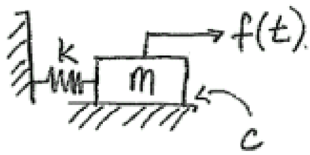
F/ Application to Harmonic Oscillators

-Let's look at second order linear constant coefficient ODEs (eg. Mech, elect., fluids, thermal systems can all be described like this.)

-Mechanical System

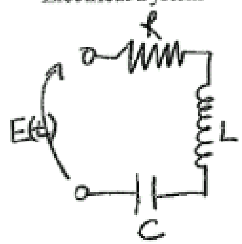


$$mx'' + Cx' + kx = f$$

$$x(0) = x_0,\ x_0'(0) = x_0'$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{f}{m} \quad \cdots (1)$$

-Electrical System



$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE(t)}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{dE(t)}{dt} \quad \cdots (2)$$

-Both have the same form $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = f(t)\omega_n^2$ with $y(0) = y_0, y'(0) = y_0'$

ω_n = natural frequency and $\xi =$ the damping ratio

This is standard notation for both electrical and mechanical systems

Note that both constants are always non-negative here

-The characteristic equation is $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$

-In this form, the roots are given by

$$\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

-Look first at the homogeneous case $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = 0$

(1) & (2) both look like

$$y'' + 2\zeta w_n y' + w_n^2 y = f(t) w_n^2, \quad f(0) = y_0$$
$$f'(0) = y_0'$$

w_n = natural frequency
ζ = dumping ratio

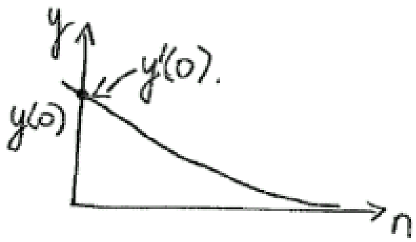
$\left.\begin{matrix} w_n = \text{natural frequency} \\ \zeta = \text{dumping ratio} \end{matrix}\right\}$ always non-negative

-Case I, $\xi > 1$ We get two real roots $\left(\omega_n\sqrt{\xi^2-1} < \xi\omega_n\right)$

-always stable since $\xi\omega_n > 0$ and the second term is less than $\xi\omega_n$

$$y_n = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

-This system is called overdamped



If ξ is large, the plot slows down.

(always stable)

-Case II, $\xi = 1$ We have two repeated roots

$\lambda_{1,2} = -\xi\omega_n$

$$y_n = c_1 e^{-\xi\omega_n t} + c_2 t e^{-\xi\omega_n t}$$

approaches 0 much faster

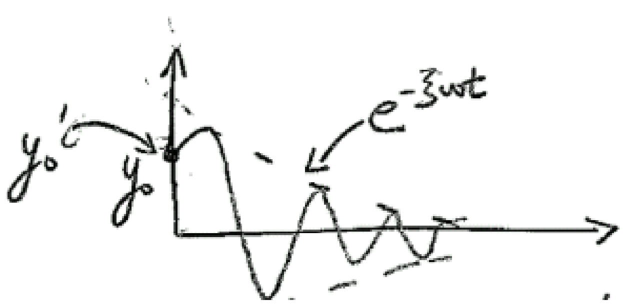
-this system is called critically damped

-again, this is always stable

-Case III, $\xi < 1$ -We have complex roots $\lambda_{1,2} = \underbrace{-\xi\omega_n}_{\text{real}} \pm \underbrace{\omega_n i\sqrt{1-\xi^2}}_{\text{complex}}$ $\alpha \pm i\omega$

-Let $\omega = \omega_n\sqrt{1-\xi^2}$

$$y_n = e^{-i\omega_n t}(c_1\cos\omega t + c_2\sin\omega t)$$



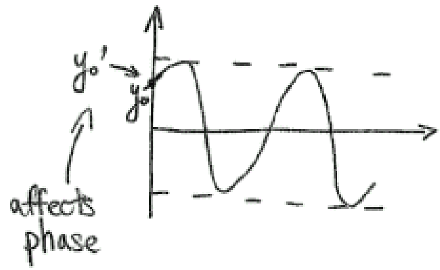
-again, always stable. This is called an under-damped system

$\lambda_1 \lambda_2 = \pm \omega_n i$

Special case $\xi = 0$

Then we have purely complex roots (no damping or resistance case)

$y_n = C_1 \cos \omega_n t + C_2 \sin \omega_n t$



-Look now at the NONHOMOGENEOUS case

-The equations are of the form $\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = f(t)\omega_n^2$

-We need to look at the three cases; overdamped, critically damped and underdamped. For all these cases, these were stable for the homogeneous case

-Suppose $f(t) = F_o \cos(\omega t)$. We will use the method of undetermined coefficients

-The form of the solution is $y(t) = \underline{y_h(t)} + \underbrace{A\cos\omega t + B\sin\omega t}_{y_p}$

