Notes on MLE, Fisher's information and robust SE

Let $X_1, ..., X_n \sim$ i.i.d. Exp(θ) (Exponential distribution with mean $1/\theta$).

The likelihood function for a sample of size n is

$$L(\theta) = \prod_{i=1}^{n} \theta \exp(-\theta x_i) = \theta^n \exp(-\theta \sum_{i=1}^{n} x_i)$$

The log-likelihood function is

$$\ell_n(\theta) = \log L(\theta) = n \log \theta - \theta \sum_{i=1}^{n} x_i$$

The score function is

$$S_n = \frac{\partial l_n(\theta)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i$$

The second derivative of the log-likelihood function is

$$\frac{\partial^2 \ell(\theta)}{\partial \theta^2} = \frac{-n}{\theta^2}$$

The MLE $\hat{\theta}$ can derived from letting the score function equal to 0:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}_n}$$

The expected Fisher's information $\mathcal{I}(\theta)$ under the true model is

$$\mathcal{I}_n(\theta) = Var(S_n) = \mathbb{E}(S_n^2) = -\left(\frac{\partial^2 \ell(\theta)}{\partial \theta^2}\right) = \frac{n}{\theta^2}$$

An estimate of the expected Fisher's information is

$$\widehat{\mathcal{I}_n(\theta)} = \frac{n}{\hat{\theta}^2} = n\bar{x}_n^2$$

The limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is

$$\begin{aligned}\sqrt{n}(\hat{\theta} - \theta) &\xrightarrow{d} N(0, \frac{1}{\mathcal{I}(\theta)}) \\ &\xrightarrow{d} N(0, \theta^2)\end{aligned}$$

The limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$ IS NOT

$$\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0, \frac{1}{\widehat{\mathcal{I}(\theta)}})$$
$$\xrightarrow{d} N(0, \hat{\theta}^2)$$
$$\xrightarrow{d} N(0, \frac{1}{\bar{x}_n^2})$$

Because there is no sample average in the limit; it turns into the population average.

An asymptotically valid 95% confidence interval for $\hat{\theta}$ is

$$\hat{\theta} \pm 1.96\sqrt{\frac{\hat{\theta}^2}{n}}$$

When $\hat{\theta}$ is replaced with MLE, we can write the confidence interval as

$$\frac{1}{\bar{x}_n} \pm \sqrt{\frac{1}{n\bar{x}_n^2}}$$

When a robust estimate is of interest, we can write down a as

$$a = -\mathbb{E}\left(\frac{\partial^2 \ell(x_i;\theta)}{\partial\theta^2}\right) = \frac{1}{\theta^2}$$

An estiamte for a would be

$$\hat{a} = -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \ell(x_i;\hat{\theta})}{\partial\theta^2} = \frac{1}{1/\bar{x}_n^2} = \bar{x}_n^2$$

Meanwhile, b is

$$b = \mathbb{E}(S_i^2) = \mathbb{E}(\frac{1}{\theta} - \bar{x}_n)^2 = \frac{1}{\theta^2} + \mathbb{E}(x_i^2) - \frac{2\mathbb{E}(x_i)}{\theta} = \frac{1}{\theta^2} + Var(x_i) + (\mathbb{E}(x_i))^2 - \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

An estiamte for b would be

$$\hat{b} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial \ell(x_i;\hat{\theta})}{\partial\theta}\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\hat{\theta}} - x_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$$

Therefore,

$$\frac{\hat{b}}{\hat{a}^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}{n\bar{x}_n^4}$$