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McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion ☆

Zhi Li [a,*], Liping Xu [a], Litan Yan [b]

[a] School of Information and Mathematics, Yangtze University, Jingzhou 434023, PR China
[b] Department of Statistics, College of Science, Donghua University, 2999 North Renmin Rd, Songjiang, Shanghai 201620, PR China

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ABSTRACT

In this paper, we study a class of McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion and impulsive McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion. We will first discuss questions of existence and uniqueness of solutions under some appropriate conditions by some fixed point theorems. Subsequently, we consider various stability properties with respect to initial data and coefficients by some time-changed Gronwall-like inequalities and time-changed Gronwall-like inequalities with jumps, generalizing some known results.

1. Introduction

Over the past few decades, McKean-Vlasov stochastic differential equations whose drift and diffusion coefficients depend not only on the state of the unknown process but also on its probability distribution have attracted the increasing attention due to the wide applications in stochastic control, queue systems, mathematical finance, multi-factor stochastic volatility and statistical physics; see, for example, [5,7,13,26]. So far, McKean-Vlasov stochastic differential equations have been investigated considerably on the existence and uniqueness of solutions [14,16,28], stability [1,12,27,33], ergodicity and invariant probability measures [4,24,32], Harnack inequalities [3,10,15], numerical algorithms [2,9,20] among others.

Recently, the time-changed semimartingale has attracted lots of attentions and became one of the most active areas in stochastic analysis thanks to model anomalous diffusions arising in physics, finance, hydrology, and cell biology (see Umarov et al. [31] for the details). Kobayashi [19] investigated stochastic integrals with

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* Corresponding author.
E-mail address: lizhi_csu@126.com (Z. Li).

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regard to the time-changed semimartingale and obtained the time-changed Itô's formula when the time change is the first hitting time process of a stable subordinator of index between 0 and 1. Following [19], the existence and stability of SDEs driven by the time-changed Brownian motion or Lévy noise received much attention. For example, Wu [34] investigated the exponential sample-path stability, p-th moment asymptotic stability and p-th moment exponential stability for SDEs driven by the time-changed Brownian motion; Wu [35] investigated the stochastic stability, stochastically asymptotic stability and globally stochastically asymptotic stability for SDEs driven by the time-changed Brownian motion; Zhang and Yuan [38] considered the exponential stability for stochastic functional differential equations with Markovian switching driven by the time-changed Brownian motion; Nane and Ni [29] studied the stability of the solution of stochastic differential equations driven by time-changed Lévy noise in both probability and moment sense; Nane and Ni [30] established the stability of the paths of stochastic differential equations driven by time-changed Lévy noise; Li et al. [22] considered Global attracting sets and exponential stability for stochastic functional differential equations driven by the time-changed Brownian motion; Yin et al. [37] discussed the stability of nonlinear stochastic differential equations driven by time-changed Lévy process with impulsive effects.

As far as we know, there is no result on McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion. In this paper, we will attempt to consider the following McKean-Vlasov stochastic differential equations driven by time-changed Brownian motion B_{E_t}

$$\begin{cases} dx_t = f(t, E_t, x_t, \mathcal{L}_{x_t})dE_t + g(t, E_t, x_t, \mathcal{L}_{x_t})dB_{E_t}, \quad t \geq 0, \\ x_0 = x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where E_t is specified as an inverse of a stable subordinator of index $\beta \in (0, 1)$, $\mathcal{L}_{x_t}$ denotes the law of x_t and f, g are some measurable functions. By using some fixed point theorems, we will discuss the existence and uniqueness of solutions for (1.1) under some appropriate conditions. The stability analysis of solutions is the particular interest in stochastic differential equation since it had a great practical and theoretical significance. The other one objective of our work is to study the stability properties with respect to initial data and coefficients by some time-changed Gronwall-like inequalities.

Impulsive phenomena can be found in a wide variety of evolutionary processes, for example, economics, biology, mechanics, electronics and telecommunications, in which many sudden and abrupt changes take place instantaneously, in the form of impulses. Recently, there has been increasing interest in the study of stochastic differential equation with impulses (see [6,8,11,21]). In this work, we will also study the existence, uniqueness and stability properties with respect to initial data and coefficients of solutions for impulses McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion by establishing some time-changed Gronwall-like inequalities with jumps.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we prove that the existence and uniqueness theorem for (1.1) under some appropriate conditions. In Section 4, we devote ourselves to the stability properties with respect to initial data and coefficients. In Section 5, we consider the existence, uniqueness and stability for impulses McKean-Vlasov stochastic differential equations driven by the time-changed Brownian motion.

2. Preliminary

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{D(t), t \geq 0\}$ be a left limit and right continuous non-decreasing Lévy process which is named a subordinator that begins with 0. In particular, $D(t)$ is called the β-stable subordinator denoted by $D_\beta(t)$ if it is a strictly increasing with the following Laplace transform

$$\mathbb{E}(e^{-\lambda D_\beta(t)}) = e^{-t\lambda^\beta}, \quad \lambda > 0, \ \beta \in (0, 1).$$

Define the generalized inverse of an adapted β-stable subordinator $D_\beta(t)$ as

$$E_t := E_t^\beta = \inf\{s > 0 : D_\beta(s) > t\},$$

which is known as the initial hitting time process. The time change process E_t is nondecreasing and continuous.

Remark 2.1. From Theorem 1 and Remark 6(2) in [17], we know that $\mathbb{E}[e^{\lambda E_t^2}]$ exists if $1/2 < \beta < 1$ while it does not if $0 < \beta < 1/2$. So, we assume that $1/2 < \beta < 1$ in this work for ensuring the existence of $\mathbb{E}[e^{\lambda E_t^2}]$.

For each fixed $t \geq 0$ the random time E_t is an $\mathcal{F}_t$-stopping time, and therefore, the time-changed filtration $\mathcal{G}_t = \mathcal{F}_{E_t}$, $t \geq 0$ is well-defined. Moreover, since the time change E is an $\mathcal{F}_{E_t}$-adapted nondecreasing process and the time-changed Brownian motion $B \circ E = B_{E_t}$, $t \geq 0$ is an $\mathcal{F}_{E_t}$-martingale, SDE (1.1) is understood within the framework of stochastic integrals driven by semimartingales. Since the Brownian motion B and the subordinator D are assumed independent, it is possible to set up B and D on a product space with product measure $\mathbb{P} = \mathbb{P}_B \times \mathbb{P}_D$ with obvious notation. Let $\mathbb{E}_B$, $\mathbb{E}_D$ and $\mathbb{E}$ denote the expectations under the probability measures $\mathbb{P}_B$, $\mathbb{P}_D$ and $\mathbb{P}$, respectively.

2.1. Wasserstein metric

For $d \in \mathbb{N}$, $\mathbb{R}^d$ denotes the d-dimensional euclidean space, $\langle \cdot, \cdot \rangle$ the inner product and $|\cdot|$ the corresponding norm. If $m \in \mathbb{N}$, $\mathbb{R}^{d\times m}$ denotes the space of all $d \times m$-matrices with the norm $\|\cdot\|$. Let $P(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ and for any $p > 1$ denote by $P_p(\mathbb{R}^d)$ the subspace of $P(\mathbb{R}^d)$ of probability measures with finite moment of order p.

For $\mu, \nu \in P_p(\mathbb{R}^d)$, define the p-Wasserstein distance $W_p(\mu, \nu)$ by

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \Big[\int\limits_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \Big]^{1/p},$$

where $\Pi(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are, respectively, μ and ν.

