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Convergence and Stability of an Explicit Method for Autonomous Time-Changed Stochastic Differential Equations with Super-Linear Coefficients

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Abstract. In this paper, numerical methods for the time-changed stochastic differential equations of the form

$$dY(t)=a(Y(t))dt+b(Y(t))dE(t)+\sigma(Y(t))dB(E(t))$$

are investigated, where all the coefficients $a(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ are allowed to contain some super-linearly growing terms. An explicit method is proposed by using the idea of truncating terms that grow too fast. Strong convergence in the finite time of the proposed method is proved and the convergence rate is obtained. The proposed method is also proved to be able to reproduce the asymptotic stability of the underlying equation in the almost sure sense. Simulations are provided to demonstrate the theoretical results.

AMS subject classifications: 60H10, 65C30, 60J60

Key words: Time-changed stochastic differential equations, explicit method, super-linear coefficients, strong convergence, asymptotic stability.

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1 Introduction

Time-changed stochastic processes have been attracting increasing attentions in recent years, since they are powerful tools to describe sub-diffusion phenomena and they have the strong connection with fractional partial differential equations [3,25,28,36].

One of the blooming topic in this area is the study on time-changed stochastic differential equations (SDEs). Compared with classical SDEs, time-changed SDEs could be

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used to describe the relatively slower diffusion of particles that may be stuck for a random period of time. In addition, time-changed SDEs have many different probabilistic properties from the classical SDEs [20]. So the analyses of both the underlying solutions and numerical solutions to time-changed SDEs need some new ideas and techniques. Many existing works have been devoted to the investigations on different properties of the underlying solutions to time-changed SDEs. Kobayashi studied the existence and uniqueness of the solution to a large class of time-changed SDEs and proposed the corresponding Itô formula in [20]. Wu, in [37] and [38], investigated different types of stability of time-changed SDEs driven by the time-changed Brownian motion when the coefficients of the underlying equations satisfy the global Lipschitz condition. Liu studied the polynomial stability in the moment sense of time-changed SDEs driven by the time-changed Brownian when some super-linear terms appear in the coefficients in [23]. Zhu et al. discussed the almost sure stability of a class of time-changed SDEs in [42]. Nane and Ni developed the Itô formula for time-changed Lévy processes and studied the stability of time-changed SDEs driven by the time-changed Lévy process in [30] and [31]. Zhang and Yuan proved the Razumikhin-type theorem for hybrid time-changed stochastic function differential equations in [40]. We refer the readers to the monograph [36] by Umarov et al. for the detailed and systematic introduction to time-changed SDEs and their related topics.

Compared with the fruitful results on the underlying equations, the amount of works on numerical methods for time-changed SDEs is relatively less. However, it should be noted that the numerical methods are important for the application of time-changed SDEs, since explicit forms of solutions to most of time-changed SDEs are rarely obtained.

To our best knowledge, the first work that directly discretizes the underlying equation to obtain the numerical approximation is [17], where the Euler-Maruyama (EM) method was proposed for time-changed SDEs of the form

$$dY(t)=b(E(t),Y(t))dE(t)+\sigma(E(t),Y(t))dB(E(t)). \tag{1.1}$$

Under the global Lipschitz condition for the state variables in the coefficients $a(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$, the convergences in both the strong and weak senses were proved in that paper. Afterwards, the semi-implicit EM method [7] and the truncated EM method [24] method were proposed for (1.1), when the global Lipschitz condition is no longer imposed. All the three papers [7,17,24] employed the duality principle that was discovered by Kobayashi in [20]. Briefly speaking, to construct numerical methods for (1.1), one could first construct numerical methods for the classical SDEs driven by the classical Brownian motion of the form

$$dy(t)=b(t,y(t))dt+\sigma(t,y(t))dB(t).$$

Then, due to $Y(t)=y(E(t))$, together with some proper discretization of $E(t)$ one could obtain numerical methods for (1.1).

However, a slight change in the structure of (1.1) could make the duality principle inapplicable. For example, time-changed SDEs of the form

$$dY(t)=b(t,Y(t))dE(t)+\sigma(t,Y(t))dB(E(t)) \tag{1.2}$$

has no dual classical SDEs. Therefore, some other approaches were searched. In [18], Jin and Kobayashi proposed the EM method for (1.2), proved the strong convergence and obtained the convergence rate, when the spatial variables in the coefficients satisfy the global Lipschitz condition. Li, Liu and Tang investigated the strong convergence of the truncated EM method for (1.2), when the spatial variables are allowed to grow super-linearly [21]. More recently, the same group of authors in [19] studied the more general type of time-changed SDEs of the form

$$dY(t)=a(E(t),Y(t))dt+b(E(t),Y(t))dE(t)+\sigma(E(t),Y(t))dB(E(t)), \tag{1.3}$$

for which the duality principle is not applicable either. Both the EM method and the Milstein method were proposed and studied in that work. This type of time-changed SDEs could be used to describe the sub-diffusion, in which for a random time period only the non-random drift term affects the state of the system. For more detailed discussions on applications of this type of time-changed SDEs, we refer the readers to [2,20] and the monograph [36].

It should be mentioned that those results obtained in [18] and [19] are under the global Lipschitz condition on the state variables in the coefficients, which excludes time-changed SDEs with super-linear terms like

$$dY(t)=(-Y(t)-Y^3(t))dt+(Y(t)-2Y^3(t))dE(t)+Y^2(t)d(B(E(t))).$$

Meanwhile, classical SDEs driven by the classical Brownian motion with super-linear coefficients have been receiving lots of attention in recent years. Due to the divergence of the classical explicit EM method [12], different approaches were developed to handle the super-linearly growing terms. One natural alternative is the implicit method [8,14,29]. As mentioned by Higham in [10], explicit methods are sometimes still preferred when a large number of samples are required to simulate. Therefore, explicit methods with different modified coefficients of the underlying equations have been developed in recent years [4,6,13,15,27,41,43]. We just mention some of the papers here and refer the readers to the references therein for more works.

On the contrary, there are few works on numerical methods for time-changed SDEs with super-linear coefficient when the duality principle is not applicable. To fill up this gap, by using the truncating idea that was initialised in [27], we are going to develop an explicit numerical method for time-changed SDEs (1.3) when the global Lipschitz condition is not satisfied by the coefficients for the state variables. To keep the notations simple, we will focus on the autonomous version of (1.3), i.e.,

$$dY(t)=a(Y(t))dt+b(Y(t))dE(t)+\sigma(Y(t))dB(E(t)), \tag{1.4}$$

and study the strong convergence of the developed method.

Apart from the finite time convergence, the ability of reproducing the stability of the underlying equations is also a key property of good numerical methods [5,11,22,35]. For underlying time-changed SDEs, as mentioned above, the stabilities in different senses have been investigated by many scholars. But there are very few works on stabilities of numerical methods. Therefore, in addition to the study of the finite time convergence, we will also discuss the ability of reproducing the stability of the underlying time-changed SDEs for our proposed method in this paper.

The main contributions of this paper are threefold.

- An explicit numerical method is developed for time-changed SDEs (1.4), whose coefficients are allowed to consist of super-linearly growing terms.
- The strong convergence in the finite time of the developed method is proved and the convergence rate is obtained.
- The asymptotic stability in the almost sure sense of the developed method is proved.

It should be mentioned that new difficulties indeed arise when both the super-linear terms and time-changed processes appear together in the equation. Compared with the classical SDEs, we need to handle two stochastic processes, $B(t)$ and $E(t)$, in both theoretical and numerical aspects. In addition, the composition, B(E(t)), requires the Gronwall-type inequality with a stochastic driver when we estimate the moments. Although the time-changed processes could be understood as sub-diffusion process, without some truncating idea to control the fast growing coefficients the moments of the numerical solutions may still blow up. So, by taking all those facts into considerations, constructing numerical methods for this type of time-changed SDEs and conducting the numerical analysis are non-trivial.

Before the end of this section, it should also be mentioned that the weak convergence is definitely important for numerical methods for time-changed SDEs. But due to the limited length of the paper, we focus on the strong convergence in this work and may investigate the weak convergence in future.

The rest of the paper is constructed as follows. The necessary mathematical preliminaries and the development of the explicit method are presented in Section 2. The strong convergence in the finite time and the convergence rate of the developed method are discussed in Section 3. The asymptotic stability of the developed method is investigated in Section 4. Numerical simulations that demonstrate those theoretical results obtained in Sections 3 and 4 are presented in Section 5.

2 Mathematical preliminaries

Throughout this paper, we let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B(t)=(B_1(t),B_2(t),\cdots,B_m(t))^T$ be an

m-dimensional $\mathcal{F}_t$-adapted standard Brownian motion. Let $D(t)$ be a one-dimensional strictly increasing Lévy process in $(\Omega,\mathcal{F},\mathbb{P})$ with Laplace transform

$$\mathbb{E}e^{-rD(t)}=e^{-t\phi(r)}, \quad r>0, \quad t\geq 0,$$

where the Laplace exponent $\phi:(0,\infty)\to(0,\infty)$ is a Bernstein function with $\phi(0+):=\lim_{r\downarrow 0}\phi(r)=0$ [34]. Set $D(0)=0$, almost surely. We refer the readers to monographs [1,32, 33] for more detailed introduction to such a process.

Now, we define the inverse sub-ordinator $E(t)$ by the inverse of $D(t)$ in the way that

$$E(t):=\inf\{u\geq 0;\, D(u)>t\}, \quad t\geq 0.$$

It is clear to see that $E(t)$ is continuous and non-decreasing almost surely. Then, $B(E(t))$ is called the time-changed Brownian motion, which is regarded as a sub-diffusion process [25,36].

In the rest of this paper, we assume that $B(t)$ and $D(t)$ are independent. Denote the expectations functioning on $B(t)$ and $E(t)$ by $\mathbb{E}_B$ and $\mathbb{E}_D$, respectively. In addition, the probability expectation of $\mathbb{P}$ is defined by $\mathbb{E}$, which has the property $\mathbb{E}(\cdot)=\mathbb{E}_B\mathbb{E}_D(\cdot)=\mathbb{E}_D\mathbb{E}_B(\cdot)$.

Denote the Euclidean norm of $x\in\mathbb{R}^d$ by $|x|$ and the x transpose by x^T. The trace norm for a matrix is denoted by

$$|A|:=\sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}.$$

Define

$$a\vee b=\max(a,b) \quad \text{and} \quad a\wedge b=\min(a,b).$$

In this paper, we consider the d-dimension time-changed SDEs of the form

$$dY(t)=a(Y(t))dt+b(Y(t))dE(t)+\sigma(Y(t))dB(E(t)), \quad Y(0)=Y_0, \quad t\geq 0, \tag{2.1}$$

with $\mathbb{E}|Y_0|^q<\infty$ for all $q>0$. Here, $a:\mathbb{R}^d\to\mathbb{R}^d$, $b:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times m}$.

