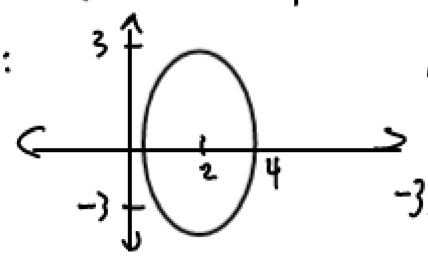
Example: Evaluate $\iint_{D_{xy}} x\,dx\,dy$, where D_{xy} is the region enclosed by the ellipse $\frac{(x-2)^2}{4} + \frac{y^2}{9} = 1$

Sol^n:



Express D_{xy}:

$0 \le x \le 4$

$-3\sqrt{1-\frac{(x-2)^2}{4}} \le y \le 3\sqrt{1-\frac{(x-2)^2}{4}}$ → Looks messy. Maybe there's an easier way.

Instead, map D_{xy} onto the unit circle, D_{uv} $(u^2+v^2=1)$

Let $u = \frac{x-2}{2}$, $v = \frac{y}{3}$

$$\iint_{D_{xy}} x\,dx\,dy = \iint_{D_{uv}} \underbrace{(2u+2)}_{x} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{1}{6}$

$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = 6$

$$= 12 \iint_{D_{uv}} (u+1)\,du\,dv$$

Use polar coordin. $u = r\cos\theta$ $v = r\sin\theta$

$$= 12\pi$$

$\int_a^b dx = b-a$ (length of the interval) $\iiint_D dV = V(D)$ (volume of D)

$\iint_D dA = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta A_i = A(D)$ (Area of D)

E.g. let $f(x,y,z)$ be the density of an object at a point in space. Then $\iiint_D f(x,y,z)\,dV$ represents the <u>mass</u> of the object

Triple Integrals

Simplest Case: iterated integrals $D = \{a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2\}$

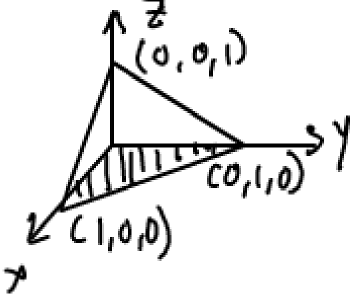
Then $\iiint_D f(x,y,z)\,dV = \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz$ ⇒ Can be written in 6 different orders.

When the bounds aren't constant, we have the following:

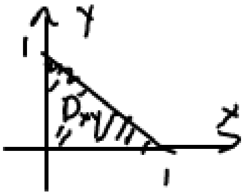
If D is a subset of R^3 defined by $z_\ell(x,y) \le z \le z_u(x,y)$ and $(x,y) \in D_{xy}$, then $\iiint_D f(x,y,z) = \iint_{D_{xy}} \int_{z_\ell(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$ don't have to start w/ z

Now we treat the double integral in the usual way.

Ex: Find $\iiint_D x\,dV$, where D is the region bounded by the planes $x+y+z=1$, $x=0$, $y=0$, $z=0$ (in the first octant $x\geq0$ $y\geq0$ $z\geq0$)



We can describe D by: $\underbrace{0}_{xy\text{plane}} \leq z \leq \underbrace{1-x-y}_{\text{plane}}$



D_{xy}: $0 \leq y \leq 1-x$

$0 \leq x \leq 1$ $\Rightarrow \iiint_D x\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx$