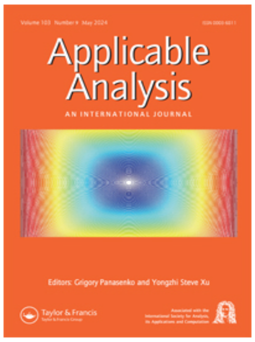

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Mean square stability of the split-step theta method for non-linear time-changed stochastic differential equations

Dongxuan Wu[a], Zhi Li[a], Liping Xu[a] and Chuanhui Peng[b]

[a]School of Information and Mathematics, Yangtze University, Jingzhou, People's Republic of China; [b]Department of Mathematics, College of Arts and Sciences, Washington University in St. Louis, St. Louis, MO, USA

ABSTRACT

This paper investigates the split-step theta (SST) method to approximate a class of time-changed stochastic differential equations, whose drift coefficient can grow super-linearly and diffusion coefficient obeys the global Lipschitz condition. The strong convergence of the SST method is proved, and the SST method attains the classical 1 of convergence. In addition, the mean square stability of the time-changed stochastic differential equations is investigated. Two examples are presented to show the consistency of the theoretical results.

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1. Introduction

Time-changed processes and time-changed stochastic differential equations (SDEs), which are the critical mathematical tools to describe subdiffusion, have been broadly investigated in recent decades [1–5]. Time-changed SDEs have been attractively applied in various areas, including biology, physics, economics, and finance, see [6–9].

In this paper, we will study a class of time-changed stochastic differential equations (SDEs) with the form

$$\mathrm{d}X(t) = f(E(t), X(t))\,\mathrm{d}E(t) + g(E(t), X(t))\,\mathrm{d}B(E(t)), \quad t \in [0, T], \tag{1}$$

here the coefficients f and g satisfy some regular conditions (to be specified in Section 2), $B(t)$ is a standard Brownian motion and $E(t)$ is the inverse of a subordinator. The composition process $B(E(t))_{t\in[0,T]}$ is called a time-changed Brownian motion which is understood as a subdiffusion (see [5,10]).

Since it is generally impossible to derive the explicit solution to some SDEs, numerical approximations become extremely important when one applies them to model uncertain phenomena in real life. To our best knowledge, Jum [11] is the first paper to study the finite time strong convergence of numerical methods for time-changed SDEs by directly discretizing the equations and studying the two types of errors involved in using the Euler-Maruyama (EM) method. Afterward, an increasing number of papers have studied the time-changed SDEs, and it is worth noting that most of them use the EM method to study the convergence and stability of time-changed stochastic differential

CONTACT Liping Xu xlp211@126.com

equations, such as [11–13]. Furthermore, to better investigate this class of SDEs, Kobayashi was proved the dual principle in [13], which provided a new way to study the time-changed SDEs. As far as we know, the semi-implicit and truncated EM method were investigated for the time-changed SDEs with coefficients obeying super-linear growth in Deng [14] and Liu [15], respectively, and strong convergence and convergence rate were obtained in both papers. All the two papers above are based on the discussion of the duality principle. However, with a slight change in (1), the duality principle will no longer work. In [16], Jin and Kobayashi studied the time-changed SDEs with time-space-dependent coefficients of the form

$$\mathrm{d}Y(t) = \mu\,(t, Y(t))\,\mathrm{d}E(t) + \sigma\,(t, Y(t))\,\mathrm{d}B(E(t)). \tag{2}$$

The classical dual Itô SDEs do not exist, so they proposed the EM method to approximate (2) and obtained the strong convergence and convergence rate.

Meanwhile, classical SDEs with the drift coefficient obeys the super-linear growth have received lots of attention in recent years. Due to the divergent nature of classical explicit EM methods [17], different approaches to deal with the super-linear growth term have been proposed. Explicit methods with different correction factors for the underlying equations have been developed in recent years [18–23]. However, only a little have considered the implicit method, such as [24,25].

To cope with such super-linearity, we propose the SST method to approximate the SDEs driven by time-changed Brownian motions in this paper. It have been studied for approximating different types of SDEs driven by Brownian motions, see [26–28] and the references therein. But few authors have used SST method to deal with such time-changed SDEs, and this paper will fill this gap.

The main contributions of this paper are as follows.

(1) The SST method is developed for time-changed SDEs (1).
(2) We establish the mean square stability of the underlying time-changed SDEs. In addition, the numerical solution is proved to be able to preserve such a property.
(3) The strong convergence in the finite time of SST is proved and the classical 1 rate of convergence in the step size is obtained.

The rest of this paper is organized as follows. Section 2 is devoted to some mathematical preliminaries for the time-changed SDEs. The main results and proofs are presented in Section 3. The strong convergence of the numerical method is proved in Section 3.1, and the mean square stabilities of both underlying and numerical solutions are shown in Section 3.2. In Section 4, we present numerical simulations to demonstrate the theoretical results derived in Sections 3.1 and 3.2.

2. Preliminaries

Throughout this paper, unless otherwise specified, we will use the following notation. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$ and $\langle x, y\rangle$ be the inner product of vectors $x, y \in \mathbb{R}^d$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. For two real numbers u and v, we use $u \wedge v = \min(u, v)$ and $u \vee v = \max(u, v)$.

Moreover, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), B_2(t), \dots, B_m(t))^T$ be an m-dimensional $\mathcal{F}_t$-adapted standard Brownian motion. Let $\mathbb{E}$ denote the expectation under the probability measure $\mathbb{P}$.

For a subordinator $D(t)$, which is a non-decreasing Lévy process on $[0, \infty)$ with Laplace transform

$$\mathbb{E}\left[\mathrm{e}^{-sD(t)}\right] = \mathrm{e}^{-t\psi(s)}, \quad s > 0, \quad t \geq 0,$$

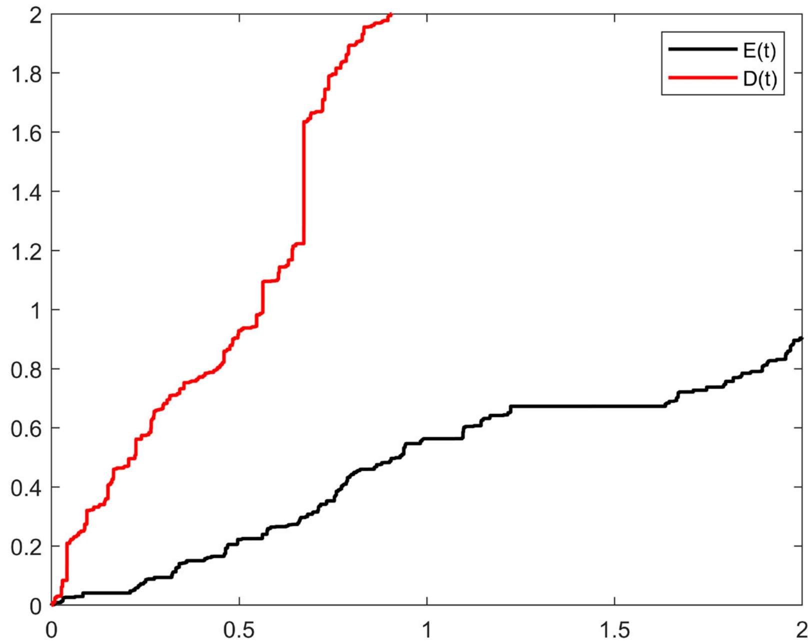


Figure 1. Sample paths of 0.8-stable subordinator D and the corresponding inverse subordinator E.

where $\psi(s) = \int_0^\infty (1 - \mathrm{e}^{-sy})\nu(\mathrm{d}y)$ with condition $\int_0^\infty (y \wedge 1)\nu(\mathrm{d}y) < \infty$. We focus on the case when Lévy measure ν is infinite (i.e. $\nu(0,\infty) = \infty$). Let $E = (E(t))_{t\geq 0}$ be the inverse of D defined by

$$E(t) := \inf\{s \geq 0; D(s) > t\}, \quad t \geq 0.$$

We call $E(t)$ an inverse subordinator. It is clear that $E(t)$ has continuous, non-decreasing paths starting at 0. Besides, the relation between D and E implies $\{E(t) > x\} = \{D(x) < t\}$ for all $t, x \geq 0$ showed in Figure 1. If the subordinator $D(t)$ is stable with index $\beta \in (0,1)$, i.e. $\psi(s) = s^\beta$, then $E(t)$ is called an inverse β-stable subordinator.

We always assume that $B(t)$ and $D(t)$ are independent. The process $B(E(t))$ is called a time-changed Brownian motion and $B(E(t))$ is understood as a subdiffusion (see [5,10]).

Consider the following time-changed SDEs

$$\mathrm{d}X(t) = f(E(t), X(t))\,\mathrm{d}E(t) + g(E(t), X(t))\,\mathrm{d}B(E(t)), \quad t \in [0,T],$$

with $\mathbb{E}|X(0)|^\gamma < 0$ for any $\gamma \in (0,\infty)$, where $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable coefficients.

The following assumptions are imposed on the coefficients of (1):

(**H1**) There exists a constant $K_1 > 0$ such that, for all $t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$\langle x - y, f(t,x) - f(t,y)\rangle \leq K_1|x - y|^2.$$

(**H2**) There exist constants $K_2 > 0$, $a \geq 2$ and $\gamma \in (0,2]$ such that, for all $t, s \geq 0$ and $x \in \mathbb{R}^d$,

$$|f(t,x) - f(s,x)|^2 \leq K_2\left(1 + |x|^a\right)|t - s|^\gamma,$$
$$|g(t,x) - g(s,x)|^2 \leq K_2\left(1 + |x|^2\right)|t - s|^\gamma.$$

(H3) Assume that there exist constants $K_3 > 0$ and $b \geq 0$ such that, for all $t \geq 0$ and $x, y \in \mathbb{R}^d$,

$$|f(t,x) - f(t,y)|^2 \leq K_3 \left(1 + |x|^b + |y|^b\right) |x - y|^2,$$
$$|g(t,x) - g(t,y)|^2 \leq K_3 |x - y|^2.$$

(H4) Assume that there exist constant $p \geq 2$ and $K_4 > 0$ such that, for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\langle x, f(t,x)\rangle + \frac{p-1}{2}|g(t,x)|^2 \leq K_4 \left(1 + |x|^2\right).$$

To avoid complicated notations, we further assume that both $|f(t,0)|$ and $|g(t,0)|$ are bounded. According to **(H2)** and **(H3)**, we can see that there exists a constant $K_5 > 0$ such that, for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$|f(t,x)|^2 \leq K_5 \left(1 + |x|^a\right),$$

and

$$|g(t,x)|^2 \leq K_5 \left(1 + |x|^2\right).$$

According to the duality principle in [13], the time-changed SDEs (1) and the classical SDEs of itô type

$$\mathrm{d}Y(t) = f(t, Y(t))\,\mathrm{d}t + g(t, Y(t))\,\mathrm{d}B(t), \quad Y(0) = X(0), \tag{3}$$

have a deep connection.

Lemma 2.1: *Let* **(H1−H4)** *hold. Then, time-changed SDEs* (1) *has a unique strong solution which is a continuous* $\mathcal{F}_{E(t)}$*-semimartingale.*

Proof: The proof of existence and uniqueness is almost similar to Lemma 4.1 of [13], so it is omitted here. ■

The next lemma states the relationship between the solution to (1) and the solution to (3).

Lemma 2.2 ([13]): *Suppose the assumptions* **(H1−H4)** *hold. If* $Y(t)$ *is the unique solution to the* SDEs (3), *then the time-changed process* $Y(E(t))$, *which is an* $\mathfrak{F}_{E(t)}$*-semimartingale, is the unique solution to the time-changed* SDEs (1). *On the other hand, if* $X(t)$ *is the unique solution to the time-changed* SDEs (1), *then the process* $X(D(t))$, *which is an* $\mathfrak{F}_t$*-semimartingale, is the unique solution to the* SDEs (3).

In the following, we approximate the time-changed SDEs (1). Firstly, we construct the numerical method for the SDEs (3). Next, we discretize the inverse subordinator $E(t)$ and time-changed Brownian motion $B(E(t))$. Then, the composition of the numerical solution of the SDEs (3) and the discretized inverse subordinator are used to approximate the solution to the time-changed SDEs (1).

Given a free parameter $\theta \in [0, 1]$, the numerical solutions approximated by the SST method are defined by

$$\begin{aligned} y_k &= \bar{y}_k + \theta \Delta t f\left(t_k + \theta \Delta t, y_k\right), \\ \bar{y}_{k+1} &= \bar{y}_k + \Delta t f\left(t_k + \theta \Delta t, y_k\right) + g\left(t_k + \theta \Delta t, y_k\right) \Delta B_k, \end{aligned} \tag{4}$$

where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$, $t_k = k\Delta t$ and $y_0 = \bar{y}_0$.

We also define the piecewise continuous numerical solution by $y(t) := y_k$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. y_k is the numerical solution of (3) under (4).

We follow the idea in [29,30] to approximate the inverse subordinator $E(t)$ in a time interval $[0, T]$ for any given $T > 0$. We simulate the path of $D(t)$ by $D_{\Delta t}(t_i) = D_{\Delta t}(t_{i-1}) + \Delta_i$ with $D_{\Delta t}(0) = 0$, where Δ_i is independently identically sequence with $\Delta t =^d D(\Delta t)$ in distribution. The procedure is stopped when

$$T \in [D_{\Delta t}(t_n), D_{\Delta t}(t_{n+1})),$$

for some n. Then the approximation $E_{\Delta t}(t)$ to $E(t)$ is generated by

$$E_{\Delta t}(t) = (\min\{n; D_{\Delta t}(t_n) > t\} - 1)\Delta t, \tag{5}$$

for $t \in [0, T]$. It is easy to see

$$E_{\Delta t}(t) = k\Delta t, \quad \text{when } t \in \left[D_{\Delta t}(t_k), D_{\Delta t}\left(t_{k+1}\right)\right).$$

Next lemma will be used as the approximation error of $E_{\Delta t}(t)$ to $E(t)$, and the proof can be found in [3,11].

Lemma 2.3: *Almost surely,*

$$E(t) - \Delta t \le E_{\Delta t}(t) \le E(t),$$

holds for all $t > 0$.

The following lemma states that the inverse subordinator $E(t)$ is known to have the finite exponential moment, which has been proved in [11,31].

Lemma 2.4: *For any $\delta > 0$, there exists $C = C(\delta) > 0$ such that*

$$\mathbb{E}e^{\delta E(t)} \le e^{Ct}, \quad \text{for all } t \ge 1.$$

Lemma 2.5 ([14]): *Suppose that the assumptions* (**H1**$-$**H4**) *hold. Then the solution of* (3) *satisfies*

$$\mathbb{E}|Y(t)|^p \le 2^{\frac{p-2}{2}}(1 + \mathbb{E}|Y(0)|^p)e^{pK_4 t}, \quad \text{for all } t \ge 0.$$

Lemma 2.6 ([14]): *Suppose that the assumptions* (**H1**$-$**H4**) *hold. Then for any $q \in (1, 2p/a]$ and $t, s \ge 0$ with $|t - s| \le 1$,*

$$\mathbb{E}|Y(t) - Y(s)|^q \le C|t - s|^{q/2}e^{Ct},$$

where $C > 0$ is a constant independent of t and s.

3. Main results

3.1. Strong convergence

When $\theta = 1$ and $\theta = 0$, the SST method is the split-step backward Euler method and the Euler method, respectively, and their strong convergence has been investigated in [32]. In this paper we consider the case of $\frac{1}{2} \le \theta \le 1$. In order to study the strong convergence of the SST method, we first discuss moment properties of the numerical solutions y_k and $\bar{y}_k$ at time $t_k = k\Delta t$, where $k = 0, 1, \ldots, N$ with $N\Delta t = T$.

We divide the proof into two steps. The following lemma give the relationship between $\mathbb{E}|y_k|^2$ and $\mathbb{E}|\bar{y}_k|^2$.

Lemma 3.1: *Suppose that f satisfies* (**H1**) *and let* $\frac{1}{2} < \theta \leq 1$. *Then there exist two positive constants* $\alpha = 1/(1-2\theta\Delta t K_1)$ *and* $\beta = 2\theta\Delta t K_1/(1-2\theta\Delta t K_1)$ *such that*

$$\mathbb{E}\left|y_k\right|^2 \leq \alpha\mathbb{E}\left|\bar{y}_k\right|^2 + \beta, \tag{6}$$

where y_k *and* $\bar{y}_k (k = 0, 1, 2, \dots, N)$ *are produced by* (4).

Proof: From (4), we have that

$$\left|y_k\right|^2 = \left|\bar{y}_k\right|^2 + \theta^2\Delta t^2\left|f\left(t_k + \theta\Delta t, y_k\right)\right|^2 + 2\theta\Delta t\left\langle\bar{y}_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle \tag{7}$$

and

$$\left\langle\bar{y}_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle = \left\langle y_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle - \theta\Delta t\left\langle f\left(t_k + \theta\Delta t, y_k\right), f\left(t_k + \theta\Delta t, y_k\right)\right\rangle. \tag{8}$$

By (7) and (8), we get

$$\begin{aligned}
\left|y_k\right|^2 &= \left|\bar{y}_k\right|^2 - \theta^2\Delta t^2\left\langle f\left(t_k + \theta\Delta t, y_k\right), f\left(t_k + \theta\Delta t, y_k\right)\right\rangle + 2\theta\Delta t\left\langle y_k, f\left(t_t + \theta\Delta t, y_k\right)\right\rangle \\
&\leq \left|\bar{y}_k\right|^2 + 2\theta\Delta t\left\langle y_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle \\
&\leq \left|\bar{y}_k\right|^2 + 2\theta\Delta t K_1\left(1 + \left|y_k\right|^2\right),
\end{aligned}$$

which gives

$$\left|y_k\right|^2 \leq \alpha\left|\bar{y}_k\right|^2 + \beta, \tag{9}$$

where $\alpha = 1/(1-2\theta\Delta t K_1)$ and $\beta = 2\theta\Delta t K_1/(1-2\theta\Delta t K_1)$. The proof is complete. ■

The second lemma shows that the numerical solutions y_k $(k = 1, 2, \dots, N)$ produced by the SST method are bounded.

Lemma 3.2: *Let assumptions* (**H1**−**H4**) *hold, Let* $p \geq 1$, $\frac{1}{2} \leq \theta \leq 1$ *and* Δt *be sufficiently small. Then the numerical solution of SDEs* (1) *satisfies*

$$\mathbb{E}\left[\sup_{0\leq k\Delta t\leq T}\left|\bar{y}_k\right|^{2p}\right] \vee \mathbb{E}\left[\sup_{0\leq k\Delta t\leq T}\left|y_k\right|^{2p}\right] \leq \widehat{C}\,\mathrm{e}^{\bar{C}\Delta t},$$

where $\widehat{C}$ *and* $\overline{C}$ *is a constant independent* Δt.

Proof: By (7) and (8), we have

$$\begin{aligned}
\left|\bar{y}_{k+1}\right|^2 &= \left|\bar{y}_k\right|^2 + \Delta t^2\left|f\left(t_k + \theta\Delta t, y_k\right)\right|^2 + \left|g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right|^2 + 2\left\langle\bar{y}_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle\Delta t \\
&\quad + 2\left\langle\bar{y}_k, g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right\rangle + 2\left\langle f\left(t_k + \theta\Delta t, y_k\right), g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right\rangle\Delta t \\
&\leq \left|\bar{y}_k\right|^2 + 2\left\langle y_k, f\left(t_k + \theta\Delta t, y_k\right)\right\rangle\Delta t + \left|g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right|^2 \\
&\quad + 2\left\langle\bar{y}_k, g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right\rangle + 2\left\langle\frac{y_k - \bar{y}_k}{\theta}, g\left(t_k + \theta\Delta t, y_k\right)\Delta B_k\right\rangle.
\end{aligned}$$

Substituting (9) into the above inequality leads to

$$\begin{aligned}\left|\bar{y}_{k+1}\right|^2 \le \left|\bar{y}_k\right|^2 + 2\alpha K_1 \Delta t \left|\bar{y}_k\right|^2 + 2(\beta+1)K_1\Delta t + \left|g\left(t_k+\theta\Delta t, y_k\right)\Delta B_k\right|^2 \\ + 2\left(1-\frac{1}{\theta}\right)\left\langle \bar{y}_k, g\left(t_k+\theta\Delta t, y_k\right)\Delta B_k\right\rangle + \frac{2}{\theta}\left\langle y_k, g\left(t_k+\theta\Delta t, y_k\right)\Delta B_k\right\rangle .\end{aligned}$$

By virtue of recursive calculation, we have

$$\begin{aligned}\left|\bar{y}_k\right|^2 \le \left|\bar{y}_0\right|^2 + 2\alpha K_1 \Delta t \sum_{j=0}^{k-1}\left|\bar{y}_j\right|^2 + 2(\beta+1)K_1 T + \sum_{j=0}^{k-1}\left|g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right|^2 \\ + 2\left(1-\frac{1}{\theta}\right)\sum_{j=0}^{k-1}\left\langle \bar{y}_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle + \frac{2}{\theta}\sum_{j=0}^{k-1}\left\langle y_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle .\end{aligned}$$

Raising both sides to the power p by the elementary inequality

$$\left|\sum_{i=1}^{n} u_i\right|^q \le n^{q-1}\sum_{i=1}^{n}|u_i|^2, u_i \in \mathbb{R}^d, \tag{10}$$

then we get

$$\begin{aligned}\left|\bar{y}_k\right|^{2p} \le 6^{p-1}\Bigg\{ & \left|\bar{y}_0\right|^{2p} + (2\alpha K_1\Delta t)^p k^{p-1}\sum_{j=0}^{k-1}\left|\bar{y}_j\right|^{2p} + (2(\beta+1)K_1T)^p \\ & + k^{p-1}\sum_{j=0}^{k-1}\left|g(t_j+\theta\Delta t, y_j)\Delta B_j\right|^{2p} + 2^p\left(\frac{1}{\theta}-1\right)^p\left|\sum_{j=0}^{k-1}\left\langle \bar{y}_j, g(t_j+\theta\Delta t, y_j)\Delta B_j\right\rangle\right|^p \\ & + \left(\frac{2}{\theta}\right)^p\left|\sum_{j=0}^{k-1}\left\langle y_j, g(t_j+\theta\Delta t, y_j)\Delta B_j\right\rangle\right|^p\Bigg\}.\end{aligned}$$

Thus, for $0 \le l \le N$, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le k\le l}\left|\bar{y}_k\right|^{2p}\right] \le 6^{p-1}\Bigg\{ & \mathbb{E}\left|\bar{y}_0\right|^{2p} + (2\alpha K_1)^p\, T^{p-1}\Delta t\sum_{j=0}^{l-1}\mathbb{E}\left|\bar{y}_j\right|^{2p} + (2(\beta+1)K_1T)^p \\ & + l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\left|g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right|^{2p}\right] \\ & + 2^p\left(\frac{1}{\theta}-1\right)^p\mathbb{E}\left[\sup_{0\le k\le l}\left|\sum_{j=0}^{k-1}\left\langle \bar{y}_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle\right|^p\right] \\ & + \left(\frac{2}{\theta}\right)^p\mathbb{E}\left[\sup_{0\le k\le l}\left|\sum_{j=0}^{k-1}\left\langle y_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle\right|^p\right]\Bigg\} \\ = {} & 6^{p-1}(I_1+I_2+I_3+I_4+I_5+I_6).\end{aligned} \tag{11}$$

To bound the fourth term on the right-hand side of (11), we note that $y_k \in \mathcal{F}_{t_k}$, ΔB_k is independent of $\mathcal{F}_{t_k}$ and $\mathbb{E}|\Delta B_j|^{2p} \le c_p(\Delta t)^p$, where c_p is a constant. Thus, it is easy to deduce that, with suitable constants $C_1 = C_1(p, T, K_1, \theta)$.

$$
\begin{aligned}
I_4 = l^{p-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\left|g\left(t_j+\theta h, y_j\right)\Delta B_j\right|^{2p}\right] &\le l^{p-1}\sum_{j=0}^{k-1}\mathbb{E}\left|g\left(t_j+\theta\Delta t, y_j\right)\right|^{2p}\mathbb{E}\left|\Delta B_j\right|^{2p}\\
&\le 2^{p-1}l^{p-1}K_5^p c_p(\Delta t)^p\sum_{j=0}^{k-1}\left[(1+\beta)^p+\alpha^p\mathbb{E}\left|\bar{y}_j\right|^{2p}\right]\\
&\le C_{41}+C_{42}\Delta t\sum_{j=0}^{l-1}\mathbb{E}\left|\bar{y}_j\right|^{2p},
\end{aligned}
$$

where $C_{41} = 2^{p-1}l^{p-1}K_5{}^p c_p(\Delta t)^p(1+\beta)^p$ and $C_{42} = 2^{p-1}l^{p-1}K_5{}^p c_p(\Delta t)^{p-1}\alpha^p$.

For I_5, by the Burkholder-Davis-Gundy's inequality (see [33], Theorem 1.73), we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le k\le l}\left|\sum_{j=0}^{k-1}\left\langle\bar{y}_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle\right|^p\right]\\
&\le C_p\mathbb{E}\left[\left|\sum_{j=0}^{l-1}\left|\bar{y}_j\right|^2\left|g\left(t_j+\theta\Delta t, y_j\right)\right|^2\Delta t\right|^{p/2}\right]\\
&\le C_p\left(K_5\Delta t\right)^{p/2}l^{p/2-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\left|\bar{y}_j\right|^p\left(1+\left|y_j\right|^2\right)^{p/2}\right]\\
&\le C_p\left(K_5\Delta t\right)^{p/2}l^{p/2-1}\mathbb{E}\left[\sum_{j=0}^{l-1}\frac{\left|\bar{y}_j\right|^{2p}+\left(1+\left|y_j\right|^2\right)^p}{2}\right]\\
&\le C_p\left(K_5\right)^{p/2}T^{p/2-1}\Delta t\mathbb{E}\left[\sum_{j=0}^{l-1}\left(2^{p-1}(1+\beta)^p+\left(2^{p-1}\alpha^p+1\right)\left|\bar{y}_j\right|^{2p}\right)\right],
\end{aligned}
$$

then

$$
I_5 = 2^p\left(\frac{1}{\theta}-1\right)^p\mathbb{E}\left[\sup_{0\le k\le l}\left|\sum_{j=0}^{k-1}\left\langle\bar{y}_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle\right|^p\right]\le C_{51}\Delta t\sum_{j=0}^{l-1}\mathbb{E}\left|\bar{y}_j\right|^{2p}+C_{52},
$$

where

$$
\begin{aligned}
C_{51} &= \frac{1}{2}2^p\left(\frac{1}{\theta}-1\right)^p C_p K_5{}^{p/2}T^{p/2-1}2^{p-1}\alpha^p+1,\\
C_{52} &= \frac{1}{2}2^p\left(\frac{1}{\theta}-1\right)^p C_p K_5{}^{p/2}T^{p/2-1}(\Delta t)2^{p-1}(1+\beta)^p.
\end{aligned}
$$

Similarly, we can get

$$
I_6 = \left(\frac{2}{\theta}\right)^p\mathbb{E}\left[\sup_{0\le k\le l}\left|\sum_{j=0}^{k-1}\left\langle y_j, g\left(t_j+\theta\Delta t, y_j\right)\Delta B_j\right\rangle\right|^p\right]\le C_{61}\Delta t\sum_{j=0}^{l-1}\mathbb{E}\left|\bar{y}_j\right|^{2p}+C_{62},
$$

where

$$
\begin{aligned}
C_{61} &= \frac{1}{2}\left(\frac{2}{\theta}\right)^p C_p {K_5}^{p/2} T^{p/2-1} \alpha^p (2^{p-1}+1), \\
C_{62} &= \frac{1}{2}\left(\frac{2}{\theta}\right)^p C_p {K_5}^{p/2} T^{p/2-1} \alpha^p 2^{p-1} (\beta+1)^p + \beta^p.
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\left[\sup_{0\le k\le l} \left|\bar{y}_k\right|^{2p}\right] \le \hat{C} + \bar{C}\Delta t \sum_{j=0}^{l-1} \mathbb{E}\left|\bar{y}_j\right|^{2p},
$$

where

$$
\begin{aligned}
\hat{C} &= 6^{p-1}[\mathbb{E}|y_0|^{2p} + (2(\beta+1)K_1 T)^p + C_{41} + C_{52} + C_{62}], \\
\bar{C} &= 6^{p-1}[(2\alpha K_1)^p T^{p-1} + C_{42} + C_{51} + C_{61}].
\end{aligned}
$$

Applying the discrete Gronwall inequality, we have

$$
\mathbb{E}\left[\sup_{0\le k\le l} \left|\bar{y}_k\right|^{2p}\right] \vee \mathbb{E}\left[\sup_{0\le k\le l} \left|y_k\right|^{2p}\right] \le \widehat{C}\, \mathrm{e}^{\overline{C}\Delta t}, \quad 0 \le l \le N. \tag{12}
$$

Hence, the proof is complete. ■

Remark 3.3: We discuss the SDEs (4) which, compared with [34,35], has drift coefficients that satisfy superlinear growth. Before proving Lemma 3.2, we had to obtain that the relation between $\mathbb{E}|y_k|^2$ and $\mathbb{E}|\bar{y}_k|^2$ (3.1) holds, which is useful for the results of Lemma 3.2 and the strong convergence (Theorem 3.4) by using the SST method.

In the following, we will study the strong convergence of (4) to (3) by the SST method of $\theta \in [1/2, 1]$. To be more precise, we need to trace the temporal variable t carefully so that no term like t^a for $a > 1$ would appear in the exponential function on the right hand side of the inequality in the statement of Theorem 3.4. Since the t will be replaced by $E(t)$ in Theorem 3.5 and an expectation will be taken on it, by Lemma 2.4 a term like t^a with $a > 1$ will lead to the unboundedness of $\mathbb{E}\,\mathrm{e}^{\delta(E(t))^a}$ with $a > 1$.