The last two terms are the assumed particular solution

-Substitute this in

$$- A\omega^2 \cos \omega t - B\omega^2 \sin \omega t + 2\xi\omega_n(-A\omega\sin \omega t + B\omega\cos \omega t) + \omega_n^2 (A\cos \omega t + B\sin \omega t) = \omega_n^2 F_0 \cos(\omega t)$$

Compare the coefficients to find A and B

for $l \neq 0, y_h \to 0$

After a bit of math, we get

$$y = y_h + \underbrace{\frac{(\omega_n^2-\omega^2)(\omega_n^2 F_0)}{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}\cos\omega t + \frac{2\xi\omega_n\omega(\omega_n^2 F_0)}{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}\sin\omega t}_{y_p}$$

Now, y_h goes to zero in all cases but the particular solution is oscillating with the same frequency as the input but with different amplitudes and phase shift

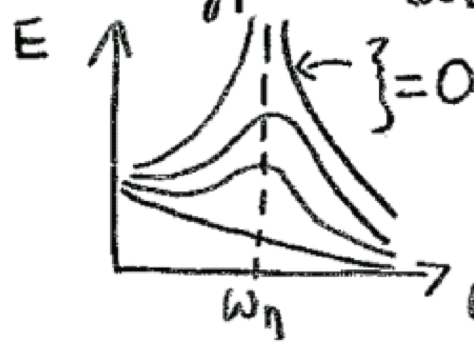
y_h is called the transient and the particular solution is called the steady state solution

$$y_p = E\cos(\omega t+\Phi) = E(\cos\omega t\cos\Phi - \sin\omega t\sin\Phi)$$
$$= E\cos\Phi\cos\omega t - E\sin\Phi\sin\omega t$$

$E\cos\Phi = u \quad -E\sin\Phi = v$

$E \Rightarrow$ square & add

$E^2\cos^2\Phi + E^2\sin^2\Phi = E^2 = u^2+v^2$



u & v are coefficients of the particular sdn (A & B in $A\cos\omega t + B\sin\omega t$)

-NOW, what happens in the case when there is no damping ($\xi = 0$) ie, when $\ddot{y} + \omega_n^2 y = f(t)\omega_n^2 = F_o\omega_n^2\cos\omega t$. This gives

$\Phi \Rightarrow$ divide

$\tan\Phi = \frac{-v}{u}$

$$y = y_h + y_p = \underbrace{A\cos\omega_n t + B\sin\omega_n t}_{y_h} + \underbrace{\frac{(\omega_n^2 F_0)}{(\omega_n^2-\omega^2)}\cos\omega t}_{\text{method of undetermined coeff. } y_p}$$

This is the superposition of two harmonic oscillations

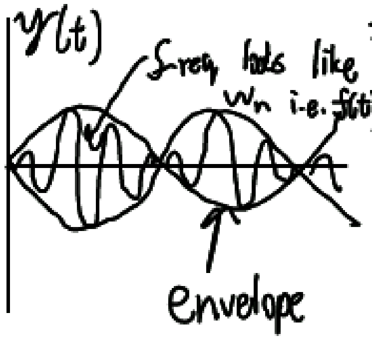
$\boxed{\xi = 0}$

i)Beating phenomenon. Suppose ω is close to ω_n, $y(0)=0, \dot{y}(0)=0$

$y(0) = 0 \Rightarrow A = -\frac{w_n^2}{w_n^2-w^2}F_0$

$y'(0) = 0 \Rightarrow B = 0$

$$y(t) = -\frac{w_n^2}{w_n^2-w^2}F_0\cos w_n t + \frac{w_n^2}{w_n^2-w^2}F_0\cos wt$$
$$= \frac{F_0 w_n^2}{w_n^2-w^2}(\cos wt - \cos w_n t)$$



$$= \frac{F_0 w_n^2}{w_n^2 - w^2} \underbrace{2\sin\left(\frac{w_n - w}{2} t\right)}_{\text{envelope}} \underbrace{\sin\left(\frac{w_n + w}{2} t\right)}_{\text{same freq as } f(t)}$$

ii)Resonance- If $\omega = \omega_n$, then we can't use this equation since the forcing function is of the same form as y_h

$$\underbrace{\ddot{y} + w_n^2 y = F_0 w_n^2 \cos w_n t}_{\text{particular sol}_n}$$

$$y_h = \underbrace{A\cos w_n t + B\sin w_n t}_{y_h}$$

What do we know about the form of the particular solution

we have to try $y_p = t(A\cos w_n t + B\sin w_n t)$

$y = y_h + y_p$ - blows up linearly

-The competition

A. Calculation of the Laplace Transform and Properties

-Definition: Let $f(t)$ be defined for all $t \geq 0$. Then

$\int_0^\infty e^{-st} \cos at\, dt$

$$\mathcal{L}\{ f(t)\} \cong F(s) = \int_0^\infty e^{-st} f(t) dt$$

s is a complex #

is called the Laplace transform of $f(t)$ for all s such that this improper integral (ie limit at ∞) exists

