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Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process

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ABSTRACT

We study the convergence of a drift implicit scheme for one-dimensional SDEs that was considered by Alfonsi (2005) for the Cox–Ingersoll–Ross (CIR) process. Under general conditions, we obtain a strong convergence of order 1. In the CIR case, Dereich et al. (2012) have shown recently a strong convergence of order 1/2 for this scheme. Here, we obtain a strong convergence of order 1 under more restrictive assumptions on the CIR parameters.

This paper analyses the strong error of a discretization scheme for the Cox–Ingersoll–Ross (CIR) process and complements a recent paper by Dereich et al. (2012). The CIR process, which is widely used in financial modelling, follows the well defined SDE:

$$dX_t = (a - kX_t)dt + \sigma\sqrt{X_t}dW_t, \qquad X_0 = x. \tag{1}$$

Here, W denotes a standard Brownian motion, $a \geq 0$, $k \in \mathbb{R}$, $\sigma > 0$ and $x \geq 0$. This SDE has a unique strong solution that is nonnegative. It is even positive when $\sigma^2 \leq 2a$ and $x > 0$, which we assume in this paper. It is well-known that the usual Euler–Maruyama scheme is not defined for (1). Different ad-hoc discretization schemes have thus been proposed in the literature (see references in Dereich et al. (2012)). Here, we focus on a drift implicit scheme that has been proposed in Alfonsi (2005). We consider a time horizon $T > 0$ and a regular time grid:

$$t_k = \frac{kT}{n}, \quad 0 \leq k \leq n.$$

By Itô's formula, $Y_t = \sqrt{X_t}$ satisfies

$$dY_t = \left(\frac{a - \sigma^2/4}{2Y_t} - \frac{k}{2}Y_t\right)dt + \frac{\sigma}{2}dW_t, \qquad Y_0 = \sqrt{x}. \tag{2}$$

We consider the following drift implicit Euler scheme

$$\hat{Y}_0 = \sqrt{x}, \qquad \hat{Y}_t = \hat{Y}_{t_k} + \left(\frac{a - \sigma^2/4}{2\hat{Y}_t} - \frac{k}{2}\hat{Y}_t\right)(t - t_k) + \frac{\sigma}{2}(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}]. \tag{3}$$

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Eq. (3) is a quadratic equation that has a unique positive solution:

$$\hat{Y}_t = \frac{\hat{Y}_{t_k} + \frac{\sigma}{2}(W_t - W_{t_k}) + \sqrt{\left(\hat{Y}_{t_k} + \frac{\sigma}{2}(W_t - W_{t_k})\right)^2 + 2\left(1+\frac{k}{2}(t-t_k)\right)\left(a - \frac{\sigma^2}{4}\right)(t-t_k)}}{2\left(1+\frac{k}{2}(t-t_k)\right)},$$

provided that the time-step is small enough ($T/n \leq 2/\max(-k, 0)$ with the convention $2/0 = +\infty$). Last, we set $\hat{X}_t = (\hat{Y}_t)^2$, $t \in (t_k, t_{k+1}]$. It is shown in Alfonsi (2005) that this scheme has uniformly bounded moments. We recall now the main result of Dereich et al. (2012) that gives a strong error of order 1/2.

Theorem 1. *Let $x > 0$, $2a > \sigma^2$ and $T > 0$. Then, for all $p \in [1, \frac{2a}{\sigma^2})$, there is a constant $K_p > 0$ such that for any $n \geq \frac{T}{2}\max(-k, 0)$,*

$$\left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{X}_t - X_t|^p\right]\right)^{1/p} \leq K_p\sqrt{\frac{T}{n}}.$$

Let us remark that, in contrary to Dereich et al. (2012), we here do not consider a linear interpolation between t_k and t_{k+1} for $\hat{X}_t$, but use a different time-continuous extension. This removes the logarithm term of Theorem 1.1 in Dereich et al. (2012).

The strong error of $\hat{X}$ is studied numerically in Alfonsi (2005, Fig. 2). This numerical study shows that the strong convergence rate depends on the parameters σ^2 and a. When σ^2/a is small enough, a strong error of order 1 is observed. The scope in this paper is to prove the following result.

