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Research paper

Strong approximation of non-autonomous time-changed McKean–Vlasov stochastic differential equations

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A B S T R A C T

In this paper, we investigate the time-changed McKean–Vlasov stochastic differential equations (MV-SDEs) via interacting particle systems, where the MV-SDEs contain two drift terms, one driven by the random time change E_t and the other driven by a regular, non-random time variable t. Strong convergence and convergence rate in the finite time of the Euler–Maruyama (EM) method on the particle system is discussed. Numerical example illustrates that "particle corruption" will not occur on the whole particle system and show the consistency with convergence result.

1. Introduction

McKean–Vlasov stochastic differential equations (MV-SDEs), which are also referred to as distribution-dependent or mean-field SDEs, describe a limiting behavior of individual particles and have been widely applied in stochastic control [1,2]; in the networked system [3–5] and mathematical finance [6] etc. In contrast to the standard SDEs, the coefficients of MV-SDEs depend not only on the state of the solution process but also on its probability distribution. So far, MV-SDEs have been investigated considerably on the existence and uniqueness of solutions [7–9], stability [10–12], Feynman–Kac Formulae [13,14], numerical methods [15–18] and so on.

In this paper, we consider a class of time-changed MV-SDEs of the form

$$dX_t = f(E_t)dt + h(E_t, X_t, \mu_t^X)dE_t + g(E_t, X_t, \mu_t^X)dB_{E_t}, \quad t \in [0, T]. \tag{1.1}$$

Here the coefficients f, h and g satisfy some regular conditions (to be specified in Section 2), $\mu_t := \mathscr{L}_{X_t}$ is the law of X at time t, B_t is a standard Brownian motion and E_t is the inverse of a subordinator. The composition process $(B_{E_t})_{t\in[0,T]}$ is called a time-changed Brownian motion which is understood as a subdiffusion (see [19,20]). The rigorous mathematical definitions are postponed to Section 2.

Since Kobayashi [21] investigated stochastic integrals with regard to the time-changed semimartingale, obtained the time-changed Itô's formula and showed deep connections (duality principle) of

$$dY_t = h(E_t, Y_t)dE_t + g(E_t, Y_t)dB_{E_t} \tag{1.2}$$

and

$$dZ_t = h(t, Z_t)dt + g(t, Z_t)dB_t, \tag{1.3}$$

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the time-changed semimartingale has been growing rapidly due to model anomalous diffusions arising in physics, finance, hydrology, cell biology among others (see [20] for more details). Following [21], Wu [22] studied various stability behaviors of SDEs driven by time-changed Brownian motion, and provided necessary conditions for stochastic stability, stochastically asymptotic stability and globally stochastically asymptotic stability in [23]. Nane and Ni considered the SDEs driven by time-changed Lévy noise, they investigated the stability in probability and moment sense in [24] and gave conditions for the solutions to be path stable and exponentially path stable in [25]. Zhu [26] studied the exponential stability and almost sure exponential stability of the solution to time-changed SDEs. Li et al. [27] considered the global attracting sets and exponential stability of time-changed stochastic functional differential equations. Liu [28] investigated the moment stability of time-changed SDEs with the super-linear coefficients.

In contrast to the fruitful results on the underlying equations, little has been concerned about numerical methods. However, it is necessary to note that the numerical methods for the time-changed SDEs are also crucial in the applications since the true solution of the SDEs is rarely obtained.

As far as we know, Jum [29] is the first work to discrete the Eq. (1.2) directly. Under the global Lipschitz condition, the convergences in both strong and weak senses were proved. After that, the semi-implicit and truncated EM method were investigated for the time-changed SDEs with coefficients obeying super-linear growth in Deng [30] and Liu [31] respectively, and strong convergence and convergence rate were obtained in both papers. All the three papers above are based on the discussion on the duality principle proposed in Kobayashi [21]. However, with a slight change in (1.2), the duality principle will no longer work. In [32], Jin and Kobayashi studied the time-changed SDEs with time-space-dependent coefficients of the form

$$dY_t = \mu(t, Y_t)dE_t + \sigma(t, Y_t)dB_{E_t}. \tag{1.4}$$

The classical dual Itô SDEs do not exist, so they proposed the EM method to approximate (1.4) and obtained the strong convergence and convergence rate. Our paper is inspired by [33], the authors studied the time-changed SDEs of the form

$$dY_t = H(E_t, Y_t)dt + F(E_t, Y_t)dE_t + G(E_t, Y_t)dB_{E_t}. \tag{1.5}$$

The convergence rate was obtained when $H(E_t, Y_t) = H(E_t)$, and the convergence rate for Milstein and Itô-Taylor-type were also investigated when $H = 0$.

The main contributions of this paper are as follows:

(1) We prove the particle system converges (propagation of chaos) with the corresponding rate without coercivity. This result is crucial in showing the convergence of the numerical scheme to the particle system rather than to the original time-changed McKean–Vlasov SDE.
(2) Different from [33], the one-sided Lipschitz condition is used in the drift coefficient $h(t, x, \mu)$ instead of the global Lipschitz condition.
(3) The strong convergence of the EM method in the finite time is proved and the classical 1/2 rate of convergence in the step size is obtained. Combining this with the propagation of chaos result gives an overall convergence rate for the EM scheme.

The rest of the paper is organized as follows. In Section 2, we introduce Wasserstein metric and some necessary preliminaries for the time-changed MV-SDEs. Propagation of chaos and strong convergence results are proved in Section 3. In Section 4, we present a numerical example to show the consistency with strong convergence result in Section 3.

2. Preliminaries

Throughout this paper, let $(\Omega_B, \mathcal{F}^B, \mathbb{P}_B)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{E}_B$ denote the expectation with respect to $\mathbb{P}_B$. Let $\{D(t), t \geq 0\}$ be a right continuous left limit increasing Lévy process that is called subordinator starting from 0 defined on a complete probability space $(\Omega_D, \mathcal{F}^D, \mathbb{P}_D)$ with a filtration $\{\mathcal{F}_t^D\}_{t\geq 0}$ satisfying the usual conditions. Let $\mathbb{E}_D$ denote the expectation under the probability measure $\mathbb{P}_D$. Denote the inner product of x, y in $\mathbb{R}^n$ by $\langle x, y\rangle$.

For a subordinator $D(t)$, which is a non-decreasing Lévy process on $[0, \infty)$ with Laplace transform

$$\mathbb{E}[e^{-sD_t}] = e^{-t\psi(s)}, \quad s > 0, t \geq 0,$$

where $\psi(s) = \int_0^\infty (1 - e^{-sy})v(dy)$ with condition $\int_0^\infty (y \wedge 1)v(dy) < \infty$. We focus on the case when Lévy measure v is infinite (i.e. $v(0, \infty) = \infty$).

Let $E = (E_t)_{t\geq 0}$ be the inverse of D defined by

$$E_t := \inf\{s \geq 0; D_s > t\}, \quad t \geq 0.$$

We call E an inverse subordinator. It is clear that E has continuous, non-decreasing paths starting at 0. Besides, the relation between D and E implies $\{E_t > x\} = \{D_x < t\}$ for all $t, x \geq 0$ showed in Fig. 1. If the subordinator D_t is stable with index $\beta \in (0, 1)$, i.e. $\psi(s) = s^\beta$, then E is called an inverse β-stable subordinator.

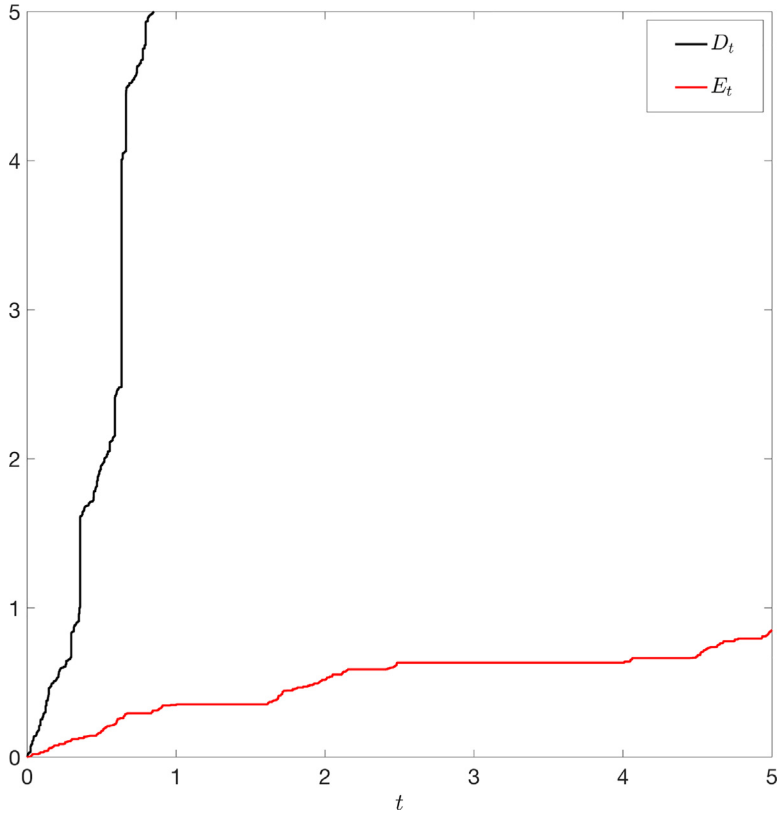

Fig. 1. Sample paths of 0.9-stable subordinator D and the corresponding inverse subordinator E.

Let B_t be a standard Brownian motion independent of E_t, define the following filtration as

$$\mathcal{F}_t = \bigcap_{s>t}\{\sigma(B_r : 0 \le r \le s) \vee \sigma(E_r : r \ge 0)\},$$

where $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. It concludes that the time-changed Brownian motion B_{E_t} is a square integrable martingale with respect to the filtration $\{\mathcal{F}_{E_t}\}_{t\ge 0}$. Define the product probability space by

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega_B \times \Omega_D, \mathcal{F}^B \times \mathcal{F}^D, \mathbb{P}_B \times \mathbb{P}_D).$$

Let $\mathbb{E}$ denote the expectation under the probability measure $\mathbb{P}$. It is clear that $\mathbb{E}(\cdot) = \mathbb{E}_D\mathbb{E}_B(\cdot) = \mathbb{E}_B\mathbb{E}_D(\cdot)$.