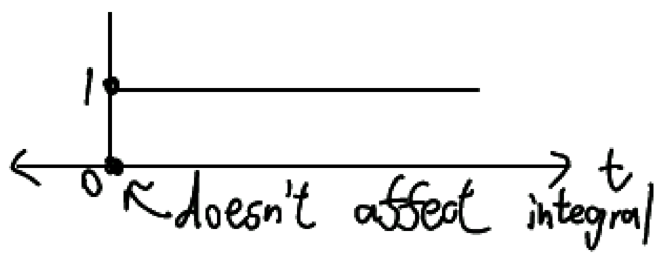
-The inverse Laplace transform is $\mathcal{L}^{-1}\{F(s)\} = f(t)$

-Lower case will generally be a function with respect to t and upper case will be the Laplace transform with respect to s

-Assume $f(t) = 0, t < 0$

-eg. Step Input

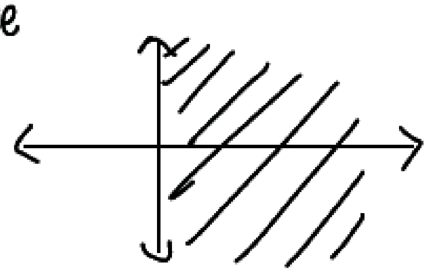
$f(t) = 1, t \geq 0$

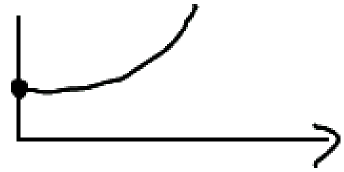


$$F(s) = \int_0^\infty e^{-st} \cdot 1 dt$$
$$= \lim_{T\to\infty} \left(-\frac{1}{s} e^{-st}\right)\Big|_0^T$$
$$= \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} - \left(-\frac{1}{s}\right)\right]$$
$$= 0 + \frac{1}{s} \text{ as } T \to \infty$$
for $Re\{s\} > 0$

$\therefore F(s) = \frac{1}{s}$

region of convergence
$re\{s\} > 0$



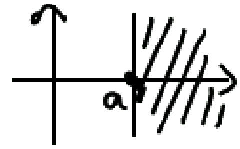


-eg/ $f(t)=e^{at}, t\geq 0$

$$F(s)=\int_0^\infty e^{-st}e^{at}\,dt=\lim_{T\to\infty}\int_0^T e^{-(s-a)t}\,dt=\lim_{T\to\infty}\frac{1}{a-s}e^{-(s-a)t}\Big|_0^T$$

if $re\{s-a\}>0$ then as $T\to\infty$, the first term goes to zero, i.e. $re\{s\}>a$

$$\therefore F(s)=0-\frac{1}{a-s}=\frac{1}{s-a},\ re\{s\}>a$$



-We can keep doing this for arbitrary useful functions to generate tables

-The Laplace transform is linear

$$\mathcal{L}[af(t)+bg(t)]=\int_0^\infty e^{-st}(af(t)+bg(t))\,dt$$
$$=a\int_0^\infty e^{-st}f(t)\,dt+b\int_0^\infty e^{-st}g(t)\,dt$$
$$=aF(s)+bG(s)$$

(Note: if each has a region of convergence, the intersection will work for the sum)

The inverse Laplace Transform is also linear.

$$\mathcal{L}^{-1}[aF(s)+bG(s)]=a\mathcal{L}^{-1}[F(s)]+b\mathcal{L}^{-1}[G(s)]$$
$$=af(t)+bg(t)$$

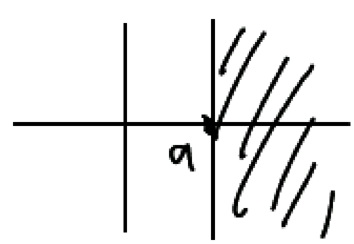
-eg/ $f(t)=\cosh at=\dfrac{e^{at}+e^{-at}}{2}=\frac{1}{2}e^{at}+\frac{1}{2}e^{-at}$

$$\mathcal{L}\{\cosh at\}=\mathcal{L}\{\tfrac{1}{2}e^{at}+\tfrac{1}{2}e^{-at}\}=\tfrac{1}{2}\mathcal{L}\{e^{at}\}+\tfrac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$=\frac{1}{2}\frac{1}{s-a}+\frac{1}{2}\frac{1}{s-a}=\frac{s}{s^2-a^2}$$

$re\{s\}>a$ & $re\{s\}>-a$

region of convergence $\Rightarrow re(s)>|a|$



-eg/ $F(s)=\frac{1}{(s-1)(s-2)}$. Find $f(t)$ PFE: $F(s)=\frac{1}{s-2}-\frac{1}{s-1}$

$f(t)=e^{2t}-e^{t}$

-We now forget about the region of convergence. The only time you ever really need it is to use integral formulas for doing the inverse transform. It is sufficient to know that there is some region where the Laplace transform exists

-These are Laplace transforms to commit to memory (we will eventually prove all of these)

$f(t), t\geq 0$	$F(s)$
1 (step)	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$

aside: $d\{uv\}=du\,v+u\,dv$ $\therefore uv=\int v\,du+\int u\,dv$ $\therefore$ integration by parts

