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LIMIT THEOREMS FOR CONTINUOUS-TIME RANDOM WALKS WITH INFINITE MEAN WAITING TIMES

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Abstract

A continuous-time random walk is a simple random walk subordinated to a renewal process used in physics to model anomalous diffusion. In this paper we show that, when the time between renewals has infinite mean, the scaling limit is an operator Lévy motion subordinated to the hitting time process of a classical stable subordinator. Density functions for the limit process solve a fractional Cauchy problem, the generalization of a fractional partial differential equation for Hamiltonian chaos. We also establish a functional limit theorem for random walks with jumps in the strict generalized domain of attraction of a full operator stable law, which is of some independent interest.

Keywords: Operator self-similar process; continuous-time random walk

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1. Introduction

Continuous-time random walks (CTRWs) were introduced in [30] to study random walks on a lattice. They are now used in physics to model a wide variety of phenomena connected with anomalous diffusion [18], [29], [37], [42]. A CTRW is a random walk subordinated to a renewal process. The random-walk increments represent the magnitude of particle jumps, and the renewal epochs represent the times of the particle jumps. If the time between renewals has finite mean, the renewal process is asymptotically equivalent to a constant multiple of the time variable, and the CTRW behaves like the original random walk for large time [2], [19]. In many physical applications, the waiting time between renewals has infinite mean [39]. In this paper, we derive the scaling limit of a CTRW with infinite mean waiting time. The limit process is an operator Lévy motion subordinated to the hitting time process of a classical stable subordinator. The limit process is operator self-similar; however, it is not a Gaussian or operator stable process, and it does not have stationary increments. Kotulski [19] and Saichev and Zaslavsky [33] computed the limit distribution for scalar CTRW models at one fixed point in time. In this paper, we derive the entire stochastic process limit in the space $\mathrm{D}([0, \infty), \mathbb{R}^d)$ and we elucidate the nature of the limit process as a subordinated operator Lévy motion. We also establish a functional limit theorem for random walks with jumps in the strict generalized domain of attraction of a full operator stable law (Theorem 4.1) which is of some independent interest.

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Zaslavsky [45] proposed a fractional kinetic equation for Hamiltonian chaos, which Saichev and Zaslavsky [33] solved in the special case of symmetric jumps on $\mathbb{R}^1$. This fractional partial differential equation defines a fractional Cauchy problem [1] on $\mathbb{R}^1$. In this paper, we show that the distribution of the CTRW scaling limit has a Lebesgue density which solves a fractional Cauchy problem on $\mathbb{R}^d$. This provides solutions to the scalar fractional kinetic equation with asymmetric jumps, as well as the vector case. Operator scaling leads to a limit process with more realistic scaling properties, which is important in physics, since a cloud of diffusing particles may spread at a different rate in each coordinate [28].

2. Continuous-time random walks

Let $J_1, J_2, \ldots$ be nonnegative independent and identically distributed (i.i.d.) random variables that model the waiting times between jumps of a particle. We set $T(0) = 0$ and $T(n) = \sum_{j=1}^n J_j$, the time of the nth jump. The particle jumps are given by i.i.d. random vectors $Y_1, Y_2, \ldots$ on $\mathbb{R}^d$ which are assumed independent of (J_i). Let $S_0 = 0$ and $S_n = \sum_{i=1}^n Y_i$, the position of the particle after the nth jump. For $t \geq 0$, let

$$N_t = \max\{n \geq 0 : T(n) \leq t\}, \tag{2.1}$$

$[ED]^{-1} = \int_{-\infty}^{\infty} \frac{g_\beta(x)}{x}\,dx$

the number of jumps up to time t, and define

$$X(t) = S_{N_t} = \sum_{i=1}^{N_t} Y_i, \tag{2.2}$$

故 $ED = \int_{-\infty}^{\infty} x\, g_\beta(x)\,dx$

the position of a particle at time t. The stochastic process $\{X(t)\}_{t\geq 0}$ is called a *continuous-time random walk* (CTRW).

Assume that J_1 belongs to the strict domain of attraction of some stable law with index $\beta \in (0, 1)$. This means that there exist $b_n > 0$ such that

$$b_n(J_1 + \cdots + J_n) \Rightarrow D, \tag{2.3}$$

随机变量D的分布ρ 有勒贝格密度 g_β

where $D > 0$ almost surely. Here $\Rightarrow$ denotes convergence in distribution. The distribution ρ of D is stable with index β, meaning that $\rho^t = t^{1/\beta}\rho$ for all $t > 0$, where ρ^t is the tth convolution power of the infinitely divisible law ρ and $(a\rho)\{dx\} = \rho\{a^{-1}\,dx\}$ is the probability distribution of aD for $a > 0$. Moreover, ρ has a Lebesgue density g_β which is a C^∞ function. Note that by Theorem 4.7.1 and (4.7.13) of [42] it follows that there exists a constant $K > 0$ such that

$$g_\beta(x) \leq K x^{(1-\beta/2)/(\beta-1)} \exp\left\{-|1-\beta|\left(\frac{x}{\beta}\right)^{\beta/(\beta-1)}\right\} \tag{2.4}$$

for all $x > 0$ sufficiently small.

For $t \geq 0$, let $T(t) = \sum_{j=1}^{\lfloor t \rfloor} J_j$ and let $b(t) = b_{\lfloor t \rfloor}$, where $\lfloor t \rfloor$ denotes the integer part of t. Then $b(t) = t^{-1/\beta}L(t)$ for some slowly varying function $L(\cdot)$ (so that $L(\lambda t)/L(t) \to 1$ as $t \to \infty$ for any $\lambda > 0$, see for example [13]) and it follows from Example 11.2.18 of [26] that

$$\{b(c)T(ct)\}_{t\geq 0} \stackrel{\text{FD}}{\Longrightarrow} \{D(t)\}_{t\geq 0} \quad \text{as } c \to \infty, \tag{2.5}$$

where $\stackrel{\text{FD}}{\Longrightarrow}$ denotes convergence in distribution of all finite-dimensional marginal distributions. The process $\{D(t)\}$ has stationary independent increments and, since the distribution ρ of $D(1)$ is strictly stable, $\{D(t)\}$ is called a strictly stable Lévy process. Moreover,

$$\{D(ct)\}_{t\geq 0} \stackrel{\text{FD}}{=} \{c^{1/\beta}D(t)\}_{t\geq 0} \tag{2.6}$$

for all $c > 0$, where $\stackrel{\text{FD}}{=}$ denotes equality of all finite-dimensional marginal distributions. Hence, by Definition 13.4 of [36], the process $\{D(t)\}$ is self-similar with exponent $H = 1/\beta > 1$. See [36] for more details on stable Lévy processes and self-similarity. Also see [12] for a nice overview of self-similarity in the one-dimensional case. Note that by Example 21.7 of [36] the sample paths of $\{D(t)\}$ are almost surely increasing. Moreover, since $D(t) \stackrel{\text{D}}{=} t^{1/\beta} D$, where $\stackrel{\text{D}}{=}$ means equal in distribution, it follows that

$$D(t) \to \infty \quad \text{in probability as } t \to \infty.$$

Then it follows from Theorem I.19 of [6] that $D(t) \to \infty$ almost surely as $t \to \infty$. Furthermore, note that since $b(c) \to 0$ as $c \to \infty$ it follows that

