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Large deviations for subordinated Brownian motion and applications

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ABSTRACT

This paper concerns the problem of large deviation for the subordinated process $Z^H(t) = W^H(T(t))$. The process $W^H = \{W^H(t), t \in \mathbb{R}\}$ is the fractional Brownian motion with Hurst index $H \in (0, 1)$ taking values in $\mathbb{R}$. $T = \{T(t), t \geq 0\}$ is the inverse α-stable subordinator. In this paper we extend the results obtained in M.M. Meerschaert et al. (2008) to the whole range of parameter $\alpha \in (0, 1)$.

1. Introduction

Fractional Brownian motion (FBM) $W^H = \{W^H(t),\ t \in \mathbb{R}\}$ is a self similar, mean-zero Gaussian process with stationary increments. Its main characteristics is determined by the so-called Hurst index $H \in (0, 1)$. Covariance of FBM is given by

$$\mathbb{E}\big(W^H(s)W^H(t)\big) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\big). \tag{1.1}$$

If $H = 1/2$ we obtain the classical Brownian motion, which will be denoted as W. Using (1.1) one can verify that W^H is self-similar with index H (i.e., for all constants $c > 0$, the process $\{W^H(ct),\ t \in \mathbb{R}\}$ and $\{c^H W^H(t),\ t \in \mathbb{R}\}$ have the same finite dimensional distributions).

Self-similar, non-Gaussian processes have inspired many researchers, see for example Samorodnitsky and Taqqu (1994), where the authors consider self-similar stable processes with stationary increments. The process W with time replaced by independent one-dimensional Brownian motion $B = \{B_t\ :\ t \geq 0\}$ is called iterated Brownian motion (IBM) and was considered by Burdzy (1993, 1994). His work inspired many researchers to explore the connections between IBM (or other iterated processes) and partial differential equations (PDEs) (Allouba, 2002; Allouba and Zheng, 2001; Nane, 2008a; Baeumer et al., 2009), potential theoretical results (DeBlassie, 2004; Bañuelos and DeBlassie, 2006; Nane, 2006a,b, 2007, 2008b), and its sample path properties (Burdzy and Khoshnevisan, 1995; Csáki et al., 1995, 1996, 1997; Hu et al., 1995; Khoshnevisan and Lewis, 1996; Xiao, 1998; Hu, 1999).

In this paper, we consider another class of iterated self-similar processes formed by subordinated FBM. Let $\{T : T(t), t \geq 0\}$ be the inverse α-stable subordinator (see Section 2 for its definition). Let $Z^H = \{Z^H(t), t \geq 0\}$ be a stochastic process defined as $Z^H(t) = W^H(T(t))$, $t \geq 0$. We call this iterated process *subordinated fractional Brownian motion*.

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Since the sample paths of W^H and T are continuous, the subordinated FBM also has continuous sample paths. Process W^H is self-similar with self-similarity index H and process T is self-similar with self-similarity index α, thus one can easily verify that Z^H is self-similar with self-similarity index $H\alpha$. However, Z^H is non-Markovian, non-Gaussian and does not have stationary or independent increments. When $H = 1/2$, we will call the process $Z^{1/2}$ subordinated Brownian motion. Its path properties were investigated in Nane (2009) and Magdziarz (2010).

The objective of this paper is to extend the results presented in paper Meerschaert et al. (2008), where authors considered local time fractional Brownian motion, namely the process $W^H(L_t)$, where L_t is the local time of real-valued, strictly stable Levy process of index $1 < \gamma \leq 2$. From Stone (1963) we have the following equality $T(\rho t) = L_t$, where $\alpha = 1 - 1/\gamma$ and $\rho > 0$ is the appropriate positive constant. Thus the results presented in Meerschaert et al. (2008) are valid for inverse α-stable subordinators with $0 < \alpha \leq 1/2$. This restriction for the parameter α follows from the methods used in Meerschaert et al. (2008), which were exploiting local times of stable processes. In this paper we generalize the results presented in Meerschaert et al. (2008) to the case of any α in the interval $(0, 1)$. The key step is the derivation of estimates of moments (Lemmas 2.1 and 2.2), which are valid for any $\alpha \in (0, 1)$. In the proof of the estimates we use different technique than Meerschaert et al. (2008), which is based on the renewal theory. The remaining results are proved in similar way as in Meerschaert et al. (2008), therefore we leave only the crucial steps.

