Induction

To prove $P(n)$ for all $n \in \mathbb{N}$ by induction,

① (Base Case) $P(1)$ is true

② (Induction Step) For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Example: Prove that $1+2+\dots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

$$\left[\begin{array}{c} 1 = \frac{1\cdot 2}{2} \\ 1+2 = \frac{2\cdot 3}{2} \\ 1+2+3 = \frac{3\cdot 4}{2} \\ \vdots \end{array}\right]$$

proof: By induction on n Base case.

When $n=1$, LHS $=1$, RHS $= \frac{1\cdot 3}{2} = 1$.

Induction Hypothesis:

Assume $1+2+\dots+k = \frac{k(k+1)}{2}$ for some $k \in \mathbb{N}$.

Induction Step:

$\left\{ \text{want to prove } 1+2+\dots+(k+1) = \frac{(k+1)(k+2)}{2} \right\}$

$$\begin{aligned} 1+2+\dots+(k+1) &= [1+2+\dots+k] + (k+1) \\ &= \frac{k(k+1)}{2} + k+1 \quad \text{by ind. hyp.} \\ &= \frac{k^2+k+2k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

By induction, our statement is true.

Why induction works...

Want $P(1), P(2), \dots$

Base case: $P(1)$ is true.

Ind. step: $k=1$, $P(1) \Rightarrow P(2)$
$k=2$ $\quad P(2) \Rightarrow P(3)$
$\vdots$
k $\quad P(k) \Rightarrow P(k+1)$
$\vdots$

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \dots P(k) \Rightarrow \dots$

Example: Prove that for all $n \in \mathbb{N}$, $5 \mid (6^n - 1)$

proof: By induction on n.

Base case: when $n=1$, $6^1 - 1 = 5$, and $5 \mid 5$.

Ind. hyp: Assume $5 \mid (6^k - 1)$ for som $k \in \mathbb{N}$.

Ind. step: (Want to prove $5 \mid (6^{k+1} - 1)$)
don't assume

$$6^{k+1} - 1 = 6 \cdot 6^k - 1 - 5 + 5$$
$$= 6(6^k - 1) + 5$$

By ind. hyp., $5 \mid (6^k - 1)$, we know $5 \mid 5$.
By divisibility of integer combinations, $5 \mid (6(6^k - 1) + 5)$,
So $5 \mid (6^{k+1} - 1)$.

By induction, $5 \mid (6^n - 1)$ for all $n \in \mathbb{N}$.

Example: Prove that $2^n < n!$ for all $n \in \mathbb{N}$, $n \geq 4$.

Proof: By induction on n.

Base case: When $n=4$, $2^4=16$, $4!=24$, and $16<24$.

Ind. Hyp: Assume $2^k<k!$ for some $k\geq 4, k\in\mathbb{N}$.

Ind. Step: (Want to prove $2^{k+1}<(k+1)!$)

$$\begin{aligned} 2^{k+1} &= 2\cdot 2^k \\ &< 2k! \quad \text{by ind. hyp.} \\ &< (k+1)k! \quad \text{since } 2<k+1 \text{ (since } k\geq 4) \\ &< (k+1)! \end{aligned}$$

By induction, it works.

Bad induction: For all $n\in\mathbb{N}$, $n+1<n$.

Assume that $k+1<k$ for some k

Add 1 to both sides to get

$$(k+1)+1<k+1$$

Bad proof since no base case.