The Remainder Theorem for Taylor Polynomials

Important Question: How good is the approximation $f(x) \approx P_{n,x_0}(x)$

we look to quantify the absolute error $|f(x) - P_{n,x_0}(x)|$

Start with the FTC: If F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$

write in a slightly different way.

$$\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$$

or

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt \quad (*)$$

since $P_{0,x_0}(x) = f(x_0)$, we can view $(*)$ as saying

$f(x) = P_{0,x_0}(x) + \underset{\substack{\uparrow \\ \text{remainder}}}{R_0(x)}$, where $R_0(x) = \int_{x_0}^{x} f'(t)\,dt$

consider the integral $\int_{x_0}^{x} f'(t)\,dt$

integrate by parts. $\quad = t \cdot f'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} t f''(t)\,dt$

$u = f'(t) \quad dv = dt$

$du = f''(t)\,dt \quad v = t$ $\quad = x f'(x) - x_0 f'(x_0) - \int_{x_0}^{x} t f''(t)\,dt$

$\left(\int u\,dv = uv - \int v\,du \right)$ $\quad$ plug this into $(*)$

$$f(x) = f(x_0) + xf'(x) - x_0 f'(x_0) - \int_{x_0}^{x} t f''(t)\,dt$$

We want $P_{1,x_0}(x) = f(x_0) + f'(x_0)(x-x_0)$ on the RHS:
Add & subtract $xf'(x_0)$ on the RHS.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{P_{1,x_0}(x)} + x\big(\underbrace{f'(x) - f'(x_0)}_{\int_{x_0}^{x} f''(t)\,dt}\big) - \int_{x_0}^{x} t f''(t)\,dt$$

$$\Rightarrow f(x) = P_{1,x_0}(x) + \underbrace{\int_{x_0}^{x} (x-t) f''(t)\,dt}_{R_1(x)}$$

Note: $R_0(x)$ depended on f'. $R_1(x)$ depends on f''.
Exercise: Integrate $R_1(x)$ by parts to show

$$f(x) = P_{2,x_0} + R_2(x), \text{ when } R_2(x) = \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt.$$

The general case is <u>Taylor's theorem with Integral Remainder</u>:

Suppose that f has $n+1$ derivatives at x_0.
Then $f(x) = P_{n,x_0}(x) + R_n(x)$, where

$$P_{n,x_0} = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \& \quad R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

The $R_n(x)$ looks daunting - that's ok - we only use it to get another result.

The key is $f^{(n+1)}(t)$. Suppose that $|f^{(n+1)}(t)| \leq K$ $\forall t \in [x_0, x]$

The error is:

$$|f(x) - P_{n,0}(x)| = |R_n(x)| = \left| \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n-1)}(t)\,dt \right|$$

$$\leq \int_{x_0}^{x} \left| \frac{(x-t)^n}{n!} \underbrace{f^{(n-1)}(t)}_{K} \right| dt \quad \text{(property of integrals)}$$

$$\leq \frac{K}{n!} \int_{x_0}^{x} |x-t|^n \, dt$$

$$= \frac{-K}{n!} \cdot \frac{|x-t|^{n+1}}{(n+1)} \Bigg|_{x_0}^{x}$$

$$= \frac{K}{(n+1)!} |x - x_0|^{n+1}$$

Taylor's Inequality

The error in using $P_{n,x_0}(x)$ to approximate $f(x)$ satisfies

$$|R_n(x)| \leq \frac{K}{(n+1)!} |x - x_0|^{n+1} \quad \forall x \in [x_0, x]$$

where $|f^{(n+1)}(t)| \leq K$

Ex. Previously, we compared $P_{2,1}(x)$ for $f(x) = \ln x$.

$P_{2,1}(x) = (x-1) - \frac{1}{2}(x-1)^2$

Estimated $\ln(1.2) \approx P_{2,1}(1.2) = \frac{9}{50} = 0.18$

Use Taylor's inequality to find a bound on the error $|R_2(x)| = |P_{2,1}(x) - f(x)|$ on the interval $[1, 1.2]$.

Sol^n $\quad |R_2(x)| \le \frac{K}{3!}|x-1|^3$

$$= \frac{2}{6}|x-1|^3$$

$$= \frac{1}{3}|x-1|^3$$

On $[1, 1.2]$, $|x-1|^3 \le |1.2-1|^3$

$$= \left(\frac{1}{5}\right)^3$$

$$|R_2(x)| \le \frac{1}{3}\cdot\frac{1}{5^3} = \frac{1}{375}$$

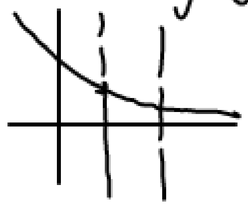
$$\frac{1}{375} \approx 0.00267$$

where $|f^{(3)}(t)| \le K$

$f'(t) = \frac{1}{t}$, $f''(t) = -\frac{1}{t^2}$

$f'''(t) = \frac{2}{t^3}$

$|f'''(t)| = \underbrace{\frac{2}{t^3}}_{\text{decreasing fn}} \le \underbrace{2}_{K}$ on $t \in [1, 1.2]$



max at lest endpoint

Since $\ln(1.2) \approx 0.1823$ we can see our estimate is within the error bound.