k}) \right) \\
&= \mathbf{E}\exp\left(i\sum_{k=1}^{m} \tilde{c_k} S_\alpha(e^{s_k}) \right) \mathbf{E}\exp\left(i\sum_{k=m+1}^{M} \tilde{c_k} R_\alpha(e^{s_k}) \right) \\
&= \mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k e^{-\frac{\alpha}{\gamma}s_k} L_\gamma(S_\alpha(e^{s_k})) \right) \mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k e^{-\frac{\alpha}{\gamma}s_k} L_\gamma(S_\alpha(e^{s_k})) \right)
\end{aligned}$$

which ends the proof. □

Corollary 2. *The process $Z(t) = e^{-\frac{\alpha}{\gamma}t} L_\gamma(S_\alpha(e^t))$ is ergodic.*

3. Increments of Lévy flight with infinitely divisible waiting times

We start with the following lemma concerning the increments of the generalized inverse subordinator.

Lemma 2. *The stationary discrete-time process $W(n) = \tilde{S}_\eta(n+1) - \tilde{S}_\eta(n)$ is mixing.*

Proof. It is enough to show that

$$\begin{aligned}
&\mathbf{E}\exp\left(\sum_{k=1}^{m} a_k \tilde{S}_\eta(s_k) + \sum_{k=m+1}^{M} a_k \left(\tilde{S}_\eta(s_k+n+1) - \tilde{S}_\eta(s_k+n) \right) \right) \\
&\xrightarrow{n\to\infty} \mathbf{E}\exp\left(\sum_{k=1}^{m} a_k \tilde{S}_\eta(s_k) \right) \mathbf{E}\exp\left(\sum_{k=m+1}^{M} a_k \left(\tilde{S}_\eta(s_k+1) - \tilde{S}_\eta(s_k) \right) \right),
\end{aligned}$$

for $m, M \in \mathbb{N}$, $1 \le m \le M$, $a_k = ib_k$ and $b_k \in \mathbb{R}$ for $k = 1, ..., M$, $0 < s_1 < ... < s_M$. The proof goes by induction on M, very similarly as in [25] and in Lemma 1. Here we only provide two key formulas. The first one concerns $\tilde{S}_\eta(t)$ [13]:

$$\begin{aligned}
&\mathbf{E}\exp\left(\sum_{k=1}^{M} c_k \tilde{S}_\eta(t_k) \right) = \mathbf{E}\exp\left(\sum_{k=2}^{M} c_k \tilde{S}_\eta(t_k) \right) \\
&+ \frac{c_1}{\sum_{k=1}^{M} c_k} \int_0^{t_1} \mathbf{E}\exp\left(\sum_{k=2}^{M} c_k S_\eta(t_k - y) \right) d_y \mathbf{E}\exp\left(\tilde{S}_\eta(y) \sum_{k=1}^{M} c_k \right),
\end{aligned}$$

where $M \geq 2$, $0 \leq t_1 \leq ... \leq t_M$ and $c_1, ..., c_M \in \mathbb{C}$. The second one is for $S_\eta(t)$ and comes from Lemma 7 in [25]:

$$\mathbf{E}\exp\left(\sum_{k=1}^{m} c_k S_\eta(t_k) + \sum_{k=m+1}^{M} c_k \left(S_\eta(t_k+n+1) - S_\eta(t_k+n)\right)\right)$$
$$\xrightarrow{n\to\infty} \mathbf{E}\exp\left(\sum_{k=1}^{m} c_k S_\eta(t_k)\right)\mathbf{E}\exp\left(\sum_{k=m+1}^{M} c_k \left(\tilde{S}_\eta(t_k+1) - \tilde{S}_\eta(t_k)\right)\right),$$

where $M, m \in \mathbb{N}$, $0 \leq m \leq M$, $0 \leq t_1 \leq ... \leq t_M$, $c_1, ..., c_m \in \mathbb{C}$ and $c_{m+1}, ..., c_M \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}[z] \leq 0\}$. $\square$

Corollary 3. *The stationary process $W(n) = \tilde{S}_\eta(n+1) - \tilde{S}_\eta(n)$ is ergodic.*

Next we move to the main theorem of this section.

