Recall: $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

$\therefore a_{-n} = a_n$

$b_{-n} = -b_n$

-Complex Fourier Series

Now, recall the form of the Fourier Series. We can slightly rewrite as

$$f(x) = \sum_{n=0}^{\infty}\left[a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L} + \frac{b_n}{2}\sin\frac{n\pi x}{L}\right] + \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L} + \frac{b_n}{2}\sin\frac{n\pi x}{L}\right]$$

$\cos\left(-\frac{n\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right)$

$\sin\left(-\frac{n\pi x}{L}\right) = -\sin\left(\frac{n\pi x}{L}\right)$

$$= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L} + \frac{b_n}{2}\sin\frac{n\pi x}{L}\right] + \sum_{n=0}^{\infty}\left[\frac{a_{-n}}{2}\cos\frac{-n\pi x}{L} + \frac{b_{-n}}{2}\sin\frac{-n\pi x}{L}\right]$$

$$= \sum_{n=0}^{\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L} + \frac{b_n}{2}\sin\frac{n\pi x}{L}\right] + \sum_{n=0}^{-\infty}\left[\frac{a_n}{2}\cos\frac{n\pi x}{L} + \frac{b_n}{2}\sin\frac{n\pi x}{L}\right]$$

Note: n is a dummy variable

-Therefore, we can easily show that this is the same as

$$f(x) = \sum_{n=-\infty}^{\infty}\left[a_n'\cos\frac{n\pi x}{L} + b_n'\sin\frac{n\pi x}{L}\right] \text{ where } a_n' = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \text{ and}$$

$$b_n' = \frac{1}{2L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

(Note what happens with the a_0)

-Now, $\cos\frac{n\pi x}{L} = \frac{e^{\frac{in\pi x}{L}} + e^{\frac{-in\pi x}{L}}}{2}$, $\sin\frac{n\pi x}{L} = \frac{e^{\frac{in\pi x}{L}} - e^{\frac{-in\pi x}{L}}}{2i} = \frac{-i\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)}{2}$

Thus, $f(x) = \sum_{n=-\infty}^{\infty} a_n'\left(\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2}\right) - \frac{ib_n'}{2}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{a_n' - ib_n'}{2}\right)e^{\frac{in\pi x}{L}} + \underbrace{\sum_{n=-\infty}^{\infty}\left(\frac{a_n' + ib_n'}{2}\right)e^{-\frac{in\pi x}{L}}}_{\text{sub } n \text{ with } -n}$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{a_n' - ib_n'}{2} + \frac{a_{-n}' + ib_{-n}'}{2}\right)e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty}\underbrace{\left(a_n' - ib_n'\right)}_{c_n}e^{\frac{in\pi x}{L}} \qquad \text{since } a_{-n}' = a_n',\ b_{-n}' = -b_n'$$

$$\boxed{\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}}$$

-but $c_n = a_n - b_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx - \frac{1}{2L}\int_{-L}^{L} f(x) i \sin\frac{n\pi x}{L}dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}dx$

Thus

$$\boxed{f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}dx}$$

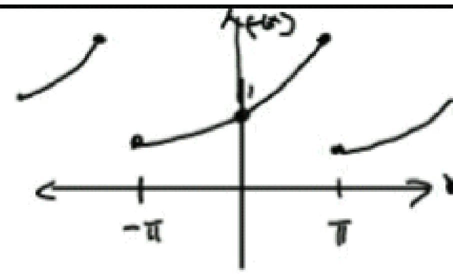
Complex Fourier Series

Note, $c_{-n} = \bar{c}_n$ (complex conjugate) guarantees $f(x)$ ends up as a real expression after evaluating the series

-eg/ $f(x) = e^x \quad -\pi < x < \pi$

Soln/ $f(x)$ is periodic

$L = \pi$



$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-in}e^{(1-in)x}\Big|_{-\pi}^{\pi}\right]$$

Recall $e^{in\pi} = e^{-in\pi} = (-1)^n$ ☆

$$\therefore c_n = \frac{1}{2\pi}\cdot\frac{(1+in)}{(1+n^2)}(-1)^n\underbrace{\left(e^{\pi} - e^{-\pi}\right)}_{2\sinh\pi}$$

$$\therefore e^x = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{(1+in)}{(1+n^2)}e^{inx}$$

