Power Series

A power series centred at x_0 is any series of the form:

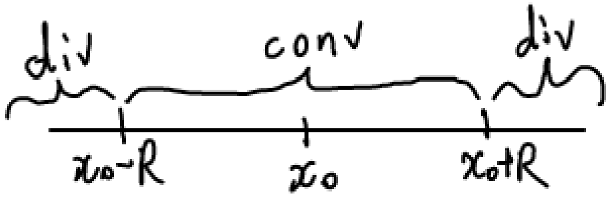
$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \ldots$$

Note: Taylor series is the specic case when we've found the $c_k = \frac{f^{(k)}(x_0)}{k!}$

For a power series, there are 3 possibilities of convergence:

1) Series converges only when $x = x_0$.
2) Series Converges for all x
3) There is a number $R>0$ such that the series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

$$\boxed{|x-x_0|<R} \Leftrightarrow \boxed{x_0-R<x<x_0+R}$$



R - radius of convergence.

The <u>interval of convergence</u> is the of x-values where the series converges. This includes endpoints, which must be checked separately.

We use the Ratio Test to find R.

<u>Ex.</u> Find the radius & interval of convergence.

1) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $(=e^x)$ Note $x_0=0$

Ratio Test: $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right| = \frac{k!}{(k+1)k!}|x| = \frac{|x|}{k+1} \to 0$ as $k\to\infty$

doesn't depend on x - ratio always less than 1.

$\Rightarrow$ The series converges for all x.

2) $\sum_{k=1}^{\infty} \frac{x^k}{k}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)}{x^k/k}\right| = \frac{k}{(k+1)} \cdot |x| = \frac{1}{1+1/k}|x| \to |x| \text{ as } k\to\infty$$

By the ratio test, the series converges if $|x|<1$ & diverges if $|x|>1 \Rightarrow \underline{R=1}$.

At $x=\pm 1$, the ratio test is inconclusive ~ must check separately.

Endpoints: $x=1$, The series is $\sum_{k=1}^{\infty}\frac{1}{k}$, which is divergent $\Rightarrow x=1$ is excluded.

$x=-1$, $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$, which is convergent (by the AST)

So the interval of convergence is $[-1,1)$.

3) Ex. $\sum_{k=1}^{\infty}\frac{(x+2)^k}{k\cdot 2^k}$ Ans. $\underline{[-4,0]}$

Manipulation of power series

Given a series $\sum C_k(X-X_0)^k$ with radius of convergence R, we can:

- differentiate - integrate - multiply by const.
- add to another series (radius $\geq R$)

and the result also has radius of convergence R.

Note: The interval could change (endpoints)

A common starting point is $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots$

This is geometric with $a=1, r=x$. So it converges to

$\frac{a}{1-r} = \frac{1}{1-x}$ when $|x|<1$.

$(*) \Rightarrow \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x|<1$.

Ex. Differentiate $(*)$ to obtain a power series for

$\frac{1}{(1-x)^2}$.

Soln: $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right)$

note index $= \sum_{k=0}^{\infty} \frac{d}{dx}(x^k)$

The series has radius of convergence 1.

$= \sum_{k=1}^{\infty} k x^{k-1}$

$= 1 + 2x + 3x^2 + \dots$