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First order strong convergence of an explicit scheme for the stochastic SIS epidemic model

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ABSTRACT

A novel explicit time-stepping scheme, called Lamperti smoothing truncation scheme, is devised in this paper to strongly approximate a stochastic SIS epidemic model, whose solution process takes values in a bounded domain and whose coefficients violate the global monotonicity condition. The proposed scheme is based on combining a Lamperti-type transformation with an explicit truncation method. The new scheme results in numerical approximations preserving the domain of the original SDEs and is proved to retain a mean-square convergence rate of order one. Numerical examples are finally reported to confirm our theoretical findings.

1. Introduction

Mathematical epidemiology has made a significant progress for better understanding of the disease transmissions. Over the past decades, many mathematical models have been developed to describe the transmission dynamics of infectious diseases (see, e.g., [1,2]). A simple but prototypical one is the susceptible–infected–susceptible (SIS) model, which is often used to describe epidemic problems. It assumes that individuals can be infected multiple times throughout their lives with no immunity after each infection. Typical examples mentioned in the literature are rota-viruses, some sexually transmitted as well as bacterial infections [3]. Recently, much attention has been paid to the random effects of the environment on the spread of epidemics in the population. Actually, Gray et al. [4] extended the classical SIS epidemic model to a stochastic counterpart. Let S_t denote the number of susceptibles and Y_t the number of infecteds at time t. The stochastic SIS epidemic model [4] is described by,

$$\begin{cases} \mathrm{d}S_t = [\mu N - \beta S_t Y_t + \nu Y_t - \mu S_t]\,\mathrm{d}t - \sigma S_t Y_t \mathrm{d}W_t, \\ \mathrm{d}Y_t = [\beta S_t Y_t - (\mu+\nu) Y_t]\,\mathrm{d}t + \sigma S_t Y_t \mathrm{d}W_t \end{cases} \tag{1.1}$$

with initial values satisfying $S_0 + Y_0 = N$. Here N is the total amount of the population, $\mu \geq 0$ the per capita death rate, $\nu \geq 0$ the rate at which infected individuals become cured, and $\beta \geq 0$ the disease transmission coefficient. Moreover, W_t is a Brownian motion defined on a complete probability space and σ is the variance of the number of potentially infectious contacts that a single infected individual makes with another individual over the small time interval $[t, t+\mathrm{d}t)$.

Given that $S_t + Y_t = N$, it is sufficient to study the stochastic differential equation (SDE) for Y_t, described by,

$$\mathrm{d}Y_t = Y_t(\beta N - \mu - \nu - \beta Y_t)\,\mathrm{d}t + \sigma Y_t (N - Y_t)\,\mathrm{d}W_t. \tag{1.2}$$

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Gray et al. [4] assert that the problem (1.2) admits a unique global solution taking values in the domain $(0, N)$, i.e.,

$$P\{Y_t \in (0, N), \forall t \geq 0\} = 1. \tag{1.3}$$

Since the analytical solution to the nonlinear SDE (1.2) is not available, seeking good numerical approximations becomes one important way to understand the behavior of the underlying model. The most popular numerical scheme is the Euler–Maruyama (EM) method [5], which, as already shown by [6], produces divergent approximations when used to solve SDEs with super-linear growing coefficients such as (1.2). In order to numerically approximate such SDEs with super-linear growing coefficients, various convergent schemes have been proposed and analyzed in the past few years, including implicit methods [7–13] and modified explicit methods [12–27]. In these works, a global monotonicity condition was usually imposed on the drift and diffusion coefficients, which was essentially used in the analysis when one attempts to prove strong convergence. Therefore, the above mentioned convergence theory cannot be applied to the stochastic SIS epidemic model (1.2) directly, as drift and diffusion coefficients of (1.2) do not obey a global monotonicity condition in $\mathbb{R}$. Analyzing strong convergence rates of numerical schemes in a nonglobal monotonicity condition, however, turns out to be a challenging task and has been very recently studied in [28], where the authors established a perturbation theory for general SDEs in $\mathbb{R}^d$, $d \in \mathbb{N}$ to help recover the strong convergence rate. In the present work we are interested in a particular class of scalar SDEs that admit solutions taking values in a bounded domain $(0, N)$ and aim to design an explicit time discretization scheme preserves the domain of the original SDEs.

More precisely, we formulate a class of more general SDEs with nonglobally monotone coefficients, described by,

$$\mathrm{d}Y_t = \gamma(Y_t)\mathrm{d}t + \sigma Y_t(N - Y_t)\mathrm{d}W_t, \quad t \in [0, T], \quad \text{with } Y_0 = y_0 \in (0, N), \tag{1.4}$$

which, under certain assumptions (Assumption 2.1) imposed on the drift coefficient γ, are well-posed in the domain $(0, N)$ and cover the stochastic SIS epidemic model (1.2) as a special case. By combining the Lamperti-type transformation with a smoothing truncation technique we devise an explicit time-stepping scheme (4.6), called Lamperti smoothing truncation (LST) scheme, to numerically solve (2.2). Specifically, by a Lamperti transformation we transform the original model (2.2) into SDEs with additive noise and exponentially growing drift coefficient. For the transformed SDEs (3.16), a new kind of smoothing truncation method is developed. Transforming the truncation method back results in numerical approximations (4.6) of the original problem. The proposed explicit scheme is able to preserve the domain $(0, N)$ of the original problem (2.2) and retains a first order of strong convergence (Theorem 5.4). Up to the best of our knowledge, this paper is the first one to develop and analyze an efficient numerical method for the stochastic SIS epidemic model, with the strong convergence rate successfully revealed.

Before closing the introduction part, we would like to add further comments on a few closely relevant works. Recently, the backward Euler method was proposed and analyzed in [9,10] after a Lamperti-type transformation applied to scalar SDEs defined in a domain, resulting in an implicit scheme called Lamperti-backward Euler (LBE) method. It was shown that the LBE is strongly convergent with rate one. As implied by the numerical comparison (see Table 1), the proposed explicit scheme in this paper is computationally much more efficient than the LBE method, which needs to implicitly solve a transcendental equation for every time step. We also note that some authors have studied convergence rates of explicit methods for SDEs with the polynomially growing coefficients, see [12–18,20,22,23,25–27]. In [24], a boundary truncated method combined with the Lamperti-type transformation was introduced for strong approximations of financial SDEs with non-Lipschitz coefficients. As shown in [24, Hy0 (2.2)], a kind of (inverse) polynomial growth condition was required for the error analysis. This excludes the considered model which, after the Lamperti-type transformation, has an exponentially growing drift coefficient and violates the (inverse) polynomial growth assumption [24, Hy0 (2.2)]. A modified truncated method in [21] is applied to the transformed SDE of (1.2). The method only obtained the logarithmic convergence rate under the case of exponentially growing coefficients (see Theorem 2.5 and Example 1 in [21]). More recently, Li et al. [29] proposed a new explicit method and obtained the convergence rate of the method under more relaxed conditions. However, the transformed SDEs (3.16) is still not satisfied with the condition (2.3) in [29]. To deal with SDEs with exponentially growing coefficients, in this paper, we propose a new truncated method which is different from those in the literature.

This article is structured as follows. In Section 2, we present some necessary notation and assumptions. In Section 3, we analysis the properties of the analytic solution of (2.2). In Section 4, an explicit scheme will be introduced and some useful lemmas will be given. The convergence of numerical method will be analyzed in Section 5: First, the strong half order convergence will be given. Second, the exponential integrability of numerical solutions will be analyzed. Finally, the strong first order convergence will be obtained. One example is given in Section 6 to illustrate our results.

2. Notation and assumptions

Throughout this paper we use the following notation. Let $\mathbb{R} = (-\infty, \infty)$ denote the set of real numbers, and let $\mathbb{R}_+ = [0, \infty)$ be the set of nonnegative real numbers. Given $a, b \in \mathbb{R}$, $a \vee b$ denotes the maximum of a and b, and $a \wedge b$ denotes the minimum of a and b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. Given a uniform step-size $\Delta = T/M$ for a positive integer M, we define the partition of the interval $[0, T]$ by $\mathbb{T}_\Delta := \{t_i = i\Delta : i = 0, 1, 2, \ldots, M\}$. And let $\underline{t} = t_i$,

Table 1
Error and runtime.