Is this real? Note that c_n & c_{-n} are complex conjugates

Also e^{inx} & e^{-inx} are complex conjugates

$\therefore$ $+|n|$ and $-|n|$ terms are complex conjugates

For the numerical example,

for $n>0$, the term $c_n e^{\frac{in\pi x}{L}}$ is $(1+in)(\cos nx + i\sin nx)$

$n<0$ $\qquad c_{-|n|}e^{-\frac{i|n|\pi x}{L}}$ is $(1-i|n|)(\cos|n|x + i\sin|n|x)$

add these to get $\underline{2(\cos nx - n\sin nx)}$ for $n = 1, 2, \ldots$

$\hookrightarrow$ REAL

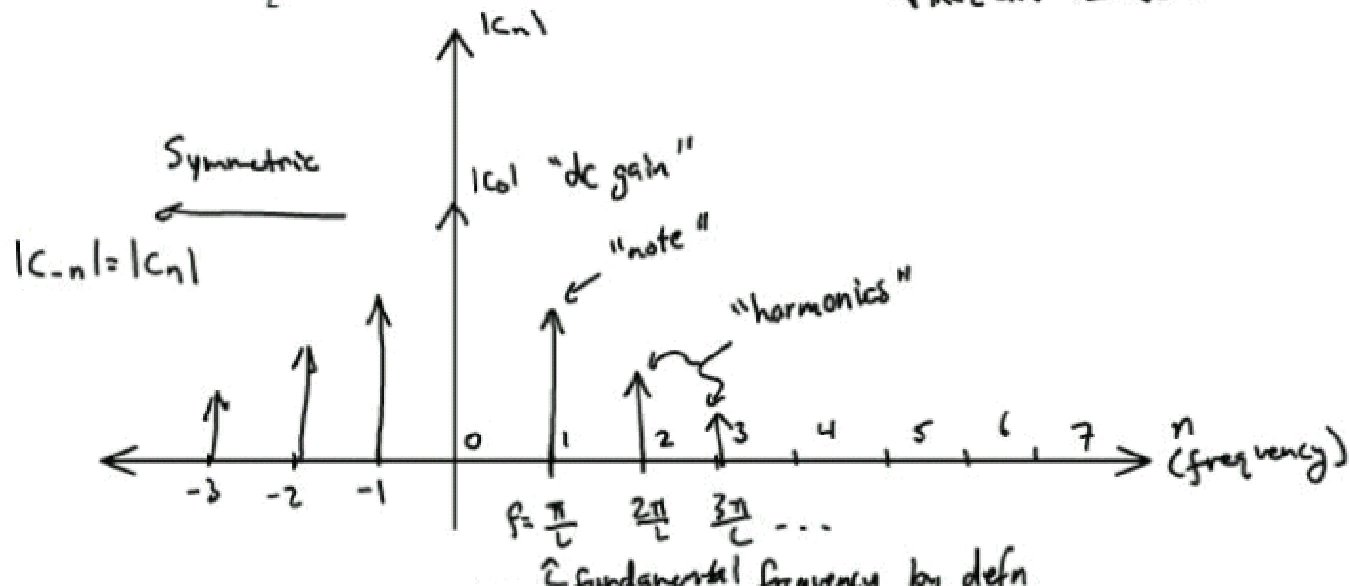
-Amplitude spectrum

The Fourier Series are very useful in finding how much signal there is at each frequency, ie, the frequency content

The plots are done either using the real representation or the complex representation (more commonly the latter)

Now, $f(x)=\sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$ $\quad c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{-in\pi x}{L}}dx$ $\quad |c_n|$ = amount of signal at each frequency $\frac{n\pi}{L}$

and let $\omega_0=\frac{\pi}{L}$ (the fundamental frequency) $\quad |c| = \sqrt{Re\{c_n\}^2 + Im\{c_n\}^2}$



-It turns out (see text for details, pg ~~214~~ 317) that the power at each frequency is given by $|c_0|^2$ for the constant component and $|c_{-n}|^2+|c_n|^2$ or $2|c_n|^2$ for the nth component

$$\text{i.e.}\quad \underbrace{\frac{1}{T}}_{}\int_{C}^{C+T} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Power = $\frac{\text{Energy}}{\text{Time}}$ $\quad$ power related to signal squared

