Last time: Polynomial Interpolation

Suppose we want to draw a smooth (differentiable) curve through $n+1$ points. A polynomial of degree n is the simplest such curve.

<u>Ex.</u> Find the cubic polynomial that passes thru $(0,1),(1,0)$ $(2,1)$ & $(3,3)$.

(* For 4 points, the interpolation formula is:

$$y = y_0 + x\Delta y_0 + \frac{1}{2}x(x-1)\Delta^2 y_0 + \frac{1}{6}x(x-1)(x-2)\Delta^3 y_0)$$

given y_0, y_1, y_2, y_3, set up the finite difference table.

$$\begin{array}{cccc} y & \Delta y & \Delta^2 y & \Delta^3 y \\ 1 & & & \\ & -1 & & \\ 0 & & 2 & \\ & 1 & & -1 \\ 1 & & 1 & \\ & 2 & & \\ 3 & & & \end{array}$$

$$\begin{aligned} y &= 1 + x(-1) + \frac{1}{2}x(x-1)(2) + \frac{1}{6}x(x-1)(x-2)(-1) \\ &= 1 - x + x(x-1) - \frac{1}{6}x(x-1)(x-2) \end{aligned}$$

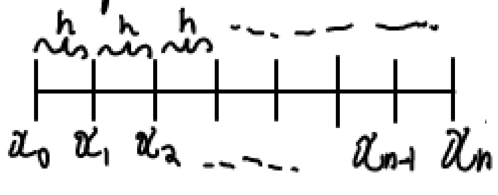
Generalization: For $n+1$ points, x-values $0,1,2,\ldots,n$.

$$y = y_0 + x\Delta y_0 + \frac{1}{2}x(x-1)\Delta^2 y_0 + \frac{1}{3!}x(x-1)(x-2)\Delta^3 y_0 + \frac{1}{4!}x(x-1)(x-2)(x-3)\Delta^4 y_0 + \ldots + \frac{1}{n!}x(x-1)(x-2)\ldots(x-(n-1))\Delta^n y_0$$

Further generalization: The nodes (x-values) are not $0, 1, 2, \ldots$.

Assume they are $x_0, x_1, \ldots, x_n$, where $x_j = x_0 + jh$, for $j = 0,1,2,\ldots$

(equidistant nodes with spacing h)



The n^{th} order polynomial is:

$$y = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{2h^2}\Delta^2 y_0 + \frac{(x-x_0)(x-x_1)(x-x_2)}{3!\,h^3}\Delta^3 y_0 + \ldots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{n!\,h^n}\Delta^n y_0$$

Ex. Estimate $f(1.75)$ if f passes thru:
$(1,3), (1.5,5), (2,9), (2.5,5)$.

Sol^n: $x_0=1$, $h=0.5$

F.D. Table

3			
	2		
5		2	
	4		-10
9		-8	
	-4		
5			

$$y = 3 + \frac{(x-1)(2)}{(0.5)} + \frac{(x-1)(x-1.5)(2)}{2(0.5)^2} + \frac{(x-1)(x-1.5)(x-2)}{3!(0.5)^3}(-10)$$

$$= 3 + 4(x-1) + 4(x-1)(x-1.5) - \frac{40}{3}(x-1)(x-1.5)(x-2).$$

Our Estimate is:

$$f(1.75) = 3 + 4\left(\tfrac{3}{4}\right) + 4\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) - \tfrac{40}{3}\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)\left(-\tfrac{1}{4}\right) \approx 7.375$$

How confident are you in this estimate? ~ no idea of the error.
not very confident.

Taylor Polynomials

Say we want to approximate a function $f(x)$ near a point x_0 with a polynomial of degree n. Based on the form of the linear approximation,

$L(x) = f(x_0) + f'(x_0)(x-x_0)$, assume the polynomial can be expressed:

$$(*) \quad p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n$$

where the coefficients $a_0, a_1, \dots, a_n$ are to be determined.

The "best" polynomial approximation will have: n^{th} derivative

$$(**) \quad f(x_0) = p(x_0),\ f'(x_0) = p'(x_0),\ f^{(n)}(x_0) = p^{(n)}(x_0),$$
for all n.

sub $x = x_0$ into $(*)$: $p(x_0) = a_0$

$(**)$: $a_0 = f(x_0)$

Differentiate $(*)$: $p'(x) = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots + na_n(x-x_0)^{n-1}$

sub $x = x_0$: $p'(x_0) = a_1$

$(**)$: $a_1 = f'(x_0)$

Differentiate $(*)$ again:

$$p''(x) = 2a_2 + 6a_3(x-x_0) + \ldots + n(n-1)a_n(x-x_0)^{n-2}$$

$$\Rightarrow p''(x_0) = 2a_2 \Rightarrow (***): a_2 = \frac{1}{2}f''(x_0).$$

Continue on... $a_3 = \frac{1}{6}f'''(x_0)$

$a_4 = \frac{1}{24}f^{(4)}(x_0)$ etc

In general, $a_n = \frac{1}{n!}f^{(n)}(x_0)$.

with the coefficients defined the n^{th} order of the Taylor Polynomial of f about x_0.

$$P_{n,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

or: $$P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$