Simulating ODE's

- Approximately a solution for an ODE using Numerical methods
- These are used when there's no closed form solns, or a really complicated ODE (non-linear)

Given $f\left(\frac{d^n y}{dt^n}, \frac{d^{n-1} y}{dt^{n-1}}, \dots, y, t\right) = 0$

State Space

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x_1 = y \quad x_2 = \frac{dy}{dt}$ $\quad x_k = \frac{d^{k-1} y}{dt^{k-1}}$ for $k = 1 \dots n$

Ex: $a_0(t) y^n(t) + \dots + a_n(t) y(t) = F(t)$

$$\frac{d\vec{x}}{dt} = g(\vec{x}, t)$$

$$x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ \frac{d^{n-1} y}{dt^{n-1}} \end{bmatrix} \quad y^n(t) = \frac{1}{a_0} \cdot \left[f(t) - a_1 y^{n-1}(t) \dots - a_n y(t)\right]$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \frac{dx}{dt} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ \left[f(t) - a_n x(t) \dots - a_1 x_{n-1}(t)\right] \frac{1}{a_0} \end{bmatrix}$$

Ex.

$$\frac{d^3 y}{dt^3} + \cos(t) y_2 = t$$

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ x_3 \\ t - \cos(t) x_4 \\ t^2 + 5x_1 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dddot{y}_1 \\ \dot{y}_2 \end{bmatrix}$$

state vector

$$x = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ \ddot{y}_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\cos(t) \\ 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \\ t^2 \end{bmatrix}$$

$$= A(t) x + B(t)$$

Euler's Method

$$\frac{dx}{dt} = \lim_{\Delta T \to 0} \frac{x(t+\Delta T) - x(t)}{\Delta T}$$

$$\frac{dx}{dt} \approx \frac{x(t+\Delta T) - x(t)}{\Delta T} \qquad \Delta T \ll 1$$

$$\Delta T \frac{dx}{dt} + x(t) = x(t+\Delta T)$$

Iterative soln

$$x(t_0 + k\Delta T) = \Delta T \cdot \frac{dx}{dt}\left(t_0 + (k-1)\Delta T\right) + x(t_0 + (k-1)\Delta T)$$

$K = 1 \ldots n$

$t_0 = 0 \qquad \Delta T = 0.1$

$x(0) = 0$

$x(t_0) = 0$

① $x(t_0) = 0$ for $t_0 = 0$

② $t = t_0 + \Delta T = 0.1$

$x(0.1)$?

$$x(t_0 + \Delta T) = \Delta T \cdot \frac{dx}{dt}(t_0) + x(t_0) = 0.1(0) + 0 = 0$$

$$\boxed{\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\cos(t) \\ 5 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ t \\ t^2 \end{bmatrix}} \qquad x(0.1) = 0$$

③ $t = t_0 + 2\Delta T$

$= 0.2$

$x(0.2)?$

$$x(t_0 + 2\Delta T) = \Delta T \cdot \frac{dx}{dt}(t_0 + \Delta T) + x(t_0 + \Delta T)$$

$$= 0.1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ (0.1)^2 \end{bmatrix} + 0$$

$$x(0.2) = \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0.001 \end{bmatrix}$$

④ $t = t_0 + 3\Delta T$

$= 0.3$

$x(0.3)?$

$$x(t_0 + 3\Delta T) = \Delta T \cdot \frac{dx}{dt}(t_0 + 2\Delta T) + x(t_0 + 2\Delta T)$$

$$= 0.1(\qquad\qquad) + \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0.001 \end{bmatrix}$$

$$\frac{dx(0.2)}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\cos(0.2) \\ 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.01 \\ 0.001 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0.04 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.01 \\ -0.001\cos(0.2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 0.04 \end{bmatrix}$$

$$x(0.3) = \begin{bmatrix} 0 \\ 0.001 \\ 0.0299 \\ 0.005 \end{bmatrix} = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ \ddot{y}_1 \\ y_2 \end{bmatrix}$$

Runge-Uutta Method

① Let $t = t_0$ $\qquad \frac{dx}{dt} = g(x,t)$

pick ΔT

Set $x(t) = x(t_0)$

② $k_1 = g(t, x(t))$

$k_2 = g(t + \frac{1}{2}\Delta T, x(t) + \frac{1}{2}\Delta T k_1)$

$k_3 = g(t + \frac{1}{2}\Delta T, x(t) + \frac{1}{2}\Delta T k_2)$

$k_4 = g(t + \Delta T, x(t) + \Delta T k_3)$

③ $x(t + \Delta T) = x(t) + \frac{\Delta T}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

④ let $t = t_0 + \Delta T$ goto ②

<u>MATLAB</u> ODE45