The following assumptions are imposed on the coefficients of (2.1).

Assumption 2.1. Assume that there exist positive constants γ and L such that

$$|a(x)-a(y)|\vee|b(x)-b(y)|\vee|\sigma(x)-\sigma(y)|\leq L(1+|x|^\gamma+|y|^\gamma)|x-y|$$

for any $x,y\in\mathbb{R}^d$.

It can be observed from Assumption 2.1 that all $x\in\mathbb{R}^d$,

$$|a(x)|\vee|b(x)|\vee|\sigma(x)|\leq H|x|^{\gamma+1}, \quad \forall |x|\geq 1, \tag{2.2}$$

where $H=2L+|a(0)|+|b(0)|+|\sigma(0)|$.

Assumption 2.2. Assume that there exists a positive constant K such that

$$(x-y)^{\mathrm{T}}(a(x)-a(y))\leq K|x-y|^2$$

for any $x,y\in\mathbb{R}^d$.

Assumption 2.3. Assume that there exists a pair of constants $q>2$ and $K_3>0$ such that

$$(x-y)^{\mathrm{T}}(b(x)-b(y))+\frac{5q-1}{2}|\sigma(x)-\sigma(y)|^2\leq K_3|x-y|^2$$

for any $x,y\in\mathbb{R}^d$.

Assumption 2.4. Assume that there exists a positive constant K_1 such that

$$x^{\mathrm{T}}a(x)\leq K_1|x|^2$$

for any $x\in\mathbb{R}^d$.

Assumption 2.5. Assume that there exists a pair of constants $r>2$ and $K_2>0$ such that

$$x^{\mathrm{T}}b(x)+\frac{5r-1}{2}|\sigma(x)|^2\leq K_2(1+|x|^2)$$

for $x\in\mathbb{R}^d$.

Now we are going to develop the explicit numerical method.

To define our numerical method, we choose a strictly increasing continuous function $\mu:\mathbb{R}_+\to\mathbb{R}_+$ such that $\mu(u)\to\infty$ as $u\to\infty$ and

$$\sup_{|x|\leq u}(|a(x)|\vee|b(x)|\vee|\sigma(x)|)\leq\mu(u),\quad \forall u\geq 1.$$

Denote by μ^{-1} the inverse function of μ. It is clear that μ^{-1} is a strictly increasing continuous function from $[\mu(1),\infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h}\geq 1\wedge\mu(1)$ and a strictly decreasing function $h:(0,1]\to[\mu(1),\infty)$ such that

$$\lim_{\Delta\to 0}h(\Delta)=\infty,\quad \Delta^{1/4}h(\Delta)\leq\hat{h},\quad \forall\Delta\in(0,1]. \tag{2.3}$$

For a given step size $\Delta\in(0,1]$, define the truncated mapping $\pi_\Delta:\mathbb{R}^d\to\{x\in\mathbb{R}^d:|x|\leq\mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x)=\Big(|x|\wedge\mu^{-1}(h(\Delta))\Big)\frac{x}{|x|},$$

where we set $x/|x|=0$ when $x=0$. Define the truncating functions by

$$a_\Delta(x)=a(\pi_\Delta(x)),\quad b_\Delta(x)=b(\pi_\Delta(x)),\quad \sigma_\Delta(x)=\sigma(\pi_\Delta(x)),$$

for $x\in\mathbb{R}^d$. It is easy to see that for any $t\in[0,T]$ and all $x\in\mathbb{R}^d$,

$$|a_\Delta(x)|\vee|b_\Delta(x)|\vee|\sigma_\Delta(x)|\le\mu(\mu^{-1}(h(\Delta)))=h(\Delta). \tag{2.4}$$

The discrete-time version of the numerical solutions $y_{\Delta,i}\approx Y(t_i)$ for $t_i=i\Delta$ are formed by setting $y_{\Delta,0}=Y(0)$ and computing

$$\begin{aligned}y_{\Delta,i+1}=&y_{\Delta,i}+a_\Delta(y_{\Delta,i})\Delta+b_\Delta(y_{\Delta,i})(E(t_{i+1})-E(t_i))\\&+\sigma_\Delta(y_{\Delta,i})(B(E(t_{i+1}))-B(E(t_i))).\end{aligned} \tag{2.5}$$

There are two versions of the continuous-time numerical solutions. The first one is defined by

$$\bar{y}_\Delta(t)=\sum_{i=0}^{\infty}y_{\Delta,i}\mathbf{1}_{[t_i,t_{i+1})}(t),\quad t\ge0, \tag{2.6}$$

which is a simple step process so its sample paths are not continuous. The other one is defined by

$$y_\Delta(t)=y_\Delta(0)+\int_0^t a_\Delta(\bar{y}_\Delta(s))ds+\int_0^t b_\Delta(\bar{y}_\Delta(s))dE(s)+\int_0^t\sigma_\Delta(\bar{y}_\Delta(s))dB(E(s)), \tag{2.7}$$

for $t\ge0$.

We borrow the following lemma from [28].

Lemma 2.1. *For any $t_i\le t\le t_{i+1}$, there exists a constant k such that*

$$|E(t)-E(t_i)|\le|E(t_{i+1})-E(t_i)|\le k\Delta.$$

The next three lemmas can be proved in the similar way as those in [9].

Lemma 2.2. *Let Assumption* 2.4 *holds. Then, for all $\Delta\in(0,1]$, we have*

$$x^{\mathrm{T}}a_\Delta(x)\le K_1|x|^2.$$

Lemma 2.3. *Let Assumption* 2.5 *holds. Then, for all $\Delta\in(0,1]$, we have*

$$x^{\mathrm{T}}b_\Delta(x)+\frac{5r-1}{2}|\sigma_\Delta(x)|^2\le\hat{K}_2(1+|x|^2),$$

where

$$\hat{K}_2=2K_2\left(1\vee\frac{1}{\mu^{-1}(h(1))}\right).$$

Lemma 2.4. *Let Assumption* 2.1 *holds. Then, for all $\Delta\in(0,1]$, we have*

$$|a_\Delta(x)-a_\Delta(y)|\vee|b_\Delta(x)-b_\Delta(y)|\vee|\sigma_\Delta(x)-\sigma_\Delta(y)|\le L(1+|x|^\gamma+|y|^\gamma)|x-y|,$$

for all $x,y\in\mathbb{R}^d$.

3 Main results: the finite time strong convergence

We start this section by proving that the pth moment of the underlying solution is bounded.

Lemma 3.1. *Let Assumptions* 2.1, 2.4 *and* 2.5 *hold. Then, for any* $p\in[2,r]$

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)|^p\right)<\infty.$$

Proof. Let $\theta_\ell=\inf\{t\ge 0;|Y(t)|>\ell\}$ for $\ell\in\mathbb{N}$. Since the solution $Y(t)$ has continuous paths, $|Y(t)|<\infty$, and hence, $\theta_\ell\uparrow\infty$ as $\ell\to\infty$. Then, it can be seen that

$$\int_0^t \mathbb{E}_B\left(\sup_{0\le s\le r\wedge\theta_\ell}|Y(s)|^p\right)dE(r)<\ell^p E(t)<\infty.$$

Fix $\ell\in\mathbb{N}$, $t\in[0,T]$. By timed-changed Itô formula, we have

$$|Y(s)|^p=|Y(0)|^p+I_1+I_2+M_s, \tag{3.1}$$

where

$$\begin{aligned}
I_1&:=\int_0^s p|Y(r)|^{p-2}Y^{\mathrm{T}}(r)a(Y(r))dr,\\
I_2&:=\int_0^s\left(p|Y(r)|^{p-2}Y^{\mathrm{T}}(r)b(Y(r))+\frac{1}{2}p(p-1)|Y(r)|^{p-2}|\sigma(Y(r))|^2\right)dE(r),\\
M_s&:=\int_0^s p|Y(r)|^{p-1}\sigma(Y(r))dB(E(r)).
\end{aligned}$$

By Assumption 2.4, we have

$$I_1\le\int_0^s pK_1|Y(r)|^{p-2}|Y(r)|^2dr\le pK_1\int_0^s|Y(r)|^p dr. \tag{3.2}$$

Noting that the stochastic integral $(M_t)_{t\ge 0}$ is a local martingale with quadratic variation

$$[M,M]_t=\int_0^t p^2|Y(r)|^{2p-2}|\sigma(Y(r))|^2dE(r).$$

For $0\le r\le t\wedge\theta_\ell$, it can be seen that

$$\begin{aligned}
p^2|Y(r)|^{2p-2}|\sigma(Y(r))|^2=&p^2|Y(r)|^p|Y(r)|^{p-2}|\sigma(Y(r))|^2\\
\le&p^2\left(1+\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\right)|Y(r)|^{p-2}|\sigma(Y(r))|^2.
\end{aligned}$$

