# Multivariable Calculus

Many quantities of interest depend on more than a single variable.

e.g. Volume of a circular cylinder

$$V(r,h) = \pi r^2 h.$$

e.g. displacement of a wave (guitar string)

$$u(x,t)$$

e.g. wind chill - function of temperature & wind speed.

e.g. Electric Field

$$\vec{E} = \vec{E}(x,y,z,t)$$

**Def$^n$**: A scalar function $f: \mathbb{R}^n \to \mathbb{R}$ is a function whose domain is a subset of $\mathbb{R}^n$ and whose range is a subset of $\mathbb{R}$.

We'll start out with functions $f: \mathbb{R}^2 \to \mathbb{R}$ which maps a point $(x,y)$ in $\mathbb{R}^2$ to a point $f(x,y)$ in $\mathbb{R}$.

The domain $D(f)$ is a subset of $\mathbb{R}^2$ and the range $R(f)$ is a subset of $\mathbb{R}$.

Ex. Find the domain & range.

1) $f(x,y) = \sqrt{1-x^2-y^2}$

Restriction: $1-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 1$

The region lying on or inside the unit circle.

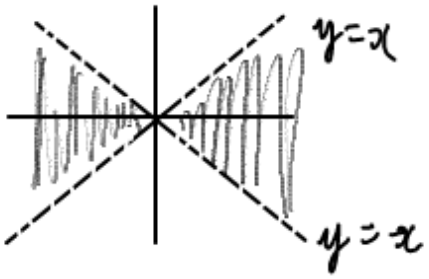
$D(f): \{(x,y) \mid x^2+y^2 \leq 1\}$

Range: $[0,1]$

on $x^2+y^2=1$ ↑ ↑ $(x,y)=(0,0)$

2) $f(x,y)=\ln(x^2-y^2)$

Restriction: $x^2-y^2>0 \Rightarrow x^2>y^2 \Rightarrow |x|>|y|$




Range: $\mathbb{R}$

Def$^n$: The graph of $f:\mathbb{R}^2\to\mathbb{R}$ is the set of points $(a,b,f(a,b))$ in $\mathbb{R}^3$ such that $(a,b)\in D(f)$.

→ Surface in $\mathbb{R}^3$ where $z=f(x,y)$ is the height above $(x,y)$.

→ Difficult to draw in general - use some 2D visualization to help.

Def$^n$: For a function $f:\mathbb{R}^2\to\mathbb{R}$. The level curves of $f$ are the curves $f(x,y)=k$,
where $k$ is a constant in the range of $f$.
($k$ takes all values in $R(f)$).

Level curves are vertical projections of the intersection of the surface $f(x,y)$ with the plane $z=k$.

Example: If elevation of Earth's surface is $z=f(x,y)$,
A contour map shows curves of constant elevation.
Weather network: curves of constant pressure. (isobars) "high pressure system"

Curves of constant temperature (isotherms)
eg. isopotentials

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**Ex.** Sketch the level curves of $f(x,y) = \sqrt{1-x^2-y^2}$

**Soln:** $D(f) : \{(x,y) \mid x^2+y^2 \leq 1\}$ & $R(f) : [0,1]$

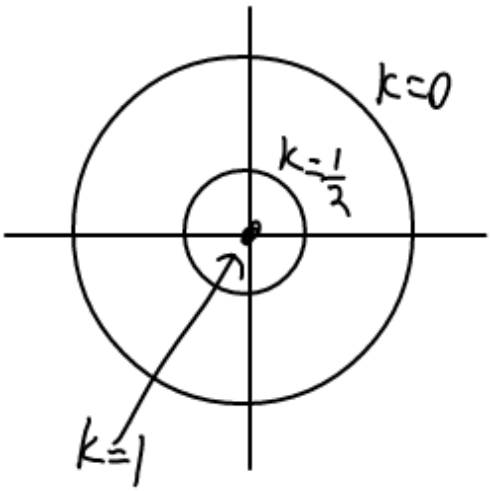
Take k-values in $[0,1]$

The level curves are $\sqrt{1-x^2-y^2} = k$.

$1-x^2-y^2 = k^2 \Rightarrow \underline{x^2+y^2 = 1-k^2}$

Level curves are circles centred at the origin with radius $\sqrt{1-k^2}$.

| k | radius |
|---|---|
| 0 | 1 |
| 1/2 | $\sqrt{3}/2$ |
| 1 | 0 |




This surface is a hemisphere.

If we write $z = \sqrt{1-x^2+y^2}$

$\rightarrow z^2 = 1 - x^2 + y^2$, $z \geq 0$

$\rightarrow \underbrace{x^2+y^2+z^2 = 1}$, $z \geq 0$

eqn of unit sphere.