Eg/ Show $\mathcal{L}\{t^n\}$ formula use induction

assume formula holds for n i.e. $\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$

we already showed this for $n=0$ i.e. step

$$\mathcal{L}\{t^{n+1}\}=\int_0^\infty e^{-st}t^{n+1}dt$$

$$=-\frac{1}{s}e^{-st}t^{n+1}\Big|_0^\infty+\frac{n+1}{s}\int_0^\infty e^{-st}t^n dt$$

if $Re\{s\}>0$ this is 0 $\mathcal{L}\{t^n\}$

$= 0 + \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$ $\quad \therefore \mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{n+2}}$

$\therefore$ by induction, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ QED

-Everything still holds in the complex domain

aside: $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

-Eg $\mathcal{L}\{\sin \omega t\}$

$$\mathcal{L}\left\{\frac{e^{jwt} - e^{-jwt}}{2j}\right\} = \frac{1}{2j}\mathcal{L}\{e^{jwt}\} - \frac{1}{2j}\mathcal{L}\{e^{-jwt}\}$$

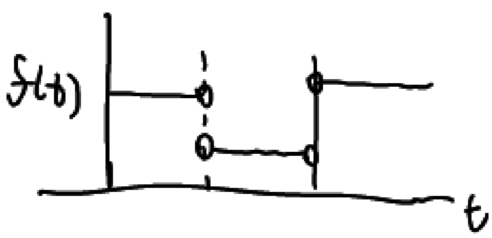
$$= \frac{1}{2j}\frac{1}{s-jw} - \frac{1}{2j}\frac{1}{s+jw} = \frac{w}{s^2+w^2}$$

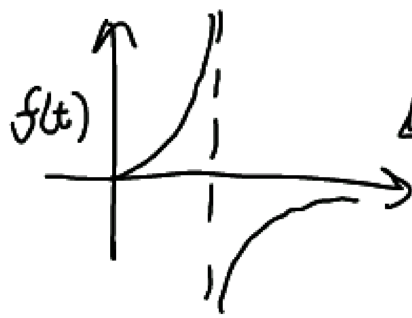
-Now, not all $f(t)$ have a Laplace transform. The integral, intuitively, needs to have the integrand, $e^{-st} f(t)$, go to zero sufficiently "fast". From the examples, we seemed to need the overall function to decay ~~faster~~ slower than e^{-kt}, $k>0$, in order for the integral to exist

eg/ $e^{-st}t^n$, $e^{-st}e^{at}$ can all be made to decay sufficiently fast but $e^{-st}e^{t^2}$ won't converge for any choice of s

-Now, $f(t)$ doesn't need to be continuous (eg. turning on a switch, pulse/square wave generator)

-Let's assume $f(t)$ is piecewise continuous. In other words, it is continuous on any finite interval. If it is discontinuous, it will have finite right and left hand limits (ie can only have finite jumps)





← not p.c.

-Theorem. A function $f(t)$ is piecewise continuous on every finite interval for $t > 0$ and $|f(t)| \le Me^{at}$ for some a (can be positive!). Then the Laplace transform exists for $re\{s\} > a$

$\}$ hold for almost all practical eng. examples

-When it exists, the Laplace transform is unique except at the discontinuities

-These are sufficient conditions

-eg. $\mathcal{L}\{\frac{1}{\sqrt{t}}\}$

↑ not continuous at $t=0$ but there is a region of convergence

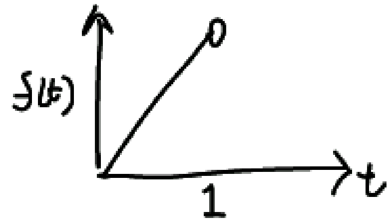
$\therefore$ There is a Laplace

eg. $\mathcal{L}\{e^{t^2}\}$

↑ doesn't have a Laplace transform

eg. What is the Laplace transform of

$$f(t) = t, 0 \le t < 1$$
$$= 0, t \ge 1$$



P.C.

$|f(t)| \le Me^{\alpha t}$ for some M & α

$\therefore$ Laplace transform exists

Sol'n/

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\, t\,dt \qquad \text{integration by parts}$$

$$= -\frac{te^{-st}}{s}\Big|_0^1 - \int_0^1 e^{-st}\,dt$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^1$$

$$= -\frac{e^{-s}}{s} + \left(-\frac{1}{s^2}\right)e^{-s} - \left(-\frac{1}{s^2}\right) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right)$$

-Inverse Laplace

-can apply $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}\,ds$

(This is an integration in the complex plane. The α depends on the region of convergence

-We can, if *F(s)* is rational, use tables and partial fraction expansions

$\longrightarrow$ ratio of polynomials

eg/ $F(s) = \frac{3s^2 + s - 7}{s^3 - 7s^2}$

$$= \frac{3s^2 + s - 7}{s^2(s-7)} = \frac{A}{s-7} + \frac{B}{s} + \frac{C}{s^2}$$

B. Application to ODEs

-We need to figure out what happens when we differentiate a function

-Theorem- suppose $f(t)$ is continuous for $t \geq 0$ and satisfies the conditions to have a Laplace transform. As well, suppose $\frac{df}{dt}(t)$ is piecewise continuous on every finite interval in $t \geq 0$. Then, there exists an α such that $\mathcal{L}\{\frac{df}{dt}(t)\}$ exists for $\mathrm{Re}\{s\} > \alpha$ and

$$\mathcal{L}\{\frac{df}{dt}(t)\} = s\ \mathcal{L}\{f(t)\} - f(0)$$

where $f(0)$ is the initial value.

