Proc. Camb. Phil. Soc. (1964), **60**, 923
Printed in Great Britain

On doubly stochastic Poisson processes

By J. F. C. KINGMAN
Pembroke College, Cambridge

(*Received* 30 *January* 1964)

The class of stationary point processes known as 'doubly stochastic Poisson processes' was introduced by Cox (2) and has been studied in detail by Bartlett (1). It is not clear just how large this class is, and indeed it seems to be a problem of some difficulty to decide of a general stationary point process whether or not it can be represented as a doubly stochastic Poisson process. (A few simple necessary conditions are known. For instance, Cox pointed out in the discussion to (1) that a double stochastic Poisson process must show more 'dispersion' than the Poisson process. Such conditions are very far from being sufficient.) The main result of the present paper is a solution of the problem for the special case of a renewal process, justifying an assertion made in the discussion to (1).

A stochastic point process on the real line R is conveniently described by the sequence $\{x_n;\ n = 0, \pm 1, \pm 2, \ldots\}$ of instants at which the events (or jumps) of the process occur, the labelling of the events being uniquely specified by requiring that

$$\ldots < x_{-2} < x_{-1} < 0 \leqslant x_0 < x_1 < \ldots. \tag{1}$$

We shall see below how the distributions of $\{x_n\}$ may be obtained for a doubly stochastic Poisson process.

Let Π be a Poisson process with unit rate on R, and let $\{x_n\}$ denote the sequence of instants at which the events of Π occur. Let ϕ be any measurable function of R into itself, and consider the point process $\phi(\Pi)$ whose events occur exactly at the points $\phi(x_n)$. Then, if $A_1, A_2, \ldots, A_m$ are disjoint measurable subsets of R, and if N_j is the number of events of $\phi(\Pi)$ occurring in A_j, then N_j is the number of events of Π occurring in $\phi^{-1}(A_j)$, and so the N_j are independent Poisson variables, the mean of N_j being the Lebesgue measure of $\phi^{-1}(A_j)$.

In particular, let λ be a non-negative measurable function on R, such that the function

$$\lambda_*(x) = \int_0^x \lambda(t)\,dt \tag{2}$$

is finite for all finite x, but tends to $\pm\infty$ as $x \to \pm\infty$. Then, since λ_* is non-decreasing, it has an inverse function λ^*, which is unique if (as we can and shall) we take it to be continuous from the right. More precisely, we define

$$\lambda^*(x) = \sup\{y; \lambda_*(y) \leqslant x\}. \tag{3}$$

It now follows that the process $\lambda^*(\Pi)$, whose events occur at the (distinct) instants $y_n = \lambda^*(x_n)$, is an inhomogeneous Poisson process, and that the expected number of events in a set A is the Lebesgue measure of $\lambda_*(A)$, which is just $\int_A \lambda(t)\,dt$. Hence

$\lambda^*(\Pi)$ is an inhomogeneous Poisson process with rate $\lambda(t)$. In other words, we can construct a Poisson process with rate $\lambda(t)$ from one of unit rate by transforming the time axis by means of the function λ^*.

A double stochastic Poisson process is constructed by replacing $\lambda(t)$ by a stationary stochastic process $\Lambda(t)$. Thus let $\Lambda(t)$ be a non-negative stationary stochastic process which is measurable in the sense of Doob ((4), Section II.2), and suppose that (with probability one), the function

$$\Lambda_*(x) = \int_0^x \Lambda(t)\,dt \tag{4}$$

is finite for all finite x, but is not identically zero. Then the doubly stochastic Poisson process generated by Λ (which will be denoted by Π_Λ) is a point process which, conditional on Λ, is an inhomogeneous Poisson process with rate $\Lambda(t)$.

It is an easy consequence of the stationarity of Λ that $\Lambda_*(x) \to \pm\infty$ as $x \to \pm\infty$, and hence almost all realizations of Λ have the properties of the function λ considered above, and in particular the right-continuous inverse function Λ^* of Λ_* exists. It follows that, if Π is a Poisson process of unit rate independent of Λ, then, conditional on Λ, the process $\Lambda^*(\Pi)$ is an inhomogeneous Poisson process with rate $\Lambda(t)$. Hence $\Lambda^*(\Pi)$ is a representation of the doubly stochastic Poisson process Π_Λ generated by Λ.

