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First order strong convergence of positivity preserving logarithmic Euler–Maruyama method for the stochastic SIS epidemic model[☆]

Hongfu Yang[a,b], Jianhua Huang[a,*]

[a] College of Liberal Arts and Science, National University of Defense Technology, Changsha, 410073, China
[b] School of Mathematics and Statistics, Guangxi Normal University, Guilin, 541004, China

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ABSTRACT

In this article, we first construct a positivity-preserving numerical method for the stochastic Susceptible–Infected–Susceptible (SIS) epidemic model by combining the logarithmic transformation and Euler–Maruyama (EM) method. Then, we show that the algorithm not only preserves the domain of the original SDE, but also has the first-order rate of the pth-moment convergence over a finite time interval for all $p > 0$. Finally, some numerical experiments are provided to illustrate the theoretical results.

1. Introduction

In this paper, we consider the following stochastic SIS epidemic model [1] described by

$$\begin{cases} \mathrm{d}S(t) = \big[\mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t)\big]\mathrm{d}t - \sigma S(t)I(t)\mathrm{d}B(t), \\ \mathrm{d}I(t) = \big[\beta S(t)I(t) - (\mu+\gamma)I(t)\big]\mathrm{d}t + \sigma S(t)I(t)\mathrm{d}B(t), \end{cases} \tag{1.1}$$

with initial values satisfying $I(0) + S(0) = S_0 + I_0 = N$, where $B(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and N, μ, γ, β, σ are non-negative numbers. Since $\mathrm{d}\big[S(t) + I(t)\big] = \big[\mu N - \mu\big(S(t) + I(t)\big)\big]\mathrm{d}t$, it is easy to obtain that $I(t) + S(t) = N$. Then we only need to consider the SDE for $I(t)$ as follows

$$\mathrm{d}I(t) = \big[\eta I(t) - \beta I(t)^2\big]\mathrm{d}t + \sigma I(t)\big(N - I(t)\big)\mathrm{d}B(t), \tag{1.2}$$

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* Corresponding author.
E-mail addresses: yanghf783@nenu.edu.cn (H. Yang), jhhuang32@hotmail.com (J. Huang).

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with initial value $I(0) = I_0 \in (0, N)$, where $\eta = \beta N - \mu - \gamma$. As a significant type of epidemic models, the stochastic SIS epidemic models have been widely applied to analyze the spread and control of many infectious diseases in the real world. In recent years, there are many studies on stochastic SIS epidemic models. In particular, Gray et al. [1] have showed that for any given initial value $I_0 \in (0, N)$, (1.2) has a unique global solution taking values in the domain $(0, N)$, i.e., $\mathbb{P}\big\{I(t) \in (0, N)\ \forall t \geq 0\big\} = 1$.

Since it is almost impossible to know the explicit solution of SDE (1.2), constructing appropriate numerical methods to approximate (1.2) and even preserve properties of (1.2) is important and necessary. It is well-known that the classical EM schemes [2] are popular for approximating SDEs but fails to simulate a large class of SDEs with superlinear growth coefficients [3]. Especially, the authors in [1] have used the classical EM method to simulate the exact solution of (1.2) but is not positivity-preserving. Moreover, several modified EM methods [4–7] have been developed to simulate nonlinear SDEs in the last decade. However, all numerical approximations mentioned above require coefficients to satisfy at least a global monotonicity condition, which are not applied directly to (1.2). For nonnegativity or positive preserving numerical methods of nonlinear SDEs, Liu and Mao [8] have studied the convergence of a stopped EM method with nonnegative numerical solutions. Mao, Wei and Wiriyakraikul [9] have established a positive preserving truncated EM method for stochastic Lotka–Volterra competition model but without any convergence rate of the algorithm. Chen, Gan and Wang [10] have proposed the Lamperti smoothing truncation scheme that can preserve the domain of the original SDEs and proved a mean-square convergence rate of order one. Nevertheless, we noted that the numerical method in [10] could be simplified, and the corresponding results could further be improved.