Theorem 2. *Let $x > 0$, $a > \sigma^2$ and $T > 0$. Then, for all $p \in [1, \frac{4a}{3\sigma^2})$, there is a constant $K_p > 0$ such that for any $n \geq \frac{T}{2}\max(-k, 0)$,*

$$\left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{X}_t - X_t|^p\right]\right)^{1/p} \leq K_p\frac{T}{n}.$$

Thus, we get a strong error of order 1 under more restrictive conditions on σ^2/a. Both theorems are complementary and are compatible with the numerical study of Alfonsi (2005), which indicates that the strong error order downgrades as long as σ^2/a increases.

The paper is structured as follows. We first prove that $\left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^p\right]\right)^{1/p} \leq K_p\frac{T}{n}$ under a general framework for Y and $\hat{Y}$ that extends (2) and (3). Then, we deduce Theorem 2 from this result. Also, we construct an analogous drift implicit scheme for other one-dimensional diffusions, and get a strong error of order one under suitable assumptions on the coefficients. This scheme has the advantage to be naturally defined in diffusion domains as $(0, +\infty)$ for the CIR case.

1. A general framework for Y and $\hat{Y}$

Let $c \in [-\infty, +\infty)$, $I = (c, +\infty)$, and $d \in I$ an arbitrary point in the interval. We consider in this section the following SDE defined on $I = (c, +\infty)$:

$$dY_t = f(Y_t)dt + \gamma dW_t, \quad t \geq 0, \qquad Y_0 = y \in I, \tag{4}$$

with $\gamma > 0$. We make the following monotonicity assumption on the drift coefficient f:

$$f : I \to \mathbb{R} \text{ is } \mathcal{C}^2, \quad \text{such that } \exists \kappa \in \mathbb{R}, \forall y, y' \in I, y \leq y', f(y') - f(y) \leq \kappa(y' - y). \tag{5}$$

Besides, we assume that

$$v(x) = \int_d^x \int_d^y \exp\left(-\frac{2}{\gamma^2}\int_z^y f(\xi)d\xi\right) dzdy \quad \text{satisfies} \lim_{x\to c+} v(x) = +\infty. \tag{6}$$

The Feller's test (see e.g. Theorem 5.29 p. 348 in Karatzas and Shreve (1991)) ensures that Y never reaches c nor $+\infty$ by (5), and the SDE (4) admits a unique strong solution on I. More precisely, assumption (5) gives $f(\xi) \leq \kappa\xi + \mu$ for $\xi \geq d$, with $\mu = f(d) - \kappa d$. It is well-known that such a sublinear growth enables to get uniform upper bounds for the moments $\mathbb{E}((Y_t^+)^p)$. This prevents in particular from explosion, and we can check by simple calculations that $\lim_{x\to+\infty} v(x) = +\infty$.

Let us now define the drift implicit scheme. Let us first observe that for $h > 0$ such that $\kappa h < 1$, $y \mapsto y - hf(y)$ is a strictly increasing bijection from I to $\mathbb{R}$. Indeed, it is continuous and we have from (4):

$$y \leq y', \quad y' - y - h(f(y') - f(y)) \geq (1 - \kappa h)(y' - y).$$

In particular, this inequality gives $\lim_{y\to+\infty} y - hf(y) = +\infty$. When $c = -\infty$, we similarly get $\lim_{y\to-\infty} y - hf(y) = -\infty$, which shows the claim in this case. For $c > -\infty$, we first remark that $\lim_{c+} f$ exists from (5) since $y \mapsto f(y) - \kappa y$ is nonincreasing. This limit is necessarily equal to $+\infty$ from (6), which gives $\lim_{y\to c+} y - hf(y) = -\infty$ and the surjectivity.