$$b(c)T(\lfloor cx \rfloor + k(c)) \Rightarrow D(x) \quad \text{as } c \to \infty$$

for any $x \geq 0$ as long as $|k(x)| \leq M$ for all $c > 0$ and some constant M. Hence, it follows along the same lines as Example 11.2.18 of [26] that

$$\{b(c)T(\lfloor cx \rfloor + k(c))\}_{x \geq 0} \stackrel{\text{FD}}{\Longrightarrow} \{D(x)\}_{x \geq 0} \quad \text{as } c \to \infty \tag{2.7}$$

as long as $|k(c)| \leq M$ for all $c > 0$ and some constant M. Furthermore, since (J_j) are i.i.d., it follows that the process $\{T(k) : k = 0, 1, \ldots\}$ has stationary increments, that is, for any nonnegative integer ℓ we have

$$\{T(k + \ell) - T(\ell) : k = 0, 1, \ldots\} \stackrel{\text{FD}}{=} \{T(k) : k = 0, 1, \ldots\}.$$

Assume that (Y_i) are i.i.d. $\mathbb{R}^d$-valued random variables independent of (J_i) and assume that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law ν, where 'full' means that ν is not supported on any proper hyperplane of $\mathbb{R}^d$. By Theorem 8.1.5 of [26] there exists a function $B \in \mathrm{RV}(-E)$ (that is, $B(c)$ is invertible for all $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-E}$ as $c \to \infty$ for any $\lambda > 0$), E being a $d \times d$ matrix with real entries, such that

$$B(n) \sum_{i=1}^{n} Y_i \Rightarrow A \quad \text{as } n \to \infty, \tag{2.8}$$

where A has distribution ν. Then $\nu^t = t^E \nu$ for all $t > 0$, where $T\nu\{\mathrm{d}x\} = \nu\{T^{-1}\,\mathrm{d}x\}$ is the probability distribution of TA for any Borel measurable function $T : \mathbb{R}^d \to \mathbb{R}^m$. Note that by Theorem 7.2.1 of [26] the real parts of the eigenvalues of E are greater than or equal to $\frac{1}{2}$.

Moreover, if we define the stochastic process $\{S(t)\}_{t \geq 0}$ by $S(t) = \sum_{i=1}^{\lfloor t \rfloor} Y_i$, it follows from Example 11.2.18 of [26] that

$$\{B(c)S(ct)\}_{t \geq 0} \stackrel{\text{FD}}{\Longrightarrow} \{A(t)\}_{t \geq 0} \quad \text{as } c \to \infty, \tag{2.9}$$

where $\{A(t)\}$ has stationary independent increments with $A(0) = 0$ almost surely and $P_{A(t)} = \nu^t = t^E \nu$ for all $t > 0$; P_X denoting the distribution of X. Then $\{A(t)\}$ is continuous in law, and it follows that

$$\{A(ct)\}_{t \geq 0} \stackrel{\text{FD}}{=} \{c^E A(t)\}_{t \geq 0},$$

so, by Definition 11.1.2 of [26], $\{A(t)\}$ is operator self-similar with exponent E. The stochastic process $\{A(t)\}$ is called an *operator Lévy motion*. If the exponent $E = aI$ is a constant multiple of the identity, then ν is a stable law with index $\alpha = 1/a$, and $\{A(t)\}$ is a classical

d-dimensional Lévy motion. In the special case $a = \frac{1}{2}$, the process $\{A(t)\}$ is a d-dimensional Brownian motion.

Since we are interested in convergence of stochastic processes in Skorokhod spaces, we need some further notation. If S is a complete separable metric space, let $\mathrm{D}([0, \infty), S)$ denote the space of all right-continuous S-valued functions on $[0, \infty)$ with limits from the left. Note that in view of [6, p. 197], we can assume without loss of generality that sample paths of the processes $\{T(t)\}$ and $\{D(t)\}$ belong to $\mathrm{D}([0, \infty), [0, \infty))$, and that sample paths of $\{S(t)\}$ and $\{A(t)\}$ belong to $\mathrm{D}([0, \infty), \mathbb{R}^d)$.

3. The time process

In this section we investigate the limiting behaviour of the counting process $\{N_t\}_{t\geq 0}$ defined by (2.1). It turns out that the scaling limit of this process is the hitting-time process for the Lévy motion $\{D(x)\}_{x\geq 0}$. This hitting-time process is also self-similar with exponent β. We will use these results in Section 4 to derive limit theorems for the CTRW.

Recall that all the sample paths of the Lévy motion $\{D(x)\}_{x\geq 0}$ are continuous from the right, with left-hand limits, strictly increasing, and that $D(0) = 0$ and $D(x) \to \infty$ as $x \to \infty$. Now, the hitting-time process

$$E(t) = \inf\{x : D(x) > t\} \tag{3.1}$$

is well-defined. If $D(x) \geq t$, then $D(y) > t$ for all $y > x$ so that $E(t) \leq x$. On the other hand, if $D(x) < t$, then $D(y) < t$ for all $y > x$ sufficiently close to x, so that $E(t) > x$. Then it follows easily that, for any $t_1, \dots, t_m$ such that $0 \leq t_1 < \cdots < t_m$ and any $x_1, \dots, x_m \geq 0$, we have

$$\{E(t_i) \leq x_i \text{ for } i = 1, \dots, m\} = \{D(x_i) \geq t_i \text{ for } i = 1, \dots, m\}. \tag{3.2}$$

Proposition 3.1. *The process $\{E(t)\}_{t\geq 0}$ defined by (3.1) is self-similar with exponent $\beta \in (0, 1)$, that is, for any $c > 0$,*

称E(t)是β自相似的.

$$\{E(ct)\}_{t\geq 0} \overset{\mathrm{FD}}{=} \{c^\beta E(t)\}_{t\geq 0}.$$

Proof. Note that it follows directly from (3.1) that $\{E(t)\}_{t\geq 0}$ has continuous sample paths and hence is continuous in probability and so in law. Now fix any $t_1, \dots, t_m$ such that $0 < t_1 < \cdots < t_m$ and any $x_1, \dots, x_m \geq 0$. Then by (2.6) and (3.2) we obtain that

$$\begin{aligned}
\mathrm{P}\{E(ct_i) \leq x_i \text{ for } i = 1, \dots, m\} &= \mathrm{P}\{D(x_i) \geq ct_i \text{ for } i = 1, \dots, m\} \\
&= \mathrm{P}\{(c^{-\beta})^{1/\beta} D(x_i) \geq t_i \text{ for } i = 1, \dots, m\} \\
&= \mathrm{P}\{D(c^{-\beta} x_i) \geq t_i \text{ for } i = 1, \dots, m\} \\
&= \mathrm{P}\{E(t_i) \leq c^{-\beta} x_i \text{ for } i = 1, \dots, m\} \\
&= \mathrm{P}\{c^\beta E(t_i) \leq x_i \text{ for } i = 1, \dots, m\}.
\end{aligned}$$

Hence by Definition 11.1.2 of [26] the assertion follows.

We now collect some further properties of the process $\{E(t)\}_{t\geq 0}$.

