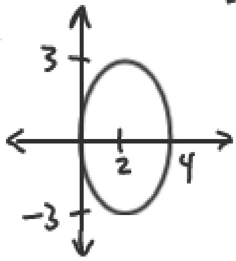
Example: Evaluate $\iint_{D_{xy}} x\,dx\,dy$, where D_{xy} is the region enclosed by $\frac{(x-2)^2}{4}+\frac{y^2}{9}=1$.

Sol^n:



Express D_{xy}:

$0 \le x \le 4$

$-3\sqrt{1-\frac{(x-2)^2}{4}} \le y \le 3\sqrt{1-\frac{(x-2)^2}{4}}$

Looks messy

Alternative: Map D_{xy} to the unit circle, D_{uv} $(u^2+v^2=1)$

Let $u=\frac{x-2}{2}$, $v=\frac{y}{3}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\frac{1}{6}} = 6$$

$$\iint_{D_{xy}} x\,dx\,dy = \iint_{D_{uv}} \underbrace{(2u+2)}_{x}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= 12\iint_{D_{uv}} (u+1)\,du\,dv$$

Use polar coordinates

$u = r\cos\theta$ $\quad D_{r\theta}$: $0 \le r \le 1$

$v = r\sin\theta$ $\qquad\quad 0 \le \theta \le 2\pi$

$$= 12\iint_{D_{r\theta}} (r\cos\theta+1)\left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= 12\int_0^{2\pi}\int_0^1 (r\cos\theta+1)\,r\,dr\,d\theta$$

$$= 12\int_0^{2\pi}\left[\int_0^1 r^2\cos\theta + r\,dr\right]d\theta$$

$$= 12\int_0^{2\pi}\left[\frac{1}{3}r^3\cos\theta+\frac{1}{2}r^2\Big|_0^1\right]d\theta$$

$$= 12\int_0^{2\pi} \frac{1}{3}\cos\theta+\frac{1}{2}\,d\theta$$

$$= 12\left[\frac{1}{3}\sin\theta+\frac{1}{2}\theta\right]_0^{2\pi}$$

$$= 12\pi$$

$$\therefore \iint_{D_{xy}} x\,dx\,dy = 12\pi$$

$\int_a^b dx = b-a$ (interval length) $\quad \iint_D dA = A(D) \quad \iiint_D dV = V(D)$

E.g. Let $f(x,y,z)$ be the density of an object at a point in space. Then $\iiint_D f(x,y,z)\,dV$ represent the <u>mass</u> of the object

<u>Triple Integrals</u>

Simplest Case: iterated integrals $D=\{a_1 \le x \le a_2,\ b_1 \le y \le b_2,\ c_1 \le z \le c_2\}$

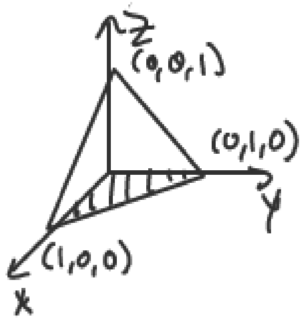
Then $\iiint_D f(x,y,z)\,dV = \int_{c_1}^{c_2}\int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y,z)\,dx\,dy\,dz \Rightarrow$ can be rewritten in 6 orders

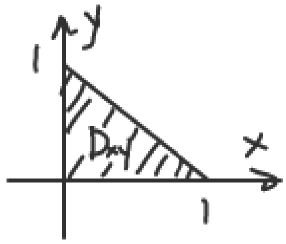
When the bounds aren't constant, we have the following:

If D is a subset of $\mathbb{R}^3$ defined by $z_\ell(x,y) \le z \le z_u(x,y)$ and $(x,y) \in D_{xy}$, then $\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}}\int_{z_\ell(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$ Note: you don't have to start with z (x, y too)

Now, we can treat and solve the double integral the usual way.

Ex: Find $\iiint_D x\,dV$, where D is the region bounded by the planes $x+y+z=1$, $x=0$, $y=0$, $z=0$ (in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$)





We can describe D by: $\underbrace{0}_{xy\text{ plane}} \leq z \leq \underbrace{1-x-y}_{\text{plane}}$

D_{xy}: $0 \leq y \leq 1-x$
$0 \leq x \leq 1$

$$\Rightarrow \iiint_D x\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx$$