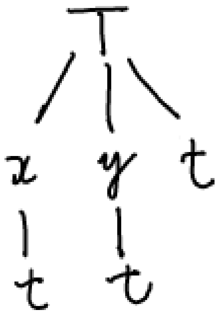
Other forms of the chain rule

Consider again the ant walking on the plate with pos. $(x(t), y(t))$. Now let the temp on the plate be a fcn. of time also, so $T = T(x, y, t)$.
Find the r.o.c. of temp wrt time felt by the ant.

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial t}$$

$\frac{\partial T}{\partial t}$ represents the r.o.c. of temp w.r.t. time at a fixed posn (x, y) on the plate.
We can draw the chain of dependence:



Another form: $z = f(x, y)$, $x = x(s, t)$, $y = y(s, t)$.

z can be expressed as a fcn. of s & t.
→ can compute $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

To find $\frac{\partial z}{\partial s}$, follow all paths from z to s.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{dy}{\partial s}$$

Exercise: If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then express $\frac{\partial z}{\partial r}$ & $\frac{\partial z}{\partial \theta}$ in terms of $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Second Partials & the Chain Rule

Recall the single variable case: $y = f(x)$, $x = g(t)$. Then y is a fcn. of t and:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

What is $\frac{d^2y}{dt^2}$?

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right)$$

product Rule $\longrightarrow$
$$= \underbrace{\frac{d}{dt}\left(\frac{dy}{dx}\right)}\frac{dx}{dt} + \frac{dy}{dx}\underbrace{\frac{d}{dt}\left(\frac{dx}{dt}\right)}_{\frac{d^3x}{dt^2}}$$

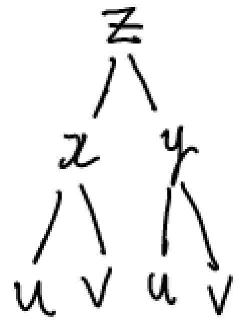
$\frac{dy}{dx}$ is a fcn. of x, which is a fcn. of t.

Let $w = \frac{dy}{dx}$ $\quad \begin{array}{c} w \\ | \\ x \\ | \\ t \end{array}$ $\quad$ Then $\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$

$$\Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\left(\frac{dx}{dt}\right) = \frac{d^3y}{dx^2}\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

Now, let $z = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$.



Find $\frac{\partial^2 z}{\partial u^2} \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

Then $\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)$

Product Rule $\rightarrow$ $= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial z}{\partial x}\underbrace{\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u}\right)} \rightarrow \frac{\partial^2 x}{\partial u^2}$

$+ \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u} + \frac{\partial z}{\partial y}\underbrace{\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right)} \rightarrow \frac{\partial^2 y}{\partial u^2}$

The partials $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)$ & $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$ require care.

$\frac{\partial z}{\partial x}$ is a fcn of x, y, which in turn are fcn. of u, v.

$\frac{\partial z}{\partial x} = w$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}(w) = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial u}$$

similarly, $\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial u}$

Put it together:

$$Z_{uu} = Z_{xx}(X_u)^2 + \underbrace{Z_{xy} X_u Y_u} + Z_x X_{uu} + Z_y Y_{uu} + \underbrace{Z_{yx} Y_u X_u} + Z_{yy}(Y_u)^2$$