Theorem 2. *The stationary discrete-time process $Y(n)$ given by Eq. (2) is mixing.*

Proof. To prove that $Y(n)$ is mixing it is enough to show that (see [32])

$$\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k Y(s_k) + i\sum_{k=m+1}^{M} z_k Y(s_k+n)\right) \xrightarrow{n\to\infty} \mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k Y(s_k)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k Y(s_k)\right),$$

for each choice of $M, m \in \mathbb{N}$, $0 < s_1 < ... < s_n \in \mathbb{N}$ and $z_1, ..., z_n \in \mathbb{R}$. In our case this convergence is equivalent to

$$\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k \left(L_\psi(\tilde{S}_\eta(s_k+1)) - L_\psi(\tilde{S}_\eta(s_k))\right) + i\sum_{k=m+1}^{M} z_k \left(L_\psi(\tilde{S}_\eta(s_k+n+1)) - L_\psi(\tilde{S}_\eta(s_k+n))\right)\right)$$
$$\xrightarrow{n\to\infty} \mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k \left(L_\psi(\tilde{S}_\eta(s_k+1)) - L_\psi(\tilde{S}_\eta(s_k))\right)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} \left(L_\psi(\tilde{S}_\eta(s_k+1)) - L_\psi(\tilde{S}_\eta(s_k))\right)\right).$$

We know that L_ψ has independent and stationary increments, moreover L_ψ and $\tilde{S}_\eta$ are independent. Therefore the above condition is tantamount to

$$\mathbf{E}\exp\left(\sum_{k=1}^{m} \psi(z_k)(\tilde{S}_\eta(s_k+1)) - \tilde{S}_\eta(s_k)) + \sum_{k=m+1}^{M} \psi(z_k)\left(\tilde{S}_\eta(s_k+n+1) - \tilde{S}_\eta(s_k+n)\right)\right)$$
$$\xrightarrow{n\to\infty} \mathbf{E}\exp\left(\sum_{k=1}^{m} \psi(z_k)\left(\tilde{S}_\eta(s_k+1) - \tilde{S}_\eta(s_k)\right)\right)\mathbf{E}\exp\left(\sum_{k=m+1}^{M} \psi(z_k)\left(\tilde{S}_\eta(s_k+1) - \tilde{S}_\eta(s_k)\right)\right),$$

which follows from Lemma 2. $\square$

Corollary 4. *The stationary discrete-time process $Y(n)$ given by Eq. (2) is ergodic.*

4. Lamperti transformation of subordinated fractional Brownian motion

The fractional Brownian motion (FBM) $B_H(t)$ is a Gaussian process with zero mean and autocovariance function given by

$$\mathbf{E}B_H(t)B_H(s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0,$$

where $H \in (0,1)$ is the selfsimilarity (Hurst) index. For $H = 1/2$ we recover the standard Brownian motion. It is a known fact that the increments of B_H are stationary. For $H \neq 1/2$ they are not independent. The variance of FBM equals $\mathbf{E}B_H^2(t) = t^{2H}$ which means that when $H > 1/2$ FBM is superdiffusive and when $H < 1/2$ FBM displays subdiffusive behavior. Moreover the process $B_H(t)$ is self-similar with index H. After changing the time of B_H by the independent inverse subordinator $S_\alpha(t)$, which is α-self-similar, we obtain the process $B_H(S_\alpha(t))$ which is αH-self-similar. This process is used to model fractional dynamics of different types [33,40]. Large deviations of $B_H(S_\alpha(t))$ were investigated in [33,11] In what follows we verify ergodic properties of the Lamperti transformation of $B_H(S_\alpha(t))$, namely we study the stationary process

$$X(t) = e^{-\alpha H t} B_H\left(S_\alpha\left(e^t\right)\right). \tag{11}$$

We are going to prove that this process is ergodic and mixing.