Δ	LBE scheme		LST scheme	
	Error	Runtime	Error	Runtime
2^{-4}	0.0949	18.157384 (s)	0.1183	0.463107 (s)
2^{-5}	0.0490	27.758453 (s)	0.0557	0.540652 (s)
2^{-6}	0.0257	47.410464 (s)	0.0265	0.671723 (s)
2^{-7}	0.0128	88.223921 (s)	0.0132	0.992514 (s)
2^{-8}	0.0064	167.646877 (s)	0.0067	1.544030 (s)
2^{-9}	0.0031	329.018970 (s)	0.0034	2.808140 (s)
2^{-10}	0.0015	636.901631 (s)	0.0017	4.341149 (s)

for any $\mathbb{T}_\Delta$ and $t \in [t_i, t_{i+1})$. Let $\emptyset$ denote the empty set, and $\operatorname{sgn}(\cdot)$ denote the sign function define on $\mathbb{R}$. By $C^2([0,N];\mathbb{R})$ we denote the collection of all continuous functions $\gamma : [0,N] \to \mathbb{R}$ that are twice continuously differentiable. And by $C^2((0,N);\mathbb{R}_+)$ we denote the family of all non-negative functions $V(y)$ on $(0,N)$ that are twice continuously differentiable. Furthermore, for each $V \in C^2((0,N);\mathbb{R}_+)$, we define an operator $\mathcal{L}$:

$$\mathcal{L}V(y) = V'(y)\gamma(y) + \tfrac{1}{2}V''(y)\sigma^2 y^2 (N-y)^2. \tag{2.1}$$

For notational simplicity, we use the convention that C (with or without subscript) represents a generic constant, whose values may be different for different appearances.

Let us focus on a class of general SDEs as follows,

$$\mathrm{d}Y_t = \gamma(Y_t)\mathrm{d}t + \sigma Y_t(N - Y_t)\mathrm{d}W_t, \quad t \in [0,T], \quad \text{with } Y_0 = y_0 \in (0,N), \tag{2.2}$$

whose drift coefficient γ satisfies the following assumptions.

Assumption 2.1. Let $\gamma \in C^2([0,N];\mathbb{R})$ and it holds that

$$\gamma(0) \ge 0 \quad \text{and} \quad \gamma(N) \le 0. \tag{2.3}$$

Remark 2.1. We highlight that Assumption 2.1 is very weak and can cover the stochastic SIS epidemic model (1.2) as a special case. And the condition $\gamma \in C^2([0,N];\mathbb{R})$ implies that the first and second derivatives of γ are bounded on the closed interval $[0,N]$.

From Assumption 2.1, the following proposition can be obtained immediately.

Proposition 2.2. *Suppose Assumption 2.1 holds. Then there exists a positive constant K_0 such that for any $y \in [0,N]$,*

$$|\gamma'(y)| \vee |\gamma''(y)| \le K_0. \tag{2.4}$$

Furthermore, we have

$$\gamma(0) - K_0 y \le \gamma(y) \le \gamma(N) + K_0(N-y), \tag{2.5}$$

$$|\gamma(y)| \le K_0 N. \tag{2.6}$$

Proof. Obviously, (2.4) follows from $\gamma \in C^2([0,N];\mathbb{R})$. Therefore, for any $y \in (0,N)$ we have

$$\gamma(y) - \gamma(0) = \int_0^y \gamma'(s)\mathrm{d}s \ge -K_0 y \tag{2.7}$$

and

$$\gamma(N) - \gamma(y) = \int_y^N \gamma'(s)\mathrm{d}s \ge -K_0(N-y). \tag{2.8}$$

By (2.3) we further deduce

$$-K_0 N \le \gamma(0) - K_0 y \le \gamma(y) \le \gamma(N) + K_0(N-y) \le K_0 N. \tag{2.9}$$

Consequently the assertions (2.5) and (2.6) are validated. □

3. Model properties

The present section begins with some properties of solutions to (2.2), including (inverse) moment boundedness of the solutions. After introducing a Lamperti-type transformation, we transform (2.2) into an SDE with additive noise, whose properties will be also provided. First of all, we prove that the analytic solutions to (2.2) remain in the interval $(0, N)$ with probability one.

Theorem 3.1. *Under Assumption 2.1, the SDE (2.2) has a unique strong solution Y_t, satisfying*

$$\mathbb{P}\left(Y_t \in (0, N),\ t \in [0, T]\right) = 1. \tag{3.1}$$

Proof. First we choose $m_0 > 0$ sufficiently large such that $1/m_0 < Y_0 < N - (1/m_0)$. For each integer $m \geq m_0$, define a stopping time

$$\tau_m = \inf\Big\{t \in [0, T] : Y_t \notin (\tfrac{1}{m}, N - \tfrac{1}{m})\Big\}, \tag{3.2}$$

where throughout this paper we set $\inf \emptyset = T$ as usual. Further, we define a function $V \in C^2((0, N); \mathbb{R}_+)$ by

$$V(y) = \frac{1}{y} + \frac{1}{N-y}. \tag{3.3}$$

In light of (2.3) and (2.5), one observes that

$$\begin{aligned}\mathcal{L}V(y) &= \left[-\tfrac{1}{y^2} + \tfrac{1}{(N-y)^2}\right]\gamma(y) + \sigma^2 y^2 (N-y)^2 \left[\tfrac{1}{y^3} + \tfrac{1}{(N-y)^3}\right] \\ &\leq -\tfrac{\gamma(0)-K_0 y}{y^2} + \tfrac{\gamma(N)+K_0(N-y)}{(N-y)^2} + \sigma^2 N^2 V(y) \\ &\leq \left(K_0 + \sigma^2 N^2\right) V(y).\end{aligned} \tag{3.4}$$

Using the Itô formula, for any $t \in [0, T]$ and $m > m_0$, we have

$$\begin{aligned}\mathbb{E}\left(V(Y_{t\wedge\tau_m})\right) =& V(Y_0) + \mathbb{E}\int_0^{t\wedge\tau_m} \mathcal{L}V(Y_s)\mathrm{d}s \\ \leq& V(Y_0) + \left(K_0 + \sigma^2 N^2\right)\int_0^t \mathbb{E}V(Y_{s\wedge\tau_m})\mathrm{d}s.\end{aligned} \tag{3.5}$$

Applying the Gronwall inequality further yields

$$\mathbb{E}V(Y_{t\wedge\tau_m}) \leq V(Y_0)e^{(K_0+\sigma^2N^2)t}. \tag{3.6}$$

Following the same lines as used in the proof of [4, Theorem 3.1], one can arrive at the desired assertion. □

It should be note that, since the process Y_t is a solution of one-dimensional SDE (2.2), Theorem 3.1 can be also proved by using "Feller's test", see e.g. [30, Theorem 5.5.29]. For the sake of readability, we present the above proof.

Evidently, any positive moment of the solution to (2.2) is bounded, since $\mathbb{P}(Y_t \in (0, N)) = 1$. Now, we give the boundedness of its inverse moments. The inverse moments play an important role in the analysis of convergence rate for the numerical scheme.

Theorem 3.2 (*Boundedness of Inverse Moments*)**.** *Let Assumption 2.1 hold. Then for any $p \geq 0$ we have*

$$\left(\sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{-p}\right]\right) \vee \left(\sup_{t\in[0,T]} \mathbb{E}\left[(N - Y_t)^{-p}\right]\right) \leq C_p, \tag{3.7}$$

where

$$C_p = \left(Y_0^{-p} \vee (N - Y_0)^{-p}\right) \exp\left\{pK_0 T + \tfrac{p(p+1)}{2}\sigma^2 N^2 T\right\}. \tag{3.8}$$

Proof. It follows from (2.5) that, for any $y \in (0, N)$,

$$\begin{aligned}\mathcal{L}\left(y^{-p}\right) &= -py^{-(p+1)}\gamma(y) + \tfrac{p(p+1)}{2}\sigma^2 y^{-p}(N-y)^2 \\ &\leq -py^{-(p+1)}\left(\gamma(0) - K_0 y\right) + \tfrac{p(p+1)}{2}\sigma^2 N^2 y^{-p} \\ &\leq \left(pK_0 + \tfrac{p(p+1)}{2}\sigma^2 N^2\right) y^{-p},\end{aligned} \tag{3.9}$$

where the last step uses the assumption $\gamma(0) \geq 0$.

