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Transportation Inequalities for Stochastic Differential Equations Driven by the Time-Changed Brownian Motion

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Abstract
In this paper, we investigate a class of stochastic differential equations driven by the time-changed Brownian motion and impulsive stochastic differential equations driven by the time-changed Brownian motion. By establishing some time-changed retarded Gronwall-like inequalities and using the Girsanov transformation argument, we establish the quadratic transportation inequalities for the law of the solution of those equations.

Keywords Transportation inequality · Girsanov transformation · Time-changed retarded Gronwall-like inequality · Time changed-Brownian motion · Impulsive

Mathematics Subject Classification (2010) 60H15 · 60G15 · 60H05

1 Introduction

Recently, the time-changed semimartingale has attracted lots of attention and become one of the most active areas in stochastic analysis thanks to model anomalous diffusions arising in physics, finance, hydrology, and cell biology (see Umarov et al. [22] for the details). Kobayashi [10] investigated stochastic integrals with regard to the time-changed semimartingale and obtained the time-changed Itô's formula when the time change is the first hitting time process of a stable subordinator of index between 0 and 1. Following [10], stochastic differential equations driven by the time-changed Brownian motion or Lévy noise received much attention, see, e.g., [13, 17, 26, 27] for the stability; [6, 9, 14] for the numerical approximation scheme; [4, 21] for averaging principle.

On the other hand, the Talagrand-type transportation cost inequalities (TCIs) for stochastic processes have attracted increasing attention owing to the wide applications in Tsirel'son-type inequality and Hoeffding-type inequality ([7, 24, 25]) and in the concentration of empirical measure ([15, 23]). One of the most effective methods to establish the transportation inequality is the Girsanov transformation argument introduced in [7]. A great quantity of results

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on the TCIs for stochastic processes are achieved by using the Girsanov transformation argument. For example, [24, 25] for diffusion processes on $\mathbb{R}^d$, [19] for multidimensional semi-martingales, [23] for SDEs with pure jump processes, [20] for SDEs driven by a fractional Brownian motion, [12] for stochastic delay evolution equations driven by fBm with Hurst parameter $1/2 < H < 1$ and [1] for neutral stochastic evolution equations driven by fBm with Hurst parameter $0 < H < 1/2$.

Now, let us recall the kinds of inequalities. Let (E, d) be a metric space equipped with a σ-field $\mathcal{B}$ such that the distance $d(\cdot, \cdot)$ is $\mathcal{B}\otimes\mathcal{B}$-measurable. Given $p \geq 1$ and two probability measure μ and ν on E, the Wasserstein distance is defined by

$$W_p^d(\mu, \nu) = \inf_{\pi\in\mathcal{C}(\mu,\nu)} \left(\int\int d(x, y)^p d\pi(x, y)\right)^{1/p},$$

where $\mathcal{C}(\mu, \nu)$ denotes the totality of probability measures on the product space $E \times E$ with the marginal μ and ν. The relative entropy of ν with respect to μ is defined as

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \ln\frac{d\nu}{d\mu}d\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise}. \end{cases}$$

The probability measure μ satisfies the T^p transportation inequality on (E, d) if there exists a constant $C \geq 0$ such that for any probability measure ν,

$$W_p^d(\mu, \nu) \leq \sqrt{2C\mathbf{H}(\nu|\mu)}.$$

We write $\mu \in T_p(C)$ for this relation. The cases "$p = 1$" and "$p = 2$" are of particular interest. $T_1(C)$ is related to concentration of measure phenomenon and well characterized. The property $T_2(C)$ is stronger than $T_1(C)$ and it is closely associated with Poincaré inequality, logarithmic Sobolev inequality and Hamilton-Jacobi equations.

As far as we know, there is no result on the TCIs for stochastic differential equations driven by the time-changed Brownian motion. In this paper, we will attempt to explore the TCIs for the following SDEs driven by time-changed Brownian motion

$$\begin{cases} dx(t) = f(t, E_t, x(t))dt + h(t, E_t, x(t))dE_t + g(t, E_t, x(t))dB_{E_t}, & t \geq 0, \\ x(0) = x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where E_t is specified as an inverse of a stable subordinator of index $\beta \in (0, 1)$, f, h, g are some measurable functions satisfying the conditions in respective section. The main difficulties come from two aspects. First, the integral with respect to the time-changed term dE_t is more complicated than dt and is more unknown. In addition, the time-changed term dE_t, together with the term dt increases the complexity of the system. To overcome these defects, we will develop some time-changed retarded Gronwall-like inequalities and assume that E_t is asymptotically slower than t.