For a real-valued random variable X, let $\mathrm{E}(X)$ denote its expectation, whenever it exists, and $\mathrm{var}(X)$ denote the variance of X, whenever it exists.

Corollary 3.1. *For any $t > 0$,*

(a) $E(t) \stackrel{\mathrm{D}}{=} (D/t)^{-\beta}$, *where D is as in (2.3);*

(b) *for any $\gamma > 0$ the γ-moment of $E(t)$ exists and there exists a positive finite constant $C(\beta, \gamma)$ such that*

$$\mathrm{E}(E(t)^{\gamma}) = C(\beta, \gamma) t^{\beta\gamma};$$

in particular,

$$\mathrm{E}(E(t)) = C(\beta, 1) t^{\beta}; \tag{3.3}$$

(c) *the random variable $E(t)$ has density*

密度函数 $$f_t(x) = \frac{t}{\beta} x^{-1-1/\beta} g_{\beta}(t x^{-1/\beta}),$$

where g_β is the density of the limit D in (2.3).

Proof. Note that $D(x) \stackrel{\mathrm{D}}{=} x^{1/\beta} D$. In view of (3.2), for any $x > 0$,

$$\mathrm{P}\{E(t) \le x\} = \mathrm{P}\{D(x) \ge t\} = \mathrm{P}\{x^{1/\beta} D \ge t\} = \mathrm{P}\{(D/t)^{-\beta} \le x\},$$

proving part (a).

For the proof of (b), let $H_t(y) = (y/t)^{-\beta}$. Since D has distribution ρ it follows from part (a) that $E(t)$ has distribution $H_t(\rho)$. Recall that ρ has a C^{∞} density g_β. For $\gamma > 0$ we then get

$$\begin{aligned}
\mathrm{E}(E(t)^{\gamma}) &= \int_0^{\infty} x^{\gamma} \,\mathrm{d}H_t(\rho)(x) \\
&= \int_0^{\infty} (H_t(x))^{\gamma} \,\mathrm{d}\rho(x) \\
&= t^{\beta\gamma} \int_0^{\infty} x^{-\beta\gamma} g_\beta(x) \,\mathrm{d}x \\
&= C(\beta, \gamma) t^{\beta\gamma},
\end{aligned}$$

$\mathbb{E}(E_t E_s) = \int_0^{\infty}\int_0^{\infty} x^{\gamma} y^{\gamma} \, dH_t(\rho)(x)\, dH_t(\rho)(y).$

$\begin{cases} D \text{ 分布 } \rho \\ E_t = (\frac{D}{t})^{-\beta} \end{cases} \longrightarrow E_t \stackrel{分}{=} (\frac{\rho}{t})^{-\beta} = H_t^{\rho}$

where $C(\beta, \gamma) = \int_0^{\infty} x^{-\beta\gamma} g_\beta(x) \,\mathrm{d}x$ is finite since g_β is a density function satisfying (2.4) for some $\beta \in (0, 1)$, so that $g_\beta(x) \to 0$ at an exponential rate as $x \to 0$.

Since $E(t) \stackrel{\mathrm{D}}{=} H_t(D)$ by (a) and $H_t^{-1}(x) = t x^{-1/\beta}$, (c) follows by a change of variables.

Corollary 3.2. *For any $t > 0$, the variance of $E(t)$ exists and* $\mathrm{var}(E(t)) = (C(\beta, 2) - C(\beta, 1)^2) t^{2\beta}$.

方差

Proof. It follows from Corollary 3.1(b) that $\mathrm{E}(E(t)^2)$ exists and the result follows.

Corollary 3.3. *The process $\{E(t)\}_{t \ge 0}$ does not have stationary increments.*

Proof. Suppose that $\{E(t)\}_{t \ge 0}$ is a process with stationary increments. Then, for any integer t,

$$\mathrm{E}(E(t)) = \mathrm{E}(E(1) + (E(2) - E(1)) + \cdots + (E(t) - E(t-1))) = t\,\mathrm{E}(E(1)),$$

which contradicts (3.3).

无平稳增量

无独立增量

Theorem 3.1. *The process* $\{E(t)\}_{t\geq 0}$ *does not have independent increments.*

Proof. Assume the contrary. The process $\{E(t)\}$ is the inverse of the stable subordinator $\{D(x)\}$, in the sense of Bingham [8]. Then Proposition 1(a) of [8] implies that, for any t_1, t_2 with $0 < t_1 < t_2$,

$$\frac{\partial^2 \operatorname{E}(E(t_1)E(t_2))}{\partial t_1 \partial t_2} = \frac{1}{\Gamma(\beta)^2[t_1(t_2 - t_1)]^{1-\beta}}. \tag{3.4}$$

Moreover, by Corollary 3.1 we know that, for some positive constant C,

$$\operatorname{E}(E(t)) = Ct^\beta \tag{3.5}$$

Since $E(t)$ has moments of all orders, it follows that, for t_1, t_2, t_3 such that $0 < t_1 < t_2 < t_3$, by independence of the increments and (3.5),

$$\begin{aligned} \operatorname{E}((E(t_3) - E(t_2)) \cdot (E(t_2) - E(t_1))) &= \operatorname{E}(E(t_3) - E(t_2)) \cdot \operatorname{E}(E(t_2) - E(t_1)) \\ &= C^2\{(t_2t_3)^\beta - (t_1t_3)^\beta - t_2^{2\beta} + (t_1t_2)^\beta\} \\ &=: R(t_1, t_2, t_3). \end{aligned}$$

On the other hand,

$$\begin{aligned} \operatorname{E}((E(t_3) - E(t_2)) \cdot (E(t_2) - E(t_1))) &= \operatorname{E}(E(t_2)E(t_3)) - \operatorname{E}(E(t_1)E(t_3)) \\ &\quad - \operatorname{E}(E(t_2)^2) + \operatorname{E}(E(t_1)E(t_2)) \\ &=: L(t_1, t_2, t_3), \end{aligned}$$

so that $R(t_1, t_2, t_3) = L(t_1, t_2, t_3)$ whenever $0 < t_1 < t_2 < t_3$.

Computing the derivatives of R directly and applying (3.4) to L gives

$$\begin{aligned} \frac{\partial^2 R(t_1, t_2, t_3)}{\partial t_1 \partial t_2} &= C^2\beta^2(t_1t_2)^{\beta-1}, \\ \frac{\partial^2 L(t_1, t_2, t_3)}{\partial t_1 \partial t_2} &= \Gamma(\beta)^{-2}(t_1t_2)^{\beta-1}\left\{1 - \frac{t_1}{t_2}\right\}^{\beta-1} \end{aligned}$$

whenever $0 < t_1 < t_2 < t_3$. Since the left-hand sides of the above equations are equal, so are the right-hand sides, which gives a contradiction.

