D/ Convolution and the Impulse Response

-Assume our system initially has zero initial conditions. The forcing function *u(t)* is considered the input to the system and the output is the variable *y(t)* that we are trying to solve for.

outputs, inputs

$$\underbrace{a_0 y^{(n)}(t) + \dots + a_n y(t)}_{\text{outputs}} = \underbrace{b_0 u^{(n-1)}(t) + \dots + b_{n-1} u(t)}_{\text{inputs}}$$

-The derivatives are essentially multiplication by s

$$a_0 s^n Y(s) + \dots + a_n Y(s) = b_0 s^{n-1} + \dots + b_{n-1} U(s)$$

-Taking the Laplace transform gives

$$Y(s)(a_0 s^n + \dots + a_n) = U(s)(b_0 s^{n-1} + \dots + b_{n-1})$$

$$Y(s) = G(s)U(s)$$

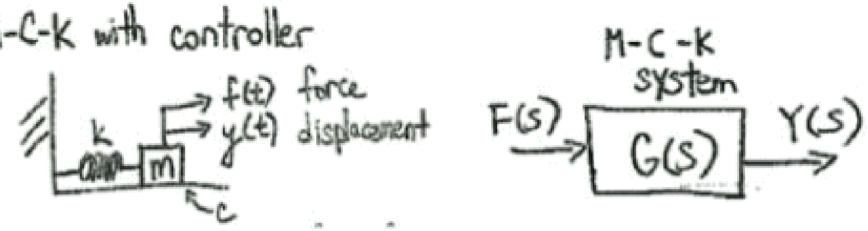
where $G(s) = \dfrac{b_0 s^{n-1} + \dots + b_{n-2} s + b_{n-1}}{a_0 s^n + \dots + a_{n-1} s + a_n}$

-*G(s)* is called the transfer function

-Used for block diagrams.

-Eg. Mass spring damper system with a controller. We want to change the damping characteristics to critical damping

M-C-K with controller



The equations are $\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 f(t)$

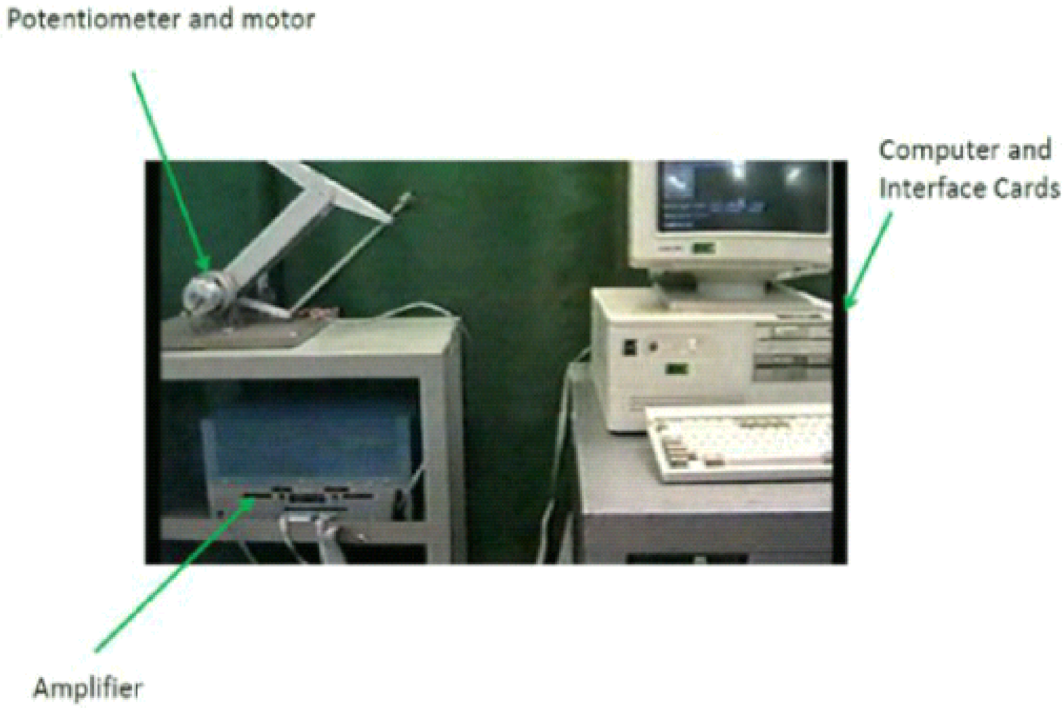
Or, in Laplace domain,

transfer function w/ no controller

$$Y(s) = \underbrace{\left(\frac{W_n^2}{s^2 + 2\xi W_n^2 s + W_n^2} \right)}_{G(s)} F(s)$$

-The force is typically created by a motor. The current to the motor comes from an amplifier which, in term, is driven by a signal from a D/A convertor connected to a computer. The computer is set up with an A/D convertor which takes signals from sensors (eg. optical encoders, voltages etc)

-WE CAN REPRESENT THE COMPUTER AS A TRANSFER FUNCTION AND CONNECT IT TO OUR SYSTEM ABOVE, CALLED THE PLANT *G(S)*, TO MODIFY ITS BEHAVIOR



-In the computer, one can use a "real-time" computer program to emulate a differential equation.

-Thus, one can represent the computer, amplifiers and amplifier, which we will refer to as the *controller*, as a "transfer function"

including sensor measurement

-The most common way to "control" or modify the behavior of a system is to use a PD controller where we make the above controller behave as the following ODE

Proportional (K_P)
Derivative (K_D)
we choose this

$$f(t) = K_p e(t) + K_D \frac{de(t)}{dt}$$

Or, in Laplace domain

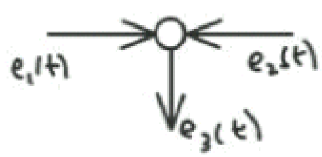
$$F(s) = \overbrace{(K_p + sK_D)}^{C(s)} E(s)$$

(Note, we can't truly do a derivative of a real signal but we can do a very good approximation)

* Know how to derive & use this

-Basic block operations:

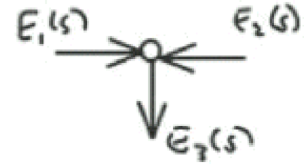
a) summing node



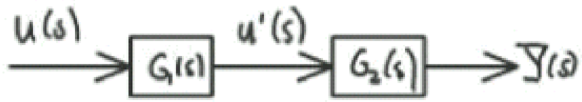
$e_3(t) = e_1(t) + e_2(t)$

By linearity,

$E_3(s) = E_1(s) + E_2(s)$

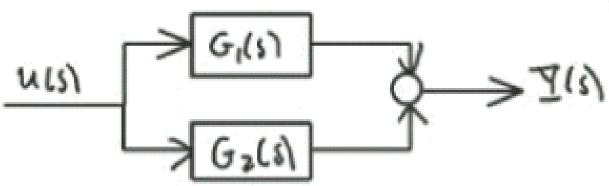


b)series



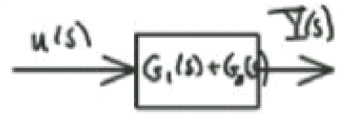
$Y(s) = G_2(s)u'(s) = G_2(s)(G_1(s)U(s)) = (G_1(s)G_2(s))u(s)$

c)parallel

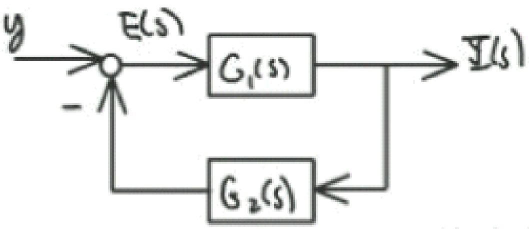


$Y(s) = G_1(s)u(s) + G_2(s)u(s)$

$= \underbrace{(G_1(s) + G_2(s))}_{G(s)}u(s)$



d)feedback



$y = G_1(s)E(s)$ but $E(s) = u(s) - G_2(s)Y(s)$

$= G_1(s)(u(s) - G_2(s)Y(s))$

$Y(s) = \frac{G_1(s)}{}$

$\frac{u(s)G_1(s) - y}{G_1(s)}$

$u(s) - G_2(s)Y(s) = \frac{y}{G_1(s)}$

-Similarly, we could do this for an electrical, thermal or fluid system

-Reminder: We assume the systems are all linear and time-invariant

-Let $e(t) = y_d(t) - y(t)$ where $e(t)$ is an error signal and $y_d(t)$ is the desired value of the signal in question (eg. a desired position in a mass-spring-damper system)

-If $e(t)$ approaches zero, then $y(t)$ approaches $y_d(t)$

-In the Laplace domain, this is

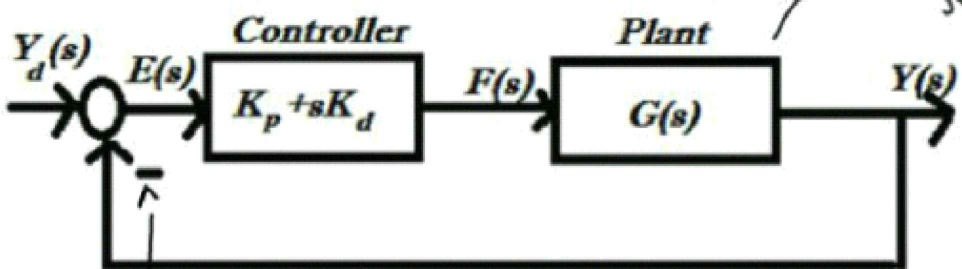
desired

$$E(s) = Y_d(s) - Y(s)$$

actual

-This gives the following block diagram → the "error" drives the computer

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



-ve feedback since +ve feedback gives -ve roots → unstable

This is called closed loop feedback. The controller $C(s)$ used is called a PD controller. The overall transfer function is

$$Y(s) = \frac{(K_p + sK_D)G(s)}{1 + (K_p + sK_D)G(s)} Y_d(s)$$ using previous formula

$$Y(s) = F(s)G(s)$$
$$E(s) = Y_d(s) - Y(s)$$
$$F(s) = C(s)E(s)$$

Thus, $E(s) = Y_d(s) - Y(s) = Y_d(s) - F(s)G(s)$

$$= Y_d(s) - C(s)G(s)E(s)$$

$$\therefore E(s)(1 + C(s)G(s)) = Y_d(s)$$

$$\therefore E(s) = \left[\frac{1}{1 + C(s)G(s)}\right] Y_d(s) = \left[\frac{s^2 + 2\zeta\omega_n + \omega_n^2}{s^2 + (2\zeta\omega_n + K_d\omega_n^2)s + (\omega_n^2 + K_p\omega_n^2)}\right] Y_d(s)$$

-If the roots of the denominator polynomial are in the left hand side of the complex plane, then $e(t)$ will go to zero!

-Revisiting stability- why do we claim that if the roots of the denominator polynomial are on the left hand side of the complex plane, that the system will have the error go to zero?

-If $Y(s) = G(s)Y_d(s)$ is some general transfer function, then we can multiply out everything to get the RHS of the equation to be a ratio of polynomials in s

E.g. $Y(s) = \underbrace{\frac{s+6}{(s+1)(s+3)}}_{G(s)} Y_d(s)$. Choose $Y_d(s) = \underbrace{\frac{1}{s+4}}_{\text{bounded}}$, i.e. $y(t) = e^{-4t}$

-When we do the PFE, all the terms on the RHS will be in terms of the roots of the denominator polynomial and those of the input $Y_d(s)$

$$Y(s) = \frac{s+6}{(s+1)(s+3)(s+4)} \overset{PFE}{\Rightarrow} Y(s) = \underbrace{\frac{A}{s+1} + \frac{B}{s+3}}_{\text{due to } G(s)} + \underbrace{\frac{C}{s+4}}_{\text{due to input}}$$

Obviously, the $\mathcal{L}^{-1}$ of this is bounded. That is, a bounded input gives a bounded output

-Recalling the definition of stability of the ODEs, these will go to zero whenever the roots are in the open left half plane!

-Definition: A transfer function ($G(s)$) is Bounded Input- Bounded output stable(BIBO) if, for ALL bounded input ($y_d(t)$), the output ($y(t)$) is always bounded

-Definition: The poles of the transfer function are the roots of the denominator polynomial ☆

-Definition: The zeros of the transfer function are the roots of the numerator polynomial (not as important for stability)

-Theorem: The transfer function is BIBO stable if and only if all the poles of the transfer function are to the left of the imaginary axis ☆

e.g. $Y(s) = \frac{s+6}{(s+1)(s+5)} Y_d(s)$ is stable as the poles of $G(s)$ are at $-1, -5$

$Y(s) = \frac{1}{s-3} Y_d(s)$ ∴ unstable.

$Y(s) = \frac{5}{(s+3)^2+2^2} Y_d(s)$ Poles @ $-3 \pm 2j$ ∴ stable

$Y(s) = \frac{1}{s} Y_d(s)$ Pole @ 0 ∴ unstable

-Note that one can run into problems with poles on the imaginary axis if the input has the same poles. That is why guaranteeing the poles are in the open left half plane (with none on the imaginary axis) guarantees stability.