Impulsive phenomena can be found in a wide variety of evolutionary processes, such as electronics, telecommunications, mechanics, economics and biology, in which many sudden and abrupt changes occur instantaneously in the form of impulses. More recently, SDEs with impulses have attracted lots of interest (see [2, 3, 8, 11, 16]). In this paper, we also consider the following impulsive time-changed SDEs

$$\begin{cases} dx(t) = f(t, E_t, x(t))dt + h(t, E_t, x(t))dE_t + g(t, E_t, x(t))dB_{E_t}, & t \geq 0,\ t \neq t_k, \\ \triangle x_{t_k} = x_{t_k^+} - x_{t_k^-} = I_k(x_{t_k}), & t = t_k,\ k = 1, 2, \cdots, \\ x_0 = x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where the fixed moments of time t_k satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{k\to+\infty} t_k = \infty$; $x_{t_k^+}$ and $x_{t_k^-}$ denote the right and left limits of x_t at time t_k; x_t at each impulsive point t_k is right continuous. And $\triangle x_{t_k} = x_{t_k^+} - x_{t_k^-}$ represents the jump in the state x at time t_k, and I_k determines the size of the jump. The second main objective of this work is to establish the TCIs for impulsive time-changed SDEs (1.2) by establishing some time-changed Gronwall-like inequalities with jumps.

The rest of this paper is structured as follows. In Section 2, we recall some necessary notations and preliminaries. In Section 3, we establish the TCIs for (1.1). In Section 4, we consider the TCIs for (1.2).

2 Preliminary

Let $(\Omega_B, \mathcal{F}^B, \mathbb{P}_B)$ be a complete probability space with a filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$ satisfying the usual conditions, i.e. the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{E}_B$ denote the probability expectation with respect to $\mathbb{P}_B$. Let $\{D(t), t \geq 0\}$ be a right continuous left limit increasing Lévy process that is called subordinator starting from 0 defined on a complete probability space $(\Omega_D, \mathcal{F}^D, \mathbb{P}_D)$ with a filtration $\{\mathcal{F}_t^D\}_{t\geq 0}$ satisfying the usual conditions. Let $\mathbb{E}_D$ denote the probability expectation under the probability measure $\mathbb{P}_D$.

For a subordinator $D(t)$, in particular, is a β-stable subordinator if it is a strictly increasing process denoted by $D_\beta(t)$ and characterized by Laplace transform

$$\mathbb{E}(e^{-\lambda D_\beta(t)}) = e^{-t\lambda^\beta}, \quad \lambda > 0,\ \beta \in (0,1). \tag{2.1}$$

For an adapted β-stable subordinator $D_\beta(t)$, define its generalized inverse as

$$E_t := E_t^\beta = \inf\{s > 0 : D_\beta(s) > t\}, \tag{2.2}$$

which is known as the first hitting time process. The time change E_t is continuous and non-decreasing; However, it is not Markovian.

Let B_t be a standard Brownian motion independent of E_t, define the following filtration as

$$\mathcal{F}_t = \bigcap_{s>t}\{\sigma(B_r : 0 \leq r \leq s) \vee \sigma(E_r : r \geq 0)\},$$

where $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. It concludes that the time-changed Brownian motion B_{E_t} is a square integrable martingale with respect to the filtration $\{\mathcal{F}_{E_t}\}_{t\geq 0}$. Define the product probability space by

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega_B \times \Omega_D, \mathcal{F}^B \times \mathcal{F}^D, \mathbb{P}_B \times \mathbb{P}_D).$$

Let $\mathbb{E}$ denote the expectation under the probability measure $\mathbb{P}$. It is clear that $\mathbb{E}(\cdot) = \mathbb{E}_D\mathbb{E}_B(\cdot) = \mathbb{E}_B\mathbb{E}_D(\cdot)$. Define a filtration $\mathcal{G}_t$ by $\mathcal{G}_t = \mathcal{F}_{E_t}$, which satisfies the usual conditions because of the right-continuity of $\mathcal{F}_t$ and E_t.

In this section, we shall impose the following assumptions:

(H1) $f, h : \mathbb{R}_+ \times \mathbb{R}_+ \times C \to \mathbb{R}^d$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \times C \to \mathbb{R}^{d\times k}$ are Borel-measurable functions and there exists a constant $L > 0$ such that for any $t_1, t_2 \geq 0$ and $x, y \in C$ such that

$$|f(t_1,t_2,x) - f(t_1,t_2,y)| \vee |h(t_1,t_2,x) - h(t_1,t_2,y)| \vee |g(t_1,t_2,x) - g(t_1,t_2,y)| \leq L|x-y|.$$

(H2) If $x(t)$ is a right continuous with left limit and $\mathcal{G}_t$-adapted process, then

$$f(t, E_t, x_t), h(t, E_t, x_t), g(t, E_t, x_t) \in \mathcal{L}(\mathcal{G}_t),$$

where $\mathcal{L}(\mathcal{G}_t)$ denotes the class of left continuous with right limit.

Remark 2.1 By [10, 27], under the hypotheses (H1)-(H2) we can immediately know that (1.1) has a unique $\mathcal{G}_t = \mathcal{F}_{E_t}$-adapted solution process $x(t)$.

3 TCIs for (1.1)

In this section, we discuss the TCIs for time-changed stochastic differential equation (1.1). For the future use, we introduce the following useful lemmas to establish connections among different kinds of time-changed integrals.

