Mistake at end of last note:

Ex. should have been $P_{3,0}(x)$ for
$f(x) = e^x + \sin x$.

$$e^x + \sin x = \left(1 + x + \frac{x^2}{2} + \cancel{\frac{x^3}{3!}} + O(x^4)\right) + \left(x - \cancel{\frac{x^3}{3!}} + O(x^5)\right)$$

$$= 1 + 2x + \frac{x^2}{2} + O(x^4) + \cancel{O(x^5)} \text{ redundant as } x \to 0$$

why is $O(x^5)$ redundant?

The $O(x^4)$ contains all powers of x from 4 and up.

<u>Products</u>: Find $P_{3,0}(x)$ for $f(x) = e^x\sqrt{1+x}$

Use binomial series on $\sqrt{1+x} = (1+x)^{1/2}$

Recall: $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + O(x^4)$ as $x \to 0$

Then:

$$e^x\sqrt{1+x} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)\right) \cdot \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right)$$

$$= 1\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right) +$$

$$x\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right) +$$

$$\frac{x^2}{2}\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right) +$$

$$\frac{x^3}{6}\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right) +$$

$$O(x^4)\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)\right)$$

$$= 1 + x\left(1+\tfrac{1}{2}\right) + x^2\left(-\tfrac{1}{8}+\tfrac{1}{2}+\tfrac{1}{2}\right) + x^3\left(\tfrac{1}{16}-\tfrac{1}{8}+\tfrac{1}{4}+\tfrac{1}{6}\right) + O(x^4)$$

as $x \to 0$

$$= 1 + \tfrac{3}{2}x + \tfrac{7}{8}x^2 + \left(\tfrac{3-6+12+8}{48}\right)x^3 + O(x^4)$$

note $O(x^5)$ up to $O(x^8)$ are contained here.

$$\Rightarrow P_{3,0}(x) = 1 + \tfrac{3}{2}x + \tfrac{7}{8}x^2 + \tfrac{17}{48}x^3$$

Exercise: Show up to order 5 that

$2\sin x \cos x = \sin 2x$ (include x^5 term)

Compositions

Find $P_{3,0}(x)$ for $f(x) = \ln(1+\sin x)$

Recall: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad |x|<1$

$$\ln(1+\sin x) = \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + \dots$$

Recall $\sin x = O(x)$

$\Rightarrow (\sin x)^4 = O(x^4)$

$$\ln(1+\sin x) = \left(x - \frac{x^3}{3!} + O(x^5)\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + O(x^5)\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + O(x^5)\right)^3 + O(x^4)$$

$$= x - \frac{x^3}{3!} - \frac{1}{2}\left(x^2 + O(x^4)\right) + \frac{1}{3}\left(x^3 + O(x^5)\right) + O(x^4)$$

$$= x - \frac{1}{2}x^2 + \left(\frac{1}{3} - \frac{1}{6}\right)x^3 + O(x^4) \text{ as } x \to 0.$$

$$P_{3,0}(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Limits

Big-O notation is useful in computing limits of indeterminate forms.

l'Hôpital's rule if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x) = \pm\infty = \lim_{x \to a} g(x)$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

e.g. $\lim_{x\to 0}\frac{\sin x}{x}$ ($0/0$ form)

use l'Hôpital $=\lim_{x\to 0}\frac{\cos x}{1}=1.$

about same difficulty in this case

Using series & Big-O:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x-x^3/3!+x^5/5!+O(x^7)}{x}$$

$$=\lim_{x\to 0} 1-x^2/3!+x^4/5!+O(x^6)=1$$

Sometimes, l'Hôpital's rule can get very complicated.

Ex. $\lim_{x\to 0}\frac{x^2\cos(x^2)-\sin^2 x}{x^4}$

$$\cos u = 1-\frac{u^2}{2}+\frac{u^4}{4!}-O(u^6)$$

$$\cos(x^2)=1-\frac{x^4}{2}+O(x^8)$$

$$\sin u = u-\frac{u^3}{3!}+O(u^5)$$

$$=\lim_{x\to 0}\frac{x^2\left(1-x^4/2+O(x^8)\right)-\left(x-\frac{x^3}{3!}+O(x^5)\right)^2}{x^4}$$

$$=\lim_{x\to 0}\frac{x^2-x^6/2+O(x^{10})-\left(x^2-\frac{2x^4}{3!}+O(x^6)\right)}{x^4}$$

$$=\lim_{x\to 0}\frac{x^4/3+O(x^6)}{x^4}$$

$$=\lim_{x\to 0}\frac{1}{3}+O(x^2)=\frac{1}{3}$$