Recall a permutation of set A is a bijection $f: A \to A$ (essentially an 'ordering' of A). If $|A| = n$, then there are $n!$ permutations of A.

<u>Binomial Coefficients</u>: $\binom{n}{k}$ is defined to be the number of k-element subsets of $[n]$.

<u>Prop.</u> $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for all $0 \le k \le n$

Pf: Let $\mathcal{L} = \{$ ordered k-tuples of distinct elements of $[n]\}$

eg. if $n=4$, $k=2$, $\mathcal{L} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

We compute the size of $\mathcal{L}$ in two ways.

In selecting some $(x_1, x_2, x_3, \dots, x_k) \in \mathcal{L}$, we have n choices for x_1, $n-1$ choices for $x_2, \dots,$ $n-k+1$ choices for x_k, so

$$|\mathcal{L}| = n(n-1)(n-2)\cdots(n-k+1)$$
$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)\cdots(1)}{(n-k)(n-k-1)\cdots(1)}$$
$$= \frac{n!}{(n-k)!}$$

Alternatively, we could first choose the set $\{x_1, \dots, x_k\}$, then choose the ordering. This gives $|L| = \binom{n}{k} \cdot k! \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Consider $(1+x_1)(1+x_2)(1+x_3) = (1+x_1)(1+x_2+x_3+x_2x_3)$

$$\Rightarrow \underset{\emptyset}{1} + \underset{\{1\}}{x_1} + \underset{\{2\}}{x_2} + \underset{\{3\}}{x_3} + \underset{\{1,2\}}{x_1x_2} + \underset{\{1,3\}}{x_1x_3} + \underset{\{2,3\}}{x_2x_3} + \underset{\{1,2,3\}}{x_1x_2x_3}$$

If we set $x_1 = x_2 = x_3 = x$, we got

$$(1+x)^3 = 1+x+x+x+x^2+x^2+x^2+x^3$$
$$= \underset{\binom{3}{0}}{1} + \underset{\binom{3}{1}}{3x} + \underset{\binom{3}{2}}{3x^2} + \underset{\binom{3}{3}}{x^3}$$

Proof of Binomial Theorem:

let $y_1, y_2, \ldots, y_n$ be 'indeterminates'.

Consider $(1+y_1)(1+y_2)(1+y_3)\ldots(1+y_n)$

For each set $S \subseteq [n]$, let $y^S = \prod_{i \in S} y_i$. eg. $y^{\{1,3,3\}} = y_1y_3y_4$

Every y^S occurs exactly once in the expansion $(1+y_1)\ldots(1+y_n)$

so $(1+y_1)(1+y_2)\ldots(1+y_n) = \sum_{S \subseteq [n]} y^S \quad (*)$

Set $y_1 = y_2 = \ldots = y_n = x$

The LHS of $(*)$ is $(1+x)^n$

If $y_1 = y_2 = \ldots = y_n = x$, then $y^S = x^{|S|}$.

Therefore $(*)$ becomes

$$(1+x)^n = \sum_{S \subseteq [n]} x^{|S|} = \sum_{k=0}^{n} (\#\text{ of } S \text{ with } |S| = k)\, x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k. \quad \blacksquare$$

<u>Corollary</u>: $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n$

$$= 2^n$$

<u>Prop</u>: If $n, k \geq 0$, then $\displaystyle\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$

NO INDUCTION PLS

$n=3,\ k=2$

$$\binom{5}{3} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

10   1   3   6

123
124
125
134
135
145

234
235
245

345