Theorem 3.4: *Let assumptions* (**H1**−**H4**) *hold, the numerical solution* (4) *approximated by SST method converges to* (3) *with*

$$
\mathbb{E}\left|Y(t_k) - y_k\right|^2 \le C\Delta t^{\gamma\wedge 1} \mathrm{e}^{Ct_{k+1}},
$$

where C is a constant.

Proof: From (3) and (4), for $k = 1, 2, \ldots, N$,

$$
\begin{aligned}
Y(t_{k+1}) - y_{k+1} = (Y(t_k) - y_k) &+ \int_{t_k}^{t_{k+1}} (1-\theta)(f(s, Y(s)) - f(t_k, y_k))\,\mathrm{d}s \\
&+ \int_{t_k}^{t_{k+1}} \theta(f(s, Y(s)) - f(t_{k+1}, y_{k+1}))\,\mathrm{d}s + \int_{t_k}^{t_{k+1}} (g(s, Y(s)) - g(t_k, y_k))\,\mathrm{d}B(s),
\end{aligned}
$$

multiplying both sides with $Y(t_{k+1}) - y_{k+1}$, yields

$$
\begin{aligned}
\left|Y(t_{k+1}) - y_{k+1}\right|^2 &= \Big\langle Y(t_{k+1}) - y_{k+1}, (Y(t_k) - y_k) + \int_{t_k}^{t_{k+1}} (1-\theta)(f(s, Y(s)) - f(t_k, y_k))\,\mathrm{d}s \\
&\quad + \int_{t_k}^{t_{k+1}} \theta(f(s, Y(s)) - f(t_{k+1}, y_{k+1}))\,\mathrm{d}s + \int_{t_k}^{t_{k+1}} (g(s, Y(s)) - g(t_k, y_k))\,\mathrm{d}B(s)\Big\rangle \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= \Big\langle Y(t_{k+1}) - y_{k+1}, (1-\theta)\int_{t_k}^{t_{k+1}} (f(s, Y(s)) - f(t_k, y_k))\,\mathrm{d}s\Big\rangle \\
&= (1-\theta)\int_{t_k}^{t_{k+1}} \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(t_k, y_k)\big\rangle \mathrm{d}s, \\
I_2 &:= \Big\langle Y(t_{k+1}) - y_{k+1}, \theta\int_{t_k}^{t_{k+1}} (f(s, Y(s)) - f(t_{k+1}, y_{k+1}))\,\mathrm{d}s\Big\rangle \\
&= \theta\int_{t_k}^{t_{k+1}} \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(t_{k+1}, y_{k+1})\big\rangle \mathrm{d}s, \\
I_3 &:= \Big\langle Y(t_{k+1}) - y_{k+1}, (Y(t_k) - y_k) + \int_{t_k}^{t_{k+1}} (g(s, Y(s)) - g(t_k, y_k))\,\mathrm{d}s\Big\rangle.
\end{aligned}
$$

In order to estimate I_1, we rewrite the integrand of I_1 into three parts

$$
\begin{aligned}
\big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(t_k, y_k)\big\rangle &= \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(s, Y(t_k))\big\rangle \\
&\quad + \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(t_k)) - f(t_k, Y(t_k))\big\rangle \\
&\quad + \big\langle Y(t_{k+1}) - y_{k+1}, f(t_k, Y(t_k)) - f(t_k, y_k)\big\rangle \\
&= I_{11} + I_{12} + I_{13}.
\end{aligned}
$$

By applying the elementary inequality

$$
\langle a, b\rangle \le \frac{1}{2}(|a|^2 + |b|^2), \quad a, b \in \mathbb{R}^d, \tag{13}
$$

and (**H3**), we have

$$
\begin{aligned}
I_{11} &= \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(s, Y(t_k))\big\rangle \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}\left|f(s, Y(s)) - f(s, Y(t_k))\right|^2 \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_3(1 + |Y(s)|^b + |Y(t_k)|^b)|Y(s) - Y(t_k)|^2.
\end{aligned}
$$

Similarly, applying (**H2**) and (**H3**)

$$
\begin{aligned}
I_{12} &= \big\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(t_k)) - f(t_k, Y(t_k))\big\rangle \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}\left|f(s, y(t_k)) - f(t_k, y(t_k))\right|^2 \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_2(1 + |Y(t_k)|^a)|s - t_k|^\gamma,
\end{aligned}
$$

and

$$\begin{aligned}
I_{13} &= \left\langle Y(t_{k+1}) - y_{k+1}, f(t_k, Y(t_k)) - f(t_k, y_k) \right\rangle \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}\left|f(t_k, Y(t_k)) - f(t_k, y_k)\right|^2 \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_3(1 + |Y(t_k)|^b + \left|y_k\right|^b)\left|Y(t_k) - y_k\right|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1 \le (1-\theta)\int_{t_k}^{t_{k+1}} &\Bigg((\frac{3}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_3(1 + |Y(s)|^b + |Y(t_k)|^b)|Y(s) - Y(t_k)|^2 \\
&+ \frac{1}{2}K_2(1 + |Y(t_k)|^a)|s - t_k|^\gamma + \frac{1}{2}K_3(1 + |Y(t_k)|^b + \left|y_k\right|^b)\left|Y(t_k) - y_k\right|^2) \Bigg)\,\mathrm{d}s,
\end{aligned}$$

by the Hölder's inequality, we obtain

$$\begin{aligned}
\mathbb{E}((1 + \left|y(s)\right|^b + \left|y(t_k)\right|^b)\left|y(s) - y(t_k)\right|^2) \le (\mathbb{E}(1 + \left|y(s)\right|^b + \left|y(t_k)\right|^b)^2)^{1/2} \\
\cdot (\mathbb{E}\left|y(s) - y(t_k)\right|^4)^{1/2},
\end{aligned}$$

applying Lemmas 2.5 and 2.6, thus

$$\begin{aligned}
\mathbb{E}I_1 \le\ & \frac{3(1-\theta)}{2}\Delta t E\left|Y(t_{k+1}) - y_{k+1}\right|^2 + C\Delta t^{\gamma+1} + C\Delta t^{\gamma+1}\,\mathrm{e}^{Ct_k} + C\Delta t^2\,\mathrm{e}^{Ct_k} \\
&+ \mathbb{E}\left[\frac{K_3}{2}(1 + |Y(t_k)|^b + \left|y_k\right|^b)\left|Y(t_k) - y_k\right|^2\right].
\end{aligned} \tag{14}$$

Next, we rewrite the integrand of I_2 into three parts

$$\begin{aligned}
\left\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(t_{k+1}, y_{k+1}) \right\rangle &= \left\langle Y(t_{k+1}) - y_{k+1}, f(t_{k+1}, Y(t_{k+1})) - f(t_{k+1}, y_{k+1}) \right\rangle \\
&\quad + \left\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(t_{k+1})) - f(t_{k+1}, Y(t_{k+1})) \right\rangle \\
&\quad + \left\langle Y(t_{k+1}) - y_{k+1}, f(s, Y(s)) - f(s, Y(t_{k+1})) \right\rangle \\
&=: I_{21} + I_{22} + I_{23},
\end{aligned}$$

by using (**H1**), we have

$$I_{21} \le K_1\left|Y(t_{k+1}) - y_{k+1}\right|^2,$$

applying the elementary inequality (13) and (**H2**), we obtain

$$\begin{aligned}
I_{22} &\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}\left|f(s, Y(t_{k+1})) - f(t_{k+1}, Y(t_{k+1}))\right|^2 \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_2(1 + \left|Y(t_{k+1})\right|^a)\left|s - t_{k+1}\right|^\gamma,
\end{aligned}$$

applying the elementary inequality (13) and (**H3**)

$$\begin{aligned}
I_{23} &\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}\left|f(s, Y(s)) - f(s, Y(t_{k+1}))\right|^2 \\
&\le \frac{1}{2}\left|Y(t_{k+1}) - y_{k+1}\right|^2 + \frac{1}{2}K_3(1 + |Y(s)|^b + \left|Y(t_{k+1})\right|^b)\left|Y(s) - Y(t_{k+1})\right|^2.
\end{aligned}$$