Proof/

-What do the conditions mean?

-Apply this twice to get

$$\mathcal{L}\{\frac{d^2 f}{dt^2}(t)\} = s\ \mathcal{L}\{\frac{df}{dt}(t)\} - f'(0)$$

-Similarly, $\mathcal{L}\{\frac{d^n f}{dt^n}(t)\} =$

-eg/ $\mathcal{L}\{\cos \omega t\} =$

-We can now solve ODEs with initial values using algebraic techniques

-Eg/ $\dot{x} = x,\ x(0) = 1$

Eg/ $\ddot{y} = 1,\ y(0) = 0,\ \dot{y}(0) = 1$

Eg/ $\ddot{y} + 9y = 1,\ y(0) = 0,\ \dot{y}(0) = 1$

Practice PFE!!!!!

C. Shifting theorems and the Heaviside Function

- s-Shifting

-Theorem: If $f(t)$ has a transform $F(s)$ (where $s > \alpha$), then $e^{at} f(t)$ has the transform $F(s-a)$ (where $s - a > \alpha$)

ie $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

-Equivalently, $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

Pf/ $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt = \int_0^\infty e^{-st}\left(e^{at} f(t)\right)dt$

$= \mathcal{L}\{e^{at} f(t)\}$ QED

-This now allows us to use PFE of all possible types of rational functions in s using our previous table

If $f(t) = e^{at} t^n$, then $F(s) = \frac{n!}{(s-a)^{n+1}} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = te^t$; $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ from table

If $f(t) = e^{at}\cos\omega t$, then $F(s) = \frac{s}{s^2+w^2}$ $\therefore F(s-a) = \frac{(s-a)}{(s-a)^2+w^2}$

If $f(t) = e^{at}\sin\omega t$, then $F(s) = \frac{w}{s^2+w^2}$ $\therefore F(s-a) = \frac{w}{(s-a)^2+w^2}$

Eg/ $F(s) = \dfrac{1}{s^2(s^2+4s+40)}$

$\hookrightarrow s^2 + 4s + 40 = (s+2)^2 + 36$

$= (s+2)^2 + 6^2$

$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C's + D'}{(s+2)^2+6^2}$ or $+ \frac{C(s+2)}{(s+2)^2+6^2} + \frac{6D}{(s+2)^2+6^2}$

from P.F.E, we get $A = -\frac{1}{100}$, $B = \frac{1}{40}$, $C = \frac{1}{400}$, $D = -\frac{1}{300}$

$f(-1) = \mathcal{L}^{-1}\left\{-\frac{1}{400s} + \frac{1}{40s^2} + \frac{1}{400}\frac{s+2}{(s+2)^2+6^2} - \frac{1}{300}\frac{6}{(s+2)^2+6^2}\right\}$

$= -\frac{1}{400} + \frac{1}{40}t + \frac{1}{400}e^{-2t}\cos 6t - \frac{1}{200}e^{-2t}\sin 6t$

Note $(s^2+4s+40)=(s+(2+6i))(s+(2-6i))$

Or solve using complex domain

$$\frac{1}{s^2(s^2+4s+40)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+2+6i)}+\frac{D}{(s+2-6i)}$$

solve for $A, B, C, D \Rightarrow C$ & D are complex (will be a conj. pair)

$$= A+Bt+Ce^{(-2-6i)t}+De^{(-2+6i)t}$$

You will get the same answer.

Eg./ $\ddot{y}+2\dot{y}+5y=0,\ y(0)=2,\ \dot{y}(0)=-4$

Take Laplace $\Rightarrow s^2Y(s)-s\,\cancel{y(0)}^{\,2}-\cancel{\dot{y}(0)}^{\,-4}+2(sY(s)-\cancel{y(0)}^{\,2})+5Y(s)=0$

$$Y(s)=\frac{2s}{(s^2+2s+5)}=\frac{2s}{(s+1)^2+2^2}=\frac{A(s+1)}{(s+1)^2+2^2}+\frac{B2}{(s+1)^2+2^2}$$

$A=2,\ B=-1$

$$Y(s)=\frac{2(s+1)}{(s+1)^2+2^2}-\frac{2}{(s+1)^2+2^2} \qquad y(t)=2e^{-t}\cos 2t-e^{-t}\sin 2t$$