In the rest of the paper, we always assume B and E are independent. Note that even though $B \circ D$ is a Lévy process, $B \circ E$ is not a Markov process (see [19,34]).

2.1. Wasserstein metric

Given $\mathcal{P}(\mathbb{R}^d)$ to denote the family of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field over $\mathbb{R}^d$. For $p > 0$, if $\mu \in \mathcal{P}(\mathbb{R}^d)$ has a finite pth moment, we then formulate $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, that is

$$\mathcal{P}_2(\mathbb{R}^d) := \left\{\mu \in \mathcal{P}(\mathbb{R}^d) | \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty\right\}.$$

For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the W_2-Wasserstein distance between μ and ν is defined by

$$W_2(\mu, \nu) := \inf_{\pi\in\mathcal{C}(\mu,\nu)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(dx, dy)\right)^{\frac{1}{2}},$$

where $\mathcal{C}(\mu, \nu)$ stands for the set of all couplings of μ and ν. Let δ_x be Dirac's delta measure centered at the point $x \in \mathbb{R}^d$. Furthermore, for $p \ge 2$, $S^p([0, T])$ refers to the space of $\mathbb{R}$-valued continuous, $\mathcal{F}_t$-adapted process, defined on the interval $[0, T]$, with finite pth moment, i.e. processes $(X_t)_{t\in[0,T]}$ satisfying $\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^p\right] < \infty$.

As W_2 is a metric, for $\mu_1, \mu_2, \mu_3 \in \mathcal{P}_2(\mathbb{R}^d)$ we always have (see [35,36])

$$W_2(\mu_1, \mu_3) \le W_2(\mu_1, \mu_2) + W_2(\mu_2, \mu_3). \tag{2.1}$$

As in [37], the empirical measure constructed form i.i.d. samples of some processes X by $\mu_s^{X,N} := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j}$. Therefore, the Wasserstein metric between two such empirical measures $\mu_s^{X,N}$ and $\mu_s^{Y,N}$ is

$$W_2(\mu_s^{X,N}, \mu_s^{Y,N}) \le \left(\frac{1}{N}\sum_{j=1}^N \left|X_s^j - Y_s^j\right|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

2.2. Time-changed McKean–Vlasov stochastic differential equations

Let E be an inverse β-stable subordinator. Let B be a d-dimensional standard Brownian motion, we consider the following time-changed MV-SDE

$$dX_t = f(E_t)dt + h(E_t, X_t, \mu_t^X)dE_t + g(E_t, X_t, \mu_t^X)dB_{E_t}, \quad t \in [0, T], \tag{2.3}$$

where $X_0 \in L_0^p(\mathbb{R}^d)$ is a non-random constant, $T > 0$ is a fixed time horizon, and $f : [0, \infty) \to [0, \infty)$, $h : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, and $g : [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$, $\mu_t := \mathscr{L}_{X_t}$ denotes the law of the process X at time t.

We make some assumptions on the coefficients throughout.

(**H1**) There exist a constant $L_h > 0$ such that for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^d$ and $\forall \mu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\langle x - y, h(t, x, \mu) - h(t, y, \mu)\rangle \le L_h|x - y|^2. \tag{2.4}$$

(**H2**) There exists a constant $L_g > 0$ such that for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^d$ and $\forall \mu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$|g(t, x, \mu) - g(t, y, \mu)| \le L_g|x - y|. \tag{2.5}$$

(**H3**) There exists a constant $L_1 > 0$ such that for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^d$ and $\forall \mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$|h(t, x, \mu) - h(t, x, \upsilon)| \vee |g(t, x, \mu) - g(t, x, \upsilon)| \le L_1 W_2(\mu, \nu). \tag{2.6}$$

(**H4**) There exists a constant $L_2 > 0$ such that for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^d$ and $\forall \mu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$|h(t, x, \mu)| \vee |g(t, x, \mu)| \le L_2 (1 + |x| + W_2(\mu, \delta_0)), \tag{2.7}$$

where δ_0 denotes the Dirac measure at 0.

(**H5**) If $x \in \mathbb{R}^d$ is a right continuous with left limit and $\mathcal{G}_t$-adapted process, then

$$h(t, x, \mu), g(t, x, \mu) \in \mathcal{L}(\mathcal{G}_t), \tag{2.8}$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of left continuous processes with right limit.

For the purposes of stability, we further assume that for $t \ge 0$,

$$f(0) \equiv 0, \quad h(t, 0, \delta_0) \equiv 0, \quad g(t, 0, \delta_0) \equiv 0. \tag{2.9}$$

Lemma 2.1. Let (**H1**)-(**H5**) hold. MV-SDE (2.3) has a unique strong solution which is a continuous $\mathcal{F}_{E_t}$-semimartingale.

Proof. The proof of existence and uniqueness is almost similar to [16,21,30], so it is omitted here. □

Compared to standard SDEs, time-changed MV-SDEs require to approximate the law μ at each time step. If the law μ^X in (2.3) is known in advance, then the time-changed MV-SDEs reduce to the classical time-changed SDEs with added time-dependency. Typically this is not the case, therefore we exploit stochastic interacting particle systems to approximate it. Before introducing the interacting particle systems, we need another assumption (**H6**) here.

(**H6**) There exists constants $L_3 > 0$ and $\theta \in (0, 1]$ such that for all $t, t' \in [0, \infty]$ and $\forall \mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\left|f(t) - f(t')\right| \vee \left|h(t, x, \mu) - h(t', x, \mu)\right| \vee \left|g(t, x, \mu) - g(t', x, \mu)\right| \le L_3\left|t - t'\right|^\theta (1 + |x| + W_2(\mu, \delta_0)). \tag{2.10}$$

We approximate (1.1) by using an N-dimensional system of interacting particles. Let $i = 1, 2, \ldots, N$ and consider N particles $X^{i,N}$ satisfying

$$dX_t^{i,N} = f(E_t)dt + h(E_t, X_t^{i,N}, \mu_t^{X,N})dE_t + g(E_t, X_t^{i,N}, \mu_t^{X,N})dB_{E_t}^i, \quad t \ge 0, \tag{2.11}$$

where $X_0^{i,N} = X_0^i$, $\mu_t^{X,N}$ means the empirical distribution corresponding to $X_t^{1,N}, X_t^{2,N}, \ldots, X_t^{N,N}$, that is

$$\mu_t^{X,N}(dx) := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}(dx),$$

where $\delta_{X_t^{j,N}}$ is the Dirac measure at point $X_t^{j,N}$, $\left(X_0^{i,N}, B^i_{E_t}\right)$ are the independent copies of $\left(X_0, B_{E_t}\right)$.

Key to the convergence as $N \to \infty$ is the concept of propagation of chaos. We further consider the stochastic non-interacting particle systems

$$dX_t^i = f(E_t)dt + h(E_t, X_t^i, \mu_t^{X^i})dE_t + g(E_t, X_t^i, \mu_t^{X^i})dB^i_{E_t}, \quad t \geq 0, \tag{2.12}$$

since X^i are independent for all $i = 1, 2, \ldots, N$, we have $\mu_t^{X_i} = \mu_t^X$. Then, pathwise propagation of chaos refers to the property (see [35,36])

$$\lim_{N\to\infty} \max_{i\in\{1,\ldots,N\}} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{i,N} - X_t^i\right|^2\right] = 0.$$

In general, we cannot simulate (2.11) directly, therefore turn to a numerical scheme such as Euler–Maruyama scheme. Let us first approximate the inverse subordinator E in a time interval $[0, T]$. Fix an equidistant step size $\delta \in (0, 1)$. We simulate a sample path of the subordinator D by $D_{i\delta} := D_{(i-1)\delta} + Z_i$, $i = 1, 2, 3, \ldots$, with $D_0 = 0$, where $\{Z_i\}_{i\in\mathbb{N}}$ is an i.i.d. sequence distributed as $Z_i \overset{d}{=} D_\delta$. The procedure is stopped when there exists the integer M satisfying

$$T \in [D_{M\delta}, D_{(M+1)\delta}).$$

Next, the approximate E_t^δ to E_t is generated by

$$E_t^\delta := (\min\{m \in \mathbb{N}; D_{m\delta} > t\} - 1)\,\delta.$$

It is easy to see that for $m = 0, 1, 2, \ldots, M$,

$$E_t^\delta = m\delta, \quad \text{when } \; t \in [D_{m\delta}, D_{(m+1)\delta}).$$

In particular, $E_T^\delta = M\delta$. The process E^δ can be used efficiently to approximate E_t (see [29,38]) for all $t \in [0, T]$, a.s.

$$E_t - \delta \leq E_t^\delta \leq E_t. \tag{2.13}$$

Now, for $m = 0, 1, 2, \ldots, M$, let

$$\tau_m = D_{m\delta}. \tag{2.14}$$

We follow the way in [32,33] to approximate the discrete and continuous time process. Because of the independence of B and D, we can approximate the Brownian motion B over the time steps $\{0, \delta, 2\delta, \ldots, M\delta\}$. Then the discrete time process $(X^\delta_{\tau_m})_{m\in\{0,1,2,\ldots,M\}}$ can be defined as

$$X^{i,N,\delta}_{\tau_{m+1}} := X^{i,N,\delta}_{\tau_m} + f(E^\delta_{\tau_m})(\tau_{m+1} - \tau_m) + h(E^\delta_{\tau_m}, X^{i,N,\delta}_{\tau_m}, \mu^{X,N,\delta}_{\tau_m})\delta + g(E^\delta_{\tau_m}, X^{i,N,\delta}_{\tau_m}, \mu^{X,N,\delta}_{\tau_m})(B^i_{E^\delta_{\tau_{m+1}}} - B^i_{E^\delta_{\tau_m}}), \tag{2.15}$$

where $X_0^{i,N,\delta} = X_0^{i,N}$. We adopt the continuous interpolation to define the continuous time process $\{X_t^{i,N,\delta}\}_{t\in[0,T]}$, when $t \in [\tau_m, \tau_{m+1})$,

$$X_t^{i,N,\delta} := X^{i,N,\delta}_{\tau_m} + \int_{\tau_m}^t f(E^\delta_{\tau_m})ds + \int_{\tau_m}^t h(E^\delta_{\tau_m}, X^{i,N,\delta}_{\tau_m}, \mu^{X,N,\delta}_{\tau_m})dE_s + \int_{\tau_m}^t g(E^\delta_{\tau_m}, X^{i,N,\delta}_{\tau_m}, \mu^{X,N,\delta}_{\tau_m})dB^i_{E_s}. \tag{2.16}$$

