NUMERICAL METHODS FOR TIME-CHANGED STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY ADDITIVE NOISE

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ABSTRACT. This paper introduces a novel numerical approach specifically tailored for a class of nonlinear time-changed stochastic differential equations (SDE). The coefficients of these SDE meet the super-linear growth condition. By employing the Lamperti transformation, we convert the multiplicative noise present in these equations into additive noise, thereby simplifying the computation of numerical solutions and enhancing the order of convergence. The study not only investigates the strong convergence of the transformed SDEs but also provides a detailed analysis of the convergence order. Furthermore, numerical simulations are conducted to substantiate the theoretical findings, showcasing the method's significant advantage in improving the convergence order. This research offers a new perspective and tool for the numerical analysis of time-changed SDEs.

keywords time-changed stochastic differential equations · Euler-Maruyama-type method · non-linear · linear · Lamperti transformation · strong convergence · Subordinator

1. INTRODUCTION

In this paper, we investigate the rate of strong convergence of a numerical approximation scheme for an SDE of the form

$$dy(t) = a(y(t))dE(t) + b(y(t))dB(E(t)) \tag{1.1}$$

where the coefficients a and b satisfy some regularity conditions. However we are not concerned with these regular conditions, but more with the regular conditions of the SDE coefficients after the transformation of the Lamperti transform. Here, $E(t)$ represents the inverse subordinate process, and $B(t)$ represents the standard Brownian motion that is independent of $E(t)$, with the precise mathematical definition provided in Section 2.

2. PRELIMINARIES

In this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ represents a complete probability space, and $D = (D_t)_{t\geq 0}$ denotes a subordinated process with Laplace exponent ψ, starting from 0, where the killing rate of ψ is 0 and it has a Lévy measure ν. That is, D is a one-dimensional, non-decreasing Lévy process starting at 0 with càdlàg paths, whose Laplace transform is:

$$\mathbb{E}[e^{-sD_t}] = e^{-t\psi(s)}, \quad \text{where} \quad \psi(s) = \int_0^\infty (1 - e^{-sy})\, \nu(\mathrm{d}y), \quad s > 0,$$

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and $\int_0^\infty (y \wedge 1)\nu(\mathrm{d}y) < \infty$. We consider the case where the Lévy measure ν is infinite, i.e., $\nu(0,\infty) = \infty$, meaning compound Poisson subordination is not considered. Let $E = (E(t))_{t\geq 0}$ be the inverse of D, i.e.:

$$E(t) := \inf\{u > 0; D_u > t\},\ t \geq 0.$$

We refer to E as the inverse subordinated process. If the subordinated process D is stable and its index is $\beta \in (0,1)$, then $\psi(s) = s^\beta$, and E is called the inverse β-stable subordinated process. Assuming $\nu(0,\infty) = \infty$ implies that D has strictly increasing paths with infinitely many jumps, thus E has continuous, non-decreasing paths starting from 0. Moreover, the inverse relationship between D and E implies that for all $t, x \geq 0$, we have $\{E_t > x\} = \{D_x < t\}$. Note that jumps in D correspond to constant (random) time intervals in E, and during these constant periods, any time-changed process of the form $X \circ E = (X_{E_t})_{t\geq 0}$ also remains constant. If B is a standard Brownian motion independent of D, particles represented by the time-changed Brownian motion $B \circ E$ are considered trapped and immobile during constant periods; note that although $B \circ D$ is a Lévy process, $B \circ E$ is not even a Markov process. (See figure 2.1)

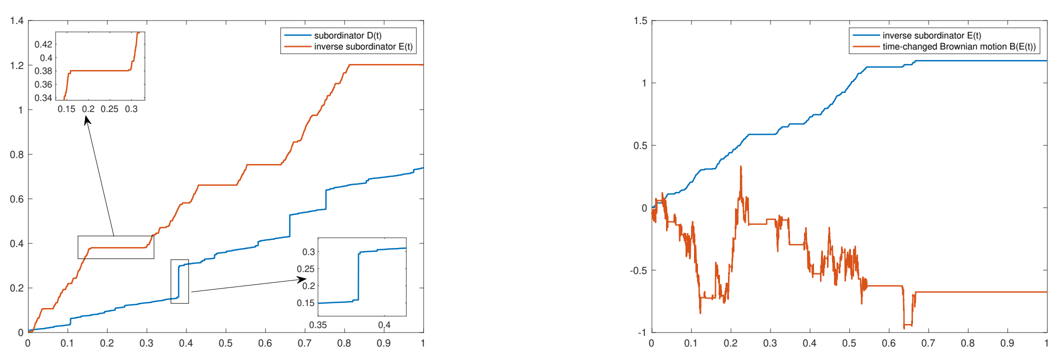


FIGURE 2.1. The left figure shows the subordinated D with stability index $\alpha = 0.8$ and the corresponding inverse subordinator E. The right figure shows the inverse subordinator E with stability index $\alpha = 0.8$ and the corresponding time-changed Brownian motion.

E and $B \circ E$ both start from 0. For the quadratic variation, we have

$$[B \circ E, B \circ E] = E \quad \text{and} \quad [E, E] = [B \circ E, E] = 0.$$

For more details on stochastic calculus for time-changed semimartingales, see Section 4 of [4].

Now, let's introduce the Lamperti transformation for time change. Let $S = (l, r)$, where $-\infty \leq l < r \leq \infty$, and let a and b be continuously differentiable functions from S to S. Consider the following SDE:

$$dy(t) = a(y(t))dE(t) + b(y(t))dB(E(t)) \tag{2.1}$$

and assume it has a unique strong solution in S, i.e.,

$$\mathbb{P}(y(t) \in S,\ t \geq 0) = 1.$$

If $b(x) > 0$ holds for all $x \in S$, then we can use the Lamperti transformation

$$F(x) = \lambda \int^x \frac{1}{b(y)} dy \tag{2.2}$$

for some $\lambda > 0$. Provided $F^{-1} : F(S) \to S$ is well-defined, let $x(t) = F(y(t))$. Using the time-change Itô formula from [8], we get:

$$dx(t) = f(x(t))dE(t) + \lambda dB(E(t)) \tag{2.3}$$

where

$$f(x) = \lambda \left(\frac{a(F^{-1}(x))}{b(F^{-1}(x))} - \frac{1}{2} b'(F^{-1}(x)) \right), \quad x \in F(S),$$

and $F(D) = (F(l), F(r))$. This transformation converts the nonlinear term of the diffusion into the drift term. In the subsequent part of this paper, we will focus on studying SDE (2.3) with additive noise.