Thus, for n such that $\kappa T/n < 1$, the following drift implicit Euler scheme is well defined

$$\hat{Y}_0 = y, \qquad \hat{Y}_t = \hat{Y}_{t_k} + f(\hat{Y}_t)(t - t_k) + \gamma(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}], 0 \le k \le n-1, \tag{7}$$

and satisfies $\hat{Y}_t \in I$, for any $t \in [0, T]$. From a computational point of view, let us remark here that in cases where $\hat{Y}_{t_{k+1}}$ cannot be solved explicitly like in the CIR case, $\hat{Y}_{t_{k+1}}$ can still be quickly computed from $\hat{Y}_{t_k}$ and $W_{t_{k+1}} - W_{t_k}$ thanks to the monotonicity of $y \mapsto y - (T/n)f(y)$ by using for example a dichotomic search or Newton's method.

The drift implicit Euler scheme (also known as backward Euler scheme) has been studied by Higham et al. (2002) for SDEs on $\mathbb{R}^d$ with a Lipschitz condition on the diffusion coefficient, a monotonicity condition that extends (5) and a polynomial Lipschitz-type condition on the drift coefficient. They show a strong error of order 1/2 in this general setting.

Proposition 3. *Let $p \ge 1$ and $n > 2\kappa T$. Let us assume that*

$$\mathbb{E}\left[\left(\int_0^T \left|f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right| du\right)^p\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\left(\int_0^T (f'(Y_u))^2 du\right)^{p/2}\right] < \infty. \tag{8}$$

Then, there is a constant $K_p > 0$ such that:

$$\left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^p\right]\right)^{1/p} \le K_p \frac{T}{n}.$$

Before proving this result, let us recall that the same result holds for the usual (drift explicit) Euler–Maruyama scheme when $I = \mathbb{R}$ (i.e. $c = -\infty$), when f is $\mathcal{C}^2$ with bounded derivatives. Said differently, the Euler–Maruyama scheme ($\bar{Y}_{t_{k+1}} = \bar{Y}_{t_k} + f(\bar{Y}_{t_k})T/n + \gamma(W_{t_{k+1}} - W_{t_k})$) coincides with the Milstein scheme when the diffusion coefficient is constant, and its order of strong error is thus equal to one. The main advantage of the drift implicit scheme is that it is well defined when $c > -\infty$ while the Euler–Maruyama is not, since the Brownian increment may lead outside I.

Proof. We may assume without loss of generality that $\kappa \ge 0$. For $t \in [0, T]$, we set $e_t = \hat{Y}_t - Y_t$. From (5), there is $\beta_t \le \kappa$, such that $f(\hat{Y}_t) - f(Y_t) = \beta_t e_t$. For $0 \le k \le n-1$, we have

$$e_{t_{k+1}} = e_{t_k} + [f(\hat{Y}_{t_{k+1}}) - f(Y_{t_{k+1}})]\frac{T}{n} + \int_{t_k}^{t_{k+1}} (f(Y_{t_{k+1}}) - f(Y_s))ds,$$

and then, by using Itô's formula and integration by parts:

$$\left(1 - \beta_{t_{k+1}}\frac{T}{n}\right) e_{t_{k+1}} = e_{t_k} + \int_{t_k}^{t_{k+1}} (u - t_k)\left[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right] du + \gamma \int_{t_k}^{t_{k+1}} (u - t_k)f'(Y_u)dW_u. \tag{9}$$

For $u \in [0, T]$, we denote by $\eta(u)$ the integer such that $t_{\eta(u)} \le u < t_{\eta(u)+1}$. We set $\Pi_0 = 1$, $\Pi_k = \prod_{l=1}^k (1 - \beta_{t_l}\frac{T}{n})$, $\tilde{e}_k = \Pi_k e_{t_k}$, $\tilde{\Pi}_k = \Pi_k/(1 - \kappa T/n)^k$ and

$$M_t = \int_0^t (1 - \kappa T/n)^{\eta(u)}(u - t_{\eta(u)})\gamma f'(Y_u)dW_u.$$

Let us remark that $\Pi_k > 0$, $\tilde{\Pi}_k \ge 1$ and $\tilde{\Pi}_k$ is nondecreasing in k since $\beta_t \le \kappa$. By multiplying Eq. (9) by Π_k, we get

$$\tilde{e}_{k+1} = \tilde{e}_k + \Pi_k\left(\int_{t_k}^{t_{k+1}} (u - t_k)\left[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right] du + \int_{t_k}^{t_{k+1}} (u - t_k)\gamma f'(Y_u)dW_u\right).$$

