# Triple Integrals

- Constant bounds (i.e. domain is rectangular)
  Just 3 iterated integrals in any order.
- Non-constant bounds:
  Find bounds on one variable in terms of two others.
  e.g. $z$ lies between the surfaces $z_l(x,y)$ & $z_u(x,y)$.

Then $\displaystyle\iiint_D f(x,y,z)\,dV = \iint_{D_{xy}} \int_{z_l(x,y)}^{z_u(x,y)} f(x,y,z)\,dz\,dA$

where $\displaystyle\iint_{D_{xy}}$ is a double integral.

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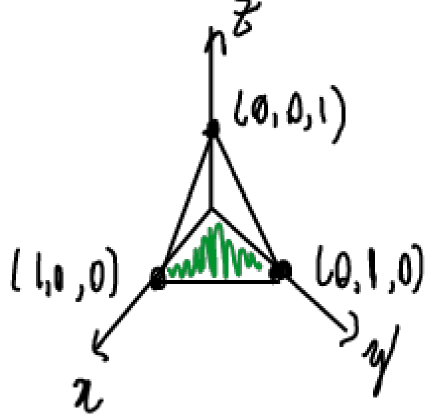
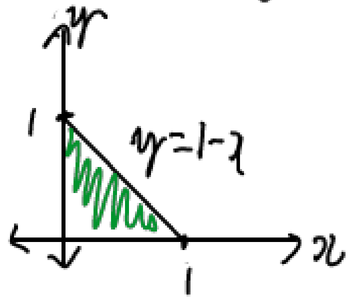
Ex: $D$ is region in first octant bounded by the plane $x+y+z=1$




Bounds on $z$: $0 \le z \le 1-x-y$

$D_{xy}$:

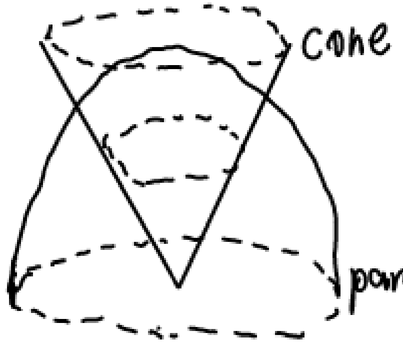



$0 \le x \le 1$
$0 \le y \le 1-x$

$$\iiint_D x\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx = \frac{1}{24} \leftarrow \text{exercise}$$

Ex. Suppose we wanted to find the volume of the solid bounded by the paraboloid $z = 2-x^2-y^2$ and the cone $z=\sqrt{x^2+y^2}$.

Soln: We know $V = \displaystyle\iiint_D dV$



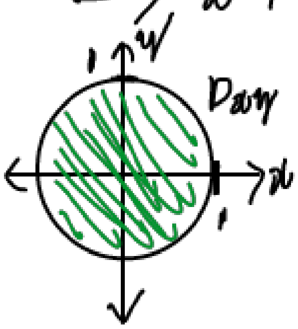

Start with the bounds on z.
z lies above the cone & below the paraboloid.

$$\Rightarrow \sqrt{x^2+y^2} \le z \le 2-x^2-y^2$$

What about $D_{xy}$?

Find the intersection (everything in the solid is included if we project the circle of intersection up & down in the z-direction).

$\Rightarrow x^2+y^2=1$ (set $\sqrt{x^2+y^2}=2-x^2-y^2$ by inspection)




$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$-1 \le x \le 1$$

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} dz\,dy\,dx$$

← looks hard
- better suited for cylindrical coordinates.

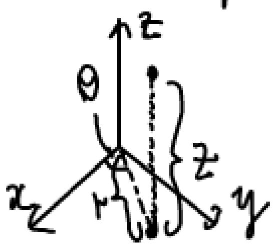
## Change of variable Theorem

Let $x=f(u,v,w)$, $y=g(u,v,w)$, $z=h(u,v,w)$ be a one-to-one mapping of $D_{uvw}$ to $D_{xyz}$. Then

$$\iiint_{D_{xyz}} H(x,y,z)\,dx\,dy\,dz = \iiint_{D_{uvw}} H\big(f(u,v,w), g(u,v,w), h(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

## Cylindrical Coordinates

- extend polar coordinates by adding an axis of symmetry




Relationship between cartesian & cylindrical coords:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

## Graphs:

- $r=1$ is an infinite cylinder of radius 1.
- $\theta = \pi/4$ is the semi-infinite plane $y=x$.
- $z=h$ is the plane $z=h$.

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Jacobian represents change in a small volume element under a transformation.

For this case, $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$ (check)

Go back to the paraboloid/cone example.

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} dz\,dy\,dx.$$

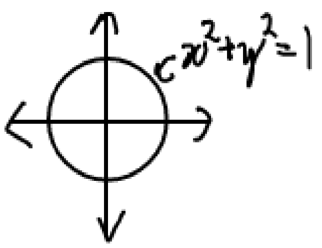
How to describe the region in cylindrical coordinates?

Start with $z$. The cone is $z = \sqrt{x^2+y^2} = r$

The paraboloid is $z = 2-(x^2+y^2) = 2-r^2$

$\Rightarrow r \le z \le 2-r^2$

What about Dom?




$\Rightarrow 0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} \overset{\text{Jacobian}}{r}\, dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \Big[ r z \Big|_r^{2-r^2} dr$$

$$= 2\pi \int_0^1 2r - r^3 - r^2\, dr = 2\pi \left[ r^2 - \tfrac{1}{4}r^4 - \tfrac{1}{3}r^3 \right]_0^1 = \frac{5\pi}{6}$$