Recursive Decompositions

S - all strings

$= \{\varepsilon\} \cup S\{0,1\}$

$$\Phi_S(x) = 1 + 2x\,\Phi_S(x)$$
$$\Phi_S(x) - 2x\,\Phi_S(x) = 1$$
$$\therefore \Phi_S(x) = \frac{1}{1-2x} = \sum_{k\geq 0} 2^k x^k$$

S - strings without 111

$= \{\varepsilon, 1, 11\} \cup S\{0\} \cup S\{01\} \cup S\{011\}$

$= \{\varepsilon\ 1, 11\} \cup S\{0, 01, 011\}$

$$\Phi_S(x) = (1+x+x^2) + \Phi_S(x)(x+x^2+x^3)$$
$$\Phi_S(x) = \frac{1+x+x^2}{1-(x+x^2+x^3)}$$

Forbidden Substrings

Problem: How many strings are there with no $11101 = L$

$M = \{$strings containing 11101 once at the time$\}$

$L \cup M = \{\varepsilon\} \cup L\{0,1\}$ & $L\{11101\} = M \cup M\{1101\}$

$$\Phi_L(x) + \Phi_M(x) = 1 + 2x\,\Phi_L(x) \qquad x^5\,\Phi_L(x) = \Phi_M(x) + x^4\,\Phi_M(x)$$

$$\Phi_L(x) + \frac{x^5}{1+x^4}\Phi_L(x) - 2x\,\Phi_L(x) = 1 \xleftarrow{\text{sub}} \Phi_M(x) = \frac{x^5}{1+x^4}\Phi_L(x)$$

$$\therefore \Phi_L(x) = \frac{1}{1 + \frac{x^5}{1+x^4} - 2x} = \frac{1-x^4}{1-2x-x^4+3x^5}$$

Partial Fraction Expansion

$$[x^n]\ \frac{1+3x}{(1-x)(1+x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-2x}$$

$$= \frac{A(1+x)(1-2x) + B(1-x)(1-2x) + C(1+x)(1-x)}{(1-x)(1+x)(1-2x)}$$

$$\begin{cases} A+B+C = 1 \\ -A-C = 3 \\ -2A+2B+C = 0 \end{cases} \Rightarrow f(x) = -\frac{2}{1-x} - \frac{1}{3}\left(\frac{1}{1+x}\right) + \frac{10}{3}\left(\frac{1}{1-2x}\right)$$

$$= -2\sum_{k\geq 0} x^k - \frac{1}{3}\sum_{k\geq 0}(-x)^k + \frac{10}{3}\sum_{k\geq 0}(2x)^k$$

$$\therefore [x^n]\, f(x) = -2 - \frac{1}{3}(-1)^n + \frac{10}{3}(2^n)$$

$$= \sum_{k\geq 0}\left(-2x^k - \frac{1}{3}(-x)^k + \frac{10}{3}(2x)^k\right)$$

$$f(x) = \sum_{k\geq 0}\left(-2 - \frac{1}{3}(-1)^k + \frac{10}{3}2^k\right)x^k$$

What if we have $\frac{g(x)}{(1-x)^m}$? Use negative binomial theorem

$$\frac{g(x)}{(1-x)^m} = g(x)\sum_{n\geq 0}\binom{n+m-1}{m-1}x^n$$

<u>Proposition</u>: Let f, g be polynomials with degree(g) < degree(f) and f has a constant term 1, then

$$\frac{g(x)}{f(x)} = \frac{h_1(x)}{(1-\theta_1(x))^{m_1}} + \frac{h_2(x)}{(1-\theta_2(x))^{m_2}} + \dots + \frac{h_\ell(x)}{(1-\theta_\ell(x))^{m_\ell}}$$

where $\deg(h_i) < m_i \ \forall i \in \{1, \dots, \ell\}$

Solving Recurrence Relations

$$\begin{cases} a_0 = 1 \\ a_1 = 4 \\ a_n = 3a_{n-1} + 2a_{n-2} \quad \forall n \geq 2 \end{cases}$$

$$a_2 x^2 - 3a_1 x^2 - 2a_0 x^2 = 0$$

$$a_3 x^3 - 3a_2 x^3 - 2a_1 x^3 = 0$$

$$a_n - 3a_{n-1} x^n - 2 a_m x^n = -1$$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$$

$$\sum_{n\geq 2} a_n x^n - 3\sum_{n\geq 2} a_{n-1} - 2\sum_{n\geq 2} a_{n-2} x^n = 0$$

$$\sum_{n\geq 2} a_n x^n - 3x\sum_{n\geq 1} a_n x^n - 2x^2 \sum_{n\geq 0} a_n x^n = 0$$