In the case $\mu = \mathcal{L}_X$ and $\nu = \mathcal{L}_Y$ are the laws of $\mathbb{R}^d$-valued random variable X and Y of order p, then

$$W_p(\mu, \nu)^p \leq \mathbb{E}[|X - Y|^p]. \tag{2.1}$$

In the literature the Wasserstein metric is restricted to W_2 while W_1 is often called the Kantorovich-Rubinstein distance because of the role it plays in optimal transport.

2.2. Assumptions

In the subsection, we make the following assumptions:

(H1) $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \times P_2(\mathbb{R}^d) \to \mathbb{R}^d$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \times P_2(\mathbb{R}^d) \to \mathbb{R}^{d\times k}$ are Borel-measurable functions and there exists a constant $L > 0$ such that for any $t_1, t_2 \geq 0$, $x, y \in \mathbb{R}^d$ and $\mu, \nu \in P_2(\mathbb{R}^d)$ such that

$$|f(t_1, t_2, x, \mu) - f(t_1, t_2, y, \nu)| \leq L(|x - y| + W_2(\mu, \nu))$$

and

$$\|g(t_1, t_2, x, \mu) - g(t_1, t_2, y, \nu)\| \leq L(|x - y| + W_2(\mu, \nu)),$$

where W_2 denotes the 2-Wasserstein metric.

(H2) If x_t is a right continuous with left limit and $\mathcal{G}_t$-adapted process, then

$$f(t, E_t, x_t, \mathcal{L}_{x_t}), g(t, E_t, x_t, \mathcal{L}_{x_t}) \in \mathcal{L}(\mathcal{G}_t),$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of left continuous with right limit.

For the purposes of stability, we shall assume that

$$f(t_1, t_2, 0, \delta_0) \equiv 0, \quad g(t_1, t_2, 0, \delta_0) \equiv 0, \quad \text{for any } t_1, t_2 \geq 0 \tag{2.2}$$

where δ_0 denotes the Dirac measure with mass at $0 \in \mathbb{R}^d$.

3. Existence and uniqueness

In this section, we consider the existence and uniqueness of a solution to (1.1) when the coefficients satisfy the assumptions (H1) and (H2). We need the following lemma provide connections among different kinds of time-changed integrals.

Lemma 3.1. *([19]) Let E_t be the $\mathcal{G}_t$-measurable time change which is the general inverse β-stable subordinator $D(t)$. Suppose $\mu(t)$ and $\delta(t)$ are $\mathcal{G}_t$-measurable and integrable. Then, for all $t \geq 0$ with probability one,*

$$\int_0^t \mu(s)dE_s + \int_0^t \delta(s)dB_{E_s} = \int_0^{E_t} \mu(D(s-))ds + \int_0^{E_t} \delta(D(s-))dB_s.$$

Furthermore, we develop the following generalized time-changed retarded Gronwall inequality which is useful in our work.

Lemma 3.2. *Suppose $D(t)$ is a β-stable subordinator and E_t is the associated inverse stable subordinator. Let $T > 0$ and $u, g : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the $\mathcal{G}_t$-measurable functions which are integrable with respect to E_t. Let $n(t)$ be a positive, monotonic, non-decreasing function. Then, the inequality*

$$u(t) \leq n(t) + \int_0^t g(s)u(s)dE_s, \quad t \geq 0 \tag{3.1}$$

implies almost surely

$$u(t) \leq n(t)exp\Big\{ \int_0^t g(s)dE_s \Big\}, \quad tt \geq 0.$$

Proof. Let

$$y(t) := n(t) + \int_0^t g(s)u(s)dE_s, \quad tt \geq 0. \tag{3.2}$$

Since $g(s)$ and $u(s)$ are positive and $n(t)$ is non-decreasing, the function $y(t)$ defined in equation (3.2) is nondecreasing. Moreover, from equations (3.1) and (3.2),

$$u(t) \leq y(t), \quad t \geq 0,$$

which implies that

$$y(t) \leq n(t) + \int_0^t g(s)y(s)dE_s, \quad t \geq 0.$$

Applying the Lemma 3.1 yields

$$y(t) \leq n(t) + \int_0^{E_t} g(D(s-))y(D(s-))ds, \quad t \geq 0. \tag{3.3}$$

Actually, for $0 \leq t \leq E_T$, $D(t-)$ is defined as

$$D(t-) = \inf\{s : s \in [0,T], E_s \geq t\} \wedge T$$

which means

$$E_{D(t-)} = t, \quad \text{and} \quad D(E_t-) \leq t. \tag{3.4}$$

Let $\tau \in [0, \infty)$, then it holds from (3.3) and (3.4) that

$$\begin{aligned} y(D(\tau-)) \leq & n(D(\tau-)) + \int_0^{E_{D(\tau-)}} g(D(s-))y(D(s-))ds \\ = & n(D(\tau-)) + \int_0^{\tau} g(D(s-))y(D(s-))ds. \end{aligned}$$

Apply generalization of Gronwall inequality (see Chapter IX.6a. Lemma 6.3 of [18]) path by path to yield

$$u(D(\tau-)) \leq y(D(\tau-)) \leq n(D(\tau-))\exp\Big\{ \int_0^{\tau} g(D(s-))ds \Big\}.$$

For any $t \geq 0$, we easily know that there is $\tau \in [0, +\infty)$ such that $D(\tau-) = t$ provided E is strictly increasing in a certain neighborhood of t. Since $n(t)$ is non-decreasing, applying (3.4) and the Lemma 3.1, we have

$$u(t) \leq y(t) \leq n(t)\exp\Big\{ \int_0^{E_t} g(D(s-))ds \Big\} = n(t)\exp\Big\{ \int_0^{t} g(s)dE_s \Big\}.$$

If E is a constant τ in a certain neighborhood $U(t;\delta)$ of t, then $t - \delta = D(\tau-)$. Since $n(t)$ is non-decreasing, applying (3.4), we have

$$u(t-\delta) \leq y(t-\delta) \leq n(t-\delta)\exp\Big\{ \int_0^{E_{t-\delta}} g(D(s-))ds \Big\}.$$

Then, for any $t \geq 0$, we have by the Lemma 3.1

$$u(t) \leq y(t) \leq n(t)\exp\Big\{\int_0^{E_t} g(D(s-))ds\Big\} = n(t)\exp\Big\{\int_0^t g(s)dE_s\Big\}.$$

The proof is complete. □

Theorem 3.1. *Under the assumptions (H1) and (H2), for any initial data $x_0 = x \in \mathbb{R}^d$, (1.1) admits a unique solution.*

Proof. For any $\mu, \nu \in P_2(\mathbb{R}^d)$, for convenience, we denote $\mu = \mathcal{L}_X$ and $\nu = \mathcal{L}_Y$, which are the laws of $\mathbb{R}^d$-valued random variable X and Y. Consider the following stochastic differential equations driven by time-changed Brownian motion B_{E_t}

$$\begin{cases} dx_t = f(t, E_t, x_t, \mu)dE_t + g(t, E_t, x_t, \mu)dB_{E_t}, \quad t \geq 0. \\ x_0 = x \in \mathbb{R}^d. \end{cases} \tag{3.5}$$

For any $T > 0$, by view of [19,38], we know that (3.5) has an unique $\mathcal{G}_t = \mathcal{F}_{E_t}$-adapted solution process $x_t(\mu)$ such that $\mathbb{E}[\sup_{0\leq t\leq T} |x_t(\mu)|^2] < +\infty$ under the assumptions (H1) and (H2). We denote $C([0,T], P_2(\mathbb{R}^d))$ the complete metric space of continuous functions from $[0,T]$ to $(P_2(\mathbb{R}^d), W_2)$ with the metric:

$$D_T(\mu, \nu) := \sup_{0\leq t\leq T} W_2(\mu_t, \nu_t), \quad \mu, \nu \in C([0,T], P_2(\mathbb{R}^d)).$$

We define an operator Ψ on $C([0,T]; P_2(\mathbb{R}^d))$ by

$$\Psi : \mu \to \mathcal{L}_{x_t(\mu)},$$

where $\mathcal{L}_{x_t(\mu)}$ is the law of process $x_t(\mu)$. Obviously, if x is a solution of equation (1.1). Then its law $\mu_t := \mathcal{L}_{x_t(\mu)}$ is a fixed point of Ψ. Thus to prove the existence and uniqueness of (1.1), it is sufficient to prove that the operator Ψ has a unique fixed point.