Then, we obtain $\tilde{e}_k = \int_0^{t_k} \Pi_{\eta(u)}(u - t_{\eta(u)})[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)]du + \sum_{l=0}^{k-1} \tilde{\Pi}_l(M_{t_{l+1}} - M_{t_l})$ by summing over k and finally get

$$e_{t_k} = \int_0^{t_k} \frac{\Pi_{\eta(u)}}{\Pi_k}(u - t_{\eta(u)})\left[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right] du + \frac{1}{\Pi_k}\sum_{l=0}^{k-1} \tilde{\Pi}_l(M_{t_{l+1}} - M_{t_l}). \tag{10}$$

Since $\frac{1}{1-x} \le \exp(2x)$ for $x \in [0, 1/2]$, we have

$$0 \le l \le k \le n, \quad 0 < \frac{\Pi_l}{\Pi_k} = \frac{1}{(1 - \kappa T/n)^{k-l}}\frac{\tilde{\Pi}_l}{\tilde{\Pi}_k} \le \exp\left(2(k-l)\kappa\frac{T}{n}\right) \le \exp(2\kappa T).$$

On the other hand, a summation by parts (Abel's lemma) gives $\sum_{l=0}^{k-1} \tilde{\Pi}_l(M_{t_{l+1}} - M_{t_l}) = \tilde{\Pi}_{k-1}M_{t_k} + \sum_{l=1}^{k-1}(\tilde{\Pi}_{l-1} - \tilde{\Pi}_l)M_{t_l}$ and thus

$$\left|\sum_{l=0}^{k-1} \tilde{\Pi}_l(M_{t_{l+1}} - M_{t_l})\right| \le \tilde{\Pi}_{k-1}|M_{t_k}| + \sum_{l=1}^{k-1}(\tilde{\Pi}_l - \tilde{\Pi}_{l-1})|M_{t_l}| \le 2\tilde{\Pi}_k \max_{1\le l\le k} |M_{t_l}|,$$

since $\tilde{\Pi}_k$ is nondecreasing. From (10) and $\frac{\tilde{\Pi}_k}{\Pi_k} = \frac{1}{(1-\kappa T/n)^k} \le \exp(2\kappa T)$, we get

$$|e_{t_k}| \le \exp(2\kappa T)\left(\frac{T}{n}\int_0^{t_k}\left|f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right| du + 2\max_{0\le l\le k}|M_{t_l}|\right).$$

Since the right hand side is nondecreasing with respect to k, we can replace the left hand side by $\max_{0\le l\le k}|e_{t_l}|$. The Burkholder–Davis–Gundy inequality gives that

$$\mathbb{E}\left[\max_{0\le l\le n}|M_{t_l}|^p\right] \le C_p\gamma^p(T/n)^p\mathbb{E}\left[\left(\int_0^T (f'(Y_u))^2du\right)^{p/2}\right],$$

since $0 \le (1-\kappa T/n)^{\eta(u)} \le 1$. Thus, there is a positive constant K depending on κ, T and p such that:

$$\mathbb{E}\left[\max_{0\le l\le n}|e_{t_l}|^p\right]$$
$$\le K\left(\frac{T}{n}\right)^p\left(\mathbb{E}\left[\left(\int_0^T\left|f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right|du\right)^p\right] + \gamma^p\mathbb{E}\left[\left(\int_0^T (f'(Y_u))^2du\right)^{p/2}\right]\right). \quad (11)$$

It remains to show the analogous upper bound for $\mathbb{E}[\max_{t\in[0,T]}|e_t|^p]$. Similarly to (9), we have for $t \in [t_k, t_{k+1}]$:

$$(1-\beta_t(t-t_k))\,e_t = e_{t_k} + \int_{t_k}^t (u-t_k)\left[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right]du + \gamma\int_{t_k}^t (u-t_k)f'(Y_u)dW_u.$$

