Last time: We estimated $\int_0^1 e^x dx$ using $P_{2,0}(x)$ for $f(x) = e^x$.

Today:

Ex. use the previous exercise to find the 4th degree Taylor Polynomial for e^{-x^2}, & find an estimate of $\int_0^1 e^{-x^2} dx$, as well as a bound on the error.

Sol^n: Computing derivatives of e^{-x^2} would be tedious. Instead, use what we know about e^x.

Let $g(u) = e^u$. We found $P_{2,0}(u) = 1 + u + \frac{u^2}{2}$.

Now, let $u = -x^2$, so $g(-x^2) = e^{-x^2}$.

Then the Taylor polynomial for e^{-x^2} is:

$$P_{2,0}(-x^2) = 1 + (-x^2) + \frac{(-x^2)^2}{2} = 1 - x^2 + \frac{x^4}{2}$$

The error satisfies:

$|g(u) - P_{2,0}(u)| = |R_2(u)| \le \frac{K}{3!}|u|^3$, where $|g^{(3)}(t)| \le K|$.

consider the interval carefully:

$x \in [0,1]$, $u = -x^2 \Rightarrow u \in [-1,0]$.

$g(t) = e^t \Rightarrow |g'''(t)| = e^t \le \underbrace{1}_{K}$ on $[-1,0]$

$$\Rightarrow |R_2(u)| \le \frac{1}{6}|u|^3$$

We can write the error in terms of x.

$$|g(-x)^2 - P_{2,0}(-x^2)| \le |R_2(-x^2)| \le \frac{1}{6}x^6$$

The estimate is:

$$\int_0^1 e^{-x^2} dx \approx \int_0^1 P_{2,0}(-x^2)dx$$

$$= \int_0^1 (1 - x^2 + \frac{x^4}{2})dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_0^1$$

$$= \frac{23}{30}$$

And the error is:

$$\left|\int_0^1 e^{-x^2}dx - \int_0^1 P_{2,0}(-x^2)dx\right| = \left|\int_0^1 e^{-x^2} - P_{2,0}(-x^2)dx\right|$$

$$\le \int_0^1 |e^{-x^2} - P_{2,0}(-x^2)|dx$$

$$\le \int_0^1 \frac{1}{6}x^6 dx$$

$$\le \frac{1}{42}$$

Therefore, $\int_0^1 e^{-x^2}dx \approx \frac{23}{30}$ with error at most $\frac{1}{42}$.

Infinite Series

To estimate e, we found $P_{n,0}(x)$ for e^x and used $e = f(1) \approx P_{n,0}(1) = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

For various values of n, we got

$P_{1,0}(1) = 2$, $P_{3,0}(1) = 2.\overline{6}$, $P_{5,0}(1) = 2.71\overline{6}$.

$P_{11,0}(1) = 2.718281826$ $\quad P_{17,0}(1) \approx 2.718281\ldots$

14 digits agreement

More terms $\rightarrow$ better approximation.

Can we get equality?

Not if we stop at a finite number

But yes, if we sum an infinite number of terms.

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$$

or $e = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}$ or $\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right)$

If this seems wierd, think about the following:

$$\frac{1}{9} = 0.\overline{1}$$

$$= \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{10^n}$$

Much the same as we can sum an infinity of numbers, we can do this for functions.

For $f(x)=e^x$, $P_{n,0}(x) = 1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}$

It turns out that we can write.

$$e^x = 1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\dots$$

or $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$ (true for all x)

This is called the Taylor Series of f (or Maclaurin series) centred at 0.

when $x_0 = 0$.

How does this work?

Use Taylor's Thm & Taylor's Inequality:

$$e^x = P_{n,0}(x) + R_n(x), \text{ where } |R_n(x)| \le \frac{K}{(n+1)!}|x|^{n+1}$$

we know K exists: $|f^{(n+1)}(t)| = e^t \; \forall t$.

Consider an arbitrary interval $[a,b]$ that contains 0.

Then $|f^{(n+1)}(t)| = e^t \le \underbrace{e^b}_{K}$ on $[a,b]$

What happens to $|R_n(x)|$ as $n\to\infty$?

View $\frac{|x|^{n+1}}{(n+1)!}$ as a sequence.

we'll show later $\lim_{n\to\infty} \frac{|x|}{(n+1)!} = 0$.

Thus, $R_n(x) \to 0$ as $n \to \infty$ for all x.

And so $P_{n,0}(x) \to e^x$ as $n \to \infty$.

sequence of Taylor Polynomials.