Notice that $x_t(\mu)$ has continuous paths and $\mathbb{E}[\sup_{0\leq t\leq T} |x_t(\mu)|^2] < +\infty$, and we deduce Ψ is well defined. On the other hand, by (2.1) and the Burkholder-Davis-Gundy inequality (Jin and Kobayashi [17]), we have

$$\begin{aligned}
\lim_{h\to 0} W_2(\mathcal{L}_{x_{t+h}(\mu)}, \mathcal{L}_{x_t(\mu)}) &\leq \lim_{h\to 0} [\mathbb{E}|x_{t+h}(\mu) - x_t(\mu)|^2]^{1/2} \\
&= \lim_{h\to 0} \Big[\mathbb{E}\Big|\int_t^{t+h} f(s, E_s, x_s, \mu)dE_s + \int_t^{t+h} g(s, E_s, x_s, \mu)dB_{E_s}\Big|^2\Big]^{1/2} \\
&\leq \lim_{h\to 0} \Big[2h\mathbb{E}\Big|\int_t^{t+h} |f(s, E_s, x_s, \mu)|^2 dE_s\Big| + 2b_2 \int_t^{t+h} |g(s, E_s, x_s, \mu)|^2 dE_s\Big]^{1/2} \\
&= 0,
\end{aligned}$$

where the positive constant b_2 comes from [17]. Thus, $t \to \mathcal{L}_{x_t(\mu)}$ is continuous in $(P_2(\mathbb{R}^d), W_2)$.

For $n \in \mathbb{N}_0$ and $\mu, \nu \in P_2(\mathbb{R}^d)$, define $x^{(n)} := \Psi^n \mu$ and $y^{(n)} := \Psi^n \nu$. Moreover define $\mu_t^{(n)} := \mathcal{L}_{x_t^{(n)}}$ and $\nu_t^{(n)} := \mathcal{L}_{y_t^{(n)}}$. By the definition of Ψ we have for $n \geq 1$ that $x^{(n)} = \Psi(x^{(n-1)})$ solves

$$\begin{cases} dx_t^{(n)} = f(t, E_t, x_t^{(n)}, \mu_t^{(n-1)})dE_t + g(t, E_t, x_t^{(n)}, \mu_t^{(n-1)})dB_{E_t}, \\ x_0^{(n)} = x \in \mathbb{R}^d, \end{cases} \tag{3.6}$$

and $y^{(n)} = \Psi(y^{(n-1)})$ solves

$$\begin{cases} dy_t^{(n)} = f(t, E_t, y_t^{(n)}, \nu_t^{(n-1)})dE_t + g(t, E_t, y_t^{(n)}, \nu_t^{(n-1)})dB_{E_t}, \\ y_0^{(n)} = x \in \mathbb{R}^d. \end{cases} \tag{3.7}$$

By using (3.6) and (3.7) we have for $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}_B\Big[\sup_{0\le s\le t} |x_s^{(n)} - y_s^{(n)}|^2\Big] \le& 2\mathbb{E}_B\Big[\sup_{0\le s\le t}\Big|\int_0^s (f(r, E_r, x_r^{(n)}, \mu_r^{(n-1)}) - f(r, E_r, y_r^{(n)}, \nu_r^{(n-1)}))dE_r\Big|^2\Big] \\ &+ 2\mathbb{E}_B\Big[\sup_{0\le s\le t}\Big|\int_0^s (g(r, E_r, x_r^{(n)}, \mu_r^{(n-1)}) - g(r, E_r, y_r^{(n)}, \nu_r^{(n-1)}))dB_{E_r}\Big|^2\Big] \\ =&: 2(I_1 + I_2). \end{aligned} \tag{3.8}$$

Since B_t is independent of E_t, by applying the assumption (H1), (2.1) and the Hölder inequality we can get

$$\begin{aligned} I_1 \le& 2E_T L^2 \mathbb{E}_B\Big[\sup_{0\le s\le t}\int_0^s (|x_r^{(n)} - y_r^{(n)}|^2 + W_2(\mu_r^{(n-1)}, \nu_r^{(n-1)})^2)dE_r\Big] \\ \le& 2E_T L^2 \int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n)} - y_r^{(n)}|^2]dE_s + 2E_T L^2\Big(\int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n-1)} - y_r^{(n-1)}|^2]dE_s\Big). \end{aligned} \tag{3.9}$$

By assumption (H1), (2.1) and the Burkholder-Davis-Gundy inequality (Jin and Kobayashi [17]),

$$\begin{aligned} I_2 \le& b_2\mathbb{E}_B\Big[\int_0^t \|g(r, E_r, x_r^{(n)}, \mu_r^{(n-1)}) - g(r, E_r, y_r^{(n)}, \nu_r^{(n-1)})\|^2 dE_r\Big] \\ \le& 2b_2L^2\mathbb{E}_B\Big[\int_0^t (|x_r^{(n)} - y_r^{(n)}|^2 + W_2(\mu_r^{(n-1)}, \nu_r^{(n-1)})^2)dE_r\Big] \\ \le& 2b_2L^2 \int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n)} - y_r^{(n)}|^2]dE_s + 2b_2L^2\Big(\int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n-1)} - y_r^{(n-1)}|^2]dE_s\Big), \end{aligned} \tag{3.10}$$

where the positive constant b_2 comes from [17].
Then, combining (3.9)-(3.10) with (3.8) we have

$$\begin{aligned} \mathbb{E}_B\Big[\sup_{0\le s\le t} |x_s^{(n)} - y_s^{(n)}|^2\Big] \le& 4(E_T L^2 + b_2L^2)\int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n)} - y_r^{(n)}|^2]dE_s \\ &+ 4(E_T L^2 + b_2L^2)\Big(\int_0^t \mathbb{E}_B[\sup_{0\le r\le s} |x_r^{(n-1)} - y_r^{(n-1)}|^2]dE_s\Big). \end{aligned}$$

Thus, from the Lemma 3.2, one has

$$\mathbb{E}_B\Big[\sup_{0\le s\le t}|x_s^{(n)}-y_s^{(n)}|^2\Big]\le 4(E_TL^2+b_2L^2)e^{4(E_TL^2+b_2L^2)E_T}\Big(\int_0^t\mathbb{E}_B[\sup_{0\le r\le s}|x_r^{(n-1)}-y_r^{(n-1)}|^2]dE_s\Big). \tag{3.11}$$