Thus we have proved that Π_Λ can be constructed by taking a Poisson process Π of unit rate and transforming the time scale by the random function Λ^*. In particular, if the events of Π occur at $\{x_n\}$ (where the x_n satisfy (1)), then the events of Π_Λ occur at the instants

$$y_n = \Lambda^*(x_n) \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Notice that $\{y_n\}$ also satisfies (1), so that $\{y_n\}$ describes Π_Λ in the same way that $\{x_n\}$ describes Π.

Equation (5) shows that the process $\{y_n\}$ can be obtained by observing the process Λ^* at the instants of a Poisson process Π (which is of unit rate and independent of Λ^*). We say that $\{y_n\}$ is obtained from Λ^* by *Poisson sampling*. Properties of stochastic processes observed at random in this way have been studied in (7), and the results of that paper may be invoked to give relations between $\{y_n\}$ and Λ^*. For instance, a knowledge of the finite-dimensional distributions of $\{y_n\}$ suffices to determine those of Λ^*. It should, however, be remarked that the theorems in (7) are stated for processes continuous in probability, and Λ^* will not in general have this property. However, since Λ^* is almost certainly right-continuous, it is right-continuous in probability in the sense that

$$\operatorname*{p\,lim}_{x \downarrow x_0} \Lambda^*(x) = \Lambda^*(x_0) \tag{6}$$

for each x_0, and it is readily seen that this weaker condition is sufficient for the validity of the theorems.

To illustrate these ideas, we consider the problem mentioned at the beginning of the paper of deciding whether a given renewal process can be represented as a doubly stochastic Poisson process. Thus suppose, if possible, that Λ is such that Π_Λ is a renewal process with lifetime distribution function F, and write

$$\mu = \int_0^\infty x\,dF(x), \quad \phi(\theta) = \int_0^\infty e^{-\theta x}\,dF(x) \quad (\theta \geqslant 0). \tag{7}$$

Then $\{y_n;\ n = 0, 1, 2, \ldots\}$ is a process with stationary independent increments, and since Π_Λ is stationary, we must have

$$\mathbf{E}\{\exp[-\theta y_0]\} = [1-\phi(\theta)]/\mu\theta, \quad \mathbf{E}\{\exp[-\theta(y_n - y_{n-1})]\} = \phi(\theta), \tag{8}$$

(cf. (3), page 28).

From (5) it follows that the process obtained by Poisson sampling of Λ^* has stationary independent increments satisfying (8). The results of (7) would lead one to expect that this implies that Λ^* has stationary independent increments. This is indeed true, but the necessary computations will be relegated to an appendix, where it will be shown that $\Lambda^*(x)$ $(x \geqslant 0)$ is a process with stationary independent increments satisfying

$$\mathbf{E}\{\exp[-\theta(\Lambda^*(x+y) - \Lambda^*(x))]\} = \exp[-y\{1-\phi(\theta)\}/\phi(\theta)] \tag{9}$$

and

$$\mathbf{E}\{\exp[-\theta\Lambda^*(0)]\} = [1-\phi(\theta)]/\mu\theta\phi(\theta). \tag{10}$$

We may now appeal to the Lévy–Khinchin representation theorem for the distributions of processes with stationary independent increments, as specialized to the case in which the increments are non-negative (cf. (5), Section 4) to show that there exists a non-negative number h and a non-decreasing function $H(z)$ $(z > 0)$ which satisfies

$$\int_0^\infty \min(z, 1)\, dH(z) < \infty, \tag{11}$$

such that

$$\mathbf{E}\{\exp[-\theta(\Lambda^*(x+y) - \Lambda^*(x))]\} = \exp\left[-y\left(h\theta + \int_0^\infty (1-e^{-\theta z})\, dH(z)\right)\right]. \tag{12}$$

It then follows from (9) that

$$\phi(\theta) = \left[1 + h\theta + \int_0^\infty (1-e^{-\theta z})\, dH(z)\right]^{-1}, \tag{13}$$

and so

$$\begin{aligned}\mu = \lim_{\theta\to 0} \theta^{-1}[1-\phi(\theta)] &= \lim_{\theta\to 0} \theta^{-1}[\phi(\theta)^{-1} - 1] \\ &= \lim_{\theta\to 0}\left[h + \int_0^\infty \theta^{-1}(1-e^{-\theta z})\, dH(z)\right] \\ &= h + \int_0^\infty z\, dH(z)\end{aligned}$$

