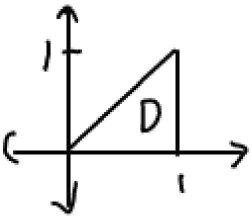
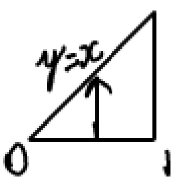
Non-rectangular domains

Suppose we want to integrate $f(x,y)$ over the triangular region D:



What are the bounds of integration?
2 ways to set this up.

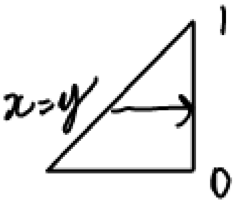
① Let x have constant bounds $0 \le x \le 1$
Then $0 \le y \le x$.



$$\iint_D f(x,y)\,dA = \int_0^1\int_0^x f(x,y)\,dy\,dx$$

Order is important!

② Let y have const. bounds $0 \le y \le 1$
Then $y \le x \le 1$

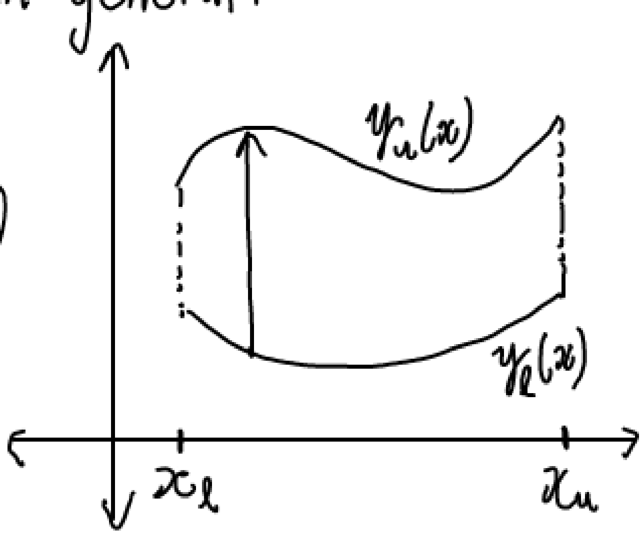


Then $\iint_D f(x,y)\,dA = \int_0^1\int_y^1 f(x,y)\,dx\,dy$

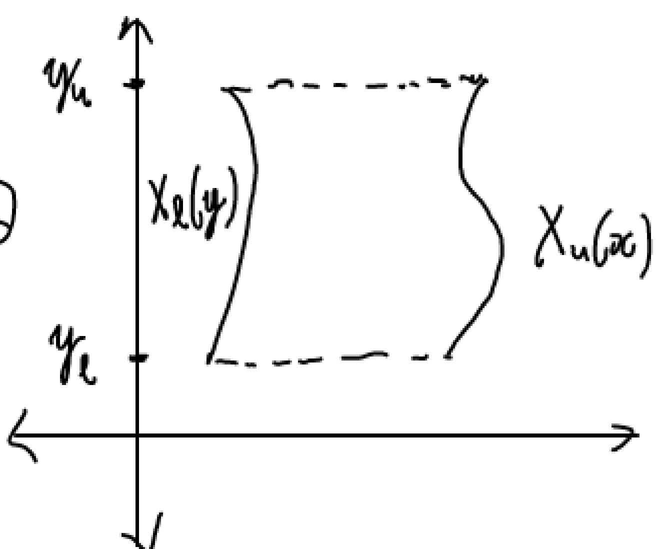
Non-trivial to switch order for non-rectangular domains.

In general:

①



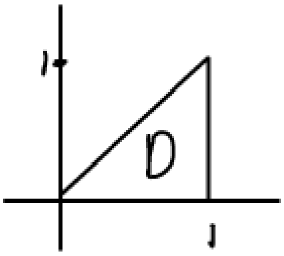
②



$$\iint_D f(x,y)\,dA \overset{①}{=} \int_{x_l}^{x_u}\int_{y_l(x)}^{y_u(x)} f(x,y)\,dy\,dx \overset{②}{=} \int_{y_l}^{y_u}\int_{x_l(y)}^{x_u(y)} f(x,y)\,dx\,dy$$

Ex. Evaluate $\iint_D x^2 y\,dA$ where D is the triangle where D is the triangle with vertices $(0,0), (1,0), (1,1)$.

Soln:



Can describe D as:

$0 \le y \le 1$
$y \le x \le 1$

$$\iint_D x^2 y\,dA = \int_0^1\int_y^1 x^2 y\,dx\,dy = \int_0^1 \left[\tfrac{1}{3}x^3 y\Big|_y^1\right] dy$$

$$= \int_0^1 \left[\tfrac{1}{3}y - \tfrac{1}{3}y^4\right] dy$$

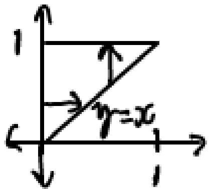
$$= \frac{1}{3}\left[\tfrac{1}{2}y^2 - \tfrac{1}{5}y^5\Big|_0^1\right]$$

$$= \frac{1}{10}$$

$\boxed{\text{Excersise: Switch Order}}$

We don't always have the choice of order.

e.g. $\iint_D e^{-y^2}\,dA$, where D is the triangle with vertices $(0,0), (0,1), (1,1)$