Defining

$$\mathcal{J}^n(t)=\mathbb{E}_B\Big[\sup_{0\le s\le t}|x_s^{(n)}-y_s^{(n)}|^2\Big],$$

we have for any $n\ge 1$ and $t\in[0,T]$

$$\mathcal{J}^n(t)\le\kappa(E_T)\int_0^t\mathcal{J}^{n-1}(s)dE_s$$

with $\kappa(E_T)=4(E_TL^2+b_2L^2)e^{4(E_TL^2+b_2L^2)E_T}$. Consequently, by iteration we obtain

$$\mathcal{J}^n(t)\le\frac{\kappa^n(E_T)(E_T)^n}{n!}\mathbb{E}_B|X-Y|^2. \tag{3.12}$$

Taking $\mathbb{E}_D$ on two sides of (3.11), we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^{(n)}-y_t^{(n)}|^2\Big]\le\frac{\mathbb{E}[\kappa^n(T,E_T)(E_T)^n]}{n!}\mathbb{E}|X-Y|^2.$$

For large n, Ψ^n is a strict contraction which implies that Ψ admits a unique fixed point in the complete metric space $C([0,T];P_2(\mathbb{R}^d))$. Since T is arbitrary, we can immediately obtain the existence and uniqueness of solution for (1.1). The proof is complete. □

4. Stability

4.1. Stability of the Picard successive approximations

Assume that f and g satisfy assumptions (H1) and (H2). We will prove the convergence of the Picard iteration scheme, which is useful for numerical computations of the unique solution of (1.1). Let $x_t^0=x\in\mathbb{R}^d$ for all $t\in[0,T]$ and define x^{n+1} as the solution of the following SDE:

$$\begin{cases} dx_t^{n+1}=f(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dE_t+g(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dB_{E_t},\\ x_0^{n+1}=x\in\mathbb{R}^d. \end{cases} \tag{4.1}$$

Theorem 4.1. *Under the assumption (H1) and (H2), the sequence x^n converges to the unique solution x of (1.1), i.e.,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|x_t^n-x_t|^2\Big]\to 0,\quad \textit{as } n\to\infty.$$

Proof. Let $n\ge 0$, by using the Hölder inequality and the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^{n+1}-x_t^n|^2\Big]\le&2E_T\mathbb{E}_B\Big[\int_0^T|f(t,E_t,x_t^n,\mathcal{L}_{x_t^n})-f(t,E_t,x_t^{n-1},\mathcal{L}_{x_t^{n-1}})|^2dE_t\Big]\\ &+2b_2\mathbb{E}_B\Big[\int_0^T\|g(t,E_t,x_t^n,\mathcal{L}_{x_t^n})-g(t,E_t,x_t^{n-1},\mathcal{L}_{x_t^{n-1}})\|^2dE_t\Big].\end{aligned}$$

Then, by view of the assumption (H1) and (2.1) we get

$$\begin{aligned}
&\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^{n+1}-x_t^n|^2\Big]\\
\le& 4E_TL^2\mathbb{E}_B\Big[\int_0^T|x_t^n-x_t^{n-1}|^2dE_t\Big]+4E_TL^2\mathbb{E}_B\Big[\int_0^T W_2(\mathcal{L}_{x_t^n},\mathcal{L}_{x_t^{n-1}})^2dE_t\Big]\\
&+4b_2L^2\mathbb{E}_B\Big[\int_0^T|x_t^n-x_t^{n-1}|^2dE_t\Big]+4b_2L^2\mathbb{E}_B\Big[\int_0^T W_2(\mathcal{L}_{x_t^n},\mathcal{L}_{x_t^{n-1}})^2dE_t\Big]\\
\le& 4(E_T+b_2)L^2\mathbb{E}_B\Big[\int_0^T|x_t^n-x_t^{n-1}|^2dE_t\Big]+4(E_T+b_2)L^2\mathbb{E}_B\Big[\int_0^T \mathbb{E}|x_t^n-x_t^{n-1}|^2dE_t\Big]\\
\le& 8(E_T+b_2)L^2\int_0^T\mathbb{E}_B\Big[\sup_{0\le s\le t}|x_s^n-x_s^{n-1}|^2\Big]dE_t.
\end{aligned}\tag{4.2}$$

Defining

$$\mathcal{K}^n(T)=\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^{n+1}-x_t^n|^2\Big],$$

we have for any $n\ge 1$ and $t\in[0,T]$

$$\mathcal{K}^n(T)\le\rho(E_T)\int_0^T\mathcal{K}^{n-1}(t)dE_t$$

with $\rho(E_T)=8(E_T+b_2)L^2$. Consequently, by iteration we obtain

$$\mathcal{J}^n(T)\le\frac{\rho^{n+1}(E_T)(E_T)^{n+1}}{(n+1)!}\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^1-x_t^0|^2\Big].\tag{4.3}$$

On the other hand, we know that

$$\begin{aligned}
\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^1-x_t^0|^2\Big]\le&4E_TL^2\int_0^T\mathbb{E}_B|x|^2dE_s+4b_2L^2\int_0^T\mathbb{E}_B|x|^2dE_s\\
=&4(E_T^2+b_2E_T)L^2\mathbb{E}_B|x|^2.
\end{aligned}\tag{4.4}$$

Combining (4.3) with (4.4) we obtain

$$\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^{n+1}-x_t^n|^2\Big]\le\frac{4\rho^{n+1}(E_T)(E_T)^{n+1}(E_T^2+b_2E_T)L^2}{(n+1)!}\mathbb{E}_B|x|^2.\tag{4.5}$$

Taking $\mathbb{E}_D$ on both sides of (4.5), we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^{n+1}-x_t^n|^2\Big]\le\frac{4\mathbb{E}[\rho^{n+1}(E_T)(E_T)^{n+1}(E_T^2+b_2E_T)]L^2}{(n+1)!}\mathbb{E}|x|^2,$$

which means that x_t^n is a Cauchy sequence in $L^2(\Omega;C([0,T]);\mathbb{R}^d)$. Therefore, x_t^n converges to a limit x_t which is the unique solution of (1.1). The proof is complete. □

4.2. Stability with respect to initial condition

In this subsection, we will consider the convergence of (1.1) with respect to small perturbations of the initial condition. To this end, we denote by x_t^x the unique solution of (1.1) with initial data x.

Theorem 4.2. *Assume that f and g satisfy the assumption (H1) and (H2), then the mapping*

$$\Phi : \mathbb{R}^d \to L^2(\Omega; C([0,T]); \mathbb{R}^d)$$

defined by $\Phi(x)_t = x_t^x$ is continuous.