2. Moment estimates

An inverse α-stable subordinator $T(t)$ is defined as

$$T(t) = \inf\{\tau : U_\alpha(\tau) > t\}. \tag{2.1}$$

U_α is the α-stable subordinator with Laplace transform $E(e^{-uU_\alpha(t)}) = e^{-tu^\alpha}$, $0 < \alpha < 1$. Inverse subordinators are commonly used in physics in the description of anomalous diffusion (Metzler and Klafter, 2000; Magdziarz et al., 2007; Magdziarz, 2009a,b). We start with the following moment estimates for inverse subordinator T.

Lemma 2.1. *Let $T = \{T(t), t \geq 0\}$ be the inverse α-stable subordinator with index $0 < \alpha < 1$. Then for all $0 < a < b < \infty$ and all integers $n \geq 1$,*

$$\left(\frac{b-a}{b}\right)^{1-\alpha} \frac{n!(b-a)^{n\alpha}}{\Gamma((n-1)\alpha+2)\Gamma(\alpha)} \leq \mathbb{E}[|T(b)-T(a)|^n] \leq \left(\frac{b-a}{b}\right)^{1-\alpha} \frac{n!(b-a)^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{2.2}$$

In the case $a = 0$ we have the equality

$$\mathbb{E}[|T(b)|^n] = \frac{b^{\alpha n}\Gamma(n+1)}{\Gamma(\alpha n+1)}. \tag{2.3}$$

Proof. The proof of equality (2.3) can be found in Piryatinska et al. (2005).

Let us consider Eq. (2.2). From Theorem 1 in Lagerås (2005), we have

$$\mathbb{E}[|T(b)-T(a)|^n] = n! \int_A \prod_{j=1}^{n} U(dx_j - x_{j-1}), \tag{2.4}$$

where $A = \{(x_0, x_1, \ldots, x_n) : x_0 = 0,\ a < x_1 < x_2 < \cdots < x_n \leq b\}$ and $U(x) = \frac{x^\alpha}{\Gamma(\alpha+1)}$. Thus we have that

$$\mathbb{E}[|T(b)-T(a)|^n] = n! \int_a^b U'(x_1) \int_{x_1}^b U'(x_2 - x_1) \ldots \int_{x_{n-1}}^b U'(x_n - x_{n-1}) dx_n \ldots dx_2 dx_1.$$

This expression can be evaluated in terms of the Gamma function. First let us observe that

$$\int_{x_{n-1}}^b U'(x_n - x_{n-1}) dx_n = \int_{x_{n-1}}^b \frac{\alpha(x_n - x_{n-1})^{\alpha-1}}{\Gamma(\alpha+1)} dx_n = \frac{(b-x_{n-1})^\alpha}{\Gamma(\alpha+1)}.$$

Integration over $x_{n-1} \in [x_{n-2}, b]$, by changing the variables twice $v = x_{n-1} - x_{n-2}$ and $w = v/(b - x_{n-2})$ yields

$$\begin{aligned}\int_{x_{n-2}}^b U'(x_{n-1} - x_{n-2}) \frac{(b-x_{n-1})^\alpha}{\Gamma(\alpha+1)} dx_{n-1} &= \int_{x_{n-2}}^b \frac{\alpha(x_{n-1}-x_{n-2})^{\alpha-1}}{\Gamma(\alpha+1)} \frac{(b-x_{n-1})^\alpha}{\Gamma(\alpha+1)} dx_{n-1} \\ &= \int_0^{b-x_{n-2}} \frac{\alpha v^{\alpha-1}(b-x_{n-2}-v)^\alpha}{\Gamma^2(\alpha+1)} dv \\ &= \int_0^1 \frac{\alpha(b-x_{n-2})^{2\alpha}(1-w)^\alpha w^{\alpha-1}}{\Gamma^2(\alpha+1)} dw\end{aligned}$$

