We showed the mapping $(u,v) = F(x,y) = (x+y, x-y)$ mapped the square S $(0 \le x \le 1, 0 \le y \le 1)$ to a tilted square $F(S)$.
refer to last note

Note: area of $S = 1$
area of $F(S) = 2$

This change in area is an important concept in the context of change of variables in double integrals. We need the Jacobian.

Defn: The Jacobian of a mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ $(u,v) = F(x,y) = (f(x,y), g(x,y))$ is denoted $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined by

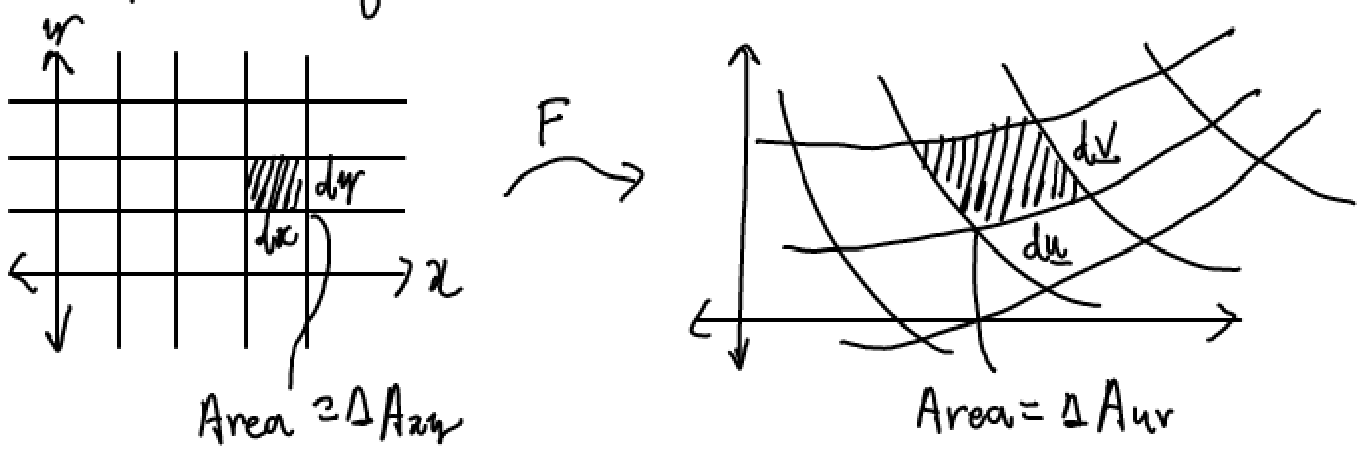
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = u_x v_y - u_y v_x$$

e.g. Previous Ex.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

Notice $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ is the area change from xy to uv coordinates.

Interpretation of Jacobian



$$\Delta A_{uv} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| \Delta A_{xy}$$

Represents the scaling factor of F for a small area element.

Recall substitution method for integrals.

$$\int x e^{x^2} dx \quad \begin{array}{l} u = x^2 \\ du = 2x\,dx \Rightarrow du = \frac{du}{dx} \cdot dx \end{array}$$

For double integrals, our differential elements are related by $du\,dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\,dy$

Change of variable Theorem

Let $(x,y) = F(u,v) = (f(u,v), g(u,v))$ be a one-to-one mapping of D_{uv} to D_{xy} (where $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$).

Then

$$\iint_{D_{xy}} H(x,y)\,dx\,dy = \iint_{D_{uv}} H(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Note: $\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)}$

Polar Coordinates

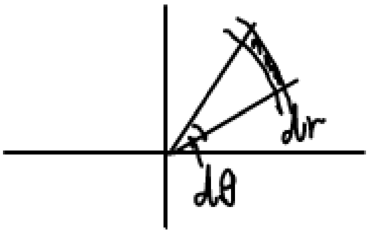
Cartesian $\to$ Polar is a mapping $\mathbb{R}^2 \to \mathbb{R}^2$.

$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad \text{Then} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} X_r & X_\theta \\ Y_r & Y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Exercise: Show $\frac{\partial(r,\theta)}{\partial(x,y)} = 1/r$ $\quad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$

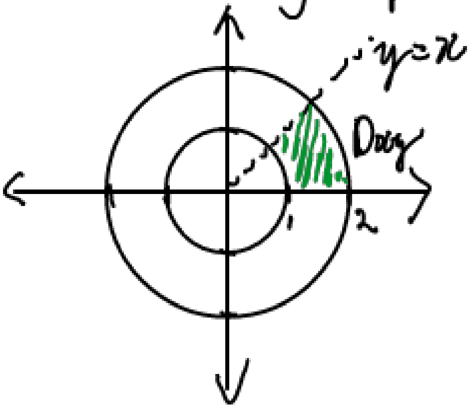


Area element change depends on distance to origin r.

Example: Find $\iint_{D_{xy}} e^{\sqrt{x^2+y^2}}\, dA$, where

$$D_{xy} = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0,\ y \le x\}$$

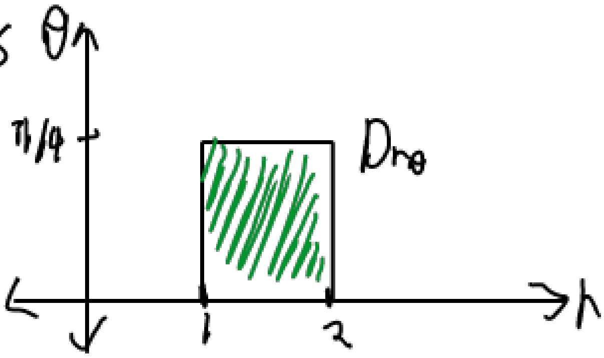
Note: both integrand & region can be simplified using polar coordinates.



We can express the domain in polar coords as

$1 \le r \le 2$

$0 \le \theta \le \pi/4$



$$\iint_{D_{xy}} e^{\sqrt{x^2+y^2}}\, dx\,dy = \iint_{D_{r\theta}} \underbrace{e^r}_{\text{domain}} \cdot \underbrace{r}_{\text{Jacobian}}\, dr\, d\theta = \int_0^{\pi/4}\int_1^2 re^r\, dr\, d\theta$$

$$= \underbrace{\left[\int_0^{\pi/4} d\theta\right]}_{\pi/4} \underbrace{\left[\int_1^2 re^r\, dr\right]}_{e^2} \quad \text{IBP } re^r - e^r$$

$$= \pi e^2/4$$

Useful result: If $f(x,y) = g(x) \cdot h(y)$,

then $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d h(y)\,dy \int_a^b g(x)\,dx$$