Proof. Let x_n be a sequence in $\mathbb{R}^d$ converging to x. Now, we prove that

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |x_t^{x_n} - x_t^x|^2\Big] = 0.$$

By using the elementary inequality, we have

$$\begin{aligned}
&\mathbb{E}_B\Big[\sup_{0\le t\le T} |x_t^{x_n} - x_t^x|^2\Big] \\
\le &3\mathbb{E}_B|x_n - x|^2 + 3\mathbb{E}_B\Big[\sup_{0\le t\le T}\Big|\int_0^t f(s, E_s, x_s^{x_n}, \mathcal{L}_{x_s^{x_n}}) - f(s, E_s, x_s^x, \mathcal{L}_{x_s^x})dE_s\Big|^2\Big] \\
&+ 3\mathbb{E}_B\Big[\sup_{0\le t\le T}\Big|\int_0^t g(s, E_s, x_s^{x_n}, \mathcal{L}_{x_s^{x_n}}) - g(s, E_s, x_s^x, \mathcal{L}_{x_s^x})dB_{E_s}\Big|^2\Big] \\
=: &3(J_1 + J_2 + J_3).
\end{aligned} \tag{4.6}$$

Applying the Hölder inequality, (2.1) and the assumption (H1) we obtain

$$\begin{aligned}
J_2 \le &2E_T L^2 \mathbb{E}_B\Big[\sup_{0\le t\le T}\int_0^t (|x_s^{x_n} - x_s^x|^2 + W_2(\mathcal{L}_{x_s^{x_n}}, \mathcal{L}_{x_s^x})^2)dE_s\Big] \\
\le &4E_T L^2 \int_0^T \mathbb{E}_B[\sup_{0\le s\le t} |x_s^{x_n} - x_s^x|^2]dE_t.
\end{aligned} \tag{4.7}$$

By view of the Burkholder-Davis-Gundy inequality, (2.1) and the assumption (H1) we obtain

$$\begin{aligned}
J_3 \le &2b_2 L^2 \int_0^T (|x_t^{x_n} - x_t^x|^2 + W_2(\mathcal{L}_{x_t^{x_n}}, \mathcal{L}_{x_t^x})^2)dE_t\Big] \\
\le &4b_2 L^2 \int_0^T \mathbb{E}_B[\sup_{0\le s\le t} |x_s^{x_n} - x_s^x|^2]dE_t.
\end{aligned} \tag{4.8}$$

Combining (4.7)-(4.8) and (4.6), we deduce that

$$\mathbb{E}_B\Big[\sup_{0\le t\le T} |x_t^{x_n} - x_t^x|^2\Big] \le 3\mathbb{E}_B|x_n - x|^2 + 12(E_T + b_2)L^2 \int_0^T \mathbb{E}_B[\sup_{0\le s\le t} |x_s^{x_n} - x_s^x|^2]dE_t.$$

Then, by the Lemma 3.2 we obtain that

$$\mathbb{E}_B\Big[\sup_{0\le t\le T}|x_t^{x_n}-x_t^x|^2\Big]\le 3\mathbb{E}_B|x_n-x|^2\exp[12(E_T^2+b_2E_T)L^2]. \tag{4.9}$$

Finally, taking $\mathbb{E}_D$ on both sides of (4.9), we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^{x_n}-x_t^x|^2\Big]\le 3\mathbb{E}|x_n-x|^2\mathbb{E}[\exp(12(E_T^2+b_2E_T)L^2)].$$

Therefore x_n converging in $\mathbb{R}^d$ to x implies that $\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^{x_n}-x_t^x|^2\Big]=0$. The proof is complete. □

4.3. Stability with respect to the coefficients

In this subsection, we will study the convergence of (1.1) with respect to small perturbations of the coefficients f and g. To this end, we consider the sequences of functions f_n and g_n and the corresponding McKean-Vlasov stochastic differential equations driven by time-changed Brownian motion B_{E_t}

$$\begin{cases} dx_t^n=f_n(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dE_t+g_n(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dB_{E_t}, & t\ge 0,\\ x_0^n=x\in\mathbb{R}^d. \end{cases} \tag{4.10}$$

Theorem 4.3. *Assume that the coefficients f, g, f_n and g_n satisfy the assumptions (H1) and (H2). Furthermore, assume that for each $T>0$ and each a compact set K*

$$\lim_{n\to\infty}\sup_{0\le t\le T}\sup_{x\in K}\sup_{\mu\in P_2(\mathbb{R}^d)}|f_n(t,E_t,x,\mu)-f(t,E_t,x,\mu)|=0 \tag{4.11}$$

and

$$\lim_{n\to\infty}\sup_{0\le t\le T}\sup_{x\in K}\sup_{\mu\in P_2(\mathbb{R}^d)}\|g_n(t,E_t,x,\mu)-g(t,E_t,x,\mu)\|=0, \tag{4.12}$$

then

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\le t\le T}|x_t^n-x_t|^2\Big]=0,$$

where x_t^n and x_t are solutions of (4.10) and (1.1), respectively.

Proof. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}|x_t^n - x_t|^2 \leq & 4\Big|\int_0^t (f_n(s, E_s, x_s^n, \mathcal{L}_{x_s^n}) - f_n(s, E_s, x_s, \mathcal{L}_{x_s}))dE_s\Big|^2 \\ & + 4\Big|\int_0^t (f_n(s, E_s, x_s, \mathcal{L}_{x_s}) - f(s, E_s, x_s, \mathcal{L}_{x_s}))dE_s\Big|^2 \\ & + 4\Big|\int_0^t (g_n(s, E_s, x_s^n, \mathcal{L}_{x_s^n}) - g_n(s, E_s, x_s, \mathcal{L}_{x_s}))dB_{E_s}\Big|^2 \\ & + 4\Big|\int_0^t (g_n(s, E_s, x_s, \mathcal{L}_{x_s}) - g(s, E_s, x_s, \mathcal{L}_{x_s}))dB_{E_s}\Big|^2 \\ =: & \, 4(I_1(t) + I_2(t) + I_3(t) + I_4(t)).\end{aligned} \tag{4.13}$$

Applying the Hölder inequality, (2.1) and the assumption (H1) we obtain

$$\begin{aligned}\mathbb{E}_B[\sup_{0\leq t\leq T} I_1(t)] \leq & 2E_T L^2 \mathbb{E}_B\Big[\int_0^T (|x_s^n - x_s|^2 + W_2(\mathcal{L}_{x_s^n}, \mathcal{L}_{x_s})^2)dE_s\Big] \\ \leq & 4E_T L^2 \int_0^T \mathbb{E}_B[\sup_{0\leq s\leq t} |x_s^n - x_s|^2]dE_t.\end{aligned} \tag{4.14}$$

For $I_2(t)$, by the Hölder inequality we get

$$\mathbb{E}_B[\sup_{0\leq t\leq T} I_2(t)] \leq E_T \mathbb{E}_B\Big[\int_0^T |f_n(s, E_s, x_s, \mathcal{L}_{x_s}) - f(s, E_s, x_s, \mathcal{L}_{x_s})|^2 dE_s\Big]. \tag{4.15}$$

By view of the Burkholder-Davis-Gundy inequality, (2.1) and the assumption (H1) we obtain

$$\begin{aligned}\mathbb{E}_B[\sup_{0\leq t\leq T} I_3(t)] \leq & 2b_2 L^2 \mathbb{E}_B\Big[\int_0^T (|x_s^n - x_s|^2 + W_2(\mathcal{L}_{x_s^n}, \mathcal{L}_{x_s})^2)dE_s\Big] \\ \leq & 4b_2 L^2 \int_0^T \mathbb{E}_B[\sup_{0\leq s\leq t} |x_s^n - x_s|^2]dE_t.\end{aligned} \tag{4.16}$$

By the Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}_B[\sup_{0\leq t\leq T} I_4(t)] \leq b_2 \mathbb{E}_B\Big[\int_0^T |g_n(s, E_s, x_s, \mathcal{L}_{x_s}) - g(s, E_s, x_s, \mathcal{L}_{x_s})|^2 dE_s\Big]. \tag{4.17}$$