e.g. $KE=\frac{1}{2}mv^2$ $\quad P=\frac{1}{2}Ri^2$

$\boxed{\text{Parseval's Theorem}}$

If you use Fourier Series,

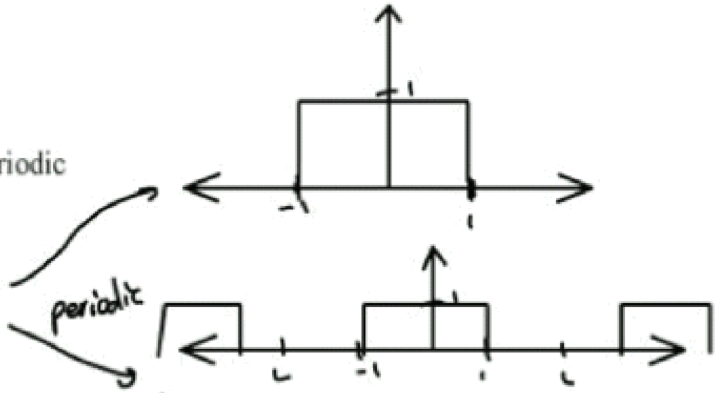
$$\frac{1}{T}\int_{C}^{C+T} |f(t)|^2\,dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

(Easy to show using orthogonality)

-Fourier Integrals

Now, what happens if the signal is aperiodic

Eg. Pulse $f(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 \leq x < 1 \\ 0 & 1 \leq x \leq L \end{cases}$



↳ This is even ∴ Fourier Cosine Series

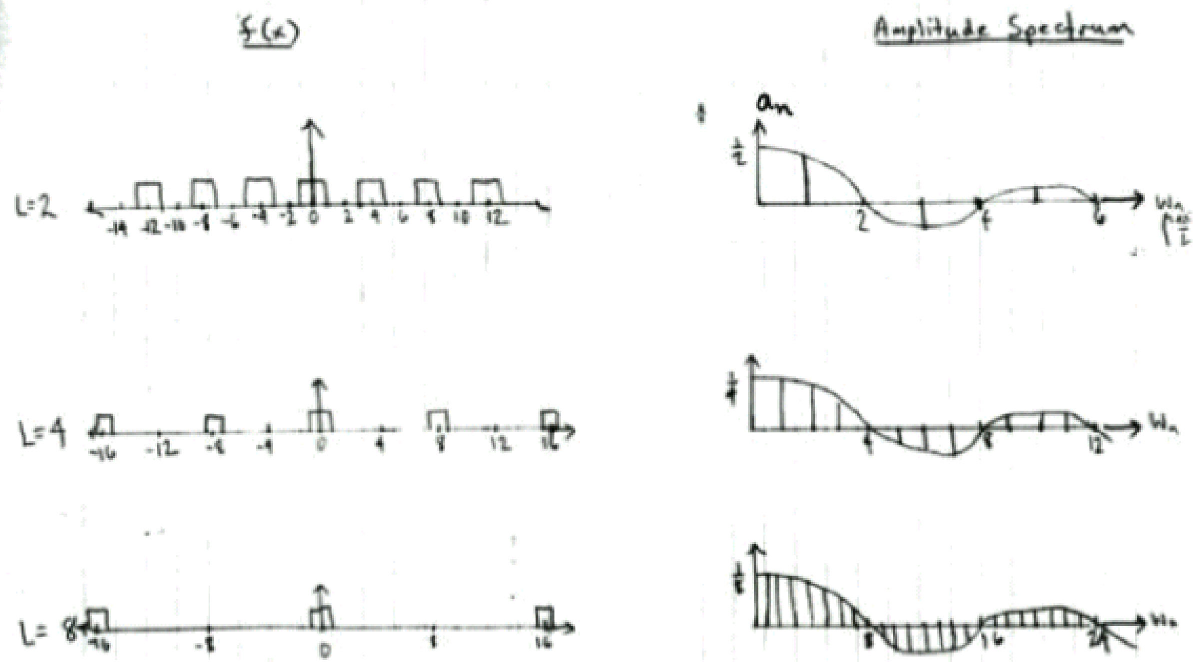
$a_0 = \frac{1}{L} \quad b_n = 0$

$a_n = \frac{2}{L}\left(\frac{\sin(\frac{n\pi}{L})}{\frac{n\pi}{L}}\right)$

What happens to the Fourier Series as $L \to \infty$?

Let $\omega_n = \dfrac{n\pi}{L}$

What happens to the amplitude spectrum?