Eg/ $\ddot{y}-2\dot{y}+y=e^t+t,\ y(0)=1,\ \dot{y}(0)=0$

$$s^2Y(s)-sy(0)-y'(0)-2(sY(s)-y(0))+Y(s)=\frac{1}{s-1}+\frac{1}{s^2}$$

$$Y(x)=\frac{s-2}{(s-1)^2}+\frac{1}{(s-1)^3}+\frac{1}{s^2(s-1)^2}$$

$\frac{A}{s-1}+\frac{B}{(s-1)^2}$

$A=1, B=-1$

$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{t^2}{2}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}=\frac{e^t t^2}{2}$

$\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$

$A=2, B=1, C=-2, D=1$

$\frac{1}{(s-1)^2}-\frac{2}{s-1}+\frac{1}{s^2}+\frac{2}{s}$

$$\therefore y(t) = e^t \cancel{- te^t} + \frac{e^t t^2}{2} \cancel{+ te^t} - 2e^t + t + 2 = -e^t + t + 2 + \frac{t^2 e^t}{2}$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

PFE: repeated real roots
complex with real part roots

Eg/ $\ddot{y} + 6\dot{y} + 13y = 1$, $y(0) = \cancel{1}\ 0$, $\dot{y}(0) = 0$

$$s^2 Y(s) + 6sY(s) + 13Y(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{1}{s(s^2+6s+13)} = \frac{1}{s((s+3)^2+2^2)} = \frac{A}{s} + \frac{Bs+C}{(s+3)^2+2^2}$$

$$= \frac{A}{s} + \frac{B(s+3)}{(s+3)^2+2^2} + \frac{C2}{(s+3)^2+2^2}$$

Solve for A, B, C

$$\therefore y(t) = A + Be^{-3t}\cos 2t + Ce^{-3t}\sin 2t$$

Extra problem:

Look at $\dfrac{1}{s^4+16}$

Solution:

$$\frac{1}{s^4+16} = \frac{1}{(s-2e^{j45})(s-2e^{-j45})(s-e^{j135})(s-e^{-135j})}$$

$$= \frac{1}{(s-2\{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}j\})(s-2\{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}j\})(s-2\{-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}j\})(s-2\{-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}j\})}$$

$$= \frac{1}{(\{s-\sqrt{2}\}-\sqrt{2}j)(\{s-\sqrt{2}\}+\sqrt{2}j)(\{s+\sqrt{2}\}-\sqrt{2}j)(\{s+\sqrt{2}\}+\sqrt{2}j)}$$

$$= \frac{1}{(\{s+\sqrt{2}\}^2+2)(\{s-\sqrt{2}\}^2+2)}$$

This can be checked by multiplying the two terms in the denominator

Thus, these are these are terms involving quadratics which will lead to terms such as

$$e^{-\sqrt{2}t}\sin(\sqrt{2}t), e^{-\sqrt{2}t}\cos(\sqrt{2}t), e^{\sqrt{2}t}\sin(\sqrt{2}t), e^{\sqrt{2}t}\cos(\sqrt{2}t),$$

-Terms to use in general $Y(s) = \frac{f(s)}{g(s)}$ where order of $f(s)$ < order of $g(s)$ (if equal, devide thru)

Terms in $g(s)$	Inverse Transform
$(s-a)$	$\mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{at}$
$(s-a)^m$	$\mathcal{L}^{-1}\left\{\frac{A_m}{(s-a)^m} + \dots + \frac{A_1}{s-a}\right\} = \frac{A_m e^{at} t^{m-1}}{(m-1)!}$
$(s-\alpha)^2+\beta^2$	$\mathcal{L}^{-1}\left\{\frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{C\beta}{(s-\alpha)^2+\beta^2}\right\} = Ae^{\alpha t}\cos\beta t + Ce^{\alpha t}\sin\beta t$
$((s-\alpha)^2+\beta^2)^m$	Look it up

e.g. $\frac{s+3}{s+5} = 1 - \underbrace{\frac{2}{s+5}}_{\text{PFE}}$

-Now look at time-shifting

Thm- Suppose $\mathcal{L}\{f(t)\} = F(s)$. Then the function

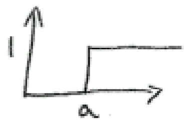
$$\tilde{f}(t) = \begin{cases} 0, & t < a \\ f(t-a), & t \ge a \end{cases}$$

has the Laplace Transform $e^{-as}F(s)$

ASK JSON

-Heaviside Step Function $u(t-a)$

$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t\geq a\end{cases}$$



- $u(t)$ is the step function. Thus, $u(t-a)$ is a delayed step

A delayed function is $\tilde{f}(t)= f(t-a)u(t-a)$

-Theorem (restated)

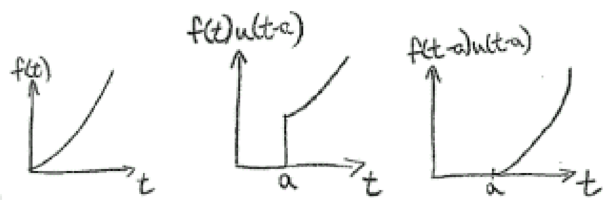
$$\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$$
$$\mathcal{L}^{-1}\{e^{-as}F(s)\}= f(t-a)u(t-a)$$