By using the inequality $(ab)^{1/2}\le a/\lambda+\lambda b$ for any $a,b>0$ and $\lambda>0$, we can see that for $0\le s\le t\wedge\theta_\ell$

$$
\begin{aligned}
([M,M]_s)^{1/2} &\le p\left((1+\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p)\int_0^s |Y(r)|^{p-2}|\sigma(Y(r))|^2 dE(r)\right)^{\frac{1}{2}}\\
&\le p\left(\frac{(1+\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p)}{2pb_1}+2pb_1\int_0^s |Y(r)|^{p-2}|\sigma(Y(r))|^2 dE(r)\right), \qquad (3.3)
\end{aligned}
$$

where b_1 is the constant appearing in the Burkholder-Davis-Gundy (BDG) inequality (Chapter 1, Theorem 7.3 in [26]). By the above estimates I_1 (3.2) and M_s (3.3), using Assumption 2.5 and the BDG inequality, we have

$$
\begin{aligned}
&1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\\
\le& 1+|Y(0)|^p+pK_1\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s|Y(r)|^p dr\Big]\\
&+\mathbb{E}_B\left(\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s\left(p|Y(r)|^{p-2}Y^{\mathrm{T}}(r)b(Y(r))+\frac{1}{2}p(p-1)|Y(r)|^{p-2}|\sigma(Y(r))|^2\right)dE(r)\right)\\
&+\mathbb{E}_B\left(\int_0^{t\wedge\theta_\ell}2p^2b_1|Y(r)|^{p-2}|\sigma(Y(r))|^2 dE(r)\right)+\frac{1}{2}\left(1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\right)\\
\le& 1+|Y(0)|^p+\frac{1}{2}\left(1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\right)+pK_1\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s|Y(r)|^p dr\Big]\\
&+b\mathbb{E}_B\left(\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s p|Y(r)|^{p-2}\left(Y^{\mathrm{T}}(r)b(Y(r))+\frac{5p-1}{2}|\sigma(Y(r))|^2\right)dE(r)\right)\\
\le& 1+|Y(0)|^p+\frac{1}{2}\left(1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\right)+pK_1\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s|Y(r)|^p dr\Big]\\
&+pbK_2\mathbb{E}_B\left(\sup_{0\le s\le t\wedge\theta_\ell}\int_0^s|Y(r)|^{p-2}(1+|Y(r)|^2)dE(r)\right),
\end{aligned}
$$

where $b=b_1\vee 1$. Note that for any non-negative process $m(r)$ the inequality

$$
\int_0^{t\wedge\theta_\ell} m(r)dE(r)\le\int_0^t m(r\wedge\theta_\ell)dE(r), \qquad (3.4)
$$

holds. Due to $s\le t\wedge\theta_\ell\le T$, we can see that

$$
\begin{aligned}
&1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\\
\le& 1+|Y(0)|^p+\frac{1}{2}\left(1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\right)+pK_1\mathbb{E}_B\left(\sup_{0\le s\le T}\int_0^s|Y(r\wedge\theta_\ell)|^p dr\right)
\end{aligned}
$$

$$
\begin{aligned}
&+pbK_2\mathbb{E}_B\left(\sup_{0\le s\le T}\int_0^s |Y(r\wedge\theta_\ell)|^{p-2}(1+|Y(r\wedge\theta_\ell)|^2)dE(r)\right)\\
\le& 1+|Y(0)|^p+\frac{1}{2}\left(1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\right)+pK_1\int_0^T\left(1+\mathbb{E}_B\Big[\sup_{0\le r\le t\wedge\theta_\ell}|Y(r)|^p\Big]\right)dt\\
&+2pbK_2\int_0^T\left(1+\mathbb{E}_B\Big[\sup_{0\le r\le t\wedge\theta_\ell}|Y(r)|^p\Big]\right)dE(t),
\end{aligned}
$$

which in turn gives

$$
\begin{aligned}
1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\le& 2(1+|Y(0)|^p)+2pK_1\int_0^T\left(1+\mathbb{E}_B\Big[\sup_{0\le r\le t\wedge\theta_\ell}|Y(r)|^p\Big]\right)dt\\
&+4pbK_2\int_0^T\left(1+\mathbb{E}_B\Big[\sup_{0\le r\le t\wedge\theta_\ell}|Y(r)|^p\Big]\right)dE(t).
\end{aligned}
$$

The Gronwall-type inequality (Chapter IX.6a, Lemma 6.3 in [16]) yields

$$
1+\mathbb{E}_B\Big[\sup_{0\le s\le t\wedge\theta_\ell}|Y(s)|^p\Big]\le 2(1+|Y(0)|^p)e^{2pK_1T+4pbK_2E(T)}
$$

for all $t\in[0,T]$. Setting $t=T$, letting $\ell\to\infty$ and using the monotone convergence theorem, we obtain

$$
1+\mathbb{E}_B\Big[\sup_{0\le t\le T}|Y(s)|^p\Big]\le 2(1+|Y(0)|^p)e^{2pK_1T+4pbK_2E(T)}.
$$

Taking $\mathbb{E}_D$ on both sides and using the facts that

$$
\mathbb{E}_D[e^{cT}]<\infty \quad \text{and} \quad \mathbb{E}_D[e^{cE(T)}]<\infty
$$

for any $c>0$ (Lemma 2.3 of [7]) give the desired result. □

The next lemma states that the difference between two versions of the continuous-time numerical methods, (2.6) and (2.7), will tend to 0 as $t\to\infty$ in the moment sense.

Lemma 3.2. *For any $\Delta\in(0,1]$ and any $\bar{p}>2$, we have*

$$
\mathbb{E}_B|y_\Delta(t)-\bar{y}_\Delta(t)|^{\bar{p}}\le C_{\bar{p}}\Delta^{\bar{p}/2}(h(\Delta))^{\bar{p}}, \quad \forall t\ge 0, \tag{3.5}
$$

where $C_{\bar{p}}=3^{\bar{p}-1}(1+k^{\bar{p}})b_{\bar{p}}$. Consequently,

$$
\lim_{\Delta\to 0}\mathbb{E}_B|y_\Delta(t)-\bar{y}_\Delta(t)|^{\bar{p}}=0, \quad \forall t\ge 0. \tag{3.6}
$$

Proof. Fix any $\Delta\in(0,1]$, $\bar{p}>2$ and $t\geq 0$, there is a unique integer $i\geq 0$ such that $t_i\leq t\leq t_{i+1}$. By the time-changed Itô formula and Lemma 2.1, we derive from (2.7) that

$$\begin{aligned}
&\mathbb{E}_B|y_\Delta(t)-\bar{y}_\Delta(t)|^{\bar{p}}=\mathbb{E}_B|y_\Delta(t)-y_{\Delta,i}|^{\bar{p}}\\
\leq&3^{\bar{p}-1}\left(\mathbb{E}_B\left|\int_{t_i}^t a_\Delta(\bar{y}_\Delta(s))ds\right|^{\bar{p}}+\mathbb{E}_B\left|\int_{t_i}^t b_\Delta(\bar{y}_\Delta(s))dE(s)\right|^{\bar{p}}+\mathbb{E}_B\left|\int_{t_i}^t \sigma_\Delta(\bar{y}_\Delta(s))dB(E(s))\right|^{\bar{p}}\right)\\
\leq&3^{\bar{p}-1}\left(\Delta^{\bar{p}-1}\mathbb{E}_B\int_{t_i}^t|a_\Delta(\bar{y}_\Delta(s))|^{\bar{p}}ds+|E(t)-E(t_i)|^{\bar{p}-1}\mathbb{E}_B\int_{t_i}^t|b_\Delta(\bar{y}_\Delta(s))|^{\bar{p}}dE(s)\right.\\
&\quad\left.+\mathbb{E}_B\left|\int_{t_i}^t \sigma_\Delta(\bar{y}_\Delta(s))dB(E(s))\right|^{\bar{p}}\right)\\
\leq&3^{\bar{p}-1}\left(\Delta^{\bar{p}-1}\mathbb{E}_B\int_{t_i}^t|a_\Delta(\bar{y}_\Delta(s))|^{\bar{p}}ds+(k\Delta)^{\bar{p}-1}\mathbb{E}_B\int_{t_i}^t|b_\Delta(\bar{y}_\Delta(s))|^{\bar{p}}dE(s)\right.\\
&\quad\left.+\mathbb{E}_B\left|\int_{t_i}^t \sigma_\Delta(\bar{y}_\Delta(s))dB(E(s))\right|^{\bar{p}}\right).
\end{aligned}\tag{3.7}$$

Then by the BDG inequality, we can see

$$\begin{aligned}
&\mathbb{E}_B\left|\int_{t_i}^t \sigma_\Delta(\bar{y}_\Delta(s))dB(E(s))\right|^{\bar{p}}\\
=&\mathbb{E}_B\left(\left|\int_{t_i}^t \sigma_\Delta(\bar{y}_\Delta(s))dB(E(s))\right|^2\right)^{\bar{p}/2}\\
\leq&b_{\bar{p}}\mathbb{E}_B\left(\int_{t_i}^t|\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right)^{\bar{p}/2}\\
\leq&b_{\bar{p}}(k\Delta)^{\bar{p}/2-1}\mathbb{E}_B\left(\int_{t_i}^t|\sigma_\Delta(\bar{y}_\Delta(s))|^{\bar{p}}dE(s)\right).
\end{aligned}\tag{3.8}$$

Substituting the estimate (3.8) into (3.7), we have

$$\begin{aligned}
&\mathbb{E}_B|y_\Delta(t)-\bar{y}_\Delta(t)|^{\bar{p}}\\
\leq&3^{\bar{p}-1}\left(\Delta^{\bar{p}-1}\Delta(h(\Delta))^{\bar{p}}+(k\Delta)^{\bar{p}-1}(k\Delta)(h(\Delta))^{\bar{p}}+(k\Delta)^{\bar{p}/2-1}(k\Delta)(h(\Delta))^{\bar{p}}\right)\\
\leq&C_{\bar{p}}\Delta^{\bar{p}/2}(h(\Delta))^{\bar{p}},
\end{aligned}$$

where (2.4) is used. This completes the proof of (3.5). Note from (2.3), we have

$$\Delta^{\bar{p}/2}(h(\Delta))^{\bar{p}}\lesssim\Delta^{\bar{p}/4}.$$

Finally, (3.6) is obtained from (3.5) immediately. □

Now, we prove the boundedness of the pth moment the numerical solution.

Lemma 3.3. *Let Assumptions* 2.1, 2.4 *and* 2.5 *hold. Then, for any* $p\in[2,r]$

$$\sup_{0\le\Delta\le 1}\mathbb{E}\left(\sup_{0\le t\le T}|y_\Delta(t)|^p\right)\le C,\quad \forall T>0, \tag{3.9}$$

where C is a positive constant independent of Δ.