As $L \to \infty \quad w_n = \frac{n\pi}{L} \to 0$

∴ We get more points in the amplitude spectrum that are more densely packed

i.e. we approach a continuum

Now,

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}[\cos\omega_n x\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv]$$

Let $\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$ or $\frac{1}{L} = \frac{\Delta w}{\pi}$

Assume $\int_{-\infty}^{\infty}|f(x)|dx$ is finite (Absolutely integrable)

Thus, the term $\frac{1}{2L}\int_{-L}^{L} f_L(v)dv \to 0$ as $L \to \infty$

$$f_L(x) = \underbrace{\frac{1}{\pi}\sum_{n=1}^{\infty}[\cos\omega_n x\int_{-L}^{L} f_L(v)\cos(\omega_n v)dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin(\omega_n v)dv]\Delta\omega}_{\text{Riemann Integral}}$$

As $L \to \infty$, $\Delta w \to dw$ and this becomes an Integral

Thus,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[\cos\omega x\underbrace{\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv}_{A(w)} + \sin\omega x\underbrace{\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv}_{B(w)}]d\omega$$

no longer periodic

The following theorem is hard to prove

Thm- If $f(x)$ is piecewise continuous in every finite interval and has a right hand and left hand derivative at each point, and if $\underline{\int_{-\infty}^{\infty}|f(x)|dx < \infty}$, then $f(x)$ can be represented by

$$f_L(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]dw$$

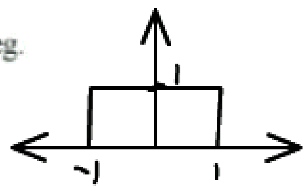
where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv$

At each point where $f(x)$ is discontinuous, then $f(x) = \frac{1}{2}[f(x_-) + f(x_+)]$

-Gibbs Phenomena also still holds at discontinuities
The peak narrows as $L \to \infty$

-When finding the coefficients, the same principle applies with respect to integrating even and/or odd functions. That is, odd functions involve only $B(w)$ and even functions only involve $A(w)$

-eg.



This is an even function

$\therefore B(w)=0$

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty}\underbrace{f(v)\cos(wv)}_{\text{even}}dv=\frac{2}{\pi}\int_0^{\infty}f(v)\cos(wv)dv$$

$$=\frac{2}{\pi}\int_0^1\cos(wv)dv$$

$$=\frac{2\sin w}{\pi w} \leftarrow \text{should only be a function of } w$$

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos wx\left(\frac{\sin w}{w}\right)dw$$

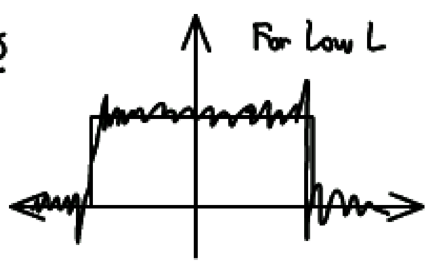
Note: at $x=\pm 1$, the function integral converges to $1/2$.

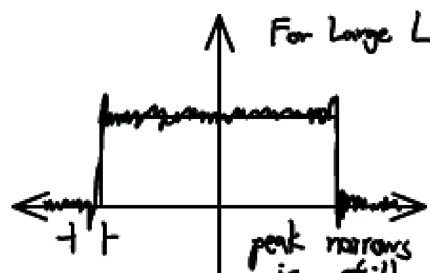
-Look at this example and, instead of integrating to infinity, let's integrate to L

The approximation is $f(x)=\frac{2}{\pi}\int_0^L \cos wx\left(\frac{\sin w}{w}\right)dw$

As $L\to\infty$, we get closer & closer to the square function

Gibbs





peak narrows but height is still periodic

-This is the Gibb's phenomenon. Thus, it also happens at discontinuities for Fourier Integrals. In other words, we get this overshoot phenomenon that gets narrower as L gets larger.

-Now, substitute the complex exponential for the sin and cos terms

$$f(x)=\int_0^{\infty}\left[A(w)\frac{1}{2}(e^{iwx}+e^{-iwk})+B(w)\frac{1}{2i}(e^{iwx}-e^{-iwx})\right]dw$$

$$=\int_0^{\infty}\left\{\underbrace{\frac{1}{2}(A(w)-iB(w))}_{c(w)}e^{iwx}+\underbrace{\frac{1}{2}(A(w)+iB(w))}_{\bar{c}(w)\text{ complex conjugate}}e^{-iwx}\right\}dw$$

$$f(x)=\int_0^{\infty}c(w)e^{iwx}dw+\underline{\int_0^{\infty}\bar{c}(w)e^{-iwx}dw}$$

Let $w'=-w$, $dw=-dw'$

$$=\int_0^{\infty}c(w)e^{iwx}dw+\int_0^{-\infty}\underbrace{\bar{c}(-w')}_{\frac{1}{2}(A(-w')+iB(-w'))=\frac{1}{2}(A(w')-iB(w'))=c(w')}e^{iw'x}(-dw')$$

cos sin

$$\therefore f(x)=\int_0^{\infty}c(w)e^{iwx}dw-\int_0^{-\infty}c(w')e^{iw'x}dw'$$

$$=\int_0^{\infty}c(w)e^{iwx}dw-\int_{-\infty}^{0}c(w')e^{iw'x}dw'$$

$$=\int_{-\infty}^{\infty}c(w)e^{iwx}dw \Rightarrow c(w)=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}f(v)\cos wv\,dv-i\int_{-\infty}^{\infty}f(v)\sin wv\,dv\right]$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(v)\underbrace{(\cos wv-i\sin wv)}_{e^{iwv}}dv)$$