$$= \frac{\alpha(b-x_{n-2})^{2\alpha}}{\Gamma^2(\alpha+1)}\int_0^1 (1-w)^{\alpha} w^{\alpha-1} dw$$
$$= \frac{(b-x_{n-2})^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Iterating this procedure we obtain

$$\mathbb{E}[|T(b)-T(a)|^n] = n!\int_a^b \frac{\alpha(b-x_1)^{(n-1)\alpha} x_1^{\alpha-1}}{\Gamma((n-1)\alpha+1)\Gamma(\alpha+1)} dx_1$$
$$= \int_0^1 \frac{\alpha[(b-a)-(b-a)v]^{(n-1)\alpha}(a+(b-a)v)^{\alpha-1}(b-a)}{\Gamma((n-1)\alpha+1)\Gamma(\alpha+1)} dv. \tag{2.5}$$

For (2.5) we have once

$$\int_0^1 \frac{\alpha[(b-a)-(b-a)v]^{(n-1)\alpha}(a+(b-a)v)^{\alpha-1}(b-a)}{\Gamma((n-1)\alpha+1)\Gamma(\alpha+1)} dv$$
$$\le \left(\frac{b-a}{b}\right)^{1-\alpha}\int_0^1 \frac{\alpha(b-a)^{n\alpha}(1-v)^{(n-1)\alpha} v^{\alpha-1}}{\Gamma((n-1)\alpha+1)\Gamma(\alpha+1)} dv$$
$$= \left(\frac{b-a}{b}\right)^{1-\alpha}\frac{(b-a)^{n\alpha}}{\Gamma(n\alpha+1)},$$

which yields the expression for the upper bound. For the lower bound we have

$$\int_0^1 \frac{\alpha[(b-a)-(b-a)v]^{(n-1)\alpha}(a+(b-a)v)^{\alpha-1}(b-a)}{\Gamma((n-1)\alpha+1)\Gamma(\alpha+1)} dv \ge \left(\frac{b-a}{b}\right)^{1-\alpha}\int_0^1 \frac{(b-a)^{n\alpha}(1-v)^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)\Gamma(\alpha)} dv$$
$$= \left(\frac{b-a}{b}\right)^{1-\alpha}\frac{(b-a)^{n\alpha}}{\Gamma((n-1)\alpha+2)\Gamma(\alpha)}. \quad \square$$

With the above results it is easy to obtain bounds for the moments of the increment $Z^H(b) - Z^H(a)$. The following lemma discusses this problem and is sufficient for proving (3.5) in Theorem 3.2.

Lemma 2.2. *Let $W^H = \{W^H(t), t \in \mathbb{R}\}$ be the fractional Brownian motion of index H and $T(t)$ be the inverse α-stable subordinator of index $0 < \alpha < 1$. Assume that $T(t)$ is independent of W^H. Then for all $0 < a < b < \infty$ and all positive integers n,*

$$c_1(n)(b-a)^{n\alpha} \le \mathbb{E}\left(\left|W^H(T(b)) - W^H(T(a))\right|^{n/H}\right) \le c_2(n)(b-a)^{\alpha n}, \tag{2.6}$$

where

$$c_1(n) = \frac{n!}{\sqrt{\pi}} 2^{n/(2H)}\Gamma\left(\frac{n}{2H}+\frac{1}{2}\right)\left(\frac{b-a}{b}\right)^{1-\alpha}\frac{1}{\Gamma((n-1)\alpha+2)\Gamma(\alpha)} \tag{2.7}$$

and

$$c_2(n) = \frac{n!}{\sqrt{\pi}} 2^{n/(2H)}\Gamma\left(\frac{n}{2H}+\frac{1}{2}\right)\left(\frac{b-a}{b}\right)^{1-\alpha}\frac{1}{\Gamma(n\alpha+1)}. \tag{2.8}$$