Look at $G(s) = \frac{1}{s}$ $\quad Y(s) = G(s)Y_d(s) = \frac{1}{s}Y_d(s)$

Note: $Y_d(s) = \frac{1}{s}$ is bounded (it's a step) but $Y = \frac{1}{s}Y_d(s) = \frac{1}{s^2}$ ∴ $y(t) = t$

An unbounded output results from a bounded output ∴ Not BIBO stable

Back to mass-spring-damper system

-Now, $Y(s) = \dfrac{(K_p + sK_d)\,G(s)}{1+(K_1+sK_d)G(s)}\,Y_d(s) = \dfrac{\omega_n^2(K_p+sK_D)}{s^2+(2\zeta\omega_n+K_D\omega_n^2)s+(K_p\omega_n^2+\omega_n^2)}$

with controller and feedback

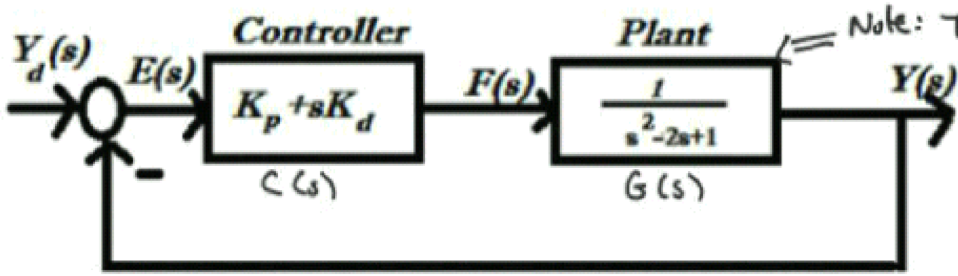
-We can compare to the original system without a controller, where the transfer function is

$$Y(s) = \underbrace{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}_{G(s)}\,F(s)$$

-THUS, we can change the behavior of the system by adjusting K_p & K_d $\leftarrow$ $C(s) = K_p + K_d s$

"spring" "damper"

-Eg/ Suppose we have the following system. Design a controller to make this system stable, critically damped and decay as fast as e^{-2t} and te^{-2t} $\Leftarrow$ we want poles at $-2, -2$



Note: This is highly unstable. Poles at $s=1$

Sol^n:

$$Y(s) = \frac{C(s)G(s)}{1+C(s)G(s)}Y_d(s)$$

$$= \frac{(K_p+sK_d)\left(\frac{1}{s^2-2s+1}\right)}{1+(K_p+sK_d)\left(\frac{1}{s^2-2s+1}\right)}Y_d(s)$$

$$= \frac{K_p+sK_d}{s^2-2s+1+K_p+sK_d}Y_d(s)$$

$$= \frac{K_p+sK_d}{s^2+(K_d-2)s+(1+K_p)}Y_d(s)$$

We want poles at $-2, -2$,

So the denominator must look like

$(s+2)(s+2) = s^2+4s+4$

By comparing coefficients

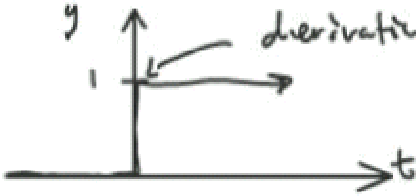
$K_p = 3$, $K_d = 6$

-Now, we have $Y(s) = G(s)U(s)$

If $U(s)=1$, then $Y(s)=G(s)$ or $y(t)=g(t)$

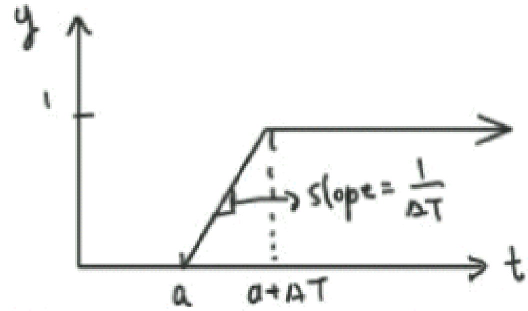
-What is this input that has a Laplace transform =1?

-Motivation- $1= 1/s * s$ (ie derivative of a step)



derivative is undefined for $t=0$

-Since the derivative of a step is undefined, we need an approximation

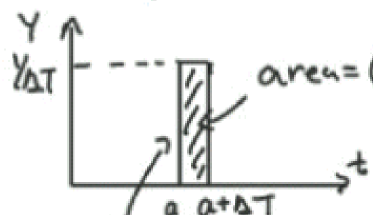


(If a=0, this approximates a step)

-As $\Delta T \to 0$ and $a \to 0$, we approach a step input

-Let us look at the derivative to this approximation of a step

$$f_{\Delta T,a}(t) = \begin{cases} \frac{1}{\Delta T} & , a \le t \le a+\Delta T \\ 0 & , otherwise \end{cases}$$



area $= (\frac{1}{\Delta T})(\Delta T) = 1$

approximate derivative

Now, $\int_0^\infty f_{\Delta T,a}(t)dt = 1$

As well, we can take the Laplace transform

$$f_{\Delta T,a}(t) = \frac{1}{\Delta T}(u(t-a) - u(t-(a+\Delta T)))$$

$$\mathcal{L}\{f_{\Delta T,a}(t)\} = \frac{1}{\Delta T}\left(\frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+\Delta T)s}\right)$$

$$= e^{-as}\left(\frac{1-e^{-\Delta Ts}}{\Delta T s}\right)$$

This gives $F_{\Delta T,a}(s) = e^{-as}\frac{1-e^{-\Delta Ts}}{\Delta Ts}$

when $\Delta T \to 0$, $a \to 0$, so we get the step function

Look at the limit as $\Delta T \to 0$

- We get $F_{\Delta T, a}(s)\Big|_{\Delta T \to 0} = \frac{0}{0}$ $\therefore$ Use L'Hopital's rule
- So, $\lim\limits_{\Delta T \to 0} e^{-as} \frac{\frac{\partial\, num}{\partial \Delta T}}{\frac{\partial\, denom}{\partial \Delta T}} = \lim\limits_{\Delta T \to 0} e^{-as}\left(\frac{s\cancel{e^{-\Delta T s}}^{\,1}}{s}\right) = e^{-as}$

-We define $\delta(t-a) \equiv \lim\limits_{\Delta T \to 0} f_{\Delta T, a}(t)$. This is called the *Dirac Delta function*

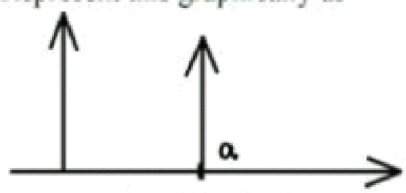
-The integral of this signal is called an impulse. We approximate this Dirac delta function with a sudden force/voltage etc over a short period of time (eg. hammer test)

Thus, $\mathcal{L}\{\delta(t-a)\} = \lim\limits_{\Delta T \to 0} F_{\Delta T, a}(s) = e^{-as}$. In particular, $\mathcal{L}\{\delta(t)\} = 1$

