Maclaurin Series = Taylor series about $x=0$.

Infinite Series

Def^n: An infinite series (or just series) of constants a_k is defined as:

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k$$

Def^n: Let $\{S_n\}$ be the sequence of partial sums, defined as:

$$S_0 = a_0$$
$$S_1 = a_0 + a_1$$
$$S_2 = a_0 + a_1 + a_2, \text{ etc.}$$
$$S_n = a_0 + a_1 + \dots + a_n = \sum_{k=0}^{n} a_k$$

If $\{S_n\}$ converges (to say S), that is, if $\lim_{n\to\infty} S_n = S$, we say that the series $\sum a_k$ is convergent, with sum S.

Otherwise, it is divergent.

Ex. $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

Form the partial sums:

$S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$ .....

In general, $S_n = 1 - \frac{1}{2^n}$ or $\frac{2^n - 1}{2^n}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k}$ converges to 1.

This is an example of a geometric series.

General Form:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots \quad \begin{pmatrix} a\text{-first term} \\ r\text{-common ratio} \end{pmatrix}$$

For which value of r does this converge?

For $|r|<1$. Result:

$$\underbrace{\sum_{k=0}^{\infty} ar^k \text{ converges to } \frac{a}{1-r} \text{ if } |r|<1.}_{\text{prev. ex: } a=\frac{1}{2},\ r=\frac{1}{2} \Rightarrow \text{sum}=1.}$$

Eg. $\sum_{k=0}^{\infty} 10^k = 1+10+100+\dots\dots$ diverges

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Recall this Calculation.

$$\begin{aligned} 0 &= (1-1)+(1-1)+(1-1)+\dots \\ &= 1-(1-1)-(1-1)-(1-1)-\dots \\ &= 1-0-0-0\dots \\ &= 1 \end{aligned}$$

Now we can see what's wrong – construct $\{S_n\}$.

for $\sum_{k=0}^{\infty} (-1)^k$

$\{S_n\} = \{1,0,1,0,1,0,\dots\} \leftarrow$ sequence has no limit.

By definition, the series is divergent.

One other important series is the harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$$

The series is divergent.

Idea of proof:

The sum is: $1 + \frac{1}{2} + \underbrace{\left(\underbrace{\frac{1}{3}}_{>\frac{1}{4}} + \frac{1}{4}\right)}_{>\frac{1}{2}} + \underbrace{\left(\underbrace{\frac{1}{5}}_{>\frac{1}{8}} + \underbrace{\frac{1}{6}}_{>\frac{1}{8}} + \underbrace{\frac{1}{7}}_{>\frac{1}{8}} + \underbrace{\frac{1}{8}}_{>\frac{1}{8}}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{>\frac{1}{2}} + \cdots$

In general, we can write an inequality for S_{2^n}:

$S_{2^n} \geq 1 + \frac{n}{2}$ since $1 + \frac{n}{2} \to \infty$ as $n \longrightarrow \infty$, then $S_{2^n} \to \infty$ as $n \to \infty$ & the series is divergent.

The requirement that $\lim_{k \to \infty} a_k = 0$ is a necessary condition for convergence of $\sum_{k=0}^{\infty} a_k$, but it is not sufficient.

Note: For any series $\sum a_k$, there are two associated sequences:

1) The sequence of terms $\{a_k\}$.

2) The sequence of partial sums $\{S_n\}$.

The test of divergence:

For a series $\sum a_k$, if $\lim_{k \to \infty} a_k \neq 0$ or does not exist, then $\sum a_k$ is divergent.

Ex: $\sum_{k=1}^{\infty} \frac{k}{2k+1} = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \cdots$

$\lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$ → series is divergent.