Let

$$n_t = \max\{m \in \mathbb{N}\cup\{0\}; \tau_m \leq t\} \quad \text{for} \quad t \geq 0.$$

Then it is clear that $\tau_{n_t} \leq t < \tau_{n_t+1}$ for any $t > 0$. Using (2.15) and the identity $X_t^{i,N,\delta} - X_0^{i,N,\delta} = \sum_{k=0}^{n_s-1}(X^{i,N,\delta}_{\tau_{k+1}} - X^{i,N,\delta}_{\tau_k}) + (X_t^{i,N,\delta} - X^{i,N,\delta}_{\tau_{n_s}})$, we can express $X_t^{i,N,\delta} - X_0^{i,N,\delta}$ as

$$\sum_{k=0}^{n_s-1}\left[f(E_{\tau_k})(\tau_{k+1} - \tau_k) + h(E_{\tau_k}, X^{i,N,\delta}_{\tau_k}, \mu^{X,N,\delta}_{\tau_k})\delta + g(E_{\tau_k}, X^{i,N,\delta}_{\tau_k}, \mu^{X,N,\delta}_{\tau_k})(B^i_{(k+1)\delta} - B^i_{k\delta})\right] + (X_t^{i,N,\delta} - X^{i,N,\delta}_{\tau_{n_s}}),$$

where we use $k\delta = E_{D_{k\delta}} = E_{\tau_k}$. According to (2.16) and the fact that $\tau_k = \tau_{n_r}$ for any $r \in [\tau_k, \tau_{k+1})$, we can rewrite the latter in the convenient form

$$X_t^{i,N,\delta} = X_0^{i,N,\delta} + \int_0^t f(E_{\tau_{n_s}})ds + \int_0^t h(E_{\tau_{n_s}}, X^{i,N,\delta}_{\tau_{n_s}}, \mu^{X,N,\delta}_{\tau_{n_s}})dE_s + \int_0^t g(E_{\tau_{n_s}}, X^{i,N,\delta}_{\tau_{n_s}}, \mu^{X,N,\delta}_{\tau_{n_s}})dB^i_{E_s}, \tag{2.17}$$

where $X_0^{i,N,\delta} = X_0$, $\mu_t^{X,N,\delta}(dx) = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N,\delta}}(dx)$.

3. Main results

Theorem 3.1. Let assumptions **(H1)-(H6)** and (2.9) hold. Suppose further $X_0 \in L^p(\Omega \to R; \mathcal{F}_0, \mathbb{P})$ for some $p > 4$. Then, for any $T > 0$, there exists a constant $C > 0$ which is independent of δ and N such that

$$\sup_{1\le i\le N} \mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N,\delta}\right|^2\right] \le C\begin{cases} N^{-1/2} + \delta^{\min\{1,2\theta\}}, & d<4,\\ N^{-1/2}\log(N) + \delta^{\min\{1,2\theta\}}, & d=4,\\ N^{-2/d} + \delta^{\min\{1,2\theta\}}, & d>4. \end{cases} \tag{3.1}$$

Before proving Theorem 3.1, we need several lemmas and propositions.

Lemma 3.2 ([21]). Let E_t be the $\mathcal{G}_t$-measurable time change which is the general inverse β-stable subordinator $D(t)$. Suppose $\mu(t)$ and $\delta(t)$ are $\mathcal{G}_t$-measurable and integrable. Then, for all $t \ge 0$ with probability one,

$$\int_0^t \mu(s)dE_s + \int_0^t \delta(s)dB_{E_s} = \int_0^{E_t} \mu(D(s-))ds + \int_0^{E_t} \delta(D(s-))dB_s.$$

Lemma 3.3. Suppose $D(t)$ is a β-stable subordinator and E_t is the associated inverse stable subordinator. Let $T > 0$ and $u, f, g : \Omega \times [0, T] \longrightarrow \mathbb{R}_+$ be the $\mathcal{G}_t$-measurable functions which are integrable with respect to E_t. Assume $u_0 \ge 0$ is a constant. Then, the inequality

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t g(s)u(s)dE_s, \quad t \in [0, T] \tag{3.2}$$

implies almost surely

$$u(t) \le u_0\exp\left\{\int_0^t f(s)ds + \int_0^t g(s)dE_s\right\}, \quad t \in [0, T].$$

Proof. Let

$$y(t) := u_0 + \int_0^t f(s)u(s)ds + \int_0^t g(s)u(s)dE_s, \quad t \in [0, T]. \tag{3.3}$$

Since $f(s)$, $g(s)$ and $u(s)$ are positive, the function $y(t)$ defined in Eq. (3.3) is non-decreasing. Moreover, from (3.2) and (3.3),

$$u(t) \le y(t), \quad t \in [0, T],$$

which implies that

$$y(t) \le u_0 + \int_0^t f(s)y(s)ds + \int_0^t g(s)y(s)dE_s, \quad t \in [0, T].$$

Applying Lemma 3.2 yields

$$y(t) \le u_0 + \int_0^t f(s)y(s)ds + \int_0^{E_t} g(D(s-))y(D(s-))ds, \quad t \in [0, T]. \tag{3.4}$$

Actually, for $0 \le t \le E_T$, $D(t-)$ is defined as

$$D(t-) = \inf\{s : s \in [0, T], E_s > t\} \wedge T$$

which means

$$E_{D(t-)} = t, \quad \text{and} \quad D(E_t-) \ge t. \tag{3.5}$$

Let $\tau \in [0, \infty)$ and $\tau \in [0, E_T]$, then it holds from (3.4) and (3.5) that

$$\begin{aligned} y(D(\tau-)) \le& u_0 + \int_0^{D(\tau-)} f(s)y(s)ds + \int_0^{E_{D(\tau-)}} g(D(s-))y(D(s-))ds\\ =& u_0 + \int_0^{D(\tau-)} f(s)y(s)ds + \int_0^{\tau} g(D(s-))y(D(s-))ds. \end{aligned}$$

Applying the retarded Gronwall-like inequality (see [39], Proposition 1) path by path to yield

$$u(D(\tau-)) \le y(D(\tau-)) \le u_0\exp\left\{\int_0^{D(\tau-)} f(s)ds + \int_0^{\tau} g(D(s-))ds\right\}.$$

Let $t = D(\tau-)$. Then, applying (3.5) and following by Lemma 3.2, we have

$$u(t) \le y(t) \le u_0 \exp\left\{\int_0^t f(s)ds + \int_0^{E_t} g(D(s-))ds\right\} = u_0 \exp\left\{\int_0^t f(s)ds + \int_0^t g(s)dE_s\right\}.$$

The proof is complete. □

Proposition 3.4. *Let the assumptions **(H1)-(H3)** hold. Let X^i be the solution to (2.12), and $X^{i,N}$ be the solution to (2.11). Then, for any $T>0$, there exists a constant $C>0$ which is independent of δ and N such that*

$$\sup_{1\le i\le N} \mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N}\right|^2\right] \le C \begin{cases} N^{-1/2}, & d<4, \\ N^{-1/2}\log(N), & d=4, \\ N^{-2/d}, & d>4. \end{cases} \tag{3.6}$$

Proof. Let us fix $1 \le i \le N$, then apply the time-changed Itô's formula (see [21], Theorem 3.3) to the difference

$$\begin{aligned} \left|X_s^i - X_s^{i,N}\right|^2 =& 2\int_0^s \left\langle X_r^i - X_r^{i,N}, h(E_r, X_r^i, \mu_r) - h(E_r, X_r^{i,N}, \mu_r^{X,N})\right\rangle dE_r \\ &+ 2\int_0^s \left\langle X_r^i - X_r^{i,N}, g(E_r, X_r^i, \mu_r) - g(E_r, X_r^{i,N}, \mu_r^{X,N})\right\rangle dB^i_{E_r} \\ &+ \int_0^s \sum_{a=1}^l \left|g_a(E_r, X_r^i, \mu_r) - g_a(E_r, X_r^{i,N}, \mu_r^{X,N})\right|^2 dE_r \\ =&: \sum_{i=1}^3 I_i, \end{aligned} \tag{3.7}$$

where g_a is the ath column of matrix g. For the first term,

$$\begin{aligned} \left\langle X_r^i - X_r^{i,N}, h(E_r, X_r^i, \mu_r) - h(E_r, X_r^{i,N}, \mu_r^{X,N})\right\rangle =& \left\langle X_r^i - X_r^{i,N}, h(E_r, X_r^i, \mu_r) - h(E_r, X_r^{i,N}, \mu_r)\right\rangle \\ &+ \left\langle X_r^i - X_r^{i,N}, h(E_r, X_r^{i,N}, \mu_r) - h(E_r, X_r^{i,N}, \mu_r^{X,N})\right\rangle. \end{aligned}$$

Taking the supremum over s and the expectation $\mathbb{E}_B$ on both sides, by using (**H1**) and (**H3**) we can easily obtain

$$\begin{aligned} \mathbb{E}_B\left[\sup_{0\le s\le t} I_1\right] \le& \ \mathbb{E}_B\left[2\int_0^t \left(L_h\left|X_r^i - X_r^{i,N}\right|^2 + \frac{1}{2}L_1\left|X_r^i - X_r^{i,N}\right|^2 + \frac{1}{2}L_1 W_2(\mu_r, \mu_r^{X,N})^2\right) dE_r\right] \\ \le& \ (2L_h + L_1)\mathbb{E}_B\left[\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] + L_1\mathbb{E}_B\left[\int_0^t W_2(\mu_r, \mu_r^{X,N})^2 dE_r\right]. \end{aligned}$$