-Thus

$$f(x)=\int_{-\infty}^{\infty}C(\omega)e^{i\omega x}d\omega \text{ where } C(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv$$

-eg $f(x)=xe^{-|x|}$

continuous & odd

Is this abs integrable? i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ bounded

$\int_{-\infty}^{\infty}|f(x)|dx=\int_{-\infty}^{\infty}\underbrace{|x|e^{-|x|}}_{\text{even}}dx=2\int_0^{\infty}xe^{-x}dx=2$ $\therefore$ bounded can have n complex F. I.

$$c(w)=\frac{1}{2\pi}\int_{-\infty}^{\infty}te^{-|t|}dt=\frac{1}{2\pi}\int_{-\infty}^{0}te^{t}e^{-iwt}dt+\frac{1}{2\pi}\int_0^{\infty}te^{-t}e^{-iwt}dt$$

$$=\frac{-2iw}{(1+w^2)^2\pi} \quad \therefore xe^{-|x|}=-\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{w}{(1+w^2)^2}e^{iwx}dw$$

-This leads directly the to Fourier Transform with just a reshuffling of constants

Given that a function f is piecewise continuous on [-L,L] for any L.

Suppose $\int_{-\infty}^{\infty}|f(t)|dt$ is bounded. The Fourier transform of f is

$$F[f(t)]=F(w)=\int_{-\infty}^{\infty}f(t)e^{-iwt}dt$$

Recalling the discussion about the amplitude spectrum, this gives an indication of the "amount of signal" at any frequency w. The amplitude spectrum is a graph of $|F(w)|$ vs w

-There are tables of Fourier transforms

-This is the second most used transform next to Laplace

-By observation, the inverse Fourier Transform is

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega = F^{-1}[F(w)]$$

-An important property is Parseval's identity

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Thus, $F(w)$ gives amplitude and the area squared gives a sense of the power

-Properties include linearity and differentiation is multiplication by iw

ie $F[d\{f(t)\}/dt] = iwF(w)$ and

-Convolution still holds as well

-Applications in optics, digital filtering, spectral analysis

Fourier Transforms

-Applications in optics, digital filtering, spectral analysis

- Definition: Given that a function f is piecewise continuous on $[-L,L]$ for any L.

Suppose $\int_{-\infty}^{\infty} | f(t) | dt$. The Fourier transform of $f(t)$ is

$$F[f(t)]=F(w)=\int_{-\infty}^{\infty} f(t)e^{-iwt}dt$$

-Definition: The inverse Fourier Transform is

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = F^{-1}[F(w)]$$

-The absolutely integrable condition rules out a number of functions

Eg. $f(t) = \text{constant}$ X
$f(t) = \sin wt$ X
$f(t) = e^{\alpha t}, \alpha < 0$ X (It blows up as $t \to -\infty$)

In practice, we look at infinite signals that start & end

-Eg. Find the Fourier Transform of $f(t) = e^{-at}$, $a > 0, t > 0$

↳ You have to show its absolutely integrable

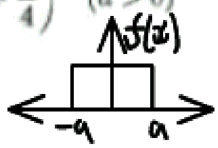
$$F(f(t)) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-iwt} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + iw)t} dt$$