by monotone convergence. Hence H satisfies

$$\int_0^\infty z\, dH(z) = \mu - h < \infty, \tag{14}$$

which is a stronger condition than (11). Equations (10) and (13) give

$$\begin{aligned}\mathbf{E}\{\exp - \theta\Lambda^*(0)\} &= \frac{h}{\mu} + \int_0^\infty \frac{1-e^{-\theta z}}{\mu\theta}\, dH(z) \\ &= \frac{h}{\mu} + \frac{1}{\mu}\int_0^\infty [H(\infty) - H(z)]\, e^{-\theta z} dz,\end{aligned}$$

which shows that

$$\mathbf{P}\{\Lambda^*(0) = 0\} = h/\mu. \tag{15}$$

https://doi.org/10.1017/S030500410003838X Published online by Cambridge University Press

We now show that h must be strictly positive. For suppose on the contrary that $h = 0$. Then (15) shows that, with probability one, $\Lambda^*(0) > 0$, so that Λ^* almost certainly has a jump at the origin. Thus, with probability one, $\Lambda_*(x) = 0$ for all sufficiently small positive x, and therefore the right-hand derivative $D_+\Lambda_*(0)$ exists and is zero. But Λ_* has stationary increments and so, for each t,

$$\mathbf{P}\{D_+\Lambda_*(t)=0\} = 1.$$

It follows from Fubini's theorem and the measurability of Λ that

$$\mathbf{P}\{D_+\Lambda_*(t)=0 \text{ for almost all } t\} = 1.$$

Hence, from (4), there is probability one that $\Lambda(t) = 0$ for almost all t, and so that Λ_* is identically zero, which is contrary to hypothesis.

The contradiction shows that $h > 0$. Now write $\lambda = h^{-1}$, $K(z) = h^{-1}H(z)$. Then K is non-decreasing, with

$$\int_0^\infty z\,dK(z) = \lambda\mu - 1 < \infty. \tag{16}$$

Moreover, from (13) we have

$$\phi(\theta) = \lambda\left[\lambda+\theta+\int_0^\infty (1-e^{-\theta z})\,dK(z)\right]^{-1}. \tag{17}$$

Therefore, for an equilibrium renewal process to be representable as a doubly stochastic Poisson process, it is necessary that the Laplace–Stieltjes transform $\phi(\theta)$ of its lifetime distribution function F to be of the form (17), where λ is a positive number and $K(z)$ $(z > 0)$ a non-decreasing function satisfying (16).

This condition is also sufficient, but to prove this we need to invoke some results from the theory of *regenerative events* summarized in (6) and developed in detail in (8) and (9). For any positive number λ, and any non-decreasing function $K(z)$ $(z>0)$ with

$$\int_0^\infty z\,dK(z) = m < \infty, \tag{18}$$

there exists ((6), Theorem III) a continuous function $p(t)$ $(t \geqslant 0)$ satisfying

$$r(\theta) = \int_0^\infty p(t)\,e^{-\theta t}dt = \left[\theta+\int_0^\infty (1-e^{-\theta z})\,dK(z)\right]^{-1}, \tag{19}$$

and, as $t \to \infty$,

$$p(t) \to \varpi = (1+m)^{-1} > 0. \tag{20}$$

Moreover ((9), Section 2), there exists a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ and a family $\mathscr{E} = \{E(t);\, t \in R\}$ of $\mathfrak{A}$-measurable subsets of Ω such that, whenever

$$k \geqslant 1 \quad\text{and}\quad t_1 \leqslant t_2 \leqslant \ldots \leqslant t_k,$$

we have

$$\mathbf{P}\{E(t_1)\,E(t_2)\ldots E(t_k)\} = \varpi p(t_2-t_1)\,p(t_3-t_2)\ldots p(t_k-t_{k-1}). \tag{21}$$

$\mathscr{E}$ is called an *equilibrium regenerative event* with p-function $p(t)$, and $E(t)$ is to be interpreted as the occurrence of $\mathscr{E}$ at time t. It is shown in (8) that $\mathscr{E}$ may be taken to be measurable in the sense that

$$\{(t,\omega);\, t\in R, \omega \in E(t)\}$$

is a measurable subset of $R \times \Omega$.

https://doi.org/10.1017/S030500410003838X Published online by Cambridge University Press