Introducing the following three lemmas on time change from [8]:

Lemma 2.1 (First Variable Change Formula)**.** *Let B be a one-dimensional standard Brownian motion. If $H \in L(B(t), \mathcal{F}_t)$, then $H_{E(t-)} \in L(B_{E(t)}, \mathcal{F}_{E_t})$. Furthermore, for all $t \geqslant 0$, almost everywhere*

$$\int_0^{E_t} H_s dB(s) = \int_0^t H_{E(s-)} dB_{E(s)}.$$

Lemma 2.2 (Second Variable Change Formula)**.** *Let B be a one-dimensional standard Brownian motion. Assume D and E satisfy $[D \longrightarrow E]$ or $[D \longleftarrow E]$. Assume B and E are synchronized. If $K \in L(B_{E(t)}, \mathcal{F}_{E_t})$, then $(K_{D(t-)}) \in L(B(t), \mathcal{F}_{E(D_t)})$. Furthermore, for all $t \geqslant 0$, almost everywhere*

$$\int_0^t K_s dB_{E(s)} = \int_0^{E_t} K_{D(s-)} dB(s).$$

Lemma 2.3 (Itô Formula)**.** *Let B be a one-dimensional standard Brownian motion. Let D and E satisfy $[D \longrightarrow E]$ or $[D \longrightarrow E]$. X is a stochastic process defined by the following SDE:*

$$X(t) := \int_0^t A(s)ds + \int_0^t F(s)dE(s) + \int_0^t G(s)dB(E(s))$$

where $A(s) \in L(t, \mathcal{F}_{E(t)})$, $F(s) \in L(E(t), \mathcal{F}_{E(t)})$, and $G(s) \in L(B(E(s)), \mathcal{F}_{E(t)})$. If $f \in C^2(\mathbb{R})$, then $f(X(t))$ is an $\mathcal{F}_{E(t)}$-semimartingale, and for all $t \geq 0$, we have

$$\begin{aligned} f(X(t)) - f(0) &= \int_0^t f'(X(s))A(s)ds + \int_0^{E(t)} f'\left(X(D(s-))\right) F(D(s-))ds \\ &+ \int_0^{E(t)} f'\big(X(D(s-))\big) G(D(s-))dB(s) + \frac{1}{2}\int_0^{E(t)} f''\big(X(D(s-))\big)\big\{G(D(s-))\big\}^2 ds. \end{aligned}$$

Next, introduce the discrete Gronwall inequality:

Lemma 2.4. *Let $\Delta t > 0, g_n, \lambda_n \in \mathbb{R}, \eta > 0, a_1 = 0$, and suppose $1 - \eta \Delta E_j > 0$, $1 + \lambda_n > 0, n \in \mathbb{N}$. If*

$$a_{n+1} \leq a_n(1 + \lambda_n) + \eta a_{n+1} \Delta E_n + g_{n+1}$$

then the following inequality holds:

$$a_n \leq \sum_{j=0}^{n-1} \prod_{i=j}^{n} (1 - \eta \Delta E_i) g_{j+1} \prod_{l=j+1}^{n-1} (1 + \lambda_l) \tag{2.4}$$

The following lemma is crucial for proving the main theorems in this paper, quoting the lemma from [7].

Lemma 2.5. *Assume that in the Laplace transform of the inverse subordinated process $E(t)$, $\psi(s) = s^\alpha$. Then*

$$\lim_{t\to\infty} \frac{E_t}{t} = 0, \ a.s. \tag{2.5}$$

Lemma 2.6. *If E is the inverse of the subordinated process D, and its Laplace exponent ψ has a regular variation index at infinity $\beta \in [0,1)$. If $\beta = 0$, further assume $\nu(0,\infty) = \infty$. Fix $\lambda > 0$, $t > 0$, and $r > 0$.*

(1) If $r < \frac{1}{1-\beta}$, then $\mathbb{E}\left[e^{\lambda E_t^r}\right] < \infty$.
(2) If $r > \frac{1}{1-\beta}$, then $\mathbb{E}\left[e^{\lambda E_t^r}\right] = \infty$.

Lemma 2.7. *For any given $0 = t_0 < ih < t_1 < t_2 < \ldots < t_n < (i+1)h$, we have:*

$$\mathbb{E}\left[\int_{ih}^{(i+1)h}\int_{ih}^{t_n}\cdots\int_{ih}^{t_2} 1dE(t_1)\ldots dE(t_{n-1})dE(t_n)\right] \le Ch^{1+(n-1)\beta} \tag{2.6}$$

where C is a constant independent of h.

Proof. Now, introduce the random measure Π on $[0,\infty)$, defined as $\Pi((s,t]) = E(t) - E(s)$, where $t > s \ge 0$. Let $\{C(t)\}_{t\ge 0}$ be a Cox process driven by Π, i.e., conditional on $\Pi = \lambda$, the distribution of $\{C(t)\}$ is equivalent to that of an inhomogeneous Poisson process with intensity λ. Note that according to [3], $\{C(t)\}$ is a renewal process with the renewal function:

$$u(t) = \mathbb{E}[C(t)] = \mathbb{E}[E(t)] = \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{2.7}$$

For the renewal process $C(t)$, see [1], we obtain

$$\mathbb{E}[\mathrm{d}C(t_n)\ldots\mathrm{d}C(t_1)] = \prod_{i=1}^{n} u'(t_i - t_{i-1})\mathrm{d}t_i$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_n$. Since the factorial moments of the Cox process $C(t)$ are equal to the ordinary moments of its driving measure Π, see [1], we get

$$\mathbb{E}[\mathrm{d}E(t_n)\ldots\mathrm{d}E(t_1)] = \prod_{i=1}^{n} u'(t_i - t_{i-1})\mathrm{d}t_i.$$

Thus:

$$\begin{aligned}
I &= \mathbb{E}\left[\int_{ih}^{(i+1)h}\int_{ih}^{t_n}\int_{ih}^{t_{n-1}}\cdots\int_{ih}^{t_2} 1dE(t_1)\ldots dE(t_{n-2})dE(t_{n-1})dE(t_n)\right] \\
&= \int_{ih}^{(i+1)h}\int_{ih}^{t_n}\int_{ih}^{t_{n-1}}\cdots\int_{ih}^{t_2} 1\mathbb{E}\left[dE(t_1)\ldots dE(t_{n-2})dE(t_{n-1})dE(t_n)\right] \\
&= \frac{\alpha^n}{\Gamma^n(\alpha+1)}\int_{ih}^{(i+1)h}\int_{ih}^{t_n}\int_{ih}^{t_{n-1}}\cdots\int_{ih}^{t_2}\prod_{i=1}^{i=n}(t_i - t_{i-1})^{\alpha-1}dt_1\ldots dt_{n-1}dt_n
\end{aligned}$$

Next, consider the integral term separately:

$$I_1 = \int_{ih}^{t_2}(t_2 - t_1)^{\alpha-1}t_1^{\alpha-1}dt_1$$

Make the following transformation: let $t_1 = ih + s_1h$ and $t_2 = ih + s_2h$, where h is the step size, hence $s_1, s_2 \in [0,1]$. Since we do not consider time at the origin, $i = \frac{T}{h}$ where T is a time range, thus

$$I_1 = \int_0^{s_2}(s_2 - s_1)^{\alpha-1}h^{\alpha-1}(ih + s_1h)^{\alpha-1}hds_1$$

$$= h^\alpha \int_0^{s_2} (s_2 - s_1)^{\alpha-1}(ih + s_1 h)^{\alpha-1} ds_1$$

Since $(ih + s_1 h)^{\alpha-1}$ is monotonically decreasing with respect to s_1 in $[0, 1]$, and the integrand and integration range in I_n are positive, we have

$$I_1 \le h^\alpha \int_0^{s_2} (s_2 - s_1)^{\alpha-1}(ih)^{\alpha-1} ds_1$$
$$= T^{\alpha-1} h^\alpha \int_0^{s_2} (s_2 - s_1)^{\alpha-1} ds_1$$

Let $w_1 = s_2 - s_1$, thus

$$I_1 \le T^{\alpha-1} h^\alpha \int_0^{s_2} (s_2 - s_1)^{\alpha-1} ds_1 = T^{\alpha-1} h^\alpha \int_0^{s_2} w_1^{\alpha-1} dw_1 = \frac{T^{\alpha-1} s_2^\alpha}{\alpha} h^\alpha$$

