Last time: Infinite Series

- A series $\sum a_k$ converges iff its sequence of partial sums $\{S_n\}$ converges.

(if $\lim_{n\to\infty} S_n = S$, then the sum of the series is S)
where $S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$.

- Geometric Series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ if $|r|<1$ and diverges otherwise.

- The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent

- Every series has two associated sequences:
  1) that of terms $\{a_k\}$
  2) that of its partial sums $\{S_n\}$

Test for Divergence

If $\lim_{k\to\infty} a_k \neq 0$ or does not exist, then $\sum a_k$ is divergent.

The Integral Test

Suppose f is cts, positive, and decreasing on $[k_0, \infty)$ and let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$

Then $\sum_{k=k_0}^{\infty} a_k$ is convergent if and only if

$$\int_{k_0}^{\infty} f(x)\,dx \text{ is convergent.}$$

Ex. Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Look at $\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} \ln|x| \Big|_1^t$

$$= \lim_{t\to\infty} \ln t - \ln 1$$

$$\to \infty$$

The improper integral diverges, and so does the series by the integral test.

Ex. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx$$

$$= \lim_{t\to\infty} -\frac{1}{x} \Big|_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{t} + 1$$

$$= 1 \Rightarrow \text{convergent}$$

By the integral test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

Note: 1 is not the sum of the series -

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots > 1$$

Test says - conv. of $\int \leftrightarrow$ conv. of $\sum$.

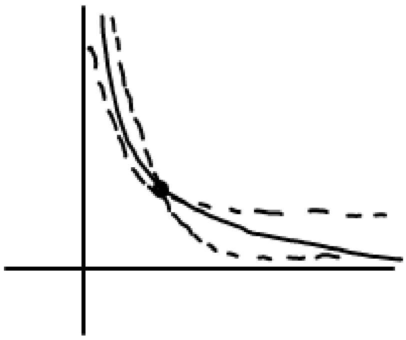
In fact, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (Euler)

P-series

For which values of p does $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge?

Recall: $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges iff $\boxed{p>1}$

$-\frac{1}{x}$ ---- $\frac{1}{x^2}$ ---- $\frac{1}{\sqrt{x}}$



$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ Converges iff } p>1}$$

The comparison test

Suppose $\sum a_k$ & $\sum b_k$ are series with positive terms & $a_k \leq b_k$ for all k Then:

1) if $\sum b_k$ is convergent, $\sum a_k$ must be convergent.
2) if $\sum a_k$ is divergent, $\sum b_k$ must be divergent.

Idea: use series that we know the convergence properties of (p-series, geometric series) as comparison.

Ex. $\sum_{k=0}^{\infty} \frac{1}{2^k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \ldots$

Series is similar to $\sum_{k=0}^{\infty} \frac{1}{2^k}$. When k is large, the '+1' in the denominator is insignificant.

Since $2^k+1 > 2^k \Rightarrow \frac{1}{2^k+1} < \frac{1}{2^k} \; \forall k$

& $\sum \frac{1}{2^k}$ is a convergent geometric series, by the Comp test, $\sum \frac{1}{2^k+1}$ is also convergent.

Ex. $\sum_{k=1}^{\infty} \frac{k+1}{k^2} = 2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{6} + \ldots$

Since $k+1 > k \Rightarrow \frac{k+1}{k^2} > \frac{k}{k^2} = \frac{1}{k}$ & $\sum \frac{1}{k}$ is a divergent series, then $\sum \frac{k+1}{k^2}$ is divergent by C.T.

Ex. $\sum_{k=1}^{\infty} \frac{k-1}{k^2}$

$\frac{k-1}{k^2} < \frac{k}{k^2} = \frac{1}{k}$ & $\sum \frac{1}{k}$ is divergent.

Inequality points in the wrong direction.
no conclusion (need another test)

Limit Comparison Test

Suppose $\sum a_k$ & $\sum b_k$ are series with positive terms. If:

$$\lim_{k \to \infty} \frac{a_k}{b_k} = c, \quad 0 < c < \infty$$

then either both series converge or both diverge.

Ex. $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{k-1}{k^2}}_{a_k}$ For the comparison term b_k, look at the dominant behaviour in numerator & denominator when k is large.

Let $b_k = \frac{k}{k^2} = \frac{1}{k}$.

$$\frac{a_k}{b_k} = \frac{k-1/k^3}{1/k} = \frac{k^2-k}{k^2} = 1-\frac{1}{k} \to 1 \text{ as } k \to \infty$$

Since $\sum \frac{1}{k}$ is divergent, by the L.C.T,

$\sum \frac{k-1}{k^2}$ is divergent as well.

Exercises: 1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+1}}$ 2) $\displaystyle\sum_{k=1}^{\infty} \frac{2k-1}{k^3+k^2+1}$ 3) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k^4-1}$

4) $\displaystyle\sum_{k=1}^{\infty} \frac{\arctan k}{k}$