Moreover when $a = 0$ we have the equality

$$\mathbb{E}\left(\left|W^H(T(b))\right|^{n/H}\right) = \frac{1}{\sqrt{\pi}} 2^{n/2H}\frac{\Gamma\left(\frac{n}{2H}+\frac{1}{2}\right)\Gamma(n+1)}{\Gamma(\alpha n+1)} b^{\alpha n}. \tag{2.9}$$

Proof. We use the facts that the inverse α-stable subordinator has non-decreasing paths and W^H is H self-similar with stationary increments. Conditioning on T we arrive at

$$\mathbb{E}\left(\left|W^H(T(b)) - W^H(T(a))\right|^{n/H}\right) = \mathbb{E}\left(\left|W^H(1)\right|^{n/H}\right)\mathbb{E}\left(|T(b)-T(a)|^n\right). \tag{2.10}$$

Since $W^H(1)$ has standard normal distribution, we get that

$$\mathbb{E}\left(\left|W^H(1)\right|^{n/H}\right) = \frac{1}{\sqrt{\pi}} 2^{n/(2H)}\Gamma\left(\frac{n}{2H}+\frac{1}{2}\right). \quad \square \tag{2.11}$$

Combining (2.10) and (2.11) with (2.2) and (2.3) gives the desired result.

3. Large deviations results

In this section we prove the main results, namely Theorems 3.1 and 3.2. These theorems extend results obtained in Meerschaert et al. (2008) (Theorems 1.1 and 1.2 respectively) to the case where the parameter α of the inverse subordinator can take all the values from the interval $(0, 1)$. We also apply the maximal inequality due to Móricz et al. (1982) to derive the upper bounds for the tail probabilities of $\max_{t\in[0,1]} Z^H(t)$ and $\max_{t\in[a,b]} |Z^H(t) - Z^H(a)|$.

Theorem 3.1. *Let $Z^H = \{Z^H(t),\ t \geq 0\}$ be the subordinated fractional Brownian motion. Then for every Borel set $D \subseteq \mathbb{R}$,*

$$\limsup_{t\to\infty} t^{-\frac{2H\alpha}{(1-2H+2H\alpha)}} \log \mathbb{P}\left\{t^{-\frac{2H\alpha}{(1-2H+2H\alpha)}} Z^H(t) \in D\right\} \leq -\inf_{x\in\overline{D}} \Lambda_1^*(x) \tag{3.1}$$

and

$$\liminf_{t\to\infty} t^{-\frac{2H\alpha}{(1-2H+2H\alpha)}} \log \mathbb{P}\left\{t^{-\frac{2H\alpha}{(1-2H+2H\alpha)}} Z^H(t) \in D\right\} \geq -\inf_{x\in D^0} \Lambda_1^*(x), \tag{3.2}$$

where $\overline{D}$ and D^0 denote respectively the closure and interior of D and

$$\Lambda_1^*(x) = x^{\frac{2}{1+2H-2H\alpha}} \frac{1+2H-2H\alpha}{2} \left(\frac{1-2H+2H\alpha}{2B_1}\right)^{\frac{1-2H+2H\alpha}{1+2H-2H\alpha}}, \tag{3.3}$$

where B_1 is the positive constant defined by

$$B_1 = \frac{(1-2H+2H\alpha)}{2}\left(\frac{H}{\alpha^{\alpha/(1-\alpha)}}\right)^{2H(1-\alpha)/(1-2H+2H\alpha)}. \tag{3.4}$$

Proof. It is easy to observe that the function $\Lambda_1(\theta) = B_1\theta^{2/(1-2H+2H\alpha)}$ in Theorem 3.1 is smooth and continuous on $\mathbb{R}$. Thus by application of the Gärtner–Ellis theorem, see Dembo and Zeitouni (1998), the pair $(t^{-\frac{2H\alpha}{(1-2H+2H\alpha)}} Z^H(t), t^{\frac{2H\alpha}{(1-2H+2H\alpha)}})$ satisfies large deviation principle with good rate function

$$\Lambda_1^*(x) = \sup_{\theta\in\mathbb{R}}(\theta x - \Lambda_1(\theta)),$$

which is the Fenchel–Legendre transform of Λ_1. Observation that Λ_1^* coincides with (3.3) finishes the proof of Theorem 3.1. □