-Calling this a function is a misnomer. Why? Not defined for $t=0$

It goes to ∞ at $t=a$

-Represent this graphically as

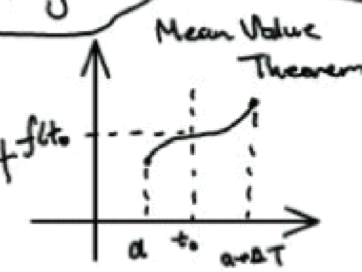


This is really a *distribution*
↳ defined in terms of an integral

-Properties

1. $\int_0^{\infty} \delta(t-a)\,dt = 1$

$\exists t_0 \in [a, a+\Delta T]$
s.t. $f(t_0)\Delta T = \int_a^{a+\Delta T} f(t)\,dt$



2. Filtering property: $\int_0^{\infty} f(t)\delta(t-a)\,dt = f(a)$

Pf/ ☆ proof won't be on exam

$$\int_0^{\infty} f(t)\,\delta(t-a)\,dt = \lim_{\Delta T \to 0} \int_0^{\infty} f(t)\frac{1}{\Delta T}\left(u(t-a) - u(t-(a+\Delta T))\right)dt$$

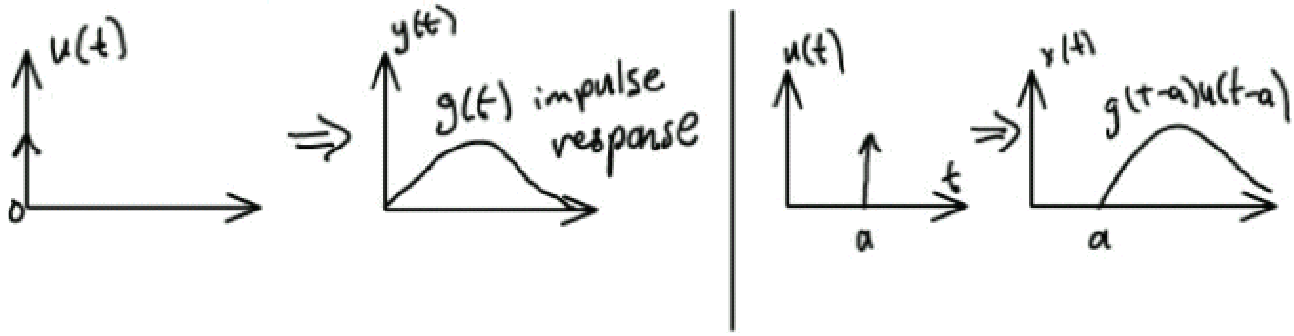
$$= \lim_{\Delta T \to 0} \frac{1}{\Delta T} \int_a^{a+\Delta T} f(t)\,dt \quad \leftarrow \text{since it is 0 for all other values}$$

$$= \lim_{\Delta T \to 0} \frac{1}{\Delta T} f(t)\,\Delta T$$

$$= f(a) \text{ as } t \to a \text{ when } \Delta T \to 0 \quad \text{QED}$$

-Now, recall $Y(s)=G(s)U(s)$. If $U(s)=1$ (in other words, we apply an impulse to our system), then $Y(s)=G(s)$ or, taking the inverse Laplace transform, $y(t)=g(t)$ is called the impulse response

-If there are zero initial conditions and $u(t)=\delta(t-a)$, then the resulting response is the inverse Laplace transform of $G(s)e^{-sa}$ or the impulse response time shifted by a to give $g(t-a)H(t-a)$ where we use $H(t-a)$ for the Heaviside function instead of u to avoid confusion with the input



-In practice, we often hit a linear time invariant system with an impulse to check this impulse response. (We, of course, use a pulse as an approximation). Don't forget we need to normalize so that the integral is one

2tt 1312

-Why is $g(t)$ important?

-We need the concept of *convolution!*

-Recall $Y(s)=G(s)U(s)$.

-Can we find $y(t)$ from $g(t)$ for any arbitrary $u(t)$?

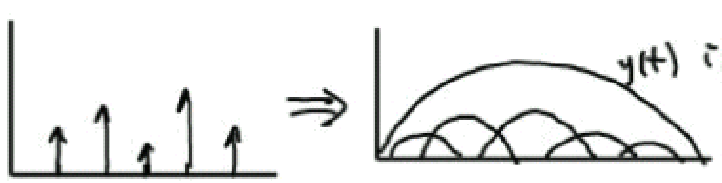
-Theorem: Let $u(t)$ and $g(t)$ satisfy the conditions that guarantee the existence of the Laplace transforms. Then, the product of their transforms $Y(s)=G(s)U(s)$ is the transform of the *convolution* of $g(t)$ and $u(t)$ which is

$$y(t)=\underset{\substack{\uparrow\\ \text{Convolution}}}{g(t)*u(t)}=\int_0^t u(\tau)g(t-\tau)d\tau \quad \Leftarrow \text{memorize}$$

$$\mathcal{L}\{(f*g)(t)\}=F(s)G(s)$$
$$(f*g)(t)=\mathcal{L}^{-1}\{F(s)G(s)\}$$

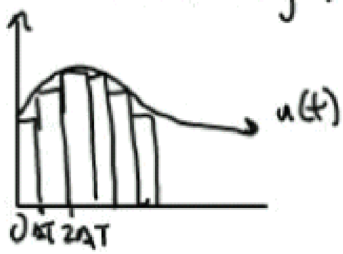
Pf/ Our system is linear, therefore superposition holds

If we have a $u(t)$ made of dirac delta functions



i.e. each $\delta_\Delta(t-n\Delta T)$ generates an impulse response at $n\Delta T$ i.e. $g(t-n\Delta T)H(t-n\Delta T)$

Arbitrary input



$$u(t) \approx \sum_{n=0}^{\infty} u(n\Delta T)\,\delta_{\Delta T}(t-n\Delta T)\,\Delta T$$

$$\approx \sum_{n=0}^{\infty} u(n\Delta T)\,\Delta T\left[g(t-n\Delta T)\,H(t-n\Delta T)\right]$$

$= \emptyset$ when $n\Delta T > t$ since $H(t-n\Delta T)$

$$= \sum_{n=0}^{n'} u(n\Delta T)\,g(t-n\Delta T)\,\Delta T$$ when n' is the smallest n s.t. $n\Delta t > t$

as $\Delta T \to 0$, you get a Riemann integral $y(t) = \int_0^t u(\tau)g(t-\tau)\,d\tau$

QED

-Second proof/ Not responsible

→ Do proof entirely in Laplace Domain

-Properties

1. $f*g=g*f$ Commutative $\Rightarrow y(t) = \int_0^t u(\tau)\,g(t-\tau)\,d\tau = \int_{2t}^{t} g(\tau)\,u(t-\tau)\,d\tau$