Lemma 3.1 ([10]) *Let E_t be the $\mathcal{G}_t$-measurable time change which is the general inverse β-stable subordinator $D(t)$. Suppose $\mu(t)$ and $\delta(t)$ are $\mathcal{G}_t$-measurable and integrable. Then, for all $t \geq 0$ with probability one,*

$$\int_0^t \mu(s)dE_s + \int_0^t \delta(s)dB_{E_s} = \int_0^{E_t} \mu(D(s-))ds + \int_0^{E_t} \delta(D(s-))dB_s.$$

To establish results on the TCIs for (1.1), we develop the following time-changed retarded Gronwall-like inequality.

Lemma 3.2 *Suppose $D(t)$ is a subordinator with infinite Lévy measure, and E_t is the associated inverse. Let $u, f, g : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the $\mathcal{G}_t$-adapted functions which are integrable with respect to E_t. Let $n(t)$ be a positive, monotonic, non-decreasing function. Then, the inequality*

$$u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t g(s)u(s)dE_s, \quad t \geq 0, \ a.s., \tag{3.1}$$

implies almost surely

$$u(t) \leq n(t)exp\Big\{\int_0^t f(s)ds + \int_0^t g(s)dE_s\Big\}, \quad t \geq 0.$$

Proof Let

$$y(t) := n(t) + \int_0^t f(s)u(s)ds + \int_0^t g(s)u(s)dE_s, \quad t \geq 0. \tag{3.2}$$

Since $f(t)$ $g(t)$ and $u(t)$ are positive and $n(t)$ is non-decreasing, the function $y(t)$ defined in equation (3.2) is non-decreasing. Moreover, from equations (3.1) and (3.2),

$$u(t) \leq y(t), \quad t \geq 0,$$

which implies that

$$y(t) \leq n(t) + \int_0^t f(s)y(s)ds + \int_0^t g(s)y(s)dE_s, \quad t \geq 0.$$

Applying Lemma 3.1 yields

$$y(t) \leq n(t) + \int_0^t f(s)y(s)ds + \int_0^{E_t} g(D(s-))y(D(s-))ds, \quad t \geq 0. \tag{3.3}$$

Actually, for $t \geq 0$, $D(t-)$ is defined as

$$D(t-) = \inf\{s, E_s \geq t\},$$

which implies that

$$E_{D(t-)} = t, \quad \text{and} \quad D(E_t-) \leq t. \tag{3.4}$$

Let $\tau \in [0, \infty)$, then it holds from (3.3) and (3.4) that

$$\begin{aligned} y(D(\tau-)) \leq & n(D(\tau-)) + \int_0^{D(\tau-)} f(s)y(s)ds + \int_0^{E_{D(\tau-)}} g(D(s-))y(D(s-))ds \\ = & n(D(\tau-)) + \int_0^{D(\tau-)} f(s)y(s)ds + \int_0^{\tau} g(D(s-))y(D(s-))ds. \end{aligned}$$

Applying the retarded Gronwall-like inequality (see Proposition 1 in [28]) path by path to yield

$$u(D(\tau-)) \leq y(D(\tau-)) \leq n(D(\tau-))\exp\Big\{\int_0^{D(\tau-)} f(s)ds + \int_0^{\tau} g(D(s-))ds\Big\}.$$

Let $t = D(\tau-)$. Since $n(t)$ is non-decreasing, applying (3.4) and the Lemma 3.1, we have

$$u(t) \leq y(t) \leq n(t)\exp\Big\{\int_0^{t} f(s)ds + \int_0^{E_t} g(D(s-))ds\Big\} = n(t)\exp\Big\{\int_0^{t} f(s)ds + \int_0^{t} g(s)dE_s\Big\}.$$

The proof is complete. □

To obtain our desired results on TCIs for (1.1) and (1.2), we need the following assumption on time-changed process E_t.

(H3) E_t is the $\mathcal{G}_t$-measurable time change and is asymptotically slower than t, i.e,

$$\lim_{t\to\infty} \frac{E_t}{t} = 0, \quad a.s.$$

Remark 3.1 Let $\{E_t\}_{t\geq 0}$ be the inverse of the mixed stable subordinator $D(t)$ with Laplace exponent, then E_t is the $\mathcal{G}_t$-measurable time change and is asymptotically slower than t (see the Lemma 2.11 in [18]).

