Mathematical Admin

$\mathbb{N} = \{1, 2, 3, \dots\}$

$[n] = \{1, 2, 3, \dots n\}$

We will often build sets from smaller sets

$A \cup B = \{x \cdot x \in A \text{ or } x \in B\}$

$A \cap B = \{x \cdot x \in A \text{ and } x \in B\}$

if A, B are finite, then $|A \cup B| = |A| + |B| - |A \cap B|$

if $A \cap B = \emptyset$, then A, B are disjoint and

$|A \cup B| = |A| + |B|$

Unions are 'sums'

The cartesian product of A & B is the set $A \times B = \{(a,b) : a \in A, b \in B\}$; i.e. the set of all pairs with first element in A and second element in B.

For $k \in \mathbb{N}$, A^k denotes $\overbrace{(A \times A) \times \dots \times A}^{k}$

$= \{(a_1, a_2, \dots, a_k) \cdot a_i \in A \text{ for each } i\}$

e.g. $\{(0,1)\}^5 = \{(0,0,0,0,0), (0,0,0,0,1) \dots\}$

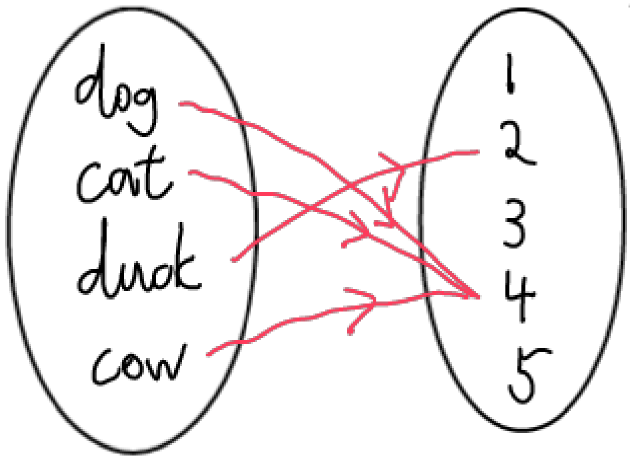
If A, B are finite, then $|A \times B| = |A| \cdot |B|$

e.g. $\{1,2\} \times \{2,3,4\} = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}$

It follows that $|A^k| = |A|^k$.

A function $f: A \to B$ is a subset of $A \times B$ in which every element of A occurs exactly once as a first element of a pair.

eg: $\{(\text{dog}, 4), (\text{cat}, 4), (\text{duck}, 2), (\text{cow}, 4)\}$ is a function from $\{\text{dog}, \text{cat}, \text{duck}, \text{cow}\}$ to $\mathbb{N}$.



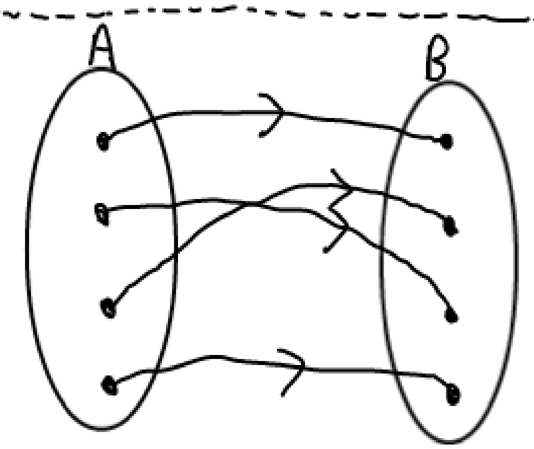
f is one-to-one if (injective)

$f(x) = f(y) \to x = y$ for all $x, y \in A$

f is onto if (subjective)

for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

If A & B are finite, then the number of functions f from A to B is $|B|^{|A|}$



A function that is both one to one and onto is bijective (a bijection)

Prop: If A, B are finite and there exists a bijection $f: A \to B$, then $|A| = |B|$.

bijective

$\mathbb{N}$ square numbers

$1 \longrightarrow 1$

$2 \longrightarrow 4$

$3 \longrightarrow 9$

$4 \longrightarrow 12$

real numbers

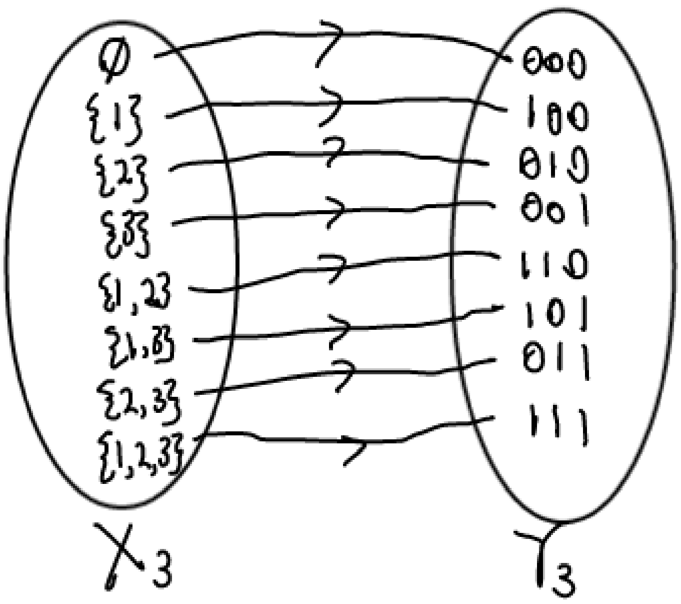
Q

Combinatorial Proofs

Prop. There are 2^n subsets of $[n]$

Pf: Let X_n be the set of subsets of $[n]$
Y_n be the set of binary strings of length n.

We showed earlier $|Y_n| = 2^n$.



We define a bijection $f: X_n \to Y_n$

For each $S \in X_n$, let $f(S) = a_1 a_2 a_3 \dots a_n$
where $a_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$
for each $i \in [n]$

For each string $b_1, b_2, \dots, b_n \in Y_n$,
let $g(b_1, b_2, \dots, b_n) = \{ i \in [n] : b_i = 1 \}$
$\therefore$ g is an inverse of f, f is a bijection.
therefore $|X_n| = |Y_n|$ since $|Y_n| = 2^n$, we have $|X_n| = 2^n$

Prop: A function $f: A \to B$ is a bijection iff it has an inverse function $g: B \to A$
ie. a function g so that $g(f(a)) = a$ for all $a \in A$
$f(g(b)) = b$ for all $b \in B$