Theorem 3.2. *Let $Z^H = \{Z^H(t),\ t \geq 0\}$ be the subordinated fractional Brownian motion. Then for any $0 \leq a \leq b < \infty$*

$$\lim_{x\to\infty} \frac{\log \mathbb{P}\left\{\left|Z^H(b) - Z^H(a)\right| > x\right\}}{x^{\frac{2}{1+2H-2H\alpha}}} = -B_2, \tag{3.5}$$

where B_2 is a positive constant defined by

$$B_2 = \frac{1+2H-2H\alpha}{2}\left(\frac{H}{\alpha^{\alpha/(1-\alpha)}}\right)^{-\frac{2H(1-\alpha)}{1+2H-2H\alpha}} (b-a)^{\frac{-2H\alpha}{1+2H-2H\alpha}}. \tag{3.6}$$

Proof. Fix $0 < \beta < \frac{2}{1+2H-2H\alpha}$. In a similar way to Meerschaert et al. (2008) we get

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(t|W^H(T(b)) - W^H(T(a))|^\beta)]}{t^\rho} = B_3, \tag{3.7}$$

where $\rho = \frac{2}{2-\beta-2H\beta+2H\beta\alpha}$ and the constant B_3 is defined by

$$B_3 = \frac{(b-a)^{\beta H\alpha\rho}}{\rho}\left(\frac{\beta^{\beta/(2H)}(\beta H)^{\beta-\beta\alpha}}{\alpha^{\beta\alpha}}\right)^{\rho H}. \tag{3.8}$$

By the application of Eq. (3.7) and Theorem 1 in Davies (1976) we obtain that

$$\lim_{u\to\infty} \frac{\log \mathbb{P}\{|W^H(T(b)) - W^H(T(a))|^\beta \geq u\}}{u^{\rho/(\rho-1)}} = -(1-\rho^{-1})(\rho B_3)^{-1/(\rho-1)}. \tag{3.9}$$

Denote $\rho = \frac{2}{2-\beta-2H\beta+2H\beta\alpha}$ and $x = u^{1/\beta}$. After simplifying the right-hand side of (3.9) we get

$$\lim_{x\to\infty} \frac{\log \mathbb{P}\{|W^H(T(b)) - W^H(T(a))| \geq x\}}{x^{2/(1+2H-2H\alpha)}} = -\frac{1+2H-2H\alpha}{2}\left(\frac{H}{\alpha^{\alpha/(1-\alpha)}}\right)^{-\frac{2H(1-\alpha)}{1+2H-2H\alpha}} (b-a)^{\frac{-2H\alpha}{1+2H-2H\alpha}},$$

which finishes the proof. □

Below we extend the results of Theorem 4.2 in Meerschaert et al. (2008).

Theorem 3.3. *Let $T = \{T(t) : t \geq 0\}$ be the inverse α-stable subordinator of index $\alpha \in (0, 1)$. Then the following two statements hold:*

(i) *For all $0 \leq a \leq b < \infty$,*

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(t(T(b) - T(a)))]}{t^{1/\alpha}} = (b - a). \tag{3.10}$$