Hence,

$$I_2 \le \theta \int_{t_k}^{t_{k+1}} (K_1+1)\big|Y(t_{k+1})-y_{k+1}\big|^2 + \frac{1}{2}K_2(1+\big|Y(t_{k+1})\big|^a)\big|s-t_{k+1}\big|^\gamma$$
$$+ \frac{1}{2}K_3(1+|Y(s)|^b+\big|Y(t_{k+1})\big|^b)\big|Y(s)-Y(t_{k+1})\big|^2)\,\mathrm{d}s.$$

Taking expectations on both sides and applying Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}\mathbb{E}I_2 &\le \theta(K_1+1)\Delta t\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2 + C\Delta t^{\gamma+1} + C\Delta t^{\gamma+1}\,\mathrm{e}^{Ct_{k+1}} + C\Delta t^2\,\mathrm{e}^{Ct_{k+1}}\\ &\le \theta(K_1+1)\Delta t\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2 + C\Delta t^{\gamma+1}\,\mathrm{e}^{Ct_{k+1}}.\end{aligned} \tag{15}$$

For I_3, by applying the elementary inequality (13), we have

$$\begin{aligned}I_3 &\le \frac{1}{2}\left|Y\left(t_{k+1}\right)-y_{k+1}\right|^2 + \frac{1}{2}\left|\left(Y\left(t_k\right)-y_k\right) + \int_{t_k}^{t_{k+1}} \left(g(s,Y(s))-g\left(t_k,y_k\right)\right)\mathrm{d}B(s)\right|^2\\ &=: \frac{1}{2}\left|Y\left(t_{k+1}\right)-y_{k+1}\right|^2 + \frac{1}{2}I_{31},\end{aligned}$$

taking expectation on both sides and using the Itô isometry, it follows that

$$\mathbb{E}I_{31} = \mathbb{E}\big|Y(t_k)-y_k\big|^2 + \mathbb{E}\int_{t_k}^{t_{k+1}} \big|g(s,Y(s))-g(t_k,y_k)\big|^2\,\mathrm{d}s.$$

By the elementary inequality (10) with $n=3$ and $q=2$,

$$\begin{aligned}\big|g(s,Y(s))-g(t_k,y_k)\big|^2 &\le 3(\big|g(s,Y(s))-g(s,Y(t_k))\big|^2 + \big|g(s,Y(t_k))-g(t_k,Y(t_k))\big|^2\\ &\quad + \big|g(t_k,Y(t_k))-g(t_k,x_k)\big|^2)\\ &\le 3(K_3|Y(s)-Y(t_k)|^2 + K_2(1+|Y(t_k)|^2)|s-t_k|^\gamma\\ &\quad + K_3\big|Y(t_k)-y_k\big|^2),\end{aligned}$$

taking expectation on both sides, then we have

$$\mathbb{E}I_3 \le \frac{1}{2}E\big|Y(t_{k+1})-y_{k+1}\big|^2 + \left(\frac{1+3K_3\Delta t}{2}\right)E\big|Y(t_k)-y_k\big|^2 + C\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_{k+1}}. \tag{16}$$

Combining (14) –(16) yields

$$\begin{aligned}\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2 &\le \frac{3(1-\theta)\Delta t}{2}E\big|Y(t_{k+1})-y_{k+1}\big|^2 + \theta(K_1+1)\Delta t\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2\\ &\quad + \frac{1}{2}\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2 + C\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_k} + \frac{K_3}{2}E(1+|Y(t_k)|^b\\ &\quad + \big|y_k\big|^b)\mathbb{E}\big|Y(t_k)-y_k\big|^2 + C\Delta t^{\gamma+1}\,\mathrm{e}^{Ct_{k+1}} + \frac{1+3K_3\Delta t}{2}\mathbb{E}\big|Y(t_k)-y_k\big|^2\\ &\quad + C\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_{k+1}}\\ &\le \frac{1+\Delta t(3-\theta+2\theta K_1)}{2}\mathbb{E}\big|Y(t_{k+1})-y_{k+1}\big|^2 + C\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_{k+1}}\\ &\quad + \mathbb{E}\big|Y(t_k)-y_k\big|^2\mathbb{E}\left(\frac{K_3}{2}(1+|Y(t_k)|^b+\big|y_k\big|^b) + \frac{1+3K_3\Delta t}{2}\right),\end{aligned}$$

from (12) and Lemma 2.5, we obtain

$$\mathbb{E}\left|Y(t_{k+1}) - y_{k+1}\right|^2 \le \frac{1 + K_4(\hat{C}\,\mathrm{e}^{\bar{C}\Delta t} + 3\Delta t) + K_3(1 + 2^{b-2/2} + 2^{b-2/2}\mathbb{E}\,|Y(0)|^b\,\mathrm{e}^{bK_4 t})}{1 - \Delta t(3 - \theta + 2\theta K_1)} \times \left(\mathbb{E}\left|Y(t_k) - y_k\right|^2 + C\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_{k+1}}\right).$$

Summing both sides of the above from 0 to $k-1$, we obtain

$$\sum_{l=1}^{k}\mathbb{E}\left|Y(t_l) - y_l\right|^2 \le \frac{1 + K_4(\hat{C}\,\mathrm{e}^{\bar{C}\Delta t} + 3\Delta t) + K_3(1 + 2^{b-2/2} + 2^{b-2/2}\mathbb{E}\,|Y(0)|^b\,\mathrm{e}^{bK_4 t})}{1 - \Delta t(3 - \theta + 2\theta K_1)} \times \left(\sum_{l=0}^{k-1}\mathbb{E}Y(t_l) - y_l|^2 + kC\Delta t^{(\gamma+1)\wedge 2}\,\mathrm{e}^{Ct_{k+1}}\right).$$

Due to the fact that $k\Delta t = t_k \le \mathrm{e}^{Ct_k}$, from combining same terms together on both sides we can derive

$$\begin{aligned}&\mathbb{E}\left|Y(t_k) - y_k\right|^2\\ &\quad\le \frac{K_4(\hat{C}\,\mathrm{e}^{\bar{C}\Delta t} + 3\Delta t) + K_3(1 + 2^{b-2/2} + 2^{b-2/2}\mathbb{E}|Y(0)|^b\,\mathrm{e}^{bK_4 t}) + \Delta t(3 - \theta + 2\theta K_1)}{1 - \Delta t(3 - \theta + 2\theta K_1)}\\ &\qquad + C\Delta t^{\gamma\wedge 1}\,\mathrm{e}^{Ct_{k+1}}.\end{aligned}$$

By the discrete version of the Gronwall inequality (see [36], Lemma 6.3), we have

$$\mathbb{E}\left|Y(t_k) - y_k\right|^2 \le C\Delta t^{\gamma\wedge 1}\mathrm{e}^{Ct_{k+1}}. \tag{17}$$