Since $(1-\beta_t(t-t_k)) \ge 1/2$, we get:

$$\max_{t\in[t_k,t_{k+1}]}|e_t| \le 2\left(|e_{t_k}| + \frac{T}{n}\int_{t_k}^{t_{k+1}}\left|f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right|du + \gamma\max_{t\in[t_k,t_{k+1}]}\left|\int_{t_k}^t (u-t_k)f'(Y_u)dW_u\right|\right),$$

and thus

$$\max_{t\in[0,T]}|e_t|^p \le 2^p3^{p-1}\left(\max_{0\le k\le n}|e_{t_k}|^p + \left(\frac{T}{n}\right)^p\left(\int_0^T\left|f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)\right|du\right)^p\right.$$
$$\left. + \gamma^p\max_{0\le s\le t\le T}\left|\int_s^t (u-t_{\eta(u)})f'(Y_u)dW_u\right|^p\right).$$

Since $\left|\int_s^t (u-t_{\eta(u)})f'(Y_u)dW_u\right|^p \le 2^p\left(\left|\int_0^t (u-t_{\eta(u)})f'(Y_u)dW_u\right|^p + \left|\int_0^s (u-t_{\eta(u)})f'(Y_u)dW_u\right|^p\right)$, we conclude by using once again Burkholder–Davis–Gundy inequality, (8) and (11). □

2. Application to the CIR process

For the CIR case, we have $c = 0$ (i.e. $I = (0, +\infty)$), $f(y) = \frac{a-\sigma^2/4}{2y} - \frac{k}{2}y$ and $\gamma = \sigma/2$. When $2a \ge \sigma^2$, we can check that both (5) and (6) are satisfied. By Jensen inequality, (8) holds if we have

$$\int_0^T \mathbb{E}[|f'(Y_u)f(Y_u)|^p + |f''(Y_u)|^p + |f'(Y_u)|^{2\vee p}]du < \infty. \quad (12)$$

The moments of the CIR process are uniformly bounded on $[0, T]$ under the following condition (see Dereich et al. (2012, Eq. (7))):

$$\sup_{t\in[0,T]}\mathbb{E}[X_t^q] < \infty \quad \text{for } q > -\frac{2a}{\sigma^2}. \quad (13)$$

Condition (12) will hold as soon as $\sup_{t\in[0,T]}\mathbb{E}[Y_t^{-(4\vee 3p)}] = \sup_{t\in[0,T]}\mathbb{E}[X_t^{-(2\vee\frac{3}{2}p)}] < \infty$. This is satisfied when $\sigma^2 < a$ and $p < \frac{4}{3}\frac{a}{\sigma^2}$, and we have $\left(\mathbb{E}\left[\max_{t\in[0,T]}|\hat{Y}_t - Y_t|^p\right]\right)^{1/p} \le K_p\frac{T}{n}$.

From now on, we assume that $\sigma^2 < a$ and consider $1 \le p < \frac{4}{3}\frac{a}{\sigma^2}$. Let $\varepsilon > 0$ such that $p(1+\varepsilon) < \frac{4}{3}\frac{a}{\sigma^2}$. Since $\hat{X}_t - X_t = (\hat{Y}_t - Y_t)(\hat{Y}_t + Y_t)$, we have by Hölder's inequality:

$$\mathbb{E}\left[\max_{t\in[0,T]} |\hat{X}_t - X_t|^p\right]^{\frac{1}{p}} \le \mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t + Y_t|^{p\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{p(1+\varepsilon)}}.$$

The moment boundedness of $\hat{Y}$ is checked in Alfonsi (2005) and Dereich et al. (2012), and the second expectation is thus finite. Proposition 3 now gives Theorem 2.

3. Application to $dX_t = (a - kX_t)dt + \sigma X_t^\alpha dW_t$, with $1/2 < \alpha < 1$

We consider this SDE starting from $X_0 = x > 0$ with parameters $a > 0$, $k \in \mathbb{R}$ and $\sigma > 0$. This SDE is known to have a unique strong positive solution X, which can be checked easily by Feller's test for explosions. We set

$$Y_t = X_t^{1-\alpha}.$$

It is defined on $I = (0, +\infty)$ and satisfies (4) with

$$f(y) = (1-\alpha)\left(ay^{-\frac{\alpha}{1-\alpha}} - ky - \alpha\frac{\sigma^2}{2}y^{-1}\right) \quad \text{and} \quad \gamma = \sigma(1-\alpha).$$