Theorem 3.1 *Let the conditions (H1)-(H3) hold and $\mathbb{P}_{x_0}$ be the law of $x(\cdot, x_0)$, solution process of (1.1). Assume further that g is bounded by $\widetilde{g} := \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}^d} \|g(t, E_t, x)\|$. Then the probability measure $\mathbb{P}_{x_0}$ satisfies $T_2(C)$ on the metric space $C([0, T]; \mathbb{R}^d)$ with:*

(a) $C = 4\tilde{g}^2 T exp^{4T^2L^2+4(L^2+TL^2)T}$ *with the metric*

$$d_\infty(\gamma_1, \gamma_2) := \sup_{0\leq t\leq T} |\gamma_1 - \gamma_2|, \quad \gamma_1, \gamma_2 \in C([0, T]; \mathbb{R}^d);$$

(b) $C = 4\tilde{g}^2 T^2 exp^{4T^2L^2+4(L^2+TL^2)T}$ *when using the metric*

$$d_2(\gamma_1, \gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2} \quad \gamma_1, \gamma_2 \in C([0, T]; \mathbb{R}^d).$$

Proof Let $\mathbb{P}_{x_0}$ be the law of $x(\cdot, x_0)$ on $\mathscr{L} := C([0, T]; \mathbb{R}^d)$ and $\mathbb{Q}$ be any probability measure on $\mathscr{L}$ such that $\mathbb{Q} \ll \mathbb{P}_{x_0}$. Define

$$\widetilde{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))\mathbb{P}, \tag{3.5}$$

which is a probability measure on $(\Omega, \mathcal{F})$. Recalling the definition of entropy and adopting a measure-transformation argument we obtain from (3.5) that

$$\begin{aligned}\mathbf{H}(\widetilde{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}\right) d\widetilde{\mathbb{Q}} = \int_\Omega \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))\right) \frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))d\mathbb{P} \\ &= \int_{\mathscr{L}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}\right) \frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}} d\mathbb{P}_{x_0} \\ &= \mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0}).\end{aligned}$$

Following [5, 7], there exists a predictable process $\{h_1(t)\}_{0\le t\le T} \in \mathbb{R}^d$ with $\int_0^T |h_1(s)|^2 ds < \infty$, $\mathbb{P}$-a.s., such that

$$\mathbf{H}(\widetilde{\mathbb{Q}}|\mathbb{P}) = \mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0}) = \frac{1}{2}\mathbb{E}^{\widetilde{\mathbb{Q}}}|h_1(t)|^2 dt.$$

Due to the Girsanov theorem, the process $\{\widetilde{B}_t\}_{t\in[0,T]}$ defined by

$$\widetilde{B}_t = B_t - \int_0^t h_1(s)ds$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t\ge 0}$ on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{Q}})$ and the process $\{\widetilde{B}_t\}_{t\in[0,T]}$ defined by

$$\widetilde{B}_{E_t} = B_{E_t} - \int_0^{E_t} h_1(s)ds$$

is a time-changed Brownian motion with respect to $\{\mathcal{F}_t\}_{t\ge 0}$ on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{Q}})$.

Consequently, under the measure $\widetilde{\mathbb{Q}}$, the process $\{x(t, x_0)\}_{t\in[0,T]}$ satisfies

$$\begin{cases} dx(t) = f(t, E_t, x(t))dt + h(t, E_t, x(t))dE_t + g(t, E_t, x(t))h_1(E_t)dE_t \\ \qquad\quad + g(t, E_t, x(t))d\widetilde{B}_{E_t}, \ t \in [0, T], \\ x_0 \quad\ = x. \end{cases} \tag{3.6}$$

We now consider the solution y (under $\widetilde{\mathbb{Q}}$) of the following equation:

$$\begin{cases} dy(t) = f(t, E_t, y(t))dt + h(t, E_t, y(t))dE_t + g(t, E_t, y(t))d\widetilde{B}_{E_t}, \ t \in [0, T], \\ y_0 = x. \end{cases} \tag{3.7}$$

By the Remark 2.1, under $\widetilde{\mathbb{Q}}$ the law of $y(\cdot)$ is $\mathbb{P}_{x_0}$. Thus (x, y) under $\widetilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_{x_0})$, and it follows that

$$\begin{aligned}[W_2^{d_2}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 &\le \mathbb{E}_{\widetilde{\mathbb{Q}}}(|d_2(x, y)|^2) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\int_0^T |x(t) - y(t)|^2 dt\right), \\ [W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 &\le \mathbb{E}_{\widetilde{\mathbb{Q}}}(|d_\infty(x, y)|^2) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\sup_{0\le t\le T} |x(t) - y(t)|^2\right).\end{aligned}$$

We now estimate the distance between x and y with respect to d_2 and d_∞. Note from (3.6) and (3.7) that

$$\begin{aligned} x(t)-y(t)=&\int_0^t [f(s,E_s,x(s))-f(s,E_s,y(s))]ds+\int_0^t [h(s,E_s,x(s))-h(s,E_s,y(s))]dE_s\\ &+\int_0^t g(s,E_s,x(s))h_1(E_s)dE_s+\int_0^t [g(s,E_s,x(s))-g(s,E_s,y(s))]d\widetilde{B}_{E_s}\\ :=&I_1+I_2+I_3+I_4. \end{aligned} \tag{3.8}$$

Firstly, applying the assumption (H1) and the Hölder's inequality to I_1,

$$\begin{aligned} \mathbb{E}_{\widetilde{B}}|I_1|^2 \le& T\mathbb{E}_{\widetilde{B}}\int_0^t |f(s,E_s,x(s))-f(s,E_s,y(s))|^2ds\\ \le& TL^2\int_0^t \mathbb{E}_{\widetilde{B}}|x(s)-y(s)|^2ds. \end{aligned} \tag{3.9}$$