Motivated by the work [10–12], we aim to propose a kind of logarithmic transformed EM scheme combining the EM scheme and the Lamperti-type transformation $y(t) = \log(I(t)) - \log(N - I(t)) = \log \frac{I(t)}{N - I(t)}$. According to (1.2), applying Itô's formula to $y(t)$ yields

$$\mathrm{d}y(t) = F(y(t))\mathrm{d}t + \sigma N \mathrm{d}B(t), \qquad t \geq 0, \tag{1.3}$$

with $y(0) = \log(I_0) - \log(N - I_0)$, where $F(x) = \eta - (\mu + \gamma)e^x + \frac{\sigma^2 N^2}{2} - \frac{\sigma^2 N^2}{1+e^x}$.

In this paper, we define the EM scheme which is different from [10] for (1.3) by

$$X_{k+1} = X_k + F(X_k)\Delta + \sigma N \Delta B_k, \quad \text{with} \quad X_0 = y(0), \tag{1.4}$$

for any $\Delta \in (0, 1]$ and $k \geq 0$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$ and $t_k = k\Delta$. Transforming back, i.e.

$$I_k = N - \frac{N}{1 + e^{X_k}} = \frac{N e^{X_k}}{1 + e^{X_k}}, \tag{1.5}$$

gives a strictly positive approximation of the original SDE (1.2). Then we can show the pth-moment convergence of the logarithmic EM scheme (1.4) for SDE (1.2) with first-order rate for all $p > 0$. Lastly, a numerical example is given to show the efficiency of the numerical scheme.

2. Main results

Throughout this paper C stands for the generic positive real constants whose values may change between occurrences and is independent of Δ and k. Additionally, $|\cdot|$ is the Euclidean norm on $\mathbb{R} := (-\infty, +\infty)$ and $\lfloor T/\Delta \rfloor$ represents the integer part of T/Δ.

Lemma 2.1 *([10, Theorem 3.2]). For any $p \geq 0$ we have*

$$\Big(\sup_{t\in[0,T]} \mathbb{E}\big[I(t)^{-p}\big]\Big) \vee \Big(\sup_{t\in[0,T]} \mathbb{E}\big[\big(N - I(t)\big)^{-p}\big]\Big) \leq K_p,$$

where $K_p = \Big(I_0^{-p} \vee \big(N - I_0\big)^{-p}\Big) \exp\Big(p(|\beta N - \mu - \gamma| + 2\beta N)T + \frac{p(p+1)}{2}\sigma^2 N^2 T\Big)$.

Lemma 2.2 *([10, Proposition 3.3]). For any $p \in \mathbb{R}$, the transformed SDE (1.3) has the following exponential integrability property*

$$\sup_{t\in[0,T]} \mathbb{E}\big[e^{py(t)}\big] \le N^{|p|}K_{|p|},$$

where K_p is given by Lemma 2.1.

Lemma 2.3. *For any $p > 0$, the EM scheme defined by (1.4) has the property that*

$$\sup_{\Delta\in(0,1]} \sup_{0\le k\le\lfloor T/\Delta\rfloor} \mathbb{E}\big[\mathrm{e}^{pX_k}\big] \le C, \quad \forall\, T > 0.$$

Proof. We know that

$$X_{k+1} \le X_k + \big(\eta + \frac{\sigma^2N^2}{2}\big)\Delta + \sigma N\Delta B_k \le Y_0 + \big(\eta + \frac{\sigma^2N^2}{2}\big)k\Delta + \sum_{i=0}^{k-1}\sigma N\Delta B_i$$

for any integer $k \ge 1$. Thus, for any $p > 0$ we have

$$\mathbb{E}\big[\mathrm{e}^{pX_k}\big] \le \exp\Big(pY_0 + p\big(\eta + \frac{\sigma^2N^2}{2}\big)T\Big)\mathbb{E}\Big[\exp\Big(p\sum_{i=0}^{k-1}\sigma N\Delta B_i\Big)\Big] \le C\exp\Big(\frac{p^2\sigma^2N^2}{2}T\Big).$$

The proof is complete. □

Theorem 2.1. *For any $q > 0$ there exists a constant $C > 0$ such that for any $\Delta \in (0,1]$*

$$\mathbb{E}\Big[\sup_{k=0,\dots,\lfloor T/\Delta\rfloor} |X_k - y(t_k)|^q\Big] \le C\Delta^q, \quad \forall\, T > 0. \tag{2.1}$$

Proof. By (1.3), we have

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} F(y(s))\mathrm{d}s + \sigma N\Delta B_k. \tag{2.2}$$

