Formal Power Series

A formal power series (FPS) is an expression of the form:

$$\sum_{k\geq 0} a_k x^k \qquad \text{equivalently } a_0 + a_1x + a_2x + \dots$$

where $a_0, a_1, a_2 \dots \in \mathbb{R}$

We often use notation like $A(x)$ for such an object

"A general function is a clothesline on which we hang a sequence of coefficients for display" - H. Wilf.

Thus, formal power series are *not* functions or polynomials, but objects in their own right.

Given formal power series $A(x) = \sum_{k\geq 0} a_k x^k$ and $B(x) = \sum_{k\geq 0} b_k x^k$, we define $(A+B)(x) = \sum_{k\geq 0} (a_k + b_k) x^k$

and $$AB(x) = \sum_{k\geq 0} \underbrace{\left(\sum_{i=0}^{k} a_i b_{k-i} \right)}_{\text{finite sum of real numbers}} x^k$$

e.g. let $A(x) = \sum_{k\geq 0} k x^k$, $B(x) = \sum_{k\geq 0} x^k$

$$(A+B)(x) = \sum_{k\geq 0} (k+1) x^k$$

$$AB(x) = \sum_{k\geq 0} \left(\sum_{i=0}^{k} a_i b_{k-1} \right) x^k = \sum_{k\geq 0} \left(\sum_{i=0}^{k} i \cdot 1 \right) x^k$$

$$= \sum_{k\geq 0} (0+1+2+\dots+k) x^k$$

$$= \sum_{k\geq 0} \frac{k(k+1)}{2} x^k$$

$$AB(x) = \sum_{k\geq 0} \binom{k+1}{2} x^k$$

We also define notation for 'coefficient extraction'.
$[x^k]A(x)$ is defined to be the coefficient of x^k in the series $A(x)$.

e.g. $[x^4](1-x+2x^2-3x^3+7x^4-\dots) = 7$

$[x^k](1+x)^n = \binom{n}{k}$
Binomial Theorem Restated

$[x^k]AB(x) = \sum_{i=0}^{k} [x^i]A(x)\,[x^{k-i}]B(x)$
from definition of power series product

Given that we have defined multiplication, we can write and solve equations whose variables are power series.

e.g. find a formal power series $A(x)$ so that
$(1+x+x^2+\dots)A(x) = 1-x$

let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$

We want $(1+x+x^2+x^3+\dots)(a_0+a_1x+a_2x^2+\dots)$
$= 1-x+0x^2+0x^3+0x^4+\dots$

$a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + \dots$
$= 1-x+0x^2+0x^3+0x^4+\dots$

We need
$$a_0 = 1 \quad \Rightarrow \quad a_0 = 1$$
$$a_0 + a_1 = -1 \quad \Rightarrow \quad a_1 = -2$$
$$a_0 + a_1 + a_2 = 0 \quad \Rightarrow \quad a_3 = 1$$
$$a_0 + a_1 + a_2 + a_3 = 0 \quad \Rightarrow \quad a_4 = 0$$
$$\vdots \qquad \Rightarrow \quad a_k = 0 \text{ for all } k \geq 3$$

So $A(x) = 1 - 2x + x^2$

eg. find a power series $B(x) = b_0 + b_1 x + b_2 x^2 + \dots$ so that $(1+2x)B(x) = 1-x$

so $(1+2x)(b_0 + b_1 x + b_2 x^2 + \dots) = 1 - x$
$$b_0 + (2b_0 + b_1)x + (2b_1 + b_2)x^2 + (2b_2 + b_3)x^3 + \dots = 1 - x + 0x^2 + 0x^3 + \dots$$
$$b_0 = 1$$
$$2b_0 + b_1 = -1$$
$$2b_k + b_{k+1} = 0 \text{ for all } k \geq 1$$
$$\Downarrow$$
$$b_0 = 1$$
$$2b_0 + b_1 = -3$$
$$b_{k+1} = -2b_k \text{ for } k \geq 1$$

$$b_2 = (-3)(-2)$$
$$b_3 = (-3)(-2)^2$$
$$b_4 = (-3)(-2)^3$$
$$\vdots$$
$$b_k = (-3)(-2)^{k-1} \text{ for } k \geq 1$$

So $$B(x) = 1 + \sum_{k \geq 1} (-3)(-2)^{k-1} x^k$$
$$= 1 - 3 \sum_{k \geq 1} (-2)^{k-1} x^k$$

Alternatively, we could solve these equations by turning reciprocals into infinite series using $\frac{1}{1-r} = 1+r+r^2+\dots$

first eqn: we had $(1+x+x^2+\dots)A(x) = 1-x$

$$\iff \frac{1}{1-x}A(x) = 1-x \quad \text{so} \quad A(x) = (1-x)^2$$
$$= 1-2x+x^2$$

For $(1+2x)B(x) = 1-x$

we have
$$B(x) = (1-x)\left(\frac{1}{1+2x}\right) = (1-x)\left(\frac{1}{1-(-2x)}\right)$$
$$= (1-x)(1+(-2x)+(-2x)^2+(-2x)^3+\dots)$$
$$= (1-x)(1-2x+4x^2-8x^3-\dots)$$
$$= (1-2x+4x^2-8x^3+\dots) - x(1-2x+4x^2-8x^3+\dots)$$
$$+ \dots$$