Fluctuation and Noise Letters


# Parameter-related strong convergence rate of the backward Euler-Maruyama method for time-changed stochastic differential equations

Wei Liu, Meng Wang, Ruchun Zuo*
Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China

The strong convergence of the backward Euler-Maruyama method for time-changed stochastic differential equations with additive noise is investigated. This work focuses on the detailed study of the relationship between the strong convergence rate and the parameter, $\alpha \in (0,1)$, of the inverse subordinator. The strong convergence rate of $\alpha$ is proved and numerical simulations are provided to illustrate such an interesting observation.

Keywords: time-changed stochastic differential equations · backward Euler-Maruyama method · strong convergence rate · superlinear drift coefficient

## 1. Introduction

One way to construct subdiffusion processes is to change the time variable of Brownian motion into some inverse subordinator, and the resulting process is called time-changed Brownian motion. Stochastic differential equations (SDEs) driven by this process and the more general time-changed SDEs have been widely applied as a mathematical tool to model the phenomenon of anomalous diffusion [14].

Since explicit forms of true solutions can hardly be found, numerical methods always play a key role in applying the time-changed SDE models. In recent years, studies on numerical methods for time-changed SDEs have been blooming. Jum and Kobayashi investigated the Euler-Maruyama method for a class of time-changed SDEs by using the duality principle and obtained the strong convergence rate of a half [7]. Deng and Liu employed the similar idea to study the semi-implicit Euler–Maruyama method and also obtained the strong convergence rate of a half [2]. By directly discretizing the equations, Jin and Kobayashi studied both the Euler-type and Milstein-type methods, and obtained the convergence rates of a half and one, respectively [5,6]. The truncated Euler-type methods were also discussed to approximate this type of SDEs [10,12]. The split-step theta method was studied in [15].

The strong convergence rates that were obtained by all the existing results mentioned above are either a half for the Euler-type method or one for the Milstein-type

*Corresponding author: ruchunzuo@outlook.com

method, which are similar to those cases where classical SDEs driven by classical Brownian motion were studied. However, if one carefully looks at the construction of the time-changed Brownian motion (see Section 2 for details), it is clear that the inverse subordinated process that is used to change the time of the Brownian motion has a parameter $\alpha \in (0, 1)$.

In this paper, we focus on such a parameter $\alpha$ and prove that the strong convergence rate of the backward Euler-Maruyama method is related to such a parameter. Our intuition of such a result came from the numerical study on SDEs driven by $\alpha$-stable process [9], where the strong convergence rate is naturally related to the parameter $\alpha$.

To avoid too complicated notations, we only look at the time-changed SDE driven by additive noise in this work. It should be mentioned that thanks to the Lamperti transformation, our result could also cover some cases of multiplicative noise.

This work is constructed as follows. The mathematical preliminary is stated in Section 2. The main theorem and its proof are presented in Section 3. Some numerical simulations are provided in Section 4.

## 2. Mathematical Preliminary

In this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ represents a complete probability space, $D = (D(t)_{t\geq 0})$ is a one-dimensional, non-decreasing Lévy process starting at 0 with càdlàg paths, whose Laplace transform is

$$\mathbb{E}\left[e^{-sD_t}\right] = e^{-t\psi(s)}, \text{ where } \psi(s) = \int_0^\infty (1-e^{-sy})\,\nu(\mathrm{d}y),\ s > 0, \text{ and } \int_0^\infty (y\wedge 1)\nu(\mathrm{d}y) < \infty. \tag{1}$$

We consider the case where the Lévy measure $\nu$ is infinite, i.e., $\nu(0, \infty) = \infty$, meaning compound Poisson subordination is not considered. Let $E = (E(t))_{t\geq 0}$ be the inverse of $D$, i.e.

$$E(t) := \inf\{u > 0; D(u) > t\},\ t \geq 0.$$

If the subordinated process $D$ is stable with the parameter $\alpha \in (0, 1)$, then $\psi(s) = s^\alpha$ and $E$ is called the inverse $\alpha$-stable subordinated process. Assuming $\nu(0, \infty) = \infty$ implies that $D$ has strictly increasing paths with infinitely many jumps, thus $E$ has continuous, non-decreasing paths starting from 0. Moreover, the inverse relationship between $D$ and $E$ implies that for all $t,\ x \geq 0$, we have $\{E(t) > x\} = \{D(x) < t\}$.

