Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Dictionary ADT

A *dictionary* is a collection of *items*,
each of which contains a *key* and some *data*
and is called a *key-value pair* (KVP).
Keys can be compared and are (typically) unique.

Operations:

- *search*(k)
- *insert*(k, v)
- *delete*(k)
- optional: *join*, *isEmpty*, *size*, *etc.*

Dictionary

↳ ADT → Data → a collection of key/value pairs

↳ operation

↳ insert
search
delete

Data Structures for Dictionaries:

An unsorted array

↳ insert: → $O(1)$ amortized, insert at the end
search: → $\Theta(n)$
delete: → $\Theta(n)$ { have to search

Linked List

↳ insert → $O(1)$
search → $\Theta(n)$
delete → $\Theta(n)$

Sorted array

↳ insert → $O(n)$ { have to shift right after insert}
search → $O(\log n)$ { binary search}
delete → $O(n)$ { have to shift left after insert}

Elementary Implementations

Common assumptions:

- Dictionary has n KVPs
- Each KVP uses constant space
- Comparing keys takes constant time

Unordered array or linked list

search $\Theta(n)$
insert $\Theta(1)$
delete $\Theta(n)$ (need to search)

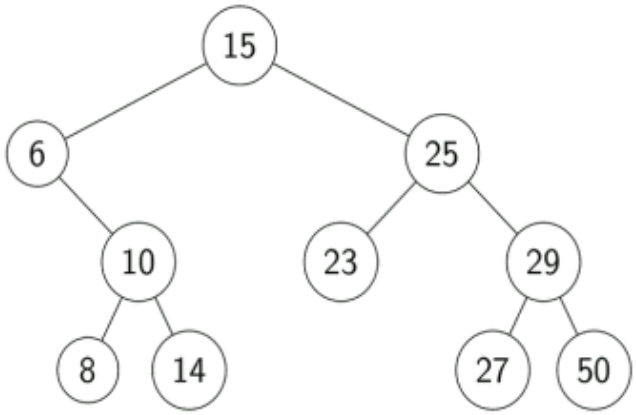
Ordered array

search $\Theta(\log n)$
insert $\Theta(n)$
delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

Ordering Every key k in $T.left$ is less than the root key.
Every key k in $T.right$ is greater than the root key.



BST Search and Insert

search(k) Compare k to current node, stop if found, else recurse on subtree unless it's empty

insert(k, v) Search for k, then insert (k, v) as new node

Example:

BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up
- Else, swap with *successor* or *predecessor* node and then delete

Height of a BST

search, *insert*, *delete* all have cost $\Theta(h)$, where
h = height of the tree = max. path length from root to leaf

If n items are *insert*ed one-at-a-time, how big is h?

- Worst-case: $n-1 = \Theta(n)$
- Best-case: $\lceil \lg(n) \rceil = \Theta(\log n)$
- Average-case: $\Theta(\log n)$
  (just like recursion depth in quick-sort!)

AVL Trees

Introduced by Adel'son-Vel'skiĭ and Landis in 1962,
an *AVL Tree* is a BST with an additional structural property:
The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be -1.)

At each non-empty node, we store $height(R) - height(L) \in \{-1, 0, 1\}$:

-1 means the tree is *left-heavy*

0 means the tree is *balanced*

1 means the tree is *right-heavy*

- We could store the actual height, but storing balances is simpler and more convenient.

$H(n) = 1 + \frac{1}{n}\sum_{i=0}^{n-1} \max(H(i), H(n-i-1))$ i is the index of root in sorted array

$= 1 + \frac{1}{n}\left(\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} H(n-i-1) + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} H(i)\right)$

$= 1 + \frac{1}{n}\left(\sum_{j=n-\lceil \frac{n}{2} \rceil}^{n-1} H(j) + \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} H(i)\right)$