Thus, combining (4.14)-(4.17) with (4.13) we have

$$\mathbb{E}_B[\sup_{0\leq t\leq T} |x_t^n - x_t|^2] \leq 16(E_T + b_2)L^2 \int_0^T \mathbb{E}_B[\sup_{0\leq s\leq t} |x_s^n - x_s|^2]dE_t + C_n,$$

where

$$C_n = 4E_T \mathbb{E}_B\Big[\int_0^T |f_n(s,E_s,x_s,\mathcal{L}_{x_s}) - f(s,E_s,x_s,\mathcal{L}_{x_s})|^2 dE_s\Big]$$
$$+ 4b_2 \mathbb{E}_B\Big[\int_0^T \|g_n(s,E_s,x_s,\mathcal{L}_{x_s}) - g(s,E_s,x_s,\mathcal{L}_{x_s})\|^2 dE_s\Big].$$

By view of the Lemma 3.2, we know that

$$\mathbb{E}_B[\sup_{0\le t\le T} |x_t^n - x_t|^2] \le C_n \exp[16(E_T^2 + b_2E_T)L^2]. \tag{4.18}$$

Taking $\mathbb{E}_D$ on both sides of (4.18), we have

$$\mathbb{E}[\sup_{0\le t\le T} |x_t^n - x_t|^2] \le \mathbb{E}[C_n \exp[16(E_T^2 + b_2E_T)L^2].$$

By using (4.11) and (4.12) it is easy see that $C_n \to 0$ as $n \to \infty$, which means that

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |x_t^n - x_t|^2\Big] = 0.$$

The proof is complete. □

5. Impulsive McKean-Vlasov SDEs with time-changed Brownian motion

In this section, we consider the following impulsive McKean-Vlasov stochastic differential equations with time-changed Brownian motion

$$\begin{cases} dx_t = f(t,E_t,x_t,\mathcal{L}_{x_t})dE_t + g(t,E_t,x_t,\mathcal{L}_{x_t})dB_{E_t}, & t\ge 0,\ t\ne t_k,\\ \triangle x_{t_k} = x_{t_k^+} - x_{t_k^-} = I_k(x_{t_k}), & t = t_k,\ k=1,2,\cdots,\\ x_0 = x \in \mathbb{R}^d, \end{cases} \tag{5.1}$$

where the fixed moments of time t_k satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < \cdots$, and $\lim_{k\to+\infty} t_k = \infty$; $x_{t_k^+}$ and $x_{t_k^-}$ denote the right and left limits of x_t at time t_k; x_t at each impulsive point t_k is right continuous. And $\triangle x_{t_k} = x_{t_k^+} - x_{t_k^-}$ represents the jump in the state x at time t_k, and I_k determines the size of the jump.

The main objective of this section is to establish the existence, uniqueness and some stability properties of solutions for (5.1) under some appropriate conditions. To this end, we need to establish some impulsive time-changed Gronwall-like inequalities.

Lemma 5.1. *Let $n(t)$ be a positive, monotonic, non-decreasing function. Let the nonnegative piecewise continuous function $u(t)$ satisfy*

$$u(t) \le n(t) + \int_0^t f(s)u(s)ds + \sum_{0<t_i<t} \beta_i u(t_i), \quad t\ge 0 \tag{5.2}$$

where $\beta_i \ge 0$, $f(s) > 0$ has discontinuous points of the first kind at t_i. Then,

$$u(t) \le \Pi_{0<t_i<t}(1+\beta_i)n(t)exp\Big\{\int_0^t f(s)ds\}, \quad t\ge 0.$$

Proof. Let $y(t) = \frac{u(t)}{n(t)}$. Since $n(t)$ is a positive, monotonic, non-decreasing function, we have for all $0 \le s \le t$

$$\frac{u(s)}{n(t)} \le \frac{u(s)}{n(s)} = y(s).$$

Then, the nonnegative piecewise continuous function $y(t)$ satisfies

$$y(t) \le 1 + \int_0^t f(s)y(s)ds + \sum_{0<t_i<t} \beta_i y(t_i), \quad t \ge 0.$$

By the Lemma 1.7.1 of [36] we have

$$y(t) \le \Pi_{0<t_i<t}(1+\beta_i)\exp\Big\{\int_0^t f(s)ds\}, \quad t \ge 0.$$

So,

$$u(t) \le \Pi_{0<t_i<t}(1+\beta_i)n(t)\exp\Big\{\int_0^t f(s)ds\}, \quad t \ge 0.$$

The proof is complete. □

Remark 5.1. By view of the Lemma 2.5 of [23], we can immediately conclude that the conclusions of the Lemma 5.1 still can be established when $u(t)$ is only integrable on $[0, +\infty)$.

Lemma 5.2. *Suppose $D(t)$ is a β-stable subordinator and E_t is the associated inverse stable subordinator. Let $u, f : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the $\mathcal{G}_t$-measurable functions which are integrable with respect to E_t. Let $n(t)$ be a positive, monotonic, non-decreasing function. Then, the inequality*

$$u(t) \le n(t) + \int_0^t f(s)u(s)dE_s + \sum_{0<t_i<t} \beta_i u(t_i), \quad t \ge 0 \tag{5.3}$$

implies almost surely

$$u(t) \le \Pi_{0<t_i<t}(1+\beta_i)n(t)\,exp\Big\{\int_0^t f(s)dE_s\Big\}, \quad t \ge 0.$$

Proof. Let

$$y(t) := n(t) + \int_0^t f(s)u(s)dE_s + \sum_{0<t_i<t} \beta_i u(t_i), \quad t \ge 0. \tag{5.4}$$

Since $f(s)$ and $u(s)$ are positive and $n(t)$ is non-decreasing, the function $y(t)$ defined in equation (5.4) is nondecreasing. Moreover, from equations (5.3) and (5.4),

$$u(t) \le y(t), \qquad t \ge 0,$$

which implies that

$$y(t) \le n(t) + \int_0^t f(s)y(s)dE_s + \sum_{0<t_i<t} \beta_i y(t_i), \quad t \ge 0.$$

Applying the Lemma 3.1 yields

$$y(t) \le n(t) + \int_0^{E_t} f(D(s-))y(D(s-))ds + \sum_{0<t_i<t} \beta_i y(t_i), \quad t \ge 0. \tag{5.5}$$

Let $\tau \in [0, \infty)$, then it holds from (5.5) and (3.4) that

$$\begin{aligned} y(D(\tau-)) \le & n(D(\tau-)) + \int_0^{E_{D(\tau-)}} f(D(s-))y(D(s-))ds + \sum_{0<t_i<D(\tau-)} \beta_i y(t_i) \\ = & n(D(\tau-)) + \int_0^{\tau} f(D(s-))y(D(s-))ds + \sum_{0<t_i<D(\tau-)} \beta_i y(t_i). \end{aligned}$$

Apply the Lemma 5.1 and the Remark 5.1 path by path to yield

$$u(D(\tau-)) \le y(D(\tau-)) \le \Pi_{0<t_i<D(\tau-)}(1+\beta_i)n(D(\tau-))\exp\Big\{\int_0^{\tau} f(D(s-))ds\Big\}.$$

For any $t \ge 0$, we have from the proof of the Lemma 3.2

$$u(t) \le y(t) \le \Pi_{0<t_i<t}(1+\beta_i)n(t)\exp\Big\{\int_0^{E_t} f(D(s-))ds\Big\} = \Pi_{0<t_i<t}(1+\beta_i)n(t)\exp\Big\{\int_0^{t} f(s)dE_s\Big\}.$$

The proof is complete. □

Let $PC(J; \mathbb{R}^d) = \varphi : J \to \mathbb{R}^d$ be continuous for all but at most a finite number of points $t \in J$ and at these points $t \in J$; $\varphi(t+)$ and $\varphi(t-)$ exist; $\varphi(t+) = \varphi(t)$, where $J \subset \mathbb{R}$ is a bounded interval, $\varphi(t+)$ and $\varphi(t-)$ denote the right-hand and left-hand limits of the function $\varphi(t)$, respectively.

In the section, we make the following assumption on the impulsive item.

(H3) There are some constants $\beta_k > 0$ such that for any $x, y \in \mathbb{R}^d$ and $M := \sum_{\beta_k} < \infty$

$$|I_k(x) - I_k(y)| \le \beta_k |x - y|, \quad k = 1, 2, \cdots.$$

Theorem 5.1. *Under the assumptions (H1)-(H3), for any initial data $x_0 = x \in \mathbb{R}^d$, (5.1) admits a unique solution.*

Proof. Let $\mu = \mathcal{L}_X \in P_2(\mathbb{R}^d)$ and $\nu = \mathcal{L}_Y \in P_2(\mathbb{R}^d)$ be fixed, which are the laws of $\mathbb{R}^d$-valued random variable X and Y, and consider the following impulsive McKean-Vlasov stochastic differential equations driven by time-changed Brownian motion B_{E_t}

$$\begin{cases} dx_t = f(t, E_t, x_t, \mu)dE_t + g(t, E_t, x_t, \mu)dB_{E_t}, & t \geq 0,\ t \neq t_k, \\ \triangle x_{t_k} = x_{t_k^+} - x_{t_k^-} = I_k(x_{t_k}), & t = t_k,\ k = 1, 2, \cdots, \\ x_0 = x \in \mathbb{R}^d. \end{cases} \tag{5.6}$$

For any $T > 0$, by view of Zhang and Yuan [38] and Mao and Yuan [25], we know that for any initial condition $x_0 \in \mathbb{R}^d$, there exists a unique $\mathcal{G}_t = \mathcal{F}_{E_t}$-adapted solution process $x_t(\mu)$ to (5.6) such that $\mathbb{E}[\sup_{0\leq t\leq T} |x_t(\mu)|^2] < +\infty$ under the assumptions (H1)-(H3). Now, let us consider the mapping $\Psi : PC([0, T]; P_2(\mathbb{R}^d)) \to PC([0, T]; P_2(\mathbb{R}^d))$:

$$\mu \to \Psi(\mu) = \mathcal{L}_{x_t(\mu)}, \quad \text{the distribution of } x_t(\mu).$$

Notice that $\mathbb{E}[\sup_{0\leq t\leq T} |x_t(\mu)|^2] < +\infty$, and we deduce Ψ is well defined.

To prove the existence and uniqueness of (5.1), it is sufficient to prove that the mapping Ψ has a unique fixed point. For $n \in \mathbb{N}_0$ and $\mu, \nu \in P_2(\mathbb{R}^d)$, define $x^{(n)} := \Psi^n\mu$ and $y^{(n)} := \Psi^n\nu$. Moreover define $\mu_t^{(n)} := \mathcal{L}_{x_t^{(n)}}$ and $\nu_t^{(n)} := \mathcal{L}_{y_t^{(n)}}$. By the definition of Ψ we have for $n \geq 1$ that $x^{(n)} = \Psi(x^{(n-1)})$ solves

$$\begin{cases} dx_t^{(n)} = f(t, E_t, x_t^{(n)}, \mu_t^{(n-1)})dE_t + g(t, E_t, x_t^{(n)}, \mu_t^{(n-1)})dB_{E_t}, & t \geq 0,\ t \neq t_k, \\ \triangle x_{t_k}^{(n)} = x_{t_k^+}^{(n)} - x_{t_k^-}^{(n)} = I_k(x_{t_k}^{(n)}), & t = t_k,\ k = 1, 2, \cdots, \\ x_0^{(n)} = x \in \mathbb{R}^d, \end{cases} \tag{5.7}$$

and $y^{(n)} = \Psi(y^{(n-1)})$ solves

$$\begin{cases} dy_t^{(n)} = f(t, E_t, y_t^{(n)}, \nu_t^{(n-1)})dE_t + g(t, E_t, y_t^{(n)}, \nu_t^{(n-1)})dB_{E_t}, & t \geq 0,\ t \neq t_k, \\ \triangle y_{t_k}^{(n)} = y_{t_k^+}^{(n)} - y_{t_k^-}^{(n)} = I_k(y_{t_k}^{(n)}), & t = t_k,\ k = 1, 2, \cdots, \\ y_0^{(n)} = x \in \mathbb{R}^d \end{cases} \tag{5.8}$$

By using (5.7) and (5.8) we have for $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}_B\Big[\sup_{0\leq s\leq t} |x_s^{(n)} - y_s^{(n)}|^2\Big] \leq & 3\mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big|\int_0^s (f(r, E_r, x_r^{(n)}, \mu_r^{(n-1)}) - f(r, E_r, y_r^{(n)}, \nu_r^{(n-1)}))dE_r\Big|^2\Big] \\ & + 3\mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big|\int_0^s (g(r, E_r, x_r^{(n)}, \mu_r^{(n-1)}) - g(r, E_r, y_r^{(n)}, \nu_r^{(n-1)}))dB_{E_r}\Big|^2\Big] \\ & + 3\mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big|\sum_{0<t_k<s} I_k(x_{t_k}^{(n)}) - \sum_{0<t_k<s} I_k(y_{t_k}^{(n)})\Big|^2\Big] \\ =: & 3(I_1 + I_2 + I_3). \end{aligned} \tag{5.9}$$

Since B_t is independent of E_t, by applying the assumption (H1), (2.1) and the Hölder inequality we can get

$$\begin{aligned} I_1 \leq & 2E_T L^2 \mathbb{E}_B\Big[\sup_{0\leq s\leq t}\int_0^s (|x_r^{(n)} - y_r^{(n)}|^2 + W_2(\mu_r^{(n-1)}, \nu_r^{(n-1)})^2)dE_r\Big] \\ \leq & 2E_T L^2 \int_0^t \mathbb{E}_B[\sup_{0\leq r\leq s} |x_r^{(n)} - y_r^{(n)}|^2]dE_s + 2E_T L^2 \int_0^t \mathbb{E}_B[\sup_{0\leq r\leq s} |x_r^{(n-1)} - y_r^{(n-1)}|^2]dE_s. \end{aligned} \tag{5.10}$$

By assumption (H1), (2.1) and the Burkholder-Davis-Gundy inequality (Jin and Kobayashi [17]),

$$\begin{aligned} I_2 \leq & b_2\mathbb{E}_B\Big[\int\limits_0^t \|g(r,E_r,x_r^{(n)},\mu_r^{(n-1)})-g(r,E_r,y_r^{(n)},\nu_r^{(n-1)})\|^2 dE_r\Big] \\ \leq & 2b_2L^2\mathbb{E}_B\Big[\int\limits_0^t (|x_r^{(n)}-y_r^{(n)}|^2+W_2(\mu_r^{(n-1)},\nu_r^{(n-1)})^2)dE_r\Big] \\ \leq & 2b_2L^2\int\limits_0^t \mathbb{E}_B[\sup_{0\leq r\leq s}|x_r^{(n)}-y_r^{(n)}|^2]dE_s+2b_2L^2\int\limits_0^t \mathbb{E}_B[\sup_{0\leq r\leq s}|x_r^{(n-1)}-y_r^{(n-1)}|^2]dE_s, \end{aligned} \tag{5.11}$$

where the positive constant b_2 comes from [17].

For I_3, by the assumption (H3) and the Hölder inequality we have

$$\begin{aligned} I_3 \leq & \mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big|\sum_{0<t_k<s}\beta_k|x_{t_k}^{(n)}-y_{t_k}^{(n)}|\Big|^2\Big] \\ \leq & \mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big(\sum_{0<t_k<s}\beta_k\Big)\cdot\Big(\sum_{0<t_k<s}\beta_k|x_{t_k}^{(n)}-y_{t_k}^{(n)}|^2\Big)\Big] \\ \leq & M\mathbb{E}_B\Big[\sup_{0\leq s\leq t}\Big(\sum_{0<t_k<s}\beta_k|x_{t_k}^{(n)}-y_{t_k}^{(n)}|^2\Big)\Big] \\ = & M\mathbb{E}_B\Big[\Big(\sum_{0<t_k<t}\beta_k|x_{t_k}^{(n)}-y_{t_k}^{(n)}|^2\Big)\Big] \\ \leq & M\sum_{0<t_k<t}\beta_k\mathbb{E}_B\Big[\sup_{0<s<t_k}|x_s^{(n)}-y_s^{(n)}|^2\Big]. \end{aligned} \tag{5.12}$$

Then, combining (5.10)-(5.12) with (5.9) we have

$$\begin{aligned} \mathbb{E}_B\Big[\sup_{0\leq s\leq t}|x_s^{(n)}-y_s^{(n)}|^2\Big] \leq & 6(E_TL^2+b_2L^2)\int\limits_0^t \mathbb{E}_B[\sup_{0\leq r\leq s}|x_r^{(n)}-y_r^{(n)}|^2]dE_s \\ & + 6(E_TL^2+b_2L^2)\int\limits_0^t \mathbb{E}_B[\sup_{0\leq r\leq s}|x_r^{(n-1)}-y_r^{(n-1)}|^2]dE_s \\ & + 3M\sum_{0<t_k<t}\beta_k\mathbb{E}_B\Big[\sup_{0<s<t_k}|x_s^{(n)}-y_s^{(n)}|^2\Big]. \end{aligned}$$

Thus, from the Lemma 5.2, one has

$$\mathbb{E}_B\Big[\sup_{0\leq s\leq t}|x_s^{(n)}-y_s^{(n)}|^2\Big] \leq 6(E_TL^2+b_2L^2)\Pi(1+3M\beta_i)e^{6(E_TL^2+b_2L^2)E_T}\int\limits_0^t \mathbb{E}_B[\sup_{0\leq r\leq s}|x_r^{(n-1)}-y_r^{(n-1)}|^2]dE_s.$$

Defining

$$\mathcal{J}^n(t)=\mathbb{E}_B\Big[\sup_{0\leq s\leq t}|x_s^{(n)}-y_s^{(n)}|^2\Big],$$

we have for any $n\geq 1$ and $t\in[0,T]$

$$\mathcal{J}^n(t)\leq \kappa(E_T)\int\limits_0^t \mathcal{J}^{n-1}(s)dE_s$$

with $\kappa(E_T) = 18M(E_T L^2 + b_2 L^2)\Pi(1+\beta_i)e^{6(E_T L^2 + b_2 L^2)E_T}$. Consequently, by iteration we obtain

$$\mathcal{J}^n(t) \leq \frac{\kappa^n(E_T)(E_T)^n}{n!}\mathbb{E}_B|X-Y|^2. \tag{5.13}$$

Taking $\mathbb{E}_D$ on both sides of (5.13), we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}|x_t^{(n)} - y_t^{(n)}|^2\Big] \leq \frac{\mathbb{E}[\kappa^n(T,E_T)(E_T)^n]}{n!}\mathbb{E}|X-Y|^2.$$

For large n, Ψ^n is a strict contraction which implies that Ψ admits a unique fixed point in the complete metric space $PC([0,T];P_2(\mathbb{R}^d))$. Since T is arbitrary, we can immediately obtain the existence and uniqueness of solution for (5.1). The proof is complete. □

For the purposes of stability, we also assume that

$$I_k(0) \equiv 0 \quad \text{for any } k = 1, 2, \cdots.$$

As the equation (1.1), we also can obtain some stability properties of solution to (5.1) with respect to initial data and coefficients by using the Lemma 5.2.

Let $x_t^0 = x \in \mathbb{R}^d$ for all $t \in [0,T]$ and define x^{n+1} as the solution of the following SDE:

$$\begin{cases} dx_t^{n+1} = f(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dE_t + g(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dB_{E_t}, & t\geq 0,\ t\neq t_k, \\ \triangle x_{t_k}^{n+1} = x_{t_k^+}^{n+1} - x_{t_k^-}^{n+1} = I_k(x_{t_k}^{n+1}), & t = t_k,\ k=1,2,\cdots, \\ x_0^{n+1} = x \in \mathbb{R}^d. \end{cases} \tag{5.14}$$

Theorem 5.2. *Under the assumption (H1)-(H3), the sequence x^n converges to the unique solution x of (5.1), i.e.,*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|x_t^n - x_t|^2\Big] \to 0, \quad \text{as } n\to\infty.$$

Next, we consider the convergence of (5.1) with respect to small perturbations of the initial condition. We denote by x_t^x the unique solution of (5.1) with initial data x.

Theorem 5.3. *Under the assumption (H1)-(H3), the mapping*

$$\Phi : \mathbb{R}^d \to L^2(\Omega; PC([0,T]); \mathbb{R}^d)$$

defined by $\Phi(x)_t = x_t^x$ is continuous.

Now, we consider the convergence of (5.1) with respect to small perturbations of the coefficients f and g. To this end, we consider the sequences of functions f_n and g_n and the corresponding impulsive McKean-Vlasov stochastic differential equations driven by time-changed Brownian motion B_{E_t}

$$\begin{cases} dx_t^n = f_n(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dE_t + g_n(t,E_t,x_t^n,\mathcal{L}_{x_t^n})dB_{E_t}, & t\geq 0,\ t\neq t_k, \\ \triangle x_{t_k}^n = x_{t_k^+}^n - x_{t_k^-}^n = I_k(x_{t_k}^n), & t=t_k,\ k=1,2,\cdots, \\ x_0^n = x\in\mathbb{R}^d. \end{cases} \tag{5.15}$$

Theorem 5.4. *Assume that the coefficients f, g, f_n and g_n satisfy the assumptions (H1) and (H2). Furthermore, assume that the assumption (H3) holds, and for each $T > 0$ and each a compact set K*

$$\lim_{n\to\infty} \sup_{0\le t\le T} \sup_{x\in K} \sup_{\mu\in P_2(\mathbb{R}^d)} |f_n(t, E_t, x, \mu) - f(t, E_t, x, \mu)| = 0 \tag{5.16}$$

and

$$\lim_{n\to\infty} \sup_{0\le t\le T} \sup_{x\in K} \sup_{\mu\in P_2(\mathbb{R}^d)} \|g_n(t, E_t, x, \mu) - g(t, E_t, x, \mu)\| = 0, \tag{5.17}$$

then

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{0\le t\le T} |x_t^n - x_t|^2\Big] = 0,$$

where x_t^n and x_t are solutions of (5.15) and (5.1), respectively.

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