Pf/ $e^{-as}F(s)=e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau=\int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau$

let $t=\tau+a \Rightarrow dt=d\tau$

$\therefore e^{-as}F(s)=\int_a^\infty e^{-st}f(t-a)\,dt=\int_0^\infty e^{-st}\underbrace{u(t-a)f(t-a)}_{\text{makes area 0 prior to } a}\,dt$

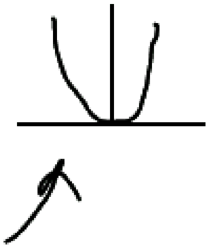
-Show that $\mathcal{L}\{u(t-a)\}=\frac{e^{-as}}{s}$ $\Rightarrow$ since $\mathcal{L}\{u(t)\}=\frac{1}{s}$, then $\mathcal{L}\{u(t-a)\}=e^{-as}\frac{1}{s}$



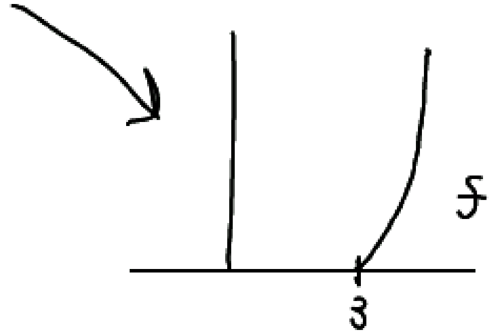
* Note: $f(t)u(t-a)\neq f(t-a)u(t-a)$

Eg/ Find $\mathcal{L}^{-1}\{e^{-3s}/s^3\}$ and sketch

$$\mathcal{L}^{-1}\{1/s^3\} = t^2/2$$

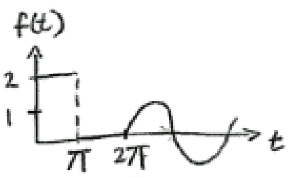


$$\mathcal{L}^{-1}\{e^{-3s}/s^3\} = \frac{1}{2}(t-3)^2 u(t-3)$$



Eg/Find the transform of

$$f(t) = \begin{cases} 2 & 0 \le t < \pi \\ 0 & \pi \le t < 2\pi \\ \sin t & t \ge 2\pi \end{cases}$$



write $f(t)$ in terms of Heavyside

from 0 to π, $f(t) = 2u(t)$

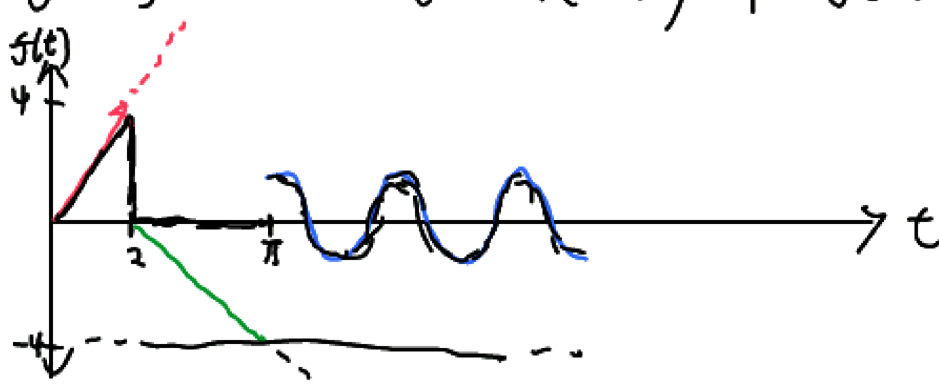
from 0 to π, $f(t) = 2u(t) - 2u(t-\pi)$

for t, $f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin(t-2\pi)$

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - e^{-\pi s}\frac{2}{s} + e^{-2\pi s}\frac{1}{s^2+1}$$

Eg/ $F(s) = \dfrac{2}{s^2} - 2\dfrac{e^{-2s}}{s^2} - 4\dfrac{e^{-2s}}{s} + s\dfrac{e^{-\pi s}}{s^2+1}$

$$\mathcal{L}^{-1}\{F(s)\} = 2t - 2u(t-2)(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi)$$



Eg. $\ddot{y} - 4\dot{y} + 4y = t + 2u(t-3)$, $y(0) = \cancel{1}\ 0$, $\dot{y}(0) = 0$

$$s^2 Y(s) - 4sY(s) + 4Y(s) = \frac{1}{s^2} + \frac{2}{s}e^{-3s}$$

all I.C. are zero

$$\therefore Y(s) = \underbrace{\frac{1}{s^2(s^2-4s+4)}} + \underbrace{\frac{2}{s(s^2-4s+4)}}e^{-3s} \leftarrow (s-2)^2$$