Theorem 3. *The process $X(t)$ defined by Eq. (11) is mixing.*

Proof. In order to prove the mixing property of $X(t)$ it is enough to show that [32]

$$\begin{aligned} &\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha H} B_H\left(S_\alpha(e^{s_k})\right) + i\sum_{k=m+1}^{M} z_k e^{-(s_k+t)\alpha H} B_H\left(S_\alpha(e^{s_k+t})\right)\right) \\ &\xrightarrow{t\to\infty} \mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha H} B_H\left(S_\alpha(e^{s_k})\right)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k e^{-s_k\alpha H} B_H\left(S_\alpha(e^{s_k})\right)\right), \end{aligned} \tag{12}$$

for every $m \in \mathbb{N}_0, M \in \mathbb{N}$, $0 \leq m \leq M$ and $z_1, ..., z_n \in \mathbb{R}$, $0 < s_1 < ... < s_n$. Now using the independence of $B_H(t)$ and $S_\alpha(t)$ together with the fact that the autocovariance function of the FBM equals:

$$\mathbf{E}B_H(t)B_H(s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0,$$

we obtain that left hand side of Eq. (12) equals

$$\mathbf{E}\exp\left(-\frac{1}{2}yAy^T\right) = \mathbf{E}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{M} a_{i,j}y_iy_j\right), \tag{13}$$

where

$$y = \left(z_1e^{-s_1\alpha H}, ..., z_me^{-s_m\alpha H}, z_{m+1}e^{-(s_{m+1}+t)\alpha H}, ...z_Me^{-(s_M+t)\alpha H}\right)$$

is a coefficient vector, y^T denotes the transposed vector and $A = [a_{i,j}]$ is a $M \times M$ covariance matrix:

$$a_{i,j} = \begin{cases} S_\alpha^{2H}(e^{s_j}) \quad \text{if } i = j \leq m \\ S_\alpha^{2H}(e^{s_j+t}) \quad \text{if } i = j > m \\ \frac{1}{2}\left(S_\alpha^{2H}(e^{s_i}) + S_\alpha^{2H}(e^{s_j}) - |S_\alpha(e^{s_i}) - S_\alpha(e^{s_j})|^{2H}\right) \quad \text{if } i \neq j \wedge i,j < m \\ \frac{1}{2}\left(S_\alpha^{2H}(e^{s_i+t}) + S_\alpha^{2H}(e^{s_j}) - (S_\alpha(e^{s_i+t}) - S_\alpha(e^{s_j}))^{2H}\right) \quad \text{if } j \leq m < i \\ \frac{1}{2}\left(S_\alpha^{2H}(e^{s_j+t}) + S_\alpha^{2H}(e^{s_i}) - (S_\alpha(e^{s_j+t}) - S_\alpha(e^{s_i}))^{2H}\right) \quad \text{if } i \leq m < j \\ \frac{1}{2}\left(S_\alpha^{2H}(e^{s_j+t}) + S_\alpha^{2H}(e^{s_i+t}) - |S_\alpha(e^{s_j+t}) - S_\alpha(e^{s_i+t})|^{2H}\right) \quad \text{if } i \neq j \wedge i,j > m. \end{cases}$$

We will deal with each of the summands $a_{i,j}y_iy_j$ separately. Firstly, we will show that when $j \leq m < i$ or $i \leq m < j$ we get $a_{i,j}y_iy_j \xrightarrow{d} 0$ when $t \to \infty$. Indeed, let us assume that $j \leq m < i$ (the proof is identical when the second inequality holds). Then

$$a_{i,j}y_iy_j = \frac{1}{2}z_iz_je^{-(s_i+s_j+t)\alpha H}\left(S_\alpha^{2H}(e^{s_i+t}) + S_\alpha^{2H}(e^{s_j}) - \left(S_\alpha(e^{s_i+t}) - S_\alpha(e^{s_j})\right)^{2H}\right).$$

When $2H < 1$ we have

$$a^{2H} - (a-b)^{2H} \leq b^{2H}$$

for $a > b$. The above inequality can be easily proven by checking that the function $f(b) = b^{2H} + (a-b)^{2H} - a^{2H}$ is increasing on an interval $[0, a/2]$ and decreasing on $[a/2, a]$ (compute f') with $f(0) = f(b) = 0$. Hence

$$\begin{aligned}
0 &\leq e^{-(s_i+s_j+t)\alpha H}\left(S_\alpha^{2H}(e^{s_i+t}) + S_\alpha^{2H}(e^{s_j}) - \left(S_\alpha(e^{s_i+t}) - S_\alpha(e^{s_j})\right)^{2H}\right) \\
&\leq 2e^{-(s_i+s_j+t)\alpha H}S_\alpha^{2H}(e^{s_j}) \xrightarrow[t\to\infty]{a.s.} 0.
\end{aligned}$$