$$= \frac{1}{\alpha + iw}$$

→ To see frequency content, Plot $|F(w)|$ vs. w ⇐ key in applications

-Fourier Tables

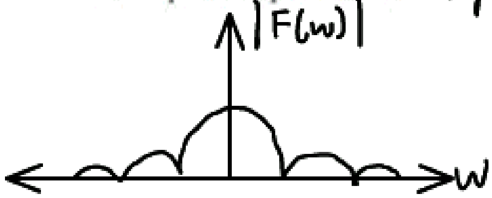
	$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
1.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\dfrac{\pi}{a} e^{-a\lvert\omega\rvert}$
* 2.	$H(x) e^{-ax} \quad (\operatorname{Re} a > 0)$	$\dfrac{1}{a + i\omega}$
3.	$H(-x) e^{ax} \quad (\operatorname{Re} a > 0)$	$\dfrac{1}{a - i\omega}$
* 4.	$e^{-a\lvert x\rvert} \quad (a > 0)$	$\dfrac{2a}{\omega^2 + a^2}$
5.	e^{-x^2}	$\sqrt{\pi} e^{-\omega^2/4}$
6.	$\dfrac{1}{2a\sqrt{\pi}} e^{-x^2/(2a)^2} \quad (a > 0)$	$e^{-a^2\omega^2}$
7.	$\dfrac{1}{\sqrt{\lvert x\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$
8.	$e^{-a\lvert x\rvert/\sqrt{2}} \sin\left(\dfrac{a}{\sqrt{2}} \lvert x\rvert + \dfrac{\pi}{4}\right) \quad (a > 0)$	$\dfrac{2a^3}{\omega^4 + a^4}$
* 9.	$H(x + a) - H(x - a)$ 	$\dfrac{2 \sin \omega a}{\omega}$
10.	$\delta(x - a)$	$e^{-i\omega a}$
11.	$f(ax + b) \quad (a > 0)$	$\dfrac{1}{a} e^{ib\omega/a} \hat{f}\left(\dfrac{\omega}{a}\right)$
12.	$\dfrac{1}{a} e^{-ibx/a} f\left(\dfrac{x}{a}\right) \quad (a > 0,\ b \text{ real})$	$\hat{f}(a\omega + b)$
13.	$f(ax) \cos cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega - c}{a}\right) + \hat{f}\left(\dfrac{\omega + c}{a}\right)\right]$
14.	$f(ax) \sin cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega - c}{a}\right) - \hat{f}\left(\dfrac{\omega + c}{a}\right)\right]$
15.	$f(x + c) + f(x - c) \quad (c \text{ real})$	$2\hat{f}(\omega) \cos \omega c$

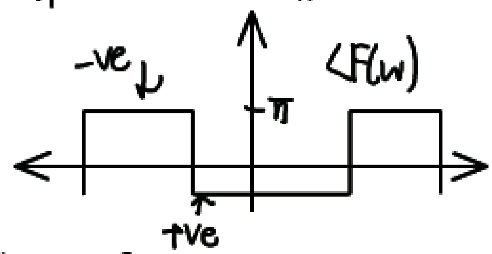
$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
16. $f(x+c) - f(x-c)$ (c real)	$2i\hat{f}(\omega)\sin \omega c$
17. $x^n f(x)$ $(n = 1, 2, \ldots)$	$i^n \dfrac{d^n}{d\omega^n}\hat{f}(\omega)$
Linearity of transform and inverse:	
18. $\alpha f(x) + \beta g(x)$	$\alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$
Transform of derivative:	
* 19. $f^{(n)}(x)$ similar to Laplace property	$(i\omega)^n \hat{f}(\omega)$ Is there a relationship between Laplace & Fourier
Transform of integral:	
20. $f(x) = \int_{-\infty}^{x} g(\xi)\,d\xi$, where $f(x) \to 0$ as $x \to \infty$	$\hat{f}(\omega) = \dfrac{1}{i\omega}\hat{g}(\omega)$
Fourier convolution theorem:	
21. $(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$	$\hat{f}(\omega)\hat{g}(\omega)$

e.g. pulse $f(t)=H(x+1)-H(x-1)$

phase

-The amplitude spectrum is a plot of $|F(w)|$ vs. w and $\angle F(w)$ vs. w





-Properties:

1. Linearity $F[af(t)+bg(t)]=aF[f(t)]+bF[g(t)]$

2. Time-shifting $F[f(t-t_0)]=\underbrace{e^{-iwt_0}}_{}F(w)$ → magnitude $=1$ & phase change

3. Frequency-shift property

If $F[f(t)]=F(w)$, then $F[e^{iw_0t}f(t)]=F(w-w_0)$ → used in AM transmission

4. Differentiation property

If $f(t)$ is continuous & $f(t)\to 0$ as $|x|\to\infty$ and $f'(t)$ is abs. integrable, then $F\{f'(t)\}=iwF(w)$

5. Convolution property

$F\{f(x)*g(x)\}=F(w)G(w)$

-An important property is Parseval's identity

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$$

I can look at power either in time or frequency domain

Thus, $F(w)$ gives amplitude and the area squared gives a sense of the power

-The relationship between Fourier and Laplace Transforms

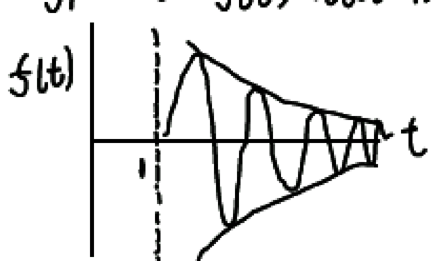
From Laplace, $f(t)=0$ for $t<0$ & $\mathcal{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}dt$

From Fourier, $f(t)=0$ for $t<0$ & abs integrable

then $F\{f(t)\}=\int_{-\infty}^{\infty}f(t)u(t)e^{-iwt}dt=\int_0^\infty f(t)e^{-iwt}dt$

The two are identical except $s=iw$

eg/ when $f(t)=u(t-1)e^{-(t-1)}\sin(t-1)$, $t\geq 0$ Find $F(w)$



This is abs. int. since it's bounded by e^{-t} which is abs. int.

$\mathcal{L}\{f(t)\}=e^{-s}\frac{1}{(s+1)^2+1^2}$, $F(w)=\underbrace{e^{-iw}}_{\text{phaseshift}}\frac{1}{(iw+1)^2+1^2}$

$e^{-as}F(s) = \mathcal{L}\{u(t-a)f(t-a)\}$

9. Partial Differential Equations

-looking at solutions that are a function of several variables

Eg. Temperature in a thin rod $u(x,t)$

-Notation for partial derivatives (x,t are the independent variables and u is the dependent)

$$\frac{\partial u(x,t)}{\partial x} \equiv u_x, \quad \frac{\partial^2 u(x,t)}{\partial x^2} \equiv u_{xx}, \quad \frac{\partial^2 u(x,t)}{\partial x \partial t} \equiv u_{xt}$$

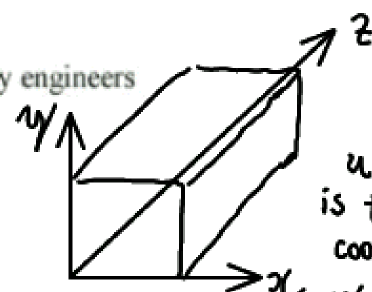
-Thus, the equations governing the behavior must be partial differential equations (PDEs) rather than ordinary differential equations

Eg. $\frac{\partial^2 u(x,t)}{\partial x^2} = -\frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$, α is a constant

-No set methodology to solve!

-Often cannot solve exactly but only approximately

-We will only examine three of the most common ones seen by engineers



u(x,y,z) is temp at a coordinate

We have the surfaces at a fixed temp

i) Laplace equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Describes the steady state heat equation, electrostatic potential in a uniform dielectric, steady state shape of an elastic membrane

ii) Wave equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u(x,y,z,t)}{\partial t^2}$$

Describes propagation of electromagnetic waves, sound vibrations

iii) Heat equation

$$\nabla^2 u = \frac{\partial^2 u(x,y,z,t)}{\partial x^2} + \frac{\partial^2 u(x,y,z,t)}{\partial y^2} + \frac{\partial^2 u(x,y,z,t)}{\partial z^2} = \frac{1}{k}\frac{\partial u(x,y,z,t)}{\partial t}$$

Describes how heat is transferred from a hot area to a cold area by conduction

-Order of a PDE is the highest partial derivative appearing in the equation

Eg all of the above examples are second order. Eg. $\frac{\partial^4 u(x,t)}{\partial x^4} = \frac{-k\partial^2 u(x,t)}{\partial t^2}$ (beam equation) is 4^{th} order

-Can use differential operators to talk about PDEs

Eg $\mathcal{L}\{u\} \hat{=} \left\{\frac{\partial^4}{\partial x^4} + \frac{k\partial^2}{\partial t^2}\right\} u(x,t)$ (beam equation)

$\mathcal{L}\{u\} \hat{=} \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u$

-Linear PDEs

A linear PDE is one that satisfies the following:

$\mathcal{L}\{\alpha u + \beta v\} = \alpha\mathcal{L}\{u\} + \beta\mathcal{L}\{v\}$ where $u(x,t)$ & $v(x,t)$ are two functions.

(all the above examples are linear) $\left\{\begin{array}{l} \text{e.g. Laplace} \to \nabla^2 \text{ is the operation} \\ \nabla^2(\alpha u + \beta v) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\alpha u + \beta v) \end{array}\right.$

$= \alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u + \beta(\sim) v$

$= \alpha\nabla^2 u + \beta\nabla^2 v$

Eg. nonlinear PDE

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$ non-linear

-Homogeneous PDEs

$\mathcal{L}\{u\} = 0$

-Non-homogeneous PDEs

$\mathcal{L}\{u\} = f(u)$

-A linear combination of solutions to a linear homogeneous PDE is also a solution

If $u_1, \ldots, u_m$ are all sol'ns to $\mathcal{L}\{u\} = 0$, then so is $\sum_{i=1}^{m} c_i u_i$ where c_i are constants

-Any solution to the nonhomogeneous PDE is called a particular solution

$\mathcal{L}\{u_p\} = f(u_p)$ ie. u_p satisfies the non-homog

-If we can the set of all solutions to the linear homogeneous ODE, then the set of all solutions to the nonhomogeneous linear PDE is $u + u_p$ for some $u \in S$ where S is the set of all solutions to the homog. and u_p is a particular soln.

