1. Nested Quantifiers
2. Functions

For all odd n, $4 \mid (n^2-1)$.

Proof: Let n be an odd integer. Then $n=2k+1$ for some $k \in \mathbb{Z}$.

So $n^2-1 = (2k+1)^2-1 = 4k^2+4k+1-1 = 4(k^2+k)$

Since $k^2+k \in \mathbb{Z}$, $4 \mid (n^2-1)$

$k^2+k = k(k+1)$ These are consecutive int, so $k(k+1)=2l$ for some $l \in \mathbb{Z}$. Then $n^2-1 = 4 \cdot 2l = 8l$. So $8 \mid (n^2-1)$.

Nested Quantifiers

Example: $\underbrace{\text{For all } x \in \mathbb{R}}_{\text{applies to any } x}$, $\underbrace{\text{there exists } y \in \mathbb{R}}_{\text{for each } x\text{, there is some } y}$ such that $y < x$.

True, Let $x \in \mathbb{R}$, Let $y = x-1$, then $y = x-1 < x$

Example: $\underbrace{\text{There exists } y \in \mathbb{R}}_{\text{find a } y}$ such that $\underbrace{\text{for all } x \in \mathbb{R}}_{\text{applies to all } x}$, $y < x$.

False. whichever y we pick, $x = y-1$ does not work.

Example: There exists $y \in \mathbb{N}$ such that for all $x \in \mathbb{N}$, $y \le x$.

True, let $y=1$. For any $x \in \mathbb{N}$, $x \ge 1 = y$.

Example: There exists $x \in \mathbb{Z}$ such that for all $y \in \mathbb{Z}$, $x \mid y$.

True, Let $x = 1$. Let $y \in \mathbb{Z}$. Then $y = 1 \cdot y$.
so $x \mid y$,

Examples: For all $X \in \mathcal{P}(\mathbb{N})$, there exists $Y \in \mathcal{P}(\mathbb{N})$ such that $X \not\subseteq Y$ and $Y \not\subseteq X$

$[\, X = \{3, 5, 19\} \quad Y = \{21\} \,]$

If $X = \mathbb{N}$, every $Y \in \mathcal{P}(\mathbb{N})$ satisfies $Y \subseteq X$

If $X = \emptyset$, " " ---- -- $X \subseteq Y$

False.

Excercise: For all $X \in \mathcal{P}(\mathbb{N})$, if $X \notin \{\emptyset, \mathbb{N}\}$, then there exists $Y \in \mathcal{P}(\mathbb{N})$ such that $X \not\subseteq Y$ and $Y \not\subseteq X$

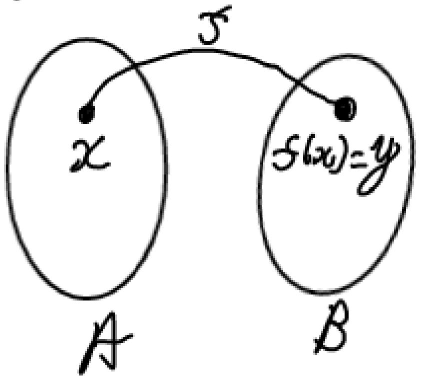
<u>Functions:</u>

$f : A \to B$

A = domain

B = codomain

definition of a function:
For all $x \in A$, there is a unique $y \in B$ such that $f(x) = y$.



Range $= \{ f(x) \mid x \in A \}$ all possible outputs

Examples: $f : \mathbb{R} \to \mathbb{R}$ — codomain

$f(x) = 2x + 1$ ①

$f(x) = x^2 - 1$ ②

$f(x) = e^x$ ③

Range: ① $\mathbb{R}$ ② $[-1, \infty)$ ③ $(0, \infty)$

range $\subseteq$ codomain

Definition: For a function $f: A \to B$,

Ⓐ It is <u>onto</u> if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$. (everything in B is mapped)

① is onto, but ② & ③ are not.

Proof that ① is onto:

(Need: for all $y \in \mathbb{R}$, there is $x \in \mathbb{R}$, such that $f(x) = y$.
$2x + 1 = y$, $x = \frac{y-1}{2}$)

Let $y \in \mathbb{R}$, Consider $x = \frac{y-1}{2}$.

Then $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1$

$= y - 1 + 1$

$= y$

So f is onto.