Theorem

Let $p(x)$ & $q(x)$ be polynomials with $\deg(p(x)) < \deg(q(x))$ and $q(x) = (1-\theta_1 x)^{m_1} \cdots (1-\theta_k x)^{m_k}$, where $m_1, m_2, \ldots \in \mathbb{N}$, $\theta_1, \theta_2, \ldots \in \mathbb{C}$ distinct

Then there exist polynomials $A_1(x), \ldots, A_k(x)$ with $\deg(A_1) < m_1, \ldots, \deg(A_k) < m_k$ and

$$[x^n]\frac{p(x)}{q(x)} = A_1(n)\theta_1^n + A_2(n)\theta_2^n + \ldots + A_k(n)\theta_k^n \text{ for all } n \geq 0.$$

(3.1.3 from notes)

General Problem

Given a recurrence $a_n = q_1 a_{n-1} + q_2 a_{n-2} + \ldots + q_k a_{n-k}$, $n \geq k$ and initial values for $a_0, a_1, \ldots, a_{k-1}$, determine a_n explicitly.

The characteristic polynomial for such a recurrence is $1 - q_1 x - q_2 x^2 - \ldots - q_k x^k$.

Lemma: Given such a recurrence, let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

Then $A(x) = \frac{p(x)}{q(x)}$, where q is the char. poly. and $\deg(p) < k$.

pf: We need to show that $A(x)q(x)$ is a polynomial of degree $< k$. let $n \geq k$, then $[x^n]A(x)q(x)$

$$\begin{aligned}
&= [x^n](a_0 + a_1 x + a_2 x^2 + \ldots)(1 - q_1 x - q_2 x^2 - \ldots - q_k x^k) \\
&= a_n - q_1 a_{n-1} - q_2 a_{n-2} - \ldots - q_k a_{n-k} \\
&= 0
\end{aligned}$$

So $\deg(A(x)q(x)) < k$ as required.

This proves the Lemma, NB. One can compute each coefficient of $p(x)$ between x^0 & x^{k-1} using the same ideas.

Combining this Lemma with thm 3.1.3, we have

$$a_n = [x^n]A(x) = [x^n]\frac{p(x)}{q(x)} \quad \text{where } \deg(p) < k$$

q is char poly

$$a_n = A_1(n)\theta_1^n + \dots + A_k(n)\theta_k^n \quad \text{where } \theta_1, \dots, \theta_j \text{ distinct,}$$

$m_1, \dots, m_j \in \mathbb{N}$.

- $q(x) = (1-\theta_1 x)^{m_1} \dots (1-\theta_j x)^{m_j}$
- A_i is a poly. of degree $< m_i$

<u>E.g.</u> Solve the recurrence defined by

$a_0 = 1$, $a_1 = -1$, $a_2 = 17$

$a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}$ (factor theorem)

<u>Soln</u>: The char. poly. is $q(x) = 1 - x - 8x^2 + 12x^3$

$$= (1-\underline{2}x)^{2}(1+\underline{3}x)^{1}$$

So $\theta_1 = \underline{2}$, $\theta_2 = \underline{-3}$, $m_1 = \underline{2}$, $m_2 = \underline{1}$

So we know there are polynomials $A_1(x)$, $A_2(x)$ where $\deg(A_1) < 2$, $\deg(A_2) < 1$

and $a_n = A_1(n)2^n + A_2(n)(-3)^n$ for all n

Let $A_1(x) = \alpha x + \beta$, $A_2(x) = \gamma$

So $a_n = (\alpha n + \beta)2^n + \gamma(-3)^n$

Using our values for a_0, a_1, a_2, we have

$$a_0 = 1 = \beta + \gamma$$
$$a_1 = -1 = 2(\alpha+\beta) - 3\gamma$$
$$a_2 = 17 = 4(2\alpha+\beta) + 9\gamma$$

$$\begin{array}{ccc|c} \alpha & \beta & \gamma & \\ 0 & 1 & 1 & 1 \\ 2 & 2 & -3 & -1 \\ 8 & 4 & 9 & 17 \end{array}$$