Using (1.2), (1.4) and (2.2) we have

$$\begin{aligned}
X_{k+1} - y(t_{k+1}) =& X_k - y(t_k) + \big(F(X_k) - F(y(t_k))\big)\Delta + \int_{t_k}^{t_{k+1}}\big(F(y(t_k)) - F(y(s))\big)\mathrm{d}s\\
=& X_k - y(t_k) - (\mu+\gamma)\big(e^{X_k} - e^{y(t_k)}\big)\Delta + \Big(\frac{\sigma^2N^2}{1+e^{y(t_k)}} - \frac{\sigma^2N^2}{1+e^{X_k}}\Big)\Delta\\
&+ (\mu+\gamma)N\int_{t_k}^{t_{k+1}} \frac{\int_{t_k}^{s}\mathrm{d}I(u)}{(N-I(s))(N-I(t_k))}\mathrm{d}s + \sigma^2N\int_{t_k}^{t_{k+1}}\int_{s}^{t_k}\mathrm{d}I(u)\mathrm{d}s\\
=& X_k - y(t_k) - (\mu+\gamma)\big(e^{X_k} - e^{y(t_k)}\big)\Delta + \sigma^2N^2\Big(\frac{e^{X_k} - e^{y(t_k)}}{(1+e^{y(t_k)})(1+e^{X_k})}\Big)\Delta + \Xi_k,
\end{aligned}$$

where $\Xi_k =: \Xi_k^{(1)} + \Xi_k^{(2)}$ and

$$\Xi_k^{(1)} = (\mu+\gamma)N\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\frac{I(u)\big(\eta - \beta I(u)\big)}{(N-I(s))(N-I(t_k))}\mathrm{d}u\mathrm{d}s + \sigma^2N\int_{t_k}^{t_{k+1}}\int_{s}^{t_k} I(u)\big(\eta - \beta I(u)\big)\mathrm{d}u\mathrm{d}s, \tag{2.3}$$

$$\Xi_k^{(2)} = \sigma N\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\Big[\frac{(\mu+\gamma)NI(u)\big(N-I(u)\big)}{(N-I(s))(N-I(t_k))} - \sigma^2I(u)\big(N-I(u)\big)\Big]\mathrm{d}B(u)\mathrm{d}s. \tag{2.4}$$

Let us define $u_k = X_k - y(t_k)$.

$$\begin{aligned}u_{k+1}^2 =& u_k^2 + (\mu+\gamma)^2\big(\mathrm{e}^{X_k} - \mathrm{e}^{y(t_k)}\big)^2\Delta^2 + \frac{\sigma^4N^4\big(e^{X_k}-e^{y(t_k)}\big)^2}{(1+e^{y(t_k)})^2(1+e^{X_k})^2}\Delta^2 + \Xi_k^2 - 2(\mu+\gamma)u_k\big(\mathrm{e}^{X_k} - \mathrm{e}^{y(t_k)}\big)\Delta\\ &+ \frac{2\sigma^2N^2u_k(e^{X_k}-e^{y(t_k)})}{(1+e^{X_k})(1+e^{y(t_k)})}\Delta + 2u_k\Xi_k - 2(\mu+\gamma)\frac{\sigma^2N^2(e^{X_k}-e^{y(t_k)})^2}{(1+e^{X_k})(1+e^{y(t_k)})}\Delta^2\\ &- 2(\mu+\gamma)\Xi_k\big(\mathrm{e}^{X_k} - \mathrm{e}^{y(t_k)}\big)\Delta + 2\Xi_k\frac{\sigma^2N^2\big(e^{X_k}-e^{y(t_k)}\big)}{(1+e^{y(t_k)})(1+e^{X_k})}\Delta.\end{aligned}$$

