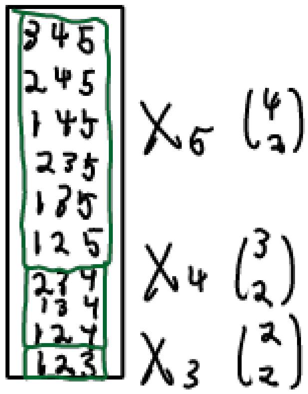
Prop: If $n, k \in \mathbb{N}_0$ then $\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$

Pf. let $X = \{n\text{-element subsets of } [n+k]\}$

Each $x \in X$ has largest element between n & $n+k$ inclusive.

For each $i \in \{0, 1, \dots, k\}$, let X_{n+i} be the set of all $x \in X$, whose largest element is $n+i$.



Clearly $|X| = \sum_{i=0}^{n} |X_{n+i}|$

and $|X| = \binom{n+k}{k}$

To construct a set in X_{n+i} we must select $n+i$ as the largest element, and then we have $\binom{n+i-1}{n-1}$ choices for the rest. So $|X_{n+i}| = \binom{n+i-1}{n-1}$.

Thus $\binom{n+k}{n} \Rightarrow |X| = \sum |X_{n+i}| = \sum \binom{n+i-1}{n-1}$ ■

Some problems we've considered:

- How many binary strings of length n are there?
- How many subsets of $[n]$ of size k are there?
- How many permutations are their of $[n]$?

Let S be a set

Let w be a 'weight' function assigning a non-negative integer weight $w(\sigma)$ to each $\sigma \in S$.

How many $\sigma \in S$ have weight k?

Given S & w, we define the generating series $\Phi_S(x)$ by $\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)}$

E.g. if $S = \{1, 3, 5\} \times \{2, 4, 6\}$

let $w(a,b) = a+b$

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = x^{w(1,2)} + x^{w(1,4)} + x^{w(1,6)} + x^{w(3,2)} + x^{w(3,4)} + x^{w(3,6)} + x^{w(5,2)} + x^{w(5,4)} + x^{w(5,6)}$$

$$= x^3 + x^5 + x^7 + x^5 + x^7 + x^9 + x^7 + x^9 + x^{11}$$

$$= x^3 + 2x^5 + 3x^7 + 2x^7 + x^{11}$$

<u>Prop:</u> $\Phi_S(x) = \sum_{k \geq 0} (\text{\# elements of } S \text{ whose weight is } k) x^k$

equivalently: The coefficient of x^k in $\Phi_S(x)$ is # elements of S of weight k.

Eg. let $S = \{\text{subsets of } [n]\}$ and $w(\sigma) = |\sigma|$ for each $\sigma \in S$.

$$\Phi_S(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

In our first example,

$S = \{1, 3, 5\} \times \{2, 4, 6\}$, $w(a,b) = a+b$

$$\Phi_S(x) = x^3 + 2x^5 + 3x^7 + 2x^9 + x^{11}$$
$$= (x^1 + x^3 + x^5)(x^2 + x^4 + x^6)$$

Another example. Let $S = \{\text{positive odd numbers}\}$ and $w(\sigma) = \sigma$

$$\Phi_S(x) = \sum_{\sigma \in S} x^{w(\sigma)} = \sum_{\substack{n \text{ odd} \\ n=0}} x^n = x^1 + x^3 + x^5 \dots$$
$$= x(1 + x^2 + x^4 \dots)$$
$$= \frac{x}{1-x^2}$$

E.g. Let $S = \{\text{permutations of } [k] \text{ for any } k\}$

$$= \{\overset{0}{()}, \overset{1}{(1)}, \overset{2}{(1,2)}, \overset{2}{(2,1)}, \overset{3}{(1,2,3)}, \overset{3}{(2,1,3)}, \overset{3}{(2,3,1)} \dots\}$$

and $w(\sigma) = \text{length}$

$$\Phi_S(x) = \sum_{k \geq 0} (\#\text{ permutations of } [k])x^k = \sum_{k \geq 0} k!\, x^k$$

For a generating series $\Phi(x)$ that is finite in length, write $\Phi(1)$ for the value obtained by subbing in $x=1$.
Write $\Phi'(x)$ for the 'derivative' of Φ.

Prop. If S is finite and w is a weight function, then

$\Phi_S(1) = |S|$

$\Phi_S'(1) = \text{total weight of elements}$