2. $f*(g+h)=f*g+f*h$ Distributive

3. $(f*g)*h=f*(g*h)$ Associative

4. $0*f=f*0=0$ Zero Element

5. $f*\delta=\delta*f=f$ Identity

Pf/ Can all be done in the Laplace domain

<u>Ex 2</u> $\mathcal{L}\{f*(g+h)\} = \mathcal{L}\{f\}\mathcal{L}\{g+h\} = \mathcal{L}\{f\}\mathcal{L}\{g\} + \mathcal{L}\{f\}\mathcal{L}\{h\}$

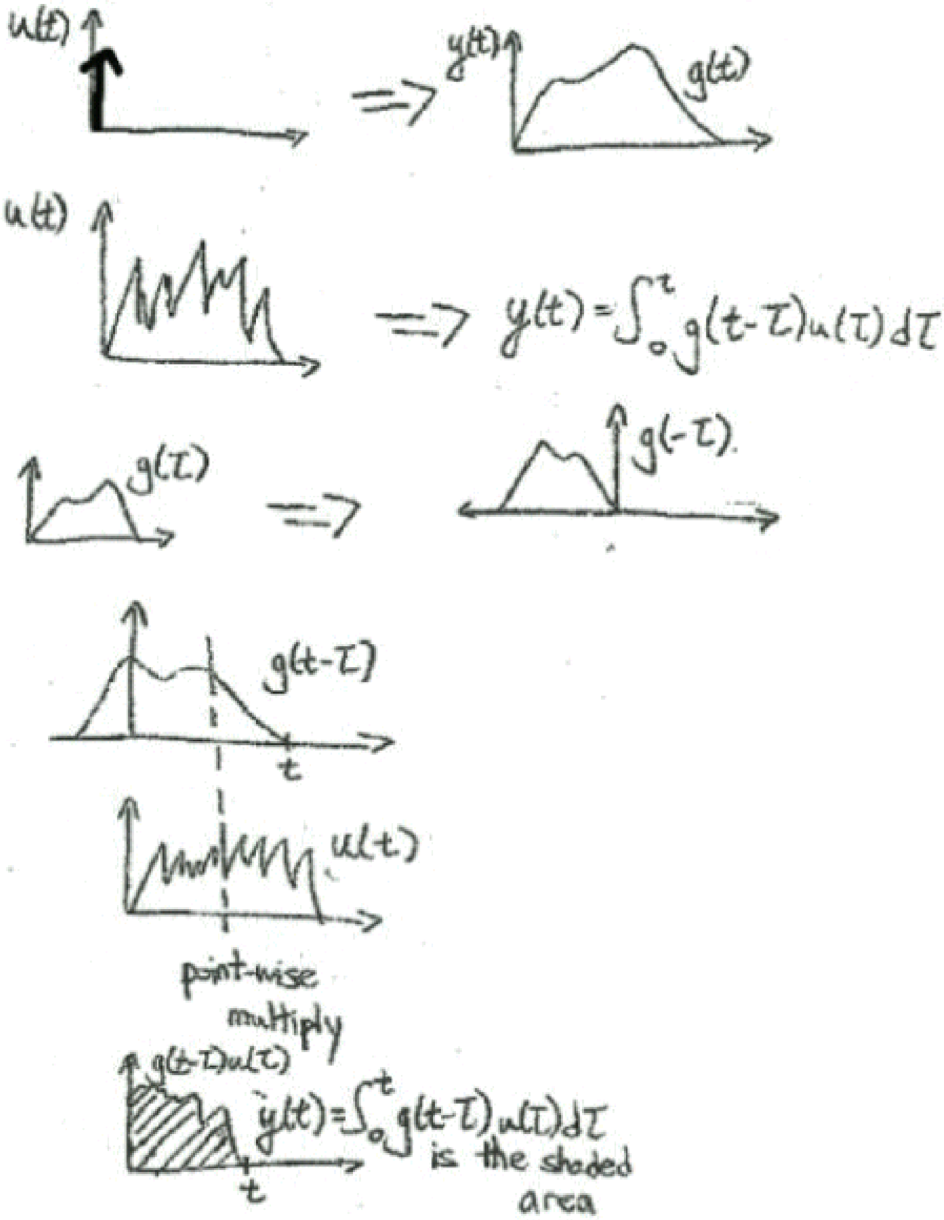
Take $\mathcal{L}^{-1} \Rightarrow \mathcal{L}^{-1}\{\mathcal{L}\{f\}\mathcal{L}\{g\} + \mathcal{L}\{f\}\mathcal{L}\{h\}\}$

$= f*g + f*h$

<u>Ex 5</u> $\mathcal{L}\{f*\delta\} = \mathcal{L}\{f\}\mathcal{L}\{\delta\} = \mathcal{L}\{f\}$

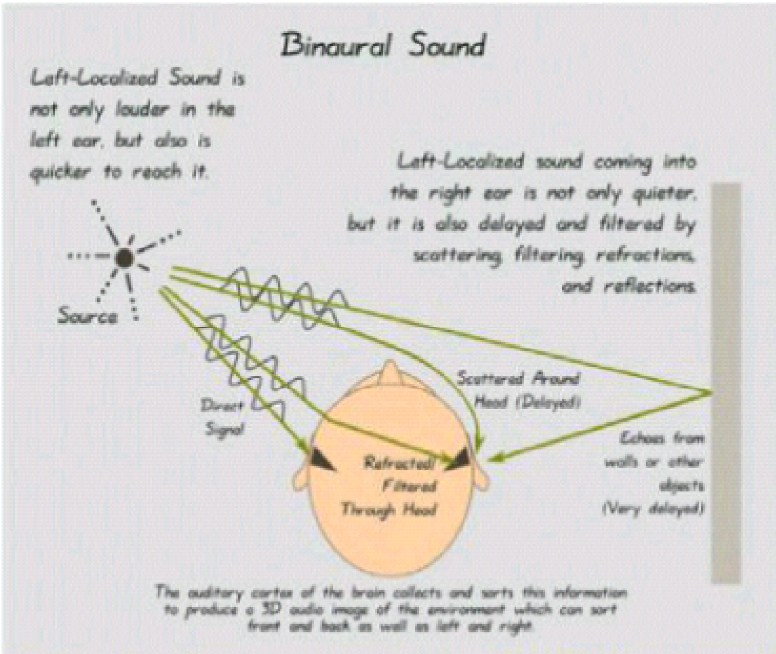
Take $\mathcal{L}^{-1} \Rightarrow f*\delta = f$

-Now, if *g(t)* is the impulse response that we obtain through a meaurement, and our system is Linear, Time-Invariant (ie constant coefficient), then we can use graphical means to calculate the output!



