Recall: for unambiguous decomposition of strings

$$S = A \cup B \Rightarrow \Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$

$$S = AB \Rightarrow \Phi_S(x) = \Phi_A(x)\Phi_B(x)$$

$$S = A^* \Rightarrow \Phi_S(x) = 1/(1-\Phi_A(x))$$

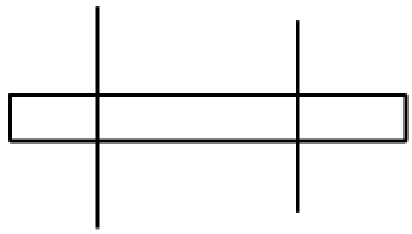
S: strings with 3 blocks

$$\Phi_S(x) = \frac{2x^3}{(1-x)^3} = 2x^3(1-x)^{-3} = 2x^3\sum_{k\ge 0}\binom{k+2}{2}x^k$$

strs of length n

$$[x^n]\Phi_S(x) = [x^{n-3}]2\sum_{k\ge 0}\binom{k+2}{2}x^k$$

$$2\binom{n-1}{2}$$



$0^*(11^*00^*)1^*$ ← all binary strs

S = strs with all blocks with length ≥ 2

$$(\varepsilon \cup \{00\}0^*)(\{11\}1^*\{00\}0^*)^*(\varepsilon \cup \{11\}1^*)$$

$$\Phi_S(x) = \left(1+\frac{x^2}{1-x}\right)^2 \Phi_{B^*} = \left(1+\frac{x^2}{1-x}\right)^2 \frac{1}{1-\left(\frac{x^2}{1-x}\right)^2}$$

$$= \left(\frac{1-x+x^2}{1-x}\right)^2\left[\frac{(1-x)^2}{(1-x)^2-x^4}\right] = \frac{(1-x+x^2)^2}{(1-x)^2-x^4} = \frac{1-x+x^2}{1-x-x^2}$$

S = Strings where an even block of 0's cannot be followed by an odd block of 1's.

$$1^*(\underbrace{0\{00\}^*11^*}_{\text{odd}} \cup \underbrace{00\{00\}^*11\{11\}^*}_{\text{even}})^* 0^*$$

$$\Phi_S = \frac{1}{1-x}\left[\frac{1}{1-\left(\frac{x}{1-x^2}x\frac{1}{1-x} + \frac{x^2}{1-x^2}\frac{x^2}{1-x^2}\right)}\right]\frac{1}{1-x}$$

$$= \frac{1}{(1-x)^2}\left[\frac{1}{1-\left(\frac{x^2}{(1-x^2)\,1-x} + \frac{x^4}{(1-x^2)^2}\right)}\right]$$

$$= \frac{1}{(1-x)^2}\left[\frac{1}{1-\frac{x^2(1+x)+x^4}{(1-x^2)^2}}\right]$$

$$= \cancel{\frac{1}{(1-x)^2}}\left[\frac{(1+x)^2\cancel{(1-x)^2}}{(1-x^2)^2-x^2(1+x)+x^4}\right]$$

$$= \frac{(1+x)^2}{x(1+x^2+x^3)}$$

S = strings w/ no l consecutive 1s & no m consecutive 0s.

$$S = (0^* \setminus 0^m 0^*)[(1^* \setminus 1^l 1^*)(00^* \setminus 0^m 0^*)]^* (1^* \setminus 1^l 1^*)$$

$$\Phi_S(x) = \left(\frac{1}{1-x} - x^m\left(\frac{1}{1-x}\right)\right)\left(\frac{1}{1-x} - \frac{x^l}{1-x}\right)\frac{1}{1-\Phi_{(1^*\setminus 1^l 1^*)}(x)\,\Phi_{(00^*\setminus 0^m 0^*)}(x)}.$$

$$\Phi_{(11^*\setminus 1^l 1^*)}(x) = \left(\frac{x}{1-x} - \frac{x^l}{1-x}\right), \quad \Phi_{(00^*\setminus 0^m 0^*)}(x) = \left(\frac{x}{1-x} - \frac{x^m}{1-x}\right)$$

$$\therefore \Phi_s(x) = \left(\frac{1-x^m}{1-x}\right)\left(\frac{1}{1-\left(\frac{x}{1-x}-\frac{x^l}{1-x}\right)\left(\frac{x}{1-x}-\frac{x^m}{1-x}\right)}\right)\left(\frac{1-x^l}{1-x}\right)$$

$$= \frac{(1-x^m)(1-x^l)\cancel{(1-x)^2}}{\cancel{(1-x)^2}\left((1-x)^2-(x-x^l)(x-x^m)\right)}$$

$$= \frac{1-x^m-x^l+x^{m+l}}{1-2x+x^{m+1}+x^{l+1}-x^{m+l}}$$

$l=1, m=1$

$$\Phi_s(x) = \frac{1-2x+x^2}{1-2x+x^2} = 1$$

$l=2, m=2$

$$\Phi_s(x) = \frac{1-2x^2+x^4}{1-2x+2x^3-x^4} = \frac{(1-x^2)^2}{(1-x^2)(1-2x+x^2)} = \frac{1+x}{1-x} = a_0+a_1x+a_2x^2+\cdots$$

$$1+x = a_0(1-x)+a_1x(1-x)+a_2x^2(1-x)+\cdots$$

$$a_0 = 1$$

$$-a_0+a_1 = 1 \Rightarrow a_1 = 2$$

$$a_i - a_{i-1} = 0 \Rightarrow a_{i+1} = a_i \quad \forall i \geq 2$$

Recursive Decompositions

S = all strings

$$S = \{\epsilon\} \cup S\{0,1\}$$

$$\Phi_S(x) = 1 + \Phi_S(x)(2x)$$

$$\Phi_S(x) - \Phi_S(x)2x = 1$$

$$\Phi_S(x) = \frac{1}{1-2x}$$

$$= \sum_{k \geq 0} 2^k x^k$$

S = strings without 111

$$S = \{\epsilon, 1, 11\} \cup S\{0\} \cup S\{01\} \cup S\{011\}$$

$$= \{\epsilon, 1, 11\} \cup \{0, 01, 011\}$$

$$\Phi_S(x) = (1 + x + x^2) + \Phi_S(x)(x + x^2 + x^3)$$

$$\Phi_S(x) = \frac{1 + x + x^2}{1 - (x + x^2 + x^3)}$$

Forbidden Substrs

Problem: How many $\overbrace{\text{strings are there w/ no 1101?}}^{L}$

$M = \{$str ~~not~~ ending w/ 11101$\}$

$L \cup M = \{\varepsilon\} \cup L\{0,1\}$

$M = L\{11101\}$