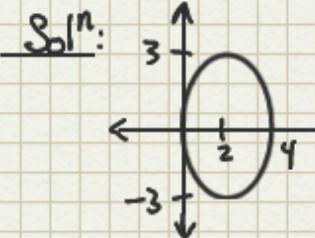


Example: Evaluate $\iint_{D_{xy}} x \, dx \, dy$, where D_{xy} is the region enclosed by $\frac{(x-2)^2}{4} + \frac{y^2}{9} = 1$.



Express D_{xy} :

$$0 \leq x \leq 4$$

$$-3\sqrt{1 - \frac{(x-2)^2}{4}} \leq y \leq 3\sqrt{1 - \frac{(x-2)^2}{4}}$$

looks messy

$$\iint_{D_{xy}} x \, dx \, dy = \iint_{D_{uv}} (2u+2) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv$$

Alternative: Map D_{xy} to the unit circle, D_{uv} ($u^2+v^2=1$)

$$\text{let } u = \frac{x-2}{2}, v = \frac{y}{3}$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} = \frac{1}{\left| \frac{u_x v_y - u_y v_x}{u_x^2 + u_y^2} \right|} = \frac{1}{\frac{1}{6}} = 6$$

$$= 12 \iint_{D_{uv}} (u+1) du \, dv$$

Use polar coordinates

$$u = r \cos \theta \quad D_{r\theta}: \quad 0 \leq r \leq 1$$

$$v = r \sin \theta \quad 0 \leq \theta \leq 2\pi$$

$$= 12 \iint_{D_{r\theta}} (r \cos \theta + 1) \left| \frac{\partial(u,v)}{\partial(r,\theta)} \right| dr \, d\theta$$

$$= 12 \int_0^{2\pi} \int_0^1 (r \cos \theta + 1) r \, dr \, d\theta$$

$$= 12 \int_0^{2\pi} \left[\int_0^1 r^2 \cos \theta + r^2 \, dr \right] d\theta$$

$$= 12 \int_0^{2\pi} \left[\frac{1}{3} r^3 \cos \theta + \frac{1}{2} r^2 \Big|_0^1 \right] d\theta$$

$$= 12 \int_0^{2\pi} \frac{1}{3} \cos \theta + \frac{1}{2} d\theta$$

$$= 12 \left[\frac{1}{3} \sin \theta + \frac{1}{2} \theta \right]_0^{2\pi}$$

$$= 12\pi$$

$$\therefore \iint_{D_{xy}} x \, dx \, dy = 12\pi$$

$$\int_a^b dx = b-a \text{ (interval length)} \quad \iint_D dA = A(D) \quad \iiint_D dV = V(D)$$

E.g. let $f(x,y,z)$ be the density of an object at a point in space. Then

$\iiint_D f(x,y,z) dV$ represent the mass of the object

Triple Integrals

Simplest Case: iterated integrals $D = \{a_1 \leq x \leq a_2, b_1 \leq y \leq b_2, c_1 \leq z \leq c_2\}$

Then $\iiint_D f(x,y,z) dV = \int_{c_1}^{c_2} \int_{b_1}^{b_2} \int_{a_1}^{a_2} f(x,y,z) dx \, dy \, dz \Rightarrow$ can be rewritten in 6 orders

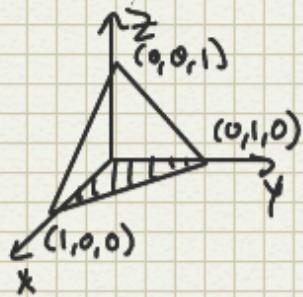
When the bounds aren't constant, we have the following:

If D is a subset of \mathbb{R}^3 defined by $z_l(x,y) \leq z \leq z_u(x,y)$ and $(x,y) \in D_{xy}$, then $\iiint_D f(x,y,z) dV = \iint_{D_{xy}} \int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z) dz \, dA$. Note: you don't have to start with z (x,y too)

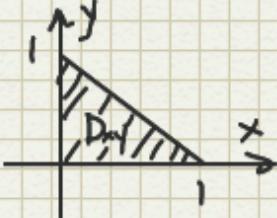
Now, we can treat and solve the double integral the usual way.

Ex: Find $\iiint_D x dV$, where D is the region bounded by the planes

$x+y+z=1$, $x=0$, $y \geq 0$, $z \geq 0$ (in the first octant $x \geq 0$, $y \geq 0$, $z \geq 0$)



We can describe D by: $0 \leq z \leq 1-x-y$ plane
xy plane



$$D_{xy}: 0 \leq y \leq 1-x \quad 0 \leq x \leq 1 \Rightarrow \iiint_D x dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx$$