For the second term, applying the Burkholder–Davis–Gundy's inequality (BDG's inequality) (see [40], Theorem 1.7.3), elementary inequality, **(H2)** and **(H3)** yields

$$\begin{aligned} &\mathbb{E}_B\left[\sup_{0\le s\le t} I_2\right] \\ \le& \ 8\sqrt{2}\mathbb{E}_B\left[\left(\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 \left|g(E_r, X_r^i, \mu_r) - g(E_r, X_r^{i,N}, \mu_r^{X,N})\right|^2 dE_r\right)^{\frac{1}{2}}\right] \\ \le& \ 8\sqrt{2}\mathbb{E}_B\left[\left(\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2 \int_0^t \sum_{a=1}^l \left|g_a(E_r, X_r^i, \mu_r) - g_a(E_r, X_r^{i,N}, \mu_r^{X,N})\right|^2 dE_r\right)^{\frac{1}{2}}\right] \\ \le& \ \frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right] + 64\mathbb{E}_B\left[\int_0^t \sum_{a=1}^l \left|g_a(E_r, X_r^i, \mu_r) - g_a(E_r, X_r^{i,N}, \mu_r^{X,N})\right|^2 dE_r\right] \\ \le& \ \frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right] + 128L_g^2\mathbb{E}_B\left[\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] + 128L_1^2\mathbb{E}_B\left[\int_0^t W_2(\mu_r, \mu_r^{X,N})^2 dE_r\right]. \end{aligned}$$

For the last term, it follows from **(H2)** and **(H3)** that

$$\mathbb{E}_B\left[\sup_{0\le s\le t} I_3\right] \le 2L_g^2\mathbb{E}_B\left[\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] + 2L_1^2\mathbb{E}_B\left[\int_0^t W_2(\mu_r, \mu_r^{X,N})^2 dE_r\right].$$

Substituting the above into (3.7), yields

$$
\begin{aligned}
\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right] \le\ & (2L_h + L_1 + 130L_g^2)\mathbb{E}_B\left[\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] \\
&+ 131L_1^2\mathbb{E}_B\left[\int_0^t W_2(\mu_r, \mu_r^{X,N})^2 dE_r\right] \\
&+ \frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right].
\end{aligned}
\tag{3.8}
$$

From (2.1) and (2.2), we have

$$
W_2(\mu_r, \mu_r^{X,N}) \le W_2(\mu_r, \mu_r^N) + W_2(\mu_r^N, \mu_r^{X,N}). \tag{3.9}
$$

Since μ_r and $\mu_r^{X,N}$ are empirical measures a standard result for the Wasserstein metric is

$$
W_2(\mu_r^N, \mu_r^{X,N}) \le \left(\frac{1}{N}\sum_{j=1}^N \left|X_r^j - X_r^{j,N}\right|^2\right)^{\frac{1}{2}}. \tag{3.10}
$$

Since all $X_s^{i,N}$ are identically distributed, we obtain

$$
\mathbb{E}_B\left[\frac{1}{N}\sum_{j=1}^N \left|X_r^j - X_r^{j,N}\right|^2\right] = \mathbb{E}_B\left[\left|X_r^i - X_r^{i,N}\right|^2\right]. \tag{3.11}
$$

Taking $\frac{1}{2}\mathbb{E}_B\left(\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right)$ to the left side of the inequality, and taking the supremum over i on (3.8), together with (3.11), Lemma 3.3 and Fubini theorem we have

$$
\begin{aligned}
&\sup_{1\le i\le N}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^i - X_s^{i,N}\right|^2\right] \\
&\text{ł}\ e2(2L_h + L_1 + 130L_g^2)\mathbb{E}_B\left[\int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] \\
&\quad + 524L_1^2\mathbb{E}_B\left[\int_0^t W_2(\mu_r, \mu_r^N)^2 dE_r + \int_0^t \left|X_r^i - X_r^{i,N}\right|^2 dE_r\right] \\
&\le\ 2(2L_h + L_1 + 130L_g^2 + 262L_1^2)\int_0^t \mathbb{E}_B\left[\sup_{0\le r\le s}\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2\right] dE_s \\
&\quad + 524L_1^2\int_0^t \mathbb{E}_B\left[W_2\left(\mu_r, \mu_r^N\right)^2\right] dE_r \\
&\le\ C_1\mathbb{E}_B\left[W_2(\mu_r, \mu_r^N)^2\right].
\end{aligned}
\tag{3.12}
$$

where $C_1 = 524L_1^2E_Te^{2(2L_h+L_1+130L_g^2+262L_1^2)E_T}$. Taking $\mathbb{E}_D$ on both side of (3.12), we have

$$
\sup_{1\le i\le N}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N}\right|^2\right] \le C_1\mathbb{E}\left[W_2(\mu_r, \mu_r^N)^2\right].
$$

Hence, the desired assertion (3.6) follows from the fact (see [41], Theorem 5.8) that for any r

$$
\mathbb{E}\left[W_2(\mu_r, \mu_r^N)^2\right] \le C\begin{cases} N^{-1/2}, & d < 4, \\ N^{-1/2}\log(N), & d = 4, \\ N^{-2/d}, & d > 4, \end{cases} \tag{3.13}
$$

where $X^i \in L^p(\mathbb{R}^d)$ for any $p > 4$. The proof is therefore complete. □

Lemma 3.5. Let the assumptions (**H1**), (**H4**), **(H6)** and (2.9) hold and $X_0 \in L^2(\mathbb{R}^d)$. Then for any $T > 0$, there exists a constant $C_4, C_5 > 0$ which is independent of δ and N such that

$$
\sup_{\delta>0}\sup_{1\le i\le N}\mathbb{E}_B\left[\sup_{0\le t\le T}\left|X_t^{i,N,\delta}\right|^2\right] \le C_4, \qquad \sup_{\delta>0}\sup_{1\le i\le N}\mathbb{E}_B\left[\sup_{0\le t\le T}\left|X_s^{i,N}\right|^2\right] \le C_5, \tag{3.14}
$$

where $C_4 = C_2e^{2T+C_3E_T}$ and $C_5 = 2\left(\mathbb{E}_B\left|X_0^i\right|^2 + L_3^2|E_T|^{2\theta}T + 195L_2^2E_T\right)e^{2T+4\left(L_h+195L_2^2\right)E_T}$.

Proof. Applying the time-changed Itô's formula to $\left|X_s^{i,N,\delta}\right|^2$ yields

$$\begin{aligned}\left|X_s^{i,N,\delta}\right|^2 =& \left|X_0^i\right|^2 + 2\int_0^s \left\langle X_r^{i,N,\delta}, f(E_{\tau_{n_r}})\right\rangle dr + 2\int_0^s \left\langle X_r^{i,N,\delta}, h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dE_r \\ &+ 2\int_0^s \left\langle X_r^{i,N,\delta}, g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dB_{E_r}^i + \int_0^s \sum_{a=1}^{l} \left|g_a(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r \\ =& \left|X_0^i\right|^2 + 2\int_0^s \left\langle X_r^{i,N,\delta}, f(E_{\tau_{n_r}})\right\rangle dr + 2\int_0^s \left\langle X_r^{i,N,\delta} - X_{\tau_{n_r}}^{i,N,\delta}, h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dE_r \\ &+ 2\int_0^s \left\langle X_r^{i,N,\delta}, g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dB_{E_r}^i + \int_0^s \sum_{a=1}^{l} \left|g_a(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r \\ &+ 2\int_0^s \left\langle X_{\tau_{n_r}}^{i,N,\delta}, h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_s}}^{X,N,\delta})\right\rangle dE_r.\end{aligned} \tag{3.15}$$

By using **(H6)** and $\tau_{n_t} \le t < \tau_{n_{t+1}}$ we have

$$\begin{aligned}&\mathbb{E}_B\left[\sup_{0\le s\le t}\int_0^s \left\langle X_r^{i,N,\delta}, f(E_{\tau_{n_r}})\right\rangle dr\right] \le \frac{1}{2}\mathbb{E}_B\left[\int_0^t \left(\left|X_r^{i,N,\delta}\right|^2 + \left|f(E_{\tau_{n_r}})\right|^2\right) dr\right] \\ \le& \frac{1}{2}\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta}\right|^2 dr\right] + \frac{1}{2}L_3^2|E_T|^{2\theta}T.\end{aligned} \tag{3.16}$$

Exploiting the BDG's inequality and **(H4)**, yields

$$\begin{aligned}&\mathbb{E}_B\left[\sup_{0\le s\le t}\int_0^s \left\langle X_r^{i,N,\delta}, g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dB_{E_r}^i\right] \\ \le& 4\sqrt{2}\mathbb{E}_B\left[\left(\int_0^t \left|X_r^{i,N,\delta}\right|^2 \left|g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right)^{\frac{1}{2}}\right] \\ \le& \frac{1}{4}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N,\delta}\right|^2\right] + 32\mathbb{E}_B\left[\int_0^t \left|g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right] \\ \le& \frac{1}{4}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N,\delta}\right|^2\right] + 96L_2^2\mathbb{E}_B\left[\int_0^t \left(1 + \left|X_{\tau_{n_r}}^{i,N,\delta}\right|^2 + W_2(\mu_{\tau_{n_r}}^{X,N,\delta}, \delta_0)^2\right) dE_r\right] \\ \le& \frac{1}{4}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N,\delta}\right|^2\right] + 96L_2^2 E_T + 192L_2^2\mathbb{E}_B\left[\int_0^t \left|X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right].\end{aligned} \tag{3.17}$$

By using elementary inequality yields

$$\begin{aligned}&\mathbb{E}_B\left[\sup_{0\le s\le t}\int_0^s \left\langle X_r^{i,N,\delta} - X_{\tau_{n_r}}^{i,N,\delta}, h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dE_r\right] \\ \le& \mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta} - X_{\tau_{n_r}}^{i,N,\delta}\right| \left|h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right| dE_r\right] \\ \le& \frac{1}{2}\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta} - X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right] + \frac{1}{2}\mathbb{E}_B\left[\int_0^t \left|h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right].\end{aligned} \tag{3.18}$$