Since $a > 0$ and $\frac{\alpha}{1-\alpha} > 1$, f is decreasing on $(0, \varepsilon)$, for $\varepsilon > 0$ small enough. It is also clearly globally Lipschitz on $[\varepsilon, +\infty)$, and (5) is thus satisfied. Also, we check easily that (6) holds. The drift implicit scheme $(\hat{Y}_t, t \in [0, T])$ given by (7) is thus well defined for n large enough and we set:

$$\hat{X}_t = (\hat{Y}_t)^{\frac{1}{1-\alpha}}.$$

To apply Proposition 3, it is enough to check that (12) holds. To do so, we recall a well known result which can be found in Berkaoui et al. (2008).

Lemma 4. *We have:* $\forall q \in \mathbb{R}$, $\sup_{t\in[0,T]} \mathbb{E}[X_t^q] < \infty$.

We can then apply Proposition 3 and get, for any $p \ge 1$ and n large enough, the existence of a constant $K_p > 0$ such that $\left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^p\right]\right)^{1/p} \le K_p \frac{T}{n}$. In particular, we get $\mathbb{E}[\max_{t\in[0,T]} \hat{Y}_t^p] < \infty$. We have $\hat{X}_t = (\hat{Y}_t)^{\frac{1}{1-\alpha}}$ and

$$|\hat{y}^{\frac{1}{1-\alpha}} - y^{\frac{1}{1-\alpha}}| = \frac{1}{1-\alpha}\left|\int_y^{\hat{y}} z^{\frac{\alpha}{1-\alpha}} dz\right| \le \frac{1}{1-\alpha}|\hat{y} - y|(\hat{y} \vee y)^{\frac{\alpha}{1-\alpha}}, \quad \hat{y}, y > 0.$$

The Cauchy–Schwarz inequality leads then to

$$\mathbb{E}\left[\max_{t\in[0,T]} |\hat{X}_t - X_t|^p\right]^{\frac{1}{p}} \le \frac{1}{1-\alpha}\mathbb{E}\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^{2p}\right]^{\frac{1}{2p}} \mathbb{E}\left[\max_{t\in[0,T]} (\hat{Y}_t \vee Y_t)^{\frac{2p\alpha}{1-\alpha}}\right]^{\frac{1}{2p}} \le \tilde{K}_p \frac{T}{n}.$$

4. Strong convergence towards X in a general framework

Let us consider a one-dimensional SDE with locally Lipschitz coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x.$$

We will consider the Lamperti transformation of this SDE. We assume that there exist $0 < \underline{\sigma} < \overline{\sigma}$ such that $\underline{\sigma} \le \sigma(x) \le \overline{\sigma}$, so that

$$\varphi(x) = \int_0^x \frac{1}{\sigma(z)} dz \quad \text{is bijective on } \mathbb{R}.$$

Lipschitz and such that φ^{-1} is Lipschitz. We also assume that b has a sublinear growth, i.e. $\exists C > 0$, $\forall x \in \mathbb{R}$, $|b(x)| \le C(1 + |x|)$, so that X has uniformly bounded moments. Besides, we assume that $\sigma \in \mathcal{C}^1$ and that $f = \left(\frac{b}{\sigma} - \frac{\sigma'}{2}\right) \circ \varphi^{-1}$ satisfies (5), (6) with $c = -\infty$, and:

$$\exists C > 0, q > 0, \forall y \in \mathbb{R}, \quad |f'(y)| + |f''(y)| \le C(1 + |y|^q). \tag{14}$$

Then $Y_t = \varphi(X_t)$ satisfies $dY_t = f(Y_t)dt + dW_t$. Since $|\varphi(x)| \leq |x|/\underline{\sigma}$, Y has uniformly bounded moments on $[0, T]$. The condition (8) is thus satisfied and the conclusion of Proposition 3 holds. Then, defining $\hat{Y}$ by (7) and setting $\hat{X}_t = \varphi^{-1}(\hat{Y}_t)$ for $t \in [0, T]$, we get that:

$$\exists K_p > 0, \quad \left(\mathbb{E}\left[\max_{t\in[0,T]} |\hat{X}_t - X_t|^p\right]\right)^{1/p} \leq K_p \frac{T}{n}. \tag{15}$$

Let us mention that the same result holds under suitable conditions on f for the scheme $\bar{X}_t = \varphi^{-1}(\bar{Y}_t)$, where $\bar{Y}$ denotes the Euler–Maruyama scheme $d\bar{Y}_t = f(\bar{Y}_{t_{\eta(t)}})dt + dW_t$. The weak convergence of this scheme has been studied by Detemple et al. (2006).