(ii) *The pair $(t^{-(1-\alpha)/\alpha}(T(b) - T(a)), t^{1/\alpha})$ satisfies LDP with good rate function $\Lambda_2^*(x) = x^{\frac{1}{1-\alpha}} \left(\frac{b-a}{\alpha}\right)^{-\alpha/(1-\alpha)} \Big(1-\alpha\Big)$, if $x > 0$ and $\Lambda_2^*(x) = \infty$, if $x \leq 0$. That is for every Borel set $F \subseteq \mathbb{R}$*

$$\limsup_{t\to\infty} t^{-1/\alpha} \log \mathbb{P}\Big\{t^{-(1-\alpha)/\alpha}(T(b) - T(a)) \in F\Big\} \leq -\inf_{x\in\bar{F}} \Lambda_2^*(x) \tag{3.11}$$

and

$$\liminf_{t\to\infty} t^{-1/\alpha} \log \mathbb{P}\Big\{t^{-(1-\alpha)/\alpha}(T(b) - T(a)) \in F\Big\} \geq -\inf_{x\in F^0} \Lambda_2^*(x). \tag{3.12}$$

Proof. Let us observe that Eq. (3.10) follows from lemma of Valiron (1949) and the moment estimates from Lemma 2.1. It follows from (i) that for all $\theta > 0$

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[\exp(t\theta(T(b) - T(a)))]}{t^{1/\alpha}} = (b - a)\theta^{1/\alpha}. \tag{3.13}$$

Denote

$$\Lambda_2(\theta) = \begin{cases} (b-a)\theta^{1/\alpha} & \text{if } \theta > 0, \\ 0 & \text{if } \theta \leq 0. \end{cases} \tag{3.14}$$

One can verify that Λ_2 is an essentially smooth continuous function taking values on $\mathbb{R}$, thus its Fenchel–Legendre transform is given by

$$\Lambda_2^*(x) = \begin{cases} x^{\frac{1}{1-\alpha}} \left(\dfrac{b-a}{\alpha}\right)^{-\alpha/(1-\alpha)} \Big(1-\alpha\Big) & \text{if } x > 0, \\ \infty & \text{if } x \leq 0. \end{cases} \tag{3.15}$$

Proof of the part (ii) follows from (3.13) and the Gärtner–Ellis theorem. □

In the following theorems we derive bounds for $\mathbb{P}\{|Z^H(b) - Z^H(a)| > x\}$ and $\mathbb{P}\{\max_{a\leq t\leq b} |Z^H(b) - Z^H(a)| > x\}$. These results will be used in Section 4, where the upper bounds for the local and uniform moduli of continuity of Z^H will be derived.

Lemma 3.4. *There exists a finite constant $A_4 > 0$, depending on H, α only, such that for all $0 \leq a < b < \infty$ and for all $x > 0$.*

$$\mathbb{P}\Big\{\big|Z^H(b) - Z^H(a)\big| > x\Big\} \leq \exp\Big(-A_4 \frac{x^{\frac{2}{1+2H-2H\alpha}}}{(b-a)^{2H\alpha/(1+2H-2H\alpha)}}\Big). \tag{3.16}$$

Proof. Consider the following random variable

$$\Lambda = \frac{|Z^H(b) - Z^H(a)|}{(b-a)^{H\alpha}}. \tag{3.17}$$

Conditioning and applying Lemma 2.2 we get for all integers $n \geq 1$

$$\mathbb{E}(\Lambda^n) \leq A_5^n n^{n(1+2H-2H\alpha)/2}, \tag{3.18}$$

where $A_5 > 0$ is a constant depending on H and α only.

The Markov inequality for any $A_6 > 0$, and (3.18) imply that for all $u > 0$

$$\mathbb{P}(\Lambda > A_6 u) \leq \left(\frac{A_5}{A_6}\right)^n \left(\frac{n^{\frac{1+2H-2H\alpha}{2}}}{u}\right)^n. \tag{3.19}$$

Taking the constant $A_6 \geq eA_5$ and $n = \left\lfloor u^{\frac{2}{1+2H-2H\alpha}} \right\rfloor$ we obtain

$$\mathbb{P}\Big\{\big|Z^H(b) - Z^H(a)\big| \geq A_6(b-a)^{H\alpha}u\Big\} \leq \exp\big(-u^{\frac{2}{1+2H-2H\alpha}}\big). \tag{3.20}$$

It is easy to show that (3.16) follows from (3.20) by substituting $x = A_6(b-a)^{H\alpha}$. □

Using Lemma 3.4 and the result of Móricz et al. (1982) we get the following theorem.