$= 1 + \frac{2}{n}\sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} H(i)$

$= 1 + \frac{2}{n}\left(\underbrace{\sum_{i=\lceil \frac{n}{2} \rceil}^{\lceil 3n/4 \rceil} H(i)}_{\frac{n}{4} * H(\frac{3n}{4})} + \underbrace{\sum_{i=\lceil \frac{3n}{4} \rceil + 1}^{n-1} H(i)}_{\frac{n}{4} * H(n)}\right)$

$H(n) \leq 1 + \frac{2}{n} * \frac{n}{4} + H(\frac{3n}{4}) + \frac{2}{n} * \frac{n}{4} * H(n)$

$\leq 2 + H(\frac{3n}{4})$

$\leq 2 + 2 + H(\frac{9n}{16})$

$\vdots$

$\leq \underbrace{2+2+2\ldots+3}_{\frac{\log_4 n}{3} \text{ times}} \in O(\log n)$

can't be better than $\log(n)$

$\therefore H(n) \in \Theta(\log n)$

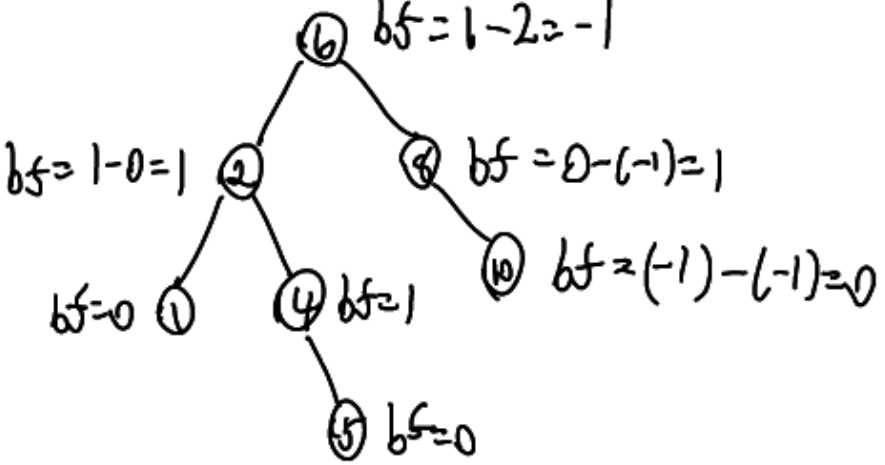
Balanced BSTs

↳ The goal is to have height ($\log n$) always

Balanced Factor (BF)

BF(n) = height(n→right) − height(n→left)

height(null) = −1



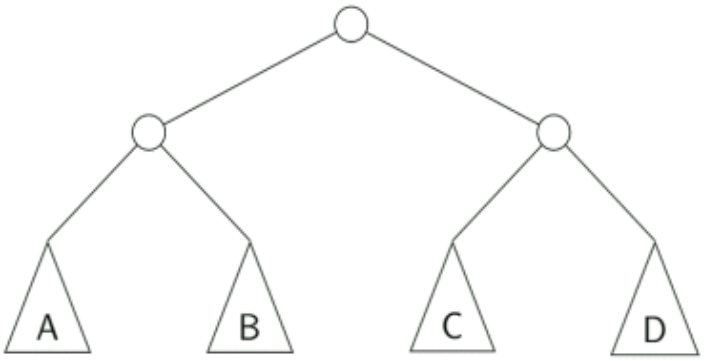
AVL insertion

To perform *insert*(T, k, v):