Let τ_m be defined by (3.2). By the Itô formula we deduce

$$\begin{aligned}\mathbb{E}\left[Y_{t\wedge\tau_m}^{-p}\right] =& Y_0^{-p} + \mathbb{E}\int_0^{t\wedge\tau_m} \mathcal{L}\left(Y_s^{-p}\right)\mathrm{d}s \\ \le& Y_0^{-p} + \left(pK_0 + \tfrac{p(p+1)}{2}\sigma^2 N^2\right)\int_0^t \mathbb{E}\left[Y_{s\wedge\tau_m}^{-p}\right]\mathrm{d}s.\end{aligned} \tag{3.10}$$

Applying the Gronwall inequality, letting $m \to \infty$ and using the Fatou lemma yield

$$\sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{-p}\right] \le C_p. \tag{3.11}$$

Owing to (2.3) and (2.5), for any $y \in (0, N)$, we also have

$$\begin{aligned}\mathcal{L}\left((N-y)^{-p}\right) =& p(N-y)^{-(p+1)}\gamma(y) + \tfrac{p(p+1)}{2}\sigma^2 y^2 (N-y)^{-p} \\ \le& p(N-y)^{-(p+1)}\left(\gamma(N) + K_0(N-y)\right) + \tfrac{p(p+1)}{2}\sigma^2 N^2 (N-y)^{-p} \\ \le& \left(pK_0 + \tfrac{p(p+1)}{2}\sigma^2 N^2\right)(N-y)^{-p}.\end{aligned} \tag{3.12}$$

According to calculations similar to those for obtaining (3.10)–(3.12), we can prove that

$$\sup_{t\in[0,T]} \mathbb{E}\left[(N-Y_t)^{-p}\right] \le C_p. \tag{3.13}$$

Therefore the desired conclusion holds. □

In the spirit of [9,10], we aim to propose a kind of Lamperti transformed scheme, instead of applying numerical methods to the original equation (2.2) directly. To do so we firstly apply the Lamperti-type transformation

$$X_t = F(Y_t) := \ln \tfrac{Y_t}{N-Y_t}, \tag{3.14}$$

or equivalently,

$$Y_t = F^{-1}(X_t) = \tfrac{Ne^{X_t}}{e^{X_t}+1}, \tag{3.15}$$

to (2.2) and by the Itô formula we obtain a transformed SDE with additive noise as follows:

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma N \mathrm{d}W_t,\ t \ge 0, \quad \text{with } x_0 = F(Y_0). \tag{3.16}$$

Here and below we always denote

$$f(x) := \tfrac{(e^x+1)^2}{Ne^x}\gamma\left(\tfrac{Ne^x}{e^x+1}\right) + \tfrac{\sigma^2 N^2(e^x-1)}{2(e^x+1)}. \tag{3.17}$$

The forthcoming results concern exponential integrability property of X_t, $t \in [0, T]$ and some estimates of the new drift function f, which will be used in convergence analysis of the numerical scheme.

Proposition 3.3. *Let Assumption 2.1 hold. Then, for any $p \in \mathbb{R}$, the transformed SDE (3.16) has the following exponential integrability property*

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{pX_t}\right] \le N^{|p|} C_{|p|}, \tag{3.18}$$

where C_p is given by (3.8).

Proof. Note that $X_t = \ln \frac{Y_t}{N-Y_t}$. Due to Theorem 3.2, for any $p \ge 0$, we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{pX_t}\right] = \sup_{t\in[0,T]} \mathbb{E}\left[Y_t^p (N-Y_t)^{-p}\right] \le N^p \sup_{t\in[0,T]} \mathbb{E}\left[(N-Y_t)^{-p}\right] \le N^p C_p. \tag{3.19}$$

Similarly, for any $p < 0$, we get

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{pX_t}\right] \le N^{|p|} C_{|p|}. \tag{3.20}$$

Hence the desired conclusion holds. □

Lemma 3.4. *Let Assumption 2.1 hold. Then there exists a positive constant C such that*

$$f'(x) \le C, \qquad \forall x \in \mathbb{R}, \tag{3.21}$$

$$|f(x)| \vee |f'(x)| \vee |f''(x)| \le C\left(e^x + e^{-x}\right), \qquad \forall x \in \mathbb{R}, \tag{3.22}$$

where f is given by (3.17).

Proof. By (2.6) and the definition of function f in (3.17), one observes that

$$\begin{aligned}|f(x)| \leq& \frac{(e^x+1)^2}{Ne^x} K_0 N + \frac{1}{2}\sigma^2 N^2 \\ \leq& C\left(e^x + e^{-x}\right).\end{aligned} \tag{3.23}$$

Using (2.3), (2.4) and (2.5), we derive that

$$\begin{aligned}f'(x) =& \frac{1}{N}(e^x - e^{-x})\gamma\left(\frac{Ne^x}{e^x+1}\right) + \gamma'\left(\frac{Ne^x}{e^x+1}\right) + \sigma^2 N^2 \frac{e^x}{(e^x+1)^2} \\ \leq& \frac{1}{N} e^x \left(\gamma(N) + \frac{K_0 N}{e^x+1}\right) - \frac{1}{N} e^{-x}\left(\gamma(0) - \frac{K_0 N e^x}{e^x+1}\right) + K_0 + \sigma^2 N^2 \\ \leq& 3K_0 + \sigma^2 N^2.\end{aligned} \tag{3.24}$$

Therefore, the conclusion (3.21) is established. Thanks to (2.6) and (3.24), one observes that

$$|f'(x)| \leq \frac{1}{N}(e^x + e^{-x}) K_0 N + K_0 + \sigma^2 N^2 \leq C\left(e^x + e^{-x}\right). \tag{3.25}$$

Noting that

$$f''(x) = \frac{1}{N}(e^x + e^{-x})\gamma\left(\frac{Ne^x}{e^x+1}\right) + \frac{e^x-1}{e^x+1}\gamma'\left(\frac{Ne^x}{e^x+1}\right) + N\frac{e^x}{(e^x+1)^2}\gamma''\left(\frac{Ne^x}{e^x+1}\right) - \sigma^2 N^2 \frac{e^x(e^x-1)}{(e^x+1)^3}, \tag{3.26}$$

one can use (2.4) and (2.6) to get

$$\left|f''(x)\right| \leq K_0\left(e^x + e^{-x}\right) + K_0 + NK_0 + \sigma^2 N^2 \leq C\left(e^x + e^{-x}\right). \tag{3.27}$$

Then (3.22) can be derived immediately from (3.23), (3.25) and (3.27) . □

4. The Lamperti smoothing truncation scheme

The goal of this section is to introduce a so-called Lamperti smoothing truncation (LST) scheme for the underlying model (2.2). This is achieved via two steps. For the first step, we introduce a smoothing truncation scheme for the transformed SDE (3.16). Secondly, we transform the resulting numerical approximation back to get the numerical scheme for the original problem.

In order to introduce the scheme, we first give a smoothing truncation function. Let $k > 0$ and $L \geq \ln(NC_1)$ be two fixed constants, where C_1 is defined by (3.8) with $p = 1$. For any $\Delta < 1$, define

$$\alpha_\Delta := L - k\ln\Delta, \tag{4.1}$$

and

$$f_\Delta(x) := \begin{cases} f(-\alpha_\Delta) - \frac{1}{2}f'(-\alpha_\Delta), & x < -\alpha_\Delta - 1, \\ f(-\alpha_\Delta) + f'(-\alpha_\Delta)\left[(x+\alpha_\Delta) + \frac{1}{2}(x+\alpha_\Delta)^2\right], & -\alpha_\Delta - 1 \leq x < -\alpha_\Delta, \\ f(x), & |x| \leq \alpha_\Delta, \\ f(\alpha_\Delta) + f'(\alpha_\Delta)\left[(x-\alpha_\Delta) - \frac{1}{2}(x-\alpha_\Delta)^2\right], & \alpha_\Delta < x \leq \alpha_\Delta + 1, \\ f(\alpha_\Delta) + \frac{1}{2}f'(\alpha_\Delta), & x > \alpha_\Delta + 1. \end{cases} \tag{4.2}$$

Obviously, the smoothing truncation function f_Δ has the first-order continuous derivative, but its second-order derivative does not exist at four piecewise points. Supplement definitions at these four points with

$$f''_\Delta(\pm\alpha_\Delta) = f''_\Delta\left(\pm(\alpha_\Delta + 1)\right) := 0. \tag{4.3}$$

Then, by the virtue of the Taylor formula with integral remainder term, we have for any $a, b \in \mathbb{R}$

$$f_\Delta(b) - f_\Delta(a) = f'_\Delta(a)(b-a) + (b-a)^2 \int_0^1 (1-\theta) f''_\Delta\Big(\theta b + (1-\theta)a\Big)\mathrm{d}\theta. \tag{4.4}$$

Now, we define the Lamperti smoothing truncation scheme. Set $x_{t_0} = X_0$ and for $i = 0, 1, \ldots, M-1$,

$$\begin{cases} x_{t_{i+1}} = x_{t_i} + f_\Delta(x_{t_i})\Delta + \sigma N \Delta W_i, \\ y_{t_i} = N\frac{e^{x_{t_i}}}{e^{x_{t_i}}+1}, \end{cases} \tag{4.5}$$

where $\Delta W_i = W_{t_{i+1}} - W_{t_i}$ is the Brownian increment. Letting $\underline{t} = t_i$ for any $t \in [t_i, t_{i+1})$, the time continuous LST approximation is defined by

$$\begin{cases} x_t = x_0 + \int_0^t f_\Delta(x_{\underline{s}})\mathrm{d}s + \sigma N \int_0^t \mathrm{d}W_s, \\ y_t = N\frac{e^{x_t}}{e^{x_t}+1}. \end{cases} \tag{4.6}$$

