Linear Diophantine

Def^n: A linear Diophantine Equation has the form:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = c$$

where $a_1, \dots, a_n, c$ are integer constants, and $x_1, \dots, x_n$ are integer variables.

Q1. Is there an integer solution?

Q2. If so, what are all integer solutions?

1-Var case. $ax = c$

① This has an int sol^n if and only if $a \mid c$.

② if $a \mid c$, then $x = \frac{c}{a}$ is the only int solution, except when $a = 0$. Since $a \mid c$, $c=0$ $0 \cdot x = 0$. Any int is a solution.

2-var case: $ax+by=c$.

we represent int solns as $\{(x_0, y_0) \in \mathbb{Z}\times\mathbb{Z} \mid ax_0 + by_0 = c\}$

Q1. Given a, b, for which values of c does $ax+by=c$ has an integer solution?

Define $S = \{c \in \mathbb{Z} \mid \exists (x_0, y_0) \in \mathbb{Z}\times\mathbb{Z}, ax_0 + by_0 = c\}$

Example: $12x + 15y = c$

By EEA, $12x+15y=3$ ($\gcd(12,15)$) has an int soln.

$(-1, 1)$ is one such solution.

$12x + 15y = 33333$ $\quad 12(-1) + 15(1) = 3$ $\quad$ multiply by 11111 on both sides.

$12(\underline{-11111}) + 15(\underline{11111}) = 33333$

$(-11111, 11111)$ is an int soln.

When $3 \mid c$, $12x + 15y = c$ has an int sol^n.

Generally: $ax + by = c$ has an int soln when

$\gcd(a, b) \mid c$.

Define $T = \{c \in \mathbb{Z} \mid \gcd(a, b) \mid c\}$

what have we "proved". $\boxed{T \subseteq S.}$

Is $S \subseteq T$? $\quad \underline{12}x + \underline{15}y = 10$

$3 \mid 12$ & $3 \mid 15$, so $3 \mid (12x + 15y)$

But $3 \nmid 10$, so this has no int sol^n.

Proposition (LDE 1): Let $a, b, c \in \mathbb{Z}$, $d = \gcd(a, b)$.
The LDE $ax + by = c$ has an integer solution if and only if $d \mid c$ $(S = T)$

Proof ($\Rightarrow$) Suppose (x_0, y_0) is an int. solution to $ax + by = C$. Then $ax_0 + by_0 = c$. Since $d = \gcd(a, b)$, $d \mid a$ and $d \mid b$. By DIC, $d \mid (ax_0 + by_0)$, so $d \mid c$.

($\Leftarrow$) suppose $d|c$. So $\exists\ k \in \mathbb{Z}$ such that $c = dk$.
By EEA, $\exists\ x_0, y_0 \in \mathbb{Z}$ such that $ax_0 + by_0 = d$.
Multiply both sides by k to get $a(kx_0) + b(ky_0) = kd$
$= c$,

So (kx_0, ky_0) is an int soln to $ax + by = c$.

Example: $119x + 84y = 777$

Use EEA on 119, 84 to get $\gcd(119, 84) = 7$.
and $119 \cdot 5 + 84(-7) = 7$.

Since $7 | 777$, there is an integer sol^n, which is $(555, -777)$

$119x + 84y = 1000$ has no int soln since $7 \nmid 1000$.