Recall from Section 2 that the function b in (2.5) is regularly varying with index $-1/\beta$. Hence b^{-1} is regularly varying with index $1/\beta > 0$ so, by Property 1.5.5 of [38], there exists a regularly varying function $\tilde{b}$ with index β such that $1/b(\tilde{b}(c)) \sim c$ as $c \to \infty$. (Here we use the notation $f \sim g$ for positive functions f, g if and only if $f(c)/g(c) \to 1$ as $c \to \infty$.) Equivalently,

$$b(\tilde{b}(c)) \sim \frac{1}{c} \quad \text{as } c \to \infty. \tag{3.6}$$

Furthermore, note that for (2.1) it follows easily that, for any integer $n \geq 0$ and any $t \geq 0$,

$$\{T(n) \leq t\} = \{N_t \geq n\}. \tag{3.7}$$

With the use of the function $\tilde{b}$ defined above, we now prove a limit theorem for $\{N_t\}_{t\geq 0}$.

Theorem 3.2. *As $c \to \infty$,*

$$\{\tilde{b}(c)^{-1} N_{ct}\} \xRightarrow{\text{FD}} \{E(t)\}_{t \geq 0}.$$

Proof. Fix any $t_1, \ldots, t_m$ such that $0 < t_1 < \cdots < t_m$ and any $x_1, \ldots, x_m \geq 0$. Note that by (3.7) we have

$$\{N_t \geq x\} = \{T(\lceil x \rceil) \leq t\},$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. This is equivalent to $\{N_t < x\} = \{T(\lceil x \rceil) > t\}$, and then (2.7) together with (3.2) and (3.6) imply

$$\begin{aligned}
&\mathrm{P}\{\tilde{b}(c)^{-1} N_{ct_i} < x_i \text{ for } i = 1, \ldots, m\} \\
&\quad = \mathrm{P}\{N_{ct_i} < \tilde{b}(c) x_i \text{ for } i = 1, \ldots, m\} \\
&\quad = \mathrm{P}\{T(\lceil \tilde{b}(c) x_i \rceil) > ct_i \text{ for } i = 1, \ldots, m\} \\
&\quad = \mathrm{P}\{b(\tilde{b}(c)) T(\lceil \tilde{b}(c) x_i \rceil) > b(\tilde{b}(c)) ct_i \text{ for } i = 1, \ldots, m\} \\
&\quad \to \mathrm{P}\{D(x_i) > t_i \text{ for } i = 1, \ldots, m\} = \mathrm{P}\{D(x_i) \geq t_i \text{ for } i = 1, \ldots, m\} \\
&\quad = \mathrm{P}\{E(t_i) \leq x_i \text{ for } i = 1, \ldots, m\} = \mathrm{P}\{E(t_i) < x_i \text{ for } i = 1, \ldots, m\}
\end{aligned}$$

as $c \to \infty$ since both $E(t)$ and $D(x)$ have a density. Hence,

$$(\tilde{b}(c)^{-1} N_{ct_i} : i = 1, \ldots, m) \Rightarrow (E(t_i) : i = 1, \ldots, m)$$

and the proof is complete.

As a corollary we get convergence in the Skorokhod space $\mathrm{D}([0, \infty), [0, \infty))$ with the J_1-topology.

Corollary 3.4. *As $c \to \infty$,*

$$\{\tilde{b}(c)^{-1} N_{ct}\}_{t \geq 0} \Rightarrow \{E(t)\}_{t \geq 0} \quad \textit{in } \mathrm{D}([0, \infty), [0, \infty)).$$

Proof. Note that the sample paths of $\{N_t\}_{t \geq 0}$ and $\{E(t)\}_{t \geq 0}$ are increasing and that by the proof of Proposition 3.1 the process $\{E(t)\}_{t \geq 0}$ is continuous in probability. Then Theorem 3.2 together with Theorem 3 of [8] yields the assertion.

Remark 3.1. The hitting time $E(t) = \inf\{x : D(x) > t\}$ is also called a *first-passage time*. A general result of Port [32] implies that $\mathrm{P}\{E(t) \geq x\} = o(x^{1-1/\beta})$ as $x \to \infty$, but in view of Corollary 3.1 that tail bound can be considerably improved in this special case. Getoor [14] computed the first and second moments of the hitting time for a symmetric stable process, but the moment results of Corollaries 3.1 and 3.2 are apparently new. The hitting-time process $\{E(t)\}$ is also an inverse process to the stable subordinator $\{D(t)\}$ in the sense of Bingham [8]. Bingham [8] and Bondesson *et al.* [10] showed that $E(t)$ has a Mittag–Leffler distribution with

$$\mathrm{E}(\mathrm{e}^{-sE(t)}) = \sum_{n=0}^{\infty} \frac{(-st^{\beta})^n}{\Gamma(1 + n\beta)},$$

which gives another proof of Corollary 3.1(b) in the special case where γ is a positive integer. The process $\{E(t)\}$ is also a *local time* for the Markov process $R(t) = \inf\{D(x) - t : D(x) > t\}$; see [6, Exercise 6.2]. This means that the jumps of the *inverse local time* $\{D(x)\}$ for the Markov process $\{R(t)\}_{t \geq 0}$ coincide with the lengths of the excursion intervals during which $R(t) > 0$. Since $R(t) = 0$ when $D(x) = t$, the lengths of the excursion intervals for $\{R(t)\}$ equal the size of the jumps in the process $\{D(x)\}$, and $E(t)$ represents the time it takes until the sum of the jump sizes (the sum of the lengths of the excursion intervals) exceeds t.

4. Limit theorems

In this section we prove a functional limit theorem for the CTRW $\{X(t)\}$ defined in (2.2) under the distributional assumptions of Section 2. The limiting process $\{M(t)\}_{t\geq 0}$ is a subordination of the operator stable Lévy process $\{A(t)\}$ in (2.9) by the process $\{E(t)\}$ introduced in Section 3. We also show that $\{M(t)\}$ is operator self-similar with exponent βE, where β is the index of the stable subordinator $\{D(t)\}$ and E is the exponent of the operator stable Lévy motion $\{A(t)\}$. Then we compute the Lebesgue density of the limit $M(t)$. In Section 5 we will use this density formula to show that $\{M(t)\}$ is the stochastic solution to a fractional Cauchy problem.

Our method of proof uses the continuous mapping theorem. In order to do so, we first need to prove $\mathrm{D}([0,\infty),\mathbb{R}^d)$-convergence in (2.9), which is apparently not available in the literature. The following result, which is of independent interest, closes this gap.

Theorem 4.1. *Under the assumptions of Section 2,*

$$\{B(n)S(nt)\}_{t\geq 0} \Rightarrow \{A(t)\}_{t\geq 0} \quad \textit{in } \mathrm{D}([0,\infty),\mathbb{R}^d)$$

as $n\to\infty$ in the J_1-topology.