Therefore:

$$I \le Ch^\alpha \int_{ih}^{(i+1)h} \int_{ih}^{t_n} \int_{ih}^{t_{n-1}} \dots \int_{ih}^{t_3} \prod_{i=3}^{n} (t_i - t_{i-1})^{\alpha-1} dt_2 \dots dt_{n-1} dt_n$$

Similarly, analyzing the following integral:

$$I_2 = \int_{ih}^{t_3} (t_3 - t_2)^{\alpha-1} dt_2 \le Ch^\alpha$$

Therefore:

$$I \le Ch^{2\alpha} \int_{ih}^{(i+1)h} \int_{ih}^{t_n} \int_{ih}^{t_{n-1}} \dots \int_{ih}^{t_4} \prod_{i=4}^{n} (t_i - t_{i-1})^{\alpha-1} dt_3 \dots dt_{n-1} dt_n$$

By iterating this process, we get:

$$I \le Ch^{(n-1)\alpha} \int_{ih}^{(i+1)h} 1 dt_1 \le Ch^{1+(n-1)\alpha}$$

□

3. Main results: the strong convergence of EM method

Assumption 3.1. *In this section, we assume that the drift coefficient f of the SDE (2.3) satisfies the global* Lipschitz *condition, that is, there exists a constant $K > 0$ such that*

$$|f(x) - f(y)| \le K|x - y|. \tag{3.1}$$

Assumption 3.2. *In this section, we assume that the drift coefficient f of the SDE (2.3) satisfies the linear growth condition, that is, there exists a constant $K > 0$ such that*

$$|f(x)| \le K(1 + |x|). \tag{3.2}$$

Assumption 3.3. *In this section, we assume that the drift coefficient f of the SDE (2.3) is twice continuously differentiable, and there exists a constant $K > 0$ such that*

$$|f(x)f'(x)| + |\sigma f'(x)| + |\sigma^2 f''(x)| \le K(1 + |x|). \tag{3.3}$$

The following proposition aims to derive sufficient conditions for the existence of the p-th moment of $\sup_{0\le r\le T} |X_r|$, where X is the solution to the SDE (2.3). These conditions are necessary for establishing the main theorems of this paper.

Proposition 3.1. *Let X be the solution to the SDE (2.3), where f satisfies assumption 3.1 and assumption 3.2. Then for any $p \ge 1$, there exist constants c_1, c_2 such that $\mathbb{E}_B[Y_T^{(p)}] < c_1 e^{c_2 E_T}$, where $Y_t^{(p)} := 1 + \sup\limits_{0\le r\le t} |X_r|^p$.*

Proof. Let $S_\ell := \inf\{t \geq 0 : Y_t^{(p)} > \ell\}$ for $\ell \in \mathbb{N}$. Since the solution X has continuous paths and $Y_t^{(p)} < \infty$ for each $t \geq 0$, $S_\ell \uparrow \infty$ as $\ell \to \infty$. For $\mathbb{P}_D$ almost every path, we first apply the Gronwall-type inequality to the function $t \mapsto \mathbb{E}_B[Y_{t\wedge S_\ell}^{(p)}]$ for a fixed ℓ, then let $t = T$ and let $\ell \to \infty$ in the resulting inequality to establish the upper bound for $\mathbb{E}_B[Y_T^{(p)}]$. Note that by the definition of S_ℓ,

$$\int_0^t \mathbb{E}_B[Y_{r\wedge S_\ell}^{(p)}]\, dE_r \leq \ell E_t < \infty,$$

which allows us to safely apply the Gronwall-type inequality.

Suppose $p \geq 2$, as the result for $1 \leq p < 2$ follows directly by applying the result for $p \geq 2$ and Jensen's inequality. By the time-changed Itô formula, we have $X_s^p = x_0^p + J_s + K_s$, where

$$J_s := \int_0^s \sigma p X_r^{p-1}\, dB_{E_r};$$

$$K_s := \int_0^s \left\{ pX_r^{p-1} f(X_r) + \frac{\sigma^2}{2} p(p-1) X_r^{p-2} \right\} dE_r.$$

Fix $t \in [0,T]$ and $\ell \in \mathbb{N}$. By assumption 3.2 and the inequality $(x+y+z)^p \leq c_p(x^p + y^p + z^p)$, where $x, y, z \geq 0$ and $c_p = 3^{p-1}$, we have

$$\mathbb{E}_B\left[\sup_{0\leq s\leq t\wedge S_\ell} |K_s|\right] \leq \left(pc_pK + \frac{1}{2}p(p-1)c_pK^2\right) \int_0^{t\wedge S_\ell} \mathbb{E}_B[Y_r^{(p)}]\, dE_r.$$

Since $(J_s)_{s\geq 0}$ is a local martingale, applying the BDG inequality gives

$$\mathbb{E}_B\left[\sup_{0\leq s\leq t\wedge S_\ell} |J_s|\right] \leq b_1 \mathbb{E}_B\left[\left(\int_0^{t\wedge S_\ell} \sigma^2 p^2 X_r^{2p-2}\, dE_r\right)^{1/2}\right],$$

thus,

$$\begin{aligned}\mathbb{E}_B\left[\sup_{0\leq s\leq t\wedge S_\ell} |J_s|\right] &\leq b_1 \mathbb{E}_B\left[pc_pK \left(Y_{t\wedge S_\ell}^{(p)} \int_0^{t\wedge S_\ell} Y_r^{(p)}\, dE_r\right)^{1/2}\right] \\ &\leq \frac{1}{2}\mathbb{E}_B\left[Y_{t\wedge S_\ell}^{(p)}\right] + 2b_1^2p^2c_p^2K^2 \int_0^{t\wedge S_\ell} \mathbb{E}_B\left[Y_r^{(p)}\right] dE_r,\end{aligned}$$

where the last inequality follows from the basic inequality $(ab)^{1/2} \leq a/\lambda + \lambda b$ for any $a, b, \lambda > 0$, with $\lambda := 2b_1pc_pK$. Note that for any non-negative process $(L_t)_{t\geq 0}$,

$$\int_0^{t\wedge S_\ell} L_r\, dE_r \leq \int_0^t L_{r\wedge S_\ell}\, dE_r.$$

Indeed, the inequality is obvious if $t \leq S_\ell$, while if $t > S_\ell$,

$$\int_0^{S_\ell} L_r\, dE_r + \int_{S_\ell}^t L_{S_\ell}\, dE_r \geq \int_0^{t\wedge S_\ell} L_r\, dE_r.$$

Thus, from the above estimates for J_s and K_s, we have

$$\mathbb{E}_B[Y_{t\wedge S_\ell}^{(p)}] \leq 2(1 + |x_0|^p) + 2K^2\left(pc_pK + \left(p(p-1)c_p/2 + 2b_1^2p^2c_p^2\right)\right) \int_0^t \mathbb{E}_B[Y_{r\wedge S_\ell}^{(p)}]\, dE_r.$$

Applying the Gronwall-type inequality, we get

$$\mathbb{E}_B[Y^{(p)}_{t\wedge S_\ell}] \le 2(1+|x_0|^p)e^{2K^2E_T\left(pc_pK+\left(p(p-1)c_p/2+2b_1^2p^2c_p^2\right)\right)}.$$

Letting $t=T$ and letting $\ell\to\infty$, since $\xi(u)$ is independent of ℓ, and by applying the monotone convergence theorem, we obtain

$$\mathbb{E}_B[Y^{(p)}_T] \le 2(1+|x_0|^p)e^{2K^2E_T\left(pc_pK+\left(p(p-1)c_p/2+2b_1^2p^2c_p^2\right)\right)}. \tag{3.4}$$

□

Next, we prove the first important theorem.

