# Local Extrema and Crit. Pts.

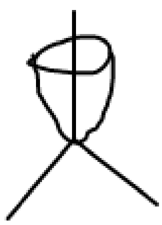
<u>Defn</u>: A fcn. $f$ has a <u>local max</u> at $(a,b)$ if $f(x,y) \le f(a,b)$ $\forall (x,y)$ near $(a,b)$

<u>Thm</u>: If $f$ has a local max/min at $(a,b)$, then $f_x(a,b) = 0 = f_y(a,b)$ (or one of $f_x, f_y$ does not exist).

Then $(a,b)$ is called a <u>critical point</u> of $f$.

<u>Ex</u> $f(x,y) = x^2 + y^2$

Paraboloid



$$\left.\begin{array}{l} f_x = 2x = 0 \Rightarrow x = 0 \\ f_y = 2y = 0 \Rightarrow y = 0 \end{array}\right\} \text{(0,0) is the only crit. pt.}$$

$f(0,0)$ is a min.

<u>Ex</u> $f(x,y) = \sqrt{1-x^2-y^2}$

Show $(0,0)$ is the only crit. pt.

<u>Ex</u> $f(x,y) = x^2 - y^2$

$$\left.\begin{array}{l} f_x = 2x = 0 \Rightarrow x = 0 \\ f_y = -2y = 0 \Rightarrow y = 0 \end{array}\right\} \text{(0,0) is the only crit. pt.}$$

Approach $(0,0)$ along $x=0$: $f(0,y) = -y^2 \rightarrow$ looks like max.

Approach $(0,0)$ along $y=0$: $f(x,0) = x^2 \rightarrow$ looks like min

$(0,0)$ is a <u>saddle point</u> (the surface is a saddle surface)

To classify a crit. pt. as a local max/min or saddle point, we need the $2^{nd}$ derivative test.

Recall for single-var fcns:

Let $f'(a)=0$, (i) if $f''(a)>0 \Rightarrow$ local min

(ii) if $f''(a)<0 \Rightarrow$ local max

(iii) if $f''(a)=0 \Rightarrow$ no conclusion

In the 2-variable case, there are 4 partials (3 unique).

Look at the Hessian Matrix.

$$Hf(\underline{x}) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The determinant is:

$$D(x,y) = f_{xx}f_{yy} - f_{xy}f_{yx}$$
$$= f_{xx}f_{yy} - (f_{xy})^2$$

## $2^{nd}$ derivative test

If $(a,b)$ is a crit. pt., then

1) If $D(a,b)>0$ & $f_{xx}(a,b)>0$ (or $f_{yy}(a,b)>0$), $(a,b)$ is a local min.
2) If $D(a,b)>0$ & $f_{xx}(a,b)<0$, $(a,b)$ is a local max.
3) If $D(a,b)<0$, $(a,b)$ is a saddle point.

Ex. Find and classify the crit. pts. of:

$$f(x,y) = 3x^2y + y^3 - 3x^2 + 1$$

$f_x = 6xy - 6x = 0 \Rightarrow 6x(y-1)=0$ ① Either $x=0$ or $y=1$

$f_y = 3x^2 + 3y^2 - 6y = 0$ ②

put $x=0$ in (2)

$$3y^2-6y=0 \Rightarrow 3y(y-2)=0 \Rightarrow y=0,2$$

$(0,0)$ & $(0,2)$ are crit. pts.

put $y=1$ in (2)

$$3x^2+3-6=0 \Rightarrow 3x^2=3 \Rightarrow x=\pm 1$$

$(1,1), (-1,1)$ are crit. pts.

To classify, compute 2nd partials:

$$f_{xx}=6y-6 \ , \ f_{xy}=6x \ , \ f_{yy}=6y-6$$

$$D(x,y)=f_{xx}f_{yy}-f_{xy}^2=(6y-6)^2-(6x)^2=36\left[(y-1)^2-x^2\right]$$

plug in crit pts.... $(0,0)$ local max

$(0,2)$ local min

$(1,1),(-1,1)$ saddle points