Ex. If $f(x,y) = xe^y$, find the direction of the max rate of change and its value from $(2,0)$.

Sketch the level curve on which $(2,0)$ lies, the tangent at $(2,0)$, and the gradient vector at $(2,0)$.

Soln. The direction of the max. rate of change is $\nabla f(2,0)$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (e^y, xe^y)$$

$$\nabla f(2,0) = (1,2)$$

The rate of change is $\|\nabla f(2,0)\| = \sqrt{1^2+2^2} = \sqrt{5}$

Level curves: $f(x,y) = k \Rightarrow xe^y = k$.

$$\Rightarrow e^y = \frac{k}{x}$$

$$\Rightarrow y = \ln\left(\frac{k}{x}\right)$$

OR $\boxed{y = \ln(k) - \ln(x)}$ $(*)$

The point $(2,0)$: $0 = \ln k - \ln 2 \Rightarrow k = 2$

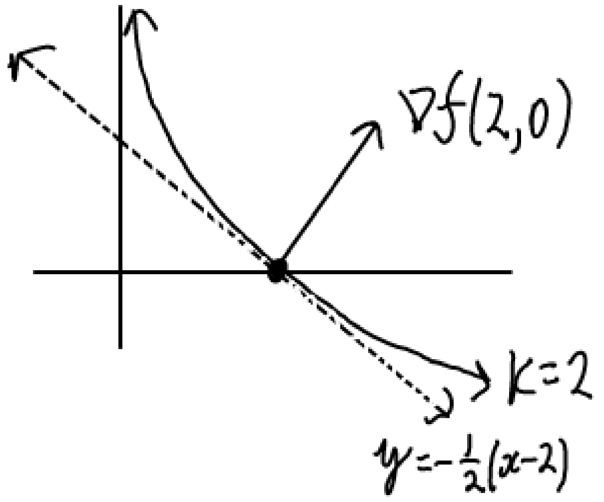
$$\Rightarrow y = \ln(2) - \ln x$$

tangent at $(2,0)$: $y - 0 = \frac{dy}{dx}(2)(x-2)$ $\qquad \frac{dy}{dx} = -\frac{1}{x}$

$$y = -\frac{1}{2}(x-2) \qquad \left(\text{slope} = -\frac{1}{2}\right)$$

Slope of $\nabla f(2,0) = (1,2) \rightarrow \frac{\text{rise}}{\text{run}} = 2$.

$\Rightarrow$ slope of tangent to L.C. at $(2,0)$ & slope of $\nabla f(2,0)$ are negative reciprocals $\Rightarrow$ orthogonal.



Suppose we want to follow a path in which we are always moving in the direction of the max. rate of change. (path of steepest ascent). Must travel on a path orthogonal to <u>all</u> level curves.

Eqn of level curves (*). $\frac{dy}{dx} = -\frac{1}{x}$

A path orthogonal to the LCs has slope satisfying

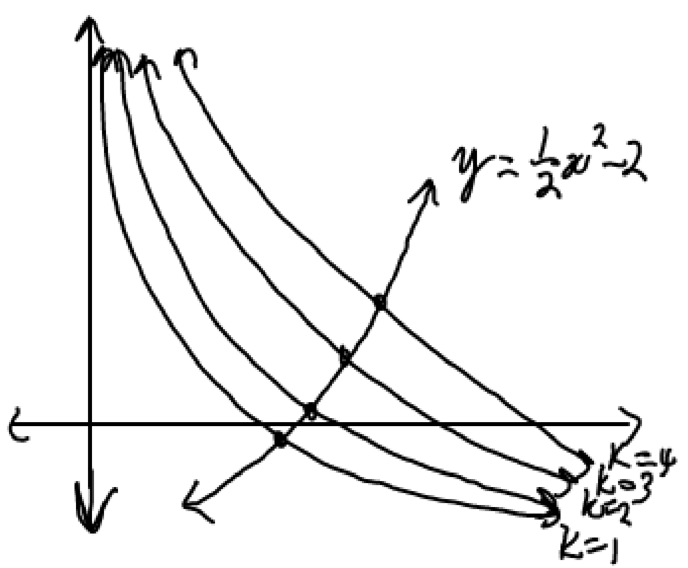
$m\left(\frac{-1}{x}\right) = -1 \Rightarrow m = x$

The path satisfies the DE $\underbrace{\frac{dy}{dx}}_{\text{slope}} = x$.

To solve: $y = \frac{1}{2}x^2 + C$

Passes through $(2,0)$: $0 = \frac{1}{2}(2)^2 + C \Rightarrow C = -2$

$$\boxed{y = \frac{1}{2}x^2 - 2}$$



$y = \ln k - \ln x$

The gradient vector as a Vector Field

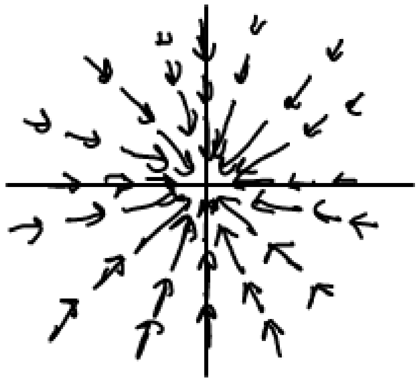
Since $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$

input x, y components f_x, f_y

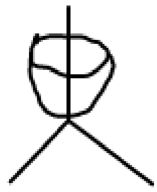
∇f is an example of a <u>vector field</u>.

At each point (x,y), there is an associated vector.

<u>ex's</u>. electric/magnetic fields, gravitational field.



Sketch the vector field
∇f for $f(x,y) = x^2 + y^2$
$\nabla f = (2x, 2y) = 2(x,y)$



At (x,y), ∇f points in direction (x,y).
(away from origin). Magnitudes get larger further from the origin.