More Complex Stuffs

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

de Moivre's Theorem: For any $n \in \mathbb{Z}$,

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Proof: For $n \geq 0$, we prove by induction.

Base case: $n=0$. $(\cos\theta + i\sin\theta)^0 = 1$

$\cos 0 + i\sin 0 = 1$

Ind. Hyp.: for some $k \geq 0$, $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Ind. Step: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

by ind hyp. $= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta)$

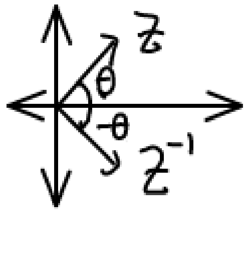
$= \cos((k+1)\theta) + i\sin((k+1)\theta)$

For $n<0$, $(\cos\theta + i\sin\theta)^n$

$= ((\cos\theta + i\sin\theta)^{-n})^{-1}$

$= (\cos(-n\theta) + i\sin(-n\theta))^{-1}$ since $-n > 0$, and we use the result above.

$= \cos(n\theta) + i\sin(n\theta)$ (negate the angle).



Complex exponentials: $\cos\theta + i\sin\theta = e^{i\theta}$

Let $f(\theta) = (\cos\theta + i\sin\theta)e^{-i\theta}$

$$\frac{df(\theta)}{d\theta} = (\cos\theta + i\sin\theta)\frac{d}{d\theta}e^{-i\theta} + \frac{d}{d\theta}(\cos\theta + i\sin\theta)$$

$$= (\cos\theta + i\sin\theta)(-i)e^{-i\theta} + (-\sin\theta + i\cos\theta)e^{-i\theta}$$

$$= e^{-i\theta}(-i\cos\theta + \sin\theta - \sin\theta + i\cos\theta) = 0$$

So $f(\theta) = C$ for some $C \in \mathbb{C}$.

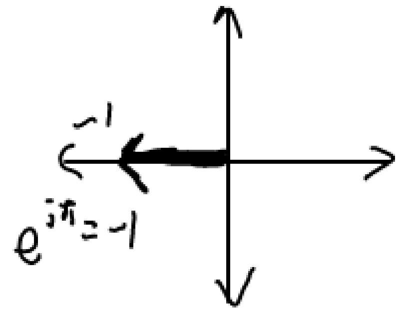
$f(0) = (\cos 0 + i\sin 0)e^{0} = 1$ So $C = 1$

$(\cos\theta + i\sin\theta)e^{-i\theta} = 1$

$\cos\theta + i\sin\theta = e^{i\theta}$

$$\boxed{\begin{aligned} z &= a+bi \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}}$$

$(e^{i\theta})^n = e^{in\theta}$



$e^{i\pi} = -1$

$e^{i\pi} + 1 = 0$

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n-th roots

$z^n = a$ for some constant $a \in \mathbb{C}$.

z is an n-th root of a.

Suppose $a = re^{i\theta}$. Let $z = se^{i\phi}$.

$$z^n = s^n e^{in\phi} = a$$

$\Rightarrow r = s^n \Rightarrow s = \sqrt[n]{r}$ (real +ve n-th root of r)

$$n\phi = \theta + 2\pi k, \quad k \in \mathbb{Z}.$$

$$\phi = \frac{\theta + 2\pi k}{n}, \quad k \in \mathbb{Z}$$

$$\boxed{\phi = \frac{\theta}{n} + \frac{2\pi}{n}k, \ k \in \mathbb{Z}}$$

k_1, k_2 give the same angle if

$$\left(\frac{\theta}{n} + \frac{2\pi}{n}k_1\right) - \left(\frac{\theta}{n} + \frac{2\pi}{n}k_2\right) = 2\pi l, \ l \in \mathbb{Z}$$

$$\frac{1}{n}(k_1 - k_2) = l$$

$$k_1 - k_2 = nl$$

$$k_1 \equiv k_2 \pmod{n}$$

We only need $k = 0, 1, 2, \ldots, n-1$.

$T = \{ re^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)} \mid k = 0, 1, \ldots, n-1 \}$ $\quad S = \{ z \mid z^n = a \}$

We proved $S \subseteq T$.

For $T \subseteq S$: $z \in T \Rightarrow z = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)}$ for some $k = 0, \ldots, n-1$

$$z^n = re^{in\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)}$$

$$= re^{i(\theta + 2\pi k)}$$

$$= re^{i\theta} = a$$