Math 213

1. Introduction to Continuous time ODEs

2. Solving linear ODEs

3. Constant coefficient ODEs

4. Laplace Transforms

5. Fourier Series, (possibly Integrals and Transforms)

6. Partial Differential Systems

Mark Breakdown:
Quiz 1: 15 marks (during one of the tutorials)
Quiz 2: 15 marks (June 29, 7pm)
Peer graded assignments: 20 marks
Peer graded Matlab simulation exercises: 15 marks
Final: 35 Marks

Textbook:
Advanced Engineering Mathematics- Michael D. Greenberg(optional but highly recommended as this is where the assignments are from)

TA:
Ahmad Bilal Asghar- abasghar@uwaterloo.ca
- Supervise the Peer graded portion of the course
-Help for Simulation exercises
Nusa Zidaric- nzidaric@uwaterloo.ca
- Tuturials, marking of quiz and final, consulting hours on the course theory

-Office hours and location: TBD. See Learn for updates.

Tutorials: Thursday 2:30 MC2065

THERE ARE PARTIAL COURSE NOTES THAT ARE ON LEARN FOR YOUR BENEFIT. PLEASE PRINT THEM OFF PRIOR TO LECTURE

As well, there is a set of course notes at https://ece.uwaterloo.ca/~math211/Lectures/
These notes complement the course well.

Assignments

There will be nine assignments in the course. THESE ARE GROUP EFFORTS. You will be randomly assigned into groups of 3 or less. The assignments will consist of the following:

1. A number of questions that are from the textbook. These do not have to be submitted and these questions will be covered during the tutorials.

2. Come up with two questions that are at the same level of difficulty as the most difficult questions that was assigned. Base the question on the topic area of the assigned questions. You need to come up with a marking rubric and solution for each question. Submit a cover page, a separate page of the question and the solutions/marking rubric on a separate page. Each group will be assigned to two other groups and each group will solve the student-created questions from the other groups (ie there will be four questions in total). These will be submitted and will be marked by the group that created the problems. In the case of any dispute, the TA will intervene to resolve the conflict, particularly if the question is poorly worded or constructed. Each group will also assign a mark between 1 and 3 for each question to the group that created the questions as to whether the difficulty level was appropriate. Thus, each group will receive a mark of 10 for each set of two questions that they solved, as well as 2-6 marks for the quality of their own questions for a total of 26 possible marks for each submission. The TA also has the option to adjust the marks for the assignments down by as much as 3 marks if he judges the question or soluton to be too simple or too poorly constructed. At the end of the term, each group will also have the ability to adjust the assignment marks up or down based on a mutually agreeable peer evaluation, as long the sum total of the marks received by all group members remain unchanged. All assignments are equally weighted and will be worth 20 percent of the final marks in total. All submitted work must be submitted via the Learn dropbox in pdf format. Handwritten is fine but must be legible.

3. There will be assigned Matlab Simulation exercises. For these, each group will mark the other two groups' simulations based on a common rubric (these will be the same groups as in 2. above). Thus, each group will receive 2X10 marks= 20 marks per simulation. At the end of the term, each group will also have the ability to adjust the assignment marks up or down based on a mutually agreeable peer evaluation, as long the sum total of the marks received by all group members remain unchanged. All simulation exercises are equally weighted and will be worth 15 percent of the final marks in total. All submitted work must be submitted via the Learn dropbox in pdf format.

4. If the group is not functioning as it should, all interactions should be documented (eg. meetings, minutes of meetins etc) to help us ascertain if someone in the group is not doing their share.

Late assignments will NOT be accepted and will receive a grade of 0.

Tutorials

The one-hour weekly tutorials will be used primarily to assist you with the assignments. The textbook questions assigned will be covered

Academic Discipline

You are expected to know what constitutes an academic offense (see Policy #17 Student AcademicDiscipline,http://www.adm.uwaterloo.ca/infosec/Policies/policy71.html. We remind you that although you are encouraged to discuss assignment problems with each other, you are expected to write up your solutions independently. Direct copying (from any source) is plagiarism, and will be treated as an academic offense if detected. Students who are unsure whether an action constitutes an offense, or who need help in learning how to avoid offenses (e.g., plagiarism, cheating) or about "rules" for group work / collaboration should seek guidance from the course professor, TA, academic advisor, or the Undergraduate Associate Dean.

Illness during exams

If you miss the midterm exam due to a documented illness, the weight will be transferred to the final exam. Be aware that we do NOT automatically grant requests for deferrals of final exams. These requests will be granted only to students who are severely ill or otherwise physically incapable of attending the examination, and whose performance in the course suggests a reasonable chance of success. Be aware that appropriate documentation (as decided by the course professor) must be provided.

Grievances

A student who believes that a decision affecting some aspect of his/her university life has been unfair or unreasonable may initiate a grievance. Read Policy 70, Student Petitions and Grievances, Section 4: http://www.adm.uwaterloo.ca/infosec/Policies/policy70.htm.

Note for students with Disabilities

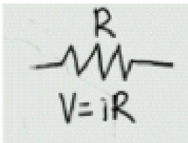
The Office for Persons with Disabilities (OPD), located in Needles Hall, Room 1132, collaborates with all academic departments to arrange appropriate accommodations for students with disabilities without compromising the academic integrity of the curriculum. If you require academic accommodations to lessen the impact of your disability, please register with the OPD at the beginning of each academic term.