Obviously, convergence with probability 1 implies convergence in distribution. In the second case $2H \geq 1$ from Taylor expansion we get

$$a^{2H} = (a-b)^{2H} + 2H\theta^{2H-1}b,$$

where $a > b$ and $\theta \in [a-b, a]$. Since $2H - 1 \geq 0$ we infer

$$a^{2H} - (a-b)^{2H} \leq 2Ha^{2H-1}b.$$

Consequently,

$$\begin{aligned}
0 &\leq e^{-(s_i+s_j+t)\alpha H}\left(S_\alpha^{2H}(e^{s_i+t}) + S_\alpha^{2H}(e^{s_j}) - \left(S_\alpha(e^{s_i+t}) - S_\alpha(e^{s_j})\right)^{2H}\right) \\
&\leq 2He^{-(s_i+s_j+t)\alpha H}S_\alpha^{2H-1}(e^{s_i+t})S_\alpha(e^{s_j}) + e^{-(s_i+s_j+t)\alpha H}S_\alpha^{2H}(e^{s_j}) \\
&= 2He^{-s_j\alpha H-(s_i+t)\alpha(1-H)}e^{-(s_i+t)\alpha(2H-1)}S_\alpha^{2H-1}(e^{s_i+t})S_\alpha(e^{s_j}) \\
&\quad + e^{-(s_i+s_j+t)\alpha H}S_\alpha^{2H}(e^{s_j}).
\end{aligned} \tag{14}$$

The process $e^{-t\alpha}S_\alpha(e^t)$ is stationary since $S_\alpha(t)$ is self-similar, thus

$$e^{-(s_i+t)\alpha(2H-1)}S_\alpha^{2H-1}(e^{s_i+t}) \stackrel{d}{=} e^{-s_i\alpha(2H-1)}S_\alpha^{2H-1}(e^{s_i})$$

for all $t > 0$. Therefore

$$2He^{-s_j\alpha H-(s_i+t)\alpha(1-H)}e^{-(s_i+t)\alpha(2H-1)}S_\alpha^{2H-1}(e^{s_i+t})S_\alpha(e^{s_j}) \xrightarrow[t\to\infty]{d} 0,$$

and thus $a_{i,j}y_iy_j \xrightarrow[t\to\infty]{d} 0$ by Eq. (14). We have already shown that the above convergence holds also in the case $2H < 1$, which implies that it is true for all $H \in (0,1)$. We now turn our attention into the rest of the summands $a_{i,j}y_iy_j$ in the exponent of Eq. (13)

$$a_{i,j}y_iy_j = \begin{cases} z_j^2 e^{-2Hs_j\alpha} S^{2H}(e^{s_j}) & \text{if } i=j\le m \\ z_j^2 e^{-2H(s_j+t)\alpha} S_\alpha^{2H}(e^{s_j+t}) & \text{if } i=j>m \\ \frac{1}{2} z_i z_j e^{-(s_i+s_j)\alpha H}\left(S_\alpha^{2H}(e^{s_i}) + S_\alpha^{2H}(e^{s_j}) - |S_\alpha(e^{s_i}) - S_\alpha(e^{s_j})|^{2H}\right) & \text{if } i\ne j \wedge i,j<m \\ \frac{1}{2} z_i z_j e^{-(s_i+s_j+2t)\alpha H}\left(S_\alpha^{2H}(e^{s_j+t}) + S_\alpha^{2H}(e^{s_i+t}) - |S_\alpha(e^{s_j+t}) - S_\alpha(e^{s_i+t})|^{2H}\right) & \\ & \text{if } i\ne j \wedge i,j>m. \end{cases}$$