We may now define a process $\Lambda(t) = \Lambda(t, \omega)$ on $(\Omega, \mathfrak{A}, \mathbf{P})$ by setting

$$\Lambda(t,\omega) = \lambda \quad \text{if} \quad \omega \in E(t), \quad \Lambda(t,\omega) = 0 \quad \text{if} \quad \omega \notin E(t). \tag{22}$$

Then Λ is stationary (by (21)) and measurable, and $\Lambda_*(x) \sim \lambda \varpi x$ as $x \to \pm\infty$. Hence Λ generates a doubly stochastic Poisson process Π_Λ. Moreover, it follows from the basic properties of regenerative events (see (8), Proposition 1) that, conditional on $\Lambda(t_0) = \lambda$, the processes $\{\Lambda(t);\ t < t_0\}$ and $\{\Lambda(t);\ t > t_0\}$ are independent. Ryll-Nardzewski (10) has shown how to define a unique probability measure $\mathbf{P}^{t_0}$ conditional on the occurrence of an event of Π_Λ at the instant t_0. Under this probability measure it is clear that the distributions of $\Lambda(t)$ are just those conditional on $\Lambda(t_0) = \lambda$. Hence, under $\mathbf{P}^{t_0}$, $\{\Lambda(t);\ t < t_0\}$ and $\{\Lambda(t);\ t > t_0\}$ are independent, so that, conditional on the occurrence of an event of Π_Λ at t_0, the events of Π_Λ before and after t_0 are independent. It follows easily that Π_Λ is an equilibrium renewal process.

If F is the lifetime distribution function of Π_Λ, F_k the k-fold convolution of F itself, and $H = \sum_{k=1}^{\infty} F_k$ the renewal function of Π_Λ, and if $\mathbf{E}^0$ denotes expectation with respect to the Ryll-Nardzewski probability measure $\mathbf{P}^0$, then we have

$$H(t) = \mathbf{E}^0\{\text{number of events of } \Pi_\Lambda \text{ in } (0,t)\} = \lambda \int_0^t p(s)\,ds.$$

Hence

$$\int_0^\infty e^{-\theta t}\,dH(t) = \lambda \int_0^\infty p(t)\,e^{-\theta t}\,dt,$$

so that

$$\frac{\phi(\theta)}{1-\phi(\theta)} = \lambda r(\theta),$$

i.e.

$$\phi(\theta) = \lambda r(\theta)\,[1+\lambda r(\theta)]^{-1} = \lambda\left[\lambda+\theta+\int_0^\infty (1-e^{-\theta z})\,dK(z)\right]^{-1},$$

from (19). We have therefore proved the following theorem.

An equilibrium renewal process with lifetime distribution function F can be expressed as a doubly stochastic Poisson process if and only if there exists $\lambda > 0$ and a non-decreasing function $K(z)$ $(z > 0)$ with

$$\int_0^\infty z\,dK(z) < \infty,$$

such that, for all $\theta > 0$,

$$\phi(\theta) = \int_0^\infty e^{-\theta x}\,dF(x) = \lambda\left[\lambda+\theta+\int_0^\infty (1-e^{-\theta z})\,dK(z)\right]^{-1}. \tag{23}$$

The corresponding process $\Lambda(t)$ takes on only the values $\lambda, 0$, and the event $\{\Lambda(t) = \lambda\}$ is an equilibrium regenerative event in the sense of (9).

It follows from Theorem 4 of (8) that there exists a continuous function $p_1(t)$ (which is in fact the p-function of some regenerative event) such that

$$\int_0^\infty p_1(t)\,e^{-\theta t}\,dt = \left[\lambda+\theta+\int_0^\infty (1-e^{-\theta z})\,dK(z)\right]^{-1}.$$

Thus, if F is given by (23), then F has a density $\lambda p_1(t)$, i.e.

$$F(x) = \lambda \int_0^x p_1(t)\,dt.$$

https://doi.org/10.1017/S030500410003838X Published online by Cambridge University Press

In fact, the reader will easily verify from (23) that an equilibrium renewal process with lifetime distribution function F is a doubly stochastic Poisson process if and only if there exists a function $p_1(t)$ in the class $\mathscr{P}$ defined in (8) such that

$$\alpha = \int_0^\infty p_1(t)\,dt < \infty \tag{24}$$

and

$$dF(x) = \alpha^{-1} p_1(x)\,dx. \tag{25}$$

It will be seen that this is a very severe restriction on the distribution function F.