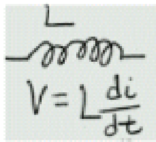
1. Introduction to Continuous time ODEs

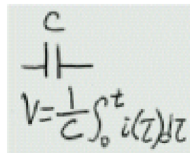
-Differential Equations

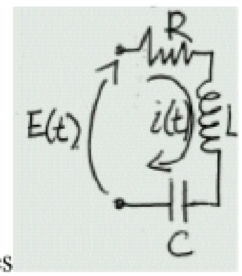
eg/ RLC circuit

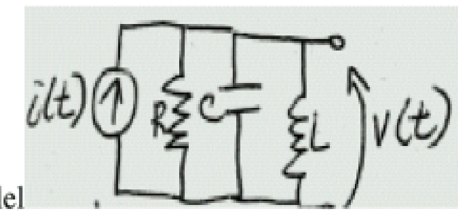
-constitutive relations







Series 

parallel 

-Kirchoff's laws

$\sum$ Voltages around a loop = 0

$\therefore E(t) - Ri - L\frac{di}{dt} - \frac{1}{C}\int_0^t i(\tau)d\tau$

$\Rightarrow L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dE(t)}{dt}$

Given $i(0)$, $\frac{di(0)}{dt}$, $E(t)$,

solve for $i(t)$

$\sum$ currents at a node = 0

$i = i_R + i_C + i_L$

$= \frac{V}{R} + C\frac{dV}{dt} + \frac{1}{L}\int_0^t V(\tau)d\tau$

$\therefore C\ddot{V} + \frac{1}{R}\dot{V} + \frac{1}{L}V = \frac{di}{dt}$ $\quad \dot{V}$ is $\frac{dv}{dt}$

Given $v(0)$, $\frac{dv(0)}{dt}$, $i(t)$, $\quad \ddot{V}$ is $\frac{d^2v}{dt^2}$

Solve for $v(t)$

-An ODE (ordinary differential equation) is an equation that contains one or several derivatives of an unknown function, say $y(x)$

-Notation: $y' = \frac{dy(x)}{dx}, y'' = \frac{d^2 y(x)}{dx^2}, \ldots, y^{[n]} = \frac{d^n y(x)}{dx^n}$

eg/ $x^2 y''' y' + 2e^x y'' = (x^2+2)y$

or $F(x, y, y', y'', y''') = 0$

-Note: A PDE (Partial Differential Equation) has several functions in the argument, e.g. $y(x,t)$, and involves partial derivatives

eg/ $$\frac{\partial^4 y(x,t)}{\partial x^4} = K \frac{\partial^2 y(x,t)}{\partial t^2}$$ beam eq'n

-Notation: $\dot{y}(t) = \frac{dy(t)}{dt}$

-Def'n- A first order ODE is one which involves y' but no higher derivatives

eg/ $y'y - y^2 = 0$ or $F(x, y, y') = 0$

-Def'n- nth order ODE involves $y^{[n]}$

eg/ $F(x, y, y', \ldots, y^{[n-1]}, y^{[n]}) = 0$

-Goal: We want a function which satisfies the ODE in some open interval ($a<x<b$) of the real line (usually, we want the whole real line). Such a function is called a solution

-eg/ $y' = \frac{y}{x} + 1$ Show that $y = x\ln|x| + Cx$ is a sol'n

Sol'n/

substitute into the LHS & RHS and show they are equal.

$$y' = \frac{d}{dx}(x\ln|x| + Cx) = \ln|x| + \frac{1}{x}\cdot x + C = \ln|x| + 1 + C$$
$$= \frac{x(\ln|x| + C)}{x} + 1 = \frac{y}{x} + 1 = RHS$$

$\therefore$ This is a sol'n.

-Types of solutions

1. If a solution contains an arbitrary constant as in the previous example, this is called a *general solution* previous example

2. If this constant has a value, then this is called a *particular solution*
choose $c=0$, i.e. $y = x\ln|x|$ } particular sol'n

3. If the solution is implicitly defined by an equation, then this is an *implicit solution*

eg/ $yy' = -x$ $x^2 + y^2 = c$ is a solution

LHS $= yy'$ to get y', differentiate wrt $x \Rightarrow 2x + 2yy' = 0$
$\therefore yy' = -x$
$\therefore$ This works

4. *Singular solutions*- sometimes, an additional solution can't be obtained by choosing C in the general solution. This is called a singular solution

eg/ $(y')^2 - xy' + y = 0$

You can show $y = cx - c^2$ is a sol'n where c is an arbitrary constant.

However, $\underbrace{y = \frac{x^2}{4}}_{\text{a singular solution}}$ is also a sol'n.

Not all ODEs have solutions
eg/ $|y'| + 3 = 0$

Note $|y'| = -3$. There is no $y(x)$ which can satisfy this

-Initial Value Problem
$F(x, y, y') = 0, y(x_0) = y_0$
This generates a particular solution
eg/ $y' + y/x = e^x, y(1) = 0$
→ you can show that $y = \frac{1}{x}(xe^x - e^x + c)$ is a sol'n,

$y(1) = 0 \Rightarrow 0 = \frac{1}{1}(1 \cdot e^1 - e^1 + c) \quad \therefore c = 0$

$\therefore y = \frac{1}{x}(xe^x - e^x)$ which is a particular solution.

-Linear nth order ODE

$$a_0(x)y^{[n]}(x) + \ldots + a_n(x)y(x) = f(x)$$

If not in this form, then this is nonlinear. Solutions will satisfy superposition when $f(x)=0$

Pf/ Superposition means any linear combination of solutions is also a solution

$$a_0(x)y^{[n]}(x) + \ldots + a_n(x)y(x) = 0$$

Let $t(x)$ & $v(x)$ be two sol'ns

$\therefore$ Superposition means $\alpha t(x) + \beta v(x)$ is also a sol'n where α, β are constants

$$a_0(x)[\alpha t^{[n]}(x) + \beta v^{[n]}(x)] + \ldots + a_0(x)[\alpha t(x) + \beta v(x)]$$

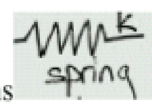
$$= [a_0(x)\alpha t^{[n]}(x) + \ldots a_n(x)\alpha t[x]] + [a_0(y)\beta v^{[n]}(x) + \ldots + a_n(x)\beta v(x)]$$

$= 0$ since $\alpha t(x) + \beta v(x)$ is a soln, linear, superposition holds

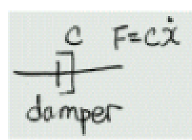
Definition:
If $f(x)=0$, this is homogeneous
If $f(x) \neq 0$, this is nonhomogeneous

Eg/ mechanical circuit (mass spring damper)

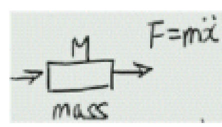
-constitutive relations



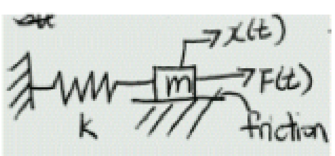
$F = kx$



↳ many models
This is a visceus friction model



- FBDs



$\underbrace{\sum \text{Forces} = m\ddot{x}}_{}$, $\sum \text{Torque} = I\ddot{\theta}$
x-direction
moment of Inertia
angular acc'n

$$F - kx - c\dot{x} = m\ddot{x}$$

$$\therefore \; m\ddot{x} + c\dot{x} + kx = F$$

Solve for $x(t)$ given $F(t)$, $x(0)$, $\dot{x}(0)$

-Simulating ODEs

-There are many tools such as Matlab for simulating ODEs.

-The ODEs must be converted in a vector valued First Order ODE. This creates what is referred to as a *state space equation.*

-There are many ways to do this. One easy way is to define a state $x_n = \frac{d^{n-1}y}{dt^{n-1}}$

Eg/ $a_0(t)y^n(t) + ... + a_n(t)y(t) = f(t)$

-This can be done for a system of equations as well

-Eg/

$$\frac{d^3 y_1}{dt^3}(t) + \cos(t)\, y_2(t) = t$$

$$\dot{y}_2(t) - 5 y_1(t) = t^2$$

-Once the equations are written in this form, many Matlab Commands can be used to automatically generate the solution

https://www.youtube.com/watch?v=5qH4bmHR1YM

https://www.youtube.com/watch?v=fx3bl4oA_0U

https://www.youtube.com/watch?v=dFF_Isz_TjU

Eg/ radioactive decay $\frac{dN}{dt} = kN$ where N = number of nuclei which is proportional to mass
k = decay rate

$\therefore \frac{dm}{dt} = -km$ soln is $m(t) = m_0 e^{-kt}$ $\}$ verify by substitution

What is m_0? It is the mass at $t=0$.

Half life and carbon dating- For living things, there is a known ration of C^{12} to C^{14}

For carbon dating C^{14} decays to N^{14} with a $\frac{1}{2}$ life of 5570 years

Suppose we have "dead" wood that has .2g of C^{14} whereas a similar piece of living wood has 2.6g of C^{14}.

How old is the wood?

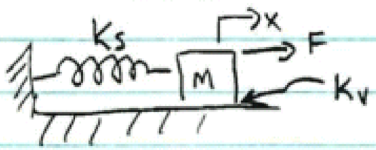
$$\frac{m_0}{2} = m_0 e^{-k\,5570\text{ years}} \Rightarrow k = \frac{\ln 2}{5570\text{ yrs}}$$

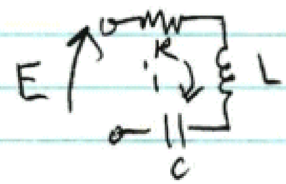
$$\therefore .2g = 2.6g\, e^{-kt}$$

$$t \approx 21000 \text{ yrs}$$

Equivalence between Mechanical and Electrical Systems

- Mechanical $\Longleftrightarrow$ Electrical





$$m\ddot{x} + K_v\dot{x} + K_s x = F$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$(\dot{Q} = i)$$

Mass m	Inductance L
Damping K_v	Resistance R
Spring K_s	Capacitance^{-1} $\frac{1}{C}$
Position x	Charge Q
Force F	Voltage E

See Lecture 4 in Tenti-Harder Notes at https://ece.uwaterloo.ca/~math211/Lectures/

2. Solving ODEs in Continuous Time Case

-There are many ways to solve ODEs

-Only in specialized cases can one find an exact solution

-In general, not very useful in "real life"

A/Separable Equations (first order)

- $\dot{x} = f(x,t)$ → does not have to be linear

-This is separable if $\dot{x} = A(t)B(x) = \dfrac{dx}{dt}$

$$\therefore \int \frac{1}{B(x)}\,dx = \int A(t)\,dt + C \quad \text{where } C \text{ is an integration constant}$$

(Note $B(x) \neq 0$)

In other words, we integrate both sides to get an implicit (general) solution

Eg. $9yy' + 4x = 0$

$$\therefore 9yy' = -4x \Rightarrow y' = \overset{\text{"y"}}{\frac{1}{9y}}\overset{\text{"x"}}{(-4x)}$$

$$9y\frac{dy}{dx} = -4x \Rightarrow 9y\,dy = -4x\,dx$$

$$\frac{9y^2}{2} = -2x^2 + C$$

Eg. $y' = \overset{\text{"y"}}{(1+y^2)}\overset{\text{"x"}}{1}$

$$\therefore \frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = dx \Rightarrow \arctan y = x + c$$

$$\therefore y = \tan(x+c)$$

Eg. $\underbrace{y' + 5x^4y^2}$ $= 0,\ y(0) = 1$

$$\frac{dy}{dx} = -5x^4y^2 \qquad \therefore \int \frac{dy}{y^2} = \int -5x^4\,dx \quad \text{note: } y \neq 0$$

$$-\frac{1}{y} = -x^5 + C,\ y \neq 0 \Leftarrow \text{general soln}$$

$$y(0) = 1 \Rightarrow -\frac{1}{1} = -0^5 + C \quad \therefore C = -1$$

$$\therefore y = \frac{1}{x^5+1} \Leftarrow \text{particular soln}$$

Note: $y=0$ satisfies the original ODE (by substitution) you can't get this from the gen. soln.
∴ singular soln

$\frac{dy}{dx} = f(y)g(x) \leftarrow$ separable

Eg. $t^2\dot{y} = 1+y \Rightarrow \frac{dy}{dx} = \frac{1}{t^2}(1+y)$ ∴ separable

$$\frac{dy}{1+y} = \frac{dx}{t^2}, \quad t\neq 0,\ y\neq -1$$

$$\ln|1+y| = -\frac{1}{t} + c$$

$$e^{\ln|1+y|} = e^{-\frac{1}{t}+c}$$

$$|1+y| = e^{-\frac{1}{t}+c} = e^{-\frac{1}{t}}\overbrace{e^{c}}^{k} = ke^{-\frac{1}{t}} \quad \text{where } k>0$$

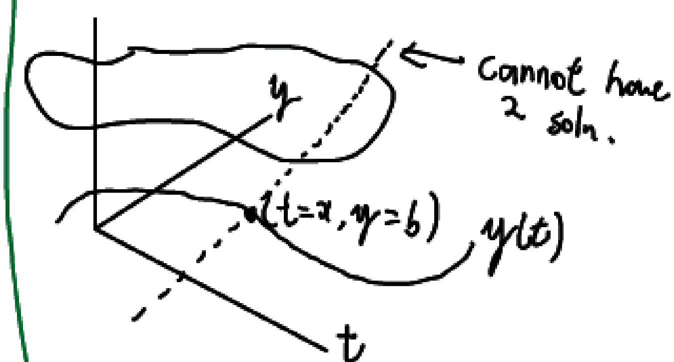
$$1+y = \pm ke^{-\frac{1}{t}} \quad \text{or} \quad y = \pm ke^{-\frac{1}{t}} - 1 \quad (k>0)$$

$$\therefore y = Be^{-1/t} - 1, \quad B \neq 0$$

Is $y=-1$ a singular soln? By substitution, $y=-1$ is a singular soln.

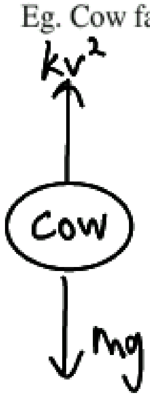
∴ $y = De^{-1/t} - 1$, $D \in \mathbb{R}$

-Theorem- If $f(t, y)$ is continuous on some rectangle R in the (t,y) plane containing the point (a,b), then $\dot{y} = f(t, y)$, $y(a) = b$ has at least one solution defined on some open interval containing $t = a$. If $\partial f/\partial y$ is continuous on R, then the solution is unique on some open interval containing $t = a$.



← sufficient cond. If it holds, soln exists and is unique.
If it doesn't hold, no conclusion.

Eg. Cow falling out of a plane.



$k \Rightarrow$ drag coefficient

$$\therefore m\frac{dv}{dt} = mg - kv^2$$

This is separable $\frac{dv}{v^2 - \frac{mg}{k}} = -\frac{k}{m}dt$

Let $A = \sqrt{\frac{mg}{k}}$, $\int \frac{dv}{v^2-A^2} = \int -\frac{k}{m}dt$ $\Rightarrow$ partial fraction expansion $\frac{1}{v^2-A^2} = \frac{C_1}{V-A} + \frac{C_2}{V+A} \Rightarrow C_1 = \frac{1}{2A}$, $C_2 = -\frac{1}{2A}$

$= \frac{1}{2A}\left(\frac{1}{V-A} + \frac{1}{V+A}\right)$

$$\int \frac{1}{2A}\left(\frac{1}{V-A} + \frac{1}{V+A}\right)dv = -\frac{k}{m}t + c$$

$\Rightarrow \ln\left|\frac{V-A}{V+A}\right| = -\frac{2Akt}{m} + 2Ac$ ∴ $\left|\frac{V-A}{V+A}\right| = e^{-\frac{2Akt}{m}}\underbrace{e^{2Ac}}$ +ve const B

Check ⇒ $V=A$ is a singular soln. ($V=-A$ is also a singular soln) $\frac{V-A}{V+A} = \pm Be^{-\frac{2Akt}{m}}$ where $V-A\neq 0$, $V+A \neq 0$

Cont: $\because$ $V=A$ is a solution. and $\frac{V-A}{V+A}=\pm Be^{\frac{-2Akt}{m}}$, $B>0$

$$\therefore \frac{V-A}{V+A} = De^{\frac{-2Akt}{m}}, \quad D \in \mathbb{R}$$

$\therefore$ the terminal velocity

as $t \to \infty$, the RHS goes to zero $\therefore$ the LHS is zero

$\therefore V = A = \sqrt{\frac{mg}{k}}$

B/Linear Equations

-Now, recall the form of a linear ODE

-A first order linear ODE is $a_0(x)y' + a_1(x)y = f(x)$

-assume $a_0(x) \neq 0$ over the interval of interest. This gives

$$y' + p(x)y = q(x)$$

-Look first at the homogeneous case! ie $q(x) = 0$. This is separable

$$\frac{dy}{dx} = -p(x)y$$
$$\frac{dy}{y} = -p(x)dx$$

$$\int \frac{1}{y} dy = -\int p(x)dx + C$$
$$\ln|y| = -\int p(x)dx + C$$
$$|y| = e^{-\int p(x)dx + C} \quad \therefore y = \underbrace{\pm e^{C}}_{\text{positive constant}} e^{-\int p(x)dx}$$

check if $y=0$ is a singular sol'n. It is.

$$\boxed{\therefore\ y = De^{-\int p(x)dx},\ D \in \mathbb{R}}$$

-eg $\underbrace{y' + \overset{p(x)}{x}y}$ $= 0,\ y(0) = 1$

$$y = De^{-\int x dx} = De^{-\frac{x^2}{2}}$$

Since $y(0)=1$ $\therefore D=1$ $\therefore y = e^{-\frac{x^2}{2}}$

-Now, we use the homogeneous solution to solve the nonhomogeneous case
(use a technique called integrating factors)
$y' + P(x)y = q(x)$

Multiply both sides by $\sigma(x)$, the integrating factor

$$\underbrace{\sigma y' + \sigma p y}_{} = \sigma q$$

Key point $\longrightarrow \dfrac{d\{\sigma y\}}{dx} = \sigma y' + \sigma' y$ This will hold if $\sigma' = \sigma p$ or $\underbrace{\sigma' - \sigma p = 0}_{\text{homogenius}}$

$\therefore \sigma = e^{\int p(x)dx}$

-Steps
1. Solve for $\sigma(x)$
2. Multiply the ODE by $\sigma(x)$
3. LHS is then $\dfrac{d\{\sigma y\}}{dx}$ $\longrightarrow \dfrac{d\{\sigma y\}}{dx} = \sigma q \quad \therefore \sigma y = \int \sigma q\, dx + c$
or $y = \sigma^{-1}\int \sigma q\, dx + \sigma^{-1} c$
4. Integrate both sides

eg. $y' + y = 1$, $y(0) = 0$

$$\sigma y' + \sigma y = \sigma$$

$$\frac{d\{\sigma y\}}{dx} = \sigma y' + \sigma' y$$ By comparison, $\sigma' = \sigma$ or $\sigma' - \sigma = 0$
$\Rightarrow \sigma = e^x$

$\therefore \dfrac{d\{e^x y\}}{dx} = \sigma = e^x$ Integrate both sides $e^x y = \int e^x dx + c$
$= e^x + c$

$\therefore y = 1 + ce^{-x}$. Finally, $y(0) = 0 \Rightarrow c = -1$

$\therefore y = 1 - e^{-x}$

Theorem:
The linear equation $y' + p(x)y = q(x)$ has a solution through an initial condition $y(a) = b$ if $p(x)$ and $q(x)$ are continuous at a. Further, the solution is unique and it exists at least on the largest interval containing $x = a$ over which $p(x)$ and $q(x)$ are continuous
Eg/ $p(x) = 1/x$

No guarantee of a solution over any interval containing $x = 0$.

C/ Second and Higher Order Linear ODEs

-Start with second order ODE

$$y'' + P(x)y' + Q(x)y = R(x) \qquad y(x_0) = A, y'(x_0) = B$$

-Theorem

If P,Q,R are continuous on an open interval and given an initial condition x_0 in that interval, the initial value problem has a **unique** solution

For a linear 2nd order homogeneous ($R(x)=0$), any linear combination of solutions on an interval is also a solution

-Def'n-

Two functions f and g are linear independent on an interval if neither is a multiple of the other. Otherwise, they are linearly dependent.

$\Rightarrow$ no constant k such that $f(x) = kg(x)$

-We have a general solution for the second order linear ODE if $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are **linearly independent**

eg, $y'' - y = 0, y(0) = 5, y'(0) = 3$

① $y = e^x$ ② $y_2 = e^{-x}$ ③ $y_3 = 2e^x$

Suppose we choose y_1 and y_3 as the two sol'ns

i.e. $y = c_1 e^x + c_2 2e^x \therefore y' = c_1 e^x + c_2 2e^x$

$$\left.\begin{array}{l} y(0) = 5 = c_1 + 2c_2 \\ y'(0) = 3 = c_1 + 2c_2 \end{array}\right\} \text{no sol'n.}$$

Suppose we choose y_1 and y_2

$y = c_1 e^x + c_2 e^{-x} \rightarrow y' = c_1 e^x - c_2 e^{-x}$

$y(0) = 5 = c_1 + c_2$

$y'(0) = 3 = c_1 - c_2$ $\qquad c_1 = 4, c_2 = 1$

$\therefore y = 4e^x + e^{-x}$

-This holds for higher order linear ODEs

-Def'n- A set $\{u_1, u_2, ..., u_n\}$ is said to be linearly dependent if at least one of them can be expressed as a linear combination of the others. Otherwise, they are linearly independent

Eg. $\{x, x^2, x^3+3x-1\}$ This is L.I.

Eg. $\{1, x, x^2, x^2+3x-1\}$ This is L.D.

Eg. $\{e^x, e^{-x}, \sinh x\}$ Since $\sinh x = \frac{e^x - e^{-x}}{2}$; these are L.D.

Eg. $\{x, 0\}$ Since $0 = 0 \cdot n$ $\therefore$ these are L.D.

-Theorem-

Let the coefficients of $a_0(x)y^{[n]}(x)+...+a_n(x)y(x)=0$ be continuous on an open interval. Then n linearly independent solutions $\{y_1, y_2, ..., y_n\}$ can be found. A general solution is $y = c_1y_1 + c_2y_2 + ... + c_ny_n$. Any set of such n linearly independent solutions is called a *basis* or *fundamental set* of solutions

(This general solution gives the entire family of solutions to the homogenous linear ODE)

Note: this set is not unique Changing the set only changes the constants

$y = c_1y_1 + c_2y_2 + ... + c_ny_n$
all sol'ns i.e. no singular solutions

-Theorem

If the coefficients of $a_0(x)y^{[n]}(x)+...+a_n(x)y(x)=0$ are all continuous on a closed interval, then the initial value problem with n initial conditions at a point a in the interval has a unique solution

Eg./ $y''' + 5y'' + 3y' - 9y = 0$, $y(0) = 0, y'(0) = 0, y''(0) = 1$

$y_1 = e^x$, $y_2 = e^{-3x}$, $y_3 = xe^{-3x}$ are sol'ns. They are also L.I.

$$y = C_1 e^x + C_2 e^{-3x} + C_3 x e^{-3x}$$

$$\left.\begin{aligned} y(0) &= C_1 + C_2 \\ y'(0) &= C_1 - 3C_2 + C_3 \\ y''(0) &= C_1 + 9C_2 - 6C_3 = 1 \end{aligned}\right] \quad \begin{aligned} C_1 &= \tfrac{1}{16} \\ C_2 &= -\tfrac{1}{16} \\ C_3 &= -\tfrac{1}{4} \end{aligned}$$

-Now we can talk about the nonhomogeneous case

-Theorem

Given $a_0(x)y^{[n]}(x) + ... + a_n(x)y(x) = f(x)$ with all the coefficients and the forcing function continuous. Let y_p be any solution to this. Then, *every* solution to this equation can be written as $y = y_p + y_h$ where y_h is a general solution to the homogeneous equation

Eg/ $y'' - 4y' + 3y = 10e^{-2x}$, $y(0) = 1$, $y'(0) = -3$

1. Look at homogenius ODE $y'' - 4y' + 3y = 0$
   you can show e^x and e^{3x} are li solutios
   $y_h = C_1 e^x + C_2 e^{3x}$
2. Find any y_p:
   Take educated guess. All derivatives of e^{-2x} gives a constant $\times$ e^{-2x}
   $\therefore$ try $y_p = Ce^{-2x}$
   $$4Ce^{-2x} + 8Ce^{-2x} + 3Ce^{-2x} = 10e^{-2x}$$
   $\therefore 15C = 10$ $\therefore C = \frac{2}{3}$ $\therefore y_p = \frac{2}{3}e^{-2x}$

$\therefore$ All solutions generated by $y = y_p + y_h = \frac{2}{3}e^{-2x} + C_1 e^x + C_2 e^{3x}$

-EVERY SOLUTION CAN BE WRITTEN IN THIS FORM. That is, if someone gets another y_p, then one can make them look the same by giving y_h different coefficients accordingly

Eg. $y'' - 4y' + 3y = 10e^{-2x}$ as in the previous example

One can show that $y_p = \frac{2}{3}e^{-2x} + \frac{1}{2}e^{x}$ is also another particular sol'n.

Then $y = \frac{2}{3}e^{-2x} + \frac{1}{2}e^{x} + c_1' e^{x} + c_2' e^{3x}$

$$= \frac{2}{3}e^{-2x} + \underbrace{\left(\tfrac{1}{2} + c_1'\right)}_{c_1} e^{x} + \underbrace{c_2'}_{c_2} e^{3x}$$

Constant coefficient case: e.g. $y'' + a_0 y' + a_1 y = f(x)$ constant

-We can use method of undetermined coefficients if the forcing function has a certain form. This is an educated trial and error

Terms in the forcing function *f(x)*	Assumed form of y_p
$e^{\gamma x}$	$Ce^{\gamma x}$
x^n	$K_1 x^n + K_2 x^{n-1} + \ldots + K_{n-1}$
$\cos(wx)$ $\sin(wx)$	$A\cos(wx) + B\sin(wx)$
$e^{\alpha x}\cos(wx)$, $e^{\alpha x}\sin(wx)$	$e^{\alpha x}(A\sin(wx) + B\cos(wx))$

-If there are several terms in *f(x)*, then use the associated assumed form of y_p for each term.

Eg/ $f(x) = e^{2x} + \cos(5x)$, you would try $y_p = Ce^{2x} + A\cos(5x) + B\sin(5x)$

-If *f(x)* is one of these terms multiplied by a polynomial of order *n*, then the assumed form of y_p is also multiplied by a polynomial of order *n* but with arbitrary constants

Eg/ $f(x) = \underbrace{e^{2x}}_{Ce^{2x}}\underbrace{(x^2 + 3x + 1)}_{\text{polynomial}}$

Try $y_p = Ce^{2x} \cdot (k_1 x^2 + k_2 x + k_3)$

don't need as it multiplies all the other constants.

$\lambda = \sqrt{11} \pm \sqrt{3}\, i$

$$\lambda = \frac{11 \pm \sqrt{121 - 4 \cdot 31}}{2} = \frac{11}{2} \pm \sqrt{3}/2$$

$\lambda^2 - 11\lambda + 31 = 0$

$y = c_1 e^{\frac{11}{2}x} \cos \sqrt{3}/2\, x$
$+ c_2 e^{\frac{11}{2}x} \sin \sqrt{3}/2\, x$

$y'' - 11y' + 31y = 0$

$y(0) = 1$

$y\left(\frac{\pi}{\sqrt{3}}\right) = e^{11\pi/2\sqrt{3}}$

$C_1 = 1$

$C_2 = 1$

-If y_p looks like y_h, then try the form above multiplied by x. If that is also a part of y_h, then multiply by another x

Eg/ $y'' + 4y' + 3y = \underbrace{e^x + x}_{f(x)}$ you can show $y_h = c_1 e^{-x} + c_2 e^{-3x}$

Normally, using the table, you try $y_p = Ce^x + k_1 x + k_0$

Sub back in the orig. eqn. $\Rightarrow y_p' = Ce^x + k_1$

$y_p = \frac{1}{8}e^x + \frac{1}{3}x - \frac{4}{9}$ $\Rightarrow y_p'' = Ce^x$

Eg/ $y'' - 4y' + 3y = e^x$ $y_h = c_1 e^x + c_2 e^{3x}$

From table, try $y_p = Ce^x \longrightarrow$ part of y_h X

Try instead $y_p = xCe^x$

$\therefore y_p' = Ce^x + Cxe^x$, $y_p'' = Ce^x + Ce^x + Cxe^x$

$\underbrace{2Ce^x + \cancel{Cxe^x}}_{y_p''} - 4\underbrace{(Ce^x + \cancel{Cxe^x})}_{y_p'} + 3\underbrace{\cancel{Cxe^x}}_{y_p} = -2Ce^x = \overbrace{e^x}^{f(x)}$ $\therefore C = -\frac{1}{2}$

$y_p = -\frac{1}{2}xe^x$

-We will return to the method of underdetermined coefficients after looking at constant coefficient equation

D/ Constant Coefficient Linear ODEs

-Many systems are described by such equations (eg. RCL or mass-spring-damper systems)

-First, let us look at the second order homogeneous case first

$y'' + ay' + by = 0$

-assume a solution of the form $y = e^{\lambda x}$. Then the derivatives are $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

and we obtain the following

$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$

-Since $e^{\lambda x}$ is always nonzero, then λ must be such that $\lambda^2 + a\lambda + b = 0$. This is called the **characteristic equation**.

-The roots to this are $\lambda_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$

-The two solutions are therefore $y = e^{\lambda_1 x}$ and $y = e^{\lambda_2 x}$

-We can show that these are linearly independent if $\lambda_1 \neq \lambda_2$

-The general solution is therefore $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

-Three possibilities for $\lambda_{1,2}$:

Case I Two real roots if $a^2 - 4b > 0$
Case II Real double roots if $a^2 - 4b = 0$
Case III Complex conjugate roots if $a^2 - 4b < 0$

Case I

Eg/ $y'' + y' - 2y = 0 \Rightarrow$ Ch. eq. $(\lambda^2+\lambda-2)=0$

or $(\lambda+2)(\lambda-1)=0$

$\therefore \lambda_1=-2,\ \lambda_2=1$

$\therefore y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$

$=c_1e^{-2x}+c_2e^{x}$

Case II

$\lambda_{1,2}=-a/2$ ie. $a^2-4b=0$ This means that

$b=\frac{1}{4}a^2 \Rightarrow y''+ay'+\frac{a^2}{4}y=0$

We know one root is $-\frac{a}{2}$ $\therefore y_1=e^{-\frac{a}{2}x}$ is a soln.

Let's try $y_2=xe^{-\frac{a}{2}x} \Rightarrow y_2'=e^{-\frac{a}{2}x}+(-\frac{a}{2})xe^{-\frac{a}{2}x}$, $y_2''=-ae^{-\frac{a}{2}x}+(\frac{a^2}{4})e^{-\frac{a}{2}x}x$

There is only one solution $e^{-\frac{a}{2}x}$. Try $xe^{-\frac{a}{2}x}$ sub in. you can show this is a soln. $y=c_1e^{-\frac{a}{2}x}+c_2xe^{-\frac{a}{2}x}$

Can show these are LI. Thus, we have a general solution of the form

Case III

Complex conjugate roots (recall $\underbrace{e^{ix}=\cos x+i\sin x}_{\text{euler}}$)

Let $\lambda_{1,2}=\frac{1}{2}(-a\pm\sqrt{a^2-4b})=\alpha\pm i\omega$ where $\alpha=-a/2$ and $i\omega=\frac{\sqrt{a^2-4b}}{2}$

$y=c_1'y_1+c_2'y_2=c_1'e^{(\alpha+j\omega)x}+c_2'e^{(\alpha-j\omega)x}$

$=c_1'e^{\alpha x}e^{j\omega x}+c_2'e^{\alpha x}e^{-j\omega x}$

$e^{j\omega x}=\cos\omega x+j\sin\omega x$

$e^{-j\omega x}=\cos\omega x-j\sin\omega x$

We can rewrite this as

$$y = (c_1' + c_2') e^{\alpha x} \cos \omega x + j(c_1' - c_2') e^{\alpha x} \sin \omega x$$

Let $c_1' = \dfrac{c_1 - jc_2}{2}$, $c_2' = \dfrac{c_1 + jc_2}{2}$

$$\therefore\ \underbrace{c_1 e^{\alpha x} \cos \omega x}_{} + \underbrace{c_2 e^{\alpha x} \sin \omega x}_{}$$

l.i. solns

Now, everything is real

Eg/ $y'' - 2y' + 10y = 0$

$$\lambda_2 - 2\lambda + 10 = 0 \Rightarrow \lambda_{1,2} = 1 \pm 3i \quad \begin{cases} \alpha = 1 \\ \omega = 3 \end{cases}$$

$$\therefore\ y = c_1 e^{x} \cos 3x + c_2 e^{x} \sin 3x$$

Eg/ $y'' + 2y' - y = 0$

$$\lambda^2 + 2\lambda - 1 = 0$$

$$\lambda_{1,2} = -1 \pm \sqrt{2} \qquad y = c_1 e^{(-1+\sqrt{2})x} + c_2 e^{(-1-\sqrt{2})x}$$

Eg/ $y'' + 2y' + 2y = 0$

$$\lambda^3 + 2\lambda + 2 = 0$$

$$\frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \qquad y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

-What about higher order ODEs? $y^{[n]} + a_{n-1}y^{[n-1]} + ... + a_1 y' + a_0 y = 0$

1. Look at the characteristic equation ($y = e^{\lambda x}$)

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0$$

2. Solve for the roots (use numerical solver eg. MAPLE) $\lambda_1, \lambda_2, ..., \lambda_n$
3. The solution is a sum of **independent** solutions of the form:
   a. Real roots (not repeated)

   $$y_i = e^{\lambda_i x}$$

   b. Complex roots

   $$y_{i,i+1} = e^{(\alpha \pm j\omega)x} \quad \text{or} \quad y_{i,i+1} = e^{\alpha x}(C_1 \sin\omega x + C_2 \cos\omega x)$$

   c. Real repeated roots of order m

   $$y_i, y_{i+1}, ..., y_{i+m-1} = (C_1 + C_2 x + ... + C_m x^{m-1})e^{\lambda_i x}$$

   d. Repeated complex roots of order m

   $\lambda = \alpha \pm i\omega$ (m of them)

$$y_i, y_{i+1}, ..., y_{i+2m-1} = e^{\alpha x}(C_1 \sin\omega x + C_2 \cos\omega x)(a_1 + a_2 x + ... a_m x^{m-1})$$

-eg. $y^{[6]} - 4y^{[5]} + 14y^{[4]} + 32y^{[3]} - 79y'' + 260y' + 676y = 0$

C.E. is $\lambda^6 - 4\lambda^5 + 14\lambda^4 + 32\lambda^3 - 79y^2 + 260y + 676 = 0$

Numerical $\lambda = -2, -2, 2 \pm 3i, 2 \pm 3i$

$$y_n = (C_1 + C_2 x)e^{-2x} + e^{2x}(C_3 \sin 3x + C_4 \cos 3x)(C_5 + C_6 x)$$

(x = time)

E/ Stability for linear, constant coefficient ODEs

-important concept in mechatronics, circuits and control systems

-given an ODE $y^{[n]} + a_{n-1}y^{[n-1]} + ... + a_1 y' + a_0 y = 0$ describing a system, can we have a qualitative description of the system behavior?