Throughout this paper, we assume that $B(t)$ and $D(t)$ are independent. Let $\mathbb{E}_B$ and $\mathbb{E}_D$ denote the expectations with respect to $B(t)$ and $D(t)$, respectively. Due to the independence between $B(t)$ and $D(t)$, the expectation $\mathbb{E}$ can be expressed as

$$\mathbb{E}(\cdot) = \mathbb{E}_B\mathbb{E}_D(\cdot) = \mathbb{E}_D\mathbb{E}_B(\cdot).$$

The time-changed SDEs studied in this paper are of the form

$$dX(t) = f(X(t))dE(t) + \sigma dB(E(t)),\ t > t_0 > 0. \tag{2}$$

We impose the following assumptions.

Assumption 1. Suppose $f : I \to \mathbb{R}$ satisfies the monotonicity condition that $\forall x, y \in I$ with $x \le y$ there exists $K \in \mathbb{R}$ such that $f(y) - f(x) \le K(y - x)$, where $I$ could be $(c_1, c_2)$ for some $c_1 < c_2$ including $(-\infty, +\infty)$.

Assumption 2. For any $T > 0$, there exists a positive constant $C$ such that the drift coefficient of (2) satisfies

$$\sup_{t\in[0,T]} \mathbb{E}_B \left|f'(X(t))\right|^2 + \sup_{t\in[0,T]} \mathbb{E}_B \left|\left(f'f\right)(X(t)) + \frac{\sigma^2}{2} f''(X(t))\right| < C(T),$$

where $C(T)$ is a constant dependent on $T$.

Assumption 2, to some extension, is artificial at first glance. It is actually often employed when one studies the equations with additive noise [1,13].

Under Assumptions 1 and 2, the existence and uniqueness of the solution to equation (2) can be guaranteed. The proof is similar to that in [8].

The corresponding BEM method for (2) is

$$X_{t_{i+1}} = X_{t_i} + f(X_{t_{i+1}})\Delta E_i + \sigma \Delta B_{E_i}, \quad i = 0, 1, 2, \ldots, \qquad X_0 = X(t_0), \tag{3}$$

where $t_i = t_0 + i\Delta t$, $\Delta E_i = E(t_{i+1}) - E(t_i)$ and $\Delta B_{E_i} = B(E(t_{i+1})) - B(E(t_i))$.

Actually, by the Lamperti transformation, our result obtained for the (2) can also cover some time-changed SDEs with the multiplicative noise. More precisely, for a time-changed SDE of the form

$$dY(t) = a(Y(t))dE(t) + b(Y(t))dB(E(t)),$$

by the Lamperti transformation

$$F(X) = \lambda \int^X \frac{1}{b(Y)} dY, \quad \lambda > 0,$$

we can derive (2) with $X(t) = F(Y(t))$ and

$$f(X) = \lambda \left( \frac{a(F^{-1}(X))}{b(F^{-1}(X))} - \frac{1}{2} b'(F^{-1}(X)) \right).$$

At the end of this section, we cite a lemma from [4], which will be used in the proof of Theorem 1 in Section 3.

Lemma 1. For all $0 < a < b < \infty$ and all integers $n \ge 1$, we have

$$\mathbb{E}\left[|E(b) - E(a)|^n\right] \le C(b - a)^{1+(n-1)\alpha},$$

where $C$ is a constant.