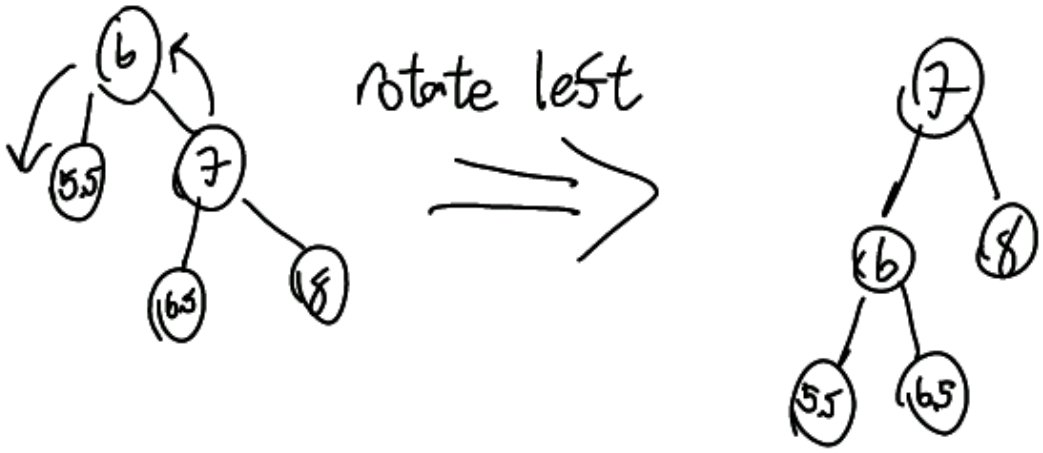
- First, insert (k, v) into T using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is -1, 0, or 1, then keep going.
- If the balance factor is ± 2, then call the *fix* algorithm to "rebalance" at that node.

How to "fix" an unbalanced AVL tree

Goal: change the *structure* without changing the *order*



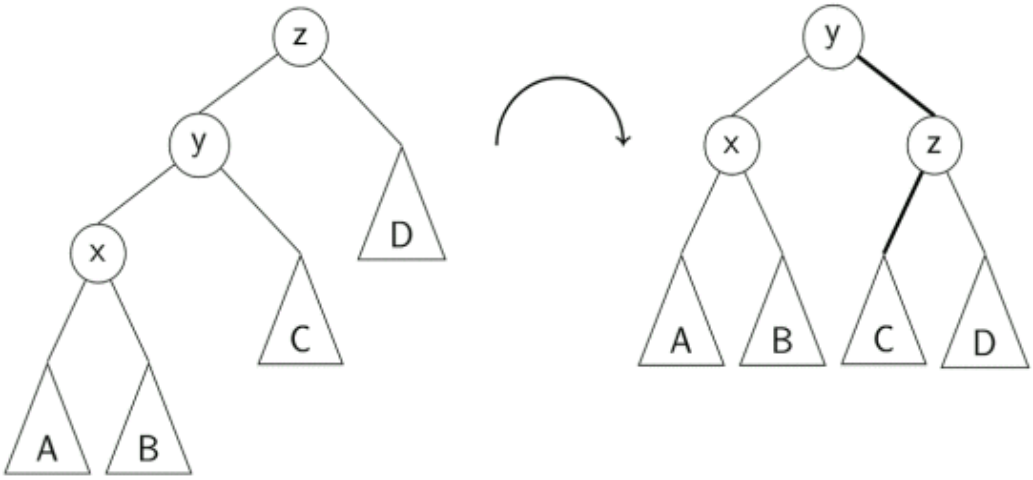
Notice that if heights of A, B, C, D differ by at most 1, then the tree is a proper AVL tree.


6
5.5
7
6.5
8
rotate left
7
6
8
5.5
6.5


10
8
15
7
9
rotate right
8
7
10
9
15

Right Rotation

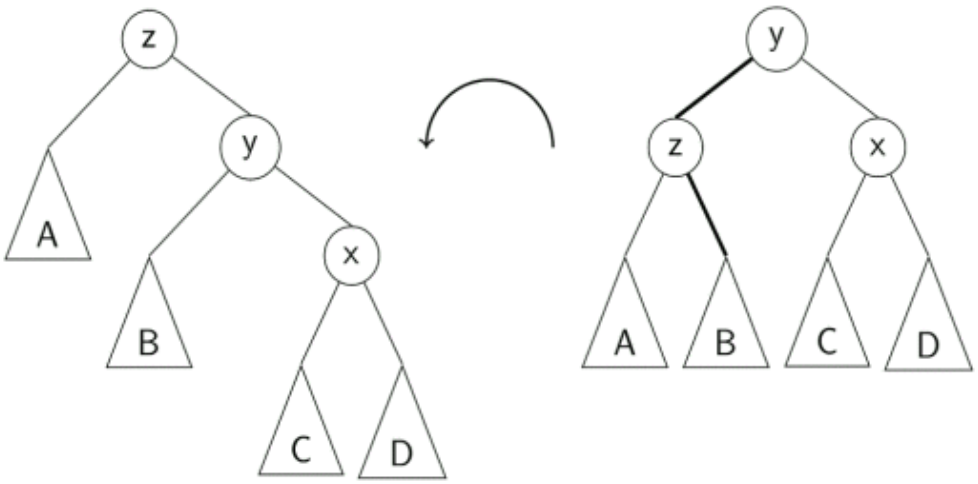
This is a *right rotation* on node z:



Note: Only two edges need to be moved, and two balances updated.

Left Rotation

This is a *left rotation* on node z:



Again, only two edges need to be moved and two balances updated.

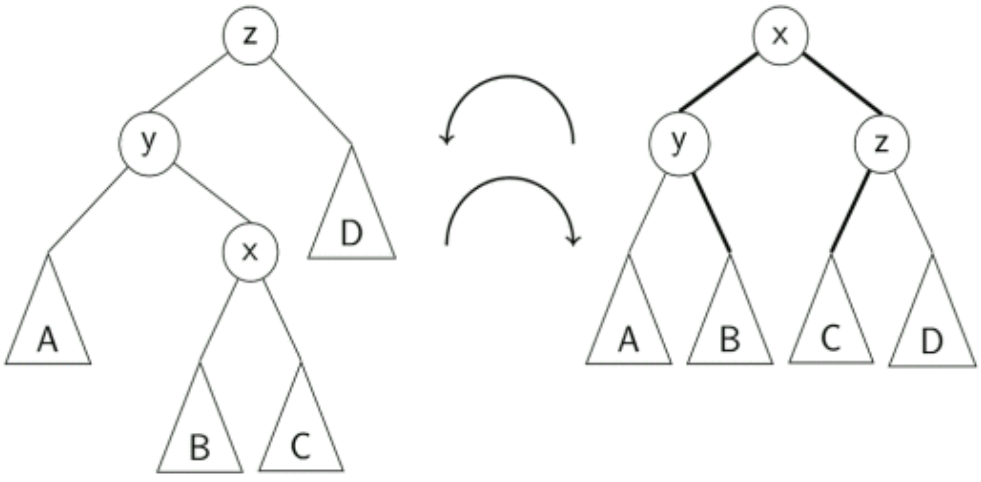
Pseudocode for rotations

```
rotate-right(T)
T: AVL tree
returns rotated AVL tree
1.   newroot ← T.left
2.   T.left ← newroot.right
3.   newroot.right ← T
4.   return newroot
```

```
rotate-left(T)
T: AVL tree
returns rotated AVL tree
1.   newroot ← T.right
2.   T.right ← newroot.left
3.   newroot.left ← T
4.   return newroot
```


Double Right Rotation

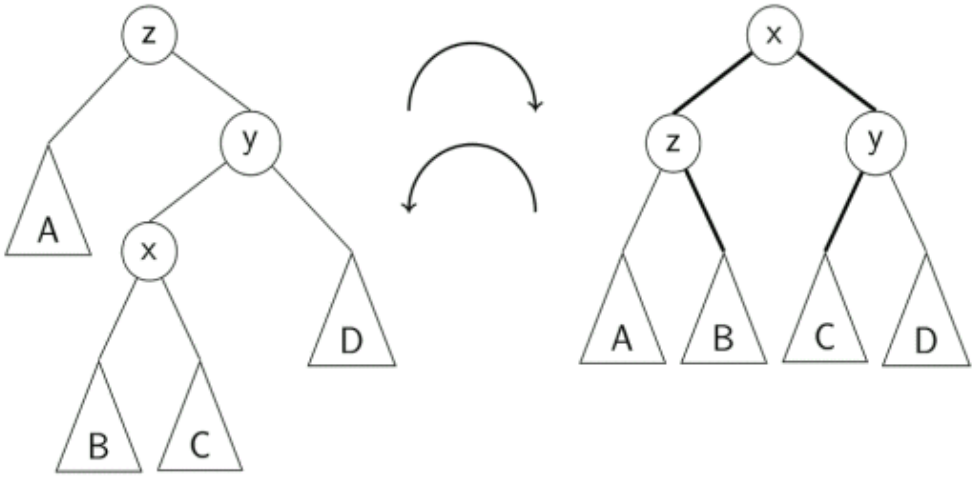
This is a *double right rotation* on node z:



First, a left rotation on the left subtree (y).
Second, a right rotation on the whole tree (z).

Double Left Rotation

This is a *double left rotation* on node z:



Right rotation on right subtree (y),
followed by left rotation on the whole tree (z).

Fixing a slightly-unbalanced AVL tree

Idea: Identify one of the previous 4 situations, apply rotations

```
fix(T)
T: AVL tree with T.balance = ±2
returns a balanced AVL tree
1.    if T.balance = −2 then
2.        if T.left.balance = 1 then
3.            T.left ← rotate-left(T.left)
4.        return rotate-right(T)
5.    else if T.balance = 2 then
6.        if T.right.balance = −1 then
7.            T.right ← rotate-right(T.right)
8.        return rotate-left(T)
```

AVL Tree Operations

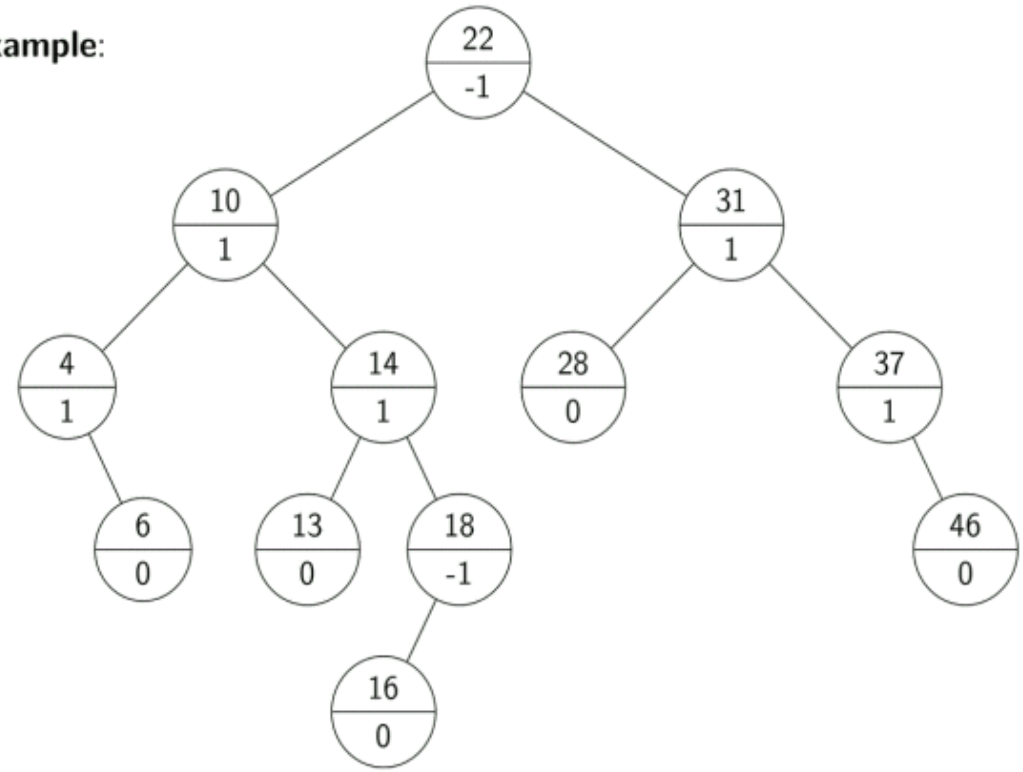
search: Just like in BSTs, costs $\Theta(height)$

insert: Shown already, total cost $\Theta(height)$
fix will be called *at most once*.

delete: First search, then swap with successor (as with BSTs), then move up the tree and apply *fix* (as with *insert*).
fix may be called $\Theta(height)$ times.
Total cost is $\Theta(height)$.