For I_2, applying the assumption (H1) and the Hölder's inequality, we can obtain

$$\begin{aligned} \mathbb{E}_{\widetilde{B}}|I_2|^2 \le& E_T\mathbb{E}_{\widetilde{B}}\int_0^t |h(s,E_s,x(s))-h(s,E_s,y(s))|^2dE_s\\ \le& E_TL^2\int_0^t \mathbb{E}_{\widetilde{B}}|x(s)-y(s)|^2dE_s. \end{aligned} \tag{3.10}$$

For I_3, by using the boundedness of g and the Hölder's inequality, we have

$$\mathbb{E}_{\widetilde{B}}|I_3|^2 \le \tilde{g}^2\mathbb{E}_{\widetilde{B}}\Big(E_T\int_0^t |h_1(E_s)|^2dE_s\Big).$$

By the Lemma 3.1, we get

$$\mathbb{E}_{\widetilde{B}}|I_3|^2 \le \tilde{g}^2\mathbb{E}_{\widetilde{B}}\Big(E_T\int_0^{E_t} |h_1(s)|^2ds\Big). \tag{3.11}$$

Since $\widetilde{B}_{E_s}$ is a $\mathcal{G}$-semimartingales, we know that by the assumption (H1)

$$\mathbb{E}_{\widetilde{B}}|I_4|^2 \le L^2\int_0^t \mathbb{E}_{\widetilde{B}}|x(s)-y(s)|^2dE_s. \tag{3.12}$$

Then, from (3.8)-(3.12), we have

$$\begin{aligned} \mathbb{E}_{\widetilde{B}}|x(t)-y(t)|^2 \le& 4TL^2\int_0^t \mathbb{E}_{\widetilde{B}}|x(s)-y(s)|^2ds+4(L^2+E_TL^2)\int_0^t \mathbb{E}_{\widetilde{B}}|x(s)-y(s)|^2dE_s\\ &+4\tilde{g}^2\mathbb{E}_{\widetilde{B}}\Big(E_T\int_0^{E_t} |h_1(s)|^2ds\Big). \end{aligned}$$

Thus, by view of the Lemma 3.2, we obtain

$$\mathbb{E}_{\widetilde{B}}|x(t)-y(t)|^2 \le 4\tilde{g}^2\mathbb{E}_{\widetilde{B}}\Big(E_T\int_0^{E_t} |h_1(s)|^2ds\Big)\exp^{4T^2L^2+4(L^2+E_TL^2)E_T}. \tag{3.13}$$

Taking $\mathbb{E}_D$ on both side of (3.13), we have

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}|x(t)-y(t)|^2 \le 4\tilde{g}^2\mathbb{E}_{\widetilde{\mathbb{Q}}}\Big(E_T\int_0^{E_t} |h_1(s)|^2ds\Big)\mathbb{E}_{\widetilde{\mathbb{Q}}}\exp^{4T^2L^2+4(L^2+E_TL^2)E_T}. \tag{3.14}$$

Since E_t is the $\mathcal{G}_t$-measurable non-decreasing time change and is asymptotically slower than t, there exists $T > 0$ such that $E_T \leq T$. Also, when $t \leq T$, we can deduce that $E_t \leq E_T \leq T$. Hence, we may write that

$$d_\infty^2(X, Y) \leq 4\tilde{g}^2 T \exp^{4T^2L^2+4(L^2+TL^2)T} \int_0^T |h_1(s)|^2 ds$$

and

$$[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq 2C_1(T, L)\mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0})$$

with $C_1(T, L) = 4\tilde{g}^2 T \exp^{4T^2L^2+4(L^2+TL^2)T}$.

Analogously for the metric d_2, we have

$$\begin{aligned} [W_2^{d_2}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq& \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T |x(t) - y(t)|^2 dt \\ \leq& 4\tilde{g}^2 T \exp^{4T^2L^2+4(L^2+TL^2)T} \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T \int_0^T |h_1(s)|^2 ds dt \\ =& 4\tilde{g}^2 T^2 \exp^{4T^2L^2+4(L^2+TL^2)T} \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T |h_1(s)|^2 ds. \end{aligned}$$

So, we can obtain

$$\begin{aligned} [W_2^{d_2}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq& 8\tilde{g}^2 T^2 \exp^{4T^2L^2+4(L^2+TL^2)T} \left(\frac{1}{2} \mathbb{E}_{\tilde{\mathbb{Q}}} \int_0^T |h_1(s)|^2 ds \right) \\ \leq& 2C_2(T, L)\mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0}) \end{aligned}$$

with $C_2(T, L) = 4\tilde{g}^2 T^2 \exp^{4T^2L^2+4(L^2+TL^2)T}$. The proof is complete. □

4 TCIs for (1.2)

In order to get the TCIs for impulsive SDEs driven by the time-changed Brownian motion (1.2), we need to develop the following time-changed retarded Gronwall-like inequality with jumps.

