# Negative Binomial Theorem

**Prop.** $(1-k)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$ $\qquad (1-x)^{-n} = \sum_{k\geq 0} \binom{n+k-}{n-1}$

equivalently $[x^n](1-x)^{-k} = \binom{n+k-1}{k-1}$

**pf.** $[x^n](1-x)^{-k}$

$$= [x^n]\left(\frac{1}{1-x}\right)^k = [x^n]\underbrace{(1+x+x^2+\dots)(1+x+x^2\dots)\dots\dots}_{k \text{ times}}$$

This coefficient is the number of solns to $a_1+a_2+\dots+a_k = n$ where $a_i \in \mathbb{N}_0$

We show this with the product lemma.

We have $1+x+x^2+x^3+\dots = \Phi_{\mathbb{N}_0}(x)$ w.r.t. the weight function $w(a) = a$.

$$(1+x+x^2+\dots)^k = \left(\Phi_{\mathbb{N}_0}(x)\right)^k \underset{\text{by product lemma}}{=} \Phi_S(x)$$

$\boxed{\begin{array}{l}\text{where } S = \mathbb{N}_0 \times \dots \times \mathbb{N}_0 \\ \qquad\quad = (\mathbb{N}_0)^k \\ \text{and } w(a_1, a_2, \dots a_k) \\ \qquad\quad = a_1+a_2+\dots+a_k\end{array}}$

$$[x^n]\left(\frac{1}{1-x}\right)^k = [x^n]\left(\Phi_{\mathbb{N}_0}(x)\right)^k = [x^n]\,\Phi_S(x)$$

$= \#$ elements of $S = (\mathbb{N}_0)^k$ of weight $n$

$= \#$ soln's to $a_1+a_2+\dots+a_k = n$ where $a_i \in \mathbb{N}_0$

Let $T = \{(a_1, a_2, \dots, a_k) \in \mathbb{N}_0^k \text{ s.t. } a_1+a_2+\dots a_k = n\}$

and $R = \{$binary strs of length $n+k-1$ w/ exactly $k-1$ ones$\}$

We know $|T| = [x^n](1-x)^{-k}$ $\quad |R| = \binom{n+k-1}{k-1}$

let $k=3$, $n=5$

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0101000
1100000

$\Downarrow$

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$2+1+2=5$
$1+1+3=5$
$0+0+5=0$

we define a bijection $f: T \to R$ by

$$f(a_1, a_2, \dots, a_k) = \underbrace{0\dots0}_{a_1}1\underbrace{0\dots0}_{a_2}1\cdots1\underbrace{0\dots0}_{a_k}$$

and its inverse by

$$g(\underbrace{0\dots0}_{b_1}1\underbrace{0\dots0}_{b_2}1\underbrace{0\dots0}_{b_3}1\dots1\underbrace{0\dots0}_{b_k}) = (b_1, \dots, b_k)$$

Clearly $f$ and $g$ are inverses so $f$ is a bijection and $|T| = |R|$ $\blacksquare$

We can use the negative binomial theorem to go between rational expressions & power series.

e.g. $(1+2x^2)^{-5} = (1-(-2x^2))^{-5} = \sum\limits_{n\geq 0} \binom{n+4}{4}(-2x^2)^n$

$$= \sum_{n\geq 0} (-2)^n \binom{n+4}{4} x^{2n}$$

$$(1-x)^{-1} = \sum_{n\geq 0} \binom{n}{0} x^n = 1 + x + x^2 + \dots$$

The ideas in this pf allude to a new type of combinatorial object.

Let $n \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, A composition of $n$ into $k$ parts is a $k$-tuple $(a_1, a_2 \dots, a_k)$ st. $a_1 + a_2 + \dots + a_k = n$ and $a_i \in \mathbb{N}_0$.

The compositions of 5 into 3 parts are
$(1,1,3)$, $(1,3,1)$, $(3,1,1)$, NB. order matters
$(1,2,2)$, $(2,1,2)$, $(2,2,1)$

<u>Prop</u>. There are $\binom{n-1}{k-1}$ compositions of $n$ into $k$ parts.

<u>Pf</u>. let $S = \{\text{compositions of } n \text{ into } k \text{ parts}\}$
$T = \{\text{solutions to } a_1 + a_2 + \dots + a_k = n-k \text{ with } a_i \in \mathbb{N}_0\}$

$f(a_1, a_2, \dots, a_k) = (a_1 - 1, a_2 - 1, \dots, a_k - 1)$ gives a bijection from $S$ to $T$.

By the meterial in the proof earlier,

$$|T| = \binom{(n-k)+k-1}{k-1}$$

$$= \binom{n-1}{k-1}$$

<u>Prop:</u> The # of compositions of $n$ - into any # of parts is $2^{n-1}$

pf. By previous prop, the number is $\sum_{k\geq 1}\binom{n-1}{k-1} = 2^{n-1}$

by binomial theorem