Recall:

n^{th} Fibonacci number

$=$ # compositions of n into parts of size 1 or 2

$=$ # comp. of $n+1$ into odd parts.

Let $T_n = \{$compositions of n into parts of odd size$\}$

Clearly $|T_1| = |T_2| = 1$. To show that T_{n+1} is the n^{th} Fib. #, it suffices to show that $|T_n| = |T_{n-1}| + |T_{n-2}|$ for $n \geq 3$. We do this by defining a bijection f between T_n and $T_{n-1} \cup T_{n-2}$.

$T_2 = \{(1,1)\}$

$T_3 = \{(1,1,1), (3)\}$

$T_4 = \{(1,1,1,1), (1,3), (3,1)\}$

$T_5 = \{(1,1,1,1,1), (1,1,3), (1,3,1), (3,1,1), (5)\}$

$T_6 = \{(1,1,1,1,1,1), (1,1,1,3), (1,1,3,1), (1,3,1,1), (3,1,1,1), (1,5),$
$(5,1), (3,3)\}$

Let $f: T_n \to T_{n-1} \cup T_{n-2}$ be defined by

$$f(a_1, a_2, \ldots, a_k) = \begin{cases} (a_1, a_2, \ldots, a_{k-1}) & \text{if } a_k = 1 \\ (a_1, a_2, \ldots, a_k - 2) & \text{if } a_k \neq 1 \end{cases}$$

and $g: T_{n-1} \cup T_{n-2} \to T_n$

by $$g(a_1, a_2, \ldots, a_k) = \begin{cases} (a_1, a_2, \ldots, a_k, 1) & \text{if } (a_1, \ldots, a_k) \in T_{n-1} \\ (a_1, a_2, \ldots, a_k + 2) & \text{if } (a_1, \ldots, a_k) \in T_{n-2} \end{cases}$$

g is the inverse of f

so f is a bijection. Thus, $|T_n| = |T_{n-1} \cup T_{n-2}| = |T_{n-1}| \cup |T_{n-2}|$ ∎

Let $T_n = \{$comps. of n into parts of odd size$\}$

Let $S_n = \{$comps. of n into parts of size 1 or 2$\}$

Can we show that $|S_n| = |T_{n-1}|$ by finding a bijection?

$$(1,3,7,5,9,3,3,1,3) \in T_{35}$$

spread each odd # into 2,2,...,1

$$(\underbrace{1}_{1},\underbrace{2,1}_{3},\underbrace{2,2,2,1}_{7},\underbrace{2,2,1}_{5},\underbrace{2,2,2,2,1}_{9},\underbrace{2,1}_{3},\underbrace{2,1}_{3},\underbrace{1}_{1},\underbrace{2,\,\mathbf{1}}_{3}) \in S_{35}$$

remove

$\downarrow$ always have 1 at the end, remove to get S_{34}

$$(1,2,1,2,2,2,1,2,2,1,2,2,2,2,1,2,1,2,1,1,2) \in S_{34}$$

Binary strings

A binary string of length k is a k-tuple $(a_1,\dots,a_k)$ where $a_i \in \{0,1\}$ (equivalently, just a member of $\{0,1\}^k$)

When writing a str, suppress commas/brackets i.e.

$$a_1a_2a_3 \cdots a_k = (a_1,a_2,a_3,\dots,a_k)$$

If $\sigma = s_1s_2\dots s_j$ and $\tau = t_1\dots t_k$ as binary strings, then

$$\sigma\tau = s_1s_2\dots s_jt_1\dots t_k$$

we write $l(\sigma)$ for the length of σ, so $l(\sigma\tau) = l(\sigma)+l(\tau)$

σ^k denotes $\underbrace{\sigma\sigma\sigma\dots\sigma}_{k \text{ times}}$, where $\sigma^0 = \varepsilon \leftarrow$ empty str.

If A, B are sets of strings then $AB = \{\alpha\beta : \alpha \in A, \beta \in B\}$ eg. $\{\varepsilon, 10, 01\}\{101, 000\}$
$= \{101, 000, 10101, 10000, 01101, 01000\}$

We also define $A^k = \underbrace{AAA\ldots A}_{k}$ (equivalently the set of all strings of the form $a_1, a_2, \ldots a_k$ where $a_i \in A$)

eg $\{0,1\}^7 = \{\text{strings of length } 7\}$

$$A^* = \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup \ldots$$
$$= \bigcup_{k \geq 0} A^k$$

eg. $\{00, 11\}^* = \{$strings where every block of zeroes or ones has even length$\}$.

<u>Substring</u> of s is a string b so that $s = abc$ for some a, c

<u>block</u> of s is a maximal substring whose members are all zero or all one

00 111 00 1 0