Theorem 3.5. *There exist positive finite constants A_7 and A_8 depending on H and α only such that for all $0 \le a < b < \infty$, for all $x > 0$ and for all $\alpha < \frac{1+2H}{4H}$*

$$\mathbb{P}\Big\{\max_{a\le t\le b}\big|Z^H(t)-Z^H(a)\big| > x\Big\} \le A_7\exp\Big(-A_8\frac{x^{\frac{2}{1+2H-2H\alpha}}}{(b-a)^{2H\alpha/(1+2H-2H\alpha)}}\Big). \tag{3.21}$$

Proof. For any integer $n \ge 2$, we divide the interval $[a, b]$ into n subintervals of length $(b-a)/n$. Denote $t_{n,i} = a+\frac{i(b-a)}{n}$, $(i \in \{0, 1, \dots, n\})$, and $t_{n,i}$ be the end points of these subintervals. Since the process Z^H has continuous sample paths it suffices to show that

$$\mathbb{P}\Big\{\max_{1\le i\le n}\big|Z^H(t_{n,i})-Z^H(a)\big| > x\Big\} \le A_7\exp\Big(-A_8\frac{x^{\frac{2}{1+2H-2H\alpha}}}{(b-a)^{2H\alpha/(1+2H-2H\alpha)}}\Big) \tag{3.22}$$

for all $n \ge 2$. Let us define the random variables $\xi_i = Z^H(t_{n,i+1}) - Z^H(t_{n,i})$ for $i \in \{0, 1, \dots, n-1\}$. Then for all integer numbers k, where $0 \le j < k \le n$ we have

$$Z^H(t_{n,k})-Z^H(t_{n,j}) = \sum_{i=j}^{k-1}\xi_i = S(j,k). \tag{3.23}$$

With the help of Lemma 3.4 we conclude that for all integers $j < k$,

$$\mathbb{P}\big\{\big|S(j,k)\big| > x\big\} \le \exp\Big(-A_4x^{\frac{2}{1+2H-2H\alpha}}\Big(\frac{n}{(k-j)(b-a)}\Big)^{2H\alpha/(1+2H-2H\alpha)}\Big). \tag{3.24}$$

We denote $\phi(x) = x^{\frac{2}{1+2H-2H\alpha}}$ and

$$g(j,k) = A_4\Big(\frac{(k-j)(b-a)}{n}\Big)^{2H\alpha/(1+2H-2H\alpha)},$$

as in Meerschaert et al. (2008). For convenience let $r = 2H\alpha/(1+2H-2H\alpha)$. Since $r \in (0, 1)$, the concavity of the function $t \mapsto t^r$ implies that for all integers $1 \le i \le j < k \le n$,

$$g(i,j) + g(j+1,k) \le 2^{1-r}g(i,k). \tag{3.25}$$

From Móricz et al. (1982) we infer that the function g satisfies the property of quasi-superadditivity with index $Q = 2^{1-r}$. Further, the functions ϕ and g satisfy all the other conditions of Theorem 2.2 in Móricz et al. (1982). This implies the existence of positive and finite constants A_7 and A_8 (depending on H, α) such that (3.22) holds. This finishes the proof of Theorem 3.5. □

4. Applications

In this section, applying the results from the previous section, we establish uniform and local moduli of continuity for Z^H. Our results extend the ones from Meerschaert et al. (2008).

