Alternating Series Test

We can use these ideas for Taylor Series as well as estimating functions or integrals. (instead of Taylor's inequalities).

Ex. Estimate $1/e$ to within $1/100$.

Idea: Let $f(x)=e^{-x}$, then $f(1)=e^{-1}=\frac{1}{e}$.

The Taylor Polynomial for e^{-x} is:

$$e^{-x}=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

Then

$$\frac{1}{e}=e^{-1}=1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\cdots$$

Since the series is alternating, we see that we can estimate $\frac{1}{e}$ using the first three terms with error less than $1/120$: $1/e \approx \frac{1}{2}-\frac{1}{6}+\frac{1}{24}=\frac{12-4+1}{24}=\frac{9}{24}$

Absolute & Conditional Convergence

Def^n: A series $\sum a_k$ is absolutely convergent if $\sum |a_k|$ is convergent.

Def^n: A series $\sum a_k$ is conditionally convergent if $\sum a_k$ is convergent but $\sum |a_k|$ is divergent.

Note: These concepts are applicable to series that contain some negative terms.

Ex. $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$ is absolutely convergent since

$\sum \left| \frac{(-1)^{k-1}}{k^2} \right| = \sum \frac{1}{k^2}$ is a convergent p-series.

$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is conditionally convergent, since it converges by AST but

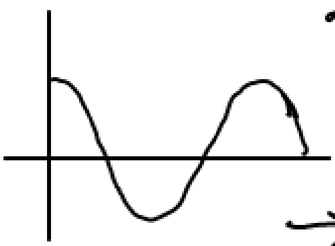
$\sum \left| \frac{(-1)^{k-1}}{k} \right| = \sum \frac{1}{k}$ is the harmonic series, ...

which is divergent.

($\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^p}$ conditionally convergent for which values of p?) $0 < p \leq 1$.

Theorem: If $\sum a_k$ is absolutely convergent, then it is convergent (in the ordinary sense).

Ex. $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$



This is not an alternating series — the sign changes irregularly.

→ Look at the absolute convergence

$\sum_{k=1}^{\infty} \left| \frac{\cos k}{k^3} \right|$ since $|\cos k| \leq 1 \Rightarrow \left| \frac{\cos k}{k^3} \right| \leq \frac{1}{k^3}$

& $\sum \frac{1}{k^3}$ is a convergent p-series.

The series $\sum \left|\frac{\cos k}{k^3}\right|$ is convergent.
By the Theorem, $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ is convergent.

The Ratio Test

Given the series $\sum a_k$, suppose that

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$, then

1) If $L<1$, $\sum a_k$ is absolutely convergent
2) If $L>1$, $\sum a_k$ is divergent.
3) If $L=1$, There is no conclusion
(could be abs. conv., cond. conv., or divergent.)

Note: The test is checking whether the series behaves like a geometric series for large K.

- useful fo factors like 2^k & $K!$
- We use the Ratio Test to determine intervals of convergence for power series.

Ex. $\sum_{k=0}^{\infty} \frac{1}{k!} = 1+1+\frac{1}{2}+\frac{1}{6}+\cdots$ (recall this $= e$)

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{1/(k+1)!}{1/k!} = \frac{k!}{(k+1)!} = \frac{\cancel{k!}}{(k+1)\cancel{k!}} = \frac{1}{k+1} \to 0 \text{ as } k\to\infty$$

$\Rightarrow$ The series is absolutely convergent.

Ex. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 3^{k+1}}{k^2 \cdot 2^k}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k-1} \cdot 3^{k+2}}{(k+1)^2 \cdot 2^{(k+1)}} \cdot \frac{k^2 \cdot 2^k}{(-1)^{k-1}\, 3^{k+1}}\right|$$

$$= \frac{3^{k+2}}{3^{k+1}} \cdot \frac{2^k}{2^{k+1}} \cdot \left(\frac{k}{k+1}\right)^2$$

$$= \frac{3}{2}\left(\frac{1}{1+1/k}\right)^2 \to \frac{3}{2} \text{ as } k \to \infty.$$

The series diverges.