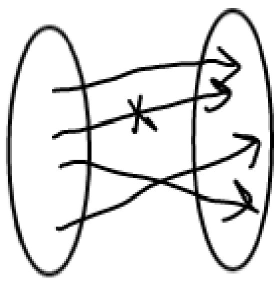
January 17 2014
1. Functions
2. Cardinality

$f: A \to B \quad f(x) = 2x+1$

① f is onto if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

② f is $1-1$ if $f(x) = f(y) \Rightarrow x = y$.
(each $x \in A$ is mapped to a different element in B)



suppose $f(x) = f(y)$, then $2x+1 = 2y+1$. So $2x = 2y$, and $x = y$. f is $1-1$.

③ f is a bijection ($1-1$ correspondence) if f is $1-1$ and onto.

Example: $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2 - 1$ $\quad$ $y = -100$. No x is mapped to y since $x^2 - 1 = -100 \Rightarrow x \notin \mathbb{R}$. So f is not onto.

$f(2) = 3 \quad f(-2) = 3$ but $2 \neq -2$, so f is not $1-1$.

Example: $f: [0, \infty) \to [5, \infty)$ where $f(x) = 5e^{2x}$.
$f: A \to B$
A function is well defined if for all $x \in A$, $f(x) \in B$

① Need $5e^{2x} \geq 5$ since $x \geq 0$, $2x \geq 0$, so $e^{2x} \geq 1$.
therefore, $f(x) = 5e^{2x} \geq 5$, and f is well defined.

② (1-1) suppose $f(x) = f(y)$ for some $x, y \in [0, \infty)$
Then $5e^{2x} = 5e^{2y}$. So $e^{2x} = e^{2y}$. Take $\ln$ on both sides to get $2x = 2y$. So $x = y$ and f is 1 to 1.

③ Let $y \in [5, \infty)$. $\left[5e^{2x} = y,\ e^{2x} = \frac{5}{y},\ 2x = \ln\left(\frac{5}{y}\right),\ x = \frac{1}{2}\ln\left(\frac{5}{y}\right) \right]$
Let $x = \frac{1}{2}\ln\left(\frac{5}{y}\right)$. Since $y \geq 5$, $\ln \frac{y}{5} \geq 0$, so $x \geq 0$.
So $x \in [0, \infty)$ Then
$f(x) = f\left(\frac{1}{2}\ln\frac{y}{5}\right) = 5e^{2\left(\frac{1}{2}\ln\frac{y}{5}\right)} = 5\left(\frac{y}{5}\right) = y$
So f is onto.

Example: Let $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$
$\{0,1\}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{0,1\}\}$

Define: $f: \mathcal{P}(\mathbb{N}_n) \to \{0,1\}^n$ where for each $X \in \mathcal{P}(\mathbb{N}_n)$,
$f(x) = (a_1, \ldots, a_n)$ where $a_i = \begin{cases} 1 & \text{if } i \in x \\ 0 & \text{if } i \notin x \end{cases}$

$n=2$: $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$\{0,1\}^2 = \{(0,0), (1,0), (0,1), (1,1)\}$

f is onto: Let $(a_1, \ldots, a_n) \in \{0,1\}^n$.
Let $X = \{i \in \mathbb{N}_n \mid a_i = 1\}$
Then $f(x) = (a_1, \ldots, a_n)$

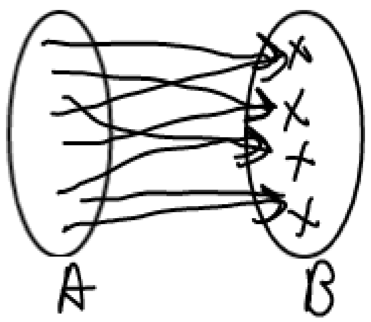
f is 1-1. Suppose $X, Y \in \mathbb{N}_n$ such that $f(X)=f(Y)$. (need to prove $X=Y$) Let $x \in X$, then in $f(X)$, the x^{th} entry is 1. Since $f(x)=f(y)$, the x^{th} entry of $f(Y)$ is 1. Then $x \in Y$, and $X \subseteq Y$. By swapping the roles of X and Y, we can prove that $Y \in X$ so $X=Y$.

Example: $f(x)=\cos x$ $\quad f(x)=f(y)$, $\cos x = \cos y \not\Rightarrow x=y$

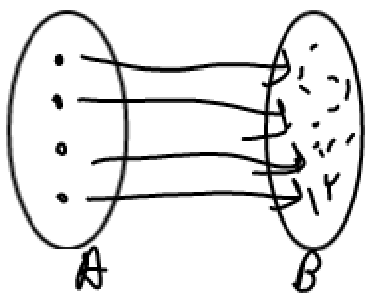
Cardinality

$f: A \to B$ $\quad A, B$ are finite

If f is onto, then $|A| \geq |B|$



If f is 1-1, then $|A| \leq |B|$



If f is a bijection, then

$|A| = |B|$

Example: $f: \mathcal{P}(\mathbb{N}_n) \to \{0,1\}^n$ is a bijection.

$$|\mathcal{P}(\mathbb{N}_n)| = |\{0,1\}^n|$$
$$= 2^n$$

Define $f: \mathbb{N} \to E$ by $f(x)=2x$. Then f is a bijection.

So $|\mathbb{N}| = |E|$?

$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$f(a,b) = 3a + 2b$

$\cancel{f(1,0) = 3}$

$\cancel{f(0,1) = 2}$

hyp: There exists a, b such that $\forall c \in \mathbb{N}$ such that $f(a,b) = c$,

base: $f(1,-1) = 1$

$\cancel{3}$

prov $c+1 = f(a,b)$

$$\left.\begin{aligned} \text{Step: } c &= f(a,b) = 3a + 2b \\ 1 &= f(1,-1) = 3(1) + 2(-1) \end{aligned}\right\}$$