Proof. In view of (2.9) and an easy extension of Theorem 5 of [15, p. 435], using the theorem in [41] in place of Theorem 2 of [15, p. 429], it suffices to check that

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\sup_{|s-t|\leq h} \mathrm{P}\{\|\xi_n(t)-\xi_n(s)\|>\varepsilon\}=0 \tag{4.1}$$

for any $\varepsilon>0$ and $T>0$, where $0\leq s,t\leq T$ and

$$\xi_n(t)=B(n)\sum_{i=1}^{\lfloor nt\rfloor} X_i.$$

Recall that $S(j)=X_1+\cdots+X_j$. Given any $\varepsilon>0$ and $\delta>0$, using the fact that $\{B(j)S(j)\}$ is uniformly tight, there exists an $R>0$ such that

$$\sup_{j\geq 1}\mathrm{P}\{\|B(j)S(j)\|>R\}<\delta. \tag{4.2}$$

In view of the symmetry in s,t in (4.1) we can assume that $0\leq s<t\leq T$ and $t-s\leq h$. Let

$$r_n(t,s)=\|B(n)B(\lfloor nt\rfloor-\lfloor ns\rfloor)^{-1}\|$$

and note that

$$\begin{aligned} r_n(t,s)&=\|B(n)B(n\cdot(\lfloor nt\rfloor-\lfloor ns\rfloor)/n)^{-1}\|\\ &\leq \sup_{0\leq\lambda\leq h+1/n}\|B(n)B(n\lambda)^{-1}\|\\ &=\|B(n)B(n\lambda_n)^{-1}\|\end{aligned}$$

for some λ_n such that $0\leq\lambda_n\leq h+1/n$.

Now choose $h_0>0$ such that $\|h^E\|<\varepsilon/2R$ whenever $0<h\leq h_0$ and assume in the following that $0<h\leq h_0$. Given any subsequence, there exists a further subsequence (n') such that $\lambda_{n'}\to\lambda\in[0,h]$ along (n'). We have to consider several cases separately.

Case 1. If $\lambda>0$, then, in view of the uniform convergence on compact sets,

$$\|B(n)B(n\lambda_n)^{-1}\|\to\|\lambda^E\|$$

and, hence, there exists an n_0 such that $r_{n'}(t,s)<\varepsilon/R$ for all $n'>n_0$.

Case 2. If $\lambda = 0$ and $n'\lambda_{n'} \le M$ for all n' in the subsequence (n'), then, since $B(n) \to 0$ as $n \to \infty$, there exists an n_0 such that

$$\|B(n')B(n'\lambda_{n'})^{-1}\| \le \|B(n')\| \sup_{1\le j\le M} \|B(j)^{-1}\| < \frac{\varepsilon}{R}$$

for all $n' \ge n_0$ and, hence, $r_{n'}(t,s) < \varepsilon/R$ for all $n' \ge n_0$.

Case 3. If $\lambda = 0$ and $n'\lambda_{n'} \to \infty$ along (n'), let $m = n\lambda_n$ and $\lambda(m) = n/m$ so that $m \to \infty$ and $\lambda(m) \to \infty$. Choose $\lambda_0 > 1$ large enough to make $\|\lambda_0^{-E}\| < \frac{1}{4}$. Then choose m_0 large enough to make $\|B(\lambda m)B(m)^{-1} - \lambda^{-E}\| < \frac{1}{4}$ for all $m \ge m_0$ and all $\lambda \in [1, \lambda_0]$. Now $\|B(m\lambda_0)B(m)^{-1}\| < \frac{1}{2}$ for all $m \ge m_0$, and since $\|\lambda^{-E}\|$ is a continuous function of $\lambda > 0$, this ensures that for some $C > 0$ we have $\|B(\lambda m)B(m)^{-1}\| \le C$ for all $m \ge m_0$ and all $\lambda \in [1, \lambda_0]$. Given $m \ge m_0$, write $\lambda(m) = \mu\lambda_0^k$ for some integer k and some $\mu \in [1, \lambda_0)$. Then

$$\begin{aligned}\|B(n)B(n\lambda_n)^{-1}\| &= \|B(m\lambda(m))B(m)^{-1}\| \\ &\le \|B(\mu\lambda_0^k m)B(\lambda_0^k m)^{-1}\| \cdots \|B(\lambda_0 m)B(m)^{-1}\| \\ &\le C(\tfrac{1}{2})^k\end{aligned}$$

and, since $\lambda(m) \to \infty$, this shows that $\|B(n)B(n\lambda_n)^{-1}\| \to 0$ as $n \to \infty$ along (n'). Hence there exists an n_0 such that

$$\|B(n')B(n'\lambda_n)^{-1}\| < \frac{\varepsilon}{R}$$

for all $n' \ge n_0$ and, hence, $r_{n'}(t,s) < \varepsilon/R$ for all $n' \ge n_0$.

Now it follows easily that there exist an $h_0 > 0$ and an n_0 such that

$$r_n(t,s) < \frac{\varepsilon}{R} \tag{4.3}$$

for all $n \ge n_0$ and s, t such that $|t-s| \le h < h_0$. Finally, in view of (4.2) and (4.3), we obtain that, for $n \ge n_0$ and s, t such that $|t-s| \le h < h_0$,

$$\begin{aligned}\mathrm{P}\{\|\xi_n(t) - \xi_n(s)\| > \varepsilon\} &= \mathrm{P}\{\|B(n)S(\lfloor nt\rfloor - \lfloor ns\rfloor)\| > \varepsilon\} \\ &\le \mathrm{P}\{r_n(t,s)\|B(\lfloor nt\rfloor - \lfloor ns\rfloor)S(\lfloor nt\rfloor - \lfloor ns\rfloor)\| > \varepsilon\} \\ &\le \sup_{j\ge 1} \mathrm{P}\{\|B_j S(j)\| > R\} < \delta,\end{aligned}$$

which proves (4.1).

Recall from the comments preceding (3.6) that $\tilde{b}$ is regularly varying with index β and that the norming function B in (2.9) is $\mathrm{RV}(-E)$. We define $\tilde{B}(c) = B(\tilde{b}(c))$. Then $\tilde{B} \in \mathrm{RV}(-\beta E)$.

Theorem 4.2. *Under the assumptions of Section 2, as $c \to \infty$,*

$$\{\tilde{B}(c)X(ct)\}_{t\ge 0} \Rightarrow \{M(t)\}_{t\ge 0} \quad \textit{in } \mathrm{D}([0,\infty), \mathbb{R}^d) \tag{4.4}$$

in the M_1-topology, where $\{M(t)\}_{t\ge 0} = \{A(E(t))\}_{t\ge 0}$ is a subordinated process with

$$P_{(M(t_1),\ldots,M(t_n))} = \int_{\mathbb{R}_+^m} P_{(A(x_i):1\le i\le m)}\, \mathrm{d}P_{(E(t_i):1\le i\le m)}(x_1,\ldots,x_m) \tag{4.5}$$

whenever $0 < t_1 < \cdots < t_m$.

Proof. Note that, since (J_i) and (Y_i) are independent, the processes $\{S(t)\}$ and $\{N_t\}$ are independent; hence, it follows from Corollary 3.4 together with Theorem 4.1 that we also have

$$\{(\tilde{B}(c)S(\tilde{b}(c)t), \tilde{b}(c)^{-1}N_{ct})\}_{t\geq 0} \Rightarrow \{(A(t), E(t))\}_{t\geq 0} \quad \text{as } c \to \infty$$

in $\mathrm{D}([0,\infty),\mathbb{R}^d) \times \mathrm{D}([0,\infty),[0,\infty))$ in the J_1-topology and, hence, also in the weaker M_1-topology. Since the process $\{E(t)\}$ is not strictly increasing, Theorem 3.1 of [44] does not apply, so we cannot prove convergence in the J_1-topology. Instead we use Theorem 13.2.4 of [43] which applies as long as $x = E(t)$ is (almost surely) strictly increasing whenever $A(x) \neq A(x-)$. This condition is easily shown to be equivalent to the statement that the independent Lévy processes $\{A(x)\}$ and $\{D(x)\}$ have (almost surely) no simultaneous jumps, which is easy to check. Then the continuous mapping theorem (see e.g. [7, Theorems 5.1 and 5.5]) together with Theorem 13.2.4 of [43] yields that

$$\{\tilde{B}(c)S(N_{ct})\}_{t\geq 0} \Rightarrow \{A(E(t))\}_{t\geq 0} \quad \text{in } \mathrm{D}([0,\infty),\mathbb{R}^d) \text{ as } c \to \infty$$

in the M_1-topology, which proves (4.4). Then (4.5) follows easily, since $\{A(t)\}$ and $\{E(t)\}$ are independent.