Moreover, by using the mean value theorem and the Young inequality we have $u_k\big(\mathrm{e}^{X_k}-\mathrm{e}^{y(t_k)}\big)\geq 0$,

$$\begin{aligned}\frac{2\sigma^2N^2u_k(e^{X_k}-e^{y(t_k)})}{(1+e^{X_k})(1+e^{y(t_k)})}\Delta \leq& \frac{2\sigma^2N^2|u_k||e^{X_k}-e^{y(t_k)}|\Delta}{(1+e^{X_k})(1+e^{y(t_k)})} \leq 2\sigma^2N^2u_k^2\Delta,\\ 2\Xi_k\frac{\sigma^2N^2\big(e^{X_k}-e^{y(t_k)}\big)}{(1+e^{y(t_k)})(1+e^{X_k})}\Delta \leq& \Xi_k^2 + \frac{\sigma^4N^4\big(e^{X_k}-e^{y(t_k)}\big)^2}{(1+e^{y(t_k)})^2(1+e^{X_k})^2}\Delta^2 \leq \Xi_k^2 + \sigma^4N^4u_k^2\Delta^2,\end{aligned}$$

and $-2(\mu+\gamma)\Xi_k\big(e^{X_k}-e^{y(t_k)}\big)\Delta \leq \Xi_k^2 + (\mu+\gamma)^2\big(e^{X_k}-e^{y(t_k)}\big)^2\Delta^2$, then we deduce that

$$\begin{aligned}u_{k+1}^2 \leq& \Big[1+2\sigma^2N^2\big(1+\sigma^2N^2\Delta\big)\Delta\Big]u_k^2 + 2(\mu+\gamma)^2\big(\mathrm{e}^{X_k}-\mathrm{e}^{y(t_k)}\big)^2\Delta^2 + 3\Xi_k^2 + 2u_k\Xi_k\\ \leq& \sum_{i=0}^k(1+C\Delta)^{k-i}\Big[3\Xi_i^2 + 2(\mu+\gamma)^2\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^2\Delta^2\Big] + 2\sum_{i=0}^k(1+C\Delta)^{k-i}u_i\Xi_i\\ \leq& C\Big[\sum_{i=0}^k\Xi_i^2 + \sum_{i=0}^k\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^2\Delta^2\Big] + 2\sum_{i=0}^k(1+C\Delta)^{k-i}u_i\Xi_i.\end{aligned}\tag{2.5}$$

Let $\mathfrak{M}_0 = 0$, and $\mathfrak{M}_k = \sum_{i=0}^{k-1}(1+C\Delta)^{k-1-i}u_i\Xi_i^{(2)}$ for any $k\geq 1$, since

$$\mathbb{E}\Big[\Xi_k^{(2)}|\mathcal{F}_{t_k}\Big] = \sigma N\mathbb{E}\Big[\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\Big[\frac{(\mu+\gamma)NI(u)\big(N-I(u)\big)}{(N-I(s))(N-I(t_k))} - \sigma^2I(u)\big(N-I(u)\big)\Big]\mathrm{d}B(u)\mathrm{d}s\Big|\mathcal{F}_{t_k}\Big] = 0.$$

It is then easy to show that $\mathbb{E}\Big[\mathfrak{M}_{k+1}|\mathcal{F}_{t_k}\Big] = \mathfrak{M}_k + u_k\mathbb{E}\Big[\Xi_k^{(2)}|\mathcal{F}_{t_k}\Big] = \mathfrak{M}_k$. This implies immediately that $\mathfrak{M}_k$ is a martingale and the Burkholder–Davis–Gundy inequality implies that

$$\mathbb{E}\Big[\sup_{k=0,\ldots,l}|\mathfrak{M}_k|^q\Big] \leq C\mathbb{E}\Big[\Big|\sum_{i=0}^{l-1}(u_i\Xi_i^{(2)})^2\Big|^{\frac{q}{2}}\Big] \leq C\mathbb{E}\Big[\Big(\sum_{i=0}^{l-1}u_i^2|\Xi_i^{(2)}|^2\Big)^{\frac{q}{2}}\Big] \leq C\mathbb{E}\Big[\big(\lfloor T/\Delta\rfloor\big)^{q/2-1}\sum_{i=0}^{l}|u_i|^q\big|\Xi_i^{(2)}\big|^q\Big]$$

for any $q\geq 2$ and $l = 0,\ldots,\lfloor T/\Delta\rfloor$. Using this and Jensen's inequality in (2.5) we now arrive at

$$\begin{aligned}&\mathbb{E}\Big[\sup_{k=0,\ldots,l}|u_{k+1}|^{2q}\Big] \leq \mathbb{E}\Big[\sup_{k=0,\ldots,l}\Big|C\Delta^2\sum_{i=0}^k\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^2 + C\sum_{i=0}^k\Xi_i^2 + 2\sum_{i=0}^k(1+C\Delta)^{k-i}u_i\Xi_i\Big|^q\Big]\\ \leq& C\mathbb{E}\Big[\Delta^{2q}\Big[\sum_{i=0}^l\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^2\Big]^q + \Big(\sum_{i=0}^l\Xi_i^2\Big)^q + \sup_{k=0,\ldots,l}\Big|\sum_{i=0}^k(1+C\Delta)^{k-i}u_i\Xi_i\Big|^q\Big]\\ \leq& C\mathbb{E}\Big[\big(\lfloor T/\Delta\rfloor\big)^{q-1}\Big(\Delta^{2q}\sum_{i=0}^l\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^{2q} + \sum_{i=0}^l\Xi_i^{2q} + \sum_{i=0}^l|u_i|^q\big|\Xi_i^{(1)}\big|^q\Big) + \sup_{k=0,\ldots,l}\big|\mathfrak{M}_{k+1}\big|^q\Big]\\ \leq& C\big(\lfloor T/\Delta\rfloor\big)^{q-1}\sum_{i=0}^l\mathbb{E}\Big[\Delta^{2q}\big(\mathrm{e}^{X_i}-\mathrm{e}^{y(t_i)}\big)^{2q} + \big|\Xi_i^{(1)}\big|^{2q} + \big|\Xi_i^{(2)}\big|^{2q} + |u_i|^q\big|\Xi_i^{(1)}\big|^q + \big(\lfloor T/\Delta\rfloor\big)^{-q/2}|u_i|^q\big|\Xi_i^{(2)}\big|^q\Big]\end{aligned}\tag{2.6}$$

for any $q \geq 2$ and $l = 0, \ldots, \lfloor T/\Delta \rfloor$. Using the mean value theorem, we infer that

$$\mathbb{E}\Big[\big(\mathrm{e}^{X_i} - \mathrm{e}^{y(t_i)}\big)^{2q}\Big] \leq \mathbb{E}\Big[\big(\mathrm{e}^{X_i} + \mathrm{e}^{y(t_i)}\big)^{q}\big|\mathrm{e}^{X_i} - \mathrm{e}^{y(t_i)}\big|^{q}\Big] \leq \mathbb{E}\Big[\big(\mathrm{e}^{X_i} + \mathrm{e}^{y(t_i)}\big)^{2q}\big|X_i - y(t_i)\big|^{q}\Big].$$

This together with Lemmas 2.2, 2.3 as well as the Young inequality implies

$$\begin{aligned}
&\mathbb{E}\Big[\Delta^{2q}\big(\lfloor T/\Delta \rfloor\big)^{q-1}\sum_{i=0}^{l}\big(\mathrm{e}^{X_i} - \mathrm{e}^{y(t_i)}\big)^{2q}\Big] \leq T^{q-1}\sum_{i=0}^{l}\mathbb{E}\Big[\Delta^{q+\frac{1}{2}}\big(\mathrm{e}^{X_i} + \mathrm{e}^{y(t_i)}\big)^{2q}|u_i|^q\Delta^{\frac{1}{2}}\Big] \\
\leq &\frac{T^{q-1}}{2}\sum_{i=0}^{l}\mathbb{E}\Big[\Delta^{2q+1}\big(\mathrm{e}^{X_i} + \mathrm{e}^{y(t_i)}\big)^{4q} + |u_i|^{2q}\Delta\Big] \leq CT^q\Delta^{2q} + CT^{q-1}\Delta\sum_{i=0}^{l}\mathbb{E}|u_i|^{2q}. \qquad (2.7)
\end{aligned}$$