$\alpha = 1$
$\beta = 0$
$\gamma = 1$
is the only soln

So $a_n = n2^n + (-3)^n$ $\blacksquare$

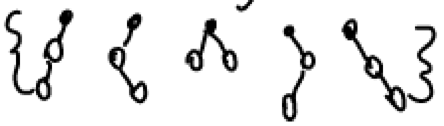
A Binary Tree is either • empty or

• a 'root' vertex together with a 'left' child & a 'right' child, each of which is a (possibly empty) binary tree.

e.g. 0 vertices: $\{\varepsilon\}$

1 vertices: $\{\bullet\}$

2 vertices: 

3 vertices: 

Q. How many binary trees exist on n vertices?

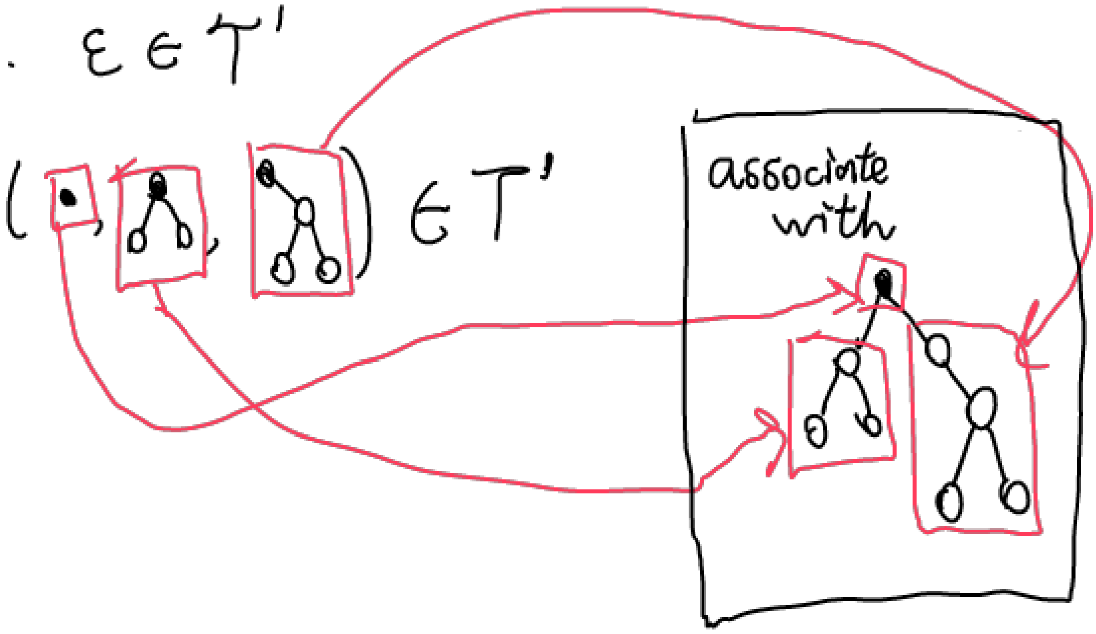
Let $\mathcal{T}$ = {binary trees}

let $w(S)$ = # vertices of S for each $S \in \mathcal{T}$.

let $T(x) = \Phi_{\mathcal{T}}(x)$. Then $[x^n]T(x)$ = # binary trees on n vertices

Let $\mathcal{T}' = \{\varepsilon\} \cup \{\bullet\} \times \mathcal{T} \times \mathcal{T}$

eg. $\varepsilon \in \mathcal{T}'$



idea: $\mathcal{T} = \{\varepsilon\} \cup \{\bullet\} \times \mathcal{T} \times \mathcal{T}$

so $\Phi_{\mathcal{T}}(x) = \Phi_{\{\varepsilon\}}(x) + \Phi_{\{\bullet\}}(x)\,\Phi_{\mathcal{T}}(x)\,\Phi_{\mathcal{T}}(x)$

$$T(x) = 1 + x\,T(x)^2$$