Spherical Coordinates

$x = \rho \sin\phi \cos\theta$ $\qquad \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

Ex.

Let $D = \{(x,y,z) \mid \underbrace{1 \le x^2+y^2+z^2 \le 4}_{\rho}, \underbrace{x \ge 0, y \ge 0}_{\theta}, \underbrace{z \ge 0}_{\phi}\}$

Evaluate $\iiint_D z\,dV$.

Soln: Part of a spherical shell of inner radius 1 and outer radius 2.

$$\Rightarrow \begin{aligned} &1 \le \rho \le 2 \\ &0 \le \phi \le \pi/2 \\ &0 \le \theta \le \pi/2 \end{aligned}$$

Then
$$\iiint_D z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \overbrace{(\rho\cos\phi)}^{z}\,\overbrace{\rho^2\sin\phi}^{\text{Jacobian}}\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\phi\sin\phi\,d\phi \int_1^2 \rho^3\,d\rho$$

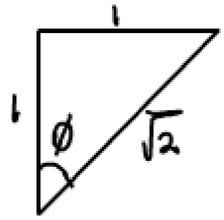
$= \dfrac{15\pi}{6}$ where the ϕ integral can be done using the double angle identity as substitution.

Cones/cylinders/spheres in each coordinate system

	Cartesian	Cylindrical	Spherical
Cone	$z = \sqrt{x^2+y^2}$	$z = r$	$\phi = \pi/4$
Cylinder (radius a)	$x^2+y^2 = a^2$	$r = a$	$\rho = a/\sin\phi$
Sphere (radius b)	$x^2+y^2+z^2 = b^2$	$r^2+z^2 = b^2$	$\rho = b$

Cones with the special angles $\phi = \frac{\pi}{4}$, $\phi = \frac{\pi}{3}$, $\phi = \frac{\pi}{6}$

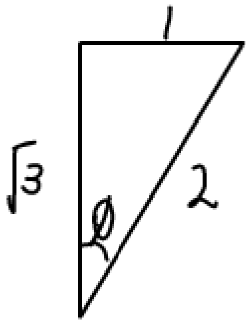
① $\phi = \frac{\pi}{4}$ Cross-section



$z = 1,\ r = 1$

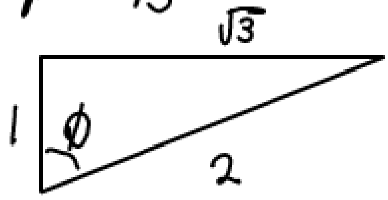
$\Rightarrow z = r \Rightarrow z = \sqrt{x^2 + y^2}$

② $\phi = \pi/6$



$\frac{z}{r} = \sqrt{3} \Rightarrow z = \sqrt{3}r \Rightarrow z = \sqrt{3x^2 + 3y^2}$

③ $\phi = \pi/3$



$\frac{z}{r} = \frac{1}{\sqrt{3}} \Rightarrow z = \frac{1}{\sqrt{3}}r \Rightarrow z = \frac{1}{\sqrt{3}}\sqrt{x^2 + y^2}$

Ex. Let S be the solid that lies below $z=\sqrt{4-x^2-y^2}$, above $z=0$ and inside $x^2+y^2=1$.

Set up the integral to find the volume of S in both cylindrical and spherical coordinates.