AVL tree examples

Example:



Height of an AVL tree

Define $N(h)$ to be the *least* number of nodes in a height-h AVL tree.

One subtree must have height at least $h-1$, the other at least $h-2$:

$$N(h) = \begin{cases} 1 + N(h-1) + N(h-2), & h \geq 1 \\ 1, & h = 0 \\ 0, & h = -1 \end{cases}$$

What sequence does this look like?

AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \cdots > 2^i N(h-2i) \geq 2^{\lfloor h/2 \rfloor}$$

Since $n > 2^{\lfloor h/2 \rfloor}$, $h \leq 2 \lg n$,
and thus an AVL tree with n nodes has height $O(\log n)$.
Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

$\Rightarrow$ *search*, *insert*, *delete* all cost $\Theta(\log n)$.

insert/delete in AVL trees

↳ 1. insert/delete as usual BST

↳ Θ(height)

2. move up, update B.F.

rotate $\Rightarrow$ $\Theta(1)$ in each level $\rightarrow$ Θ(height)

Let $N(h)$ be the min # of nodes in an AVL of height h

$$N(h) = \begin{cases} 0 & h = -1 \\ 1 & h = 0 \\ 1 + N(h-1) + N(h-2) & h > 0 \end{cases}$$

h	-1	0	1	2	3	4	5
$N(h)$	0	1	2	4	7	12	20
Fibo		0	1	1	2	3	5	8	13	21

$$N(h) = Fibo(h+3) - 1 = \left[\frac{\varphi^{h+3}}{5}\right] - 1 \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow h \in \Theta(\log n)$$

2-3 Trees

A 2-3 Tree is like a BST with additional structual properties:

- Every internal node either contains *one KVP* and *two children*, or *two KVPs* and *three children*.
- The leaves are *NIL* (do not store keys)
- All the leaves are at the same level.

Searching through a 1-node is just like in a BST.
For a 2-node, we must examine both keys and follow the appropriate path.

Insertion in a 2-3 tree

First, we search to find the lowest internal node where the new key belongs.

If the node has only 1 KVP, just add the new one to make a 2-node.

Otherwise, order the three keys as $a < b < c$.
Split the node into two 1-nodes, containing a and c,
and (recursively) insert b into the parent along with the new link.

2-3 Tree Insertion

Example:

Deletion from a 2-3 Tree

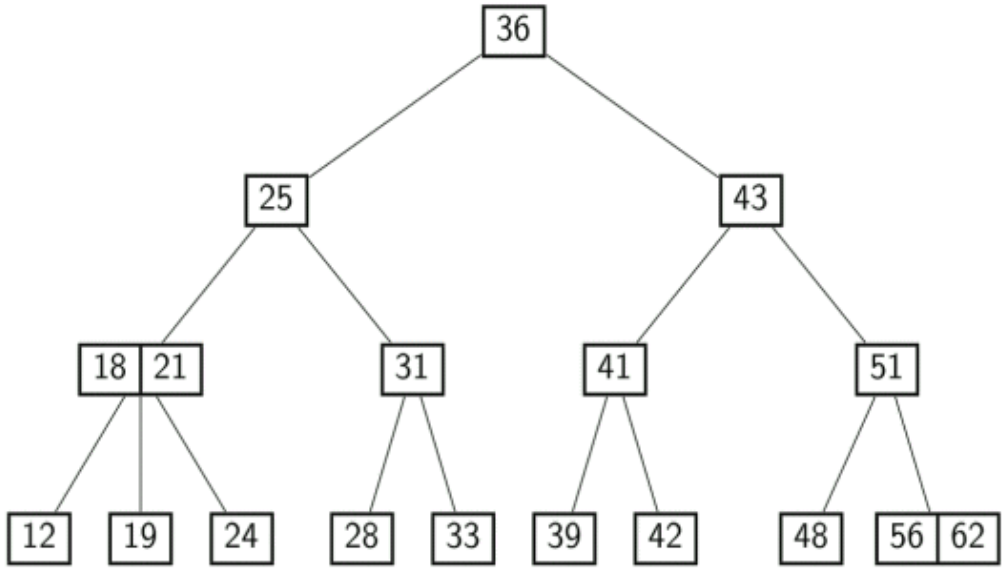
As with BSTs and AVL trees, we first swap the KVP with its successor, so that we always delete from a leaf.