By using **(H4)** and Fubini theorem, we obtain

$$\begin{aligned}&\mathbb{E}_B\left[\int_0^t \left|h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right] \\ \le& 3L_2^2\mathbb{E}_B\left[\int_0^t \left(1 + \left|X_{\tau_{n_r}}^{i,N,\delta}\right|^2 + W_2(\mu_{\tau_{n_r}}^{X,N,\delta}, \delta_0)^2\right) dE_r\right] \\ \le& 3L_2^2 E_T + 6L_2^2\mathbb{E}_B\left[\int_0^t \left|X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right] \\ \le& 3L_2^2 E_T + 6L_2^2\int_0^t \mathbb{E}_B\left[\sup_{0\le u\le r}\left|X_u^{i,N,\delta}\right|^2\right] dE_r.\end{aligned} \tag{3.19}$$

In addition, by using Hölder's inequality, **(H4)** and **(H6)**, we obtain

$$\begin{aligned}
&\left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2\\
\leq&\left|\int_{\tau_{n_r}}^r f\left(E_{\tau_{n_u}}\right)du+\int_{\tau_{n_r}}^r h\left(E_{\tau_{n_u}},X_{\tau_{n_u}}^{i,N,\delta},\mu_{\tau_{n_u}}^{X,N,\delta}\right)dE_u+\int_{\tau_{n_r}}^r g\left(E_{\tau_{n_u}},X_{\tau_{n_u}}^{i,N,\delta},\mu_{\tau_{n_u}}^{X,N,\delta}\right)dB_{E_u}^i\right|^2\\
\leq&3\left(r-\tau_{n_r}\right)\int_{\tau_{n_r}}^r\left|f\left(E_{\tau_{n_u}}\right)\right|^2du+3L_2^2\delta\int_{\tau_{n_r}}^r\left|h\left(E_{\tau_{n_u}},X_{\tau_{n_u}}^{i,N,\delta},\mu_{\tau_{n_u}}^{X,N,\delta}\right)\right|^2dE_u\\
&+3\left|\int_{\tau_{n_r}}^r g(E_{\tau_{n_u}},X_{\tau_{n_u}}^{i,N,\delta},\mu_{\tau_{n_u}}^{X,N,\delta})dB_{E_u}^i\right|^2\\
\leq&\ q3L_3^2E_T^{2\theta}(r-\tau_{n_r})^2+9L_2^2\delta\int_{\tau_{n_r}}^r\left(1+\left|X_{\tau_{n_u}}^{i,N,\delta}\right|^2+W_2\left(\mu_{\tau_{n_u}}^{X,N,\delta},\delta_0\right)^2\right)dE_u\\
&+3\left|\int_{\tau_{n_r}}^r g(E_{\tau_{n_u}},X_{\tau_{n_u}}^{i,N,\delta},\mu_{\tau_{n_u}}^{X,N,\delta})dB_{E_u}^i\right|^2.
\end{aligned}\tag{3.20}$$

Then,

$$\begin{aligned}
&\mathbb{E}_B\left[\int_0^t\left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2dE_r\right]\\
\leq&\ 3L_3^2E_T^{2\theta}\mathbb{E}_B\left[\int_0^t(r-\tau_{n_r})^2dE_r\right]+9L_2^2(1+\delta)\mathbb{E}_B\left[\int_0^t\int_{\tau_{n_r}}^r\left(1+\left|X_{\tau_{n_u}}^{i,N,\delta}\right|^2+W_2\left(\mu_{\tau_{n_u}}^{X,N,\delta},\delta_0\right)^2\right)dE_udE_r\right]\\
\leq&\ 3L_3^2E_T^{2\theta}\mathbb{E}_B\left[\int_0^t(r-\tau_{n_r})^2dE_r\right]+9L_2^2(1+\delta)\delta E_T+18L_2^2(1+\delta)\mathbb{E}_B\left[\int_0^t\int_{\tau_{n_r}}^r\left|X_{\tau_{n_u}}^{i,N,\delta}\right|^2dE_udE_r\right]\\
\leq&\ 3L_3^2E_T^{2\theta}\mathbb{E}_B\left[\int_0^t(r-\tau_{n_r})^2dE_r\right]+9L_2^2(1+\delta)\delta E_T+18L_2^2(1+\delta)\delta\int_0^t\mathbb{E}_B\left[\sup_{0\leq u\leq r}\left|X_u^{i,N,\delta}\right|^2\right]dE_r.
\end{aligned}\tag{3.21}$$

We here use the technique of Theorem 4.1 in [33],

$$\begin{aligned}
\int_0^t(r-\tau_{n_s})^2dE_r=&\sum_{i=0}^{n_t-1}\int_{\tau_i}^{\tau_{i+1}}(r-\tau_i)^2dE_r+\int_{\tau_{n_t}}^T(r-\tau_{n_t})^2dE_r\\
\leq&\ \delta\left(\sum_{i=0}^{n_t-1}(\tau_{i+1}-\tau_i)^2+(T-\tau_{n_t})^2\right)\\
\leq&\ 2T\delta\left(\sum_{i=0}^{n_t-1}(\tau_{i+1}-\tau_i)+(T-\tau_{n_t})\right)\leq 2T^2\delta.
\end{aligned}\tag{3.22}$$

Thus, according to (3.18)–(3.22), one arrives at

$$\begin{aligned}
&\mathbb{E}_B\left[\sup_{0\leq s\leq t}\int_0^s\left\langle X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta},h(E_{\tau_{n_r}},X_{\tau_{n_r}}^{i,N,\delta},\mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dE_r\right]\\
\leq&\ \frac{3}{2}\left(2L_3^2E_T^{2\theta}T^2\delta+3L_2^2(1+\delta)\delta E_T+L_2^2E_T\right)+3L_2^2(3(1+\delta)\delta+1)\int_0^t\mathbb{E}_B\left[\sup_{0\leq u\leq r}\left|X_u^{i,N,\delta}\right|^2\right]dE_r.
\end{aligned}\tag{3.23}$$

Combining (3.15), (3.17) and (3.23) together, by using **(H1)** and (2.9) yields

$$\begin{aligned}
\mathbb{E}_B\left[\sup_{0\leq s\leq t}\left|X_s^{i,N,\delta}\right|^2\right]\leq&\left(\mathbb{E}_B\left|X_0^i\right|^2+(1+6T\delta)L_3^2|E_T|^{2\theta}T+9((1+\delta)\delta+22)L_2^2E_T\right)\\
&+\frac{1}{2}\mathbb{E}_B\left[\sup_{0\leq s\leq t}\left|X_s^{i,N,\delta}\right|^2\right]+\int_0^t\mathbb{E}_B\left[\sup_{0\leq u\leq r}\left|X_u^{i,N,\delta}\right|^2\right]dr\\
&+2(L_h+9L_2^2(1+\delta)\delta+198L_2^2)\int_0^t\mathbb{E}_B\left[\sup_{0\leq u\leq r}\left|X_u^{i,N,\delta}\right|^2\right]dE_r.
\end{aligned}$$

Taking the $\frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N,\delta}\right|^2\right]$ to the left side, yields

$$\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N,\delta}\right|^2\right] \le C_2 + 2\int_0^t \mathbb{E}_B\left[\sup_{0\le r\le s}\left|X_r^{i,N,\delta}\right|^2\right]ds + C_3\int_0^t \mathbb{E}_B\left[\sup_{0\le r\le s}\left|X_r^{i,N,\delta}\right|^2\right]dE_s,$$

where $C_2 = 2\left(\mathbb{E}_B\left|X_0^i\right|^2 + (1+6T\delta)L_3^2|E_T|^{2\theta}T + 9((1+\delta)\delta+22)L_2^2E_T\right)$, $C_3 = 4(L_h + 9L_2^2(1+\delta)\delta + 198L_2^2)$.

Therefore, by using Lemma 3.3, and then taking the supremum over i on both sides, we have

$$\sup_{1\le i\le N}\mathbb{E}_B\left[\sup_{0\le t\le T}\left|X_t^{i,N,\delta}\right|^2\right] \le C_4,$$

where $C_4 = C_2e^{2T+C_3E_T}$.

Also the proof process of the uniform boundness of the mean square of $X_s^{i,N}$ is similar to $X_s^{i,N,\delta}$, we omit it here. Hence, the proof is complete. □

Proposition 3.6. *Let the assumptions* (**H1**) − (**H6**) *and* (2.9) *hold and* $X_0 \in L^2(\mathbb{R}^d)$. *Let* $X^{i,N}$ *be the solution to* (2.11), *and* $X^{i,N,\delta}$ *be the solution to* (2.17). *Then for any* $T > 0$, *there exists a constant* $C_7 > 0$ *which is independent of* δ *and* N *such that*

$$\sup_{1\le i\le N}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{i,N} - X_t^{i,N,\delta}\right|^2\right] \le C_7\delta^{\min\{1,2\theta\}}, \tag{3.24}$$

where $C_7 = 2\Lambda_1e^{2(T+\Lambda_2E_T)}$.