Equation (23) may be inverted by a method due to Kendall (5). If we write

$$L(z) = \lambda^{-1}\int_0^{z+} (1-e^{-y})\,dK(y),$$

then L is finite and non-decreasing in $z \geqslant 0$, and continuous at $z = 0$, and it follows at once from (23) that, for $\theta > 0$,

$$[\phi(\theta+1)]^{-1} - [\phi(\theta)]^{-1} = \lambda^{-1} + \int_0^\infty e^{-\theta z}\,dL(z). \tag{26}$$

Hence λ and L, and hence K, may be obtained by a Laplace transform inversion.

It is possible to give a heuristic argument for the fact that, if Π_Λ is a renewal process, then Λ only takes on two values. Suppose that Π_Λ is a renewal process with mean lifetime μ, and suppose that its renewal function $H(t)$ has a continuous derivative $h(t)$. Then, if $t_1 < t_2 < \ldots < t_n$, we have, on equating two expressions for the probability of an event in each of the intervals $(t_r, t_r + dt_r)$,

$$\mu^{-1} dt_1\, h(t_2 - t_1)\,dt_2 \ldots h(t_n - t_{n-1})\,dt_n = \mathbf{E}\{\Lambda(t_1)\,dt_1\,\Lambda(t_2)\,dt_2 \ldots \Lambda(t_n)\,dt_n\}.$$

Hence, if we let $t_r \to t$ $(r = 1, 2, \ldots, n)$ we obtain

$$\mu^{-1}\{h(0)\}^{n-1} = \mathbf{E}\{\Lambda(t)\}^n.$$

This implies that $\Lambda(t)$ takes on only the values $h(0)$ and 0, as can be seen either from the theory of moments or perhaps more simply from

$$\mathbf{E}\{[\Lambda(t)]^2[\Lambda(t)-h(0)]^2\} = \mu^{-1}\{h(0)\}^3 - 2\mu^{-1}\{h(0)\}^2 h(0) + \mu^{-1}h(0)\{h(0)\}^2 = 0.$$

The description of the set of time instants on which $\Lambda(t) = \lambda$ is necessarily complicated in the general case, but it simplifies very considerably if we restrict attention to the special case in which

$$k = K(\infty) - K(0) < \infty. \tag{27}$$

If $k = 0$, we have the Poisson process with rate λ. On the other hand, if $k > 0$, then we can write

$$K(z) - K(0) = kG(z), \tag{28}$$

where G is a distribution function on $(0, \infty)$. Then the corresponding regenerative event is stable (see (8), Section 6), and we can therefore give the following description of Λ.

The process $\Lambda(t)$ is equal to λ and 0 alternatively on intervals whose lengths are independent random variables, the lengths of the intervals on which $\Lambda(t) = \lambda$ having a negative exponential distribution with mean $1/k$, and the lengths of the intervals on which $\Lambda(t) = 0$ having distribution function G.

APPENDIX

Let $\{x_n;\ n = 0,\ 1, 2, \ldots\}$ be the instants of a Poisson process of unit rate in $t \geqslant 0$, let Λ^* be a non-decreasing process right-continuous in probability and independent of $\{x_n\}$, and suppose that the process $\{y_n\} = \{\Lambda^*(x_n)\}$ has stationary independent increments with

$$\mathbf{E}\{\exp[-\theta y_0]\} = \phi_0(\theta), \quad \mathbf{E}\{\exp[-\theta(y_n - y_{n-1})]\} = \phi(\theta).$$

We have to prove that Λ^* has stationary independent increments satisfying (9) and (10). Write

$$\Phi(\theta_1, \theta_2, \ldots, \theta_n;\ t_1, t_2, \ldots, t_n) = \mathbf{E}\left\{\exp\left[-\sum_{j=1}^{n} \theta_j(\Lambda^*(t_1 + \ldots + t_j) - \Lambda^*(t_1 + \ldots + t_{j-1}))\right]\right\},$$

where $\theta_j, t_j \geqslant 0$. Then, if $r_1, r_2, \ldots, r_n$ are integers satisfying $0 \leqslant r_1 < r_2 < \ldots < r_n$, and if $s_j = r_j - r_{j-1}$, we have

$$\begin{aligned} &\phi_0(\theta_1)\{\phi(\theta_1)\}^{r_1} \prod_{j=2}^{n} \{\phi(\theta_j)\}^{s_j} \\ &= \mathbf{E}\left\{\exp\left[-\sum_{j=1}^{n} \theta_j(y_{r_j} - y_{r_{j-1}})\right]\right\} \\ &= \mathbf{E}\{\Phi(\theta_1, \ldots, \theta_n;\ x_{r_1}, x_{r_2} - x_{r_1}, \ldots, x_{r_n} - x_{r_{n-1}})\} \\ &= \int_0^\infty \ldots \int_0^\infty \Phi(\theta_1, \ldots, \theta_n; t_1 \ldots, t_n) \frac{t_1^{r_1} e^{-t_1}}{r_1!} \prod_{j=2}^{n} \frac{t_j^{s_j - 1} e^{-t_j}}{(s_j - 1)!} dt_1 dt_2 \ldots dt_n. \end{aligned}$$

