Cylindrical Coordinates

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

Jacobian: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

<u>Ex.</u> Let $D = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le 2,\ 0 \le y \le \sqrt{4-x^2},\ -2 \le x \le 2\}$

Evaluate $\displaystyle\iiint_D x^2+y^2\,dV$.

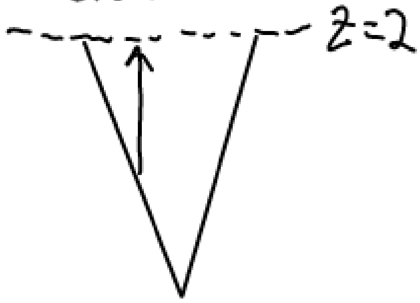
<u>Soln</u>: As an iterated integral (in cartesian coords)

$$\iiint_D x^2+y^2\,dV = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} x^2+y^2\,dz\,dy\,dx$$

Looks tedious, Easier way → transform to cylindrical coords.

What is the region D?

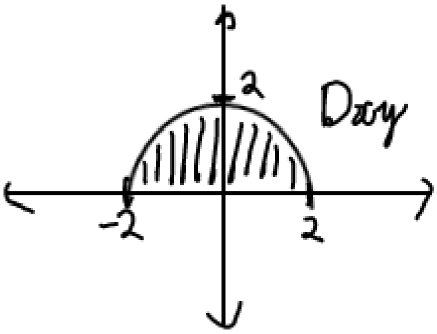
Cross-section



$z = \underbrace{\sqrt{x^2+y^2}}_{r}$ is a cone.

$x^2+y^2=4$

Now, look at D_{xy} $\quad 0 \le y \le \sqrt{4-x^2}$
$\quad -2 \le x \le 2$



D is half of a cone.

To describe in cylindrical coordinates, we have:

$r \le z \le 2,$ $\quad$ ($z = r$ is the cone)
$0 \le r \le 2,$
$0 \le \theta \le \pi$

Then, $\displaystyle\iiint_D \underbrace{x^2+y^2}_{r^2}\,dV = \int_0^{\pi}\int_0^{2}\int_r^{2} r^2 \cdot \underbrace{r}_{\text{Jacobian}}\,dz\,dr\,d\theta$

$$= \int_0^{\pi} d\theta \int_0^{2} \left[r^3 z \Big|_r^2 \right] dr = \pi \int_0^{2} 2r^3 - r^4 \, dr$$

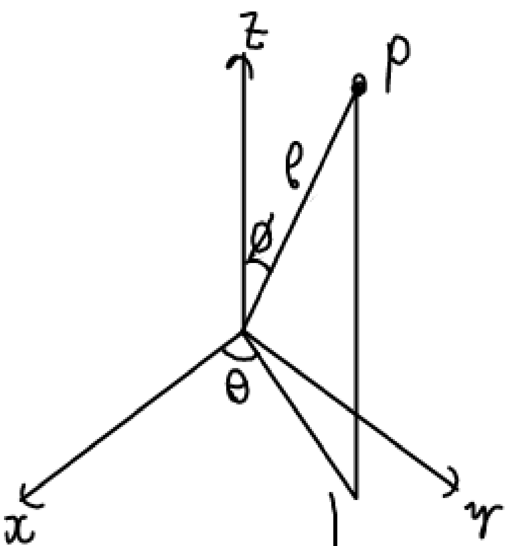
$$= \pi \left[\frac{1}{2} r^4 - \frac{r^5}{5} \Big|_0^2 \right]$$

$$= \frac{8\pi}{5}$$

Spherical Coordinates

Represent a point P by (ρ, ϕ, θ)

rho phi theta



ρ - distance from origin to P.

ϕ - angle btwn. positive z-axis and line from origin to P.

θ - same as cylindrical polar.

Restrictions: $\rho \geq 0$

$0 \leq \phi \leq \pi$

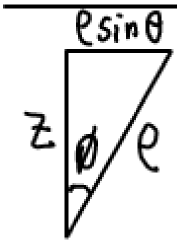
$0 \leq \theta \leq 2\pi$

$\phi = 0 \to$ pos. z-axis

$\theta = \pi/2 \to xy$-plane

$\phi = \pi \to$ neg. z-axis

Relationship btwn. Cartesian & spherical coords.



$\cos\phi = \frac{z}{\rho} \Longrightarrow \boxed{z = \rho\cos\phi}$

$\boxed{\rho\sin\theta = r} \leftarrow$ from cylindrical

$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \end{aligned}}$$

Also, $\boxed{\rho = \sqrt{x^2+y^2+z^2}}$, $\cos\phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

, $\tan\theta = y/x$

Note: $\rho = c$, (c const) represents a sphere of radius c.

$\theta = c$, is a plane in $\mathbb{R}^3$

$\phi = c$, represents a <u>cone</u>.

(in particular, $\phi = \pi/4$ is $z = \sqrt{x^2+y^2}$)

$z = \rho\cos\frac{\pi}{4} = \rho/\sqrt{2}$

$$\left.\begin{array}{l} x = \rho\sin\pi/4\cos\theta = \rho/\sqrt{2}\cos\theta \\ y = \rho\sin\pi/4\sin\theta = \rho/\sqrt{2}\sin\theta \end{array}\right\} \quad x^2+y^2 = \frac{\rho^2}{2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) \Rightarrow \sqrt{x^2+y^2} = \frac{\rho}{\sqrt{2}} = z$$

<u>Ex</u>: Show that the volume of a ball of radius R is $\frac{4\pi R^3}{3}$.

Take the ball centered at the origin.

eqn. $x^2+y^2+z^2 = R^2$

In spherical coordinats: $\rho = R$

The ball is described by: $0 \le \rho \le R$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$\boxed{\text{Jacobian: } \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi}$$

The volume is $V = \iiint_D dV$

Jacobian

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{R} \rho^2 \, d\rho$$

$$= \frac{4\pi R^3}{3}$$

Excercise: Describe

$D = \{(x, y, z) \mid 1 \le x^2 + y^2 + z^2 \le 4, x \ge 0, y \ge 0, z \ge 0\}$

in spherical coordinates.