Ind. hyp.: Assume any set of size k has 2^k subsets, for some $k \in \mathbb{N}$

Ind. step: Let A be a set of size $k+1$. Let $x \in A$.
Partition the subsets of A into X and Y where X consists of subsets of A that include x and Y consits of subsets of A that do not have x.
Y is the set of all subsets of $A \setminus \{x\}$. Since $|A \setminus \{x\}| = k$, by ind. hyp, $|Y| = 2^k$. Each element in X consists of $\{x\}$ union an element in Y. So $|X| = |Y| = 2^k$.
So A has $|X| + |Y| = 2^k + 2^k = 2^{k+1}$ subsets.
By induction, the result holds $\square$

Visualization: $A = \{1,2,3\}$ $\quad X = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$
$\qquad\qquad\qquad\qquad\qquad\quad Y = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

<u>Strong Induction</u>

Example: Let $\{a_n\}_{n\geq 0}$ be the sequence $a_1 = 4$, $a_2 = 10$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = -2 + 3 \cdot 2^n$ for all $n \in \mathbb{N}$

Proof: By induction on n.

Base case: When $n=1$, $a_1 = 4$, $-2+3\cdot 2^1 \stackrel{?}{=} 4$ ✓

Check $n=2$
$a_2 = 10$
$-2+3\cdot 2^2 = 10$ ✓

Ind hyp: Assume for some $k \in \mathbb{N}$, $a_k = \cancel{-2+3\cdot 2^k}$ $a_i = -2+3\cdot 2^i$ $\forall\, 1 \leq i \leq k$
$k \geq 2$

Ind. step: $a_{k+1} = 3a_k - 2a_{k-1}$

only valid if $k+1 \geq 3$, $k \geq 2$

$$= 3(-2+3\cdot 2^k) - 2(-2+3\cdot 2^{k-1})$$
only possible with strong induction
$$= -6 + 9\cdot 2^k + 4 - 3\cdot 2^k$$
$$= -2 + 3\cdot 2^k(3-1)$$
$$= -2 + 3\cdot 2^{k+1}$$

By strong ind., the result holds $\square$

To prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction...

1. (Base cases) $P(1), P(2), \ldots, P(b)$ are true for some $b \in \mathbb{N}$
[Anything that the ind. step is not applied to goes in base cases]
2. (Ind. step) For all integer $k \geq b$, if $P(1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

Base Case $\quad P(1), \ldots, P(b)$ True

Ind Step $\quad (k=b)\ P(1), \ldots, P(b) \Rightarrow P(b+1)$
$\qquad\qquad (k=b+1)\ P(1), \ldots, P(b+1) \Rightarrow P(b+2)$
$\qquad\qquad\qquad \vdots$
$\qquad\qquad\qquad P(1), \ldots, P(k) \Rightarrow P(k+1)$

Example: Prove that any collection of at least 8 candies can be divided into piles of 3's and/or 5's.

Base cases: 8 candies $\rightarrow$ 3, 5

9 candies $\rightarrow$ 3, 3, 3

10 candies $\rightarrow$ 5, 5

Ind. hyp: Assume any set of i candies $8 \leq i \leq k$ can be divided into 3's and 5's, for some $k \geq 10$

Ind. step: $k+1$ candies put aside 3, $k-2$ candies left. By ind. hyp, $k-2$ candies can be split into 3's and 5's $\square$ (Note: only valid when $k-2 \geq 8$, $k \geq 10$)

Example: Prove that every integer at least 2 is a product of at least one primes.

Proof: Base case: When $n=2$, 2 is a prime

No more base cases needed

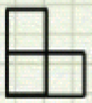
Ind hyp: For some $k \in \mathbb{N}$, $k \geq$, every integer between 2 and k is a product of primes

Ind. step: Consider $k+1$. If $k+1$ is prime, then it is already a product of primes.

If $k+1$ is not prime, then $k+1 = ab$ where $a, b \in \mathbb{N}$, $2 \leq a, b$ and $a, b < k+1$. By ind. hyp, both a and b are products of primes. Since $k+1$ is a product of a, b, $k+1$ is a product of primes $\square$

Ind. step applies to all int ≥ 3 since each int is either prime or composite. So only one base case is needed.

Example: A unit square has been removed from a $2^n \times 2^n$ board. Prove that the board can be tiled using L-shaped triominoes

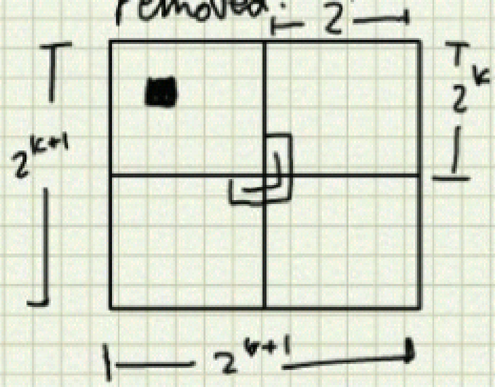


Proof: By induction on n,

Base case: When $n=1$, any 2×2 board with a square removed is already a triomino.

Ind. hyp: Assume true for some $k \in \mathbb{N}$.

Ind step: Suppose we have a $2^{k+1} \times 2^{k+1}$ board with a square removed.



Square A is a $2^k \times 2^k$ board minus a square. So A can be tiled by ind hyp.
Put a ⊞ in the center, the remaining three boards can be tiled by ind. hyp. $\square$