Last time: We defined the n^{th} degree Taylor polynomial of f at x_0:

$$P_{n,x_0} = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$= \sum_{k=0}^{n} \frac{F^{(k)}(x_0)}{k!}(x-x_0)^k$$

Note: $P_{1,x_0}(x) = L(x)$, the linear approximation.

Ex.

Find $P_{2,1}(x)$ for $f(x) = \ln x$ and use it to estimate $\ln(1.2)$. Compare with the linear approximation.

Sol^n $P_{2,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$

$f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$\Rightarrow P_{2,1}(x) = (x-1) - \frac{1}{2}(x-1)^2$

Then our estimate is $\ln(1.2) \approx P_{2,1}(1.2)$

$= (1.2-1) - \frac{1}{2}(1.2-1)^2$

$= \frac{1}{5} - \frac{1}{50}$

$= \frac{9}{50}$

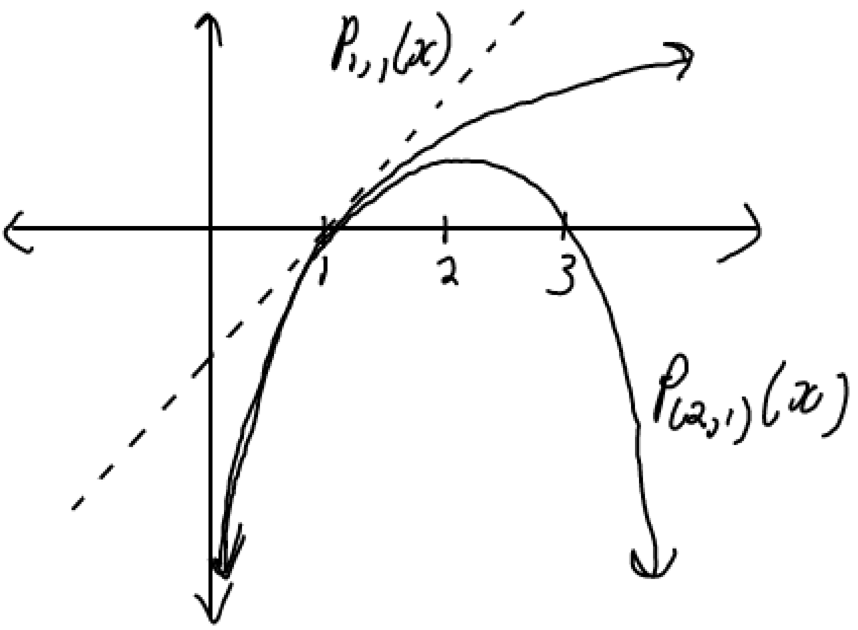
$= 0.18$

calculator value: $\ln 1.2 = 0.1823$

$L(x) = P_{1,1}(x) = x-1$

$P_{1,1}(1.2) = \frac{1}{5} = 0.2$

$\Rightarrow P_{2,1}$ gives a better estimate. (not always the case)



Turns out we get a better approx. with higher degree polynomial for $x \in (0, 2]$

without a discussion of the error, this is still not very useful.

Before the error discussion, let's compute some taylor polynomials for familiar functions.

<u>Ex.</u> ① $f(x) = e^x$ about $x = 0$

Find $P_{n,0}(x)$

$$P_{n,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(0) = 1$, $f'(x) = e^x = f''(x) = f^{(n)}(x)$ for all n.

$\Rightarrow f'(0) = 1 = f^{(n)}(0)$, for all n.

$$\Rightarrow P_{n,0}(x) = 1 + x + \frac{x^3}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\text{or} \quad = \sum_{k=0}^{n} \frac{x^k}{k!} \quad (\text{where } 0! = 1)$$

② Find $P_{2n+1,0}(x)$ for $f(x) = \sin x$.

$$\left.\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \\ f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \end{array}\right\} \Rightarrow f^{(2n)}(0) = 0 \ \forall n$$

repeats in a cycle of 4

$f^{(2n+1)}(0) = (-1)^n$ for all n.

$$P_{2n+1,0}(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

or $$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

③ Find $P_{2n,0}(x)$ for $f(x) = \cos x$

$f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{(4)}(x) = \cos x$ $\quad f^{(4)}(0) = 1$

$f^{(2n+1)}(0)$ $\forall n$

$f^{2n}(0) = (-1)^n$

$$\therefore P_{2n,0}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$$

or $$= \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{k!}$$

④ Find $P_{n,0}(x)$ for $f(x) = \frac{1}{1-x}$

$f(x) = (1-x)^{-1}$ $\quad f(0) = 1$

$f'(x) = (-1)(1-x)^{-2}(-1) = (1-x)^{-2}$ $\quad f'(0) = 1$

$f''(x) = 2(1-x)^{-3}$ $\quad f''(0) = 2$

$f'''(x) = 6(1-x)^{-4}$ $\quad f'''(0) = 6$

$f^{(4)}(x) = 24(1-x)^{-5}$ $\quad f^{(4)}(0) = 24$

In general, $f^{(k)}(0) = k!$

$$P_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \dots + x^n$$

Note: We can get $P_{n,0}(x)$ for $f(x) = \frac{1}{1+x}$ easily.

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

Then $P_{n,0}(x) = \sum_{k=0}^{n} (-x)^k = \sum_{k=0}^{n} (-1)^k x^k$

$$= 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^n x^n$$

We can also integrate and differentiate to obtain Taylor polynomials for new functions.

Since $\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$, then

$$\int \frac{1}{1+x} dx = \ln(1+x)$$