Proof. Applying the time-changed Itô's formula to the difference of $X_s^{i,N}$ and $X_s^{i,N,\delta}$, for $0\le s\le t\le T$, yields

$$\begin{aligned}
\left|X_s^{i,N} - X_s^{i,N,\delta}\right|^2 =& 2\int_0^s \left\langle X_r^{i,N} - X_r^{i,N,\delta}, f(E_r) - f(E_{\tau_{n_r}})\right\rangle dr \\
&+ 2\int_0^s \left\langle X_r^{i,N} - X_r^{i,N,\delta}, h(E_r, X_r^{i,N}, \mu_r^{X,N}) - h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dE_r \\
&+ 2\int_0^s \left\langle X_r^{i,N} - X_r^{i,N,\delta}, g(E_r, X_r^{i,N}, \mu_r^{X,N}) - g(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle dB_{E_r}^i \\
&+ \int_0^s \sum_{a=1}^{l}\left|g_a(E_r, X_r^{i,N}, \mu_r^{X,N}) - g_a(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r \\
=:& \sum_{i=1}^4 J_i.
\end{aligned} \tag{3.25}$$

For J_2, by using **(H1)**, **(H3)**, **(H6)** and elementary inequality, we obtain

$$\begin{aligned}
&\left\langle X_r^{i,N} - X_r^{i,N,\delta}, h(E_r, X_r^{i,N}, \mu_r^{X,N}) - h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle \\
=& \left\langle X_r^{i,N} - X_r^{i,N,\delta}, h(E_r, X_r^{i,N}, \mu_r^{X,N}) - h(E_{\tau_{n_r}}, X_r^{i,N}, \mu_r^{X,N})\right\rangle \\
&+ \left\langle X_r^{i,N} - X_r^{i,N,\delta}, h(E_{\tau_{n_r}}, X_r^{i,N}, \mu_r^{X,N}) - h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_r^{X,N})\right\rangle \\
&+ \left\langle X_r^{i,N} - X_r^{i,N,\delta}, h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_r^{X,N}) - h(E_{\tau_{n_r}}, X_{\tau_{n_r}}^{i,N,\delta}, \mu_{\tau_{n_r}}^{X,N,\delta})\right\rangle \\
\le& L_3\left|X_r^{i,N} - X_r^{i,N,\delta}\right|\left|E_r - E_{\tau_{n_r}}\right|^\theta\left(1 + \left|X_r^{i,N}\right| + W_2\left(\mu_r^{X,N}, \delta_0\right)\right) + L_h\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2 \\
&+ L_1\left|X_r^{i,N} - X_r^{i,N,\delta}\right| W_2\left(\mu_r^{X,N}, \mu_{\tau_{n_r}}^{X,N,\delta}\right) \\
\le& \frac{L_3}{2}\left[\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2 + \delta^{2\theta}\left(1 + \left|X_r^{i,N}\right| + W_2\left(\mu_r^{X,N}, \delta_0\right)\right)^2\right] + L_h\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2 \\
&+ \frac{L_1}{2}\left[\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2 + W_2\left(\mu_r^{X,N}, \mu_{\tau_{n_r}}^{X,N,\delta}\right)^2\right].
\end{aligned} \tag{3.26}$$

Hence, together with Lemma 3.5 we obtain

$$
\begin{aligned}
&\mathbb{E}_B\left[\sup_{0\le s\le t} J_2\right]\\
\le\ &(L_3+2L_h+L_1)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 dE_r\right]+L_3\delta^{2\theta}\mathbb{E}_B\left[\int_0^t \left(1+\left|X_r^{i,N}\right|+W_2\left(\mu_r^{X,N},\delta_0\right)\right)^2 dE_r\right]\\
&+L_1\mathbb{E}_B\left[\int_0^t W_2\left(\mu_r^{X,N},\mu_{\tau_{n_r}}^{X,N,\delta}\right)^2 dE_r\right]\\
\le\ &(L_3+2L_h+3L_1)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 dE_r\right]+2L_1\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right]\\
&+3(1+2C_5)L_3E_T\delta^{2\theta}.
\end{aligned}
\tag{3.27}
$$

For J_4, similar to (3.26), by using **(H2)**, **(H3)** and **(H6)**, we get

$$
\begin{aligned}
&\left|g_a(E_r,X_r^{i,N},\mu_r^{X,N})-g_a(E_{\tau_{n_r}},X_{\tau_{n_r}}^{i,N,\delta},\mu_{\tau_{n_r}}^{X,N,\delta})\right|^2\\
&\le 3L_3^2\delta^{2\theta}\left(1+\left|X_r^{i,N}\right|+W_2\left(\mu_r^{X,N},\delta_0\right)\right)^2+3L_g^2\left|X_r^{i,N}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2+3L_1^2W_2\left(\mu_r^{X,N},\mu_{\tau_{n_r}}^{X,N,\delta}\right)^2.
\end{aligned}
\tag{3.28}
$$

Together with Lemma 3.5, yield

$$
\begin{aligned}
\mathbb{E}_B\left[\sup_{0\le s\le t} J_4\right]\le\ &9(1+2C_5)L_3^2E_T\delta^{2\theta}+6(L_g^2+L_1^2)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 dE_r\right]\\
&+6(L_g^2+L_1^2)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right].
\end{aligned}
\tag{3.29}
$$

For J_3, according to BDG's inequality and (3.28), yields

$$
\begin{aligned}
&\mathbb{E}_B\left[\sup_{0\le s\le t} J_3\right]\\
\le\ &8\sqrt{2}\mathbb{E}_B\left[\left(\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2\sum_{a=1}^{l}\left|g_a(E_r,X_r^{i,N},\mu_r^{X,N})-g_a(E_{\tau_{n_r}},X_{\tau_{n_r}}^{i,N,\delta},\mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right)^{\frac{1}{2}}\right]\\
\le\ &8\sqrt{2}\mathbb{E}_B\left[\left(\sup_{o\le s\le t}\left|X_s^{i,N}-X_s^{i,N,\delta}\right|^2\int_0^t\sum_{a=1}^{l}\left|g_a(E_r,X_r^{i,N},\mu_r^{X,N})-g_a(E_{\tau_{n_r}},X_{\tau_{n_r}}^{i,N,\delta},\mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right)^{\frac{1}{2}}\right]\\
\le\ &\frac{1}{2}\mathbb{E}_B\left[\sup_{o\le s\le t}\left|X_s^{i,N}-X_s^{i,N,\delta}\right|^2\right]+64\mathbb{E}_B\left[\int_0^t\sum_{a=1}^{l}\left|g_a(E_r,X_r^{i,N},\mu_r^{X,N})-g_a(E_{\tau_{n_r}},X_{\tau_{n_r}}^{i,N,\delta},\mu_{\tau_{n_r}}^{X,N,\delta})\right|^2 dE_r\right]\\
\le\ &\frac{1}{2}\mathbb{E}_B\left[\sup_{o\le s\le t}\left|X_s^{i,N}-X_s^{i,N,\delta}\right|^2\right]+576(1+2C_5)L_3^2E_T\delta^{2\theta}+384(L_g^2+L_1^2)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 dE_r\right]\\
&+384(L_g^2+L_1^2)\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right].
\end{aligned}
\tag{3.30}
$$

From (3.21), (3.22) and Lemma 3.5, we have

$$
\begin{aligned}
\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N,\delta}-X_{\tau_{n_r}}^{i,N,\delta}\right|^2 dE_r\right]\le\ &6L_3^2E_T^{2\theta}T^2\delta+9L_2^2E_T(1+\delta)\delta+18L_2^2C_4E_T(1+\delta)\delta\\
\le\ &C_6\delta,
\end{aligned}
\tag{3.31}
$$

where $C_6=3\left(L_3^2E_T^{2\theta}T^2+6L_2^2E_T(1+2C_4)\right)$.

Hence, substituting **(H6)**, (3.27), (3.29) and (3.30) into (3.25), yields

$$
\begin{aligned}
&\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N}-X_s^{i,N,\delta}\right|^2\right]\\
\le\ &\Lambda_1\delta^{\min\{1,2\theta\}}+\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 ds\right]+\Lambda_2\mathbb{E}_B\left[\int_0^t \left|X_r^{i,N}-X_r^{i,N,\delta}\right|^2 dE_r\right]\\
&+\frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N}-X_s^{i,N,\delta}\right|\right],
\end{aligned}
\tag{3.32}
$$

where

$$\Lambda_1 = L_3^2 T + 2L_1 C_6 + 3(1+2C_5)L_3 E_T + 585(1+2C_5)L_3^2 E_T + 390(L_g^2 + L_1^2)C_6,$$
$$\Lambda_2 = L_3 + 2L_h + 3L_1 + 390(L_g^2 + L_1^2).$$

Taking $\frac{1}{2}\mathbb{E}_B\left[\sup_{0\le s\le T}\left|X_s^{i,N} - X_s^{i,N,\delta}\right|^2\right]$ to the left side, by using Fubini theorem and Lemma 3.3 we obtain

$$\begin{aligned}\mathbb{E}_B\left[\sup_{0\le s\le t}\left|X_s^{i,N} - X_s^{i,N,\delta}\right|\right] \le\ & 2\Lambda_1\delta^{\min\{1,2\theta\}} + 2\int_0^t \mathbb{E}_B\left[\sup_{0\le r\le s}\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2\right]ds \\ & + 2\Lambda_2\int_0^t \mathbb{E}_B\left[\sup_{0\le r\le s}\left|X_r^{i,N} - X_r^{i,N,\delta}\right|^2\right]dE_s \\ \le\ & 2\Lambda_1 e^{2T+2\Lambda_2 E_T}\delta^{\min\{1,2\theta\}}.\end{aligned} \tag{3.33}$$

Hence, taking the supremum over i on both sides, then we have

$$\sup_{0\le i\le N}\mathbb{E}_B\left[\sup_{0\le t\le T}\left|X_t^{i,N} - X_t^{i,N,\delta}\right|^2\right] \le C_7\delta^{\min\{1,2\theta\}},$$

where $C_7 = 2\Lambda_1 e^{2(T+\Lambda_2 E_T)}$. Consequently, taking the expectation $\mathbb{E}_D$ on both sides, then we get the desired result. The proof is complete. □

Proof of Theorem 3.1. It is easy to obtain

$$\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N,\delta}\right|^2 \le 2\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N}\right|^2 + 2\sup_{0\le t\le T}\left|X_t^{i,N} - X_t^{i,N,\delta}\right|^2.$$

Then, from Propositions 3.4 and 3.6 we have

$$\begin{aligned}\sup_{1\le i\le N}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N,\delta}\right|^2\right] \le\ & C\sup_{1\le i\le N}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^i - X_t^{i,N}\right|^2\right] + C\sup_{1\le i\le N}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{i,N} - X_t^{i,N,\delta}\right|^2\right] \\ \le\ & C\begin{cases} N^{-1/2} + \delta^{\min\{1,2\theta\}}, & d<4, \\ N^{-1/2}\log(N) + \delta^{\min\{1,2\theta\}}, & d=4, \\ N^{-2/d} + \delta^{\min\{1,2\theta\}}, & d>4. \end{cases}\end{aligned} \tag{3.34}$$

Therefore, the proof is complete. □

Remark 3.7.