Theorem 4.1. *Let $Z^H = \{Z^H(t), t \ge 0\}$ be an α-stable subordinated fractional Brownian motion. Then there exists a finite constant $A_9 > 0$ such that for all constants $0 \le a < b < \infty$ we have*

$$\limsup_{h\downarrow 0}\sup_{a\le t\le b-h}\sup_{0\le s\le h}\frac{|Z^H(t+s)-Z^H(t)|}{h^{H\alpha}(\log 1/h)^{(1+2H-2H\alpha)/2}} \le A_9 \quad a.s. \tag{4.1}$$

Proof. From (3.20) we have that for every $t \ge 0$ and $h > 0$

$$\mathbb{P}\big\{\big|Z^H(t+h)-Z^H(h)\big| \ge A_6h^{H\alpha}u\big\} \le \exp\big(-u^{\frac{2}{1+2H-2H\alpha}}\big). \tag{4.2}$$

Since $Z^H = \{Z^H(t), t \ge 0\}$ satisfies conditions of Lemmas 2.1 and 2.2 in Csáki and Csörgő (1992) with $\sigma(h) = h^{H\alpha}$ and $\beta = \frac{2}{1+2H-2H\alpha}$ the result (4.1) follows from Theorem 3.1 in Csáki and Csörgő (1992). □

Theorem 4.2. *Let $Z^H = \{Z^H(t),\ t \ge 0\}$ be the subordinated fractional Brownian motion. The following two statements hold:*

(i) *Almost surely,*

$$\limsup_{t\to\infty}\frac{\max_{0\le s\le t}|Z^H(s)|}{t^{H\alpha}(\log\log t)^{(1+2H-2H\alpha)/2}} \le A_8^{-(1+2H-2H\alpha)/2}. \tag{4.3}$$

(ii) *For every $t > 0$, almost surely*

$$\limsup_{h\to 0} \frac{\max_{|s|\le h} |Z^H(t+s) - Z^H(t)|}{h^{H\alpha}(\log\log 1/h)^{(1+2H-2H\alpha)/2}} \le A_8^{-(1+2H-2H\alpha)/2}. \tag{4.4}$$

In the above A_8 is the constant defined in (3.21).

Proof. Let us observe that it is sufficient to prove only (4.3) since both (4.3) and (4.4) follow from Theorem 3.5 and the Borel–Cantelli lemma. Let $\gamma > A_8^{-1}$ and $\rho > 1$ be two arbitrary constants. For every $n \ge 1$, let $T_n = \rho^n$ and consider the event

$$E_n = \left\{\omega : \max_{0\le t\le T_n} \left|Z^H(s)\right| > T_n^{H\alpha} U(T_n)\right\},$$

where $U(t) = (\gamma \log\log t)^{(1+2H-2H\alpha)/2}$. From Theorem 3.5 we conclude that

$$\mathbb{P}(E_n) \le A_7 \exp\left(-A_8 \frac{(T_n^{H\alpha}U(T_n))^{2/(1+2H-2H\alpha)}}{T_n^{2H\alpha/(1+2H-2H\alpha)}}\right) \le A_{10} n^{-A_8\gamma}. \tag{4.5}$$

Since $A_8\gamma > 1$, we have $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$. From the Borel–Cantelli lemma we have that

$$\limsup_{n\to\infty} \frac{\max_{0\le s\le T_n} |Z^H(s)|}{T_n^{H\alpha} U(T_n)} \le 1 \quad \text{a.s.} \tag{4.6}$$

Observe that $T_{n+1}/T_n = \rho$ for every $n \ge 1$. One can easily verify that for all $t \in [T_n, T_{n+1}]$,

$$\frac{\max_{0\le s\le t} |Z^H(s)|}{t^{H\alpha} U(t)} \le \rho^{H\alpha} \frac{\max_{0\le s\le T_{n+1}} |Z^H(s)|}{T_{n+1}^{H\alpha} U(T_{n+1})} \frac{U(T_{n+1})}{U(T_n)}. \tag{4.7}$$

Eqs. (4.6) and (4.7) imply

$$\limsup_{t\to\infty} \frac{\max_{0\le s\le t} |Z^H(s)|}{t^{H\alpha}(\log\log t)^{(1+2H-2H\alpha)/2}} \le \rho^{H\alpha} \gamma^{(1+2H-2H\alpha)/2}, \quad \text{a.s.} \tag{4.8}$$

Eq. (4.3) follows from (4.8) by letting $\gamma \downarrow A_8^{-1}$ and $\rho \downarrow 1$ along rational numbers. □

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