Moreover, when $t \in [t_k, t_{k+1})$ for some $k = 1, 2, \ldots, N$, from Lemma 2.6 and (17), we have

$$\begin{aligned}\mathbb{E}|Y(t) - y(t)|^2 &= \mathbb{E}\left|Y(t) - y_k\right|^2\\ &\le 2\mathbb{E}\,|Y(t) - Y(t_k)|^2 + 2\mathbb{E}\left|Y(t_k) - y_k\right|^2\\ &\le C\Delta t\mathrm{e}^{Ct} + C\Delta t^{\gamma\wedge 1}\mathrm{e}^{Ct_{k+1}}\\ &\le C\Delta t^{\gamma\wedge 1}\mathrm{e}^{Ct}.\end{aligned}$$

Therefore, the proof is complete. ■

Theorem 3.5: *Let assumptions* (**H1**−**H4**) *hold with* $p > 2(a \vee b)$. *Then the composition of the numerical solution* $y(t)$ *and the discretized inverse subordinator* $E_{\Delta t}(t)$, *i.e.* $y(E_{\Delta t}(t))$, *converges strongly to the solution of* (1) *with*

$$\mathbb{E}\left|X(T) - y(E_{\Delta t}(T))\right|^2 \le C\Delta t^{\gamma\wedge 1}\mathrm{e}^{CT},$$

where C *is a constant independent of* T *and* Δt.

Proof: By Lemma 2.2 and the elementary inequality with $n = 2$ and $q = 2$, we have

$$\begin{aligned}\mathbb{E}\left|X(T) - y(E_{\Delta t}(T))\right|^2 &= \mathbb{E}\left|Y(E(T)) - y(E_{\Delta t}(T))\right|^2\\ &\le 2\mathbb{E}|Y(E(T)) - Y(E_{\Delta t}(T))|^2 + 2\mathbb{E}\left|Y(E_{\Delta t}(T)) - y(E_{\Delta t}(T))\right|^2.\end{aligned}$$

By Lemmas 2.3, 2.4 and 2.6, we can see

$$\mathbb{E}|Y(E(T)) - Y(E_{\Delta t}(T))|^2 \leq C\Delta t \mathbb{E}e^{CE(T)} \leq C\Delta t e^{C(T\vee 1)}. \tag{18}$$

On the other hand, from Lemmas 2.3 and 2.4 and Theorem 3.4, we obtain

$$\mathbb{E}\left|Y(E_{\Delta t}(T)) - y(E_{\Delta t}(T))\right|^2 \leq C\Delta t^{\gamma\wedge 1}e^{CE_{\Delta t}(T)} \leq C\Delta t^{\gamma\wedge 1}e^{CE(T)} \leq C\Delta t^{\gamma\wedge 1}e^{C(T\vee 1)}. \tag{19}$$

Combining (18) and (19), we obtain the required assertion. The proof is complete. ■

3.2. Mean square stability

Theorem 3.6: *Let assumptions* (**H1−H4**) *hold. Assume that there exists a constant* $L_1 > 0$ *such that*

$$\left\langle x, f(t,x)\right\rangle + \frac{1}{2}\left|g(t,x)\right|^2 \leq -L_1|x|^2, \quad (x,t) \in \mathbb{R}^d \times [0,\infty). \tag{20}$$

Then the solution of the SDE (1) *satisfies*

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|X(t)|^2}{t} < 0.$$

Proof: Given (20) and by Lemma 2.2 with $p = 2$, we have

$$\mathbb{E}|Y(t)|^2 \leq e^{-L_1 t}\mathbb{E}|Y(0)|^2.$$

By Doob's martingale inequalities (see [31], Theorem 3.8), let $\varepsilon \in (0, L_1)$ be arbitrary.

$$\mathbb{P}\left\{\sup_{t_0+k-1<t<t_0+k} |Y(t)|^2 > e^{-(L_1-\varepsilon)t}\right\} \leq e^{-\varepsilon t}\mathbb{E}|Y(0)|^2.$$

By the Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$\sup_{t_0+k-1<t<t_0+k} |Y(t)|^2 \leq e^{-(L_1-\varepsilon)t}, \tag{21}$$

holds for all but finitely many k. Hence, there exists a $k_0 = k_0(\omega)$, for all $\omega \in \Omega$ excluding a $\mathbb{P}$-null set, for which (21) holds whenever $k \geq k_0$. Consequently, for almost all $\omega \in \Omega$,

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|Y(t)|^2}{t} < -(L_1 - \varepsilon) < 0,$$

and the theorem follows by letting $\varepsilon \to 0$. By using Lemma 2.2, we obtain

$$\mathbb{E}|X(t)|^2 = \mathbb{E}|Y(E(t))|^2 \leq \mathbb{E}\, e^{-L_1 E(t)} \cdot \mathbb{E}|Y(0)|^2.$$

Hence, the proof is complete. ■

Remark 3.7: It is interesting to observe that the time-changed SDEs (1) is mean square stability while the dual SDEs (3) is stable in the exponential rate. This may be due to the effect of the time-changed Brownian, which slows down the diffusion.

Now, we present our result about the exponential stability in mean square sense of the numerical solution.

Theorem 3.8: *Let assumptions* (**H1**−**H4**) *hold and satisfy* $2(1+K_1\alpha) < -K_5C_p\alpha + (1-\frac{2}{\theta})K_5{}^{-1/2}C_pT^{-1/2}\alpha$. *Then the numerical solution of the SDEs* (4) *satisfies*

$$\limsup_{k\to\infty}\frac{\log\mathbb{E}\big|y(E_{\Delta t}(t))\big|^2}{k\Delta t} < 0.$$

Proof: From Lemma 3.2 with $p = 1$, we have

$$\mathbb{E}\big|y_k\big|^2 \le \hat{C}\,\mathrm{e}^{-\bar{C}\Delta t}.$$

By using Lemma 2.3, we have

$$\mathbb{E}\big|y(E_{\Delta t}(t))\big|^2 \le \mathbb{E}\left|\widehat{C}\right|\cdot\mathbb{E}\,\mathrm{e}^{-\overline{C}E_{\Delta t}(t)/k} \le \mathbb{E}\left|\widehat{C}\right|\cdot\mathrm{e}^{\overline{C}\Delta t/k}\cdot\mathbb{E}\,\mathrm{e}^{-\overline{C}E(t)/k}.$$

By using Doob's martingale inequality, let $\varepsilon\in(0,L_1)$ be arbitrary, then we obtain

$$\mathbb{P}\left\{\sup_{t_0+k-1<t<t_0+k}\big|y(E_{\Delta t}(t))\big|^2 > \mathrm{e}^{-\overline{C}(E(t)-\Delta t-\varepsilon)/k}\right\} \le \widehat{C}\,\mathrm{e}^{-\overline{C}(E(t)-\Delta t)/k}.$$

By the Borel-Cantelli lemma, we see that for almost all $\omega\in\Omega$,

$$\sup_{t_0+k-1<t<t_0+k}\big|y(E_{\Delta t}(t))\big|^2 \le \mathrm{e}^{-\overline{C}(E(t)-\Delta t-\varepsilon)/k}, \tag{22}$$

holds for all but finitely many k. Hence, there exists a $k_0 = k_0(\omega)$, for all $\omega\in\Omega$ excluding a $\mathbb{P}$-null set, for which (22) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$,

$$\frac{1}{k\Delta t}\log\big|y(E_{\Delta t}(t))\big|^2 \le -\frac{\overline{C}(E(t)-\Delta t-\varepsilon)}{k^2\Delta t}.$$

Hence,

$$\limsup_{k\to\infty}\frac{\log\mathbb{E}\big|y(E_{\Delta t}(t))\big|^2}{k\Delta t} < 0.$$

The proof is complete. ∎

4. Numerical simulations

In this section, we provide two examples to illustrate the theorems obtained above. The first example is used to demonstrate the strong convergence of the time-changed SDEs (1) as well as the rate of convergence. The second example is used to capture the mean square stability of numerical stability at different values of $\theta = 0.6$ and $\theta = 0.8$. Throughout this section, we focus on the case that $E(t)$ is an inverse 0.9-stable subordinator.