-we consider the system to be *stable* if the solution $y(t)$ to the ODE is bounded for any initial condition. That is, there exists a constant M such that $|y(t)| \leq M, \quad \forall t > 0$. We consider the system to be *unstable* if the solution is unbounded. For a linear system, we want to see if the system "blows up"

regardless of I.C.

Eg./ $y' - 5y = 0$ Homog. Soln is $y_h = C_1 e^{5x}$. with the I.C., $y_h = ae^{5x}$

$y(0) = a$

This is unstable

-Recall that, given an nth order linear constant coefficient ODE, the solution is $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ where the y_i are a set of n linearly independent solutions. It can be shown that, no matter what solutions are used as the fundamental set of solutions, the following is true: *The system is* ***stable*** *if and only if there are no roots of the characteristic equation to the right of the imaginary axis and any roots on the imaginary axis are nonrepeated.*

The proof would require looking at the following cases.

Case a) roots to the right of the imaginary axis

Gives exponential $e^{\lambda t}$ which is unbounded

Case b) roots the left of the imaginary axis

Give exponentials $e^{-\lambda t}$, $\lambda > 0$, possibly multiplied by polynomial, which goes to zero. For complex, we get $e^{\alpha t}$ (oscillating term)

Case c) roots on the imaginary axis

If complex, get terms like $(A\cos wt + B\sin wt)\underbrace{(\text{polynomial})}_{\text{unbounded if repeated}}$

∴ Complex roots on imag. axis can't be repeated.

-eg. Is $y^{[6]} - 4y^{[5]} + 14y^{[4]} + 32y^{[3]} - 79y'' + 260y' + 676y = 0$ stable or unstable?

unstable because of the $2 \pm 3i$ term

↑ Homogeneous

↓ Non-Homogeneous

-Now look back at the nonhomogeneous case (recall the method of undetermined coefficients)

eg. $y'' + 4y = 8x^2$ - - - (1)

Look @ homog. case $y''+4y=0 \Rightarrow$ Ch. eq. $\lambda^2+4=0$

$\lambda = \pm 2i$

$y = C_1 \sin 2x + C_2 \cos 2x$

Try $y_p = k_2x^2 + k_1x + k_0$, $y_p' = 2k_2x + k_1$, $y_p'' = 2k_2$. Sub in (1)

$2k_2 + 4(k_2x^2 + k_1x + k_0) = 8x^2 = \begin{cases} 4k_0 + 2k_2 = 0 \\ 4k_1 = 0 \\ 4k_2 = 8 \end{cases}$ $\quad k_0 = -1$, $k_1 = 0$, $k_2 = 2$

$\therefore y = C_1 \sin 2x + C_2 \cos 2x + 2x^2 - 1$

Eg $y'' - 3y' + 2y = e^x$

look at homog. case, C.E. $= \lambda^2 - 3\lambda + 2 = 0$

$\lambda_{1,2} = 1, 2$

$y_h = C_1e^x + C_2e^{2x}$. Try $y_p = ke^x$ (part of y_h). Try instead $y_p = kxe^x$

$y_p' = k(e^x + xe^x)$

$y_p'' = k(2e^x + xe^x)$

Sub back, match coefficients

$k = -1$ $\quad \therefore y = C_1e^x + C_2e^{2x} - xe^x$

Eg/ $y'' - 2y' + y = e^x + x$, * $y(0) = 1, y'(0) = 0$

Homog. CE $\Rightarrow \lambda^2 - 2\lambda + 1 = 0$, $\lambda_{1,2} = 1, 1$

$y_h = c_1e^x + c_2xe^x$ Look for $y_p = \underbrace{ke^x}_{\text{part of homog. soln}} + k_2x + k_3$

$= \underbrace{k_1xe^x}_{\text{also part of homog.}} + k_2x + k_3$

$\therefore$ try $y_p = k_1x^2e^x + k_2x + k_3$

Find y_p', y_p'', and sub into * and compare coefficient

$k_1 = \frac{1}{2}$, $k_2 = 1$, $k_3 = 2$

$\therefore y = y_h + y_p$
$= c_1e^x + c_2xe^x + x + 2 + \frac{1}{2}x^2e^x$

initial conditions

Find y, y' and use the i.c. $y(0)=1$, $y'(0)=0$ find

$c_1 = -1$, $c_2 = 0$

$\therefore y = -e^x + \frac{1}{2}x^2e^x + x + 2$

Eg/ $y'' + 2y' + 5y = 16e^x + \sin 2x$

Soln: $y = e^{-x}(A\cos 2x + B\sin 2x)$
$+ 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$

$y'' - 2y' + 5y = 16e^{x} + \sin 2x$

homog.

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$

$= -1 \pm 2i$

$\therefore y_h = C_1 e^{-x}\cos 2x + C_2 e^{-x}\sin 2x$

particular:

$y_p = ke^{x} + A\sin 2x + B\cos 2x$

$y_p' = ke^{x} + 2A\cos 2x - 2B\sin 2x$

$y''_p = ke^{x} - 4A\sin 2x - 4B\cos 2x$

$+ 2ke^{x} + 4A\cos 2x - 4B\sin 2x$

$+ 5ke^{x} + 5A\sin 2x + 5b\cos 2x$

$8ke^{x} + A\sin 2x + B\cos 2x - 4A\cos 2x - 4B\sin 2x$

$k = 4$

$(A - 4B) = 1$

$B - 4A = 0$

$B = 4A$

$A + 16A = 1$

$A = \frac{1}{17}$

$B = \frac{4}{17}$

$\therefore y_p = 4e^{x} - \frac{2}{15}\cos 2x + \frac{4}{15}\sin 2x$