Theorem 3.1. *For any constant $\epsilon>0$, let $\epsilon<T_1<T_2$, $\lceil T_1/\Delta t\rceil = m$ and $\lceil T_2/\Delta t\rceil = n$. Under the conditions of* assumption 3.1, assumption 3.2, *and* assumption 3.3, *there exists a constant C such that the following inequality holds:*

$$\mathbb{E}\left[\sup_{i=m,m+1,\ldots,n}|X(t_i)-X_{t_i}|\right]\le C\Delta t^\alpha$$

Proof. Consider the integral of the SDE (2.3) over $[t_i,t_{i+1})$:

$$\int_{t_i}^{t_{i+1}} dX(s) = \int_{t_i}^{t_{i+1}} f(X(s))dE(s)+\int_{t_i}^{t_{i+1}}\sigma dB(E(s)). \tag{3.5}$$

By the change of variable formula for time transformations [4], the above is equivalent to

$$\int_{t_i}^{t_{i+1}} dX(s) = \int_{E_{t_i}}^{E_{t_{i+1}}} f(X(D(s-)))ds+\int_{t_i}^{t_{i+1}}\sigma dB(E(s)). \tag{3.6}$$

For the drift term $f(X(D(s-)))$, the following equation always holds:

$$\int_{E(t_i)}^{E(t_{i+1})} f(X(D(t-)))-f(X(D(t_i-)))dt = \int_{E(t_i)}^{E(t_{i+1})}\int_{D(t_i-)}^{D(t-)} df(X(s))dt. \tag{3.7}$$

By the change of variable Itô formula lemma 2.3, (3.7) becomes

$$\begin{aligned}
&\int_{E(t_i)}^{E(t_{i+1})} f(X(D(t-)))-f(X(D(t_i-)))dt\\
&=\int_{E(t_i)}^{E(t_{i+1})}\int_{t_i}^{t}\left(f(X(D(s-)))f'(X(D(s-)))+\frac{1}{2}\sigma^2f''(X(D(s-)))\right)ds\,dt\\
&\quad+\int_{E(t_i)}^{E(t_{i+1})}\int_{t_i}^{t}\sigma f'(X(D(s-)))\,dB(s)\,dt.
\end{aligned} \tag{3.8}$$

By (3.6) and (3.8), and the change of variable formula for time transformations, we get

$$\begin{aligned}
X(t_{i+1}) &= X(t_i)+\int_{E(t_i)}^{E(t_{i+1})} f(X(D(t_i-)))\,dt+\int_{t_i}^{t_{i+1}}\sigma\,dB(E(t))\\
&\quad+\int_{E(t_i)}^{E(t_{i+1})}\int_{t_i}^{t}\left(f(X(D(s-)))f'(X(D(s-)))+\frac{1}{2}\sigma^2f''(X(D(s-)))\right)ds\,dt\\
&\quad+\int_{E(t_i)}^{E(t_{i+1})}\int_{t_i}^{t}\sigma f'(X(D(s-))))\,dB(s)\,dt\\
&= X(t_i)+\int_{t_i}^{t_{i+1}} f(X(t_i))\,dE(t)+\int_{t_i}^{t_{i+1}}\sigma\,dB(E(t))
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_i}^{t_{i+1}}\int_{E(t_i)}^{E(t)}\left(f(X(D(s-)))f'(X(D(s-)))+\frac{1}{2}\sigma^2 f''(X(D(s-)))\right)ds\,dE(t)\\
&+\int_{t_i}^{t_{i+1}}\int_{E(t_i)}^{E(t)}\sigma f'(X(D(s-)))\,dB(s)\,dE(t)\\
=\,&X(t_i)+\int_{t_i}^{t_{i+1}} f(X(t_i))\,dE(t)+\int_{t_i}^{t_{i+1}}\sigma\,dB(E(t))\\
&+\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\left(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\right)dE(s)\,dE(t)\\
&+\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\sigma f'(X(s))\,dB(E(s))\,dE(t).
\end{aligned}
$$

Thus,

$$
X(t_{i+1})=X(t_i)+\int_{t_i}^{t_{i+1}} f(X(t_i))\,dE(t)+\int_{t_i}^{t_{i+1}}\sigma\,dB(E(t))+R_i. \tag{3.9}
$$

Decompose R_i as $R_i=R_i^{(1)}+R_i^{(2)}$, where:

$$
\begin{aligned}
R_i^{(1)}&=\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\left(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\right)dE(s)\,dE(t),\\
R_i^{(2)}&=\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\sigma f'(X(s))\,dB(E(s))\,dE(t).
\end{aligned}
$$

Using (3.6) and (3.9), we obtain:

$$
X(t_{i+1})-X(t_{i+1})=X(t_i)-X(t_i)+(f(X(t_i))-f(x^{\delta}_{t_i}))\Delta E_i+R_i. \tag{3.10}
$$

Let $e_i=X(t_i)-X_{t_i}$, from assumption 3.1 we get:

$$
|e_{s+1}|\le(1+K\Delta E_s)|e_s|+R_s,\quad \text{where } s=m,\dots,n. \tag{3.11}
$$

From lemma 2.4, we obtain $|e_n|\le\sum_{j=m-1}^{n-1}|R_{j+1}|\prod_{l=j+1}^{n-1}|1+K\Delta E_l|$.

To estimate this inequality, we use the fact that the path of the inverse subordinated process $E(t)$ is almost everywhere Hölder continuous. Specifically, for each $0<\beta<1$, there exists a constant $C>0$, such that for all t,s (when $t\neq s$), we have

$$
|E(t)-E(s)|\le C|t-s|^{\beta}\quad\text{a.s.} \tag{3.12}
$$

Here, β is related to the stability index α of the Lévy process $D(t)$, and typically $\beta\le\alpha$. This Hölder continuity implies that the increments of the inverse process $E(t)$ decay at a rate of α, so $E(t)$ exhibits smooth continuous paths. Hence,

$$
\sup_{k=m,\dots,n}\prod_{l=1}^{k}(1+K\Delta E_l)\le\sup_{k=m,\dots,n}\prod_{l=1}^{k}(1+C\Delta t^{\beta})
$$

Taking the limit as $\Delta t\to 0$, we get

$$
\sup_{k=m,\dots,n}\prod_{l=1}^{k}(1+K\Delta E_l)<\infty
$$

Thus,

$$
\mathbb{E}\left[\sup_{k=m,\dots,n}|e_k|\right]\le C\mathbb{E}\sum_{j=0}^{n-1}|R_{j+1}|\le C\mathbb{E}\sum_{j=0}^{n-1}|R^{(1)}_{j+1}|+C\mathbb{E}\sum_{j=0}^{n-1}|R^{(2)}_{j+1}|. \tag{3.13}
$$

Thus, for the first term, from assumption 3.3, proposition 3.1, and lemma 2.7, we can obtain

$$\begin{aligned}\mathbb{E}\left[|R_{j+1}^{(1)}|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\mathbb{E}_B\left(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\right)dE(s)\,dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\mathbb{E}_B\left[1+|X(s)|\right]dE(s)\,dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}e^{E(T)}dE(s)\,dE(t)\right]\end{aligned}$$

