Last time: The linear approximation of: $f(x)$ at $x=a$ is:

$$L(x) = f(a) + f'(a)(x-a)$$

e.g. Use $L(x)$ at $x=1$ for $f(x)=\sqrt{x}$ to estimate $\sqrt{1.1}$

<u>Sol$^n$</u>

$$L(x) = f(1) + f'(1)(x-1) \qquad f(x)=\sqrt{x}$$

$$L(x) = \sqrt{1} + \frac{1}{2\sqrt{1}}(x-1) \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(1.1) = 1 + \frac{1}{2}(1.1-1)$$

$$= 1 + \frac{1}{20}$$

$$= 1.05 \qquad (\text{actual value} \approx 1.0488)$$

How to estimate $\sqrt{4.6}$? Find $L(x)$ at $x=4$ (since $\sqrt{4}=2$)

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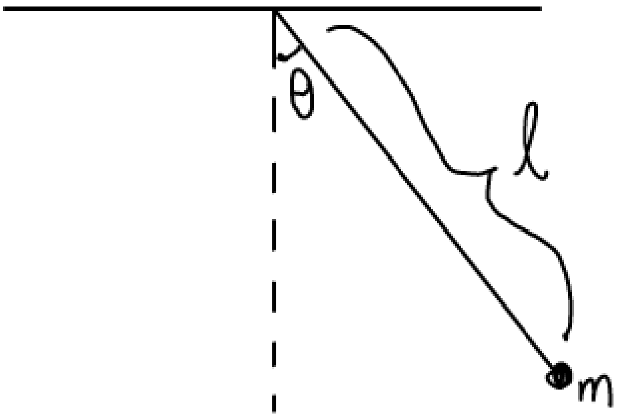
Example from physics where the linear approximation is used: the simple pendulum




The motion obeys the DE:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

This is not solveable
$\therefore$ make an approx.

Near $\theta=0$, the lin. approx. of $\sin\theta$ is $\sin\theta \approx \theta$.

The DE $\frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta$ can be solved (simple harmonic motion)

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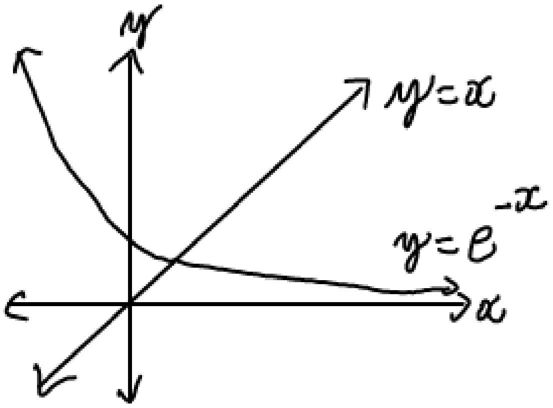
## Estimating Roots

Suppose we want to solve $x = e^{-x}$.




Clearly there is a solution.
But we can't solve in analytically.
$\therefore$ approximate.

## The Bisection Method

Based on IVT. If $f(a)<0$ and $f$ is continuous on $[a,b]$, then $\exists c \in (a,b)$ s.t. $f(c)=0$.

Let $f(x)=e^{-x}-x$. $f(x)=0$ corresponds to the root of $x=e^{-x}$.

$f(0)=1>0$, $f(1)=e^{-1}-1<0$

Since $f$ is cts on $(0,1)$, $\exists\, c \in (0,1)$ s.t. $f(c)=0$.

(i.e. the root lies in $(0,1)$)

bisect $[0,1]$ into $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$.

$f(\frac{1}{2})=e^{-\frac{1}{2}}-\frac{1}{2}\approx 0.11>0$.

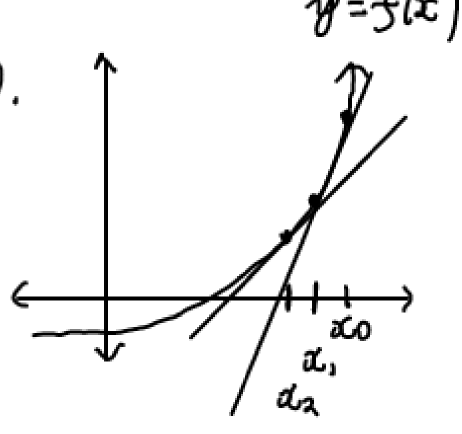
Apply IVT on $[\frac{1}{2},1] \to$ root lies in $(\frac{1}{2},1)$.

Continue until desired accuracy is reached.

Advantage: easy   Disadvantage: slow convergence.

## Newton's Method

suppose we want to solve $f(x)=0$.
(which we can't solve analytically)
Instead solve $L(x)=0$ for $x_1$.
the next approximation.



At $x = x_0$, $L(x) = f(x_0) + f'(x_0)(x - x_0) \underset{\text{set}}{=} 0$

Let $x = x_1 \Rightarrow x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Use $x_1$ and iterate again:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$x_{n-1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the sequence of $x_n$'s has a limit, then the method converges.

e.g. To solve $x = e^{-x}$, let $f(x) = x - e^{-x}$ to 8 decimal places, then $f'(x) = 1 + e^{-x}$.

For the initial guess $x_0$, choose $x_0 = 0$ (based on earlier work)

$$x_1 = x_0 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{(-1)}{2} = \frac{1}{2}$$

$x_2 \approx 0.56631100$

$x_3 = 0.56714317$

$x_4 = 0.56714329$

$x_5 = 0.56714329$ stop

## Fixed Point Iteration

Instead of solving $f(x) = 0$, we solve $x = g(x)$ & use the recurrence relation $x_{n+1} = g(x_n)$.

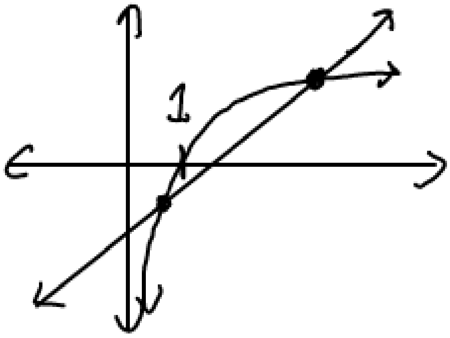
Note. A solution of $x = g(x)$ is called a fixed point of $g$.

e.g. Solve $\ln x = 2x - 3$ using fixed point iteration by writing $x = g(x)$ in 2 different ways.

① take exponent $x = e^{2x-3}$ (isolate $x$ on left)

② $\ln x + 3 = 2x \Rightarrow x = \frac{1}{2}(\ln x + 3)$ (isolate $x$ on right)

Initial guess: Start with a sketch.




$\rightarrow$ 2 solutions

$\rightarrow$ one in $(0,1)$

$\rightarrow$ one in $(1,\infty)$

let $x_0 = 1$ in ① & ②