Proof. Define $\zeta_\ell:=\inf\{t\ge 0;|y_\Delta(t)|>\ell\}$ for some positive integer ℓ. So, we have

$$\int_0^t \mathbb{E}_B\Big(\sup_{0\le s\le t\wedge\zeta_\ell}|y_\Delta(s)|^p\Big)dE(s)\le \ell^p E(t)<\infty.$$

Fix any $\Delta\in(0,1]$ and $T\ge 0$. By the time-changed Itô formula, we derive from (2.7) that, for $0\le u\le t\wedge\zeta_\ell$,

$$|y_\Delta(u)|^p=|y_\Delta(0)|^p+J_1+J_2+M_u,$$

where

$$\begin{aligned}
J_1:=&\int_0^u p|y_\Delta(s)|^{p-2}y_\Delta^{\rm T}(s)a_\Delta(\bar{y}_\Delta(s))ds,\\
J_2:=&\int_0^u\left(p|y_\Delta(s)|^{p-2}y_\Delta^{\rm T}(s)b_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}p(p-1)|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2\right)dE(s),\\
M_u:=&\int_0^u p|y_\Delta(s)|^{p-1}\sigma_\Delta(\bar{y}_\Delta(s))dB(E(s)).
\end{aligned}$$

By Lemma 2.2 and the Young inequality

$$a^{p-2}b\le\frac{p-2}{p}a^p+\frac{2}{p}b^{p/2},\quad \forall a,b\ge 0, \tag{3.10}$$

we get

$$\begin{aligned}
J_1\le&\int_0^u p|y_\Delta(s)|^{p-2}\bar{y}_\Delta^{\rm T}(s)a_\Delta(\bar{y}_\Delta(s))ds\\
&+\int_0^u p|y_\Delta(s)|^{p-2}(y_\Delta(s)-\bar{y}_\Delta(s))^{\rm T}a_\Delta(\bar{y}_\Delta(s))ds\\
\le& pK_1\int_0^u|y_\Delta(s)|^{p-2}|\bar{y}_\Delta(s)|^2ds+(p-2)\int_0^u|y_\Delta(s)|^pds\\
&+2\int_0^u|y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|a_\Delta(\bar{y}_\Delta(s))|^{p/2}ds.
\end{aligned}$$

In order to estimate M_u, we notice that the stochastic integral $(M_u)_{u\ge 0}$ is a local martingale with quadratic variation

$$[M,M]_t=\int_0^t p^2|y_\Delta(s)|^{2p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s).$$

For $0\le u\le t\wedge\zeta_\ell$, we have

$$\begin{aligned}&p^2|y_\Delta(s)|^{2p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2\\ \le&p^2|y_\Delta(s)|^p|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2\\ \le&p^2\left(\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\right)|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2.\end{aligned}$$

Using the inequality $(ab)^{1/2}\le a/\lambda+\lambda b$ for any $a,b\ge 0$ and $\lambda\ge 0$, we derive that

$$\begin{aligned}([M,M]_t)^{1/2}\le&p\left(\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\int_0^t|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right)^{1/2}\\ \le&p\left(\frac{\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p}{2pb_1}+2pb_1\int_0^t|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right).\end{aligned}$$

Therefore, for any $0\le u\le t\wedge\zeta_\ell$, by Lemmas 2.2, 2.3 and (3.10), we have

$$\begin{aligned}&\mathbb{E}_B\Big[\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\Big]\\ \le&|y_\Delta(0)|^p+pK_1\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)|^{p-2}|\bar{y}_\Delta(s)|^2ds\right)+(p-2)\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)|^pds\right)\\ &+2\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|a_\Delta(\bar{y}_\Delta(s))|^{p/2}ds\right)\\ &+\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u\left(p|y_\Delta(s)|^{p-2}y_\Delta^{\mathrm{T}}(s)b_\Delta(\bar{y}_\Delta(s))+\frac{1}{2}p(p-1)|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2\right)dE(s)\right)\\ &+\frac{1}{2}\mathbb{E}_B\Big[\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\Big]+\mathbb{E}_B\left(\int_0^{t\wedge\zeta_\ell}2p^2b_1|y_\Delta(s)|^{p-2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right)\\ \le&|y_\Delta(0)|^p+\frac{1}{2}\mathbb{E}_B\Big[\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\Big]+pK_1\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)|^{p-2}|\bar{y}_\Delta(s)|^2ds\right)\\ &+(p-2)\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)|^pds\right)\\ &+2\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^u|y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|a_\Delta(\bar{y}_\Delta(s))|^{p/2}ds\right)\\ &+b\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^up|y_\Delta(s)|^{p-2}\left(\bar{y}_\Delta^{\mathrm{T}}(s)b_\Delta(\bar{y}_\Delta(s))+\frac{5p-1}{2}|\sigma_\Delta(\bar{y}_\Delta(s))|^2\right)dE(s)\right)\\ &+\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}\int_0^up|y_\Delta(s)|^{p-2}(y_\Delta(s)-\bar{y}_\Delta(s))^{\mathrm{T}}b_\Delta(\bar{y}_\Delta(s))dE(s)\right)\end{aligned}$$

$$
\begin{aligned}
&\leq |y_\Delta(0)|^p + \frac{1}{2}\mathbb{E}_B\Big[\sup_{0\leq u\leq t\wedge\zeta_\ell}|y_\Delta(u)|^p\Big] + pK_1\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)|^{p-2}|\bar{y}_\Delta(s)|^2 ds\right)\\
&\quad +(p-2)\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)|^p ds\right)\\
&\quad +2\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|a_\Delta(\bar{y}_\Delta(s))|^{p/2}ds\right)\\
&\quad +pb\hat{K}_2\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)|^{p-2}(1+|\bar{y}_\Delta(s)|^2)dE(s)\right)\\
&\quad +(p-2)\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)|^p dE(s)\right)\\
&\quad +2\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}\int_0^u |y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|b_\Delta(\bar{y}_\Delta(s))|^{p/2}dE(s)\right),
\end{aligned}
$$

where $b=b_1\vee 1$. Thus, using the equality (3.4) and for any $0\leq u\leq t\wedge\zeta_\ell\leq T$, we have

$$
\begin{aligned}
&\mathbb{E}_B\left(\sup_{0\leq u\leq t\wedge\zeta_\ell}|y_\Delta(u)|^p\right)\\
\leq& 2|y_\Delta(0)|^p + 2pK_1\mathbb{E}_B\left(\sup_{0\leq t\leq T}\int_0^{t\wedge\zeta_\ell}|y_\Delta(s)|^{p-2}|\bar{y}_\Delta(s)|^2ds\right)\\
&\quad +2(p-2)\mathbb{E}_B\left(\sup_{0\leq t\leq T}\int_0^{t\wedge\zeta_\ell}|y_\Delta(s)|^p ds\right)\\
&\quad +4\mathbb{E}_B\int_0^T |y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|a_\Delta(\bar{y}_\Delta(s))|^{p/2}ds\\
&\quad +2pb\hat{K}_2\mathbb{E}_B\left(\sup_{0\leq t\leq T}\int_0^t |y_\Delta(s\wedge\zeta_\ell)|^{p-2}(1+|\bar{y}_\Delta(s\wedge\zeta_\ell)|^2)dE(s)\right)\\
&\quad +2(p-2)\mathbb{E}_B\left(\sup_{0\leq t\leq T}\int_0^t \mathbb{E}_B|y_\Delta(s\wedge\zeta_\ell)|^p dE(s)\right)\\
&\quad +4\mathbb{E}_B\left(\int_0^T |y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|b_\Delta(\bar{y}_\Delta(s))|^{p/2}dE(s)\right).
\end{aligned}
$$

By Lemma 3.2, inequalities (2.3) and (2.4), we get

$$
\begin{aligned}
&\mathbb{E}_B|y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}|b_\Delta(\bar{y}_\Delta(s))|^{p/2}\\
\leq&(h(\Delta))^{p/2}\mathbb{E}_B\left(|y_\Delta(s)-\bar{y}_\Delta(s)|^{p/2}\right)\\
\leq&(h(\Delta))^{p/2}\left(\mathbb{E}_B|y_\Delta(s)-\bar{y}_\Delta(s)|^p\right)^{1/2}\\
\leq& C_p\Delta^{p/4}(h(\Delta))^p\leq C_p.
\end{aligned}
$$

Thus, we have

$$\begin{aligned}&\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\right)\\ \le& C_1+C_2\int_0^T\mathbb{E}_B\left(\sup_{0\le s\le t\wedge\zeta_\ell}|y_\Delta(s)|^p\right)dt+C_3\int_0^T\mathbb{E}_B\left(\sup_{0\le s\le t\wedge\zeta_\ell}|y_\Delta(s)|^p\right)dE(t),\end{aligned}$$

where

$$C_1=2|y_\Delta(0)|^p+4C_p(T+E(T)),\quad C_2=2pK_1\vee(2p-2),\quad C_3=4pb\hat{K}_2\vee 2(p-2).$$

Then applying the Gronwall-type inequality yields that

$$\mathbb{E}_B\left(\sup_{0\le u\le t\wedge\zeta_\ell}|y_\Delta(u)|^p\right)\le C_1e^{C_2T+C_3E(T)}.$$

Due to $\zeta_\ell\to\infty$ as $\ell\to\infty$. Setting $t=T$ and letting $\ell\to\infty$, we get

$$\mathbb{E}_B\left(\sup_{0\le t\le T}|y_\Delta(u)|^p\right)\le C_1e^{C_2T+C_3E(T)}.$$

Taking $\mathbb{E}_D$ on both sides and using the facts that $\mathbb{E}_D(e^{cT})$ and

$$\mathbb{E}_D(E(T)e^{cE(T)})<\mathbb{E}_D(e^{(c+1)E(T)})<\infty,$$

we can show that

$$\mathbb{E}\left(\sup_{0\le t\le T}|y_\Delta(t)|^p\right)\le C.$$

As this holds for any $\Delta\in(0,1]$ and C is a positive constant independent of Δ, the assertion (3.9) is proved. □

Now, we are ready to prove the main theorem that states the strong convergence of the proposed numerical method.

