Last Time: $(*)\ \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \dots,\ |x| < 1$

Ex: Use this to find the Maclaurin series for $\arctan x$ & find the interval of convergence.

Sol^n: Use $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \Leftrightarrow \arctan x = \int \frac{1}{1+x^2}\,dx$

Sub $-x^2$ for x in $(*)$: $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - \dots$

which is true for $|-x^2| < 1 \Leftrightarrow |x| < 1$.

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + C$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

To get C, let $x=0$, $\arctan 0 = C \Rightarrow C = 0$.

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

The radius of convergence doesn't change with integration $\Rightarrow R = 1$.

(Series in $(*)$ is divergent at endpoints).

Check endpoints:

$\underline{x=1}$: $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ which converges by the AST.

$\underline{x=-1}$: $\sum_{k=0}^{\infty} \frac{(-1)^k(-1)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ which converges by AST.

Put $x=1$ into the series

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots$$

Exercise: Find the Maclaurin Series for $\ln(1+x)$ and its interval of convergence.

The Binomial Series

If k is any real number, then $\boxed{(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \text{ for } |x|<1}$

(This is the Taylor Series about 0 for $(1+x)^k$)

We can use the binomial series expansion for $g(u) = \frac{1}{\sqrt{1+u}}$ to find the Taylor Polynomial $P_{7,0}(x)$ for $f(x) = \arcsin x$.

$$g(u) = (1+u)^{-1/2} = 1 - \frac{1}{2}u + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}u^2 + \dots$$

Since $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, then $\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx$,

So first substitute $-x^2$ for u above and then integrate.

$$\ldots\ P_{7,0}(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{117}x^7$$

(converges for $|-x^2|<1 \Rightarrow |x|<1$)

The Big-O Order Symbol

- Gives us a concise way to express Taylor Series.
- Helps compute Taylor Polynomial for complicated functions (products, compositions)

<u>Defn</u>: Given two functions f & g, we say:

"f is of order g as $x \to x_0$" and write:

$f(x) = O(g(x))$ as $x \to x_0$.

If there exists a constant $A>0$ such that:

$$|f(x)| \le A|g(x)|$$

on some interval around x_0.

<u>Examples</u>: 1) Since $|x^2| \le |x|$ $(A=1)$ on $[-1,1]$, then $x^2 = O(x)$ as $x \to 0$.

Also, $x^2 = O(x^2)$ as $x \to 0$,

$x^2 = O(1)$ as $x \to 0$

const.

But $|x^2| \leq |x^3|$ on $[-1,1]$ so $x^2 \neq O(x^3)$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\quad |\sin x| \leq |x| \; \forall x.$

$\sin x = O(x)$ as $x \to 0$ (all higher terms $\to 0$ faster than the first).

<u>Note</u>: $10 \sin x$, $10^{10} \sin x$ are also of order x.

Magnitude irrelevant - only rate of growth.

Big-O & Taylor's Inequality

Recall $f(x) = P_{n,x_0}(x) + R_n(x)$,

where $|R_n(x)| \leq \frac{K}{(n+1)!}|x-x_0|^{n+1}$

We can write this as $R_n(x) = O\left[(x-x_0)^{n+1}\right]$

Has advantage of identifying the power of $(x-x_0)$ without computing K.

<u>Ex.</u> 1) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

$$e^x = 1 + x + \frac{x^3}{2} + \frac{x^3}{6} + O(x^4) \text{ as } x \to 0$$

$\sin x = x - \frac{x^3}{3!} + O(x^5)$ as $x \to 0$

Then $P_{4,0}(x)$ for $e^x + \sin x$ is:

$$e^x + \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4)\right) + \left(x - \frac{x^3}{3!} + O(x^5)\right)$$

$$= 1 + 2x + \frac{x^2}{2} + O(x^4) + O(x^5) \leftarrow \text{redundant.}$$

$$= 1 + 2x + \frac{x^3}{2} + O(x^4) \text{ as } x \to 0.$$