This is how we use the hammer test!

-Binaural recordings. The three dimension effect is caused by sound reaching the ears at different times BUT also from sound reflecting off the ear, the head, the shoulders etc. Your brain decodes these signals and creates the 3-D effect



http://www.freesoftwaremagazine.com/files/nodes/3536/fig_binaural_sound.jpg

-Assume linearity. There is a transfer function from a sound input at a location to the ears!

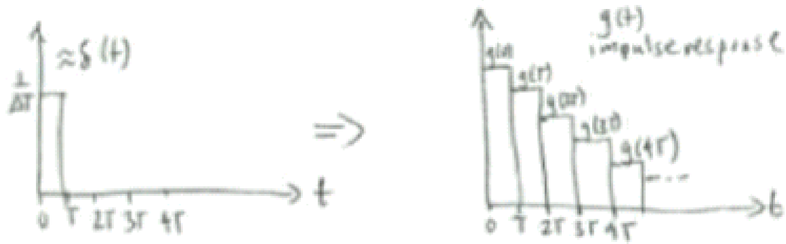
-If we record an impulse at any location, then we can take any other sound recorded on a single microphone, do a convolution and then the result will appear to be playing back from that location. REMINDER: The impulse needs to be normalized

-Now, suppose you have a wav file captured at the same sample rate. NOTE: the result is a discrete time convolution (http://www.youtube.com/watch?v=yyTu0SXeW1M)

-A primer on how to input sound in Matlab can be found at http://class.ee.iastate.edu/mmina/ee186/labs/Audio.htm whereas to send the file back out into a wav format, use
http://www.mathworks.com/help/matlab/ref/wavwrite.html

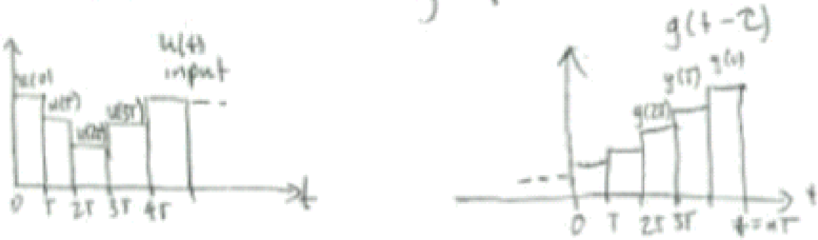
-Audacity or Goldwave are freeware that can be used to create mp3 files so that you can submit your assignment!

Suppose we have a sampled wav file of our impulse. We also have a wav file of our input.



(need to normalize)

We now have an arbitrary input



$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$$

$$y(nT) \approx [g(0)u(nT) + g(T)u((n-1)T) + \dots g(nT)u(0)]$$

$$= \sum_{i=0}^{n} g(iT)u((n-i)T) \quad *$$

Discrete time convolution.

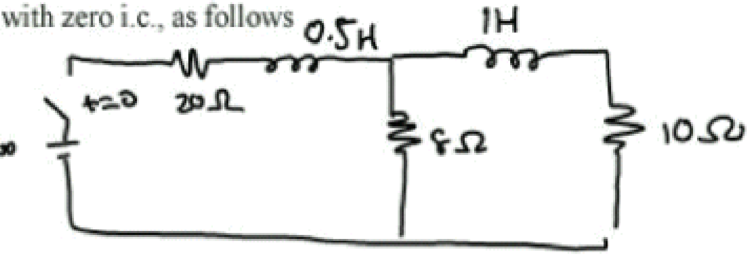
$$y(n) = \sum_{i=-\infty}^{\infty} g(i)u(n-i)$$ but $u(i)=0,\ i<0$; $g(i)=0,\ i<0$

which gives *

E/ Simultaneous Differential Equations

-There are times where there are two or more ODEs that are coupled together

eg/ Suppose we have a circuit, with zero i.c., as follows



$$\frac{di_1}{dt}+\frac{di_2}{dt}+56i_1+40i_2=400$$
$$\frac{di_2}{dt}-8i_1+10i_2=0$$

-One can use Laplace transforms and, since we solve using algebra, we simply have n equations with n unknowns which allows us to solve the problem

eg/
$$(s+56)I_1(s)+(s+40)I_2(s)=400/s$$
$$-8I_1(s)+(s+10)I_2(s)=0$$

-Use algebra to solve:
Eg/
$$I_2(s)=\frac{3200}{s(s+59.1)(s+14.9)}$$ which, using PFE and the inverse Laplace transform, gives

$$i_2(t)=3.64+1.22e^{-59.1t}-4.86e^{-14.9t}$$

Similarly, we can find

$$i_1(t)=4.55-7.49e^{-59.1t}+2.98e^{-14.9t}$$

8. Fourier Series, Integrals and Transforms

-Even, odd and periodic functions

-Periodic functions appear in many applications eg square waves, vibrating strings

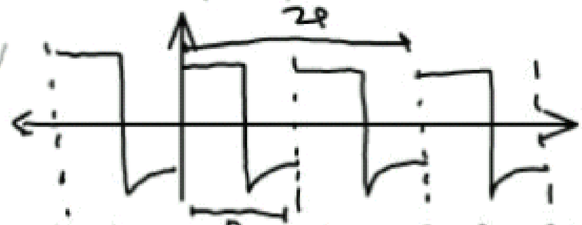
-periodic $f(t) = f(t+p)$, p is the period

-Note- a constant function is periodic with arbitrary period p

-If a function $f(t)$ is periodic with period p, it is also periodic with period $2p$, $3p$, $4p$ etc

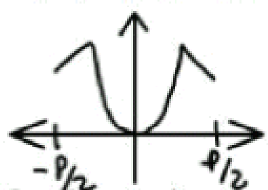
-The smallest such period p is called the fundamental period

eg/



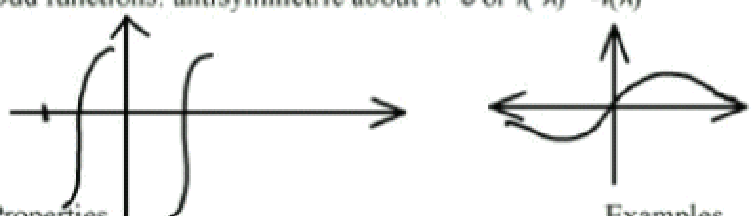
Note: This extends from $-\infty$ to ∞

-Even functions: symmetric about $x=0$ or $f(-x)=f(x)$



e.g. $x^2, x^4, \cos x$

-Odd functions: antisymmetric about $x=0$ or $f(-x)= -f(x)$



e.g. $x, x^3, \sin x$

-Properties	Examples
even+ even= even	$x^2 + x^4$
even X even= even	$x^2 \cdot \cos x$
odd+ odd= odd	$x + x^3$
odd X odd= even	$x \cdot x^3 = x^4$
even X odd= odd	$x \cdot x^2 = x^3$