Theorem 3.1. *Let Assumptions* 2.1, 2.2 *and* 2.5 *hold. In addition, suppose Assumption* 2.3 *holds with* $r>(\gamma+1)q$. *Then, for any* $p\in[2,q)$ *and* $\Delta\in(0,1]$,

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-y_\Delta(t)|^p\right)\le C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big),\tag{3.11}$$

and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\le C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big).\tag{3.12}$$

Proof. Fix $p\in[2,q)$ and $\Delta\in(0,1]$ arbitrarily. Let $e_\Delta(t)=Y(t)-y_\Delta(t)$ for $t\geq 0$. For any integral $\ell>|Y(0)|$, define the stopping time

$$\tau_\ell=\inf\{t\geq 0:|Y(t)|\vee|y_\Delta(t)|\geq\ell\},$$

where we set $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). By the time-changed Itô formula, we have that for any $0\leq t\leq T$,

$$|e_\Delta(t\wedge\tau_\ell)|^p=R_1+R_2+M_{t\wedge\tau_\ell}, \tag{3.13}$$

where

$$\begin{aligned}
R_1:=&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-1}(a(Y(s))-a_\Delta(\bar{y}_\Delta(s)))ds,\\
R_2:=&\int_0^{t\wedge\tau_\ell}\Big(p|e_\Delta(s)|^{p-1}(b(Y(s))-b_\Delta(\bar{y}_\Delta(s)))+\frac{1}{2}p(p-1)|e_\Delta(s)|^{p-2}\\
&\qquad\times|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\Big)dE(s),\\
M_{t\wedge\tau_\ell}:=&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-1}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|dB(E(s)).
\end{aligned}$$

Noting that the stochastic integral $(M_t)_{t\geq 0}$ is a local martingale with quadratic variation

$$[M,M]_{t\wedge\tau_\ell}=\int_0^{t\wedge\tau_\ell} p^2|e_\Delta(s)|^{2p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s).$$

For $0\leq s\leq t\wedge\tau_\ell$, we can show that

$$\begin{aligned}
&p^2|e_\Delta(s)|^{2p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\\
=&p^2|e_\Delta(s)|^p|e_\Delta(s)|^{p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\\
\leq&p^2\Big(\sup_{0\leq u\leq t\wedge\tau_\ell}|e_\Delta(u)|^p\Big)|e_\Delta(s)|^{p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2.
\end{aligned}$$

Then, using the inequality $(ab)^{1/2}\leq a/\lambda+\lambda b$ for any $a,b>0$ and $\lambda>0$, we get

$$\begin{aligned}
&([M,M]_{t\wedge\tau_\ell})^{1/2}\\
\leq&p\left(\sup_{0\leq u\leq t\wedge\tau_\ell}|e_\Delta(u)|^p\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^{p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right)^{1/2}\\
\leq&p\left(\frac{\sup_{0\leq u\leq t\wedge\tau_\ell}|e_\Delta(u)|^p}{2pb_1}+2pb_1\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^{p-2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2dE(s)\right).
\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13) and using the BDG inequality, we derive that

$$\begin{aligned}
&\mathbb{E}_B\left(\sup_{0\le t\le T}|e_\Delta(t\wedge\tau_\ell)|^p\right)\\
\le&\mathbb{E}_B\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}p|e_\Delta(s)|^{p-1}(a(Y(s))-a_\Delta(\bar{y}_\Delta(s)))ds\right)+\frac{1}{2}\mathbb{E}_B\Big[\sup_{0\le u\le t\wedge\tau_\ell}|e_\Delta(u)|^p\Big]\\
&+b\mathbb{E}_B\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}p|e_\Delta(s)|^{p-2}\bigg(e_\Delta^{\mathrm{T}}(s)(b(Y(s))-b_\Delta(\bar{y}_\Delta(s)))\right.\\
&\left.+\frac{5p-1}{2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\bigg)dE(s)\right),
\end{aligned}\tag{3.15}$$

where $b=b_1\vee 1$. Noting that

$$\begin{aligned}
&\frac{5p-1}{2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\\
=&\frac{5p-1}{2}|\sigma(Y(s))-\sigma(y_\Delta(s))+\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\\
\le&\frac{5p-1}{2}\bigg(|\sigma(Y(s))-\sigma(y_\Delta(s))|^2+2|\sigma(Y(s))-\sigma(y_\Delta(s))||\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|\\
&+|\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\bigg).
\end{aligned}\tag{3.16}$$

Recalling the Young inequality $ab\le(\delta/2)a^2+1/(2\delta)b^2$ for any $a,b\ge 0$ and $\delta>0$, by choosing $\delta=(5q-5p)/(5p-1)$, we get

$$\begin{aligned}
&2|\sigma(Y(s))-\sigma(y_\Delta(s))||\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|\\
\le&2\left(\frac{1}{2}\cdot\frac{5q-5p}{5p-1}|\sigma(Y(s))-\sigma(y_\Delta(s))|^2+\frac{1}{2}\cdot\frac{5p-1}{5q-5p}|\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\right)\\
=&\frac{5q-5p}{5p-1}|\sigma(Y(s))-\sigma(y_\Delta(s))|^2+\frac{5p-1}{5q-5p}|\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2.
\end{aligned}\tag{3.17}$$

Substituting (3.17) into (3.16), we get

$$\begin{aligned}
&\frac{5p-1}{2}|\sigma(Y(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\\
\le&\frac{5q-1}{2}|\sigma(Y(s))-\sigma(y_\Delta(s))|^2+\frac{(5p-1)(5q-1)}{2(5q-5p)}|\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2.
\end{aligned}$$

Thus, we can obtain from (3.15)

$$\begin{aligned}
&\mathbb{E}_B\left(\sup_{0\le t\le T}|e_\Delta(t\wedge\tau_\ell)|^p\right)\\
\le&\frac{1}{2}\mathbb{E}_B\Big[\sup_{0\le u\le t\wedge\tau_\ell}|e_\Delta(u)|^p\Big]+\mathbb{E}_B\Big[\sup_{0\le t\le T}S_1\Big]+\mathbb{E}_B\Big[\sup_{0\le t\le T}S_2\Big]
\end{aligned}$$

$$+\mathbb{E}_B\Big[\sup_{0\le t\le T} S_3\Big]+\mathbb{E}_B\Big[\sup_{0\le t\le T} S_4\Big], \tag{3.18}$$

where

$$\begin{aligned}
S_1:=&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-1}(a(Y(s))-a(y_\Delta(s)))ds,\\
S_2:=&b\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-2}\bigg(e_\Delta^{\mathrm{T}}(s)(b(Y(s))-b(y_\Delta(s)))\\
&\quad+\frac{5q-1}{2}|\sigma(Y(s))-\sigma(y_\Delta(s))|^2\bigg)dE(s),\\
S_3:=&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-1}(a(y_\Delta(s))-a_\Delta(\bar{y}_\Delta(s)))ds,\\
S_4:=&b\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-2}\bigg(e_\Delta^{\mathrm{T}}(s)(b(y_\Delta(s))-b_\Delta(\bar{y}_\Delta(s)))\\
&\quad+\frac{(5p-1)(5q-1)}{2(5q-5p)}|\sigma(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\bigg)dE(s).
\end{aligned}$$

By Assumption 2.2, we have

$$S_1\le C_1\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds, \tag{3.19}$$

where $C_1=pK$. By Assumption 2.3, we get

$$S_2\le C_2\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s), \tag{3.20}$$

where $C_2=pbK_3$. Rearranging S_3, we can show that

$$\begin{aligned}
S_3\le&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-2}\Big(e_\Delta^{\mathrm{T}}(s)(a(y_\Delta(s))-a_\Delta(y_\Delta(s)))\Big)ds\\
&+\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-2}\Big(e_\Delta^{\mathrm{T}}(s)(a_\Delta(y_\Delta(s))-a_\Delta(\bar{y}_\Delta(s)))\Big)ds\\
=:&S_{31}+S_{32}.
\end{aligned}$$

By Assumption 2.1 and the Young inequality $a^{p-2}b\le(p-2)a^p/p+2b^{p/2}/p$ for any $a,b\ge 0$ and $0\le t\wedge\tau_\ell\le t\le T$, we can derive that

$$\begin{aligned}
S_{31}\le&\int_0^{t\wedge\tau_\ell} p|e_\Delta(s)|^{p-2}\left(\frac{1}{2}|e_\Delta(s)|^2+\frac{1}{2}|(a(y_\Delta(s))-a_\Delta(y_\Delta(s)))|^2\right)ds\\
\le& p(p-1)\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds+\int_0^{t\wedge\tau_\ell}|(a(y_\Delta(s))-a_\Delta(y_\Delta(s)))|^p ds
\end{aligned}$$

$$
\begin{aligned}
&\le p(p-1)\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds\\
&\quad +L^p3^{p-1}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\pi_\Delta(y_\Delta(s))|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p ds\\
&\le p(p-1)\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds+2L^p3^{p-1}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p ds.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
S_{32}\le& p(p-1)\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds\\
&+L^p3^{p-1}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_3\le& C_3\bigg(\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds+\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p ds\\
&+\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p ds\bigg),
\end{aligned}
\tag{3.21}
$$

where $C_3=\max\{2p(p-1),2L^p3^{p-1}\}$. Rearranging S_4 gets

$$
\begin{aligned}
S_4\le& b\int_0^{t\wedge\tau_\ell}p|e_\Delta(s)|^{p-2}\bigg(e_\Delta^{\mathrm{T}}(s)(b(y_\Delta(s))-b_\Delta(y_\Delta(s)))\\
&+\frac{(5p-1)(5q-1)}{(5q-5p)}|\sigma(y_\Delta(s))-\sigma_\Delta(y_\Delta(s))|^2\bigg)dE(s)\\
&+b\int_0^{t\wedge\tau_\ell}p|e_\Delta(s)|^{p-2}\bigg(e_\Delta^{\mathrm{T}}(s)(b_\Delta(y_\Delta(s))-b_\Delta(\bar{y}_\Delta(s)))\\
&+\frac{(5p-1)(5q-1)}{(5q-5p)}|\sigma_\Delta(y_\Delta(s))-\sigma_\Delta(\bar{y}_\Delta(s))|^2\bigg)dE(s)\\
=:& S_{41}+S_{42}.
\end{aligned}
\tag{3.22}
$$

In the same way as S_{31}, we get

$$
\begin{aligned}
S_{41}\le&\frac{(p-1)(5q-5p)+(p-2)(5p-1)(5q-1)}{(5q-5p)}b\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)\\
&+b\int_0^{t\wedge\tau_\ell}|b(y_\Delta(s))-b_\Delta(y_\Delta(s))|^p dE(s)\\
&+\frac{2(5p-1)(5q-1)}{(5q-5p)}b\int_0^{t\wedge\tau_\ell}|\sigma(y_\Delta(s))-\sigma_\Delta(y_\Delta(s))|^p dE(s)\\
\le& C_{41}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)+S_{411},
\end{aligned}
$$

where

$$S_{411}=C_{41}\int_0^{t\wedge\tau_\ell}\left(|b(y_\Delta(s))-b_\Delta(y_\Delta(s))|^p+|\sigma(y_\Delta(s))-\sigma_\Delta(y_\Delta(s))|^p\right)dE(s),$$

and

$$C_{41}=\max\Big\{\frac{(p-1)(5q-5p)+(p-2)(5p-1)(5q-1)}{5q-5p},1,\frac{2(5p-1)(5q-1)}{(5q-5p)}\Big\}b.$$

By Assumption 2.1, we obtain that

$$\begin{aligned}S_{411}\le& 2L^p3^{p-1}C_{41}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\pi_\Delta(y_\Delta(s))|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p dE(s)\\ \le& 4L^p3^{p-1}C_{41}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p dE(s).\end{aligned}$$

So, we have

$$\begin{aligned}S_{41}\le C_{41}\bigg(&\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)\\ &+4L^p3^{p-1}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p dE(s)\bigg).\end{aligned}$$

