1. Sets
2. Quantifiers
3. Nested Quantifiers

$\frac{3x}{2} \in \mathbb{Z}$ so $2 | x$ ✓

$\frac{12x}{8} \in \mathbb{Z}$ so $8 | x$ ✗

let $x=4$, $8 \nmid x$

when can we say $\frac{ax}{b} \in \mathbb{Z} \Rightarrow b | x$?
Only true if a, b have no common factors other than 1.

Power Sets:
$P(S)$ is the set of all subsets of S.

$P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$P(\mathbb{N})$ = all subsets of positive integers.

$\{1,3\} \subseteq P(\{1,2,3\})$ ✗ $\quad \{\{1,3\}\} \subseteq P(\{1,2,3\})$ ✓

$\{1,3\} \in P(\{1,2,3\})$ ✓

$P(\{1,3\}) \subseteq P(\{1,2,3\}) \subseteq P(\mathbb{N})$

S of size n, $P(S)$ has size 2^n.
(each element is either in the set or not in the set)

Cardinality $|A|$ = # of elements in the set.

$|P(S)| = 2^{|S|}$

Quantifiers

- Universal quantifier "for all" $\forall$
- Existential quantifier "there exists" $\exists$

Need a universe

$\forall x, x=3 \longrightarrow \forall x \in \mathbb{R}, x=3$ F

$\exists y, y<1 \longrightarrow \exists y \in \mathbb{R}, y<1$ T

$\exists y \in \mathbb{N}, y<1$ F

$\forall x \in \{3\}, x=3$ T

Example: Defn of "divides": $a|b$ if there exists $k \in \mathbb{Z}$ such that $b=ak$.

Example: DIC "if $a|b$ and $a|c$, then for all $x, y \in \mathbb{Z}$, $a|(bx+cy)$"

Proving things with quantifiers...

① To prove "$\exists x \in S, p(x)$", construct/find x in S that works.

Example: There exists $x \in \mathbb{R}$ such that $x^2-x-1=0$.

By quadratic formula, $x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$

$x=\frac{1+\sqrt{5}}{2}$ is real.

This proof is not correct since we assumed the conclusion.

Proof: consider $x = \frac{1 \pm \sqrt{5}}{2}$. Then

$$x^2 - x - 1 = \left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1+\sqrt{5}}{2} - 1 = \frac{3+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} - 1 = 0$$

② To prove "$\forall x \in S, p(x)$", pick an arbitrary instance of x in S, show that $p(x)$ is true.

Example: For all odd integer n, $4 \mid (n^2-1)$.

proof: Let n be any odd integer. (arbitrary instance can rep all odd integers.)