Lemma 4.1 *Suppose $D(t)$ is a subordinator with infinite Lévy measure, and E_t is the associated inverse. Let $u, f, g : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the $\mathcal{G}_t$-adapted functions which are integrable with respect to E_t. Let $n(t)$ be a positive, monotonic, non-decreasing function. Then, the inequality*

$$u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t g(s)u(s)dE_s + \sum_{0<t_i<t} \beta_i u(t_i), \quad t \geq t_0, \ a.s., \tag{4.1}$$

implies almost surely

$$u(t) \leq \Pi_{0<t_i<t}(1 + \beta_i)n(t)exp\left\{ \int_0^t f(s)ds + \int_0^t g(s)dE_s \right\}, \quad t \geq t_0.$$

Proof The proof of this lemma can be given by induction with respect to $k \in \mathbb{N}$, applying for $t \in (t_k, t_{k+1}]$ Lemma 3.2 to the inequality (4.1)

$$u(t) \leq a_k + \int_{t_k}^t f(s)u(s)ds + \int_{t_k}^t g(s)u(s)dE_s, \quad t_k < t \leq t_{k+1}, \ a.s.,$$

where

$$a_k = u(t_k^+) \le n(t) + \int_{t_0}^{t_k} f(s)u(s)ds + \int_{t_0}^{t_k} g(s)u(s)dE_s + \sum_{i=1}^{k} \beta_i u(t_i)$$
$$\le n(t)\Pi_{i=1}^{k}(1+\beta_i)\exp\Big\{\int_{t_0}^{t_k} f(s)ds + \int_{t_0}^{t_k} g(s)dE_s\Big\}.$$

The proof is complete. □

Let $J \subset \mathbb{R}$ is a bounded interval. $\varphi(t+)$ and $\varphi(t-)$ is the right-hand limit of the function $\varphi(t)$ and left-hand limit of the function $\varphi(t)$, respectively. The piecewise function $\varphi(t) : J \to \mathbb{R}^d$ is continuous except at a finite number of points in its domain, and $\varphi(t+) = \varphi(t)$.

In the section, we make the following assumption on the impulsive item.

(H4) There are some constants $\beta_k > 0$ such that for any $x, y \in \mathbb{R}^d$ and $M := \sum_{\beta_k} < \infty$

$$|I_k(x) - I_k(y)| \le \beta_k|x-y|, \quad k = 1, 2, \cdots.$$

Remark 4.1 According to [26], under the hypotheses (H1)-(H2) and (H4) we can immediately know that (1.2) has a unique $\mathcal{G}_t = \mathcal{F}_{E_t}$-adapted solution process $x(t)$.

Theorem 4.1 *Let the conditions (H1)-(H4) hold and $\mathbb{P}_{x_0}$ be the law of $x(\cdot, x_0)$, solution process of (1.2). Assume further that g is bounded by $\widetilde{g} := \sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} \|g(t, E_t, x)\|$. Then the probability measure $\mathbb{P}_{x_0}$ satisfies $T_2(C)$ on the metric space $PC([0, T]; \mathbb{R}^d)$ with:*

(a) $C = 5\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T exp^{5T^2L^2+5(L^2+TL^2)T}$ with the metric

$$d_\infty(\gamma_1, \gamma_2) := \sup_{0\le t\le T} |\gamma_1 - \gamma_2|, \quad \gamma_1, \gamma_2 \in PC([0, T]; \mathbb{R}^d);$$

(b) $5\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T^2 exp^{5T^2L^2+5(L^2+TL^2)T}$ when using the metric

$$d_2(\gamma_1, \gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2} \quad \gamma_1, \gamma_2 \in PC([0, T]; \mathbb{R}^d).$$

Proof Let $\mathbb{P}_{x_0}$ be the law of $x(\cdot, x_0)$ on $\mathscr{L} := PC([0, T]; \mathbb{R}^d)$ and $\mathbb{Q}$ be any probability measure on $\mathscr{L}$ such that $\mathbb{Q} \ll \mathbb{P}_{x_0}$. Define

$$\overline{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))\mathbb{P}, \tag{4.2}$$

which is a probability measure on $(\Omega, \mathcal{F})$. Recalling the definition of entropy and adopting a measure-transformation argument we obtain from (4.2) that

$$\begin{aligned}
\mathbf{H}(\overline{\mathbb{Q}}|\mathbb{P}) &= \int_\Omega \ln\left(\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}}\right) d\overline{\mathbb{Q}} = \int_\Omega \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))\right)\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}(x(\cdot, x_0))d\mathbb{P} \\
&= \int_{\mathscr{L}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}}\right)\frac{d\mathbb{Q}}{d\mathbb{P}_{x_0}} d\mathbb{P}_{x_0} \\
&= \mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0}).
\end{aligned}$$