For any $q \geq 2$, by (2.3), (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}
\mathbb{E}\big|\Xi_i^{(1)}\big|^{2q} \leq &C\mathbb{E}\Bigg[\Big(\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\frac{I(u)|\eta - \beta I(u)|}{(N - I(s))(N - I(t_k))}\mathrm{d}u\mathrm{d}s\Big)^{2q} + \Big(\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}I(u)|\eta - \beta I(u)|\mathrm{d}u\mathrm{d}s\Big)^{2q}\Bigg] \\
\leq &C\Delta^{2q}\mathbb{E}\Bigg[\Big(\int_{t_k}^{t_{k+1}}1^{\frac{2q}{2q-1}}ds\Big)^{2q-1}\Big(\int_{t_k}^{t_{k+1}}(N - I(s))^{-2q}(N - I(t_k))^{-2q}\mathrm{d}s\Big)\Bigg] + C\Delta^{4q} \\
\leq &C\Delta^{4q-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[(N - I(s))^{-2q}(N - I(t_k))^{-2q}\Big]\mathrm{d}s + C\Delta^{4q} \\
\leq &C\Delta^{4q-1}\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}\big[(N - I(s))^{-4q}\big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}\big[(N - I(t_k))^{-4q}\big]\Big)^{\frac{1}{2}}\mathrm{d}s + C\Delta^{4q} \leq C\Delta^{4q}, \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\big|\Xi_i^{(2)}\big|^{2q} \leq &C\mathbb{E}\Bigg(\int_{t_k}^{t_{k+1}}\Big|\int_{t_k}^{s}\Big[\frac{(\mu+\gamma)NI(u)\big(N - I(u)\big)}{(N - I(s))(N - I(t_k))} - \sigma^2 I(u)\big(N - I(u)\big)\Big]\mathrm{d}B(u)\Big|\mathrm{d}s\Bigg)^{2q} \\
\leq &C\Delta^{2q-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\Big|\int_{t_k}^{s}\Big[\frac{(\mu+\gamma)NI(u)\big(N - I(u)\big)}{(N - I(s))(N - I(t_k))} - \sigma^2 I(u)\big(N - I(u)\big)\Big]\mathrm{d}B(u)\Big|^{2q}\Big]\mathrm{d}s \\
\leq &C\Delta^{3q-2}\int_{t_k}^{t_{k+1}}\mathbb{E}\Big[\int_{t_k}^{s}\Big|\frac{(\mu+\gamma)NI(u)\big(N - I(u)\big)}{(N - I(s))(N - I(t_k))} - \sigma^2 I(u)\big(N - I(u)\big)\Big|^{2q}\mathrm{d}u\Big]\mathrm{d}s \leq C\Delta^{3q}. \qquad (2.9)
\end{aligned}$$

Thus, the Cauchy–Schwarz inequality gives that

$$\mathbb{E}\big[|u_i|^q\big|\Xi_i^{(1)}\big|^q\big] \leq \Big(\mathbb{E}\big[|u_i|^{2q}\big]\Big)^{1/2}\Big(\mathbb{E}\big[\big|\Xi_i^{(1)}\big|^{2q}\big]\Big)^{1/2} \leq C\Big(\mathbb{E}\big[|u_i|^{2q}\big]\Big)^{1/2}\Delta^{2q}, \qquad (2.10)$$

$$\mathbb{E}\big[|u_i|^q\big|\Xi_i^{(2)}\big|^q\big] \leq \Big(\mathbb{E}\big[|u_i|^{2q}\big]\Big)^{1/2}\Big(\mathbb{E}\big[\big|\Xi_i^{(2)}\big|^{2q}\big]\Big)^{1/2} \leq C\Big(\mathbb{E}\big[|u_i|^{2q}\big]\Big)^{1/2}\Delta^{3q/2}. \qquad (2.11)$$

Thus, for any integer $k \geq 0$, substituting (2.7)–(2.11) into (2.6), we know that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{k=0,\ldots,l}|u_{k+1}|^{2q}\Big] \leq &C\Bigg\{T^q\Delta^{2q} + T^{q-1}\Delta\sum_{i=0}^{l}\mathbb{E}|u_i|^{2q} + \big(\lfloor T/\Delta \rfloor\big)^{q-1}\sum_{i=0}^{l}\mathbb{E}\Big[\big|\Xi_i^{(1)}\big|^{2q}\Big] \\
&+ \big(\lfloor T/\Delta \rfloor\big)^{q-1}\sum_{i=0}^{l}\Big(\mathbb{E}\Big[\big|\Xi_i^{(2)}\big|^{2q}\Big] + \mathbb{E}\Big[|u_i|^q\big|\Xi_i^{(1)}\big|^q\Big] + \mathbb{E}\Big[|u_i|^q\big|\Xi_i^{(2)}\big|^q\Big]\Big)\Bigg\} \\
\leq C\Bigg\{&T^q\Delta^{2q} + T^{q-1}\Delta\sum_{i=0}^{l}\mathbb{E}|u_i|^{2q} + \big(\lfloor T/\Delta \rfloor\big)^{q}\Delta^{4q} + \big(\lfloor T/\Delta \rfloor\big)^{q}\Delta^{3q} + \big(\lfloor T/\Delta \rfloor\big)^{q-1}\sum_{i=0}^{l}\Big(\mathbb{E}|u_i|^{2q}\Big)^{1/2}\Delta^{2q} \\
&+ \big(\lfloor T/\Delta \rfloor\big)^{q/2-1}\sum_{i=0}^{l}\Big(\mathbb{E}|u_i|^{2q}\Big)^{1/2}\Delta^{3q/2}\Bigg\} \leq C\Big(T^q\Delta^{2q} + T^{q-1}\Delta\sum_{i=0}^{l}\mathbb{E}|u_i|^{2q}\Big)
\end{aligned}$$