From Lemma 1 we have

$$\begin{aligned} \mathbf{W}_t &= \left(e^{-s_1\alpha}S_\alpha(e^{s_1}), ..., e^{-s_m\alpha}S_\alpha(e^{s_m}), e^{-(s_{m+1}+t)\alpha}S_\alpha(e^{s_{m+1}+t}), ..., e^{-(s_M+t)\alpha}S_\alpha(e^{s_M+t})\right) \\ &\xrightarrow[t\to\infty]{d} \left(e^{-s_1\alpha}S_\alpha(e^{s_1}), ..., e^{-s_m\alpha}S_\alpha(e^{s_m}), e^{-s_{m+1}\alpha}R_\alpha(e^{s_{m+1}}), ..., e^{-s_M\alpha}R_\alpha(e^{s_M})\right) = \mathbf{W}, \end{aligned}$$

where $R_\alpha(t)$ is an inverse α-stable subordinator independent of $S_\alpha(t)$, they have the same finite dimensional distributions. Let us consider a function $f = \sum_{(0\le i,j\le m)\vee(m<i,j\le M)} f_{i,j}$, where $f_{i,j} : \mathbb{R}_+^M \to \mathbb{R}$ for $\{i,j\in\mathbb{N} : (0\le i,j\le m)\vee(m<i,j\le n)\}$ are given by

$$f_{i,j}(\mathbf{x}) = \begin{cases} -\frac{1}{2} z_j^2 x_j^{2H} & \text{if } i=j \\ -\frac{1}{4} z_i z_j e^{-(s_j+s_j)\alpha H}\left(e^{2Hs_i\alpha}x_i^{2H} + e^{2Hs_j\alpha}x_j^{2H} - |e^{s_i\alpha}x_i - e^{s_j\alpha}x_j|^{2H}\right) & \text{if } i\ne j \end{cases}$$

and $\mathbf{x} = (x_1, ..., x_M)$ with $x_1, ..., x_M > 0$. The function $f : \mathbb{R}_+^M \to \mathbb{R}$ is clearly continuous. From Theorem 5.1 in [5] we obtain

$$f(\mathbf{W}_t) \xrightarrow[t\to\infty]{d} f(\mathbf{W}).$$

Observe that $f(\mathbf{W}_t) = \sum_{(0\le i,j\le m)\vee(m<i,j\le M)} \left(-\frac{1}{2}a_{i,j}y_iy_j\right)$. Since it was already shown that when $j\le m<i$ or $i\le m<j$ we have $a_{i,j}y_iy_j \xrightarrow[t\to\infty]{d} 0$ the valid conclusion is

$$-\frac{1}{2}yAy^T = \xrightarrow[t\to\infty]{d} f(\mathbf{W}).$$

Recall that A is a covariance matrix hence it is positive semi-definite and $-\frac{1}{2}yAy \le 0$. Therefore

$$\mathbf{E}\exp\left(-\frac{1}{2}yAy^T\right) \xrightarrow{t\to\infty} \mathbf{E}\exp\left(-\sum_{(0\le i,j\le m)\vee(m<i,j\le n)} f_{i,j}(\mathbf{W})\right). \tag{15}$$

Notice that the function $f_{i,j}(\mathbf{x})$ depends only on the values x_i and x_j. Furthermore $f_{i,j}$ is defined only for $0\le i,j\le m$ and $m<i,j\le n$ which implies that

$$f_{i,j}\left(e^{-s_1\alpha}S_\alpha(e^{s_1}), ..., e^{-s_m\alpha}S_\alpha(e^{s_m}), e^{-s_{m+1}\alpha}R_\alpha(e^{s_{m+1}}), ..., e^{-s_M\alpha}R_\alpha(e^{s_M})\right)$$

depends either on $(e^{-s_1\alpha}S_\alpha(e^{s_1}), ..., e^{-s_m\alpha}S_\alpha(e^{s_m}))$ when $0\le i,j\le m$ or on $(R_\alpha(e^{s_{m+1}}), ..., e^{-s_M\alpha}R_\alpha(e^{s_M}))$ when $m<i,j\le M$. However $S_\alpha(t)$ and $R_\alpha(t)$ independent thus the right hand side of Eq. (15)

equals

$$\begin{aligned}
&\mathbf{E}\exp\left(-\sum_{0\le i,j\le m} f_{i,j}(\mathbf{W})\right)\mathbf{E}\exp\left(-\sum_{m<i,j\le n} f_{i,j}(\mathbf{W})\right)\\
&=\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k e^{-s_k\alpha H}B_H\left(S_\alpha(e^{s_k})\right)\right)\mathbf{E}\exp\left(i\sum_{k=m+1}^{M} z_k e^{-s_k\alpha H}B_H\left(S_\alpha(e^{s_k})\right)\right).
\end{aligned}\tag{16}$$