Corollary 4.1. *The limiting process $\{M(t)\}_{t\geq 0}$ obtained in Theorem 4.2 is operator self-similar with exponent βE, that is, for all $c > 0$,*

$$\{M(ct)\}_{t\geq 0} \overset{\mathrm{FD}}{=} \{c^{\beta E} M(t)\}_{t\geq 0}.$$

Proof. We first show that $\{M(t)\}_{t\geq 0}$ is continuous in law. Assume that $t_n \to t \geq 0$ and let f be any bounded continuous function on $\mathbb{R}^d$. Since $\{A(x)\}_{x\geq 0}$ is continuous in law, the function $x \mapsto \int f(y)\,\mathrm{d}P_{A(x)}(y)$ is continuous and bounded. Recall from Section 3 that $\{E(t)\}_{t\geq 0}$ is continuous in law and hence $E(t_n) \Rightarrow E(t)$ as $n \to \infty$. Then

$$\begin{aligned}\int f(y)\,\mathrm{d}P_{M(t_n)}(y) &= \int \left(\int f(y)\,\mathrm{d}P_{A(x)}(y) \right) \mathrm{d}P_{E(t_n)}(x) \\ &\to \int \left(\int f(y)\,\mathrm{d}P_{A(x)}(y) \right) \mathrm{d}P_{E(t)}(x) = \int f(y)\,\mathrm{d}P_{M(t)}(y)\end{aligned}$$

as $n \to \infty$, showing that $\{M(t)\}_{t\geq 0}$ is continuous in law. It follows from Theorem 4.2 that, for any $c > 0$,

$$\{\tilde{B}(s)X(s(ct))\}_{t\geq 0} \overset{\mathrm{FD}}{\Longrightarrow} \{M(ct)\}_{t\geq 0}$$

as $s \to \infty$ and, since $\tilde{B} \in \mathrm{RV}(-\beta E)$, we also get

$$\{\tilde{B}(s)X((sc)t)\}_{t\geq 0} = \{(\tilde{B}(s)\tilde{B}(sc)^{-1})\tilde{B}(sc)X((sc)t)\}_{t\geq 0} \overset{\mathrm{FD}}{\Longrightarrow} \{c^{\beta E} M(t)\}_{t\geq 0}$$

as $s \to \infty$. Hence,

$$\{M(ct)\}_{t\geq 0} \overset{\mathrm{FD}}{=} \{c^{\beta E} M(t)\}_{t\geq 0}$$

and the proof is complete.

Recall from [17, Theorem 4.10.2] that the distribution ν^t of $A(t)$ in (2.9) has a C^∞ density $p(x,t)$, so $\mathrm{d}\nu^t(x) = p(x,t)\,\mathrm{d}x$ and g_β is the density of the limit D in (2.3).

Corollary 4.2. *Let $\{M(t)\}_{t\geq 0}$ be the limiting process obtained in Theorem 4.2. Then, for any $t > 0$,*

$$P_{M(t)} = \int_0^\infty \nu^{(t/s)^\beta} g_\beta(s)\,\mathrm{d}s = \frac{t}{\beta}\int_0^\infty \nu^\xi g_\beta(t\xi^{-1/\beta})\xi^{-1/\beta-1}\,\mathrm{d}\xi. \tag{4.6}$$

Moreover, $P_{M(t)}$ has the density

$$h(x,t) = \int_0^\infty p(x,(t/s)^\beta) g_\beta(s)\,\mathrm{d}s = \frac{t}{\beta}\int_0^\infty p(x,\xi) g_\beta(t\xi^{-1/\beta})\xi^{-1/\beta-1}\,\mathrm{d}\xi. \tag{4.7}$$

Proof. Let $H_t(y) = (y/t)^{-\beta}$. Then by Corollary 3.1 we know that $E(t) \stackrel{\mathrm{D}}{=} H_t(D)$ and, since $P_{A(s)} = \nu^s$, we obtain that

$$\begin{aligned} P_{M(t)} &= \int_0^\infty P_{A(s)}\,\mathrm{d}P_{E(t)}(s) = \int_0^\infty \nu^s \mathrm{d}P_{H_t(D)}(s) \\ &= \int_0^\infty \nu^{H_t(s)}\,\mathrm{d}P_D(s) = \int_0^\infty \nu^{(t/s)^\beta} g_\beta(s)\,\mathrm{d}s. \end{aligned}$$

The second equality of (4.6) follows from a simple substitution. The assertion (4.7) on the density follows immediately.

Corollary 4.3. *The limiting process $\{M(t)\}_{t\geq 0}$ obtained in Theorem 4.2 does not have stationary increments.*

Proof. Suppose that $t > 0$ and $h > 0$. In view of (3.1) and (3.2),

$$\{E(t+h) = E(h)\} = \{E(t+h) \leq E(h)\} = \{D(E(h)) \geq t+h\}.$$

Thus,

$$\begin{aligned} \mathrm{P}\{M(t+h) - M(h) = 0\} &= \mathrm{P}\{A(E(t+h)) = A(E(h))\} \\ &\geq \mathrm{P}\{E(t+h) = E(h)\} = \mathrm{P}\{D(E(h)) \geq t+h\} \\ &> 0 \end{aligned}$$

for all $t > 0$ sufficiently small, since $D(E(h)) > h$ almost surely by Theorem III.4 of [6]. But $\mathrm{P}\{M(t) = 0\} = 0$ since $M(t)$ has a density, hence $M(t)$ and $M(t+h) - M(h)$ are not identically distributed.