## 3. Main Result

**Theorem 1.** *Under Assumptions 1 and 2, there exists a constant $C > 0$ independent of $\Delta t$ such that*

$$\sup_{i=0,1,\ldots,N} \mathbb{E}\left[|X(t_i) - X_{t_i}|\right] \le C\Delta t^{\alpha},$$

*where $N = T/\Delta t$. That is, the convergence rate is $\alpha$ in the $L^1$ sense.*

*Proof.* According to Theorem 3.3 in [8], we have the following expansion

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} f(X(t_{i+1}))\, dE(t) + \int_{t_i}^{t_{i+1}} \sigma\, dB(E(t)) + R_{i+1}^{(1)} + R_{i+1}^{(2)}, \tag{4}$$

where

$$R_{i+1}^{(1)} = -\int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} \left( f(X(s))f'(X(s)) + \frac{1}{2}\sigma^2 f''(X(s)) \right) dE(s)\, dE(t),$$
$$R_{i+1}^{(2)} = -\int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} \sigma f'(X(s))\, dB(E(s))\, dE(t).$$

From (3) and (4), we derive

$$X(t_{i+1}) - X_{t_{i+1}} = X(t_i) - X_{t_i} + (f(X(t_{i+1})) - f(X_{t_{i+1}}))\Delta E_i + R_{i+1}^{(1)} + R_{i+1}^{(2)}.$$

Set $e_i = X(t_i) - X_{t_i}$. By Assumption 1, we obtain

$$(1 - K\Delta E_s)e_{s+1} \le e_s + R_{s+1}^{(1)} + R_{s+1}^{(2)}, \quad \text{where } s = 0, 1, \ldots, N-1. \tag{5}$$

By iteration, we have

$$|e_N| \le \sum_{j=0}^{N-1} \left| R_{j+1}^{(1)} + R_{j+1}^{(2)} \right| \prod_{l=j+1}^{N-1} |1 - K\Delta E_l|\,. \tag{6}$$

By Lemma 1, for some constant $L_1 > 0$ we have

$$\mathbb{E}\left[ |E(t) - E(s)|^2 \right] \le L_1 \, |t-s|^{1+\alpha}\,.$$

Then by Theorem 7.1.3 of [3], for some constant $L_2 > 0$ we have

$$|E(t) - E(s)| \le L_2 \, |t-s|^{\gamma} \quad \text{a.s.} \tag{7}$$

Now we can estimate (6) by

$$\sup_{k=1,2,\ldots,N} \prod_{l=0}^{k} (1 - K\Delta E_l)^{-1} < \sup_{k=1,2,\ldots,N} \prod_{l=0}^{k} (1 - KL_2\Delta t^{\gamma})^{-1} < C,$$

where the last inequality is obtained by taking the $\Delta t \to 0$. Here C represents any $\Delta t$-independent constant whose exact value is unimportant. Taking expectation and supreme on both sides of (6), we obtain

$$\sup_{k=0,1,\ldots,N} \mathbb{E}\left[|e_k|\right] \le C\mathbb{E} \sum_{j=0}^{N-1} \left| R_{j+1}^{(1)} \right| + C\mathbb{E} \sum_{j=0}^{N-1} \left| R_{j+1}^{(2)} \right|. \tag{8}$$

Since the Brownian motion $B$ and the subordinator $D$ are independent, it is clear that $\mathbb{E}(\cdot) = \mathbb{E}_D(\mathbb{E}_B(\cdot))$.