Following [5, 7], there exists a predictable process $\{h_2(t)\}_{0\le t\le T} \in \mathbb{R}^d$ with $\int_0^T |h_2(s)|^2 ds < \infty$, $\mathbb{P}$-a.s., such that

$$\mathbf{H}(\overline{\mathbb{Q}}|\mathbb{P}) = \mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0}) = \frac{1}{2}\mathbb{E}^{\overline{\mathbb{Q}}}|h_2(t)|^2 dt.$$

Due to the Girsanov theorem, the process $\{\overline{B}_t\}_{t\in[0,T]}$ defined by

$$\overline{B}_t = B_t - \int_0^t h_2(s)ds$$

is a Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \overline{\mathbb{Q}})$ and the process $\{\overline{B}_t\}_{t\in[0,T]}$ defined by

$$\overline{B}_{E_t} = B_{E_t} - \int_0^{E_t} h_2(s)ds$$

is a time-changed Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \overline{\mathbb{Q}})$.

Consequently, under the measure $\overline{\mathbb{Q}}$, the process $\{x(t, x_0)\}_{t\in[0,T]}$ satisfies

$$\begin{cases} dx(t) = f(t, E_t, x(t))dt + h(t, E_t, x(t))dE_t + g(t, E_t, x(t))h_2(E_t)dE_t \\ \qquad + g(t, E_t, x(t))d\overline{B}_{E_t}, & t \neq t_k,\ t \in [0, T], \\ \triangle x_{t_k} = x_{t_k^+} - x_{t_k^-} = I_k(x_{t_k}), & t = t_k,\ k = 1, 2, \cdots, \\ x_0 = x \in \mathbb{R}^d. \end{cases} \tag{4.3}$$

We now consider the solution y (under $\overline{\mathbb{Q}}$) of the following equation:

$$\begin{cases} dy(t) = f(t, E_t, y(t))dt + h(t, E_t, y(t))dE_t + g(t, E_t, y(t))d\overline{B}_{E_t}, & t \neq t_k,\ t \in [0, T], \\ \triangle y_{t_k} = y_{t_k^+} - y_{t_k^-} = I_k(y_{t_k}), & t = t_k,\ k = 1, 2, \cdots, \\ y_0 = x \in \mathbb{R}^d. \end{cases} \tag{4.4}$$

By the Remark 4.1, under $\overline{\mathbb{Q}}$ the law of $y(\cdot)$ is $\mathbb{P}_{x_0}$. Thus (x, y) under $\overline{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_{x_0})$, and it follows that

$$[W_2^{d_2}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq \mathbb{E}_{\overline{\mathbb{Q}}}(|d_2(x, y)|^2) = \mathbb{E}_{\overline{\mathbb{Q}}}\left(\int_0^T |x(t) - y(t)|^2 dt\right),$$

$$[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq \mathbb{E}_{\overline{\mathbb{Q}}}(|d_\infty(x, y)|^2) = \mathbb{E}_{\overline{\mathbb{Q}}}\left(\sup_{0\leq t\leq T} |x(t) - y(t)|^2\right).$$

We now estimate the distance between x and y with respect to d_2 and d_∞. Note from (4.3) and (4.4) that

$$\begin{aligned} x(t) - y(t) = &\int_0^t [f(s, E_s, x(s)) - f(s, E_s, y(s))]ds + \int_0^t [h(s, E_s, x(s)) - h(s, E_s, y(s))]dE_s \\ &+ \int_0^t g(s, E_s, x(s))h_1(E_s)dE_s + \int_0^t [g(s, E_s, x(s)) - g(s, E_s, y(s))]d\overline{B}_{E_s} \\ &+ \sum_{0<t_k<t} (I_k(x_{t_k}) - I_k(y_{t_k})) \\ :=&I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.5}$$

Firstly, applying the assumption (H1) and the Hölder's inequality to I_1,

$$\begin{aligned} \mathbb{E}_{\overline{B}}|I_1|^2 \leq &T\mathbb{E}_{\overline{B}} \int_0^t |f(s, E_s, x(s)) - f(s, E_s, y(s))|^2 ds \\ \leq &TL^2 \int_0^t \mathbb{E}_{\overline{B}}|x(s) - y(s)|^2 ds. \end{aligned} \tag{4.6}$$

For I_2, applying the assumption (H1) and the Hölder's inequality, we can obtain

$$\begin{aligned}\mathbb{E}_{\overline{B}}|I_2|^2 \leq & E_T \mathbb{E}_{\overline{B}} \int_0^t |h(s, E_s, x(s)) - h(s, E_s, y(s))|^2 dE_s \\ \leq & E_T L^2 \int_0^t \mathbb{E}_{\overline{B}}|x(s) - y(s)|^2 dE_s.\end{aligned} \tag{4.7}$$

For I_3, by using the boundedness of g and the Hölder's inequality, we have

$$\mathbb{E}_{\overline{B}}|I_3|^2 \leq \tilde{g}^2 \mathbb{E}_{\overline{B}}\Big(E_T \int_0^t |h_2(E_s)|^2 dE_s\Big).$$