It's really hard to find S ie. all soln

-There are constant and variable coefficient coefficient (u and its partial derivatives appear in the first power only) linear PDEs. There can also be systems of PDEs. We will only look at the three examples, however. We will also not look at modelling as these are the subjects of other courses.

-We often cannot find ALL the solutions of the linear homogeneous ODE. In some cases, there will be an infinite number of linearly independent solutions. We have no guarantee that these are ALL of the solutions but if they allow us to satisfy the initial conditions and boundary conditions, everything still works practically

-Classification of second order linear PDEs

$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$ where A, B, C are functions of x, y and D can be a function of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

Parabolic if $B^2 - AC = 0$
Eg. Heat equation

Hyperbolic if $B^2 - AC > 0$
Eg Wave Equation

Elliptic if $B^2 - AC < 0$
Eg. Laplace Equation

WAVE EQUATION

In general, we have $c^2\nabla^2 u = u_{tt}$ where c is the speed of propagation

Let's consider a simpler example which is one dimensional. Let us look at a vibrating string where the speed of propagation is given by

$$c^2 u_{xx} = u_{tt}$$

We will solve for $0 < x < \pi$. We need conditions at both these ends.

$$u(0,t) = 0, \qquad u(\pi,t) = 0$$

We also need initial conditions. Suppose the string starts at rest with the following condition.

$$u(x,0) = \begin{matrix} x \\ \pi - x \end{matrix}$$

A trivial solution of this is $u(x,t) = 0$, which satisfies the BC but not the initial conditions

Use separation of variables $u(x,t) = X(x)T(t)$

$$c^2 X_{xx} T = X T_{tt} \text{ or } \frac{X_{xx}}{X} = \frac{T_{tt}}{c^2 T} = \lambda$$

This can now be written as

The BC translate to

The X equation is second order. Depending on the value of λ, are are three cases

$\lambda = 0$

$\lambda > 0$

$\lambda < 0$

Now for the last case, we have a nontrivial solution. Let us use that to now solve the T equations

This gives the following solutions for T_n

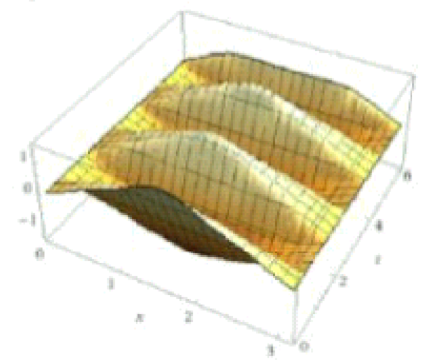
Combining the T and X equations gives the following infinite summation

We have two initial conditions, ie, the initial profile starts at rest and looks like

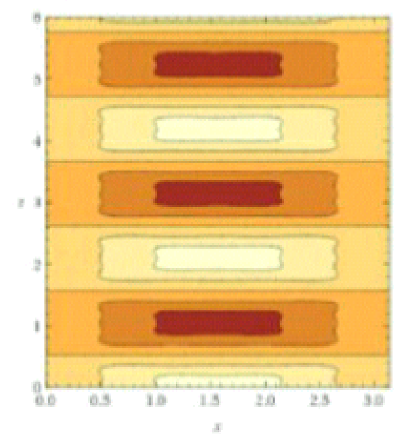
Input interpretation:

plot	$\frac{1}{\pi}4\left(\cos(3t)\sin(x) - \frac{1}{9}\cos(9t)\sin(3x) + \frac{1}{25}\cos(15t)\sin(5x) - \frac{1}{49}\cos(21t)\sin(7x)\right)$	$x = 0$ to π $t = 0$ to 6

3D plot:



Contour plot:



Recall that $c = f\lambda$

LAPLACE EQUATION

$u(x, y)$ is the temperature in a 2-D environment. Look at steady state behavior (that is why there is no dependence on time)

Infinite bar

$$u_{xx} + u_{yy} = 0$$

$$u(x, 0) = 0$$