For the first term on the right hand side of the inequality in (8), by Assumption 2 we can obtain

$$\begin{aligned}
\mathbb{E}\left[\left|R_{j+1}^{(1)}\right|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\mathbb{E}_B\left(f(X(s))f'(X(s)) + \frac{1}{2}\sigma^2 f''(X(s))\right)dE(s)\,dE(t)\right] \\
&\le C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}C(T)dE(s)\,dE(t)\right] \\
&\le C\Delta t^{1+\alpha},
\end{aligned}$$

which results in

$$\sum_{j=0}^{N-1}\mathbb{E}\left|R_j^{(1)}\right| \le C\sum_{j=0}^{N-1}\Delta t^{1+\alpha} \le C\Delta t^{\alpha}. \tag{9}$$

For the second term on the right hand side of the inequality in (8), by Assumption 2 and Lemma 1, we can get

$$\begin{aligned}
\mathbb{E}\left[\left|R_{j+1}^{(2)}\right|\right] &= \mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\mathbb{E}_B\left[\int_{t_i}^{t}\sigma f'(X(s))dB_{E(s)}\right]dE(t)\right] \\
&\le C\mathbb{E}_D\left[\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}\mathbb{E}_B\left(\sigma f'(X(s))\right)^2 dE(s)\right]^{\frac{1}{2}}dE(t)\right] \\
&\le C\mathbb{E}_D\int_{t_i}^{t_{i+1}}\left[\int_{t_i}^{t}C(T)dE(s)\right]^{\frac{1}{2}}dE(t) \\
&\le C\Delta t^{\frac{1}{2}+\alpha}.
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\sum_{j=0}^{N-1}\left|R_j^{(2)}\right| \le C\mathbb{E}\left|\sum_{j=0}^{N-1}(R_j^{(2)})^2\right|^{\frac{1}{2}} \le C\left|\sum_{j=0}^{N-1}\mathbb{E}(R_j^{(2)})^2\right|^{\frac{1}{2}} \le C\left|\sum_{j=0}^{N-1}\Delta t^{1+2\alpha}\right|^{\frac{1}{2}} \le C\Delta t^{\alpha}. \tag{10}$$

Finally, by (9) and (10), we complete the proof. □

## 4. Simulations

Consider a time-changed CIR model

$$dY(t) = \kappa(\theta - Y(t))dE(t) + \sigma Y^{\frac{1}{2}}(t)dB(E(t)),$$

where $\kappa$, $\theta$ and $\sigma$ are positive constant. By the Lamperti transformation, we can rewrite it into

$$dX(t) = \frac{1}{2}\kappa\left(\theta_v X^{-1}(t) - X(t)\right)dE(t) + \frac{1}{2}\sigma dB(E(t)), \tag{11}$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{4\kappa}\right)$

It is not hard to see that the drift coefficient of (11) satisfies Assumption 1 on $[0, \infty)$.

Now, we focus on Assumption 2. It is clear that

$$f'(X) = -\frac{1}{2}\kappa(\theta_v X^{-2} + 1)$$

and

$$f(X)f'(X) + \frac{\sigma^2}{2}f''(X) = \frac{\kappa^2}{4}(\theta_v^2 X^{-3} - X) + \frac{1}{2}\kappa\theta_v\sigma^2 X^{-3}.$$

Hence, to make Assumption 2 satisfied, it suffices to show

$$\sup_{t\in[0,T]} \mathbb{E}[X^{-3}(t)] = \sup_{t\in[0,T]} \mathbb{E}[Y^{-\frac{3}{2}}(t)] < C(T). \tag{12}$$

The following proposition clarifies that (12) is true.

Proposition 1. For the time-changed CIR process with $Y_0 > 0$, there exists a constant $C(T)$ dependent on $T$ such that

$$\sup_{t\in[0,T]} \mathbb{E}_B\left[Y^{-p}(t)\right] \le C(T),$$

where $1 < p < \frac{2\kappa\theta}{\sigma^2} - 1$.

