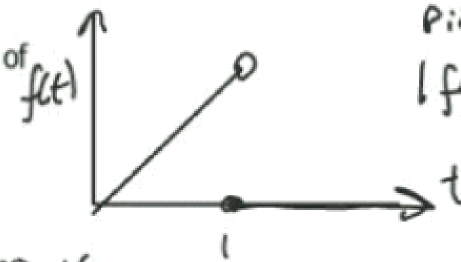
eg. What is the Laplace transform of

$$f(t) = t, 0 \le t < 1$$
$$= 0, t \ge 1$$



Piecewise continuous

$|f(t)| \le Me^{\alpha t}$ for some M and α

$\Rightarrow$ Laplace Transform exists

Sol'n/

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_0^1 e^{-st} \cdot t$$
$$= -\frac{1}{s} e^{-st} t \Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st} dt$$
$$= -\frac{1}{se^s} \quad -\frac{1}{s^2}\left(e^{-st}\Big|_0^1 \right)$$
$$= -\frac{1}{se^s} \quad - \frac{1}{s^2 e^s} + \frac{1}{s^2}$$
$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

-Inverse Laplace

-can apply $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(s)e^{st}\,ds$

(This is an integration in the complex plane. The α depends on the region of convergence

-We can, if $F(s)$ is rational, use tables and partial fraction expansions

↳ ratio of polynomials

eg/ $F(s) = \frac{3s^2 + s - 7}{s^3 - 7s^2}$

$$= \frac{3s^2 + s - 7}{s^2(s-7)} = \frac{(A+B)s^2 + (-7B + C)s - 7C}{s^3 - 7s}$$
$$= \frac{A}{s-7} + \frac{B}{s} + \frac{C}{s^2}$$
$$\therefore C = 1 \quad B = 0 \quad A = 3$$
$$F(s) = \frac{3}{s-7} + \frac{1}{s^2} \Rightarrow f(t) = 3e^{7t} + t, \; t \ge 0$$

B. Application to ODEs

-We need to figure out what happens when we differentiate a function

-Theorem- suppose $f(t)$ is continuous for $t \geq 0$ and satisfies the conditions to have a Laplace transform. As well, suppose $\frac{df}{dt}(t)$ is piecewise continuous on every finite interval in $t \geq 0$. Then, there exists an α such that $\mathcal{L}\{\frac{df}{dt}(t)\}$ exists for $\text{Re}\{s\} > \alpha$ and

$$\mathcal{L}\{\frac{df}{dt}(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

where $f(0)$ is the initial value.

Proof/ $\mathcal{L}\{\frac{df}{dt}\} = \int_0^\infty e^{-st} f'(t)\,dt$ Use integration by parts

$$= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= -f(0) + s \underbrace{\int_0^\infty f(t) e^{-st}\,dt}_{\mathcal{L}\{f(t)\}}$$

$$\therefore \mathcal{L}\{\tfrac{df}{dt}\} = s\mathcal{L}\{f(t)\} - f(0) \quad \text{QED}$$

In the actual proof, you need to track the region of Convergence

-What do the conditions mean?

-Apply this twice to get

$$\mathcal{L}\{\frac{d^2 f}{dt^2}(t)\} = s\,\underbrace{\mathcal{L}\{\frac{df}{dt}(t)\}}_{s\mathcal{L}\{f(t)\} - f(0)} - f'(0)$$

$$\therefore \mathcal{L}\{\frac{d^2 f}{dt^2}(t)\} = s^2\mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

-Similarly, $\mathcal{L}\{\frac{d^n f}{dt^n}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{[n-1]}(0)$

-eg/ $\mathcal{L}\{\cos\omega t\}$ = If $f = \sin\omega t$, $f(0)=0$, $f' = \omega\cos\omega t$

use $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$$\therefore \mathcal{L}\{\omega\cos\omega t\} = s\frac{\omega}{s^2+\omega^2} - 0$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

-We can now solve ODEs with initial values using algebraic techniques

-Eg/ $\dot{x} = x$, $x(0) = 1$ $\mathcal{L}\{\dot{x}\} = \mathcal{L}\{x\}$

$$s\overline{X}(s) - \underbrace{1}_{x(0)} = \overline{X}(s)$$

$$\overline{X}(s) = \frac{1}{s-1}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\therefore x(t) = e^t, \quad t \geq 0$$

Eg/ $\ddot{y} = 1$, $y(0) = 0$, $\dot{y}(0) = 1$

$$\mathcal{L}\{\ddot{y}\} = \mathcal{L}\{1\}$$

$$s^2\overline{Y}(s) - sy(0) - y'(0) = \frac{1}{s}$$

$$\therefore s^2\overline{Y}(s) = \frac{1}{s} + 1$$

$$\overline{Y}(s) = \frac{1}{s^3} + \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^3}\right\}$$

$$= t + \frac{t^2}{2}, \quad t \geq 0$$

Eg/ $\ddot{y} + 9y = 1$, $y(0) = 0$, $\dot{y}(0) = 1$

$$\mathcal{L}\{\ddot{y} + 9y\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\{\ddot{y}\} + 9\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$s^2\overline{Y}(s) - sy(0) - y'(0) + 9(s\overline{Y}(s) - y(0)) = \frac{1}{s}$$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$A(s^2+9) + s(Bs+C) = 1$$

$$A + B = 0$$

$$9A = 1 \qquad B = -\frac{1}{9}$$

$$A = \frac{1}{9} \qquad C = 0$$

$$\overline{Y}(s) = \frac{1}{s(s^2+9)} + \frac{1}{s^2+9}$$

$$\overline{Y}(s) = \frac{1}{9s} - \frac{1}{9}\left(\frac{s}{s^2+3^2}\right) + \frac{1}{3}\left(\frac{3}{s^2+3^2}\right)$$

$$\therefore y(t) = \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{1}{3}\sin 3t, \quad t \geq 0$$