Finally after combining Eqs. (13), (15) and (16) we get Eq. (12). □

Corollary 5. *The process $X(t) = e^{-\alpha Ht}B_H\left(S_\alpha\left(e^t\right)\right)$ is ergodic.*

5. Increments of FBM with infinitely divisible waiting times

We can replace the inverse α-stable subordinator inside the process $B_H(S_\alpha(t))$ with the generalized inverse subordinator $\tilde{S}_\eta(t)$ which has stationary increments. We assume that $B_H(t)$ and $\tilde{S}_\eta(t)$ are independent. Then the discrete-time process of increments

$$Y(n) = B_H(S_\eta(n+1)) - B_H(S_\eta(n)) \qquad n \in \mathbb{N} \tag{17}$$

is stationary. One of the most important examples of $\tilde{S}_\eta(t)$ which can be used here is the inverse Gaussian process (IG) [40,42]. Possible applications of the process Y with such subordinator include mathematical finance and hydraulics [16].

Theorem 4. *The stationary discrete-time process $Y(n)$ given by Eq. (17) is mixing.*

Proof. It is sufficient to show that

$$\begin{aligned}
&\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k\left(B_H(\tilde{S}_\eta(k+1)) - B_H(\tilde{S}_\eta(k))\right)\right.\\
&\qquad\left.+\,i\sum_{k=m+1}^{M} z_k\left(B_H(\tilde{S}_\eta(k+n+1)) - B_H(\tilde{S}_\eta(k+n))\right)\right)\\
&\xrightarrow{n\to\infty}\mathbf{E}\exp\left(i\sum_{k=1}^{m} z_k\left(B_H(\tilde{S}_\eta(k+1)) - B_H(\tilde{S}_\eta(k))\right)\right)\\
&\qquad\times\mathbf{E}\exp\left(i\sum_{k=m+1}^{M}\left(B_H(\tilde{S}_\eta(k+1)) - B_H(\tilde{S}_\eta(k))\right)\right).
\end{aligned}\tag{18}$$

We know that B_H has stationary increments, moreover B_H and $\tilde{S}_\eta$ are independent. The formula for covariance of the increments of $B_H(t)$ is also known

$$\operatorname{cov}(B_H(t_2) - B_H(t_1), B_H(t_4) - B_H(t_3)) = \frac{1}{2}\left(|t_4 - t_1|^{2H} + |t_3 - t_2|^{2H} - |t_4 - t_2|^{2H} - |t_3 - t_1|^{2H}\right).$$

Therefore the characteristic function given by the left hand side of Eq. (18) is equal to $\mathbf{E}\exp\left(-\frac{1}{2}zAz^T\right)$, where $z = (z_1, ..., z_M)$ and $A = [a_{i,j}]$ is a $M \times M$ covariance matrix:

$$a_{i,j} = \begin{cases} \left(\tilde{S}(j+1) - \tilde{S}(j)\right)^{2H} & \text{if } i = j \le m \\ \left(\tilde{S}(j+n+1) - \tilde{S}(j+n)\right)^{2H} & \text{if } i = j > m \\ \frac{1}{2}\Big(\left|\tilde{S}(j+1) - \tilde{S}(i)\right|^{2H} + \left|\tilde{S}(j) - \tilde{S}(i+1)\right|^{2H} - \left|\tilde{S}(j+1) - \tilde{S}(i+1)\right|^{2H} & \\ \qquad - \left|\tilde{S}(j) - \tilde{S}(i)\right|^{2H}\Big) & \text{if } i \ne j \wedge i,j < m \\ \frac{1}{2}\Big(\left|\tilde{S}(i+n+1) - \tilde{S}(j)\right|^{2H} + \left|\tilde{S}(i+n) - \tilde{S}(j+1)\right|^{2H} - \left|\tilde{S}(i+n+1) - \tilde{S}(j+1)\right|^{2H} & \\ \qquad - \left|\tilde{S}(i+n) - \tilde{S}(j)\right|^{2H}\Big) & \text{if } j \le m < i \\ a_{j,i} & \text{if } i \le m < j \\ \frac{1}{2}\Big(\left|\tilde{S}(i+n+1) - \tilde{S}(j+n)\right|^{2H} + \left|\tilde{S}(i+n) - \tilde{S}(j+n+1)\right|^{2H} & \\ \qquad - \left|\tilde{S}(i+n+1) - \tilde{S}(j+n+1)\right|^{2H} - \left|\tilde{S}(i+n) - \tilde{S}(j+n)\right|^{2H}\Big) & \text{if } i \ne j \wedge i,j > m. \end{cases}$$

