Taylor's Polynomial

If $P_{n,x_0}(x)$ is used to estimate $f(x)$, then the error satisfies $|f(x) - P_{n,x_0}(x)| = |R_n(x)| \leq \frac{k}{(n+1)!}|x-x_0|^{n+1}$
where $|f^{(n+1)}(t)| \leq k \quad \forall t \in [x_0, x]$

3 things to do:

1) determine interval (e.g. $[1, 1.2]$ in $\ln(1.2)$ example)
2) Find K
3) put a bound on $|x-x_0|^{n+1}$

Example:

Use a Taylor Polynomial about 0 for $f(x) = e^x$ to estimate the number e within 10^{-8}.

Sol^n:

To estimate the number e, use the fact that

$$f(1) = e^1 = e$$

So, our estimate is $e \approx P_{n,0}(1)$.

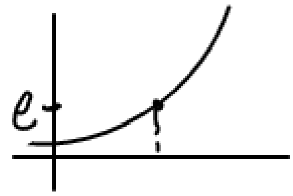
We are asked to find the n such that

$$|R_n(x)| < 10^{-8}$$

$$|R_n(x)| \leq \frac{k}{(n+1)!}|x|^{n+1}, \text{ where } |f^{(n+1)}(t) \leq k| \text{ on } [0,1]$$

Since $f(t) = e^t$, then $f^{(n)}(t) = e^t \ \forall t$.

$$\Rightarrow |f^{(n+1)}(t)| = e^t \leq e^1 \Rightarrow e \leq \underbrace{3}_{K}$$



$$\Rightarrow |R_n(x)| \leq \frac{3}{(n+1)!}|x|^{n+1} \leq 1 \text{ on } [0,1]$$

$$|R_n(x)| \leq \frac{3}{(n+1)!}|x|^{n+1} \leq 1 \text{ on } [0,1]$$
$$\leq \frac{3}{(n+1)!}$$

We need to find n such that $\frac{3}{(n+1)!} < 10^{-8}$

$$\Rightarrow (n+1)! > 3 \times 10^{8}$$

Can't solve explicitly $\rightarrow$ trial and error.

Find $12! \approx 4.79 \times 10^{8} > 3 \times 10^{8}$

Therefore, $|R_n(x)| < 10^{-8}$, and to estimate e,

use $P_{11,0}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{11!}$

evaluate at $x = 1$:

$e \approx P_{11,0}(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{11!}$

$P_{11,0}(1) \approx 2.718281826$

$e \approx 2.718281828$

error in the 9^{th} decimal place.

Estimating Integrals using Taylor Polynomials

Being able to integrate $\int_a^b f(x)$ requires knowledge of the antiderivative of f; however, this might not be readily available e.g. $\int_0^1 e^{-x^2} dx$.

Idea: polynomials are easy to integrate, so approximate f by its Taylor Polynomial.

Start with an example where we can calculate the integral by FTC.

<u>Ex.</u> Estimate $\int_0^1 e^x dx$ using $P_{2,0}(x)$ & find an upper bound on the error.

<u>Sol^n:</u> $P_{2,0}(x) = 1+x+\frac{x^2}{2}$

since $e^x \approx P_{2,0}(x)$, our estimate is

$$\int_0^1 e^x dx \approx \int_0^1 (1+x+\frac{x^2}{2}) dx = x+\frac{x^2}{2}+\frac{x^3}{6}\Big|_0^1$$

$$= 1+\frac{1}{6}+\frac{1}{6} = \frac{6+3+1}{6} = \frac{5}{3}$$

Taylor's Inequality:

$$|R_2(x)| \le \frac{K}{3!}|x|^3, \text{ where } |f'''(t)| \le K \text{ on } \underline{\underline{[0,1]}}$$

$$f(t) = e^t \Rightarrow |f'''(t)| = e^t \le e \le \underbrace{3}_{K} \quad \text{on } [0,1].$$

$$\Rightarrow |R_2(x)| \le \frac{3}{3!}|x|^3 = \frac{1}{2}x^3 \text{ on } [0,1].$$

The error satisfies:

$$\left|\int_0^1 e^x dx - \int_0^1 P_{2,0}(x)dx\right| = \left|\int_0^1 (e^x - P_{2,0}(x))dx\right|$$

$$\le \int_0^1 |e^x - P_{2,0}(x)| dx \longrightarrow R_2(x)$$

$$\leq \int_0^1 \frac{1}{2}x^3 dx = \frac{1}{8}x^4 \Big|_0^1 = \frac{1}{8}$$

$\Rightarrow \int_0^1 e^x dx \approx \frac{5}{3}$ with an error of at most $\frac{1}{8}$.

check $\frac{5}{3} = 1.\bar{6}$ $\quad \frac{1}{8} = 0.125$

$$\int_0^1 e^x dx = e^x \Big|_0^1 = e-1 \approx 1.718$$