Using P.F.E

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-2)} + \frac{D}{(s-2)^2} \qquad \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$B = \frac{1}{4}$, $A = \frac{1}{4}$, $C = -\frac{1}{4}$, $D = \frac{1}{4}$ $\qquad$ $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = 1$

$$\therefore Y(s) = \frac{1}{4s} + \frac{1}{4s^2} + \frac{(-1/4)}{(s-2)} + \frac{(1/4)}{(s-2)^2} + e^{-3s}\left(\frac{1}{2s} - \frac{1}{2(s-2)} + \frac{1}{(s-2)^2}\right)$$

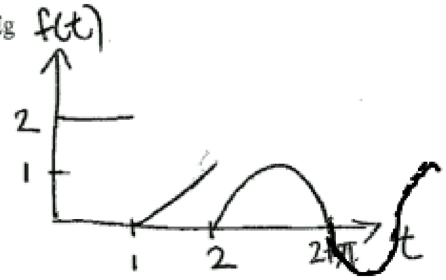
$$\therefore y(t) = \frac{1}{4} + \frac{t}{4} + \left(\frac{1}{4}\right)t\underbrace{e^{2t}}_{\text{s-shift}} + \overbrace{\frac{1}{2}u(t-3) - \frac{1}{2}u(t-3)e^{2(t-3)} + u(t-3)(t-3)e^{2(t-3)}}^{}$$

timeshift

From left to right:

$$f(t) = 2 - 2u(t-1) + (t-1)u(t-1) - (t-2)u(t-2) - 1\cdot u(t-1) + \sin(t-2)u(t-2)$$

Eg



$$\mathcal{L}\{f(t)\} = \frac{2}{s} - e^{-s}\frac{2}{s} + e^{-s}\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + e^{-2s}\frac{1}{s^2+1}$$

Eg/ $\mathcal{L}\{u(t-1)t\}$

$$f(t) = u(t-1)t = u(t-1)(t-1) + u(t-1)$$

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

-Handling periodic signals

-Theorem- if f is periodic with period T and piece-wise continuous over this period, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \overbrace{\int_0^T f(t)e^{-st}\,dt}^{\text{Laplace Transform of one period}}$$

The integral is the Laplace transform over one interval

Pf/

periodic means: $f(t) = f(t+T)$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \underbrace{\int_T^{2T} e^{-st} f(t)\,dt}_{\substack{\text{Let } t = \tau + T \\ dt = d\tau}} + \underbrace{\int_{2T}^{3T} e^{-st} f(t)\,dt}_{\substack{\text{let } t = \tau + 2T \\ dt = d\tau}} + \dots$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(\tau+T)} \underbrace{f(\tau+T)}_{f(\tau)}\,d\tau + \int_0^T e^{-s(\tau+2T)} \underbrace{f(\tau+2T)}_{f(\tau)}\,d\tau + \dots$$

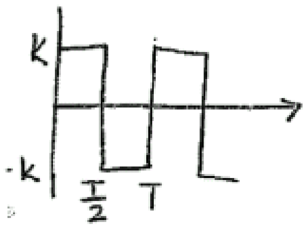
$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \dots$$

next page

$$= \int_0^T e^{-st} f(t)\,dt \underbrace{\left(1 + e^{-sT} + e^{-2sT} + \dots\right)}_{\text{geometric series } |e^{-sT}| < 1}$$

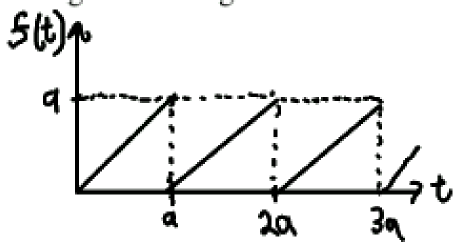
$$= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt \qquad \text{QED}$$

Eg/ Pulse generator

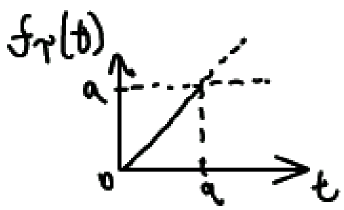


Ask Jason

Eg/ Sawtooth generator



period $\tau = a$



$$f_\tau(t) = t - (t-a)u(t-a) - au(t-a)$$

$$\mathcal{L}\{f_\tau\} = \frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{a}{s}e^{-as}$$

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sa}}\mathcal{L}\{f_\tau\}$$

$$= \frac{1}{1-e^{-sa}}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{a}{s}e^{-as}\right)$$

-Other properties:

Integration of $f(t)$

Theorem: If $f(t)$ is piecewise-continuous and satisfies $|f(t)| \le Me^{\lambda t}$, then

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

Pf/ one can show that if $|f(t)| \le ne^{\lambda t}$, the $\int_0^t f(\tau)d\tau$ would have a Laplace Transform

Let $g(t) = \int_0^t f(\tau)d\tau$ $\therefore f(t) = \frac{d\,g(t)}{dt}$

Now, one can show that $|g(t)| \le \frac{Me^{\lambda t}}{\lambda}$ which means there is a Laplace transform

$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$

$\hookrightarrow g(0) = \int_0^0 f(t)dt = 0$

The last term is zero

QED