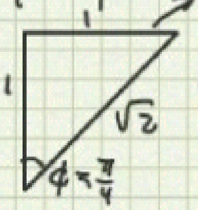
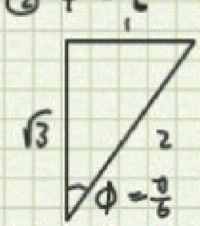
Cones with special angles

① $\phi = \frac{\pi}{4}$



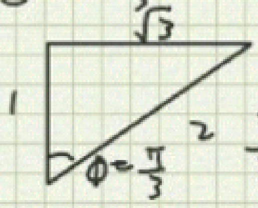
cross-section
$z=1$ $r=1$
$\Rightarrow z = r$
$= \sqrt{x^2+y^2}$

② $\phi = \frac{\pi}{6}$



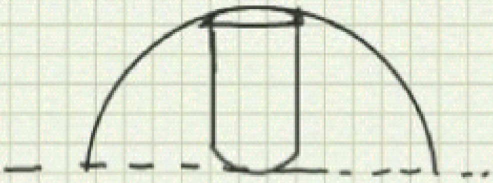
$z = \sqrt{3}$
$r = 1$
$\frac{z}{r} = \sqrt{3}$
$z = \sqrt{3}r$
$\Rightarrow z = \sqrt{3x^2+3y^2}$

③ $\phi = \frac{\pi}{3}$



$z = 1$
$r = \sqrt{3}$
$\frac{z}{r} = \frac{1}{\sqrt{3}}$
$\Rightarrow z = \frac{1}{\sqrt{3}} r$
$z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$

Ex: Let S be the solid that lies below $z = \sqrt{4-x^2-y^2}$, above $z=0$, and inside $x^2+y^2=1$. Find the volume of S in both cylindrical and spherical coordinates.



① cylindrical coords. $V = \iiint_D dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

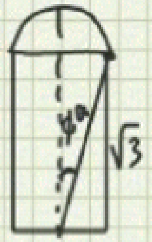
$0 \le z \le \sqrt{4-r^2}$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$V = \frac{2\pi}{3}(8-3\sqrt{3})$

② Spherical Coordinates

$0 \le \theta \le 2\pi$
$0 \le \phi \le \frac{\pi}{2}$ since $z \ge 0$

What about ρ?



$\phi^* = \frac{\pi}{6}$

For $\phi \in [0, \frac{\pi}{6}]$, $\rho \in [0,2]$
$\phi \in [\frac{\pi}{6}, \frac{\pi}{2}]$, $\rho \in [0, \frac{1}{\sin\phi}]$
since $r = \rho\sin\phi$ & $r = 1$

$V = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\sin\phi}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$= 2\pi\left(\frac{8}{3} - \sqrt{3}\right)$

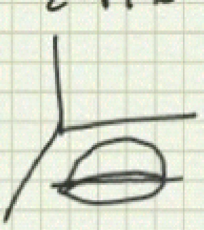
My Method

$z = \sqrt{3}$

$D_{r\theta}$: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$V_1 = \int_0^1\int_0^{2\pi} \sqrt{3}r\,d\theta\,dr$
$= \int_0^1 \sqrt{3}r\,dr \int_0^{2\pi} d\theta$
$= 2\pi\left[\frac{\sqrt{3}}{2}r^2\Big|_0^1\right]$
$= 2\pi\left(\frac{\sqrt{3}}{2}\right)$
$= \sqrt{3}\pi$

$z = \sqrt{4-x^2-y^2} - \sqrt{3}$



$D_{r\theta}$: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$u = 4-r^2$
$du = -2r\,dr$

$V_2 = \int_0^1\int_0^{2\pi}(\sqrt{4-r^2}-\sqrt{3})\,d\theta\,dr$
$= \int_0^1 \sqrt{4-r^2} - \sqrt{3}r \int_0^{2\pi} d\theta$
$= 2\pi\left[-\frac{1}{2}\int_4^3 \sqrt{u}\,du - \left[\frac{\sqrt{3}}{2}r^2\right]_0^1\right]$
$= 2\pi\left[\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_3^4\right] - \frac{\sqrt{3}}{2}\right]$
$= 2\pi\left(\frac{1}{2}\left(\frac{16}{3} - \frac{2}{3}(3)\right) - \frac{\sqrt{3}}{2}\right)$
$= 2\pi\left(\frac{8}{3} - \sqrt{3} - \frac{\sqrt{3}}{2}\right)$
$= 2\pi\left(\frac{8}{3} - \frac{3\sqrt{3}}{2}\right)$

$V_1 + V_2 = \sqrt{3}\pi + 2\pi\left(\frac{8}{3} - \frac{3\sqrt{3}}{2}\right)$
$= 2\pi\left(\frac{8}{3} - \frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$
$= 2\pi\left(\frac{8}{3} - \sqrt{3}\right)$