Say we're deleting KVP x from a node V:

- If V is a 2-node, just delete x.
- Elself V has a 2-node *immediate* sibling U, perform a *transfer*: Put the "intermediate" KVP in the parent between V and U into V, and replace it with the adjacent KVP from U.
- Otherwise, we *merge* V and a 1-node sibling U: Remove V and (recursively) delete the "intermediate" KVP from the parent, adding it to U.

2-3 Tree Deletion

Example:



B-Trees

The 2-3 Tree is a specific type of (a, b)-tree:

An (a, b)-*tree of order* M is a search tree satisfying:

- Each internal node has at least a children, unless it is the root. The root has at least 2 children.
- Each internal node has at most b children.
- If a node has k children, then it stores $k - 1$ key-value pairs (KVPs).
- Leaves store no keys and are at the same level.

A *B-tree of order* M is a $(\lceil M/2 \rceil, M)$-tree.
A 2-3 tree has $M = 3$.

search, *insert*, *delete* work just like for 2-3 trees.

Height of a B-tree

What is the least number of KVPs in a height-h B-tree?
(Height = # levels **not** counting the dummy-level - 1)

Level	Nodes	Links/node	KVP/node	KVPs on level
0	1	2	1	1
1	2	$M/2$	$M/2-1$	$2(M/2-1)$
2	$2(M/2)$	$M/2$	$M/2-1$	$2(M/2)(M/2-1)$
3	$2(M/2)^2$	$M/2$	$M/2-1$	$2(M/2)^2(M/2-1)$
...	...	...	...	...
h	$2(M/2)^{h-1}$	$M/2$	$M/2-1$	$2(M/2)^{h-1}(M/2-1)$

$$\text{Total: } n \geq 1 + 2\sum_{i=0}^{h-1}(M/2)^i(M/2-1) = 2(M/2)^h - 1$$

Therefore height of tree with n nodes is $\Theta\big((\log n)/(\log M)\big)$.

Analysis of B-tree operations

Assume each node stores its KVPs and child-pointers in a dictionary that supports $O(\log M)$ search, insert, and delete.

Then *search*, *insert*, and *delete* work just like for 2-3 trees, and each require $\Theta(\textit{height})$ node operations.

Total cost is $O\left(\dfrac{\log n}{\log M} \cdot (\log M)\right) = O(\log n)$.

Dictionaries in external memory

Tree-based data structures have poor *memory locality*:
If an operation accesses m nodes, then it must access
m spaced-out memory locations.

Observation: Accessing a single location in *external memory*
(e.g. hard disk) automatically loads a whole block (or "page").

In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

If M is small enough so an M-node fits into a single page,
then a B-tree of order M only loads $\Theta\big((\log n)/(\log M)\big)$ pages.

This can result in a *huge* savings:
memory access is often the largest time cost in a computation.

B-tree variations

Max size M: Permitting one additional KVP in each node
allows *insert* and *delete* to avoid *backtracking* via
pre-emptive splitting and *pre-emptive merging*.

Red-black trees: Identical to a B-tree with minsize 1 and maxsize 3,
but each 2-node or 3-node is represented by 2 or 3 binary nodes,
and each node holds a "color" value of red or black.

B^+-trees: All KVPs are stored at the leaves
(interior nodes just have keys),
and the leaves are linked sequentially.