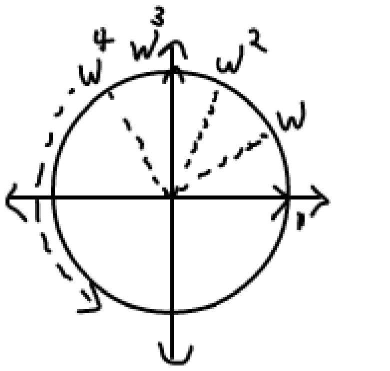
$T=\{\sqrt[n]{r}\, e^{i(\theta/n + 2\pi k/n)} \mid k=0,1,\dots,n-1\}$

$S=\{z\in\mathbb{C} \mid z^n = re^{i\theta}\}$ $\quad S=T$

Roots of Unity

z is an <u>n-th root of unity</u> if $z^n=1$.



12-th root of unity $\Rightarrow w = e^{i2\pi/12}$
$= e^{i\pi/6}$

$w = e^{i2\pi/n}$
$w^2 = e^{i4\pi/n}$

The n n-th roots of unity are
$1, w, w^2, \dots, w^{n-1}$ where $w = e^{i2\pi/n}$

Finding the sum of certain coefficients in a polynomial.

$f(x) = 3+5x-2x^2+x^3 \qquad g(x)=(1+x)^{314}$

① Sum of all coefficients $f(1)=3+5-2+1=7$
$g(1)=2^{314}$

② Sum of coefficients of even powers of x: $f(-1)=3-5-2-1$

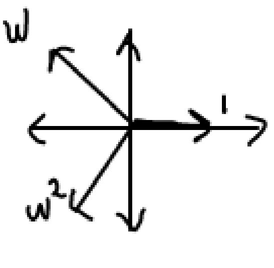
$\frac{f(1)+f(-1)}{2} = \frac{7+(-5)}{2} = 1$

$\frac{g(1)+g(-1)}{2} = \frac{2^{314}+0}{2} = 2^{313}$

Coeff of x^n:

$\frac{c(1)^n + c(-1)^n}{2} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ c & \text{if } n \text{ is even} \end{cases}$

③ Sum of coefficients of all x^{3k} for $k\in\mathbb{N}$.
3rd root of unity: $1,\ w=e^{i\frac{2\pi}{3}},\ w^2=e^{i\frac{4\pi}{3}}$



$1+w+w^2=0$
$f(1)+f(w)+f(w^2)$ $\qquad cx^n$ is the n-th term in $f(x)$
$c1^n + cw^n + cw^{2n} = c(1+w^n+w^{2n})$

$C1^n + Cw^n + Cw^{2n} = C(1 + w^n + w^{2n})$

① $3|n : n = 3k \Rightarrow 1 + w^{3k} + w^{6k} = 3$

② $n \equiv 1 \pmod 3 : n = 3k+1 \Rightarrow 1 + w^{3k+1} + w^{6k+2} = 1 + w + w^2 = 0$

③ $n \equiv 2 \pmod 3 : n = 3k+2 \Rightarrow 1 + w^{3k+2} + w^{6k+4} = 1 + w^2 + w^4$
$= 1 + w^2 + w = 0$

$\frac{f(1) + f(w) + f(w^2)}{3}$ is the sum we want.

$$\frac{g(1) + g(w) + g(w^2)}{3} = \frac{2^{314} + (1+w)^{314} + (1+w^2)^{314}}{3}$$

$$= \frac{2^{314} + (-w^2)^{314} + (-w)^{314}}{3}$$

$$= \frac{2^{314} + w^{628} + w^{314}}{3}$$

$$= \frac{2^{314} + w + w^2}{3}$$

$$= \frac{2^{314} - 1}{3}$$

Polynomials

$x^2 + 1$ cannot be factored over $\mathbb{R}$

$= (x+i)(x-i)$ can be factored over $\mathbb{C}$

$= (x+1)(x+1)$ can be factored over $\mathbb{Z}_2$

$x^2 + \underbrace{2x}_{0} + 1$

Definition: A polynomial in x over a field $\mathbb{F}$ (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_p$ for prime p, any number system that is closed under $+ - \times \div$) has the form:

$a_nX^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ where $n \geq 0$ is an integer and $a_i \in \mathbb{F}$ for each i. The set of all polynormals in X over $\mathbb{F}$ is denoted $\mathbb{F}[x]$.

$$p(x) = \sum_{j=0}^{n} a_j x^j$$