We gave a method for solving $A(x)Q(x)=B(x)$ for $Q(x)$, for any formal power series $A(x)$, $B(x)$.

e.g. $x^2 Q(x) = x$ can be 'solved' algebraically (as an equation about numbers), but has no formal power series $Q(x)$ as a soln.

Consider the case $B(x)=1$.

Let $P(x)$ be a formal power series. If $Q(x)$ is another f.p.s. such that $P(x)Q(x)=1$, then $Q(x)$ is the reciprocal (inverse) of $P(x)$.

We write $Q(x) = \frac{1}{P(x)} = P(x)^{-1}$

e.g. $(1-x)(1+x+x^2+x^3+\dots) = 1+(1-1)x+(1-1)x^2+\dots$
$= 1$

So $1+x+x^2+x^3+\dots = \frac{1}{1-x}$

Writing a FPS as a reciprocal is often useful for calculations.

e.g. solve $(1+x+x^2+\dots)Q(x) = (1+x)^2$

$\Rightarrow \frac{1}{1-x}Q(x) = (1+x)^2$

$Q(x) = (1+x)^2(1-x)$

Which formal power series have reciprocals?

$P(x) = 1-x$ does. $P(x) = x$

<u>Prop:</u> $P(x) = \sum_{k\geq 0} p_k x^k$ has a reciprocal iff $p_0 \neq 0$.

Pf: If $p_0 = 0$ then $P(x)Q(x) = (p_1 x + p_2 x^2 + \dots)(q_0 + q_1 x + q_2 x^2 + \dots)$ which has constant term 0 for any Q.

So $P(x)$ has no reciprocal.

Suppose that $p_0 \neq 0$.

$P(x)Q(x) = 1 \iff (p_0 + p_1 x + p_2 x^2 + \dots)(q_0 + q_1 x + q_2 x^2 + \dots) = 1$

$$p_0 q_0 = 1 \implies q_0 = \frac{1}{p_0}$$

$$p_0 q_1 + p_1 q_0 = 0 \implies q_1 = \frac{-p_1 q_0}{p_0}$$

$$p_0 q_2 + p_1 q_1 + p_2 q_0 = 0 \implies q_2 = \frac{-p_1 q_1 - p_2 q_0}{q_0}$$

In general $q_0 = \frac{1}{p_0}$

$$q_k = \frac{-1}{p_0} \sum_{i=1}^{k} p_i q_{k-1}$$ gives a soln

So $p(x)$ has a reciprocal

Corollary: If $A(x)$ has nonzero constant term, then $A(x)Q(x) = B(x)$ can be solved for any $B(x)$. (Since $Q(x) = A(x)^{-1} B(x)$ is a soln)

The usual way to compute inverses is:

Prop: If $A(x)$ has zero constant term, then

$$\frac{1}{1 - A(x)} = \sum_{i \geq 0} A(x)^i = 1 + A(x) + A(x)^2 + \dots$$

eg. $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{i\geq 0} (-x)^i = 1 - x + x^2 - x^3 + \dots$

$\frac{1}{3-x} = \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}x}\right) = \frac{1}{3}\sum_{i\geq 0}\left(\frac{1}{3}x\right)^i = \sum_{i\geq 0}\left(\frac{1}{3}\right)^{i+1} x^i$

$\frac{1}{1-x-x^2} = \sum_{i\geq 0}(x+x^2)^i = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \dots$

$= 1 + (x+x^2) + (x^2+2x^3+x^4) + (x^3+3x^4+3x^5+x^6) + (x^4+\dots$

$= 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots$

appears to be Fibonacci Sequence

We want to know for sure what the sequence of coefficients is.

Let $Q(x) = q_0 + q_1 x + q_2 x^2 + \dots$ be the inverse of $1-x-x^2$

$(1-x-x^2)(q_0 + q_1 x + q_2 x^2 + \dots) = 1$

$q_0 = 1$

$q_1 - q_0 = 0$

$q_2 - q_1 - q_0 = 0$

generally, $q_k - q_{k-1} - q_{k-2} = 0$ for $k \geq 2$

thus, $q_0 = 1$, $q_1 = 1$, and $q_k - q_{k-1} - q_{k-2}$, $k \geq 2$.

So $q_0, q_1, q_2, \dots$ is the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$

Also useful for simplifying is the following:

Prop: If $A(x)$ has zero constant term & $n \in \mathbb{N}$, then

$$\sum_{i=0}^{n} A(x)^i = \frac{1 - A(x)^{n+1}}{1 - A(x)}$$

Prop: If $A(x)$, $B(x)$ are FPS and $A(x)B(x) = 0$ then $A(x) = 0 \,\|\, B(x) = 0$.

Corollary: Inverses where they exist, are unique.

pf: suppose that $Q(x) \neq 0$ and $Q(x)P(x) = 1$, $Q(x)R(x) = 1$, ...

then $0 = 1 - 1 = Q(x)P(x) - Q(x)R(x)$

$$= Q(x)\,(P(x) - R(x))$$

We know $Q(x) \neq 0$ by prop, $P(x) - R(x) = 0$

so $P(x) = R(x)$.

So $Q(x)$ only has one inverse (at most).