1. Complete Solutions to LDEs

Recall: $ax+by=c$ has int. soln if and only if $\gcd(a,b) \mid c$

CAD: If $a \mid bc$ and $\gcd(a,b)=1$, then $a \mid c$.

DB GCD: If $d=\gcd(a,b)>0$, then $\gcd\left(\frac{a}{d},\frac{b}{d}\right)=1$

What are all solutions to $ax+by=c$? assume $\gcd(a,b) \mid c$.

Define $S=\{(x,y)\in \mathbb{Z}\times\mathbb{Z} \mid ax+by=c\}$

Example: $12x+15y=3$ $(-1,1)$ $(4,-3)$ $(-6,5)$ ....

Example: $12x+15y=333$ $(444,-333)$.

Suppose (x,y) is an arbitrary int sol^n.

① $12x+15y=333$

② $12\cdot 444+15(-333)=333$

①−② $12(x-444)+15(y+333)=0$

$12(x-444)=-15(y+333)$ ④

Divide gcd on both sides: $4(x-444)=-5(y+333)$

$5 \mid 4(x-444)$ By CAD, $5 \mid (x-444)$

So $x-444=5n$ for some $n\in\mathbb{Z}$. So $\underline{x=444+5n}$. ③

Sub ③ into ④: $12\cdot 5n=-15(y+333)$

$y+333=-4n$. So $\underline{y=-333-4n}$.

So $(x,y)=(\underbrace{444}_{}\ \underbrace{+5n}_{b/d},\ \underbrace{-333}_{}\ \underbrace{-4n}_{a/d})$ for some $n\in\mathbb{Z}$.

one specific solution

Generalize: Let (x_0, y_0) be one solution.

Let $T = \{(x_0 + \frac{b}{d}n), (y_0 - \frac{a}{d}n) \mid n \in \mathbb{Z}\}$

What have we illustrated?

$S \subseteq T$

Example: $12x + 15y = 12(444 + 5n) + 15(-333 - 4n)$
$= 12(444) + 60n - 15(333) - 60n$
$= 333$

$S \subseteq T$: Any sol^n has the form in T.

$T \subseteq S$: Anything with the form in T is an int sol^n.

$\therefore S = T$. So T is the complete int sol^n.

Proposition (LDE 2): Let $a, b, c \in \mathbb{Z}$, $d = \gcd(a, b) \neq 0$.

If (x_0, y_0) is one int sol^n to $ax + by = c$, then the complete set of int sol^n is $\{(x_0 + \frac{b}{d}n, y_0 - \frac{a}{d}n) \mid n \in \mathbb{Z}\}$

Proof: Let (x, y) be an int soln. Since

$ax + by = c$ and $ax_0 + by_0 = c$,

$a(x - x_0) + b(y - y_0) = 0$

Therefore $a(x - x_0) = -b(y - y_0)$ ①

Divide both sides by d to get $\frac{a}{d}(x - x_0) = -\frac{b}{d}(y - y_0)$

Then $\frac{b}{d} \mid \frac{a}{d}(x - x_0)$. Using DB GCD, $\gcd(\frac{b}{d}, \frac{a}{d}) = 1$.

By CAD, $\frac{b}{d} \mid (x - x_0)$. So $x - x_0 = \frac{b}{d}n$ ② for some $n \in \mathbb{Z}$.

So $x = x_0 + \frac{b}{d}n$.

Sub ② ⇒ ① to get

$$\frac{a}{d}\left(\frac{\cancel{b}}{\cancel{d}}n\right) = -\frac{\cancel{b}}{\cancel{d}}(y-y_0)$$

So $y - y_0 = -\frac{a}{d}n$, so $y = y_0 - \frac{a}{d}n$

Let (x,y) be an element of the given set.
Then there is $n \in \mathbb{Z}$ such that $x = x_0 + \frac{b}{d}n$, $y = y_0 - \frac{a}{d}n$.

Then $ax + by = a\left(x_0 + \frac{b}{d}n\right) + b\left(y_0 - \frac{a}{d}n\right)$

$$= ax_0 + \frac{ab}{d}n + by_0 - \frac{ab}{d}n$$

$$= ax_0 + by_0$$

$$= c$$

So (x,y) is an int soln.