D can be described as:

① $0 \le x \le 1$
$x \le y \le 1$

② $0 \le y \le 1$
$0 \le x \le y$

① $\iint_D f(x,y)\,dA = \int_0^1\int_x^1 e^{-y^2}\,dy\,dx = \int_0^1 \Big[\underbrace{\int_x^1 e^{-y^2}\,dy}_{\text{no elementary anti-derivative}} \Big] dx$ → hard

② $\iint_D e^{-y^2}\,dA = \int_0^1\int_0^y e^{-y^2}\,dx\,dy$

$= \int_0^1 \left[x e^{-y^2} \Big|_0^y \right] dy$

$= \int_0^1 y e^{-y^2}\,dy$ → substitution

let $u = y^2$

$du = 2y\,dy$

$\frac{du}{2} = y\,dy$

$= \int_0^1 \frac{1}{2} e^{-u}\,du$

$\int_{0^2=0}^{1^2=1}$

$= -\frac{1}{2} e^{-u} \Big|_0^1$

$= \frac{1}{2}\left(1 - \frac{1}{e}\right)$

Mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$

If we have a complicated domain or complicated integrand, it can be very difficult (or impossible) to compute a double integral.

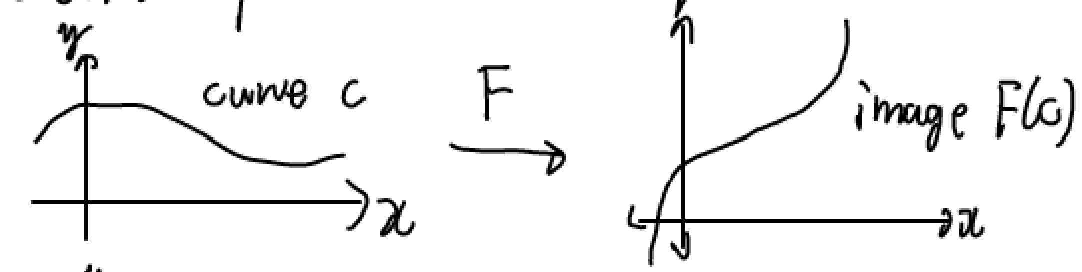
Idea: change eqns to simplify.

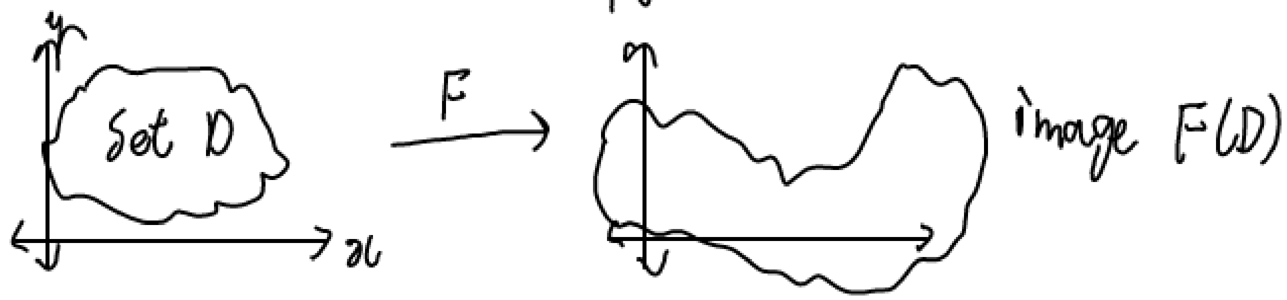
The eqns $u = f(x,y)$
$v = g(x,y)$
associates with each point $(x,y) \in \mathbb{R}^2$ a point $(u,v) \in \mathbb{R}^2$ and thus defines a vector function F

$$(u,v) = F(x,y), \text{ where}$$
$$F(x,y) = (f(x,y), g(x,y))$$

F is a <u>mapping</u> (or transformation) of $\mathbb{R}^2$ into $\mathbb{R}^2$.

Geometrically.



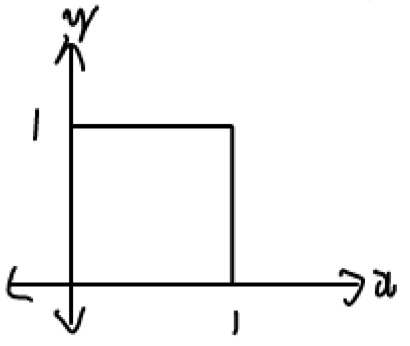


<u>Ex.</u> Let F be defined by

$$(u,v) = F(x,y) = (x+y, x-y)$$

Find the image of the square $S = \{0 \le x \le 1, 0 \le y \le 1\}$

<u>Soln</u>:



Map each of the four lines:

<u>$x=0$</u>: $u = y$, $v = -y \Rightarrow v = -u$

<u>$x=1$</u>: $u = 1+y$, $v = 1-y$ add eqns:

$$u + v = 2 \Rightarrow v = 2 - u$$

$y=0: (u,v)=(x,x) \Rightarrow v=u$

$y=1: (u,v)=(x+1, x-1). \Rightarrow$ subtract $u-v=2 \Rightarrow v=u-2$

Image: