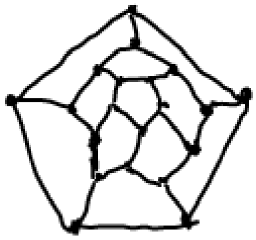
Applications of Euler's Formula

C_{60} "Buckminsterfullerene"

Defn: A <u>fullerene</u> is a planar 3-regular graph with an embedding containing only degree 5 or 6 faces.

Eg. F_n



Dodecahedron ?

12 faces

Lemma: All Fullerenes have exactly 12 degree 5 faces.

proof: let f_5 = # of deg. 5 faces
f_6 = # of deg. 6 faces

$$f_5 + f_6 = f$$

$$V - e + f_5 + f_6 = 2 \qquad 3V = 2e$$

$$5f_5 + 6f_6 = 2e$$

$$5f_5 + 6f_6 = 3V \Rightarrow f_6 = \frac{3V - 5f_5}{6}$$

$$-\frac{1}{2}V + f_5 + f_6 = 2 \Rightarrow f_6 = 2 + \frac{1}{2}V - f_5$$

$$\frac{3V - 5f_5}{6} = 2 + \frac{1}{2}V - f_5$$

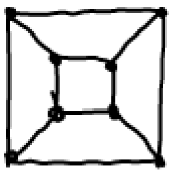
$$f_5 = 12$$

∎

Defn: A graph is platonic if it is d-regular ($d \geq 3$) and has an embedding in the plane where all faces have degree d^* ($d^* \geq 3$)

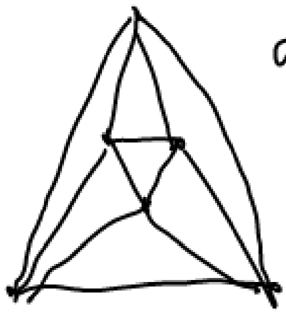
Eg. Tetrahedron

$d = d^* = 3$

Eg. Cube  $d=3$

$d^*=4$

Octahedron



$d=4, d^*=3$

Theorem 7.4.1: There are exactly 5 platonic graphs.

Lemma 7.4.2: Let G be a planar embedding with v vertices, e edges, and f faces. All vertices have degree $d \geq 3$, and all faces have degree $d^* \geq 3$. Then $(d, d^*) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$

proof: $v-e+f=2$

$$dv=2e \Rightarrow v=\frac{2e}{d}$$

$$d^*f=2e \Rightarrow f=\frac{2e}{d^*}$$

$$\frac{2e}{d}-e+\frac{2e}{d^*}=2$$

$$\frac{2}{d}+\frac{2}{d^*}=\frac{2}{e}+1$$ for any e, $\frac{2}{e}+1>1$

If $d, d^* \geq 4$, then $\frac{2}{d}+\frac{2}{d^*} \leq 1$, Contradiction.

If $d=3$, $d^* \geq 6$, then $\frac{2}{d}+\frac{2}{d^*} \leq 1$, Contradiction. ∎

Lemma 7.4.3: If G is platonic with vertex degree d and face degree d^*.

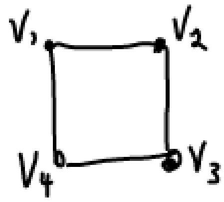
$$e = \frac{2dd^*}{2d+2d^*-dd^*}, \quad v = \frac{2e}{d}, \quad f = \frac{2e}{d^*}$$

So for each (d, d^*) we have v, e, f as determined.

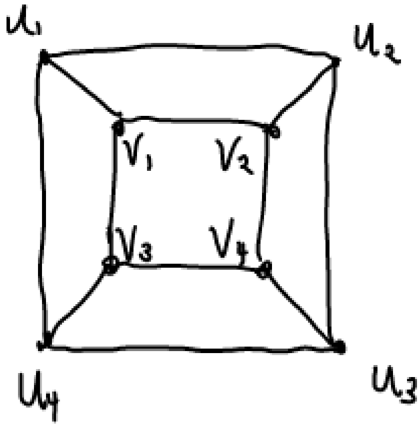
Case #1: $(d,d^*) = (3,3) \Rightarrow e = \frac{2\cdot3\cdot3}{2\cdot3+2\cdot3+3\cdot3} = 6$, $v=4$, $f=4$

the only graph w/ 4 vertices and 6 edges is K_4 (the tetrahedron)

#2: $(d,d^*) = (3,4) \Rightarrow e = \frac{2\cdot3\cdot4}{2\cdot3+2\cdot4+3\cdot4} = 12$, $v=8$, $f=6$



Graph can't have any triangles $\Rightarrow v_i \, u_i$



Case #3: $d=4$, $d^*=3$ Octahedron

#4: $d=3$, $d^*=5$ Dodecahedron

$d=5$, $d^*=3$ Icosahedron

$\sum \deg(f) = 2E$

$5f = 2E$

$n - e + f = 2$

$n - e + \frac{2E}{5} = 2$

$\frac{2e}{5} - e = 2 - n$

$e = 10 - 5n$