Last time: <u>infinite series</u>

ex. $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}$$

This also works for functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

In general, any fcn can be exporsde:
(Taylor's Thm)

$$f(x) = P_{n,x_0}(x) + R_n(x)$$

$$\Rightarrow f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_n(x)$$

If $R_n(x) \to 0$ as $n \to \infty$, then $P_{n,x_0}(x) \to f(x)$

or $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

We use the notation

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

This is the <u>Taylor series of $f$ about $x_0$</u>. Condition for convergence. ($R_n \to 0$ as $n \to \infty$).

If $\exists$ M such that $|f^{(n)}(x)| \le M$ $\forall n = 0, 1, 2, \dots$ and all $x$ in an interval $I$ containing $x_0$, then the Taylor series of $f(x)$ converges to $f(x)$ for all $x \in I$.

For $f(x)=e^x$. Consider an arbitrary interval $[a,b]$ that contains 0.

$|f^{(n)}(x)| = e^x \le e^b$ on $[a,b]$

Let $M=e^b$. Since any number can be placed in an arbitrary interval, containing 0, the Taylor Series of $e^x$ about 0 converges for all $x$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for all } x.$$

- $f(x)=\sin x$ about 0.

$f^{(n)}(x) = \pm\sin x$ or $\pm\cos x$

$\Rightarrow |f^{(n)}(x)| \le 1$ for all $n$ and all $x \in \mathbb{R}$.

$\Rightarrow$ The Taylor Series for $\sin x$ converges for all $x$.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \text{ for all } x.$$

Similarly for $\cos x$ about 0,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \text{ for all } x$$

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Not all Taylor Series converge for all values of $x$.

e.g. $f(x) = \ln x$ about $x=1$

The Taylor Series is:

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

Suppose we want to use this to estimate $\ln 3$.

$$\Rightarrow \ln 3 = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(3-1)^k}{k} = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \ldots$$

$$= 2 - 2 + \frac{8}{3} - \frac{16}{4} + \frac{32}{5} - \ldots$$

terms are getting larger in magnitude.

$\longrightarrow$ diverges

In fact, the interval of convergence of the above Taylor Series is $(0, 2]$.

Note: $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

If we look at $|f^{(n)}(x)|$:

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $f^{(4)}(x) = \frac{-6}{x^4}$

Observe the pattern.

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \Rightarrow |f^{(n)}(x)| = \frac{(n-1)!}{x^n}$$

Bounding this quantity depends on $x$ & $n$ ($x=0$ is a problem) We can't find a single value of $K$.

Doesn't necessarily mean it never converges, we just have to do more work.