The last inequality follows from (3.4). Since E_t is $\mathcal{G}_t$-measurable, and satisfies lemma 2.5, there exists a constant $T>0$ such that $E_T<T$. Similarly, when $t\leq T$, we can obtain

$$E_t \leq E_T \leq T, \tag{3.14}$$

Thus, we can obtain

$$\mathbb{E}\left[|R_{j+1}^{(1)}|\right] \leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}e^{T}dE(s)\,dE(t)\right]\leq C\Delta t^{1+\alpha}.$$

Using the linearity of expectation, we can obtain

$$\mathbb{E}\sum_{j=0}^{n-1}|R_{j+1}^{(1)}| = \sum_{j=0}^{n-1}\mathbb{E}|R_{j+1}^{(1)}| \leq C\sum_{j=0}^{n-1}\Delta t^{1+\alpha}\leq C\Delta t^{\alpha}. \tag{3.15}$$

For the second term $\mathbb{E}\left|\sum_{j=0}^{n-1}R_{j+1}^{(2)}\right|$, using the BDG inequality and lemma 2.7, we obtain Thus,

$$\begin{aligned}\mathbb{E}\left[|R_{j+1}^{(2)}|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\mathbb{E}_B\left[\int_{t_i}^{t}\sigma f'(X(s))dB_{E(s)}\right]dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}\mathbb{E}_B\left(\sigma f'(X(s))\right)^2dE(s)\right]^{\frac{1}{2}}dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}\mathbb{E}_B\left[1+|X(s)|\right]^2dE(s)\right]^{\frac{1}{2}}dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}e^{2E(T)}dE(s)\right]^{\frac{1}{2}}dE(t)\right]\\ &\leq C\left\{\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}e^{2E(T)}dE(s)\right]^{\frac{1}{2}}dE(t)\right]^2\right\}^{\frac{1}{2}}\\ &\leq C\Delta t^{\frac{1}{2}+\alpha}.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we can obtain:

$$\mathbb{E}\left|\sum_{j=0}^{n-1}R_{j+1}^{(2)}\right| \leq C\mathbb{E}\left|\sum_{j=0}^{n-1}(R_{j+1}^{(2)})^2\right|^{\frac{1}{2}} \leq C\sqrt{\sum_{j=0}^{n-1}\mathbb{E}(R_{j+1}^{(2)})^2}\leq C\sqrt{\sum_{j=0}^{n-1}\Delta t^{1+2\alpha}}\leq C\Delta t^{\alpha}. \tag{3.16}$$

Thus, combining (3.15) and (3.16), we get

$$\mathbb{E}\left[\sup_{k=m\ldots n}|e_k|\right]\leq C\Delta t^{\alpha}.$$

□

4. Main results: the strong convergence of BEM method

Assumption 4.1. *In this section, we assume that the drift coefficient f of the SDE (2.3) satisfies the super-linear growth condition, that is, there exists a constant $K > 0$ and $\gamma > 1$ such that*

$$|f(x)| \le K(1 + |x|^\gamma). \tag{4.1}$$

Assumption 4.2. *In this section, we assume that the drift coefficient f of the SDE (2.3) is twice continuously differentiable, and there exists a constant $K > 0$ and $\gamma > 1$ such that*

$$|f(x)f'(x)| + |\sigma f'(x)| + |\sigma^2 f''(x)| \le K(1 + |x|^\gamma). \tag{4.2}$$

Assumption 4.3. *In this section, we let $c \in [-\infty, +\infty), I = (c, +\infty)$, and $d \in I$ be any point in the interval, and assume that the drift coefficient f of the SDE (2.3) satisfies the following monotonicity condition:*

$$f : I \to \mathbb{R}, \text{ such that there exists } K \in \mathbb{R}, \forall x, y \in I, x \le y, \ f(y) - f(x) \le K(y - x). \tag{4.3}$$

For the existence and uniqueness proof of the solution to the SDE (2.3) under the condition of the drift coefficient f being monotone, refer to the proof in [8] for the case where the drift satisfies the global Lipschitz condition. For the stochastic differential equation (2.3), its BEM numerical scheme is:

$$X_{t_{i+1}} = X_{t_i} + f(X_{t_{i+1}})\Delta E_i + \sigma \Delta B_{E_i}, \quad i = 0, 1, 2, \ldots, \qquad X_0 = X(0) \tag{4.4}$$

where $\Delta E_i = E(t_{i+1}) - E(t_i)$ and $\Delta B_{E_i} = B(E(t_{i+1})) - B(E(t_i))$.

Similar to proposition 3.1, we can derive the following proposition.

Proposition 4.1. *Let X be the solution to the SDE (2.3), where f satisfies assumption 4.3 and assumption 4.1. Then for any $p \ge 1$, there exist constants c_1, c_2 such that $\mathbb{E}_B[Y_T^{(p)}] < c_1 e^{c_2 E_T}$, where $Y_t^{(p)} := 1 + \sup_{0 \le r \le t} |X_r|^p$.*

Next, we prove the second important theorem.

Theorem 4.1. *For any constant $\epsilon > 0$, let $\epsilon < T_1 < T_2$, with $\lceil T_1/\Delta t \rceil = m$ and $\lceil T_2/\Delta t \rceil = n$. Under the conditions of assumption 4.3, assumption 4.1, and assumption 4.2, there exists a constant C such that the following inequality holds:*

$$\mathbb{E}\left[\sup_{i=m,m+1,\ldots,n} |X(t_i) - X_{t_i}|\right] \le C\Delta t^\alpha.$$

Proof. Due to the validity of the first and second variable transformation formulas lemma 2.1 and lemma 2.2, we can consider the integral of the SDE (2.3) over the interval $[t_i, t_{i+1})$:

$$\int_{t_i}^{t_{i+1}} dX(s) = \int_{t_i}^{t_{i+1}} f(X(s))dE(s) + \int_{t_i}^{t_{i+1}} \sigma dB(E(s)) \tag{4.5}$$

which is equivalent to

$$\int_{t_i}^{t_{i+1}} dX(s) = \int_{E_{t_i}}^{E_{t_{i+1}}} f(X(D(s-)))ds + \int_{E_{t_i}}^{E_{t_{i+1}}} \sigma dB(s) \tag{4.6}$$

For the drift term $f(X(D(s-)))$, the following equality always holds:

$$\int_{E(t_i)}^{E(t_{i+1})} f(X(D(t_{i+1}-))) - f(X(D(t-)))dt = \int_{E(t_i)}^{E(t_{i+1})} \int_{D(t-)}^{D(t_{i+1}-)} df(X(s))dt \tag{4.7}$$

For $df(X(s))$, from the time transformation Itô formula lemma 2.3, we get:

$$f(X(t))-f(0)=\int_0^{E(t)} f(X(D(s-)))f'(X(D(s-)))+\frac{\sigma^2}{2}f''(X(D(s-)))ds \\ +\int_0^{E(t)}\sigma f'(X(D(s-)))dB(s)$$