By the Lemma 3.1, we get

$$\mathbb{E}_{\overline{B}}|I_3|^2 \leq \tilde{g}^2 \mathbb{E}_{\overline{B}}\Big(E_T \int_0^{E_t} |h_2(s)|^2 ds\Big). \tag{4.8}$$

Since $\overline{B}_{E_s}$ is a $\mathcal{G}$-semimartingales, we know that by the assumption (H1)

$$\mathbb{E}_{\overline{B}}|I_4|^2 \leq L^2 \int_0^t \mathbb{E}_{\overline{B}}|x(s) - y(s)|^2 dE_s. \tag{4.9}$$

By using the assumption (H4), we have

$$\mathbb{E}_{\overline{B}}|I_5|^2 \leq \sum_{0<t_k<t} \beta_k^2 \mathbb{E}_{\overline{B}}\Big(|x(t_k) - y(t_k)|^2\Big). \tag{4.10}$$

Then, from (4.6)-(4.10), we have

$$\begin{aligned}\mathbb{E}_{\overline{B}}|x(t) - y(t)|^2 \leq & 5TL^2 \int_0^t \mathbb{E}_{\overline{B}}|x(s) - y(s)|^2 ds + 5(L^2 + E_T L^2) \int_0^t \mathbb{E}_{\overline{B}}|x(s) - y(s)|^2 dE_s \\ & + 5\tilde{g}^2 \mathbb{E}_{\overline{B}}\Big(E_T \int_0^{E_t} |h_2(s)|^2 ds\Big) + 5 \sum_{0<t_k<t} \beta_k^2 \mathbb{E}_{\overline{B}}\Big(|x(t_k) - y(t_k)|^2\Big).\end{aligned}$$

Thus, by view of the Lemma 4.1, we obtain

$$\mathbb{E}_{\overline{B}}|x(t) - y(t)|^2 \leq 5\tilde{g}^2 \Pi_{0<t_i<t}(1 + 5\beta_i^2)\mathbb{E}_{\overline{B}}\Big(E_T \int_0^{E_t} |h_2(s)|^2 ds\Big)\exp^{5T^2L^2+5(L^2+TL^2)E_T}. \tag{4.11}$$

Taking $\mathbb{E}_D$ on both side of (4.11), we have

$$\mathbb{E}_{\overline{\mathbb{Q}}}|x(t) - y(t)|^2 \leq 5\tilde{g}^2 \Pi_{0<t_i<t}(1 + 5\beta_i^2)\mathbb{E}_{\overline{\mathbb{Q}}}\Big(E_T \int_0^{E_t} |h_2(s)|^2 ds\Big)\mathbb{E}_{\overline{\mathbb{Q}}}\exp^{5T^2L^2+5(L^2+TL^2)E_T}. \tag{4.12}$$

Since E_t is the $\mathcal{G}_t$-measurable time change and is asymptotically slower than t, there exists $T > 0$ such that $E_T \leq T$. Also, when $t \leq T$, we can deduce that $E_t \leq E_T \leq T$. Hence, we may write that

$$d_\infty^2(X, Y) \leq 5\tilde{g}^2 \Pi_{0<t_i<t}(1 + 5\beta_i^2) T \exp^{5T^2L^2+5(L^2+TL^2)T} \int_0^T |h_2(s)|^2 ds$$

and

$$[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_{x_0})]^2 \leq 2C_3(T, L)\mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0})$$

with $C_3(T, L) = 5\tilde{g}^2 \Pi_{0<t_i<t}(1 + 5\beta_i^2) T \exp^{5T^2L^2+5(L^2+TL^2)T}$.

Analogously for the metric d_2, we have

$$\begin{aligned}
[W_2^{d_2}(\mathbb{Q},\mathbb{P}_{x_0})]^2 \leq & \mathbb{E}_{\overline{\mathbb{Q}}}\int_0^T |x(t)-y(t)|^2 dt \\
\leq & 5\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T\exp^{5T^2L^2+5(L^2+TL^2)T}\mathbb{E}_{\overline{\mathbb{Q}}}\int_0^T\int_0^T |h_1(s)|^2 ds dt \\
= & 5\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T^2\exp^{5T^2L^2+5(L^2+TL^2)T}\mathbb{E}_{\overline{\mathbb{Q}}}\int_0^T |h_2(s)|^2 ds.
\end{aligned}$$

So, we can obtain

$$\begin{aligned}
[W_2^{d_2}(\mathbb{Q},\mathbb{P}_{x_0})]^2 & \leq 10\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T^2\exp^{5T^2L^2+5(L^2+TL^2)T}\left(\frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}}\int_0^T |h_2(s)|^2 ds\right) \\
& \leq 2C_4(T,L)\mathbf{H}(\mathbb{Q}|\mathbb{P}_{x_0})
\end{aligned}$$

with $C_4(T,L) = 5\tilde{g}^2\Pi_{0<t_i<t}(1+5\beta_i^2)T^2\exp^{5T^2L^2+5(L^2+TL^2)T}$. The proof is complete. □

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Declarations

Competing Interests The authors declare that they have no competing interests.

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