Remark 4.1. For any fixed $k > 0$, we have $-k\ln\Delta \to +\infty$, as $\Delta \to 0$. It means the event $\{X_t \in (k\ln\Delta, -k\ln\Delta)\}$ occurs with sufficiently big probability, if Δ is sufficiently small. But the interval $(k\ln\Delta, -k\ln\Delta)$ may not be large enough, when Δ is fixed. In fact, if $k = 1/9$ and $\Delta = 10^{-3}$, then $-k\ln\Delta \approx 0.7675$. So we introduce the parameter $L > 0$ and expand the interval to $(-L+k\ln\Delta, L-k\ln\Delta)$. Moreover, we let $L \geq \ln(NC_1)$ to make sure that $e^L \geq \sup_{t\in[0,T]}\mathbb{E}e^{\pm X_t}$ (by Proposition 3.3 for $p = \pm 1$). In this way, the performance of the scheme will be also good for large step-sizes. Only when the step-size is not sufficiently small, the parameter L works. And the convergence order of the LST method will not be affected by the value of L.

Now, we introduce a function h_Δ to connect the function f with the function f_Δ.

Lemma 4.2. *For any $\Delta < 1$ and $x \in \mathbb{R}$, define $f'_\Delta(x) := (f_\Delta(x))'$, $f''_\Delta(x) := (f_\Delta(x))''$ and*

$$h_\Delta(x) := \begin{cases} x, & |x| < \alpha_\Delta + 1, \\ \mathrm{sgn}(x)(\alpha_\Delta + 1), & |x| \geq \alpha_\Delta + 1. \end{cases} \tag{4.7}$$

Then, for any $x \in \mathbb{R}$, we have

$$f'_\Delta(x) \leq C \quad \text{and} \quad |f_\Delta(x)| \vee \left|f'_\Delta(x)\right| \vee \left|f''_\Delta(x)\right| \leq Ce^{|h_\Delta(x)|}. \tag{4.8}$$

Proof. If $|x| < \alpha_\Delta$, then we have $h_\Delta(x) = x$, and thus

$$f'_\Delta(x) = f'(x), \quad f_\Delta(x) = f(h_\Delta(x)), \quad f'_\Delta(x) = f'(h_\Delta(x)), \quad \text{and} \quad f''_\Delta(x) = f''(h_\Delta(x)). \tag{4.9}$$

If $|x| \geq \alpha_\Delta$, then we have $|h_\Delta(x)| \geq \alpha_\Delta$. And by the definition of $f_\Delta(x)$, it is easy to show that

$$f'_\Delta(x) \leq f'(\mathrm{sgn}(x)\alpha_\Delta) \vee 0 \tag{4.10}$$

and

$$|f_\Delta(x)| \leq |f(\mathrm{sgn}(x)\alpha_\Delta)| + \left|f'(\mathrm{sgn}(x)\alpha_\Delta)\right|, \qquad |f'_\Delta(x)| \vee |f''_\Delta(x)| \leq |f'(\mathrm{sgn}(x)\alpha_\Delta)|. \tag{4.11}$$

Combining the above results and Lemma 3.4 yields the desired assertion immediately. □

Now, we give some propositions and lemmas, which will be used in convergence analysis of the LST scheme. Obviously, the smoothing drift coefficient f_Δ satisfies the following Lipschitz continuous and one-side Lipschitz continuous form (3.21).

Proposition 4.3. *Suppose Assumption 2.1 holds. Let $k > 0$ be given. Then, for any $x, \hat{x} \in \mathbb{R}$,*

$$(x - \hat{x})\left(f_\Delta(x) - f_\Delta(\hat{x})\right) \leq C|x - \hat{x}|^2. \tag{4.12}$$

$$\left|f_\Delta(x) - f_\Delta(\hat{x})\right| \leq C\Delta^{-k}|x - \hat{x}|, \tag{4.13}$$

where, and from now on, C stands for generic positive real constants independent of Δ and its values may change between occurrences.

Using Lemma 4.2, we can easily carry out the proof of this proposition, and hence is omitted here.

Lemma 4.4. *Let Assumption 2.1 hold. Then, for any $p \geq 1$, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|f(X_t) - f\left(X_{\underline{t}}\right)\right|^p\right] \leq C\Delta^{p/2}. \tag{4.14}$$

Proof. By (3.22), for any $x, \hat{x} \in \mathbb{R}$, there exists ξ between x and $\hat{x}$ such that

$$\begin{aligned} \left|f(x) - f(\hat{x})\right|^p &\leq |f'(\xi)|^p|x - \hat{x}|^p \\ &\leq C\left(e^{p\xi} + e^{-p\xi}\right)|x - \hat{x}|^p \\ &\leq C\left(e^{px} + e^{p\hat{x}} + e^{-px} + e^{-p\hat{x}}\right)|x - \hat{x}|^p. \end{aligned} \tag{4.15}$$

By Proposition 3.3 and (3.22), for any $s, t \in [0, T]$, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{\pm pX_s}\left|X_t - X_{\underline{t}}\right|^p\right] \leq & \left(\mathbb{E}\left[e^{\pm 2pX_s}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|X_t - X_{\underline{t}}\right|^{2p}\right]\right)^{\frac{1}{2}} \\ \leq & C\left(\mathbb{E}\left[\left|\int_{\underline{t}}^{t} f(X_u)\mathrm{d}u + \sigma N\int_{\underline{t}}^{t}\mathrm{d}W_u\right|^{2p}\right]\right)^{\frac{1}{2}} \\ \leq & C\left[\left(\sup_{t\in[0,T]}\mathbb{E}\left[e^{2pX_t}\right] + \sup_{t\in[0,T]}\mathbb{E}\left[e^{-2pX_t}\right]\right)\Delta^{2p} + \Delta^p\right]^{\frac{1}{2}} \\ \leq & C\Delta^{\frac{p}{2}}.\end{aligned} \tag{4.16}$$

The desired conclusion follows from (4.15) and (4.16) immediately. □

Lemma 4.5. *Let Assumption 2.1 hold. Then*

$$\sup_{t\in[0,T]}\mathbb{E}\left[|f(X_t) - f_\Delta(X_t)|^2\right] \leq C\Delta^2. \tag{4.17}$$

Proof. By (3.22) and Lemma 4.2, for any $t \in [0, T]$, one observes that

$$\begin{aligned}|f(X_t) - f_\Delta(X_t)|^2 \leq & \left(|f(X_t)| + |f_\Delta(X_t)|\right)^2\left(I_{\{X_t > \alpha_\Delta\}} + I_{\{X_t < -\alpha_\Delta\}}\right) \\ \leq & C\left(e^{2X_t}I_{\{X_t > \alpha_\Delta\}} + e^{-2X_t}I_{\{X_t < -\alpha_\Delta\}}\right).\end{aligned} \tag{4.18}$$

Let $p = 4/k$, then an application of Markov's inequality together with Proposition 3.3 leads to

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{P}\{X_t > \alpha_\Delta\} = & \sup_{t\in[0,T]}\mathbb{P}\left\{e^{pX_t} > e^{pL}\Delta^{-pk}\right\} \\ \leq & \frac{\sup_{t\in[0,T]}\mathbb{E}\left[e^{pX_t}\right]}{e^{pL}\Delta^{-pk}} \leq C\Delta^4.\end{aligned} \tag{4.19}$$

Therefore, we deduce

$$\sup_{t\in[0,T]}\mathbb{E}\left[e^{2X_t}I_{\{X_t > \alpha_\Delta\}}\right] \leq \sup_{t\in[0,T]}\left[\left(\mathbb{E}\left[e^{4X_t}\right]\right)^{\frac{1}{2}}\left[\mathbb{P}\{X_t > \alpha_\Delta\}\right]^{\frac{1}{2}}\right] \leq C\Delta^2. \tag{4.20}$$

Similarly, we can prove that

$$\sup_{t\in[0,T]}\mathbb{E}\left[e^{-2X_t}I_{\{X_t < -\alpha_\Delta\}}\right] \leq C\Delta^2. \tag{4.21}$$

Thus, the desired conclusion follows from (4.18), (4.20) and (4.21). □

5. Strong convergence rate of the proposed scheme

In this section, the main convergence result of the proposed scheme is established from the following three steps. Firstly, the strong order 1/2 convergence will be given. Then, we will analyze the exponential integrability of numerical solutions. Finally, the strong order 1 convergence is obtained.