Similarly, we have

$$\begin{aligned}S_{42}\le C_{42}\bigg(&\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)\\ &+2L^p3^{p-1}\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p dE(s)\bigg).\end{aligned}$$

Substituting these estimates into (3.22) yields

$$\begin{aligned}S_4\le C_4\bigg(&\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)+\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p dE(s)\\ &+\int_0^{t\wedge\tau_\ell}(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p dE(s)\bigg),\end{aligned}\tag{3.23}$$

where

$$C_4=(C_{41}\vee C_{42})\times\max\{2,4L^p3^{p-1}\}.$$

Combining (3.19), (3.20), (3.21) and (3.23) together, we can show that

$$\begin{aligned}&\mathbb{E}_B\left(\sup_{0\le t\le T}|e_\Delta(t\wedge\tau_\ell)|^p\right)\\ \le& 2C_1\mathbb{E}_B\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds\right)+2C_2\mathbb{E}_B\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)\right)\\ &+\mathbb{E}_B[T_3]+\mathbb{E}_B[T_4],\end{aligned}\tag{3.24}$$

where

$$
\begin{aligned}
T_3 := & 2C_3\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds+\int_0^T(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p ds\right.\\
&\left.+\int_0^T(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p ds\right)\\
= & 2C_3\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds+T_{31}+T_{32}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
T_4 := & 2C_4\left(\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p dE(s)+\int_0^T(1+|y_\Delta(s)|^{\gamma p})|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^p dE(s)\right.\\
&\left.+\int_0^T(1+|y_\Delta(s)|^{\gamma p}+|\bar{y}_\Delta(s)|^{\gamma p})|y_\Delta(s)-\bar{y}_\Delta(s)|^p dE(s)\right).
\end{aligned}
$$

For

$$
\mathbb{E}_B[T_3]\le 2C_3\left(\mathbb{E}_B\Big[\sup_{0\le t\le T}\int_0^{t\wedge\tau_\ell}|e_\Delta(s)|^p ds\Big]+\mathbb{E}_B[T_{31}]+\mathbb{E}_B[T_{32}]\right). \tag{3.25}
$$

Using the Hölder inequality, Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
\mathbb{E}_B[T_{31}]\le & \int_0^T(\mathbb{E}_B(1+|y_\Delta(s)|^r))^{\frac{\gamma p}{r}}\left(\mathbb{E}_B|y_\Delta(s)-\pi_\Delta(y_\Delta(s))|^{\frac{rp}{r-\gamma p}}\right)^{\frac{r-\gamma p}{r}}ds\\
\le & (C+1)^{\frac{\gamma p}{r}}\int_0^T\left(\mathbb{E}_B(I_{\{|y_\Delta(s)|>\mu^{-1}(h(\Delta))\}}|y_\Delta(s)|^{\frac{rp}{r-\gamma p}})\right)^{\frac{r-\gamma p}{r}}ds\\
\le & (C+1)^{\frac{\gamma p}{r}}\int_0^T\left((\mathbb{P}\{|y_\Delta(s)|>\mu^{-1}(h(\Delta))\})^{\frac{r-(\gamma+1)p}{r-\gamma p}}(\mathbb{E}_B|y_\Delta(s)|^r)^{\frac{p}{r-\gamma p}}\right)^{\frac{r-\gamma p}{r}}ds\\
\le & (C+1)^{\frac{\gamma p}{r}}\int_0^T\left(\frac{\mathbb{E}_B|y_\Delta(s)|^r}{(\mu^{-1}(h(\Delta)))^r}\right)^{\frac{r-(\gamma+1)p}{r}}ds\\
\le & (C+1)(\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}T,
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
\mathbb{E}_B[T_{32}]\le & 2(C+1)^{\frac{\gamma p}{r}}\int_0^T(\mathbb{E}_B|y_\Delta(s)-\bar{y}_\Delta(s)|^r)^{\frac{p}{r}}ds\\
\le & 2(C+1)^{\frac{\gamma p}{r}}C_r\Delta^{p/2}(h(\Delta))^pT.
\end{aligned} \tag{3.27}
$$

Substituting (3.26) and(3.27) into (3.25), and using inequality (3.4) yields

$$
\begin{aligned}
\mathbb{E}_B[T_3]\le & 2C_3\left(\int_0^T\mathbb{E}_B\Big[\sup_{0\le t\le T}|e_\Delta(s\wedge\tau_\ell)|^p\Big]ds+(C+1)(\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}T\right.\\
&\left.+2(C+1)^{\frac{\gamma p}{r}}C_r\Delta^{p/2}(h(\Delta))^pT\right).
\end{aligned}
$$

Similarly, we show that

$$\begin{aligned}\mathbb{E}_B[T_4]\leq & 2C_4\bigg(\int_0^T \mathbb{E}_B\Big[\sup_{0\leq t\leq T}|e_\Delta(s\wedge\tau_\ell)|^p\Big]dE(s)+(C+1)(\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}E(T)\\ &+2(C+1)^{\frac{\gamma p}{r}}C_r\Delta^{p/2}(h(\Delta))^pE(T)\bigg).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathbb{E}_B\left(\sup_{0\leq t\leq T}|e_\Delta(t\wedge\tau_\ell)|^p\right)\\ \leq & C_5\bigg(\int_0^T \mathbb{E}_B\Big[\sup_{0\leq t\leq T}|e_\Delta(s\wedge\tau_\ell)|^p\Big]ds+\int_0^T \mathbb{E}_B\Big[\sup_{0\leq t\leq T}|e_\Delta(s\wedge\tau_\ell)|^p\Big]dE(s)\\ &+(\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\bigg),\end{aligned}$$

where

$$\begin{aligned}C_5=\max\bigg\{&2(C_1+C_3),2(C_2+C_4),(C+1)T,(C+1)E(T),2(C+1)^{\frac{\gamma p}{r}}C_rT,\\ &2(C+1)^{\frac{\gamma p}{r}}C_rE(T)\bigg\}.\end{aligned}$$

Applying the Gronwall-type inequality yields that

$$\mathbb{E}_B\left(\sup_{0\leq t\leq T}|e_\Delta(t\wedge\tau_\ell)|^p\right)\leq C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big).$$

Using the Fatou Lemma, and let $\ell\to\infty$ to obtain

$$\mathbb{E}_B\left(\sup_{0\leq t\leq T}|e_\Delta(t)|^p\right)\leq C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big),$$

where C depends on $e^{\lambda T}$ and $e^{\lambda E(T)}$ for some $\lambda>0$, but is independent of Δ.

Taking $\mathbb{E}_D$ on both sides gives the required assertion (3.11). By (3.11) and Lemma 3.2, we can show that assertion (3.12). Therefore, the proof is completed. □

In order to display the convergence rate more clearly, we give the following corollary, which shows that the explicit numerical method has the rate of arbitrarily close to 1/2.

Corollary 3.1. *Let Assumptions* 2.1, 2.2 *and* 2.3 *hold. In addition, suppose Assumption* 2.5 *holds for all* $r>2$. *Particularly, recalling* (2.2), *define*

$$\mu(u)=Hu^{\gamma+2},\quad u\geq 1, \tag{3.28}$$

and let

$$h(\Delta)=\Delta^{-\varepsilon} \quad \textit{for some } \varepsilon\in\Big(0,\frac{1}{4}\Big] \quad \textit{and} \quad \hat{h}\geq 1, \tag{3.29}$$

Then, for any $p>2$, $\Delta\in(0,1]$ *and* $\varepsilon\in(0,\frac{1}{4}]$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-y_\Delta(t)|^p\right)\leq C\Delta^{(1/2-\varepsilon)p}, \tag{3.30}$$

and

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\leq C\Delta^{(1/2-\varepsilon)p}. \tag{3.31}$$

Proof. (3.28) implies that

$$\mu^{-1}(u)=\Big(\frac{u}{H}\Big)^{\frac{1}{\gamma+2}}.$$

Next, according to Theorem 3.1 yields

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-y_\Delta(t)|^p\right)\leq C\left(\Delta^{\frac{\varepsilon(r-(\gamma+1)p)}{\gamma+2}}+\Delta^{(1/2-\varepsilon)p}\right),$$

and

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\leq C\left(\Delta^{\frac{\varepsilon(r-(\gamma+1)p)}{\gamma+2}}+\Delta^{(1/2-\varepsilon)p}\right).$$

Choosing r sufficiently large for

$$\frac{\varepsilon(r-(\gamma+1)p)}{\gamma+2}>\Delta^{(1/2-\varepsilon)p},$$

we can get the assertions (3.30) and (3.31) easily. $\square$

4 Main results: almost surely asymptotic stability

In this section, we consider the asymptotic stability of the numerical solution. The semi-martingale convergence theorem is used.

We assume that $a(0)=0$, $b(0)=0$ and $\sigma(0)=0$ throughout this section. Let $\mathcal{K}$ denote the family of continuous non-decreasing function $\kappa:\mathbb{R}_+\to\mathbb{R}_+$ such that $\kappa(0)=0$ and $\kappa(u)>0$ for all $u>0$.

Assumption 4.1. Assume that there are two functions $\kappa_1\in\mathcal{K}$ and $\kappa_2\in\mathcal{K}$ such that

$$2x^{\mathrm{T}}a(x)\leq-\kappa_1(|x|),$$

and

$$2x^{\mathrm{T}}b(x)+|\sigma(x)|^2\leq-\kappa_2(|x|),$$

for all $x\in\mathbb{R}^d$.

The following lemma states that the truncating functions a_Δ, b_Δ and σ_Δ satisfy Assumption 4.1.

Lemma 4.1. *Let Assumption* 4.1 *holds. Then, for any* $\Delta \in (0,1]$,

$$2x^{\mathrm{T}} a_\Delta(x) \le -\kappa_1(|\pi_\Delta(x)|), \quad x \in \mathbb{R}^d, \tag{4.1}$$

and

$$2x^{\mathrm{T}} b_\Delta(x) + |\sigma_\Delta(x)|^2 \le -\kappa_2(|\pi_\Delta(x)|), \quad x \in \mathbb{R}^d. \tag{4.2}$$

Proof. For $x \in \mathbb{R}^d$ with $|x| \le \mu^{-1}(h(\Delta))$, (4.1) holds obviously. For $x \in \mathbb{R}^d$ with $|x| > \mu^{-1}(h(\Delta))$, by Assumption 4.1, we have

$$\begin{aligned} 2x^{\mathrm{T}} a_\Delta(x) &= 2(x-\pi_\Delta(x))^{\mathrm{T}} a(\pi_\Delta(x)) + 2(\pi_\Delta(x))^{\mathrm{T}} a(\pi_\Delta(x)) \\ &\le 2(x-\pi_\Delta(x))^{\mathrm{T}} a(\pi_\Delta(x)) - \kappa_1(|\pi_\Delta(x)|). \end{aligned} \tag{4.3}$$

But, by Assumption 4.1 again,

$$2(x-\pi_\Delta(x))^{\mathrm{T}} a(\pi_\Delta(x)) = 2\left(\frac{|x|}{\mu^{-1}(h(\Delta))} - 1\right)(\pi_\Delta(x))^{\mathrm{T}} a(\pi_\Delta(x)) \le 0.$$

Substituting this into (4.3) yields (4.1) as desired. The proof of (4.2) is similar to that of (4.1), which we omit. □

The following theorem shows that the proposed numerical method retains almost surely asymptotic stability.

Theorem 4.1. *Let Assumption* 4.1 *holds. Assume that*

$$\limsup_{|x|\downarrow 0} \frac{|a(x)|^2}{\kappa_1(|x|)} < \infty \qquad \textit{and} \qquad \limsup_{|x|\downarrow 0} \frac{|b(x)|^2}{\kappa_2(|x|)} < \infty. \tag{4.4}$$

Set

$$H_1 = \sup_{0<|x|\le \mu^{-1}(h(1))} \frac{|a(x)|^2}{\kappa_1(|x|)}, \qquad H_2 = \sup_{0<|x|\le \mu^{-1}(h(1))} \frac{|b(x)|^2}{\kappa_2(|x|)}, \tag{4.5}$$

and

$$\tilde{\Delta} = \min\left(1, 0.5/H_1, 0.5/H_2, 0.25\left(\kappa_1(\mu^{-1}(h(1)))/\hat{h}\right)^2, 0.25\left(\kappa_2(\mu^{-1}(h(1)))/\hat{h}\right)^2\right). \tag{4.6}$$

Then, for every $\Delta \in (0,\tilde{\Delta}]$ *and any initial value* $Y_0 \in \mathbb{R}^d$*, the solution of the proposed numerical method* (2.5) *satisfies*

$$\lim_{i\to\infty} y_{\Delta,i} = 0 \qquad a.s. \tag{4.7}$$

Proof. According to the condition (4.4), $H_1<\infty$ and $H_2<\infty$, combined with the continuity of $a(\cdot)$, $b(\cdot)$ and the property of $\kappa_1(\cdot)$, $\kappa_2(\cdot)$, we have $\tilde{\Delta}\in(0,1]$. Fix any $\Delta\in(0,\tilde{\Delta}]$ and $Y_0\in\mathbb{R}^d$. By Lemma 4.1, we can show that

$$\begin{aligned}|y_{\Delta,i+1}|^2\le&|y_{\Delta,i}|^2+|a_\Delta(y_{\Delta,i})\Delta|^2+2y_{\Delta,i}^{\mathrm{T}}|a_\Delta(y_{\Delta,i})|\Delta+|b_\Delta(y_{\Delta,i})|^2(k\Delta)^2\\&+2a_\Delta(y_{\Delta,i})\Delta|b_\Delta(y_{\Delta,i})|(E(t_{i+1})-E(t_i))+2y_{\Delta,i}^{\mathrm{T}}|b_\Delta(y_{\Delta,i})|(E(t_{i+1})-E(t_i))\\&+|\sigma_\Delta(y_{\Delta,i})|^2(E(t_{i+1})-E(t_i))+\Delta M_i,\end{aligned}\tag{4.8}$$

where

$$\begin{aligned}\Delta M_i:=&2(y_{\Delta,i}+a_\Delta(y_{\Delta,i})\Delta+b_\Delta(y_{\Delta,i})k\Delta)^{\mathrm{T}}\sigma_\Delta(y_{\Delta,i})(B(E(t_{i+1}))-B(E(t_i)))\\&+|\sigma_\Delta(y_{\Delta,i})(B(E(t_{i+1}))-B(E(t_i)))|^2-|\sigma_\Delta(y_{\Delta,i})|^2(E(t_{i+1})-E(t_i)).\end{aligned}$$

Note that

$$\mathbb{E}\Big[(B(E(t_{i+1}))-B(E(t_i)))^2\big|\mathcal{F}_{t_i}\Big]=\mathbb{E}[E(t_{i+1})-E(t_i)].$$

It is easy to see that

$$\mathbb{E}(\Delta M_i\big|\mathcal{F}_{t_i})=0.$$

This implies that

$$M_i:=\sum_{j=0}^{i}\Delta M_j,\quad i=0,1,2,\cdots,$$

is a martingale. Recalling that $\mu^{-1}(h(\Delta))\ge\mu^{-1}(h(1))$ and using (4.5), if $0\le|x|\le\mu^{-1}(h(1))$, we get

$$|a_\Delta(x)|^2=|a(x)|^2\le H_1\kappa_1(|\pi_\Delta(x)|),\qquad |b_\Delta(x)|^2=|b(x)|^2\le H_2\kappa_2(|\pi_\Delta(x)|).$$

On the other hand, if $|x|>\mu^{-1}(h(1))$, we have

$$\begin{aligned}|a_\Delta(x)|^2\le(h(\Delta))^2\le\frac{(h(\Delta))^2}{\kappa_1(\mu^{-1}(h(1)))}\kappa_1(|\pi_\Delta(x)|),\\|b_\Delta(x)|^2\le(h(\Delta))^2\le\frac{(h(\Delta))^2}{\kappa_2(\mu^{-1}(h(1)))}\kappa_2(|\pi_\Delta(x)|).\end{aligned}$$

Consequently, using (2.3) and (4.6), for all $x\in\mathbb{R}^d$ and any $\Delta\in(0,\tilde{\Delta}]$, we can see that

$$\begin{aligned}|a_\Delta(x)|^2\Delta^{0.9}\le&\kappa_1(|\pi_\Delta(x)|)\max\left\{H_1\Delta^{0.9},\frac{(h(\Delta))^2\Delta^{0.9}}{\kappa_1(\mu^{-1}(h(1)))}\right\}\\\le&\kappa_1(|\pi_\Delta(x)|)\max\left\{H_1\Delta^{0.9},\frac{\hat{h}^2\Delta^{0.4}}{\kappa_1(\mu^{-1}(h(1)))}\right\}\\\le&0.5\kappa_1(|\pi_\Delta(x)|).\end{aligned}$$

Similarly, we have

$$|b_\Delta(x)|^2\Delta^{0.9}\leq 0.5\kappa_2(|\pi_\Delta(x)|).$$

Substituting these into (4.8), using Lemma 4.1 we derive

$$\begin{aligned}|y_{\Delta,i+1}|^2\leq&|y_{\Delta,i}|^2+0.5\kappa_1(|\pi_\Delta(y_{\Delta,i})|)\Delta^{1.1}-\kappa_1(|\pi_\Delta(y_{\Delta,i})|)\Delta\\&+0.5\kappa_2(|\pi_\Delta(y_{\Delta,i})|)k^2\Delta^{1.1}+0.5\kappa_1(|\pi_\Delta(y_{\Delta,i})|)k\Delta^{1.1}\\&+0.5\kappa_2(|\pi_\Delta(y_{\Delta,i})|)k\Delta^{1.1}-\kappa_2(|\pi_\Delta(y_{\Delta,i})|)k\Delta+\Delta M_i\\\leq&|y_{\Delta,i}|^2-\Delta[1-0.5\Delta^{0.1}(1+k)]\kappa_1(|\pi_\Delta(y_{\Delta,i})|)\\&-k\Delta[1-0.5\Delta^{0.1}(1+k)]\kappa_2(|\pi_\Delta(y_{\Delta,i})|)+\Delta M_i.\end{aligned}$$

It means that

$$\begin{aligned}|y_{\Delta,i+1}|^2\leq&|Y_0|^2-\Delta[1-0.5\Delta^{0.1}(1+k)]\sum_{j=0}^{i}\kappa_1(|\pi_\Delta(y_{\Delta,j})|)\\&-k\Delta[1-0.5\Delta^{0.1}(1+k)]\sum_{j=0}^{i}\kappa_2(|\pi_\Delta(y_{\Delta,j})|)+M_i,\quad i\geq 0.\end{aligned}$$

Applying the non-negative semi-martingale convergence theorem (see, e.g., [26, Theorem 3.9]), we get

$$\sum_{j=0}^{\infty}\kappa_1(|\pi_\Delta(y_{\Delta,j})|)<\infty\qquad a.s.\quad\text{and}\quad\sum_{j=0}^{\infty}\kappa_2(|\pi_\Delta(y_{\Delta,j})|)<\infty\qquad a.s.$$

This implies

$$\lim_{j\to\infty}\kappa_1(|\pi_\Delta(y_{\Delta,j})|)=0\qquad a.s.\quad\text{and}\quad\lim_{j\to\infty}\kappa_2(|\pi_\Delta(y_{\Delta,j})|)=0\qquad a.s.$$

Consequently, we must have

$$\lim_{j\to\infty}\pi_\Delta(y_{\Delta,j})=0\qquad a.s.$$

and the assertion (4.7) follows. The proof is completed. □

5 Numerical simulations

In this section, we give three numerical examples. Examples 5.1 and 5.2 demonstrate the strong convergence and the convergence rate. Example 5.3 displays the asymptotic stability of the numerical solution.

Example 5.1. Considering

$$dY(t)=(-Y(t)-Y^3(t))dt+(Y(t)-2Y^3(t))dE(t)+Y^2(t)d(B(E(t))),$$

with $Y(0)=1$.

For any $x,y\in\mathbb{R}^d$, we have

$$\begin{aligned}(x-y)^{\mathrm{T}}(a(x)-a(y))&=(x-y)(-x-x^3+y+y^3)\\&\leq(x-y)^2(-1-3xy)\\&\leq C|x-y|^2.\end{aligned}$$

This indicates that Assumption 2.2 holds. In the similar manner, for any $q>1$ we can see that

$$\begin{aligned}&(x-y)^{\mathrm{T}}(b(x)-b(y))+\frac{5q-1}{2}|\sigma(x)-\sigma(y)|^2\\&\leq(x-y)^2\left(1-2(x^2+xy+y^2)+\frac{5q-1}{2}(x+y)^2\right)\\&\leq(x-y)^2\left(1-(5q-4)(x^2+y^2)\right)\\&\leq C|x-y|^2,\end{aligned}$$

where the Young inequality is used. Note that the last inequality is due to the fact that polynomials with the negative coefficients for the highest order term can always be bounded from above. Further, we have

$$x^{\mathrm{T}}b(x)+\frac{5r-1}{2}|\sigma(x)|^2\leq C(1+|x|^2),$$

which means that Assumptions 2.3 and 2.5 hold.