1. We note here that if we use the drift term $f(E_t, X_t, \mu_t^X)$ in (1.1) instead of $f(E_t)$, by using the N-dimensional interacting particles systems and constructing the continuous-time Euler–Maruyama scheme, we could still get Proposition 3.4 and Lemma 3.5. However, the strong convergence result in Proposition 3.6 no longer holds. In the calculation process there exists a term $\int_0^T (t - \tau_{n_s})ds$, which is dependent on T rather than step size δ, so we could not get the strong convergence result.
2. Once we use the numerical method in [33] to discretize the time-changed SDEs containing two drift terms and a diffusion term such as (2.3), the drift coefficient before the dt term should not contain the state of the process or state-related such as its probability distribution. Otherwise the strong convergence result will not hold.

4. Numerical example

In this section, we present a numerical example to verify the rate of convergence obtained in Proposition 3.6. Firstly, we show the relation of B_{E_t} and one path of 1-dimensional system of non-interacting particles. Then, under the number of particles of 5000 and 100,000, the "particle corruption" mentioned in [17] will not occur. At last, we show the strong convergence of the Euler–Maruyama method to the McKean–Vlasov. Throughout this section, we focus on the case that E_t is an inverse 0.9-stable subordinator, i.e. $\psi(s) = s^{0.9}$.

Consider a time-changed McKean–Vlasov SDE with two drift terms and a diffusion term as follows:

$$dX_t = \sqrt{1+E_t}dt + \left(\sqrt{1+E_t}X_t - \mathbb{E}[X_t]\right)dE_t + \sqrt{1+2E_t}X_t dB_{E_t}, \quad X_0 = 1. \tag{4.1}$$

It is easy to verify that the assumptions (**H1**)-(**H6**) and (2.9) hold for (4.1) with $\theta = 1$. For the simulation of the inverse subordinator E, time-changed Brownian motion B_{E_t} and time-changed MV-SDEs Eq. (4.1), we refer to reader [20,29,30,33,42,43].

Take $t \in [0, 1]$, according to Fig. 2 we see, if B is a standard Brownian motion independent of the inverse subordinator E, the jumps of B_{E_t} correspond to the jumps of X_t, and when the B_{E_t} is a constant during a period, X_t also remains a constant.

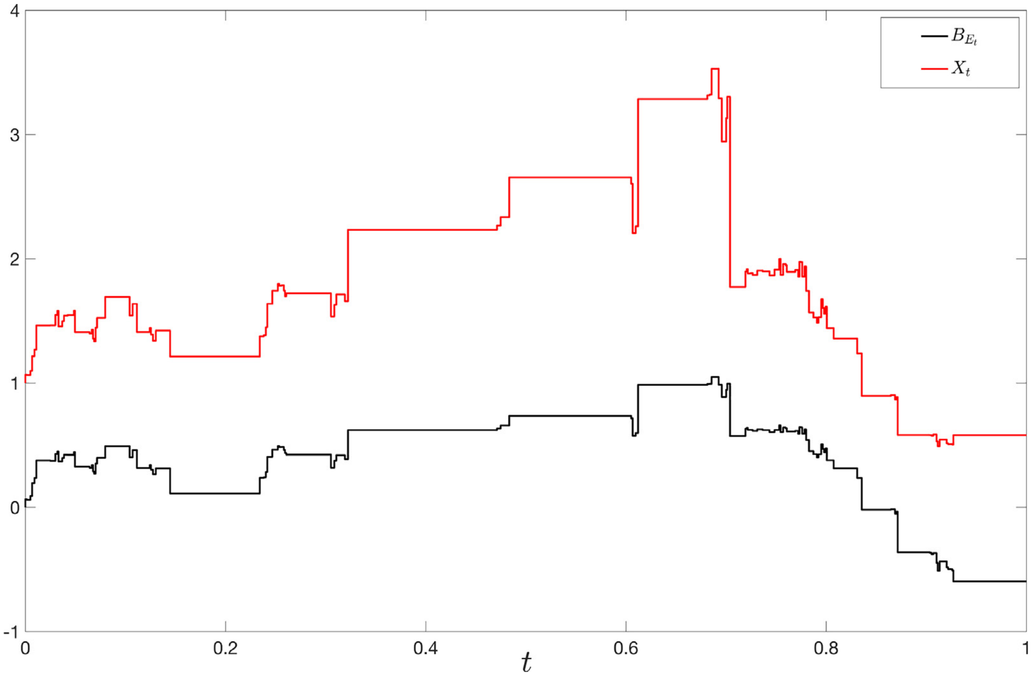

Fig. 2. Sample paths of B_{E_t} and X_t.

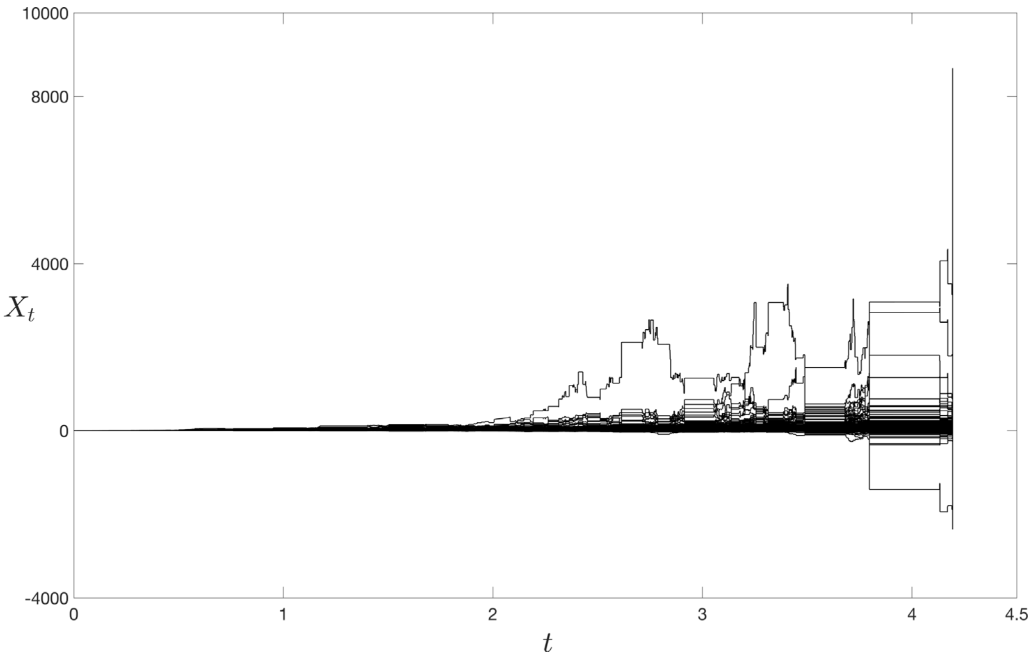

Fig. 3. Realizations of 5000 particle paths.

Then, for our example we simulate $N = 5000$ and $N = 10000$ with a time step $\delta = 2^{-14}$, according to Figs. 3 and 4 we see the "particle corruption" mentioned in [17] does not occur in the whole particle path simulation, namely all particles are reasonably well behaved.

Now we show the strong convergence and the convergence rate. Since the true solution of (4.1) is hard to obtain, we approximate the true solution with the numerical solution when the step size is sufficiently small, $\delta_0 = 2^{-14}$. Other step sizes $\delta = 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ are used to calculate the numerical solutions. For the given step size δ, time horizon

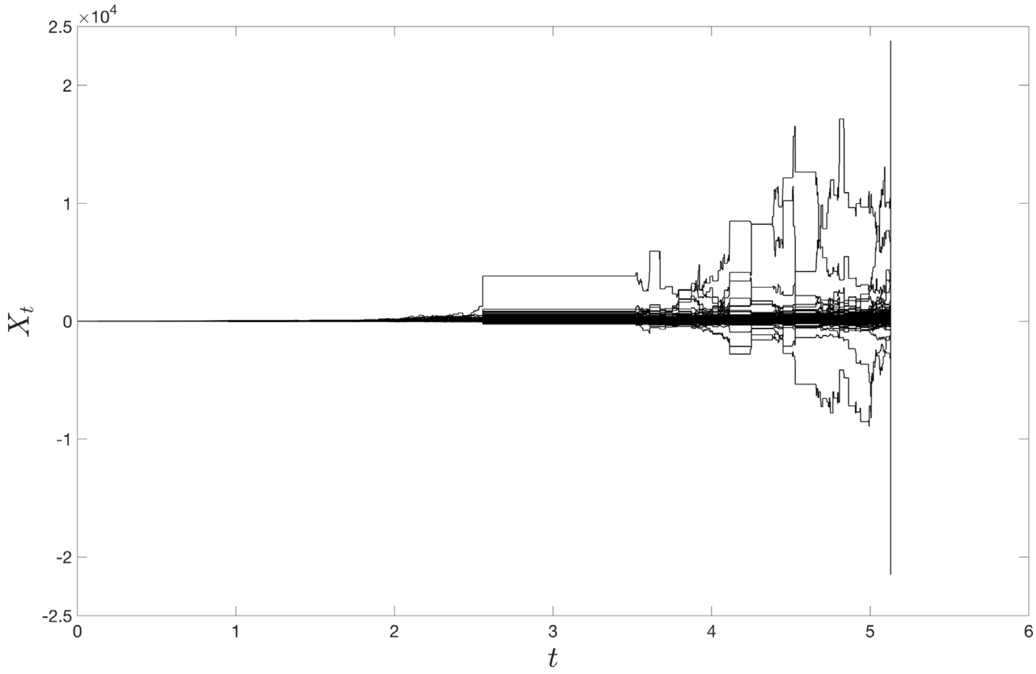

Fig. 4. Realizations of 10,000 particle paths.