for any $q \geq 2$ and $l = 0, \ldots, \lfloor T/\Delta \rfloor$, and applying Gronwall's inequality we obtain

$$\mathbb{E}\Big[\sup_{k=0,\ldots,l}|u_{k+1}|^{2q}\Big] \leq CT^q\Delta^{2q}\exp(CT^q)$$

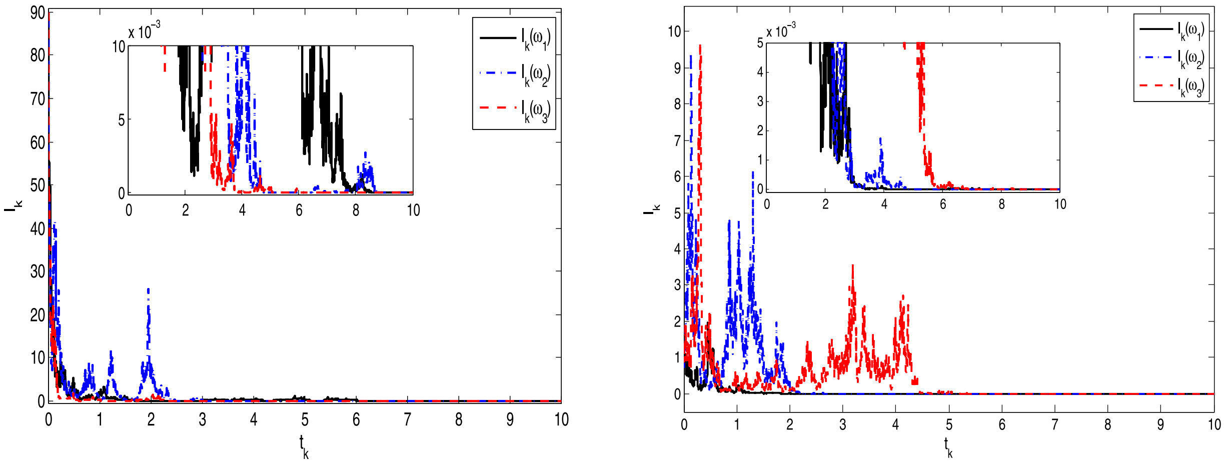

Fig. 1. The sample paths of the numerical solutions to model (3.1) by the logarithmic EM scheme with $\Delta = 0.001$ and $t \in [0, 10]$. Left: $I_0 = 90$. Right: $I_0 = 1$.

for any $q \geq 2$ and $l = 0, \ldots, \lfloor T/\Delta \rfloor$. This completes now the proof of the assertion for $q \geq 2$. The case $q \in (0, 2)$ follows now by Lyapunov's inequality. □

Theorem 2.2. *For any $p > 0$ there exists a constant $C > 0$ such that for any $\Delta \in (0, 1]$*

$$\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|I(t_k) - I_k\right|^p\right] \leq C\Delta^p, \quad \forall\, T > 0.$$

Proof. By the Lamperti-transformation and (1.5), we have

$$\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|I(t_k) - I_k\right|^p\right] = N\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|\frac{\mathrm{e}^{y(t_k)}}{1+\mathrm{e}^{y(t_k)}} - \frac{\mathrm{e}^{X_k}}{1+\mathrm{e}^{X_k}}\right|^p\right] \leq N\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|y(t_k) - X_k\right|^p\right].$$

Thus, by applying Theorem 2.1, we infer that

$$\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|I(t_k) - I_k\right|^p\right] \leq N\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor T/\Delta\rfloor} \left|y(t_k) - X_k\right|^p\right] \leq C\Delta^p \tag{2.12}$$

for any $\Delta \in (0, 1]$. The proof is complete. □

3. Numerical examples

In order to illustrate the efficiency of the logarithmic Euler–Maruyama scheme, we introduce an example and some simulations.