-Theorem $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & f(x) \text{ is even} \\ 0 & f(x) \text{ is odd} \end{cases}$

Pf/ Even: $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$ Let $u=-x$, $du=-dx$

$$= \int_{a}^{0} f(-u)(-du) + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(x)dx \quad \text{QED}$$

-Now, the following functions all have a period $2L$

$$1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, ..., \cos\frac{m\pi x}{L}, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, ..., \sin\frac{m\pi x}{L}$$

Pf/ $\sin\left(\frac{m\pi x}{L}\right)$ is periodic $\Rightarrow \sin\left(\frac{m\pi(x+2L)}{L}\right) = \sin\left(\frac{m\pi x}{L} + \frac{m\pi 2L}{L}\right)$

$= \sin\left(\frac{m\pi x}{L} + 2\pi m\right)$

Assume $m \in \mathbb{Z}$

$= \sin\left(\frac{m\pi x}{L}\right)$ QED

-Thus, the following also has period $2L$

$$a_0 + a_1\cos\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + ... + a_m\cos\frac{m\pi x}{L} + ... + b_1\sin\frac{\pi x}{L} + b_2\sin\frac{2\pi x}{L} + ... + b_m\sin\frac{m\pi x}{L} + ...$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$$

-If this converges, this will be periodic with period $2L$

-Let's not worry about convergence. The proofs are really, really hard. We will just assume that it does.

-Now, suppose we want to represent some arbitrary function of period $2L$ with such an infinite summation

$\boxed{\text{Aside: } \int_{-L}^{L} f(x)g(x) \text{ is the dot product}}$

-Theorem-Orthogonality properties

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \qquad n \neq m$$

$$\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \qquad n \neq m$$

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 0 \qquad \forall n, m$$

$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$

$= \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$

$= \frac{1}{2}\left[\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right) + \frac{L}{(m-n)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\Big|_{-L}^{L}\right]$

$= 0$ since $\sin(n'\pi) = 0 \ \forall n' \in \mathbb{Z}$

(You need to look at $n=m$ case seperately)

Pf/Use

$$\sin x\sin y = \frac{1}{2}(-\cos(x+y) + \cos(x-y)), \cos x\cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$$

$$\sin x\cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$

$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left[\overbrace{\sin\left(\frac{(n+m)\pi x}{L}\right)}^{\text{odd}} + \overbrace{\sin\left(\frac{(n-m)\pi x}{L}\right)}^{\text{odd}}\right]dx$ $\quad$ odd + odd = odd

$= 0$

-Theorem

$$\underbrace{\int_{-L}^{L} \cos^2 \frac{n\pi x}{L} = L}_{} \qquad \int_{-L}^{L} \sin^2 \frac{n\pi x}{L} = L$$

Pf/ $\int_{-L}^{L} \cos^2 \frac{n\pi x}{L} = \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{2n\pi x}{L}\right)+1\right)dx$

$$= \frac{1}{2}\left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)+x\right]_{-L}^{L}$$

$$= L$$

-Now assume the function $f(x)$ we are trying to represent is periodic.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

This is called a Fourier series. We will assume that the equality holds and that we can interchange an integration and an infinite summation (not necessarily the case)

-Find the coefficients

multiply both sides by $\cos\left(\frac{0\pi x}{L}\right)=1$

a_0 /integrate both sides from $-L$ to L

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L}\{a_0 + \sum_{n=1}^{\infty}\left[\underbrace{a_n \cos\frac{n\pi x}{L}}_{\substack{=0 \text{ (where } m=0 \\ \text{in orthog. theorem)}}} + \underbrace{b_n \sin\frac{n\pi x}{L}}_{}\right]\}dx$$

odd function so integral is 0

$$= 2La_0$$

$$\boxed{\therefore\ a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx}$$

Note: This is the average value of the function and you can often "eyeball" the value.

a_m/ we multiply both sides by $\cos\frac{m\pi x}{L}$ and integrate both sides from $-L$ to L

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}a_0\,dx}_{\substack{=0 \text{ (use}\\ \text{orthog. prop.}\\ \text{when } n=0)}} + \sum_{n=1}^{\infty}\left[\underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}a_n\cos\frac{n\pi x}{L}dx}_{\substack{=0 \quad n\neq m\\ =La_m \quad n=m}} + \underbrace{\int_{-L}^{L}\cos\frac{m\pi x}{L}b_n\sin\frac{n\pi x}{L}dx}_{=0 \text{ always}}\right]$$

$$= La_m$$

$$\boxed{\therefore\ a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx} \qquad m\neq 0$$

b_m/ we multiply both sides by $\sin\frac{m\pi x}{L}$ and integrate both sides from $-L$ to L

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \underbrace{\int_{-L}^{L}\sin\frac{m\pi x}{L}a_0\,dx}_{=0} + \sum_{n=1}^{\infty}\left[\underbrace{\int_{-L}^{L}\sin\frac{m\pi x}{L}a_n\cos\frac{n\pi x}{L}dx}_{=0 \text{ always}} + \underbrace{\int_{-L}^{L}\sin\frac{m\pi x}{L}b_n\sin\frac{n\pi x}{L}dx}_{\substack{=0 \quad n\neq m\\ =Lb_m \quad n=m}}\right]$$

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = b_m\cdot L$$

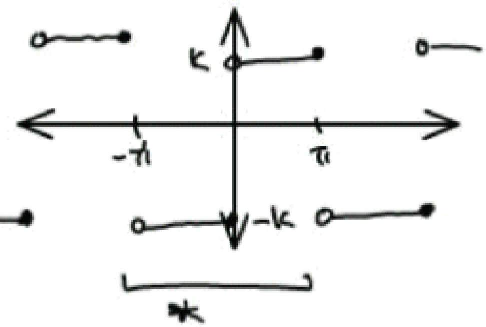
$$\boxed{\therefore\ b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$

-The coefficients are called the Fourier coefficients and the resulting summation is the Fourier Series

-Theorem- Let f be $2L$ periodic and let f and f' be piecewise continuous on the interval from $-L$ to L. Then the Fourier Series converse to $f(x)$ at every point of x where f is continuous and to the mean value $[\,f(x^+) + f(x^-)]/2$ if it is discontinuous

Recall: $f(x)$ is piecewise continuous if it has right & left hand limits that are finite

Eg/ $f(x) = \begin{cases} -k & , -\pi < x \leq 0 \\ k & 0 < x \leq \pi \end{cases}$ and periodic with period 2π



<u>Soln:</u> $f(x+2\pi) = f(x) \therefore$ period $= 2\pi$

so $L = \pi$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$ (Look at the "average value"