Thus, (4.7) becomes

$$\begin{aligned} &\int_{E(t_i)}^{E(t_{i+1})} f(X(D(t_{i+1}-)))-f(X(D(t-)))dt \\ &=\int_{E(t_i)}^{E(t_{i+1})}\int_t^{t_{i+1}}\Big(f(X(D(s)))f'(X(D(s)))+\frac{1}{2}\sigma^2 f''(X(D(s)))\Big)\,ds\,dt \\ &+\int_{E(t_i)}^{E(t_{i+1})}\int_t^{t_{i+1}}\sigma f'(X(D(s)))\,dB(s)\,dt \end{aligned} \tag{4.8}$$

Therefore, we have

$$X(t_{i+1})=X(t_i)+\int_{t_i}^{t_{i+1}} f(X(t_{i+1}))\,dE(t)+\int_{t_i}^{t_{i+1}}\sigma\,dB(E(t))+R_i \tag{4.9}$$

where

$$\begin{aligned} R_i=&-\int_{t_i}^{t_{i+1}}\int_t^{t_{i+1}}\Big(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\Big)\,dE(s)\,dE(t) \\ &-\int_{t_i}^{t_{i+1}}\int_t^{t_{i+1}}\sigma f'(X(s))\,dB(E(s))\,dE(t) \end{aligned}$$

Decomposing R_i as $R_i=R_i^{(1)}+R_i^{(2)}$, where:

$$\begin{aligned} R_i^{(1)}&=-\int_{t_i}^{t_{i+1}}\int_t^{t_{i+1}}\Big(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\Big)\,dE(s)\,dE(t), \\ R_i^{(2)}&=-\int_{t_i}^{t_{i+1}}\int_t^{t_{i+1}}\sigma f'(X(s))\,dB(E(s))\,dE(t) \end{aligned}$$

Subtracting the discrete scheme, i.e., (4.9) - (4.4), we get:

$$X(t_{i+1})-X_{t_{i+1}}=X(t_i)-X_{t_i}+(f(X(t_{i+1}))-f(X_{t_{i+1}}))\Delta E_i+R_i \tag{4.10}$$

Let $e_i=X(t_i)-X_{t_i}$, and from assumption 4.3, we obtain:

$$(1-K_1\Delta E_s)e_{s+1}\le e_s+R_s,\quad \text{where } s=m,m+1,\dots,n \tag{4.11}$$

From the Hölder continuity of the inverse subordinate process $E(t)$, i.e., (3.12), we get:

$$\sup_{k=m,\dots,n}\prod_{l=m}^{k}(1-K_1\Delta E_l)^{-1}<\sup_{k=m,\dots,n}\prod_{l=m}^{k}(1-C\Delta t^{\gamma})^{-1} \tag{4.12}$$

Taking the limit as $\Delta t\to 0$, we have:

$$\sup_{k=m,\dots,n}\prod_{l=m}^{k}(1-K_1\Delta E_l)^{-1}<\infty$$

Combining (4.12) with lemma 2.4, we obtain:

$$\mathbb{E}\left[\sup_{k=m,\dots,n}|e_k|\right]\le C\mathbb{E}\sum_{j=m}^{n}|R_j^{(1)}|+C\mathbb{E}\sum_{j=m}^{n}|R_j^{(2)}|$$

For the first term, by assumption 4.2, proposition 4.1, and (3.14), we can obtain:

$$\begin{aligned}\mathbb{E}\left[|R_{j+1}^{(1)}|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\mathbb{E}_B\left(f(X(s))f'(X(s))+\frac{1}{2}\sigma^2 f''(X(s))\right)dE(s)\,dE(t)\right]\\ &= C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\mathbb{E}_B\left[1+|X(s)|^{\gamma}\right]dE(s)\,dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}e^{\gamma E(s)}dE(s)\,dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}e^{\gamma T}dE(s)\,dE(t)\right]\\ &\leq C\Delta t^{1+\alpha}.\end{aligned}$$

Thus,

$$\sum_{m}^{n}\mathbb{E}\left|R_j^{(1)}\right| \leq C\sum_{j=m}^{n}\Delta t^{1+\alpha} \leq C\Delta t^{\alpha} \tag{4.13}$$

because

$$\begin{aligned}\mathbb{E}\left[R_k^{(2)}|\mathcal{F}_{k\Delta t}\right] &= \mathbb{E}_D\mathbb{E}_B\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\sigma f'(X(s))\,dB(E(s))\,dE(t)\right]\\ &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\mathbb{E}_B\int_{t_i}^{t}\sigma f'(X(s))\,dB(E(s))\,dE(t)\right]\\ &= 0\end{aligned}$$

Therefore,

$$\sum_{j=m}^{n}R_j^{(2)}$$

is a martingale. We know from the BDG inequality and lemma 2.7 that:

$$\mathbb{E}[dB_E dE]^2 = \mathbb{E}[(dB_E)^2(dE)^2] = \mathbb{E}_D[(dE)^2\mathbb{E}_B(dB_E)^2] \leq C\mathbb{E}_D[dE]^3 \leq C\Delta t^{1+2\alpha}$$

Thus, for the second term:

$$\begin{aligned}\mathbb{E}\left[|R_{j+1}^{(2)}|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\mathbb{E}_B\left[\int_{t_i}^{t}\sigma f'(X(s))dB_{E(s)}\right]dE(t)\right]\\ &\leq C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}\mathbb{E}_B\left(\sigma f'(X(s))\right)^2 dE(s)\right]^{\frac{1}{2}}dE(t)\right]\\ &\leq C\mathbb{E}_D\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}1dE(s)\right]^{\frac{1}{2}}dE(t)\\ &\leq C\Delta t^{\frac{1}{2}+\alpha}.\end{aligned}$$

By the BDG inequality and the Cauchy-Schwarz inequality, we obtain:

$$\mathbb{E}\sum_{j=m}^{n}\left|R_j^{(2)}\right| \leq C\mathbb{E}\left|\sum_{j=m}^{n}(R_j^{(2)})^2\right|^{\frac{1}{2}} \leq C\sqrt{\sum_{j=m}^{n}\mathbb{E}(R_j^{(2)})^2} \leq C\sqrt{\sum_{j=m}^{n}\Delta t^{1+2\alpha}} \leq C\Delta t^{\alpha}$$

In conclusion:

$$\mathbb{E}[e_n] \leq C\Delta t^{\alpha}$$

□

5. Numerical simulations

Example 1 (Time-Changed OU Process)**.** Consider the time-changed Ornstein-Uhlenbeck process driven by Brownian motion

$$dX(t) = \mu X(t)dE_t + \sigma dB_{E_t}. \tag{5.1}$$

For assumption 3.1, assumption 3.2, and assumption 3.3, it is clear that this example satisfies these conditions. The corresponding EM numerical scheme for this equation is:

$$X_{i+1} = (1 + \mu\Delta E_i)X_i + \sigma\Delta B_{E_i}.$$

In our numerical experiments, we focus on the L_1 error at the endpoint $T = 1$, so we define

$$e_T^i = \mathbb{E}\left|X_T^{\delta_4} - X_T^{\delta_i}\right|,$$

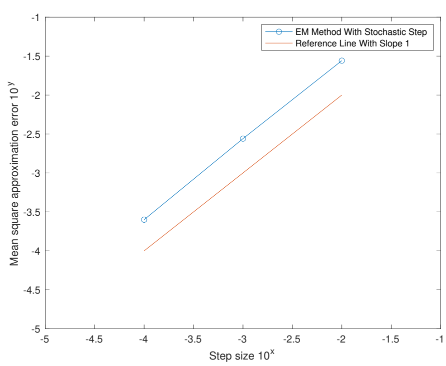
where $X_T^{\delta_i}$ is the simulated value at T with step size δ_i, $\delta_i = 2^{-i}$. For our numerical experiments, we set $\mu = 1$ and $\sigma = 1$, and use the Monte Carlo method:

$$e_T^i \approx \frac{1}{10^3}\sum_{j=1}^{10^3}\left|X_T^{\delta_{14}} - X_T^{\delta_i}\right|, \quad \text{where } i = 6, 7, 8, 9.$$

We use the step size 2^{-14} as a reference and estimate the L_1 error using step sizes $2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$. The table below compares the convergence rates and errors for different α values.

α	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
Convergence Order	0.9937	1.0345	1.0195	1.0204	1.0261	1.0318	1.0283	1.0281

Table 5.1. Convergence rates and errors for different α values



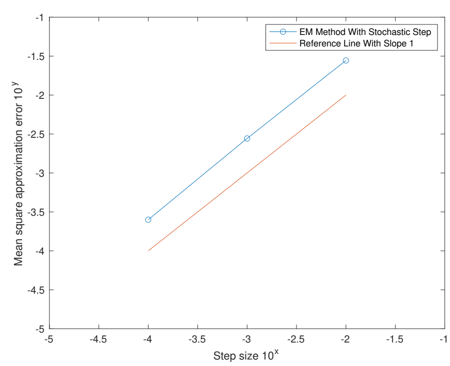


Figure 5.1. L_1 error of the EM method with stochastic steps. The left figure shows $\alpha = 0.4$ and the right figure shows $\alpha = 0.8$.

Example 2. Consider the time-changed CIR process driven by Brownian motion

$$dy(t) = \kappa(\theta - y(t))dE(t) + \sigma\sqrt{y(t)}dB(E(t)), \quad t \geq 0, \quad y(0) > 0. \tag{5.2}$$

If $2\kappa\theta \geq \sigma^2$, then $D = (0, \infty)$ and **??** holds in $(\alpha, \beta) = (0, \infty)$. Moreover, using the Itô formula with $X(t) = F(y(t))$, where F is defined by (2.2), we can obtain the Lamperti transformation of the time-changed CIR process:

$$dX(t) = f(X(t))dE(t) + \frac{1}{2}\sigma dB(E(t)), \quad t \geq 0, \quad X(0) = \sqrt{y(0)}. \tag{5.3}$$

where

$$f(X) = \frac{1}{2}\kappa\left(\theta_v X^{-1} - X\right), \quad X > 0 \tag{5.4}$$

with $\theta_v = \theta - \frac{\sigma^2}{4\kappa}$. The BEM numerical scheme is as follows:

$$X_{t_{i+1}} = X_{t_i} + f(X_{t_{i+1}})\Delta E_i + \frac{1}{2}\sigma\Delta B_{E_i}, \quad k = 0, 1, \dots \tag{5.5}$$

We observe that

$$f'(X) = -\frac{1}{2}\kappa(\theta_v X^{-2} + 1) \tag{5.6}$$

and

$$f(X)f'(X) + \frac{\sigma^2}{2}f''(X) = -\frac{\kappa^2}{4}(\theta_v^2 X^{-3} - X) + \frac{1}{2}\kappa\theta_v X^{-3}\sigma^2. \tag{5.7}$$

Therefore, to satisfy **??**, it is sufficient to have

$$\sup_{0\le t\le T}\mathbb{E}[X(t)^{-3}] = \sup_{0\le t\le T}\mathbb{E}[y(t)^{-\frac{3}{2}}] < \infty. \tag{5.8}$$

For the bounded moments of the time-changed CIR process $y(t)$, we have

$$\sup_{0\le t\le T}\mathbb{E}[y(t)^q] < \infty \quad \text{for} \quad q > -\frac{2k\theta}{\sigma^2}. \tag{5.9}$$

Next, we verify the bounded moments for the exact solution of the time-changed CIR process (5.2).

Proposition 5.1. *For the time-changed CIR process driven by Brownian motion (5.2) with* $y_0 > 0$ *and* $1 < p < \frac{2K\theta}{\sigma^2} - 1$, *there exists a constant* C *such that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[(y(t))^{-p}\right] \le C(1 + y(0)^{-p}).$$

Proof. Define the stopping time $\tau_n = \inf\{0 < s \le T; y(s) \le 1/n\}$. By applying the Itô formula, we obtain

$$\begin{aligned}
\mathbb{E}_B\left[(y(t\wedge\tau_n))^{-p}\right] &= y(0)^{-p} - p\mathbb{E}_B\left[\int_0^{t\wedge\tau_n}\frac{K(\theta - y(s))}{(y(s))^{p+1}}dE(s)\right] \\
&\quad + p(p+1)\frac{\sigma^2}{2}\mathbb{E}_B\left[\int_0^{t\wedge\tau_n}\frac{1}{(y(s))^{p+1}}dE(s)\right] \\
&\le y(0)^{-p} + pK\int_0^t\mathbb{E}_B\left(\frac{1}{(y(s\wedge\tau_n))^p}\right)dE(s) \\
&\quad + \mathbb{E}_B\left[\int_0^{t\wedge\tau_n}\frac{p\left(\frac{(p+1)\sigma^2}{2} - K\theta\right)}{(y(s))^{p+1}}dE(s)\right].
\end{aligned}$$

We can find a positive constant $\underline{C}$ such that when $\frac{(p+1)\sigma^2}{2} - K\theta < 0$, for any $y(0) = x > 0$, we have

$$\frac{p\left(\frac{(p+1)\sigma^2}{2} - K\theta\right)}{x^{p+1}} \le \underline{C}.$$

Therefore,

$$\mathbb{E}_B\left[(y(t\wedge\tau_n))^{-p}\right] \le y(0)^{-p} + \underline{C}E(T) + pK\int_0^t\sup_{r\in[0,s]}\mathbb{E}_B\left[(y(r\wedge\tau_n))^{-p}\right]dE(s).$$

Using the Gronwall inequality, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}_B\left[(y(t\wedge\tau_n)^x)^{-p}\right] \le \left(y(0)^{-p} + \underline{C}E(T)\right)\exp(pKE(T)).$$

Taking $\mathbb{E}_D$ on both sides and using the Cauchy-Schwarz inequality, we get

$$\sup_{t\in[0,T]} \mathbb{E}\left[(y(t\wedge\tau_n)^x)^{-p}\right] \le \mathbb{E}\left[\left(y(0)^{-p}+\underline{C}E(T)\right)\exp(pKE(T))\right]$$
$$\le \sqrt{\mathbb{E}\left[\left(y(0)^{-p}+\underline{C}E(T)\right)^2\right]\mathbb{E}\left[\exp(2pKE(T))\right]}.$$

From [2], we have

$$\mathbb{E}[E^n(t)] = \frac{n!}{\Gamma(n\alpha+1)}t^{n\alpha}, \tag{5.10}$$

$$\mathbb{E}[e^{\lambda E(t)}] < \infty, \tag{5.11}$$

where $\lambda\in\mathbb{R}, t>0$. Finally, letting $n\to+\infty$, we complete the proof. ☐

Remark 5.1. Here, we have only proved the existence of moments for $1<p<\frac{2K\theta}{\sigma^2}-1$. Actually, it can be proven that $p<\frac{2K\theta}{\sigma^2}$, but the proof is too complicated and not necessary for our results.

