The Alternating Series Test

Defⁿ: An alternating series is a series whose terms alternate in sign.

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

The Alternating Series Test:

If the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - \cdots\cdots$, $a_k > 0$

satisfies (i) $\lim_{k\to\infty} a_k = 0$

(ii) $a_{k+1} \leq a_k$ for all $k \geq k_0$ (ultimately decreasing)

then the series is convergent.

Ex.

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ ← alternating harmonic series

$a_k = \frac{1}{k}$

(i) $\lim_{k\to\infty} \frac{1}{k} = 0$ ✓

(ii) $a_{k+1} = \frac{1}{k+1} \leq \frac{1}{k} = a_k$ ✓

The series is convergent by the AST.

Ex. $\sum_{k=1}^{\infty} (-1)^k \frac{k^2-1}{2k^2+1}$ $\quad a_k = \frac{k^2-1}{2k^2+1} = \frac{1-1/k^2}{2+1/k^2} \to \frac{1}{2}$ as $k \to \infty$

Let $b_k = (-1)^k a_k$

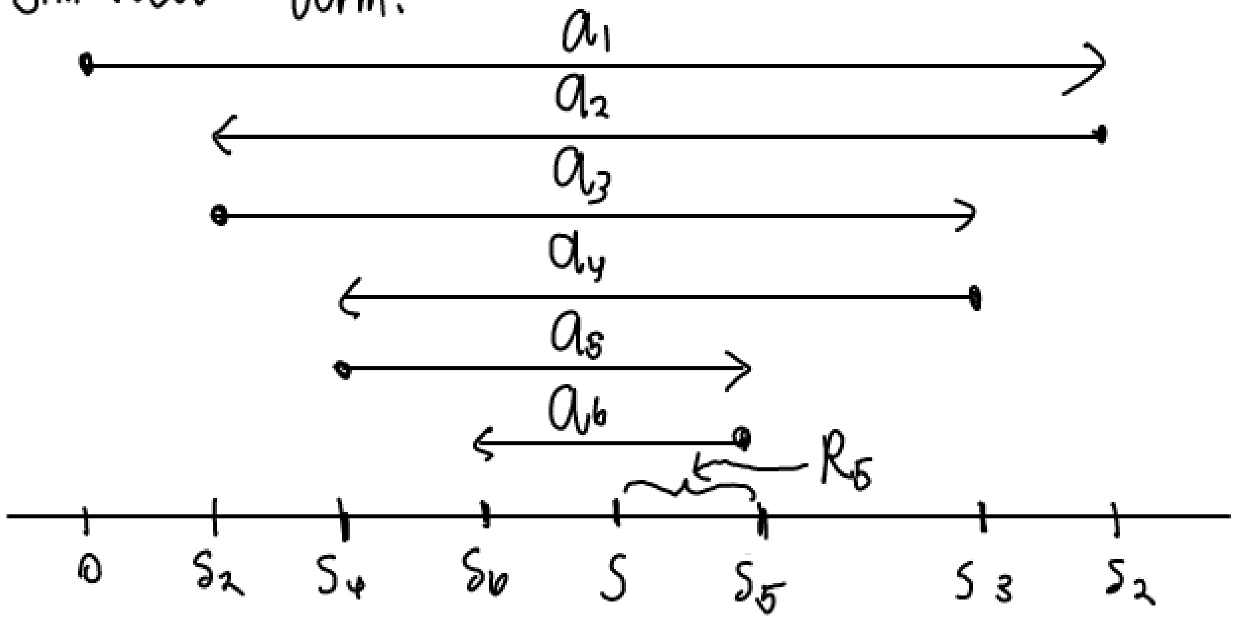
As $K \to \infty$, b_K has no limit. This condition is just checking the Test for Divergence.

Alternating Series Estimation Theorem

Consider a convergent alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$. If we use the n^{th} partial sum S_n to estimate the true sum S, the error satisfies

$$|R_n| = |S - S_n| \leq a_{n+1}$$

The error is less than the first ommited term.



e.g. Use S_5 to estimate S: $|R_5| = |S - S_5| \leq a_6$ by the picture.

- The odd sums overestimate & even sum underestimate. (Vice Versa if nagative term first)

Ex. Estimate the sum of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ to with 10^{-4}.

Sol^n $a_k = \frac{1}{k^4}$, note $\frac{1}{k^4} \to 0$ as $k \to \infty$ &

a_k is a decreasing sequence.

$\Rightarrow$ converge by AST.

By the ASET

$$|R_n| = |S - S_n| < a_{n+1}$$

Need to find n such that $\frac{1}{(n+1)^4} < 10^{-4}$

$$\Rightarrow 10^4 < (n+1)^4$$

$$\Rightarrow n > 9$$

If we take 10 terms, our estimate will be within 10^{-4}.

$$S_{10} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \cdots + \frac{1}{10^4}$$

$$\approx 0.946992$$

Computational value: ≈ 0.947032

Additional Notes:

$P_{2,0}(t)$ for $f(t)=t^2$

$P_{2,0}(t) = t^2$

$$t^2 e^{-t^2} = t^2\left(1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\dots\right) = \underbrace{t^2-t^4+\frac{t^6}{2!}-\dots}_{t^2\cdot P_{n,0}(-t^2)}$$

$$e^x : P_{n,0}(x) = 1+x+\dots+\frac{x^n}{n!}$$

$$\left|e^x - P_{n,0}(x)\right| = \left|R_n(x)\right| \leq \dots$$

$$\text{Error} = \left|\int_0^1 t^2 e^{-t^2}\,dt - \int_0^1 t^2\cdot P_{n,0}(-t^2)\,dt\right|$$

$$\leq \int_0^1 \left|t^2 e^{-t^2} - t^2 P_{n,0}(-t^2)\right| dt$$

$$\leq \int_0^1 t^2 \underbrace{\left|e^{-t^2} - P_{n,0}(-t^2)\right|}_{R_n(-t^2)} dt$$