Theorem 5.1. *Suppose Assumption 2.1 holds. Let $0 < k \leq 1/2$ be given, and let the LST scheme x_t be defined by (4.6). Then the following strong order 0.5 convergence holds*

$$\sup_{t\in[0,T]}\mathbb{E}\left[|X_t - x_t|^2\right] \leq C\Delta. \tag{5.1}$$

Proof. For any $t \in (0, T]$, there exists $t_i = \underline{t}$ such that $t \in (t_i, t_{i+1}]$, and we derive from (3.16), (4.6) and the Hölder inequality that

$$\begin{aligned}& |X_t - x_t|^2 \\ = & \left|X_{t_i} - x_{t_i} + \int_{t_i}^{t}\left(f(X_s) - f_\Delta(x_{t_i})\right)\mathrm{d}s\right|^2 \\ \leq & \left|X_{t_i} - x_{t_i}\right|^2 + 2\left(X_{t_i} - x_{t_i}\right)\int_{t_i}^{t}\left(f(X_s) - f_\Delta(x_{t_i})\right)\mathrm{d}s + \Delta\int_{t_i}^{t}\left|f(X_s) - f_\Delta(x_{t_i})\right|^2\mathrm{d}s.\end{aligned} \tag{5.2}$$

Due to (4.12) and the Young inequality, we show that

$$\begin{aligned}
&\left(X_{t_i}-x_{t_i}\right)\int_{t_i}^{t}\left(f(X_s)-f_\Delta(x_{t_i})\right)\mathrm{d}s\\
=&\left(X_{t_i}-x_{t_i}\right)\int_{t_i}^{t}\left(f(X_s)-f_\Delta(X_{t_i})+f_\Delta(X_{t_i})-f_\Delta(x_{t_i})\right)\mathrm{d}s\\
=&\left(X_{t_i}-x_{t_i}\right)\left(f_\Delta(X_{t_i})-f_\Delta(x_{t_i})\right)(t-t_i)+\left(X_{t_i}-x_{t_i}\right)\int_{t_i}^{t}\left(f(X_s)-f_\Delta(X_{t_i})\right)\mathrm{d}s\\
\leq&C\left[\Delta\left|X_{t_i}-x_{t_i}\right|^2+\frac{1}{\Delta}\left|\int_{t_i}^{t}\left(f(X_s)-f_\Delta(X_{t_i})\right)\mathrm{d}s\right|^2\right]\\
\leq&C\left[\Delta\left|X_{t_i}-x_{t_i}\right|^2+\int_{t_i}^{t}\left|f(X_s)-f_\Delta(X_{t_i})\right|^2\mathrm{d}s\right].
\end{aligned}\tag{5.3}$$

And by (4.13) and $0<k\leq 1/2$, we obtain

$$\begin{aligned}
&\Delta\int_{t_i}^{t}\left|f(X_s)-f_\Delta(x_{t_i})\right|^2\mathrm{d}s\\
=&\Delta\int_{t_i}^{t}\left|f(X_s)-f_\Delta(X_{t_i})+f_\Delta(X_{t_i})-f_\Delta(x_{t_i})\right|^2\mathrm{d}s\\
\leq&C\left(\Delta^2\left|f_\Delta(X_{t_i})-f_\Delta(x_{t_i})\right|^2+\Delta\int_{t_i}^{t}\left|f(X_s)-f_\Delta(X_{t_i})\right|^2\mathrm{d}s\right)\\
\leq&C\left(\Delta^{2-2k}\left|X_{t_i}-x_{t_i}\right|^2+\Delta\int_{t_i}^{t}\left|f(X_s)-f_\Delta(X_{t_i})\right|^2\mathrm{d}s\right)\\
\leq&C\left(\Delta\left|X_{t_i}-x_{t_i}\right|^2+\Delta\int_{t_i}^{t}\left|f(X_s)-f_\Delta(X_{t_i})\right|^2\mathrm{d}s\right).
\end{aligned}\tag{5.4}$$

Inserting (5.3), (5.4) into (5.2), and applying Lemmas 4.4 and 4.5, we have

$$\begin{aligned}
&\mathbb{E}\left[|X_t-x_t|^2\right]\\
\leq&(1+C\Delta)\mathbb{E}\left[\left|X_{t_i}-x_{t_i}\right|^2\right]+C\left(\int_{t_i}^{t}\mathbb{E}\left[\left|f(X_s)-f_\Delta(X_{t_i})\right|^2\right]\mathrm{d}s\right)\\
=&(1+C\Delta)\mathbb{E}\left[\left|X_{t_i}-x_{t_i}\right|^2\right]+C\left(\int_{t_i}^{t}\mathbb{E}\left[\left|f(X_s)-f(X_{t_i})+f(X_{t_i})-f_\Delta(X_{t_i})\right|^2\right]\mathrm{d}s\right)\\
\leq&(1+C\Delta)\mathbb{E}\left[\left|X_{t_i}-x_{t_i}\right|^2\right]+C\left(\int_{t_i}^{t}\mathbb{E}\left[\left|f(X_s)-f(X_{t_i})\right|^2\right]\mathrm{d}s+\Delta\mathbb{E}\left[\left|f(X_{t_i})-f_\Delta(X_{t_i})\right|^2\right]\right)\\
\leq&(1+C\Delta)\mathbb{E}\left[\left|X_{t_i}-x_{t_i}\right|^2\right]+C\Delta^2,
\end{aligned}\tag{5.5}$$

for all $t\in(0,T]$ and $t_i=\underline{t}$. Let $t=t_{i+1}$ and an iteration yields

$$\max_{i=0,1,\ldots,N}\mathbb{E}\left[|X_{t_i}-x_{t_i}|^2\right]\leq C\Delta.\tag{5.6}$$

Combining (5.5) and (5.6) concludes the proof. □

Remark 5.2. The strong convergence rate of order 1/2 has been obtained for the scheme. However, this is not optimal for SDEs with additive noise. To get the improved convergence rate, we require the exponential integrability of numerical solutions as follows.

Lemma 5.3. *Suppose Assumption 2.1 holds and let $0<k\leq 1/9$ be given. Then the LST scheme x_t defined by (4.6) has the property that*

$$\sup_{0<\Delta<1}\sup_{t\in[0,T]}\mathbb{E}\left[e^{8|h_\Delta(x_t)|}\right]\leq C.\tag{5.7}$$

Proof. For any $t\in[0,T]$, denoting $A_t:=\{|X_t|\geq\alpha_\Delta\}$ and $B_t:=\{|X_t-x_t|\geq\Delta^k\}$, we have

$$\mathbb{E}\left[e^{8|h_\Delta(x_t)|}\right]=\mathbb{E}\left[e^{8|h_\Delta(x_t)|}\left(I_{A_t}+I_{A_t^c\cap B_t^c}+I_{A_t^c\cap B_t}\right)\right].\tag{5.8}$$

Firstly, by Lemma 4.2, on the set A_t we show

$$|h_\Delta(x_t)| \leq 1 + \alpha_\Delta \leq 1 + |X_t|. \tag{5.9}$$

This together with Proposition 3.3 implies that

$$\mathbb{E}\Big[e^{8|h_\Delta(x_t)|} I_{A_t}\Big] \leq e^8 \mathbb{E}\left[e^{8|X_t|} I_{A_t}\right] \leq C\mathbb{E}\Big[e^{8|X_t|}\Big] \leq C. \tag{5.10}$$

Secondly, on the set $A_t^c \cap B_t^c$, we have

$$|X_t| < \alpha_\Delta \quad \text{and} \quad |X_t - x_t| < \Delta^k. \tag{5.11}$$

This together with the fact $|h_\Delta(x_t)| = (\alpha_\Delta + 1) \wedge |x_t|$ yields that

$$\begin{aligned}\Big||X_t| - |h_\Delta(x_t)|\Big| =& I_{\{|x_t|>(\alpha_\Delta+1)\}}\Big((\alpha_\Delta + 1) - |X_t|\Big) + I_{\{|x_t|\leq(\alpha_\Delta+1)\}}\Big||X_t| - |x_t|\Big| \\ \leq& I_{\{|x_t|>(\alpha_\Delta+1)\}}\Big(|x_t| - |X_t|\Big) + I_{\{|x_t|\leq(\alpha_\Delta+1)\}}\Big||X_t| - |x_t|\Big| \\ \leq& \Big||X_t| - |x_t|\Big| \\ \leq& |X_t - x_t| \\ \leq& \Delta^k. \end{aligned} \tag{5.12}$$

