Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)

Dynamic Arrays

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Dynamic arrays offer a compromise:
$O(1)$ element access, and $O(1)$ insertion/deletion *at the end*.

Two realizations of dynamic arrays:

- Allocate one HUGE array, and only use the first part of it.
- Allocate a small array initially, and double its size as needed. (*Amortized analysis* is required to justify the $O(1)$ cost for insertion/deletion at the end — take CS 341/466!)

Stack ADT

Stack: an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.
We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited sites in a Web browser, procedure calls

Realizations of Stack ADT

- using arrays
- using linked lists

Queue ADT

Queue: an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.
Items enter the queue at the *rear* and are removed from the *front*.
We can have extra operations: *size*, *isEmpty*, and *front*

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest priority*

deleteMax is also called *extractMax*.

Applications: typical "todo" list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*.

Using a Priority Queue to Sort

$PQ - Sort(A)$
1. initialize PQ to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n-1$ **do**
3. $\quad PQ.insert(A[i], A[i])$
4. **for** $i \leftarrow 0$ **to** $n-1$ **do**
5. $\quad A[n-1-i] \leftarrow PQ.deleteMax()$

Realizations of Priority Queues

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Attempt 2: Use *sorted arrays*

- insert: $O(n)$ ← Need to shift rest of array over after Binary search
- deleteMax: $O(1)$

Using sorted linked-lists is identical.
This realization used for sorting yields *insertion sort*.

Insertion Sort

- ↳ Not too bad
  - ↳ If the list is almost sorted, then it is much faster
  - ↳ It is an online algorithm
  - ↳ Ez 2 implement & "Natural"

Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:
A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, *etc.* .

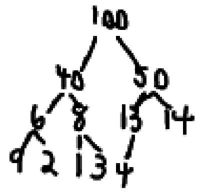
Heaps

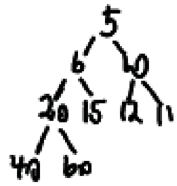
A *max-heap* is a binary tree with the following two properties:

1. Structural Property: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
2. Heap-order Property: For any node i, *key* (priority) of parent of i is larger than or equal to key of i.

A *min-heap* is the same, but with opposite order property.

Heaps





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A *min-heap* is the same, but with opposite order property.

Lemma: Height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Let H be a heap (binary tree) of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

Assume height of a heap is h

$\Rightarrow$ It has $(h-1)$ full levels

\# of nodes in the heap $\rightarrow$ $n > 1 + 2 + \cdots + 2^{h-1} = 2^h - 1$

$$n > 2^h - 1$$

$$n + 1 > 2^h$$

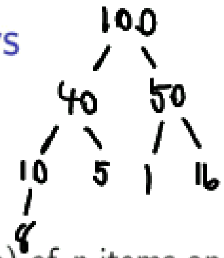
$$\log(n+1) > h$$

$$h \in O(\log n)$$

$1 + 2 + 4 + 8 = 2^4 - 1$

$= 15$

Storing Heaps in Arrays



100	40	50	10	5	1	16	8
0	1	2	3	4	5	6	7

A[0]=100
A[0] → left = 40 = A[1]
A[0] → right = 50 = A[2]

Let H be a heap (binary tree) of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i+1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i+2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).

40 = A[1]
A[1] → left = 10 = A[3]
A[2] → right = 5 = A[4]

A[i] → left = A[2i+1]
A[i] → right = A[2i+2]

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- Place the new key at the first free leaf
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bubble-up(v)
v: a node of the heap
while *parent*(v) exists **and** *key*(*parent*(v)) $<$ *key*(v) **do**
    swap v and *parent*(v)
    $v \leftarrow$ *parent*(v)

The new item bubbles up until it reaches its correct place in the heap.

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Time: O(height of heap) $= O(\log n)$.

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
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```
bubble-down(v)
v: a node of the heap
    while v is not a leaf do
        u ← child of v with largest key
        if key(u) > key(v) then
            swap v and u
            v ← u
        else
            break
```

Time: O(height of heap) $= O(\log n)$.

Priority Queue Realization Using Heaps

heapInsert(A, x)
A: an array-based heap, x: a new item
1. $size(A) \leftarrow size(A) + 1$
2. $A[size(A) - 1] \leftarrow x$
3. $bubble{-}up(A, size(A) - 1)$

heapDeleteMax(A)
A: an array-based heap
1. $max \leftarrow A[0]$
2. $swap(A[0], A[size(A) - 1])$
3. $size(A) \leftarrow size(A) - 1$
4. $bubble{-}down(A, 0)$
5. **return** *max*

Insert and deleteMax: $O(\log n)$

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Problem statement: Given n items (in $A[0 \cdots n-1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
    initialize H as an empty heap
    for i ← 0 to size(A) − 1 do
        heapInsert(H, A[i])
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This corresponds to going from $0 \cdots n-1$ in A and doing *bubble-ups*
Worst-case running time: $\Theta(n \log n)$.

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Solution 2: Using *bubble-downs* instead:

heapify(A)
A: an array
1. $n \leftarrow size(A) - 1$
2. **for** $i \leftarrow \lfloor n/2 \rfloor$ **downto** 0 **do**
3. $\quad$ *bubble−down*(A, i)

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A: an array
    n ← size(A) − 1
    for i ← ⌊n/2⌋ downto 0 do
        bubble−down(A, i)
```

A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

HeapSort

```
HeapSort(A)
    initialize H to an empty heap
    for i ← 0 to n − 1 do
        heapInsert(H, A[i])
    for i ← 0 to n − 1 do
        A[n − 1 − i] ← heapDeleteMax(H)
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Running time of HeapSort: $O(n \log n)$

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Selection

Problem Statement: The kth-max problem asks to find the *kth largest item* in an array A of n numbers.

Solution 1: Make k passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the kth largest number.
Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the k largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling *heapify*(A). Call *deleteMax*(A) k times.
Complexity: $\Theta(n + k \log n)$.

Sol 3.

↳ Form a min heap of size k (first k items) $\rightarrow \Theta(k)$

↳ For any $\underline{a}$ among $(n-k)$ remaining item:

heap-insert $(a) \rightarrow \log(k) \longrightarrow \Theta(\log k)$

heap-extract min $\rightarrow \log(k) \longrightarrow \Theta(\log k)$

$\Theta((n-k)\log k)$

↳ Final heap includes k largest items

root of that heap (min among k largest)

$\therefore \Theta(n \log k)$