Q. How many ways can you eat n pieces of fruit given that you must eat:

- most 5 apples
- least 3 bananas
- even # of cherrie

<u>Claim</u>: Answer is $[x^n](1+x^2+x^3+x^4+x^5)(x^3+x^4+x^5+\dots)(1+x^2+x^4+\dots)$

$$\underbrace{(1+x^2+x^3+x^4+x^5)}_{\substack{0\ 1\ 2\ 3\ 4\ 5 \\ \text{apples}}}\underbrace{(x^3+x^4+x^5+\dots)}_{\substack{3\ 4\ 5\ \cdots \\ \text{bananas}}}\underbrace{(1+x^2+x^4+\dots)}_{\text{cherries}}$$

$$= [x^n]\left(\frac{1-x^6}{1-x}\right)\left(\frac{x^3}{1-x}\right)\left(\frac{1}{1-x^2}\right)$$

$$= [x^n]\,\frac{x^3(1-x^6)}{(1-x)^3(1+x)}$$

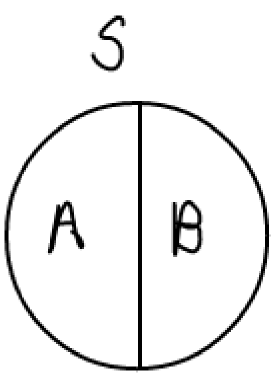
Counting problems involving multiple selections can thus be encoded as coefficients.

We now make this formal.

<u>Sum Lemma</u>:

If S is a set w/ weight function w, and A, B are sets so that $A \cap B = \emptyset$, $A \cup B = S$, then

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x)$$



Product Lemma Let A, B be sets with weight function α, β respectively. Then $\Phi_A(x)\,\Phi_B(x) = \Phi_S(x)$, where $S = A \times B$ & $w(a,b) = \alpha(a) + \beta(b)$ is the weight function on S.

Eg. Binomial Theorem

Let $S = \{\text{subsets of } [n]\}$, and $w(A) = |A|$ for $A \in S$

$$\Phi_S(x) = \sum_{k \geq 0} (\#\text{ elements of } S \text{ of weight } k)\, x^t$$

$$= \sum_{k \geq 0} \binom{n}{k} x^k$$

Induction to show $(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$

Base case: $(1+x)^0 = \binom{0}{0} x^0$ trivial. Suppose it is true for $n-1$. $(n \geq 1)$

Let $T = \{\text{elements of } S \text{ containing } n\} = \{y \cup \{n\} : y \subseteq [n-1]\}$

$R = \{\text{elements of } S \text{ not containing } n\} = \{y : y \subseteq [n-1]\}$

$T \cap R \neq \emptyset$, $T \cup R = S$.

By Sum Lemma, $\Phi_S(x) = \Phi_R(x) + \Phi_T(x)$

$$\Phi_R(x) = \sum_{y \subseteq [n-1]} x^{|y|} = \sum_{k \geq 0} \binom{n-1}{k} x^k \underset{IH}{=} (1+x)^{n-1}$$

$$\Phi_T(x) = \sum_{y \subseteq [n-1]} x^{|y \cup [n]|} = \sum_{y \subseteq [n-1]} x^{|y|+1} = x \sum_{y \subseteq [n-1]} x^{|y|} = x(1+x)^{n-1}$$

So $\Phi_S(x) = (1+x)^{n-1} + x(1+x)^{n-1} = (1+x)^n$ $\blacksquare$

≤ 5 apples, ≥ 3 bananas, even # of cherries

Set $A = \{0,1,2,3,4,5\}$ $\quad \alpha(a) = a \quad \Phi_A(x) = 1 + x + x^2 + x^3 + x^4 + x^5$

$B = \{3,4,5,6,\dots\}$ $\quad \beta(b) = b \quad \Phi_B(x) = x^3 + x^4 + \dots$

$C = \{0,2,4,6,\dots\}$ $\quad \gamma(c) = c \quad \Phi_C(x) = 1 + x^2 + x^4 + \dots$

Product Lemma gives

$\Phi_A(x)\Phi_B(x)\Phi_C(x) = \Phi_S(x)$ where $S = A \times B \times C$

$w(a,b,c) = a+b+c$

# of valid sel. of n fruits

$= $ # elements of S of weight n

$= [x^n]\,\Phi_S(x)$

$= [x^n]\,\Phi_A(x)\Phi_B(x)\Phi_C(x)$

$= [x^n]\,\dfrac{x^3(1-x^6)}{(1-x)^3(1+x)}$ from page 1

$$\boxed{= [x^{n-3}]\,\frac{(1-x^6)}{(1-x)^3(1+x)}}$$

Q. How many ways to change \$1?

a 'change of \$1' is a selection $(a,b,c,d) \in (\mathbb{N}_0)^4$

s.t. $5a + 10b + 25c + 100d = 100$

let $w_1(a) = 5c$

$w_2(b) = 10b$

$w_3(c) = 25c$

$w_4(d) = 100d$

$$\Phi^{w_1}_{\mathbb{N}_0}(x)\,\Phi^{w_2}_{\mathbb{N}_0}(x)\,\Phi^{w_3}_{\mathbb{N}_0}(x)\,\Phi^{w_4}_{\mathbb{N}_0}(x) = \Phi^{w}_{\mathbb{N}_0^4}(x)$$

where $w(a,b,c,d) = w_1(a) + w_2(b) + w_3(c) + w_4(d)$
$= 5a + 10b + 25c + 100d$

Note:
$\Phi^{w} \leftarrow$ wrt w,
$\mathbb{N}_0 \leftarrow$ set

ways to change $1
= # elements of $\mathbb{N}_0^4$ with weight 100
$= [x^{100}]\,\Phi^{w}_{\mathbb{N}_0^4}(x)$