1. Rational Roots Thm
2. Conjugate Roots Thm

$f(x) \in \mathbb{C}[x]$ of deg. n.

$f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$

Rational roots theorem (RRT)

Proposition: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0 \in \mathbb{Z}$, $a_n \neq 0$. If $\frac{p}{q}$ is a root of $f(x)$, where $\gcd(p,q)=1$, then $p \mid a_0$ and $q \mid a_n$.

Example: $f(x) = 3x^3 + x^2 + 6x + 2$. If $\frac{p}{q}$ is a rational root, then $p \mid 2$, $q \mid 3$. Possibilities for $\frac{p}{q}$ are $\pm 1/3$, $\pm 2/3$, ± 1, ± 2.

X	$1/3$	$-1/3$	$2/3$	- - - - - -
$f(x)$	~	0	~	~

$(x+\frac{1}{3})$ is a factor.

$-\frac{1}{3}$ is the only rational root of $f(x)$.

$f(x) = (x+\frac{1}{3})(3x^2+6) = 3(x+\frac{1}{3})(x^2+2) = 3(x+\frac{1}{3})(x-\sqrt{2}i)(x+\sqrt{2}i)$

Proof: $f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right)^1 + a_0 = 0$

Multiply q^n on both sides to get:

① $a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$

Rearranging ① gives $a_n p^n = q(\underbrace{\quad\quad}_{\text{some int}})$ so $q \mid a_n p^n$

Since $\gcd(p,q)=1$, by repeatedly applying CAD, $q \mid a_n$.

Rearranging ① gives $a_0 q^n = p(\underbrace{\sim\sim}_{\text{some int}})$. So $p \mid a_0 q^n$.

By CAD again, $p \mid a_0$.

Example: Prove $\sqrt{2}$ is irrational. Consider x^2-2. $\sqrt{2}$ is a root. If $\sqrt{2}=\frac{p}{q}$ where $p, q>0$, then by RRT, $p \mid (-2)$, $q \mid 1$.

So $q=1$. $p=1$ or 2. $\therefore \sqrt{2}=\frac{1}{1}$ or $\frac{2}{1}$. Contradiction.

Conjugate Roots Theorem

Proposition: Let $f(x) \in \mathbb{R}[x]$. If $c \in \mathbb{C}$ where $f(c)=0$, then $f(\bar{c})=0$.

Proof: Let $f(x)=a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, $a_i \in \mathbb{R}$.

Suppose $f(c)=0$, then $f(c)=0=a_n c^n + a_{n-1}c^{n-1} + \dots + a_0$.

Take conj of both sides to get

$$0 = \overline{a_n c^n + a_{n-1}c^{n-1} + \dots + a_0}$$

$$0 = \overline{a_n c^n} + \overline{a_{n-1}c^{n-1}} + \dots + \overline{a_0}$$

$$0 = a_n \bar{c}^n + a_{n-1}\bar{c}^{n-1} + \dots + a_0 \quad \text{since } a_i \in \mathbb{R}$$

$$= f(\bar{c})$$

Example: $f(x) = x^3 - 5x^2 + 4x + 10$ where $f(3-i) = 0$

Then $3+i$ is also a root.

$$f(x) = \underbrace{(x-(3-i))(x-(3+i))}(x+1)$$

$$(x-(3-i))(x-(3+i)) = x^2 - [(3-i)+(3+i)]x + (3-i)(3+i)$$
$$= x^2 - 6x + 10$$

Let $f(x) \in \mathbb{R}[x]$

$f(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ $\quad r_i \in \mathbb{C}$;

$$= a\underbrace{(x-r_1)\cdots(x-r_k)}_{\text{real roots}}\ \underbrace{(x-c_1)(x-\overline{c_1})(x-c_2)(x-\overline{c_2})\cdots}_{\text{Imag. roots (comes in pairs)}}$$

$$\boxed{\begin{aligned}&(x-c_1)(x-\overline{c_1}) \Rightarrow x^2 - (c_1+\overline{c_1})x + c_1\overline{c_1}\\ &= x^2 - 2\,\mathrm{Re}(c)x + |c_1|^2\end{aligned}}$$ This is in $\mathbb{R}[x]$

Prop: Any $f(x) \in \mathbb{R}[x]$ can be factored into linear and quadratic form in $\mathbb{R}[x]$.