Example 3.1. Consider the stochastic SIS epidemic model (see, e.g., [1, Example 4.2])

$$\mathrm{d}I(t) = \left[5I(t) - 0.5I(t)^2\right]\mathrm{d}t + 0.035I(t)\big(100 - I(t)\big)\mathrm{d}B(t). \tag{3.1}$$

Under different initial values ($I(0) = 90$ and $I(0) = 1$, respectively), Fig. 1 plots the sample paths of the numerical solutions of the logarithmic EM scheme (1.5). It is clear to see from Fig. 1 that the logarithmic EM scheme reproduces the dynamic properties of the underlying SDE (3.1). Moreover, due to no closed-form of the solution of (3.1), we regard the scheme (1.5) with small step size $\Delta = 2^{-19}$ as the replacement

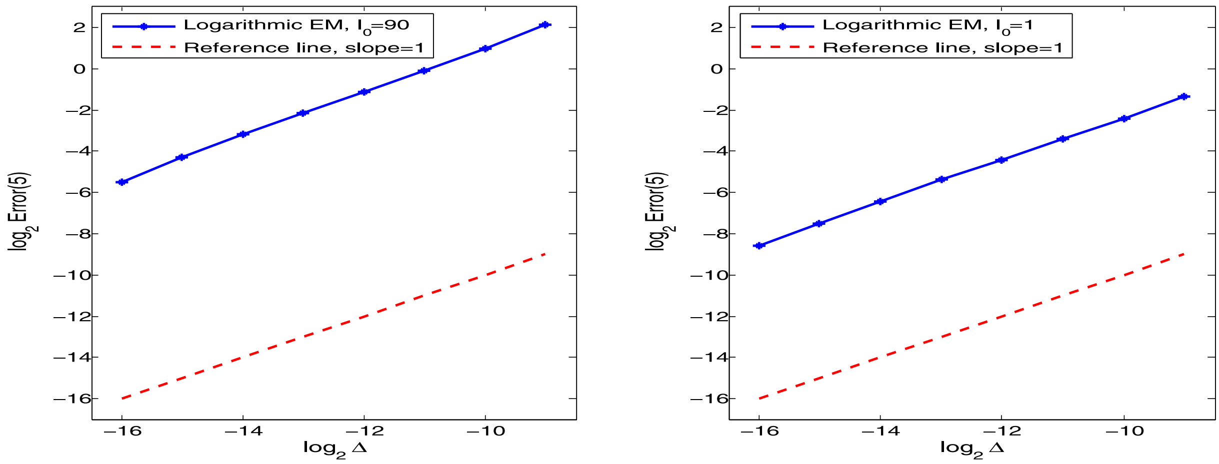

Fig. 2. The approximation error of the exact solution and the numerical solution by the logarithmic EM scheme (1.5) as the function of step size $\Delta \in \{2^{-9}, 2^{-10}, \ldots, 2^{-16}\}$, with initial values (left) $I_0 = 90$ and (right) $I_0 = 1$.

of the exact solution $I(t)$ of (3.1). Fig. 2 shows the $\log_2$-$\log_2$ approximation error Error(5) between the exact solution $I(t)$ and the numerical solution I_k with different step sizes $\Delta = 2^{-9}, 2^{-10}, \ldots, 2^{-16}$ for 1000 simulations, where the blue solid line depicts $\log_2$-$\log_2$ error while the red dashed is a reference line of slope 1, and $\mathrm{Error}(5) := \left(\mathbb{E}\left[\sup_{k=0,\ldots,\lfloor 2/\Delta \rfloor} |I(t_k) - I_k|^5\right]\right)^{1/5} \approx \left(\frac{1}{1000}\sum_{j=1}^{1000}\left[\sup_{k=0,\ldots,\lfloor 2/\Delta \rfloor} |I^{(j)}(t_k) - I_k^{(j)}|^5\right]\right)^{\frac{1}{5}}$, where j stands for the jth sample path.

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