Remark 5. Let $\gamma > 0$, $\varphi_\gamma(x) = \gamma\varphi(x)$ and $f_\gamma(y) = \gamma f(y/\gamma)$. Then, $Y'_t = \varphi_\gamma(X_t)$ solves $dY'_t = f_\gamma(Y'_t)dt + \gamma dW_t$. The associated drift implicit scheme

$$\hat{Y}'_0 = \varphi_\gamma(X_0), \qquad \hat{Y}'_t = \hat{Y}'_{t_k} + f_\gamma(\hat{Y}'_t)(t - t_k) + \gamma(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}], 0 \leq k \leq n-1,$$

clearly satisfies $\hat{Y}'_t = \gamma\hat{Y}_t$. Thus, $\hat{X}_t = \varphi_\gamma^{-1}(\hat{Y}'_t)$: the scheme $\hat{X}$ is unchanged when the transformation between X and Y is multiplied by a positive constant.

The main goal of this section was to give the general construction of the scheme $\hat{X}$ and to show that it still has a strong error of order one under some assumptions on the SDE coefficients. However, as we have seen for the CIR case, the nice and specific feature of the scheme $\hat{X}$ is to be well defined on the diffusion domain. For general diffusions on $\mathbb{R}$, the scheme $\hat{X}$ has no longer this advantage, and should be compared to other schemes with strong error one, such as the Milstein scheme. Unless cases where φ can be calculated explicitly, the Milstein scheme seems generally easier to implement in practice and should be then preferred.

Let us now compare in detail the theoretical assumptions required by both schemes to get a strong error of order one. We will see that none of them implies the other one. We know from Milstein (1995) and Jourdain and Sbai (in press) that the Milstein scheme satisfies the same estimate as (15) for any $p \geq 1$ when b and σ are $\mathcal{C}^2$ with bounded derivatives. The assumptions for $\hat{X}$ do not require this. Instead, σ has to be bounded and uniformly elliptic, b should have a sublinear growth and f should satisfy (5) and (6). Besides, to satisfy condition (14), b and σ have a priori to be respectively $\mathcal{C}^2$ and $\mathcal{C}^3$ to ensure $f \in \mathcal{C}^2$. In this case, we have

$$f' = \left(b' - \frac{b\sigma'}{\sigma} - \frac{\sigma\sigma''}{2}\right) \circ \varphi^{-1} \quad \text{and} \quad f'' = \left(\sigma b'' - \frac{b'\sigma' + b\sigma'' - b(\sigma')^2}{\sigma} - \sigma\frac{\sigma'\sigma'' + \sigma\sigma'''}{2}\right) \circ \varphi^{-1}.$$

Since $\underline{\sigma} \leq \sigma \leq \overline{\sigma}$, $|\varphi^{-1}(x)| \leq \overline{\sigma}|x|$ and b has a sublinear growth, condition (14) is satisfied as soon as b', b'', σ', σ'' and σ''' have a polynomial growth. However, condition (14) is more tractable if we set $\psi = \frac{b}{\sigma} - \frac{\sigma'}{2}$. Then, we have $f = \psi \circ \varphi^{-1}$, $f' = (\psi'\sigma) \circ \varphi^{-1}$ and $f'' = (\psi''\sigma^2 + \psi'\sigma'\sigma) \circ \varphi^{-1}$. Since σ is bounded, condition (14) holds as soon as $\sigma \in \mathcal{C}^1$ and $\psi \in \mathcal{C}^2$ are such that σ', ψ' and ψ'' have a polynomial growth.

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