## Last Time: Level Curves

$f(x,y)=k$, where $k$ is in the range of $f$.

## Cross Sections

Slicing our surface with the $x=$ constant and $y=$ constant planes can also help with visualization.

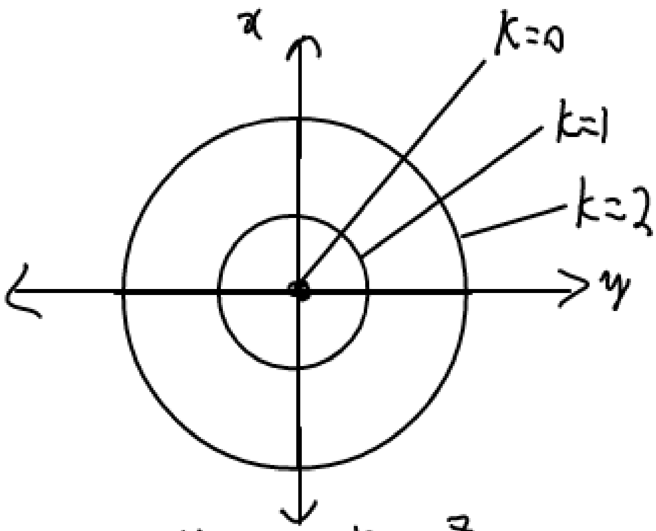
**Ex.** Draw the level curves $k=0,1,2$ and the cross-sections $x=c\ (c=0,\pm1,\pm2)$, $y=d\ (d=0,\pm1,\pm2)$ of $f(x,y)=\sqrt{x^2+y^2}$.

**Sol$^n$.** Domain $\mathbb{R}^2$ (no restriction)

Range $[0,\infty)$

Level curves: $\sqrt{x^2+y^2}=k$

$\Rightarrow x^2+y^2=k^2$



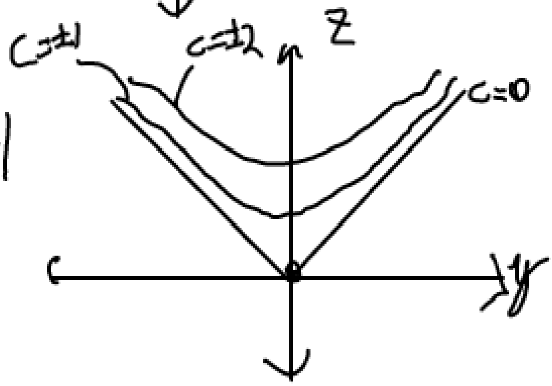

**Cross-sections: $x=c$**

$z=\sqrt{c^2+y^2}$

$c=0: z=\sqrt{y^2}=|y|$

$c=\pm1: z=\sqrt{1+y^2}$

$c=\pm2: z=\sqrt{4+y^2}$



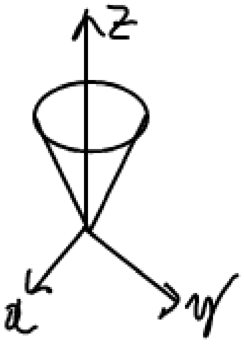

**$y=d$** $z=\sqrt{x^2+d^2}$

$d=0: z=|x|$ (symmetric)

$d=\pm1: z=\sqrt{x^2+1}$

$d=\pm2: z=\sqrt{x^2+4}$

To sketch the surface, put it all together (esp. the $c=0$, $d=0$, cross-sections)



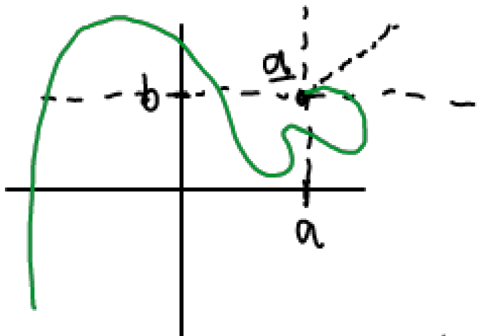

## Limits

Key difference for limits of functions $f: \mathbb{R} \to \mathbb{R}$ && functions $f: \mathbb{R}^2 \to \mathbb{R}$:

$f: \mathbb{R} \to \mathbb{R}$: For a limit as $x \to a$, you can approach either from the left or from the right.
(limits exist iff one-sided limits are equal)

$f: \mathbb{R}^2 \to \mathbb{R}$: Let $\underline{x} = (x, y)$, $\underline{a} = (a, b)$

How can we approach $\underline{a}$?




- along $x=a$ or $y=b$
- any line that passes through $\underline{a}$.
- any curve that passes thru $\underline{a}$.

$\to$ Infinite number of paths.

Only if the limit is $L$ along every path can we say:

$$\lim_{\underline{x} \to \underline{a}} f(\underline{x}) = L.$$

To prove - use defn (not in this course).

To show a limit does not exist, simply show a different value is obtained along any two paths.

Ex. $\lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3-y^3}{x^2+y^2}}_{f(x,y)}$

along $x=0$, $f(0,y) = \frac{-y^3}{y^2} = -1$.
Limit is $-1$.
Along $y=0$, $f(x,0) = \frac{x^3}{x^2} = 1$
Limit is 1.

$\therefore$ Limit D.N.E.

For continuous functions, just plug in the value

e.g. $\lim_{(x,y)\to(0,0)} \sqrt{1-x^2-y^2} = 1$

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## Partial derivatives

Defⁿ: Let $f: \mathbb{R}^2 \to \mathbb{R}$. The partial derivatives of $f$ at $(a,b)$ are:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

Notation: $\frac{\partial f}{\partial x}$, $f_x$, $D_1 f$

symbol is "die" or "dyo" or "partial with respect to"

Note: Standard differentiation rules apply.

To compute $\frac{\partial f}{\partial x}$, treat $y$ as a constant and differentiate in the usual way.

e.g. 1) $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 4y$$

2) $f(x,y) = e^{xy}\cos(x^2+y^2)$ (product && chain rule)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy}\right)\cos(x^2+y^2) + e^{xy}\cdot\frac{\partial}{\partial x}\left(\cos(x^2+y^2)\right)$$

$$= ye^{xy}\cos(x^2+y^2) + e^{xy}\left(-\sin(x^2+y^2)\cdot 2x\right)$$

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<u>Ex.</u> Determine whether $f_x(0,0)$ & $f_y(0,0)$ exist for $f(x,y) = \sqrt{x^2+y^2}$

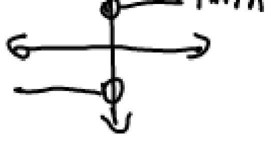
<u>Sol$^n$</u> $f_x = \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$ ; $f_y = \frac{y}{\sqrt{x^2+y^2}}$

$\therefore$ exists everywhere except $(0,0)$

At $(0,0)$, gets an indeterminate form

use def$^n$.

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{h^2}-0}{h} = \lim_{h\to 0}\frac{|h|}{h} \leftarrow \text{DNE}$$




$f_x(0,0)$ does not exist.