For **??**, it can be verified that it is sufficient to ensure that $1<\frac{4}{3}\frac{\kappa\theta}{\sigma^2}$ holds. This interval is covered by $1<p<\frac{2K\theta}{\sigma^2}-1$, so it suffices to ensure that p is within this range. As for assumption 4.3, it can be easily verified that there exists such a κ in $(0,\infty)$ to satisfy this condition. Therefore, from theorem 4.1, it follows that for the time-changed CIR process, the strong convergence order of the BEM numerical scheme is α.

Example 3. Consider the time-changed CEV process driven by Brownian motion

$$dy(t) = \kappa(\theta - y(t))dE(t) + \sigma y(t)^{\alpha}dB_{E(t)}. \tag{5.12}$$

Here, $0.5<\alpha<1, \kappa,\theta,\sigma>0$. By transforming $X(t)=F(y(t))$ where F is defined in (2.2), we get that assumption 4.3 holds in $(\alpha,\beta)=(0,\infty)$, and we have

$$dX(t) = f(X(t))dE(t) + (1-\alpha)\sigma dB(E(t)),$$

where

$$f(X) = (1-\alpha)\left(\kappa\theta X^{-\frac{\alpha}{1-\alpha}} - \kappa X - \frac{\alpha\sigma^2}{2}X^{-1}\right), \quad X>0.$$

We also need to verify **??**. Since $\alpha>0.5$, we have $\frac{1}{1-\alpha}>2$, so

$$f'(X) = -\alpha\kappa\theta X^{-\frac{1}{1-\alpha}} - (1-\alpha)\kappa + (1-\alpha)\frac{\alpha\sigma^2}{2}X^{-2}, \quad X>0,$$

and

$$f''(X) = \frac{\alpha}{1-\alpha}\kappa\theta X^{-\frac{2-\alpha}{1-\alpha}} - (1-\alpha)\alpha\sigma^2X^{-3}.$$

The following proposition verifies the bounded moments for the exact solution of the time-changed CEV process (5.12).

Proposition 5.2. *For the time-changed CEV process (5.12) driven by Brownian motion, with $X_0>0$, for any $\frac{1}{2}<\alpha<1$ and any $p>0$, there exists a constant C such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[(y(t))^{-p}\right] \le C(1+y(0)^{-p}).$$

Proof. Define the stopping time $\tau_n = \inf\{0 < s \leq T; y(s) \leq 1/n\}$. Using the Itô formula, we get

$$\begin{aligned}
\mathbb{E}_B\left[(y(t\wedge\tau_n))^{-p}\right] &= y(0)^{-p} - p\mathbb{E}_B\left[\int_0^{t\wedge\tau_n} \frac{\kappa(\theta - y(s))}{(y(s))^{p+1}} dE(s)\right] \\
&+ p(p+1)\frac{\sigma^2}{2}\mathbb{E}_B\left[\int_0^{t\wedge\tau_n} \frac{1}{(y(s))^{p+2(1-\alpha)}} dE(s)\right] \\
&\leq y(0)^{-p} + p\kappa\int_0^t \mathbb{E}_B\left(\frac{1}{(y(s\wedge\tau_n))^p}\right) dE(s) \\
&+ \mathbb{E}_B\left[\int_0^{t\wedge\tau_n}\left(p(p+1)\frac{\sigma^2}{2}\frac{1}{(y(s))^{p+2(1-\alpha)}} - p\frac{\kappa\theta}{(y(s))^{p+1}}\right) dE(s)\right]
\end{aligned}$$

We can find a positive constant C such that, for any $y(0) = x > 0$,

$$\left(p(p+1)\frac{\sigma^2}{2}\frac{1}{x^{p+2(1-\alpha)}} - p\frac{\kappa\theta}{x^{p+1}}\right) \leq C.$$

From the calculations, we get that $\underline{C} = p(2\alpha-1)\frac{\sigma^2}{2}\left[(p+2(1-\alpha))\frac{\sigma^2}{2\kappa\theta}\right]^{\frac{p+2(1-\alpha)}{2\alpha-1}}$ is the smallest upper bound. Therefore,

$$\mathbb{E}_B\left[(y(t\wedge\tau_n))^{-p}\right] \leq y(0)^{-p} + \underline{C}E(T) + p\kappa\int_0^t \sup_{r\in[0,s]} \mathbb{E}_B\left[(y(r\wedge\tau_n))^{-p}\right] dE(s).$$

Applying the Gronwall inequality, we get

$$\sup_{t\in[0,T]} \mathbb{E}_B\left[(y(t\wedge\tau_n)^x)^{-p}\right] \leq \left(y(0)^{-p} + \underline{C}E(T)\right)\exp(p\kappa E(T)).$$

Taking the expectation $\mathbb{E}_D$ on both sides and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\left[(y(t\wedge\tau_n)^x)^{-p}\right] &\leq \mathbb{E}\left[\left(y(0)^{-p} + \underline{C}E(T)\right)\exp(p\kappa E(T))\right] \\
&\leq \sqrt{\mathbb{E}\left[\left(y(0)^{-p} + \underline{C}E(T)\right)^2\right]\mathbb{E}\left[\exp(2p\kappa E(T))\right]}
\end{aligned}$$

From [2], we have

$$\mathbb{E}[E^n(t)] = \frac{n!}{\Gamma(n\alpha+1)}t^{n\alpha} \tag{5.13}$$

$$\mathbb{E}[e^{\lambda E(t)}] < \infty \tag{5.14}$$

where $\lambda \in \mathbb{R}, t > 0$. Finally, letting $n \to +\infty$, we complete the proof. □

From the Lamperti transformation, we know that the inverse moments of $X(t)$ can be controlled by the inverse moments of $y(t)$, so **??** is satisfied. Therefore, according to theorem 4.1, we conclude that the convergence order of the BEM numerical scheme for the time-changed CEV process driven by Brownian motion is α.

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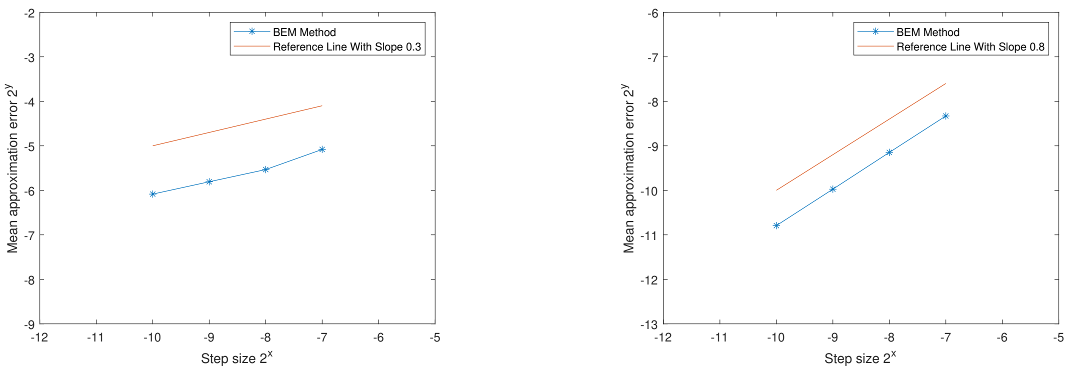

FIGURE 5.2. Absolute error estimation between the numerical solution and the analytical solution of the time-changed CIR process. The left figure shows $\alpha = 0.3$, and the right figure shows $\alpha = 0.8$.

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