By the virtue of the Cauchy mean-value theorem, we show $|e^b - e^a| \leq (e^a + e^b)|b - a|$, for any $a, b \in \mathbb{R}$. Letting $b = |h_\Delta(x_t)|$ and $a = |X_t|$, and combining Proposition 3.3, we find

$$\begin{aligned}\mathbb{E}\left[e^{8|h_\Delta(x_t)|} I_{A_t^c \cap B_t^c}\right] =& \mathbb{E}\left[\left|e^{|X_t|} + e^{|h_\Delta(x_t)|} - e^{|X_t|}\right|^8 I_{A_t^c \cap B_t^c}\right] \\ \leq& C\mathbb{E}\left[e^{8|X_t|} + \left|\left(e^{|h_\Delta(x_t)|} + e^{|X_t|}\right)\Big(|h_\Delta(x_t)| - |X_t|\Big)\right|^8 I_{A_t^c \cap B_t^c}\right] \\ \leq& C\mathbb{E}\left[e^{8|X_t|} + \left|\left(e^{\alpha_\Delta+1} + e^{\alpha_\Delta}\right)\Delta^k\right|^8 I_{A_t^c \cap B_t^c}\right] \\ \leq& C\left(1 + \left|\left(e^{1+L}\Delta^{-k} + e^L\Delta^{-k}\right)\Delta^k\right|^8\right) \\ \leq& C. \end{aligned} \tag{5.13}$$

Thirdly, on the set $A_t^c \cap B_t$, we get

$$|X_t| < \alpha_\Delta \quad \text{and} \quad \Big||X_t| - |h_\Delta(x_t)|\Big| \leq |X_t - x_t|. \tag{5.14}$$

And an application of Markov's inequality together with Theorem 5.1 leads to

$$\mathbb{E}\left[I_{A_t^c \cap B_t}\right] \leq \mathbb{P}\Big\{|X_t - x_t| \geq \Delta^k\Big\} \leq \frac{\mathbb{E}\Big[|X_t - x_t|^2\Big]}{\Delta^{2k}} \leq C\Delta^{1-2k}. \tag{5.15}$$

Note that $0 < k \leq 1/9$ and $|h_\Delta(x_t)| \leq 1+\alpha_\Delta$. Using the Cauchy mean-value theorem with Proposition 3.3 and Theorem 5.1, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{8|h_\Delta(x_t)|} I_{A_t^c \cap B_t}\right] \leq& \mathbb{E}\Big[e^{8|X_t|} + \left|e^{8|h_\Delta(x_t)|} - e^{8|X_t|}\right| I_{A_t^c \cap B_t}\Big] \\ \leq& C\mathbb{E}\left[1 + 8\left(e^{8|h_\Delta(x_t)|} + e^{8|X_t|}\right)\Big||h_\Delta(x_t)| - |X_t|\Big| I_{A_t^c \cap B_t}\right] \\ \leq& C\left\{1 + \left(e^{8+8L}\Delta^{-8k} + e^{8L}\Delta^{-8k}\right)\mathbb{E}\left[|X_t - x_t| \cdot I_{A_t^c \cap B_t}\right]\right\} \\ \leq& C\left\{1 + \Delta^{-8k}\left(\mathbb{E}\left[|X_t - x_t|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[I_{A_t^c \cap B_t}\right]\right)^{\frac{1}{2}}\right\} \\ \leq& C\left(1 + \Delta^{-8k}\Delta^{\frac{1}{2}}\Delta^{\frac{1-2k}{2}}\right) \\ \leq& C. \end{aligned} \tag{5.16}$$

Therefore, inserting (5.10), (5.13) and (5.16) into (5.8) yields

$$\mathbb{E}\left[e^{8|h_\Delta(x_t)|}\right] \leq C. \tag{5.17}$$

Then the desired conclusion follows from taking supremum on both sides of (5.17). □

Theorem 5.4 (*Main Result*). *Suppose Assumption 2.1 holds and let $0 < k \le 1/9$ be given. Then the LST scheme x_t, y_t defined by (4.6) have the following strong first order convergence*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t - x_t|^2\right] \le C\Delta^2 \quad \textit{and} \quad \sup_{t\in[0,T]} \mathbb{E}\left[|Y_t - y_t|^2\right] \le C\Delta^2. \tag{5.18}$$

Proof. Using the Itô formula and (4.12), one observes that

$$\begin{aligned}
&|X_t - x_t|^2 \\
=&2\int_0^t (X_s - x_s)\left(f(X_s) - f_\Delta(x_{\underline{s}})\right) \mathrm{d}s \\
=&2\int_0^t (X_s - x_s)\left(f(X_s) - f_\Delta(X_s)\right) \mathrm{d}s + 2\int_0^t (X_s - x_s)\left(f_\Delta(X_s) - f_\Delta(x_s)\right) \mathrm{d}s + 2J \\
\le&C\int_0^t |X_s - x_s|^2 \,\mathrm{d}s + C\int_0^t |f(X_s) - f_\Delta(X_s)|^2 \,\mathrm{d}s + 2J,
\end{aligned} \tag{5.19}$$

where we denote

$$J := \int_0^t (X_s - x_s)\left(f_\Delta(x_s) - f_\Delta(x_{\underline{s}})\right) \mathrm{d}s. \tag{5.20}$$

Moreover, it follows from Lemma 4.5 that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t - x_t|^2\right] \le C\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right] \mathrm{d}s + C\Delta^2 + 2\sup_{t\in[0,T]} \mathbb{E}[J]. \tag{5.21}$$

Using (4.4) with $b = x_s$ and $a = x_{\underline{s}}$ gives

$$\begin{aligned}
J =& \int_0^t (X_s - x_s)\left[f'_\Delta(x_{\underline{s}})\left(x_s - x_{\underline{s}}\right)\right]\mathrm{d}s \\
&+ \int_0^t (X_s - x_s)\left[(x_s - x_{\underline{s}})^2 \int_0^1 (1-\theta) f''_\Delta\left(\theta x_s + (1-\theta) x_{\underline{s}}\right)\mathrm{d}\theta\right]\mathrm{d}s \\
=& \int_0^t (X_s - x_s) f'_\Delta(x_{\underline{s}}) f_\Delta(x_{\underline{s}})(s - \underline{s})\mathrm{d}s + \sigma N\int_0^t (X_s - x_s) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
&+ \int_0^t (X_s - x_s)(x_s - x_{\underline{s}})^2 \int_0^1 (1-\theta) f''_\Delta\left(\theta x_s + (1-\theta) x_{\underline{s}}\right)\mathrm{d}\theta\mathrm{d}s \\
:=& J_1 + J_2 + J_3.
\end{aligned} \tag{5.22}$$

By Lemmas 4.2 and 5.3, we derive

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}[J_1] \le& C\left(\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right] \mathrm{d}s + \Delta^2 \int_0^T \mathbb{E}\left[\left|f'_\Delta(x_{\underline{s}}) f_\Delta(x_{\underline{s}})\right|^2\right] \mathrm{d}s\right) \\
\le& C\left(\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right] \mathrm{d}s + \Delta^2 \int_0^T \mathbb{E}\left[e^{4|h_\Delta(x_{\underline{s}})|}\right] \mathrm{d}s\right) \\
\le& C\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right] \mathrm{d}s + C\Delta^2.
\end{aligned} \tag{5.23}$$

By (3.16) and (4.6), we have

$$\begin{aligned}
J_2 =& \sigma N\int_0^t \left(X_{\underline{s}} - x_{\underline{s}} + \int_{\underline{s}}^s \left(f(X_u) - f_\Delta(x_{\underline{s}})\right) \mathrm{d}u\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
=& \sigma N\int_0^t \left(X_{\underline{s}} - x_{\underline{s}}\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
&+ \sigma N\int_0^t \left(\int_{\underline{s}}^s \left(f(X_u) - f(X_{\underline{s}})\right) \mathrm{d}u\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
&+ \sigma N\int_0^t \left(\int_{\underline{s}}^s \left(f(X_{\underline{s}}) - f_\Delta(x_{\underline{s}})\right) \mathrm{d}u\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s
\end{aligned} \tag{5.24}$$