Example 4.1: Consider a one-dimensional nonlinear time-changed SDEs

$$\mathrm{d}X(t) = (-X(t) - X^3(t))\,\mathrm{d}E(t) + X(t)\,\mathrm{d}B(E(t)), \tag{23}$$

with the initial data $X(0) = 1$. For $t\in[0,1]$ and $\Delta = 2^{-10}$, Paths of $X(t)$ are plotted in Figure 2.

Now we demonstrate the strong convergence rate. Since the explicit solution is hard to obtain, we regart the numerical solution with $\Delta t_0 = 2^{-10}$ as the true solution. Three hundreds ($M = 300$)

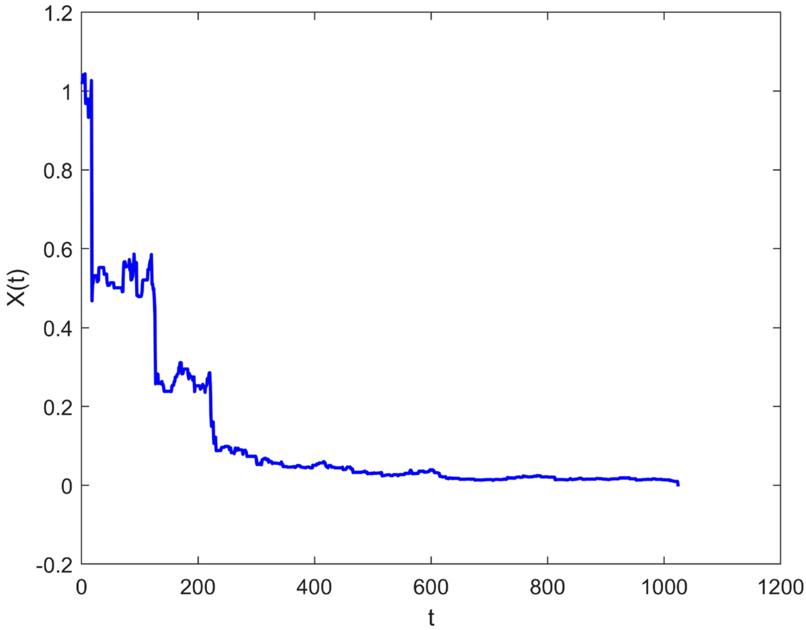


Figure 2. One path of $X(t)$.

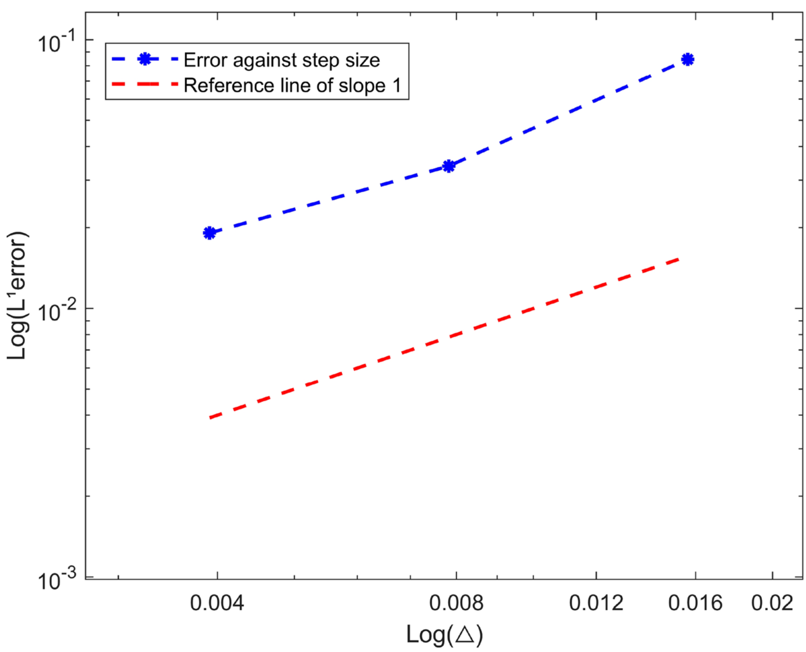
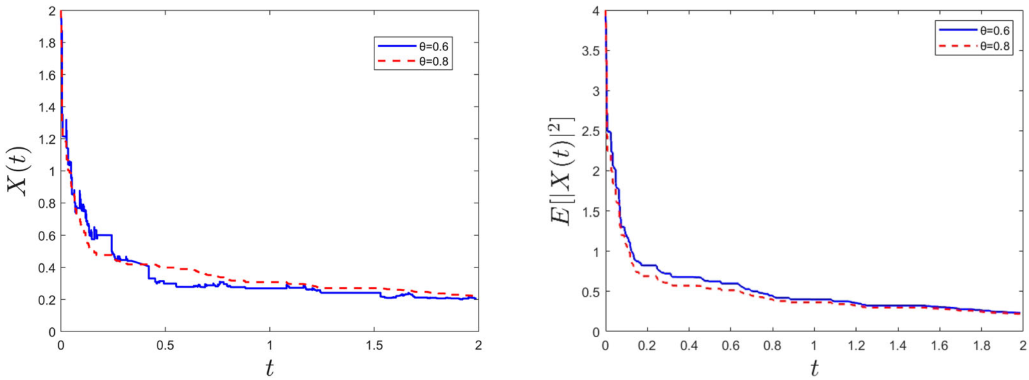


Figure 3. Blue line: Loglog plot of the strong L^1 error against the step size. Red Line: The reference line with the slope of 1.

samples are used to compute the strong convergence with the step sizes $\Delta t = 2^{-7}, 2^{-6}$ and 2^{-5}. For the given step size Δt, the L^1 strong error is calculated by

$$\text{Error} := \frac{1}{M} \sum_{k}^{M} \left| y_k(E_{\Delta t_0}(T)) - y_k(E_{\Delta t}(t)) \right|.$$



(a) One path of $X(t)$ (b) Mean square of $y(E_{\Delta t}(t))$

Figure 4. Numerical simulations of with different values of θ for Example 4.2. (a) One path of $X(t)$ and (b) Mean square of $y(E_{\Delta t}(t))$.

By Theorem 3.5, a strong convergence rate that is close to 1 is expected. The Loglog plot of the L^1 errors against the step sizes is shown in Figure 3. By linear regression, the slope is 1.0734, which is not far away from the theoretical result.

Example 4.2: Consider a one-dimensional time-changed SDEs

$$\mathrm{d}X(t) = -10X^3(t)\,\mathrm{d}E(t) + X^2(t)\,\mathrm{d}(B(E(t)), \tag{24}$$

is considered with the initial data $X(0) = 2$.

By Theorem 3.6, the time-changed SDEs (24) is mean square stability. The path to simulate numerical solutions of (24) using the SST method is shown in Figure 4(a), and chosen the fixed step size $\Delta t = 2^{-12}$ with different values of $\theta = 0.6$ and $\theta = 0.8$. Meanwhile, we simulate the mean square exponentially stable of the numerical solution of Equation (24) at different values of $\theta = 0.6$ and $\theta = 0.8$. Consequently, the computer simulation clearly supports our theoretical result.

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