Recall: For A, B sets of strings,

$$AB = \{ab : a \in A, b \in B\}$$

$$A^* = \{\varepsilon\} \cup A \cup A^2 \cup A^3 \cup \ldots$$

eg. $\{00,1\}^* = \{$string where every block of zeroes has even length$\}$

$$1\underbrace{00}\underbrace{00}111\underbrace{00}11\underbrace{00}1$$

moreover, each such string can only be obtained from $\{00,1\}^*$ in one way. Thus $\{00,1\}^*$ is <u>unambiguous</u>.

This isn't the case for every expression.

Eg. $\{101, 10\}\{110, 10, 0\} = \{101110, 10110, 1010, 10110, 1010, 100\}$

$= \{101110, 10110, 1010, 100\}$

Because the strings 10110 and 1010 are obtained twice, the expression AB is <u>ambiguous</u>.

Are these ambiguous?

- $\{10, 01\}^*$ unambiguous, each str can be expressed in at most 1 way as $s_1 s_2 \ldots s_k$ where $s_i \in \{10, 01\}$
- $\{1, 01, 111\}^*$ ambiguous, because $\underbrace{1}\underbrace{1}\underbrace{1} = \underbrace{111}$
- $\{101, \varepsilon, 01\}^*$ ambiguous because $\varepsilon^4 = \varepsilon^7$ (for example)

- $\{100, 1000\}\{1\}^* \cup \{1\}\{00\}^*\{111\}$ ambiguous

  $\underbrace{100}\underbrace{1}\underbrace{1}\underbrace{1}$ $\quad$ $\underbrace{1}\underbrace{00}\underbrace{111}$ (Union must be disjoint to be unambiguous)

- $\{1\}^*\{\{0\}\{0\}^*\{1\}\{1\}^*\}^*\{0\}^*$ unambiguous

  $10100 = 1(0\varepsilon 1\varepsilon)(00) \qquad 111 = (111)\varepsilon\varepsilon$

  unambiguous-expression for the set of all binary strings. S_0 is $\{0\}^*\{\{1\}\{1\}^*\{0\}\{0\}^*\}\{1\}^*$

  S_0 is $\{0, 1\}^*$

Strings & generating series

Let S be a set of binary strings, $w(\sigma) = \text{length}(\sigma)$

\# strings in S of length n

$= [x^n]\,\Phi_S(x)$.

Sum/Product/Star lemma for strings

S set of strings, $w(\sigma) = \text{length}(\sigma)$ for $\sigma \in S$.

- If $S = A \cup B$ unambiguously, then $\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$
- If $S = AB$ unambiguously, then $\Phi_S(x) = \Phi_A(x)\Phi_B(x)$
- If $S = A^*$ unambiguously, then $\Phi_S(x) = \sum_{k \geq 0} (\Phi_A(x))^k = \frac{1}{1 - \Phi_A(x)}$

eg. Let $S = \{$binary strings where each block of zeroes has even length$\}$

we know $S = \{00, 1\}^*$ unambiguously.

So, # strings of length n in S

$$= [x^n]\Phi_S(x) = [x^n]\frac{1}{1-\Phi_A(x)}$$, where $A = \{00, 1\}$

$$\Phi_A(x) = x + x^2$$

$$\text{So } [x^n]\Phi_S(x) = [x^n]\frac{1}{1-x-x^2}$$

$$= f_n \text{ (Fibonacci \#)}$$

eg. let $S = \{$strings with exactly three blocks$\}$

$S = \{$strings of the form $111\ldots1000\ldots011\ldots1\}$

$\cup\ \{$strings of the form $00\ldots011\ldots100\ldots0\}$

$$= \{\{1\}\{1\}^*\{0\}\{0\}^*\{1\}\{1\}^*\} \cup \{\{0\}\{0\}^*\{1\}\{1\}^*\{0\}\{0\}^*\}$$

$$\Phi_{A_0}(x) = \Phi_{\{1\}}(x)\,\Phi_{\{1\}^*}(x)\,\Phi_{\{0\}}(x)\,\Phi_{\{0\}^*}(x)\,\Phi_{\{1\}}(x)\,\Phi_{\{1\}^*}(x)$$

$$= x\left(\frac{1}{1-x}\right)x\left(\frac{1}{1-x}\right)x\left(\frac{1}{1-x}\right)$$

$$= \frac{x^3}{(1-x)^3}$$

Similarly, $\Phi_{A_1}(x) = \frac{x^3}{(1-x)^3}$

So $\Phi_S(x) = \Phi_{A_0}(x)\,\Phi_{A_1}(x) = \frac{2x^3}{(1-x)^3} = 2x^3(1-x)^{-3}$

$$= 2x^3 \sum_{n\geq 0} \binom{n+2}{2} x^n$$

So # of strs in S of length n

$$= [x^n]\, 2x^3 \sum_{k\geq 0} \binom{k+2}{2} x^k$$

$$= 2[x^{n-3}] \sum_{k\geq 0} \binom{k+2}{2} x^k = 2\binom{n+1}{2}$$

wrong