Proof. Define the stopping time $\tau_n = \inf\{0 < s \le T; Y(s) \le 1/n\}$. By Itô's formula, we have

$$\begin{aligned}
\mathbb{E}_B\left[Y^{-p}(t\wedge\tau_n)\right] &= Y^{-p}(0) - p\mathbb{E}_B\left[\int_0^{t\wedge\tau_n} \frac{\kappa(\theta - Y(s))}{Y^{p+1}(s)}\mathrm{d}E(s)\right] \\
&\quad + p(p+1)\frac{\sigma^2}{2}\mathbb{E}_B\left[\int_0^{t\wedge\tau_n} \frac{1}{Y^{p+1}(s)}\mathrm{d}E(s)\right] \\
&\le Y^{-p}(0) + p\kappa\int_0^t \mathbb{E}_B\left(\frac{1}{Y^p(s\wedge\tau_n)}\right)\mathrm{d}E(s) \\
&\quad + \mathbb{E}_B\left[\int_0^{t\wedge\tau_n} \frac{p\left(\frac{(p+1)\sigma^2}{2} - \kappa\theta\right)}{Y^{p+1}(s)}\mathrm{d}E(s)\right].
\end{aligned}$$

Since $\frac{(p+1)\sigma^2}{2} - \kappa\theta < 0$, we have

$$\mathbb{E}_B\left[Y^{-p}(t\wedge\tau_n)\right] \le Y^{-p}(0) + p\kappa\int_0^t \sup_{r\in[0,s]} \mathbb{E}_B\left[Y^{-p}(r\wedge\tau_n)\right]\mathrm{d}E(s).$$

Applying Gronwall's inequality yields

$$\sup_{t\in[0,T]} \mathbb{E}_B\left[Y^{-p}(t\wedge\tau_n)\right] \le Y^{-p}(0)\exp(p\kappa E(T)) \le Y^{-p}(0)\exp(p\kappa T),$$

where $E(t) \le E(T) \le T$ (see for example [11]) is used to obtain the second inequality. Letting $n \to +\infty$ completes the proof. □

In our numerical experiment, we set $\theta = 0.125$, $\kappa = 8$ and $\sigma = 0.5$. Then, we have $\frac{2\kappa\theta}{\sigma^2} = 4 > 2.5$. By Proposition 1, we can see $\sup_{t\in[0,T]} \mathbb{E}[Y^{-\frac{3}{2}}(t)] < C(T)$, which indicates that Assumption 2 is satisfied. Consequently, by Theorem 1, the BEM method is convergent with the rate of $\alpha$.

1000 sample paths are simulated with step sizes $2^{-7}$, $2^{-8}$, $2^{-9}$, $2^{-10}$ and the reference solution is computed using step size $2^{-13}$. The empirical convergence rates at $T = 1$ are displayed in Table 1, which are in line with the theoretical result. We also present 5 sample paths generated by our algorithm in Figure 1 to display the behavior of the solution.

Table 1: Convergence rates for different values of $\alpha$

| $\alpha$ | 0.3000 | 0.4000 | 0.5000 | 0.6000 | 0.7000 | 0.8000 | 0.9000 |
|---|---|---|---|---|---|---|---|
| rate | 0.2923 | 0.4057 | 0.5002 | 0.5834 | 0.6931 | 0.8086 | 0.9317 |