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{f(x)}\, \underset{\text{even}}{\cos\left(\frac{n\pi x}{\pi}\right)}\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\,\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

odd × even = odd

$\int$ odd $= 0$

$$= 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\Big|_{-\pi}^{0} + (-k)\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{2k}{n\pi}\left(1 - \frac{\cos n\pi}{2} - \frac{\cos(-n\pi)}{2}\right)$$

$$= \frac{2k}{n\pi}\left(1 - \cos n\pi\right)$$

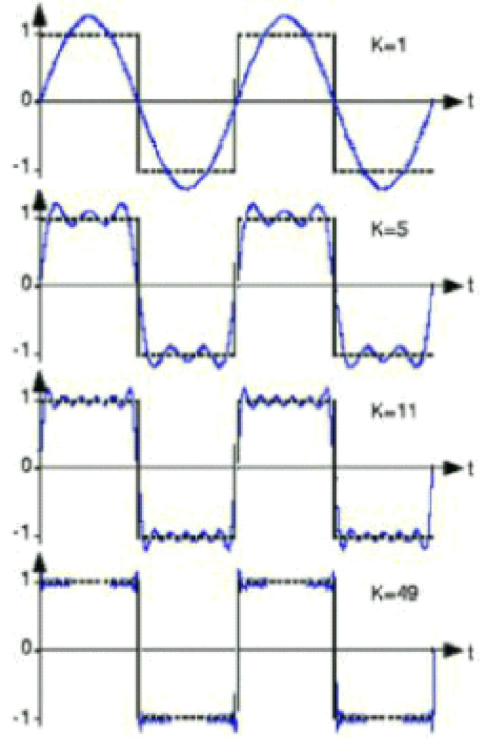
$\cos n\pi = \begin{cases} 1 & \text{when } n \text{ is even} \\ -1 & \text{when } n \text{ is odd} \end{cases}$

$\therefore b_n = \dfrac{4k}{n\pi}$ when n is odd

$$\therefore f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

S_1, S_2, S_3

$$= \frac{4k}{\pi}\sum_{r=0}^{\infty}\frac{\sin(2r+1)x}{(2r+1)}$$



S_1

S_5

S_{11}

Aside:

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \dots\right)$$

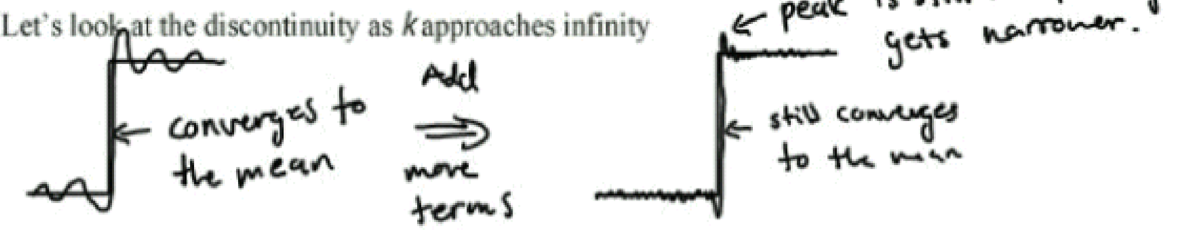
$$= \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

formula for π

-Gibbs Phenomenon

Let's look at the discontinuity as k approaches infinity

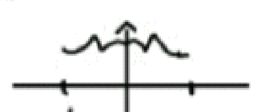


We always get an overshoot! The overshoot narrows but never goes down to zero. Thus, even though it does converge, the convergence can be very slow.

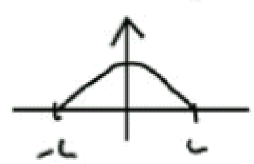
-If we can observe whether whether the function is an even or odd function, many of the Fourier coefficients become zero

-Theorem- A Fourier series of an even function of period $2L$ is a Fourier Cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad , a_0 = \frac{1}{L}\int_0^L f(x)dx \quad , a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$



Pf/ $\cos\frac{n\pi x}{L}$ is even and $\sin\frac{n\pi x}{L}$ is odd



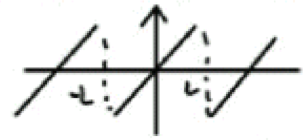
If $f(x)$ is even, $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L f(x)dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{\text{even} \times \text{even} = \text{even}}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{\text{even} \times \text{odd} = \text{odd}}dx = 0$$

-Theorem- A Fourier series of an odd function of period 2L is a Fourier Sine series

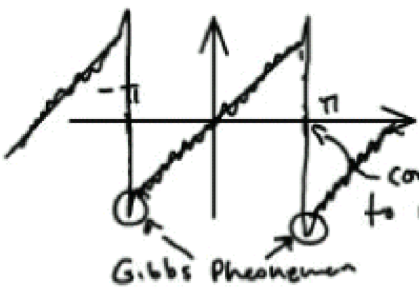
$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad , b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$



Pf/ Similarly, $a_0=0$ and $a_n=0$ as we are integrating an odd function

$$b_n=\frac{1}{L}\int_{-L}^{L} \underset{\text{odd}}{f(x)}\, \underset{\times\ \text{odd}\ =\ \text{even}}{\sin\frac{n\pi x}{L}}\,dx=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Eg. Sawtooth



$f(x)=x \quad , -\pi\le x\le\pi$ and is periodic

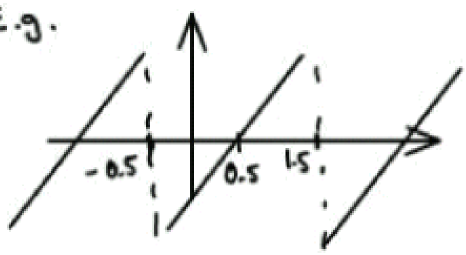
This is odd $\therefore a_0=0 \ , \ a_n=0$

$$b_n=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx=\frac{2}{\pi}\left[\frac{\sin nx-nx\cos nx}{n^2}\Bigg|_0^{\pi}\right]$$

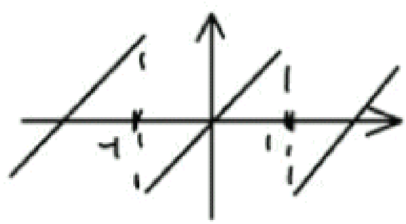
$$=-\frac{2}{n}\cos n\pi \quad =-\frac{2}{n}(-1)^n$$

$$\therefore f(x)= 2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\dots\right)$$

E.g.



Let $x'=x-0.5$ $\longrightarrow$



Look for fourier series $f(x')\Rightarrow$ only has b_n non-zero term

$$f(x')=\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x')\sin\frac{n\pi x'}{L}\,dx\ \sin\frac{n\pi x'}{L}$$