Last time: Parametric Representation

$\vec{r}(t) = (x(t), y(t))$ is the <u>position</u> vector.

Note: $\vec{r}: \mathbb{R} \to \mathbb{R}^3$

To find the velocity & acceleration.

$\vec{v}(t) = \vec{r}'(t) = (x'(t), y'(t))$

$\vec{a}(t) = \vec{r}''(t) = (x''(t), y''(t))$

## <u>The Chain Rule</u>

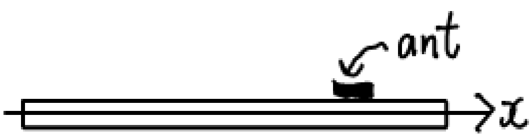
- multiple versions
- focus on the simplest for now.

Recall single variable case.

If $T = T(x)$, $x = x(t)$, then $t$ is a fn. of $t$, &

$$\frac{dT}{dt} = \frac{dT}{dx}\frac{dx}{dt}$$

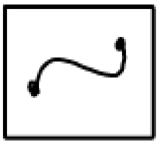
Imagine a rod lying on the $x$-axis with temperature $T(x)$. An ant crawls along the rod with position $x(t)$.




Then $\frac{dt}{dx}$ represents rate of change (r.o.c.) of temp of the rod wrt position (doesn't depend on ant). $\frac{dx}{dt}$ represents r.o.c. of pos. wrt time, or velocity. (doesn't depend on rod)

$\frac{dT}{dt}$ represents r.o.c. of temp w.r.t. time felt by the ant.

Now, Imagine a plate. Let the temp. be $T(x,y)$.



Let the ant have position $(x(t), y(t))$.
What is the r.o.c. of temp wrt time felt by the ant?

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

can write as:

$$\frac{dT}{dt} = \underbrace{\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right)}_{\text{gradient vector}} \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}_{\text{velocity vector}}$$

dot product

$$\frac{dT}{dt} = \nabla T \cdot \vec{v}$$ (note analogy to single variable case)

Def$^n$: The gradient vector for a function $f: \mathbb{R}^2 \to \mathbb{R}$ is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

"nabla", "del", or "grad"

For a definition $f: \mathbb{R}^3 \to \mathbb{R}$ $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

## Directional Derivatives

How many directions does it make sense to talk about a rate of change in?

- infinite #

**Def$^n$:** The <u>directional derivative</u> of $f: \mathbb{R}^2 \to \mathbb{R}$ at a point $\underline{a} = (a, b)$ in the direction of a unit vector $\hat{u}$ is:

$$D_{\hat{u}} f(\underline{a}) = \lim_{h \to 0} \frac{f(\underline{a} + h\hat{u}) - f(\underline{a})}{h}$$

Note: $\underline{a} + h\hat{u} = (a + hu_1, b + hu_2)$

where $\hat{u} = (u_1, u_2)$

We'll rarely use this, except to show the following:

If $\hat{u} = (1, 0)$ ($x$-direction)

$$D_{\hat{u}} f(\underline{a}) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \frac{\partial f}{\partial x}(a, b)$$

Easier way to compute:

$$D_{\hat{u}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \hat{u}$$

<u>Ex.</u> Find $D_{\hat{u}} f(\underline{a})$ of $f(x, y) = x^2 y^3 - 4y$ at the point $\underbrace{(2, -1)}_{\underline{a}}$ in the direction of the origin.

<u>Sol$^n$.</u> $\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2xy^3, 3x^2y^2 - 4)$

$\nabla f(2, -1) = (-4, 8)$

A vector pointing from $(2, -1)$ to $(0, 0)$ is $\vec{v} = (-2, 1)$.

A unit vector is $\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(-2, 1)}{\sqrt{2^2 + 1^2}} = \left( \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$

$$D_{\hat{u}} f(\underline{a}) = \nabla f(\underline{a}) \cdot \hat{u} = (-4, 8) \cdot \left( -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$= \frac{16}{\sqrt{5}}$$

# Physical Interpretation of the Gradient in 2D

- Recall the property of the dot product

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\theta$, where $\theta$ is the angle between $\vec{a}$ & $\vec{b}$.

Then,

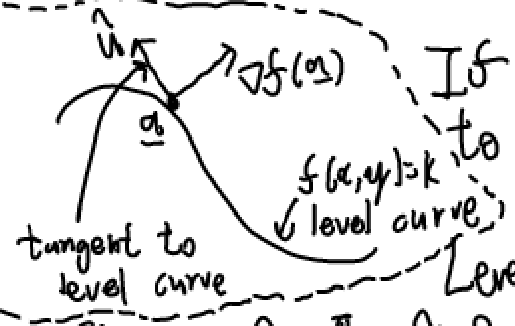
$D_{\hat{u}}f(\underline{a}) = \|\nabla f(\underline{a})\|\cdot\underbrace{\|\hat{u}\|}_{1}\cos\theta$, where $\theta$ is the angle between $\nabla f(\underline{a})$ and $\hat{u}$.

$$D_{\hat{u}}f(\underline{a}) = \|\nabla f(\underline{a})\|\cdot\cos\theta$$

For which values of $\theta$ is $D_{\hat{u}}f(\underline{a})$ maximized, minimized, and zero?

- $\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow D_{\hat{u}}f(\underline{a})$ is maximized (value $\|\nabla f\|$)
- $\theta = \pi \Rightarrow \cos\theta = -1 \Rightarrow D_{\hat{u}}f(\underline{a})$ is minimized (value $-\|\nabla f\|$)
- $\theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0 \Rightarrow D_{\hat{u}}f(\underline{a})$ is zero.

How does this relate to level curves?




If $\hat{u}$ is in the direction of the tangent to the level curve, $D_{\hat{u}}f(\underline{a}) = 0$.

Level curve $\rightarrow f$ does not change.

Since $\theta = \frac{\pi}{2}$, $\hat{u}$ & $\nabla f$ are orthogonal.

So the gradient points in the direction of the maximum rate of change of $f$.

i.e. $D_{\hat{u}}f(\underline{a})$ is maximized when $\hat{u}$ points in the direction of $\nabla f(\underline{a})$.

$\nabla f(\underline{a})$ is orthogonal to the level curve at $\underline{a}$.
$\nabla f(\underline{a})$ represents the direction of the maximum rate of change of $f$, with magnitude $\|\nabla f(\underline{a})\|$.