According to Theorem 3.1, we know that for any $p\in[2,q]$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-y_\Delta(t)|^p\right)\leq C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big),$$

and

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\leq C\Big((\mu^{-1}(h(\Delta)))^{(\gamma+1)p-r}+\Delta^{p/2}(h(\Delta))^p\Big).$$

Due to that

$$\sup_{|x|\leq u}(|a(x)|\vee|b(x)|\vee|\sigma(x)|)\leq 3u^3,\quad\forall u\geq 1,$$

where we choose $\mu(u)=3u^3$ and $h(\Delta)=\Delta^{-\varepsilon}$, for any $\varepsilon\in(0,1/4]$. As a result,

$$\mu^{-1}(u)=(u/3)^{1/3}\quad\text{and}\quad\mu^{-1}(h(\Delta))=\left(\Delta^{-\varepsilon}/3\right)^{1/3}.$$

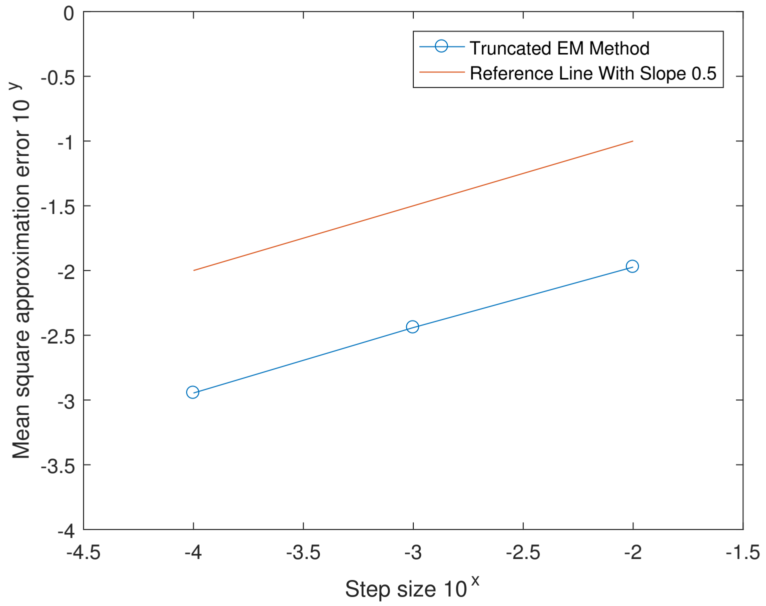

Figure 1: Convergence rate of Example 5.1.

Choosing r sufficiently large, we can get from Corollary 3.1 that

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-y_\Delta(t)|^p\right)\le\mathcal{O}\left(\Delta^{(1/2-\varepsilon)p}\right),$$

and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\le\mathcal{O}\left(\Delta^{(1/2-\varepsilon)p}\right).$$

That is, the order of convergence can be arbitrarily close to $1/2$.

Run one hundred independent trajectories for every different step sizes, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}. The numerical solution of step size 10^{-6} is used as the true solution.

The truncation accuracy is selected as $\varepsilon=0.01$, and the order of convergence can be obtained as 0.49 from Corollary 3.1. Fig. 1 shows that the rate of strong convergence is close to 0.5. An application of the linear regression shows the slope is around 0.4866.

Example 5.2. Consider a two-dimensional time-changed SDE

$$\begin{cases} dY_1(t)=(-Y_1(t)-Y_2^3(t))dt+(Y_1^2(t)-2Y_2^5(t))dE(t)+Y_2^2(t)d(B(E(t))),\\ dY_2(t)=(-Y_2(t)-Y_1^3(t))dt+(Y_2^2(t)-2Y_1^5(t))dE(t)+Y_1^2(t)d(B(E(t))), \end{cases}$$

with $Y_1(0)=1$, $Y_2(0)=2$.

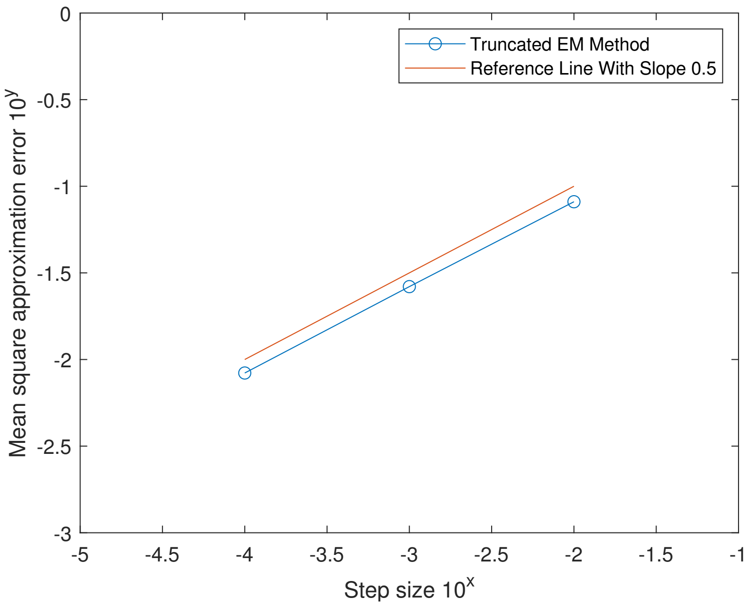

Figure 2: Convergence rate of Example 5.2.

It is clear that

$$a(x)=\begin{pmatrix}-x_1-x_2^3\\-x_2-x_1^3\end{pmatrix},\qquad b(x)=\begin{pmatrix}x_1^2-2x_2^5\\x_2^2-2x_1^5\end{pmatrix}\quad\text{and}\quad\sigma(x)=\begin{pmatrix}x_2^2\\x_1^2\end{pmatrix}.$$

Similar to Example 5.1, it is not hard to verify that the coefficients $a(x)$, $b(x)$ and $\sigma(x)$ satisfy Assumptions 2.1-2.5.

Due to that

$$\sup_{|x|\le u}\left(|a(x)|\vee|b(x)|\vee|\sigma(x)|\right)\le 3u^5,\quad \forall u\ge 1,$$

we choose $\mu(u)=3u^5$ and $h(\Delta)=\Delta^{-\varepsilon}$, for any $\varepsilon\in(0,1/4]$. As a result, we have $\mu^{-1}(u)=(u/3)^{1/5}$ and $\mu^{-1}(h(\Delta))=(\Delta^{-\varepsilon}/3)^{1/5}$. Choosing r sufficiently large, we can get from Corollary 3.1 that

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-y_\Delta(t)|^p\right)\le\mathcal{O}\left(\Delta^{(1/2-\varepsilon)p}\right),$$

and

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y(t)-\bar{y}_\Delta(t)|^p\right)\le\mathcal{O}\left(\Delta^{(1/2-\varepsilon)p}\right).$$

That is, the order of convergence can be arbitrarily close to $1/2$. We run one hundred independent trajectories for four different step sizes, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}. The numerical solution of step size 10^{-5} is used as the true solution.

The truncation accuracy is selected as $\varepsilon = 0.001$, and the order of convergence can be obtained as 0.499 from Corollary 3.1. Fig. 2 shows that the rate of strong convergence is close to 0.5. An application of the linear regression shows the slope is around 0.4943.

Example 5.3. Consider the time-changed SDE

$$dY(t) = -5Y^3(t)dt + (-Y(t) - 2Y^3(t))dE(t) + Y(t)dB(E(t)) \quad \text{with } Y(0) = 5. \tag{5.1}$$

It is not hard to see that

$$2xa(x) = -10x^4 \quad \text{and} \quad 2xb(x) + |\sigma(x)|^2 \le -4|x|^4, \quad \forall x \in \mathbb{R},$$

which imply that Assumption 4.1 is satisfied with $\kappa_1(u) = 10u^4$ and $\kappa_2(u) = 3u^4$. It can be seen that

$$\limsup_{|x|\downarrow 0} \frac{|a(x)|^2}{\kappa_1(|x|)} = 0 \qquad \text{and} \qquad \limsup_{|x|\downarrow 0} \frac{|b(x)|^2}{\kappa_2(|x|)} = 0,$$

which mean that the condition (4.4) is satisfied. We choose $h(\Delta) = 5\Delta^{-1/4}$ and $\mu(u) = 5u^3$ to define the numerical solution $y_\Delta(t)$.

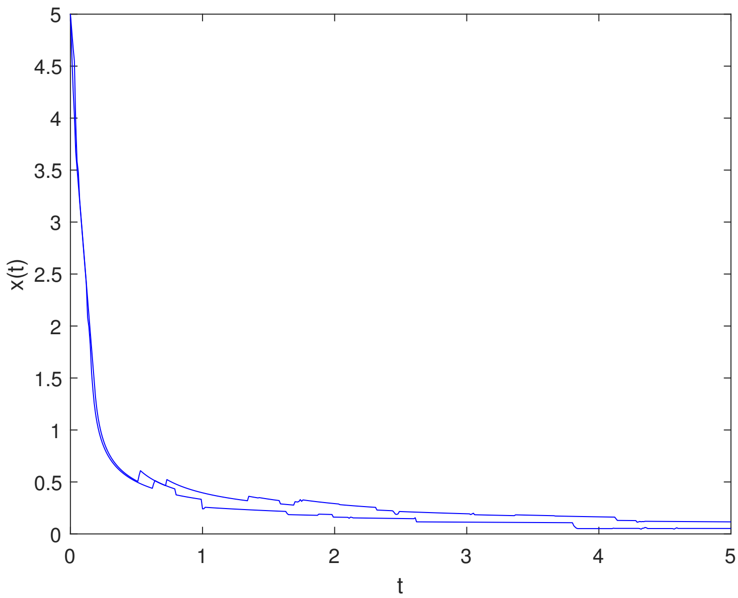

Figure 3: Two sample paths of $y_\Delta(t)$ in Example 5.3.

Applying Theorem 4.1, we can obtain that for every $\Delta \in (0, 4/25]$ and any initial value $x_0 \in \mathbb{R}$, the proposed numerical solution of the time-changed SDE (5.1) satisfies

$$\lim_{k\to\infty} y_{\Delta,k} = 0 \qquad a.s.$$

Fig. 3 shows two sample paths of the numerical solution. It is clear that the numerical solution is asymptotically stable.

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