Multiply by $z_1^{r_1} \prod_{j=2}^{n} z_j^{s_j - 1}$ (where z_j are any complex numbers with $|z_j| < 1$), and sum over all admissible values of $r_1, \ldots, r_n$, to give

$$\begin{aligned} &\frac{\phi_0(\theta_1)}{1 - z_1\phi(\theta_1)} \prod_{j=2}^{n} \frac{\phi(\theta_j)}{1 - z_j\phi(\theta_j)} \\ &= \int_0^\infty \ldots \int_0^\infty \Phi(\theta_1, \ldots, \theta_n;\ t_1, \ldots, t_n) \exp\left\{-\sum_{j=1}^{n} (1 - z_j)\, t_j\right\} dt_1 \ldots dt_n. \end{aligned}$$

Writing $p_j = 1 - z_j$, we have

$$\begin{aligned} &\frac{\phi_0(\theta_1)}{\phi(\theta_1)} \prod_{j=1}^{n} \left[\frac{1 - \phi(\theta_j)}{\phi(\theta_j)} + p_j\right]^{-1} \\ &= \int_0^\infty \ldots \int_0^\infty \Phi(\theta_1, \ldots, \theta_n;\ t_1, \ldots, t_n) \exp\left[-\sum_{j=1}^{n} p_j t_j\right] dt_1 \ldots dt_n, \end{aligned}$$

which holds whenever $|1 - p_j| < 1$, and hence by analytic continuation whenever $\Re p_j > 0$. But since Λ^* is right-continuous in probability, it follows that, for each j, Φ is right-continuous in t_j, and hence we may invert the multiple Laplace transform to give

$$\Phi(\theta_1, \ldots, \theta_n;\ t_1, \ldots, t_n) = \frac{\phi_0(\theta_1)}{\phi(\theta_1)} \prod_{j=1}^{n} \exp\left[-t_j \frac{1 - \phi(\theta_j)}{\phi(\theta_j)}\right].$$

It follows that Λ^* has independent increments, with

$$\mathbf{E}\{\exp[-\theta(\Lambda^*(x + y) - \Lambda^*(x))]\} = \exp[-y\{1 - \phi(\theta)\}/\phi(\theta)]$$

and, putting $\theta_2 = \ldots = \theta_n = 0$, that

$$E\{\exp[-\theta\Lambda^*(0)]\} = \phi_0(\theta)/\phi(\theta).$$

This suffices to prove the required result.

REFERENCES

(1) BARTLETT, M. S. The spectral analysis of point processes. *J. Roy. Statist. Soc. Ser. B*, **25** (1963), 264–296.

(2) COX, D. R. Some statistical models connected with series of events. *J. Roy. Statist. Soc. Ser. B*, **17** (1955), 129–164.

(3) COX, D. R. *Renewal theory* (Methuen; London, 1962).

(4) DOOB, J. L. *Stochastic processes* (Wiley; New York, 1953).

(5) KENDALL, D. G. Extreme-point methods in stochastic analysis. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **1** (1963), 295–300.

(6) KINGMAN, J. F. C. A continuous time analogue of the theory of recurrent events. *Bull. American Math. Soc.* **69** (1963), 268–272.

(7) KINGMAN, J. F. C. Poisson counts for random sequences of events. *Ann. Math. Statist.* **34** (1963), 1217–1232.

(8) KINGMAN, J. F. C. The stochastic theory of regenerative events. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **2** (1964), 180–224.

(9) KINGMAN, J. F. C. Linked systems of regenerative events. *Proc. London Math. Soc.* (to appear).

(10) RYLL-NARDZEWSKI, C. Remarks on processes of calls, from *Proc. 4th Berkeley Sympos. Math. Statist. and Prob.*, vol. II (Berkeley, 1961).