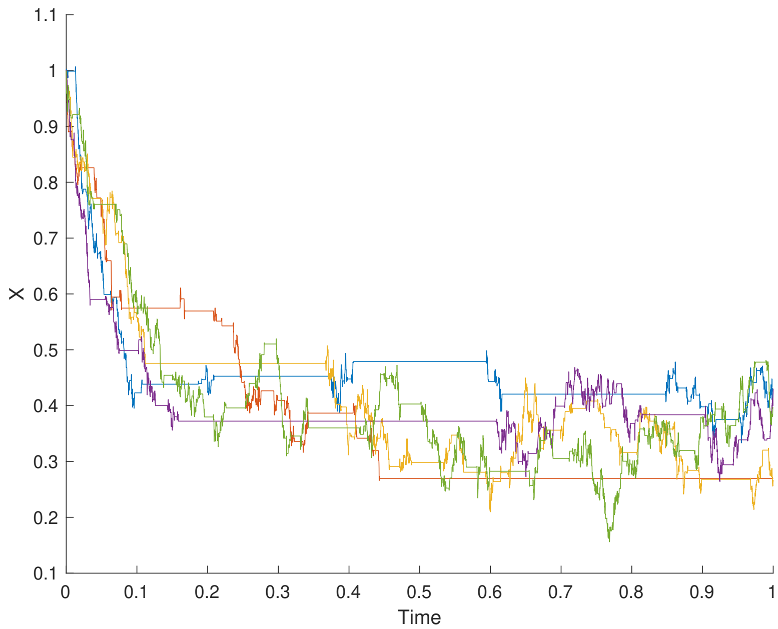


Figure 1: Sample paths with step size $2^{-13}$ and $\alpha = 0.8$

Data availability

Data are generated using codes that can be found in our GitHub repository.

## Acknowledgment

The authors appreciate the valuable comments and suggestions from the reviewers and the editor.

Wei Liu is supported in part by Shanghai Rising-Star Program under Grant No. 22QA1406900 and the National Natural Science Foundation of China under Grant No. 12271368.

## Bibliography

[1] A. Alfonsi. Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process. Stat. Probab. Lett., 83(2):602 – 607, 2013. Cited by: 67; All Open Access, Green Open Access.

[2] C.-S. Deng and W. Liu. Semi-implicit Euler-Maruyama method for non-linear time-changed stochastic differential equations. BIT Num. Math., 60:1133–1151, 2020.

[3] R. Durrett. Probability: Theory and Examples, volume 49. Cambridge university press, 2019.

[4] J. Gajda and M. Magdziarz. Large deviations for subordinated Brownian motion and applications. Statist. Probab. Lett., 88:149–156, 2014.

[5] S. Jin and K. Kobayashi. Strong approximation of stochastic differential equations driven by a time-changed Brownian motion with time-space-dependent coefficients. J. Math. Anal. Appl., 476(2):619–636, 2019.

[6] S. Jin and K. Kobayashi. Strong approximation of time-changed stochastic differential equations involving drifts with random and non-random integrators. BIT Num. Math., 61(3):829–857, 2021.

[7] E. Jum and K. Kobayashi. A strong and weak approximation scheme for stochastic differential equations driven by a time-changed Brownian motion. Probab. Math. Statist, 36(2):201–220, 2016.

[8] K. Kobayashi. Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations. J. Theor. Probab., 24:789–820, 2011.

[9] F. Kühn and R. L. Schilling. Strong convergence of the Euler-Maruyama approximation for a class of Lévy-driven SDEs. Stochastic Process. Appl., 129(8):2654–2680, 2019.

[10] X. Li, J. Liao, W. Liu, and Z. Xing. Convergence and stability of an explicit method for autonomous time-changed stochastic differential equations with super-linear coefficients. Adv. Appl. Math. Mech., 15(3):651–683, 2023.

[11] Z. Li, B. Huang, J. Zhao, and L. Xu. Transportation inequalities for stochastic differential equations driven by the time-changed Brownian motion. J. Dyn. Control Syst., 29(4):1571–1583, 2023.

[12] W. Liu, X. Mao, J. Tang, and Y. Wu. Truncated Euler-Maruyama method for classical and time-changed non-autonomous stochastic differential equations. Appl. Numer. Math., 153:66–81, 2020.

[13] A. Neuenkirch and L. Szpruch. First order strong approximations of scalar SDEs defined in a domain. Numer. Math., 128:103–136, 2014.

[14] S. Umarov, M. Hahn, and K. Kobayashi. Beyond the Triangle: Brownian Motion, Itô Calculus, and Fokker-Planck Equation-Fractional Generalizations. World Scientific, 2018.

[15] D. Wu, Z. Li, L. Xu, and C. Peng. Mean square stability of the split-step theta method for non-linear time-changed stochastic differential equations. Appl. Anal., 103(9):1733–1750, 2024.