Theorem 4.3. *Let $\{M(t)\}_{t\geq 0}$ be the limiting process obtained in Theorem 4.2. Then the distribution of $M(t)$ is not operator stable for any $t > 0$.*

Proof. The distribution ν of the limit A in (2.8) is infinitely divisible, hence its characteristic function is $\hat{\nu}(k) = \mathrm{e}^{-\psi(k)}$ for some continuous complex-valued function $\psi(\cdot)$; see for example [26, Theorem 3.1.2]. Since $|\hat{\nu}(k)| = |\mathrm{e}^{-\operatorname{Re}\psi(k)}| \leq 1$, we must have $F(k) = \operatorname{Re}\psi(k) \geq 0$ for all k. Since ν is operator stable, Corollary 7.1.2 of [26] implies that $|\hat{\nu}(k)| < 1$ for all $k \neq 0$, so in fact $F(k) > 0$ for all $k \neq 0$. Since $\nu^t = t^E\nu$, we also have $tF(k) = F(t^{E^*}k)$ for all $t > 0$ and $k \neq 0$, which implies that F is a regularly varying function in the sense of [26, Definition 5.1.2]. Then Theorems 5.3.14 and 5.3.18 of [26] imply that, for some positive real constants a, b_1, c_1, b_2, c_2,

$$c_1\|k\|^{b_1} \leq F(k) \leq c_2\|k\|^{b_2} \tag{4.8}$$

whenever $\|k\| \geq a$. In fact, we can take $b_1 = 1/a_p - \delta$ and $b_2 = 1/a_1 + \delta$, where $\delta > 0$ is arbitrary and the real parts of the eigenvalues of E are $a_1 < \cdots < a_p$. In view of (4.6), the random vector $M(t)$ has characteristic function

$$\varphi_t(k) = \int_0^\infty \mathrm{e}^{-(t/s)^\beta \psi(k)} g_\beta(s)\,\mathrm{d}s, \tag{4.9}$$

where g_β is the density of the limit D in (2.3). Using the well-known series expansion for this stable density [42, Equation (4.2.4)], it follows that, for some positive constants c_0, s_0,

$$g_\beta(s) \geq c_0 s^{-\beta-1} \tag{4.10}$$

when $s \geq s_0$. Take $r_0 = \max\{a, s_0^{\beta/b_2} t^{-\beta/b_2} c_2^{-1/b_2}\}$. Then, if $\|k\| \geq r_0$ and $s_1 = t c_2^{1/\beta} \|k\|^{b_2/\beta}$, (4.8) holds and, since $s_1 \geq s_0$, (4.10) also holds for all $s \geq s_1$. Then in view of (4.9) and the fact that $\mathrm{e}^{-u} \geq 1 - u$ for all real u it follows that

$$\begin{aligned}
\operatorname{Re} \varphi_t(k) &= \int_0^\infty \mathrm{e}^{-(t/s)^\beta \operatorname{Re}\psi(k)} g_\beta(s)\,\mathrm{d}s \\
&\geq \int_{s_1}^\infty \mathrm{e}^{-(t/s)^\beta F(k)} g_\beta(s)\,\mathrm{d}s \\
&\geq \int_{s_1}^\infty \left[1 - \left(\frac{t}{s}\right)^\beta F(k)\right] c_0 s^{-\beta-1}\,\mathrm{d}s \\
&\geq \int_{s_1}^\infty \left[1 - \left(\frac{t}{s}\right)^\beta c_2 \|k\|^{b_2}\right] c_0 s^{-\beta-1}\,\mathrm{d}s \\
&= \left(\frac{c_0}{\beta}\right) s_1^{-\beta} - \left(\frac{c_0}{2\beta}\right) t^\beta c_2 \|k\|^{b_2} s_1^{-2\beta} \\
&= \left(\frac{c_0}{\beta}\right) s_1^{-\beta} \left[1 - \frac{t^\beta c_2 \|k\|^{b_2} s_1^{-\beta}}{2}\right] \\
&= \left(\frac{c_0}{\beta}\right) t^{-\beta} c_2^{-1} \|k\|^{-b_2} [1 - \tfrac{1}{2}] \\
&= C \|k\|^{-b_2},
\end{aligned}$$

where $C > 0$ does not depend on the choice of k such that $\|k\| > r_0$. But if $M(t)$ is operator stable, then the same argument as before shows that $\operatorname{Re} \varphi_t(k) = \mathrm{e}^{-F(k)}$ for some F satisfying (4.8) whenever $\|k\|$ is large (for some positive real constants a, b_1, c_1, b_2, c_2), so that $\operatorname{Re} \varphi_t(k) \leq \mathrm{e}^{-c_1 \|k\|^{b_1}}$ whenever $\|k\|$ is large, which is a contradiction.

For a one-dimensional CTRW with infinite mean waiting time, Kotulski [19] derived the results of Theorem 4.2 and Corollary 4.2 at one fixed time $t > 0$. Our stochastic process results, concerning $\mathrm{D}([0, \infty), \mathbb{R}^d)$-convergence, seem to be new even in the one-dimensional case. Kotulski [19] also considered a coupled CTRW model in which the jump times and lengths are dependent. Coupled CTRW models occur in many physical applications [9], [18], [20], [39]. The authors are currently working to extend the results of this section to coupled models.

In the one-dimensional situation $d = 1$, Corollary 4.2 implies that $M(t) \stackrel{\mathrm{D}}{=} (t/D)^{\beta/\alpha} A$, where D is the limit in (2.3) and A is the limit in (2.8). If A is a nonnormal stable random variable with index $\alpha \in (0, 2)$, then $M(t)$ has a ν-stable distribution [22], i.e. a random mixture of stable laws. Corollary 3.1 shows that the mixing variable $(t/D)^{\beta/\alpha}$ has moments of all orders, and then Proposition 4.1 of [22] shows that, for some $D > 0$ and some $q \in [0, 1]$, $\mathrm{P}(M(t) > x) \sim qDx^{-\alpha}$ and $\mathrm{P}(M(t) < -x) \sim (1-q)Dx^{-\alpha}$ as $x \to \infty$. Proposition 5.1 of [22] shows that $\mathrm{E}\,|M(t)|^p$ exists for $p \in (0, \alpha)$ and diverges for $p \geq \alpha$. If A is a nonnormal stable random vector, then $M(t)$ has a multivariate ν-stable distribution [23], and again the moment and tail behaviour of $M(t)$ are similar to that of A. The ν-stable laws are the limiting distributions of random sums, so their appearance in the limit theory for a CTRW is natural. It may also be possible to consider ν-operator stable laws, but this is an open problem.

If A is normal, then the density $h(x, t)$ of $M(t)$ is a mixture of normal densities. In some cases, mixtures of normal densities take a familiar form. If A is normal and D is the limit in (2.3), then the density of $D^{\gamma/2}A$ is stable with index γ when $0 < \gamma < 2$; see [34]. If D is exponential, then $D^{1/2}A$ has a Laplace distribution. More generally, if A is any stable random variable or random vector and D is exponential, then $D^{1/2}A$ has a geometric stable distribution [25]. Geometric stable laws have been applied in finance [21], [24].

5. Anomalous diffusion

Let $\{A(t)\}_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^1$, so that $A(t)$ is normal with mean zero and variance $2Dt$. The density $p(x, t)$ of $A(t)$ solves the classical diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}. \tag{5.1}$$

We call $\{A(t)\}$ the *stochastic solution* to (5.1) and we say that (5.1) is the *governing equation* for $\{A(t)\}$. This useful connection between deterministic and stochastic models for diffusion allows a cloud of diffusing particles to be represented as an ensemble of independent Brownian motion particles whose density functions represent relative concentration. Because Brownian motion is self-similar with Hurst index $H = \frac{1}{2}$, particles spread like t^H in this model. In some cases, clouds of diffusing particles spread faster (superdiffusion, where $H > \frac{1}{2}$) or slower (subdiffusion, where $H < \frac{1}{2}$) than the classical model predicts. This has led physicists to develop alternative diffusion equations based on fractional derivatives. Fractional space derivatives model long particle jumps, leading to superdiffusion, while fractional time derivatives model sticking and trapping, causing subdiffusion.

