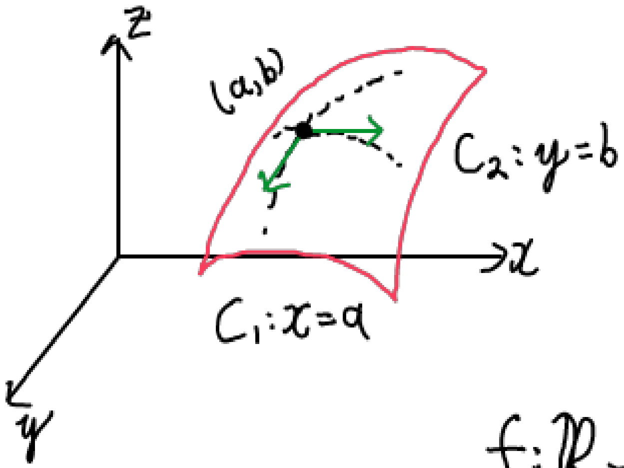
Partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

second partial derivatives $\frac{\partial^2 f}{\partial x^2}, \underbrace{\frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}}_{\text{equal}}, \frac{\partial^2 f}{\partial y^2}$

The Tangent Plane, Differentials & Taylor Polynomials.

The Tangent Plane



Two vectors in the direction of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ form a basis for the plane:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$f : \mathbb{R} \to \mathbb{R} \quad L(x) = f(a) + f'(a)(x-a)$

Can write:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Let $\Delta f = f(x,y) - f(a,b)$, $\Delta x = x-a$, $\Delta y = y-b$.

$\Rightarrow \Delta f \approx f_x \Delta x + f_y \Delta y$ (increment form) $\Delta x, \Delta y$ small.

As <u>differentials</u>, $df = f_x dx + f_y dy$

<u>Ex.</u> A company makes cylindrical drums. The radius can be controlled to within 2% & the height to within 0.5%. What is the largest percent error in the volume?

$V = \pi r^2 h$

$dV = V_r dr + V_h dh$

$dV = 2\pi r h\, dr + \pi r^2 dh$

We want $\left|\frac{dV}{V}\right|$, given $\left|\frac{dr}{r}\right| \leq 0.02$, $\left|\frac{dh}{h}\right| \leq 0.005$

$$\Rightarrow \frac{dV}{V} = \frac{2\pi r h\, dr + \pi r^2 dh}{\pi r^2 h} = 2\cdot\frac{dr}{r} + \frac{dh}{h}$$

$$\left|\frac{dV}{V}\right| = \left|2\cdot\frac{dr}{r} + \frac{dh}{h}\right| \leq 2\left|\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| = 0.045$$

$\therefore$ V has error at most 4.5%

Taylor Polynomials - Two Variable Case

Linear Approximation about $x = a$.

$f: \mathbb{R} \to \mathbb{R}$ $\quad L(x) = P_{1,a}(x) = f(a) + f'(a)(x-a)$ (tangent line)

$f: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ Let $\underline{x} = (x, y)$ and $\underline{a} = (a, b)$

$$P_{1,\underline{a}}(\underline{x}) = f(\underline{a}) + f_x(\underline{a})(x-a) + f_y(\underline{a})(y-a)$$

(tangent plane)

$\underline{P_2}$: $f: \mathbb{R} \to \mathbb{R}$

$$P_{2,a}(x) = P_{1,a}(x) + \frac{f''(a)}{2}(x-a)^2$$

$f: \mathbb{R}^2 \to \mathbb{R}$:

since $f_{xy} = f_{yx}$

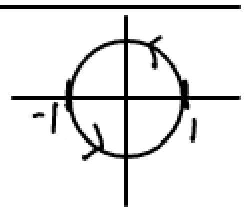
$$P_{2,\underline{a}}(\underline{x}) = P_{1,\underline{a}}(\underline{x}) + \frac{1}{2}\Big[f_{xx}(\underline{a})(x-a)^2 + 2f_{xy}(\underline{a})(x-a)(y-b) + f_{yy}(\underline{a})(y-b)^2\Big]$$

$$P_{3,\underline{a}}(\underline{x}) = P_{2,\underline{a}}(\underline{x}) + \frac{1}{3!}\Big[f_{xxx}(x-a)^3 + 3f_{xxy}(\underline{a})(x-a)^2(y-b) + 3f_{xyy}(\underline{a})(x-a)(y-b)^2 + f_{yyy}(y-b)^3\Big]$$

* note: $f_{xxy} = f_{xyx} = f_{yxx}$
$f_{xyy} = f_{yxy} = f_{yyx}$ *

Parametric Representation of Curves

$y = \cos t$, $t \in [0, 2\pi]$
$y = \sin t$



Represents the unit circle $x^2 + y^2 = 1$

Clear advantage of using parametric form:
- orientation $\to$ counter-clockwise.

We can view $\vec{r}(t) = (x(t), y(t))$ as a vector function, where $\vec{r}: \mathbb{R} \to \mathbb{R}^2$.

$\vec{r}$ is a position vector with base at the origin & tip tracing out the path.