Induction
Strong Induction

To prove $P(n)$ for all $n \in \mathbb{N}$,

① $P(1)$ is true

② for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Example: For any $n \in \mathbb{N}$, if A is a set of size n, then A has 2^n subsets. ($|\mathcal{P}(A)| = 2^n$)

proof: By induction on n.

Base case: When $n=1$, A has size 1, say $A = \{x\}$. Then the subsets of A are $\emptyset$ and $\{x\}$. There are 2^1 of thme.

Ind. hyp: Assume any set of size k has 2^k subsets, for some $k \in \mathbb{N}$.

Ind. Step: Let A be a set of size $k+1$. Let $x \in A$. Partition the subsets of A into X and Y where X consists of subsets of A that include x and Y " " " " " " do not have x.

Y is the set of all subsets of $A \setminus \{x\}$, Since $|A \setminus \{x\}| = k$, by ind. hyp., $|Y| = 2^k$. Each element in X consists of $\{x\}$ union an element of Y. So $|X| = |Y| = 2^k$.

So A has $|X|+|Y| = 2^k + 2^k = 2^{k+1}$ subsets
By induction, the result holds.

$A = \{1, 2, 3\}$ $\quad x = 1$

$X = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ contains !

$Y = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$

Strong Induction

Example:

Let $\{a_n\}_{n\geq 0}$ be the sequence $a_1 = 4$, $a_2 = 10$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = -2 + 3 \cdot 2^n$ for all $n \in \mathbb{N}$.

[$a_1 = 4$, $a_2 = 10$, $a_3 = 3a_2 - 2a_1 = 22$, $a_4 = 3a_3 - 2a_2 = 46$, ...]

Proof: By induction on n.

Base case: When $\boxed{n=1,\ a_1 = 4,\ -2+3\cdot 2^1 = 4.}$ $\boxed{n=2,\ a_2=10,\ -2+3\cdot 2^2 = 10}$

Ind. hyp: Assume for some $k \in \mathbb{N}$, $k \geq 2$, $a_i = -2 + 3\cdot 2^i$ for all int $1 \leq i \leq k$.

only valid when $k+1 \geq 3$, $k \geq 2$

Ind. Step: $a_{k+1} = 3a_k - 2a_{k-1}$

$= 3(-2+3\cdot 2^k) - 2(-2+3\cdot 2^{k-1})$ (by ind. hyp.)

$= -6 + 9\cdot 2^k + 4 - 3\cdot 2^k$

$= -2 + 3\cdot 2^k(3-1)$

$= -2 + 3\cdot 2^{k+1}$

only possible with the strong ind.

By strong ind, the result holds.

To prove $P(n)$ for all $n \in \mathbb{N}$ by strong induction.

1. (Base cases) $P(1), P(2), \dots, P(b)$ are true for some $b \in \mathbb{N}$

   [Anything that the ind. step is not applied to goes into base cases]

2. (Ind. step) For all int $k \geq b$, if $P(1), \dots, P(k)$ are true, then $P(k+1)$ is true.

Example

prove that any collection of at least 8 candies can be divided into piles of 3's and/or 5's.