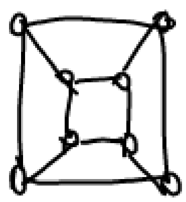
$n$-cube

vertex set $V = \{\text{binary strings of length } n\}$
two vertices (strings) are adjacent iff they differ in exactly one position.

Prop. The $n$-cube has $\underline{2^n}$ vertices & $\underline{n \cdot 2^{n-1}}$ edges

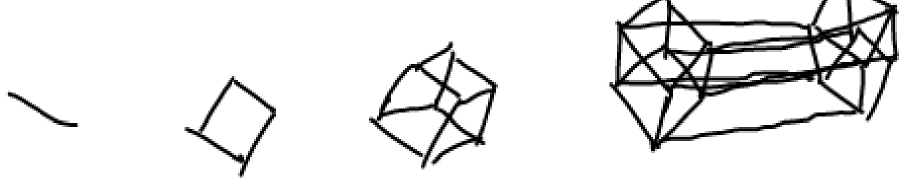


$3 \cdot 2^{3-1} = 12$

Pf: There are $2^n$ vertices because there are $2^n$ binary strings of length $n$. For each string $s$ of length $n$, there are exactly $n$ strings that differ from $s$ in one position, so each vertex of the $n$-cube has degree $n$. By the handshake theorem, $2|E| = \sum_{v \in V} \deg(v) = |V| \cdot n = n \cdot 2^n$

$$\text{So } |E| = n \cdot 2^{n-1}$$

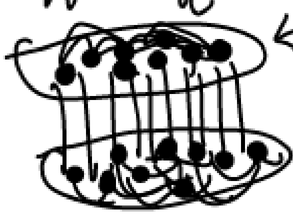
In general, for a $d$-regular graph $G$, we have
$$2|E| = \sum_{v \in V} \deg(v) = 2 \cdot |V|, \text{ so } |E| = \frac{d}{2}|V|.$$



The $n$-cube can be constructed recursively from the $(n-1)$-cube by taking two "copies" of the $(n-1)$-cube & joining pairs of corresponding vertices by an edge.

$(n-1)$-cube

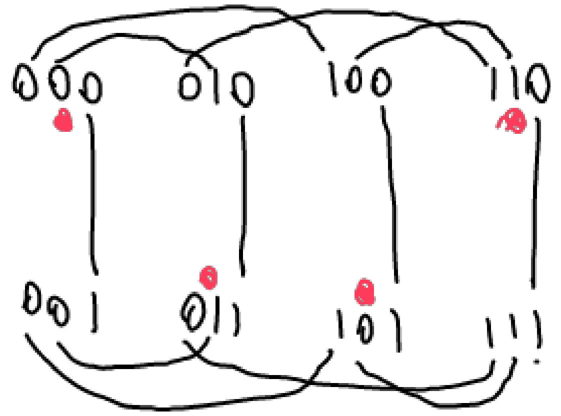
$n$-cube



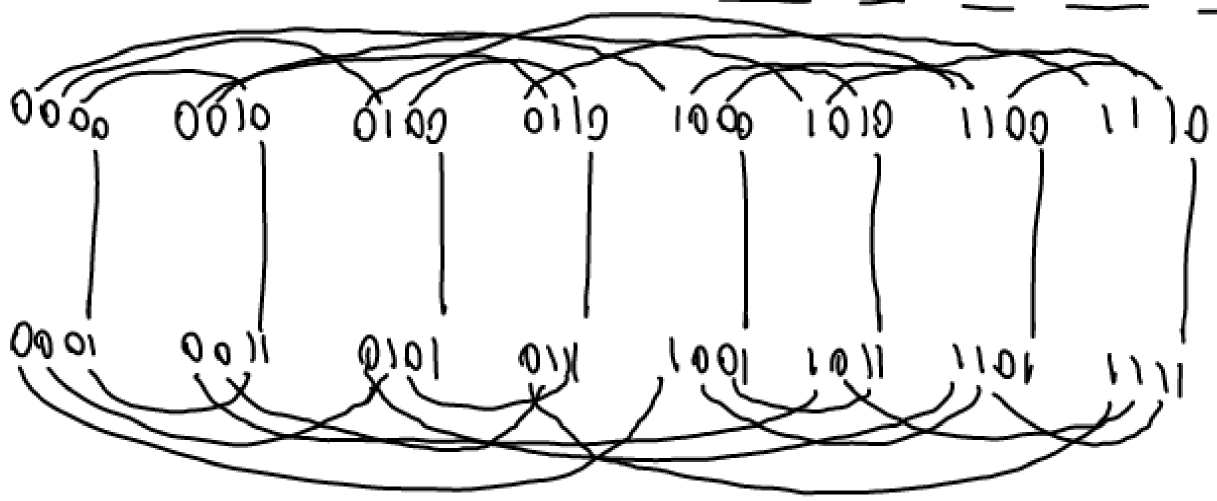
$\{$binary strings of length $n$ ending in $0\}$

$\{$binary strings of length $n$ ending in $1\}$


2-cube




3-cube




4-cube

Q. For which $n$ is the $n$-cube bipartite?

A. All $n$. Given a string with an even number of 1's, every neighbour will have an odd # of 1's. Therefore ({strings w/ an even # of 1's}, {strings w/ an odd number of 1's}) is a bipartition of the $n$-cube for any $n$.

Midterm ends here.

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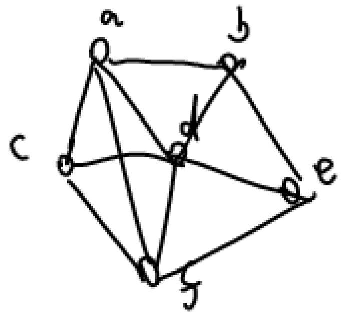
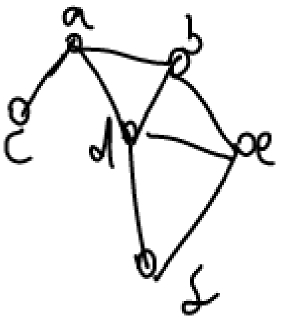
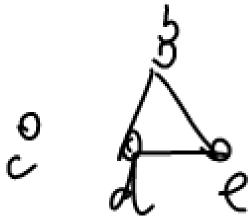
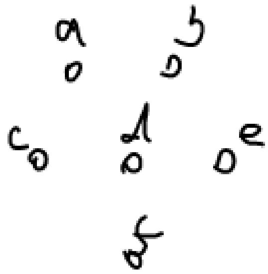
A <u>subgraph</u> of a graph $G = \{V, E\}$ is a graph $G' = \{V', E'\}$ where $V' \subseteq V$ & $E' \subseteq E$.

(essentially, this is a graph obtained by removing any number of edges and/or vertices from $G$).

 $= G$

eg. $H_1 =$  $H_2 =$ 

$H_3 =$  $H_4 =$ 

$H_1, H_2, H_3, H_4$ are subsets of $G$

$H_5 =$ 

$H_5$ is not, even though $G$ does have subgraph isomorphic to $H_5$
i.e. names of vertices matter

A subgraph $G' = (V', E')$ of $G = (V, E)$ is a spanning subgraph if $V' = V$.

a <u>walk</u> of a graph $G$ is an alternating sequence of vertices and eges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$ so that $v_0, v_1, \ldots, v_k \in V$ and each $e_i$ is an edge of $G$ from $v_{i-1}$ to $v_i$. The <u>length</u> of this walk is $k$. (number of edges).