Note that $X_{\underline{s}}$ and $x_{\underline{s}}$ are both $\mathcal{F}_{\underline{s}}$-measurable, for any $t \in [0,T]$, there exists $t_i = \underline{t}$ such that $t \in (t_i, t_{i+1}]$, and we have

$$\begin{aligned}
&\mathbb{E}\int_0^t \left(X_{\underline{s}} - x_{\underline{s}}\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
=&\sum_{j=0}^{i-1}\mathbb{E}\int_{t_j}^{t_{j+1}} \left(X_{t_j} - x_{t_j}\right) f'_\Delta(x_{t_j})(W_s - W_{t_j})\mathrm{d}s + \mathbb{E}\int_{t_i}^{t} \left(X_{t_i} - x_{t_i}\right) f'_\Delta(t_i)(W_s - W_{t_i})\mathrm{d}s \\
=&\sum_{j=0}^{i-1}\mathbb{E}\left[\left(X_{t_j} - x_{t_j}\right) f'_\Delta(x_{t_j})\mathbb{E}\left(\int_{t_j}^{t_{j+1}}(W_s - W_{t_j})\mathrm{d}s\middle|\mathcal{F}_{t_j}\right)\right] \\
&+\mathbb{E}\left[\left(X_{t_i} - x_{t_i}\right) f'_\Delta(t_i)\mathbb{E}\left(\int_{t_i}^{t}(W_s - W_{t_i})\mathrm{d}s\middle|\mathcal{F}_{t_i}\right)\right] \\
=&0
\end{aligned} \tag{5.25}$$

Similarly, for any $t \in [0,T]$, we get

$$\mathbb{E}\int_0^t \left(\int_{\underline{s}}^{s} \left(f(X_{\underline{s}}) - f_\Delta(x_{\underline{s}})\right)\mathrm{d}u\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s = 0 \tag{5.26}$$

Then, by (5.24), (5.25) and (5.26), one employs Lemmas 4.2, 4.4, 5.3 and Young's inequality to obtain

$$\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}[J_2] =&\sigma N \sup_{t\in[0,T]}\mathbb{E}\int_0^t \left(\int_{\underline{s}}^{s} \left(f(X_u) - f(X_{\underline{s}})\right)\mathrm{d}u\right) f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\mathrm{d}s \\
\le&\sigma N\mathbb{E}\int_0^T \left|\int_{\underline{s}}^{s} \left(f(X_u) - f(X_{\underline{s}})\right)\mathrm{d}u\right| \left|f'_\Delta(x_{\underline{s}})(W_s - W_{\underline{s}})\right|\mathrm{d}s \\
\le&\sigma N\int_0^T \left(\mathbb{E}\left[\left|\int_{\underline{s}}^{s} \left(f(X_u) - f(X_{\underline{s}})\right)\mathrm{d}u\right|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left|f'_\Delta(x_{\underline{s}})\right|^4\right]\right)^{\frac{1}{4}} \left(\mathbb{E}\left[\left|W_s - W_{\underline{s}}\right|^4\right]\right)^{\frac{1}{4}}\mathrm{d}s \\
\le&C\int_0^T \Delta^{\frac{3}{2}}\left(\mathbb{E}\left[e^{4|h_\Delta(x_{\underline{s}})|}\right]\right)^{\frac{1}{4}}\Delta^{\frac{1}{2}}\mathrm{d}s \\
\le&C\Delta^2.
\end{aligned} \tag{5.27}$$

Thanks to Lemma 4.2 and Young's inequality , we have

$$\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}[J_3] \\
\le&2\mathbb{E}\int_0^T |X_s - x_s|^2\,\mathrm{d}s + C\mathbb{E}\int_0^T \left(\left|f_\Delta(x_{\underline{s}})\Delta\right|^4 + \left|W_s - W_{\underline{s}}\right|^4\right)\left(e^{4|h_\Delta(x_{\underline{s}})|} + e^{4|h_\Delta(x_s)|}\right)\mathrm{d}s \\
\le&2\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right]\mathrm{d}s + C\int_0^T \left(\mathbb{E}\left[\left|f_\Delta(x_{\underline{s}})\Delta\right|^8 + \left|W_s - W_{\underline{s}}\right|^8\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[e^{8|h_\Delta(x_{\underline{s}})|} + e^{8|h_\Delta(x_s)|}\right]\right)^{\frac{1}{2}}\mathrm{d}s \\
\le&2\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right]\mathrm{d}s + C\int_0^T \left(\mathbb{E}\left[e^{8|h_\Delta(x_{\underline{s}})|}\Delta^8\right] + \mathbb{E}\left[\left|W_s - W_{\underline{s}}\right|^8\right]\right)^{\frac{1}{2}}\mathrm{d}s \\
\le&2\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right]\mathrm{d}s + C\Delta^2.
\end{aligned} \tag{5.28}$$

Substituting (5.23), (5.27) and (5.28) into (5.22) yields

$$\sup_{t\in[0,T]}\mathbb{E}[J] \le C\int_0^T \mathbb{E}\left[|X_s - x_s|^2\right]\mathrm{d}s + C\Delta^2. \tag{5.29}$$

It follows from (5.21) and (5.29) that

$$\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}\,|X_t - x_t|^2 \le& C\int_0^T \mathbb{E}\,|X_s - x_s|^2\,\mathrm{d}s + C\Delta^2 \\
\le& C\int_0^T \sup_{u\in[0,s]}\mathbb{E}\,|X_u - x_u|^2\,\mathrm{d}s + C\Delta^2.
\end{aligned}$$

With the help of the Gronwall inequality, we have

$$\sup_{t\in[0,T]}\mathbb{E}\,|X_t - x_t|^2 \le C\Delta^2. \tag{5.30}$$

At last, we establish the convergence for the original SDE (2.2). By the Lamperti-transformation (3.15) and (4.6), we prove that

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}\left[|Y_t-y_t|^2\right]=&N\sup_{t\in[0,T]}\mathbb{E}\left[\left|\tfrac{e^{X_t}}{e^{X_t}+1}-\tfrac{e^{x_t}}{e^{x_t}+1}\right|^2\right]\\ \leq&N\sup_{t\in[0,T]}\mathbb{E}\left[|X_t-x_t|^2\right]\\ \leq&C\Delta^2,\end{aligned}\tag{5.31}$$

as required. □

6. Numerical experiments

In this section we provide some numerical experiments to illustrate the previous theoretical findings. Let us focus on the aforementioned stochastic SIS epidemic model:

$$\mathrm{d}Y_t=Y_t\,(a-bY_t)\,\mathrm{d}t+\sigma Y_t(N-Y_t)\mathrm{d}W_t,\quad 0\leq t\leq T\tag{6.1}$$

with $Y_0=5$ and $T=1$. Evidently, Assumption 2.1 holds if

$$a\geq 0\quad\text{and}\quad a-bN\leq 0.\tag{6.2}$$

Hence, the results of Proposition 2.2 hold with

$$K_0=|a|\vee|2b|\vee|a-2bN|.\tag{6.3}$$

Moreover, by Theorems 3.1 and 3.2, we have $\mathbb{P}\,(Y_t\in(0,N),\ t\geq 0)=1$ and

$$\left(\sup_{t\in[0,T]}\mathbb{E}\left[Y_t^{-p}\right]\right)\vee\left(\sup_{t\in[0,T]}\mathbb{E}\left[(N-Y_t)^{-p}\right]\right)\leq C_p,\tag{6.4}$$

where

$$C_p=\left(5^{-p}\vee(N-5)^{-p}\right)\exp\left\{pK_0+\tfrac{p(p+1)}{2}\sigma^2N^2\right\}.\tag{6.5}$$

Hence, we have

$$\ln(NC_1)=\ln N-\ln(5\wedge(N-5))+K_0+\sigma^2N^2.\tag{6.6}$$

Applying the transformation (3.14), that is $X_t=F(Y_t)=\ln\frac{Y_t}{N-Y_t}$, we get the following transformed SDE:

$$\mathrm{d}X_t=f(X_t)\mathrm{d}t+\sigma N\mathrm{d}W_t,\ t\geq 0,\quad\text{with}\ \ X_0=0,\tag{6.7}$$

where

$$f(x)=a-(bN-a)e^x+\sigma^2N^2\tfrac{(e^x-1)}{2(e^x+1)}.\tag{6.8}$$

In the following, we perform numerical experiments to three concrete models.

Model 1: $a=5$, $b=0.8$, $\sigma=0.1$ and $N=10$.

By (6.3) and (6.6), we get $K_0=11$ and $\ln(NC_1)\approx 12.6931$. Hence, we choose $k=1/9$, $L=13$ in the LST scheme. To test the efficiency of the LST scheme we carry out numerical experiments using MATLAB. We compare the LST scheme with the Lamperti-backward Euler (LBE) method in [10] as follows:

$$\begin{cases}x_{t_{i+1}}=x_{t_i}+f(x_{t_{i+1}})\Delta+\sigma N\Delta W_i,\\ y_{t_i}=N\frac{e^{x_{t_i}}}{e^{x_{t_i}}+1},\end{cases}\tag{6.9}$$

The LBE scheme is an implicit method and had been proved to have first order strong convergence.

In our experiments, the approximation errors will be tested in terms of $\left(\mathbb{E}\left[|Y_T-y_T|^2\right]\right)^{\frac{1}{2}}$. The "true" solution Y_T is identified as the LBE approximation with a small step-size (2^{-14}). And the expectation is approximated by the Monte Carlo approximation, using 2000 Brownian paths. Consider the errors of the two methods and the runtime on our computer with Intel Core 4 duo CPU 3.1 GHz.

It can be seen from Fig. 1, both the LST scheme and the LBE scheme are first order convergent. With the same step-size, the error of the LST scheme is slightly larger than the LBE scheme. But the runtime of our scheme is significantly less than the LBE scheme (see Table 1). This is because our method is explicit while the LBE scheme is implicit, and the latter needs to solve a transcendental equation. Obviously, the LST scheme is more efficient than the LBE scheme.

Model 2: $a=8$, $b=1.0$, $\sigma=0.1$ and $N=10$.

By (6.3) and (6.6), we have $K_0=12$ and $\ln(NC_1)\approx 13.6931$. Hence, we choose $k=1/9$, $L=14$ in LST scheme. Let $X^j(t)$ denote the jth path of $X(t)$ in Monte Carlo approximation. In order to illustrate the influence of parameters L on the

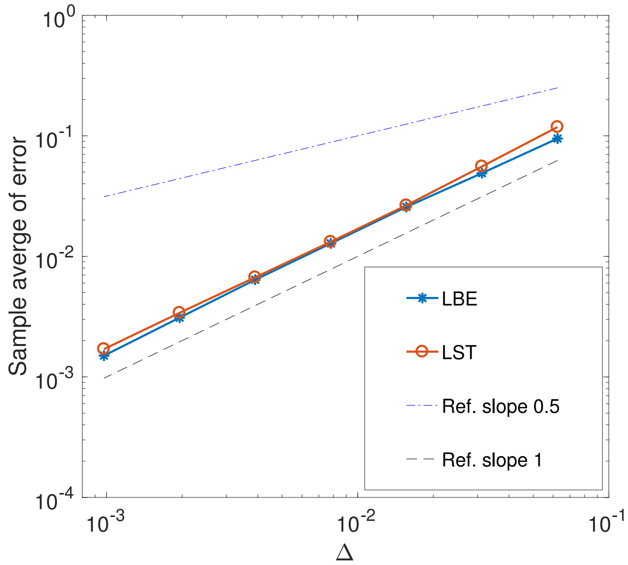

Fig. 1. Model 1, a = 5, b = 0.8, σ = 0.1 and N = 10.

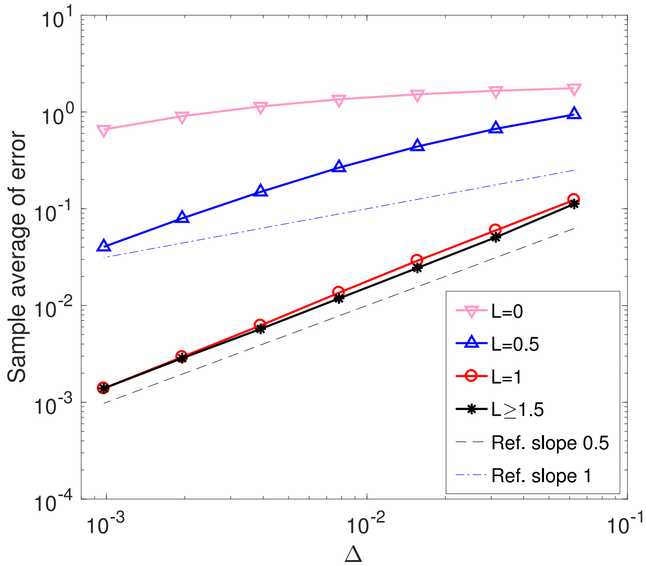

Fig. 2. Model 2, a = 8, b = 1, σ = 0.1 and N = 10.

Table 2
Error comparison for different L of Model 2.

Δ	$L=0$	$L=\frac{1}{2}$	$L=1$	$L=\frac{3}{2}$	$L=2$	$L=14$
2^{-4}	1.7610	0.9412	0.1233	0.1123	0.1140	0.1141
2^{-5}	1.6571	0.6701	0.0598	0.0507	0.0508	0.0508
2^{-6}	1.5225	0.4394	0.0290	0.0245	0.0245	0.0245
2^{-7}	1.3511	0.2660	0.0136	0.0118	0.0118	0.0118
2^{-8}	1.1421	0.1494	0.0062	0.0057	0.0057	0.0057
2^{-9}	0.9045	0.0796	0.0029	0.0029	0.0029	0.0029
2^{-10}	0.6597	0.0404	0.0014	0.0014	0.0014	0.0014

error, we choose several different L for comparison. We calculated that the sample average $\ln\left(\frac{1}{2000}\sum_{j=1}^{2000}\exp\left(|X^j(T)|\right)\right)\approx 1.4147$. The computational errors are shown in Table 2 and Fig. 2.

Table 3
Error comparison for different L of Model 3.

Δ	$L=0$	$L=1$	$L=1.3$	$L=1.8$	$L=2$	$L=9$
2^{-4}	1.2459	0.2755	0.0574	0.0361	0.0361	0.0361
2^{-5}	1.2039	0.1587	0.0251	0.0168	0.0167	0.0167
2^{-6}	1.1528	0.0843	0.0104	0.0081	0.0081	0.0081
2^{-7}	1.0899	0.0400	0.0043	0.0039	0.0039	0.0039
2^{-8}	1.0122	0.0169	0.0020	0.0019	0.0019	0.0019
2^{-9}	0.9167	0.0061	0.0010	0.0010	0.0010	0.0010
2^{-10}	0.8017	0.0018	0.0005	0.0005	0.0005	0.0005

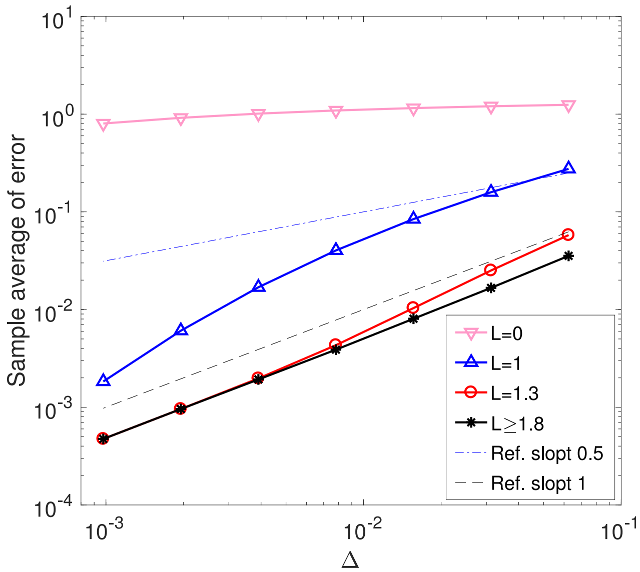

Fig. 3. Model 3, a = 6, b = 0.7, σ = 0.05 and N = 10.

Model 3: $a = 6$, $b = 0.7$, $\sigma = 0.05$ and $N = 10$.

By (6.3) and (6.6), we get $K_0 = 8$ and $\ln(NC_1) \approx 8.7031$. Hence, let $k = 1/9$, $L = 9$ in LST scheme. We calculated that the sample average $\ln\left(\frac{1}{2000}\sum_{j=1}^{2000}\exp\left(|X^j(T)|\right)\right) \approx 1.7960$. And we also select several values of L for comparison. The computational errors are shown in Table 3 and Fig. 3.

Using the small step-size 2^{-16} and 2000 paths, we calculate that

$$\ln \mathbb{E}\left[e^{|X_T|}\right] \approx \ln\left(\frac{1}{2000}\sum_{j=1}^{2000}\exp\left(|X^j(T)|\right)\right). \tag{6.10}$$

In this way we get $\ln \mathbb{E}\left[e^{|X_T|}\right] \approx 1.4147$ in Model 2 and $\ln \mathbb{E}\left[e^{|X_T|}\right] \approx 1.7960$ in Model 3.

In summary, it can be seen from Tables 2 and 3, the errors could be serious for small L even when the used step-sizes are small enough. For example, for $L = 1/2$ in Model 2 or $L = 1$ in Model 3, the errors are much larger than those for the cases $L \geq \ln \mathbb{E}\left[e^{|X_T|}\right]$ when the step-sizes are small enough. Increasing L then reduces the errors for the same step-size. As $L \geq \ln \mathbb{E}\left[e^{|X_T|}\right]$, the errors remain stable and are little affected by the value of L. The choice of the value of L is very important but complicated, which depends on the model properties. As indicated in the above numerical experiment results, one can choose the L larger than $\mathbb{E}\left[e^{|X_T|}\right]$, so that the LST scheme gives good convergence performance.

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