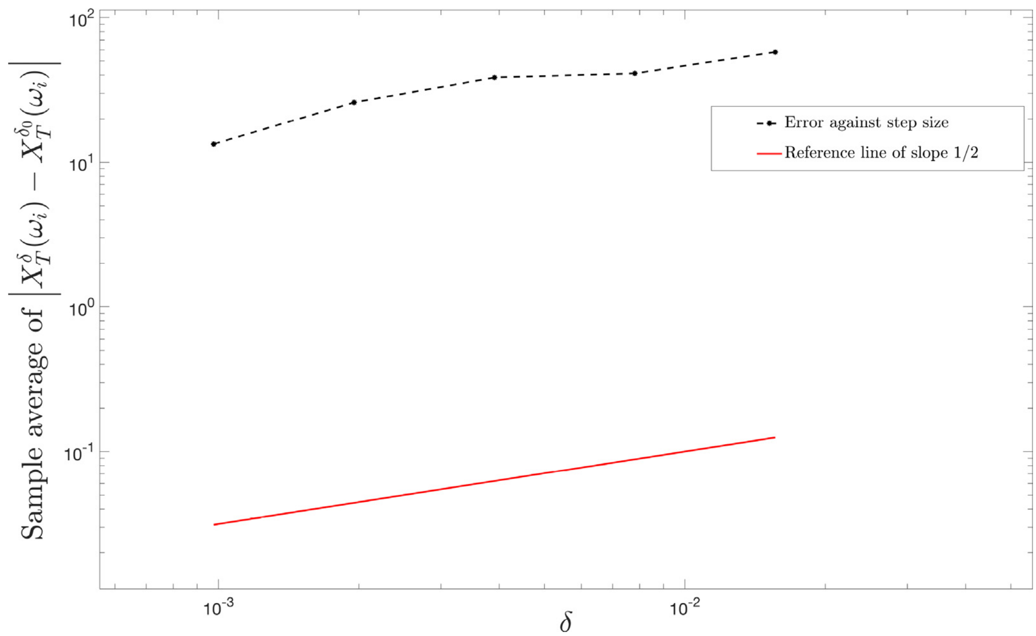

Fig. 5. The L^1 errors between the exact solution and the numerical solutions for step size $\delta = 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ with the least squares line $y = 0.4885x + 0.3232$.

$T = 1$, number of particles $N = 1000$, the L^1 strong error is calculated by

$$Error(X, \delta) := \frac{1}{N}\sum_{i=1}^{N}\left|X_T^\delta(\omega_i) - X_T^{\delta_0}(\omega_i)\right|.$$

In Fig. 5, the red solid line is the reference line with the slope of $1/2$, the slope of the black line is 0.4885. We can see that the slopes of the two curves appear to match well. Hence, the numerical results are consistent with Proposition 3.6.

5. Conclusions

In this paper, we investigate the convergence result of time-changed MV-SDEs by using interacting particle systems, where the MV-SDEs are driven by two drift terms and one diffusion term. By using the approximate method in [33], we show the propagation of chaos result and strong convergence between the stochastic N-interacting particle systems and its continuous time EM scheme. Besides, the EM scheme attains the classical $1/2$ rate of convergence in the step size. However, the approximate method in [33] obtained the continuous time EM scheme through discretizing the random time change E_t instead of the traditional non-random time variable t, it has obvious deficiencies as stated in Remark 3.7, we wonder if there are better ways to obtain the continuous time process. And also we are interested in the applications of the duality principle to MV-SDEs.

CRediT authorship contribution statement

Xueqi Wen: Strong convergence, Convergence rate, Writing – original draft. **Zhi Li:** Ensuring that the descriptions are accurate and agreed by all authors. **Liping Xu:** Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

[1] Carmona R, Delarue F, Lachapelle A. Control of McKean–Vlasov dynamics versus mean field games. Math Financial Econ 2013;7(2):131–66.
[2] Pham H, Wei X. Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics. SIAM J Control Optim 2017;55(2):1069–101.
[3] Delarue F, Inglis J, Rubenthaler S, Tanré E. Global solvability of a networked integrate-and-fire model of McKean–Vlasov type. Ann Appl Probab 2015;25(4):2096–133.
[4] Mehri S, Scheutzow M, Stannat W, Zangeneh BZ. Propagation of chaos for stochastic spatially structured neuronal networks with delay driven by jump diffusions. Ann Appl Probab 2020;30(1):175–207.
[5] Germain M, Mikael J, Warin X. Numerical resolution of McKean-Vlasov FBSDEs using neural networks. Methodol Comput Appl Probab 2022;1–30.
[6] Carmona R, Laurière M. Deep learning for mean field games and mean field control with applications to finance. 2021, arXiv preprint arXiv:2107.04568.
[7] Hammersley WR, Šiška D, Szpruch Ł. McKean–Vlasov SDEs under measure dependent Lyapunov conditions. In: Annales de l'institut henri poincaré, probabilités et statistiques. vol. 57, (2):Institut Henri Poincaré; 2021, p. 1032–57.
[8] Mishura Y, Veretennikov A. Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. Theory Probab Math Statist 2020;103:59–101.
[9] Hammersley WR, Šiška D, Szpruch Ł. Weak existence and uniqueness for McKean–Vlasov SDEs with common noise. Ann Probab 2021;49(2):527–55.
[10] Bahlali K, Mezerdi MA, Mezerdi B. Stability of McKean–Vlasov stochastic differential equations and applications. Stoch Dyn 2020;20(01):2050007.
[11] Kalinin A, Meyer-Brandis T, Proske F. Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs: a multidimensional Yamada-Watanabe approach. 2021, arXiv preprint arXiv:2107.07838.
[12] Ding X, Qiao H. Stability for stochastic McKean–Vlasov equations with Non-Lipschitz coefficients. SIAM J Control Optim 2021;59(2):887–905.
[13] Crisan D, Xiong J. Approximate McKean–Vlasov representations for a class of SPDEs. Stoch Int J Probab Stoch Process 2010;82(1):53–68.
[14] Bayraktar E, Cosso A, Pham H. Randomized dynamic programming principle and Feynman-Kac representation for optimal control of McKean-Vlasov dynamics. Trans Amer Math Soc 2018;370(3):2115–60.
[15] Chassagneux J-F, Crisan D, Delarue F. Numerical method for FBSDEs of McKean–Vlasov type. Ann Appl Probab 2019;29(3):1640–84.
[16] Li Y, Mao X, Song Q, Wu F, Yin G. Strong convergence of Euler-Maruyama schemes for McKean-Vlasov stochastic differential equations under local Lipschitz conditions of state variables. IMA J Numer Anal 2022. http://dx.doi.org/10.1093/imanum/drab107.
[17] dos Reis G, Engelhardt S, Smith G. Simulation of McKean–Vlasov SDEs with super-linear growth. IMA J Numer Anal 2022;42(1):874–922.
[18] Reisinger C, Stockinger W. An adaptive Euler–Maruyama scheme for McKean–Vlasov SDEs with super-linear growth and application to the mean-field FitzHugh–Nagumo model. J Comput Appl Math 2022;400:113725.
[19] Meerschaert MM, Scheffler H-P. Limit theorems for continuous-time random walks with infinite mean waiting times. J Appl Probab 2004;41(3):623–38.
[20] Umarov S, Hahn M, Kobayashi K. Beyond the triangle: Brownian motion, Itô calculus, and Fokker-Planck equation-fractional generalizations. World Scientific; 2018.
[21] Kobayashi K. Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations. J Theoret Probab 2011;24(3):789–820.
[22] Wu Q. Stability analysis for a class of nonlinear time-changed systems. Cogent Mathe 2016;3(1):1228273.

[23] Wu Q. Stability of stochastic differential equations with respect to time-changed Brownian motions. 2016, arXiv preprint arXiv:1602.08160.
[24] Nane E, Ni Y. Stability of the solution of stochastic differential equation driven by time-changed Lévy noise. Proc Amer Math Soc 2017;145(7):3085–104.
[25] Nane E, Ni Y. Stability of stochastic differential equation driven by time-changed Lévy noise. 2016, arXiv preprint arXiv:1604.07382.
[26] Zhu M, Li J, Liu D. Exponential stability for time-changed stochastic differential equations. Acta Math Appl Sin Engl Ser 2021;37(3):617–27.
[27] Li Z, Xu L, Xu L. Global attracting sets and exponential stability of stochastic partial functional differential equations. Systems Control Lett 2021;148:104859.
[28] Liu W. Polynomial stability of highly non-linear time-changed stochastic differential equations. Appl Math Lett 2021;119:107233.
[29] Jum E, Kobayashi K. A strong and weak approximation scheme for stochastic differential equations driven by a time-changed Brownian motion. Probab Math Statist 2016;36(2):201–20.
[30] Deng C, Liu W. Semi-implicit Euler–Maruyama method for non-linear time-changed stochastic differential equations. BIT Numer Math 2020;60(4):1133–51.
[31] Liu W, Mao X, Tang J, Wu Y. Truncated Euler-Maruyama method for classical and time-changed non-autonomous stochastic differential equations. Appl Numer Math 2020;153:66–81.
[32] Jin S, Kobayashi K. Strong approximation of stochastic differential equations driven by a time-changed Brownian motion with time-space-dependent coefficients. J Math Anal Appl 2019;476(2):619–36.
[33] Jin S, Kobayashi K. Strong approximation of time-changed stochastic differential equations involving drifts with random and non-random integrators. BIT Numer Math 2021;61(3):829–57.
[34] Meerschaert MM, Scheffler H-P. Triangular array limits for continuous time random walks. Stochastic Process Appl 2008;118(9):1606–33.
[35] Sznitman A-S. Topics in propagation of chaos. In: Ecole d'été de probabilités de Saint-Flour XIX-1989. Springer; 1991, p. 165–251.
[36] Méléard S. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. Probabilistic Model Nonlinear Partial Differ Equ 1996;42–95.
[37] Carmona R. Lectures on BSDEs, stochastic control, and stochastic differential games with financial applications. SIAM Book Series in Financial Mathematics I; 2016.
[38] Magdziarz M. Stochastic representation of subdiffusion processes with time-dependent drift. Stochastic Process Appl 2009;119(10):3238–52.
[39] Lipovan O. A retarded Grönwall-like inequality and its applications. J Math Anal Appl 2000;252(1):389–401.
[40] Mao X. Stochastic differential equations and applications. Chichester: Horwood; 2007.
[41] Carmona R, Delarue F. Probability theory of mean field games with applications I, probability theory and stochastic modelling. vol. 84, Springer International Publishing; 2017.
[42] Jum E. Numerical approximation of stochastic differential equations driven by Lévy motion with infinitely many jumps [Ph.D. thesis], University of Tennessee, Knoxville; 2015.
[43] Higham DJ. An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev 2001;43(3):525–46.