Continuous-time random walks are used by physicists to derive anomalous diffusion equations [29]. Assuming that both the waiting times and the particle jumps have a Lebesgue density, Montroll and Weiss [30] gave a formula for the Laplace–Fourier transform

$$\int_0^\infty \mathrm{e}^{-st} \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}kx} P(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

of the Lebesgue density $P(x, t)$ for the CTRW variable X_t in (2.2). Rescaling in time and space and taking limits gives the Laplace–Fourier transform of the CTRW scaling limit. Using properties of Laplace and Fourier transforms, we get a partial differential equation, which may involve fractional derivatives. In some cases, we can invert the Laplace–Fourier transform to obtain solutions to this partial differential equation. If we can recognize these solutions as density functions of a stochastic process, we also obtain stochastic solutions.

This method has been successful for scalar models of anomalous diffusion. If particle jumps belong to the strict domain of attraction of a stable law with index α, and waiting times have a finite mean, then we obtain an α-stable Lévy motion $\{A(t)\}$ as the stochastic solution of the fractional diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = qD\frac{\partial^\alpha p(x,t)}{\partial(-x)^\alpha} + (1-q)D\frac{\partial^\alpha p(x,t)}{\partial x^\alpha}, \tag{5.2}$$

where $D > 0$ and $0 \le q \le 1$ [4], [11]. Using the Fourier transform

$$\hat{p}(k,t) = \int \mathrm{e}^{-\mathrm{i}kx} p(x,t)\,\mathrm{d}x,$$

so that $\hat{p}(-k,t)$ is the characteristic function of $A(t)$, the fractional space derivative

$$\frac{\partial^\alpha p(x,t)}{\partial(\pm x)^\alpha}$$

is defined as the inverse Fourier transform of $(\pm \mathrm{i}k)^\alpha \hat{p}(k,t)$, extending the familiar formula where α is a positive integer (see e.g. [35]). The equation (5.2) has been applied to problems in physics [40] and hydrology [5], [3] where empirical evidence indicates superdiffusion. Since α-stable Lévy motion is self-similar with Hurst index $H = 1/\alpha$, densities $p(x,t)$ for the random particle location $A(t)$ spread faster than the classical model predicts when $\alpha < 2$. When $\alpha = 2$, (5.2) reduces to (5.1), reflecting the fact that Brownian motion is a special case of Lévy motion.

If $\{A(t)\}$ is an operator Lévy motion on $\mathbb{R}^d$ and if ν^t is the probability distribution of $A(t)$, then the linear operators $T_t f(x) = \int f(x-y)\nu^t(\mathrm{d}y)$ form a convolution semigroup [13], [16] with generator $L = \lim_{t\downarrow 0} t^{-1}(T_t - T_0)$. Then $q(x,t) = T_t f(x)$ solves the *abstract Cauchy problem*

$$\frac{\partial q(x,t)}{\partial t} = Lq(x,t), \qquad q(x,0) = f(x)$$

for any initial condition $f(x)$ in the domain of the generator L (see e.g. [31], Theorem I.2.4). If ν^t has Lebesgue density $p(x,t)$ for $t > 0$, then $q(x,t) = \int f(x-y)p(y,t)\,\mathrm{d}y$ and $\{p(x,t) : t > 0\}$ is called the *Green's function solution* to this abstract Cauchy problem. In this case, $\{A(t)\}$ is the stochastic solution to the abstract partial differential equation $\partial p(x,t)/\partial t = Lp(x,t)$. An α-stable Lévy motion on $\mathbb{R}^1$ has generator

$$L = qD\frac{\partial^\alpha}{\partial(-x)^\alpha} + (1-q)D\frac{\partial^\alpha}{\partial x^\alpha} \tag{5.3}$$

and then (5.2) yields an abstract Cauchy problem whose Green's function solution $p(x,t)$ is the Lebesgue density of $A(t)$. If $\{A(t)\}$ is an α-stable Lévy motion on $\mathbb{R}^d$, then L is a multidimensional fractional derivative of order α [27]. If $\{A(t)\}$ is an operator Lévy motion, then L represents a generalized fractional derivative on $\mathbb{R}^d$ whose order of differentiation can vary with coordinate [28].

Zaslavsky [45] proposed a fractional kinetic equation,

$$\frac{\partial^\beta h(x,t)}{\partial t^\beta} = Lh(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}, \tag{5.4}$$

for Hamiltonian chaos, where $0 < \beta < 1$ and $\delta(\cdot)$ is the Dirac delta function. The fractional derivative $\partial^\beta h(x,t)/\partial t^\beta$ is defined as the inverse Laplace transform of $s^\beta \tilde{h}(x,s)$, where $\tilde{h}(x,s) = \int_0^\infty \mathrm{e}^{-st} h(x,t)\,\mathrm{d}t$ is the usual Laplace transform. In the special case where L is given by (5.3) with $q = \frac{1}{2}$, Saichev and Zaslavsky [33] used the Montroll–Weiss method to show that (5.4) is the governing equation of a CTRW limit with symmetric particle jumps and infinite mean waiting times, but this method does not identify the limit process.

Theorem 5.1. *Suppose that $\{A(t)\}$ is an operator Lévy motion on $\mathbb{R}^d$. Let $p(x, t)$ denote the Lebesgue density of $A(t)$ and let L be the generator of the convolution semigroup $T_t f(x) = \int f(x-y)p(y,t)\,\mathrm{d}y$. Then the function $h(x, t)$ defined by (4.7) solves the fractional kinetic equation (5.4). Since this function is also the density of the CTRW scaling limit $\{M(t)\}$ obtained in Theorem 4.2, this limit process is the stochastic solution to (5.4).*

Proof. Baeumer and Meerschaert [1] showed that, whenever $p(x, t)$ is the Green's function solution to the abstract Cauchy problem $\partial p(x,t)/\partial t = Lp(x,t)$, the formula

$$h(x,t) = \frac{t}{\beta}\int_0^\infty p(x,\xi)g_\beta(t\xi^{-1/\beta})\xi^{-1/\beta-1}\,\mathrm{d}\xi$$

solves the fractional Cauchy problem (5.4), where g_β is the density of a stable law with Laplace transform $\exp(-s^\beta)$. But Corollary 4.2 shows that this function is also the density of the CTRW scaling limit $\{M(t)\}$.

If $\{A(t)\}$ is a Brownian motion on $\mathbb{R}^d$, then the stochastic solution to (5.4) is self-similar with Hurst index $H = \beta/2$ in view of Corollary 4.1, since $\{A(t)\}$ is operator self-similar with exponent $E = \frac{1}{2}I$. Since $H < \frac{1}{2}$, this process is subdiffusive. Saichev and Zaslavsky [33] stated that, in the case where $\{A(t)\}$ is scalar Brownian motion, the limiting process $\{M(t)\}$ is 'fractal Brownian motion'. Theorem 4.3 shows that the limiting process is not Gaussian, since $M(t)$ cannot have a normal distribution, and in view of Corollary 4.3 the process $\{M(t)\}$ does not have stationary increments. Therefore, this process cannot be a fractional Brownian motion, but rather a completely new stochastic process that merits further study.

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