If the non-negative, non-decreasing function $g(t) \xrightarrow{t\to\infty} \infty$ we have

$$\frac{1}{2}\left(|g(t+t_4) - g(t_1)|^{2H} + |g(t+t_3) - g(t_2)|^{2H} - |g(t+t_4) - g(t_2)|^{2H} - |g(t+t_3) - g(t_1)|^{2H}\right) \xrightarrow{t\to\infty} 0.$$

Since $\tilde{S}(t) \xrightarrow[t\to\infty]{a.s.} \infty$ we obtain that $a_{i,j} \xrightarrow[t\to\infty]{a.s.} 0$ for $\{i,j \in \mathbb{N} : (0 \le i,j \le m) \vee (m < i,j \le M)\}$. It follows from Lemma 2

$$\begin{aligned} \tilde{\mathbf{S}}_\eta(n) &= \big(\tilde{S}_\eta(2) - \tilde{S}_\eta(1), \tilde{S}_\eta(3) - \tilde{S}_\eta(2), ..., \tilde{S}_\eta(m+1) - \tilde{S}_\eta(m), \\ &\qquad \tilde{S}_\eta(m+2+n) - \tilde{S}_\eta(m+1+n), ..., \tilde{S}_\eta(M+n+1) - \tilde{S}_\eta(M+n)\big) \\ &\xrightarrow[n\to\infty]{d} \big(\tilde{S}_\eta(2) - \tilde{S}_\eta(1), \tilde{S}_\eta(3) - \tilde{S}_\eta(2), ..., \tilde{S}_\eta(m+1) - \tilde{S}_\eta(m), \\ &\qquad \tilde{R}_\eta(m+2) - \tilde{R}_\eta(m+1), ..., \tilde{R}_\eta(M+1) - \tilde{R}_\eta(M)\big) = \tilde{\mathbf{S}}_\eta, \end{aligned}$$

where $\tilde{R}_\eta(t)$ is an independent copy of the process $\tilde{S}_\eta(t)$. We define a continuous function $f : \mathbb{R}^M \to \mathbb{R}$, $f = \sum_{(0\le i,j\le m)\vee(m<i,j\le M)} f_{i,j}$, where $f_{i,j} : \mathbb{R}^M \to \mathbb{R}$ for $\{i,j \in \mathbb{N} : (0 \le i,j \le m) \vee (m < i,j \le M)\}$ and

$$f_{i,j}(\mathbf{x}) = \begin{cases} -\frac{1}{2} z_j^2 x_j^{2H} & \text{if } i = j \\ -\frac{1}{4} z_i z_j \, |x_i + x_{i+1} + ... + x_j|^{2H} + |x_{i+1} + x_{i+2} + ... + x_{j-1}|^{2H} - |x_{i+1} + x_{i+2} + ... + x_j|^{2H} & \\ \qquad - |x_i + x_{i+1} + ... + x_{j-1}|^{2H} & \text{if } i < j \\ f_{j,i}(\mathbf{x}) & \text{if } j < i, \end{cases}$$

where $\mathbf{x} = (x_1, ..., x_M)$. From Theorem 5.1 in [5] we get $f(\tilde{\mathbf{S}}_\eta(n)) \xrightarrow[n\to\infty]{d} f(\tilde{\mathbf{S}}_\eta)$ and the rest of the proof mimics the steps of the proof of Theorem 3. $\square$

Corollary 6. *The stationary discrete-time process $Y(n)$ given by Eq. (17) is ergodic.*

Acknowledgment

This research was partially supported by the Ministry of Science and Higher Education of Poland program Iuventus Plus no. IP2014 027073.

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