Recall: A composition of $n$ into $k$ parts is a $k$-tuple $(a_1, a_2, \dots a_k) \in \mathbb{N}^k$ so that $a_1 + a_2 + \dots + a_k = n$

NB: By this def, $a_i \geq 1$ for each $i$, and order matters

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## Restricted Compositions

Often we will need to compute the number of compositions of $n$ with various restrictions on the number of parts, or their sizes. The sum/product Lemmas do this.

### Small parts

How many compositions of $n$ have each part equal to 1 or 2?

- w/ $k$ parts?
- w/ any # of parts?

Let $S = \{1,2\}$ and $w(\sigma) = \sigma$ for each $\sigma \in S$

$$\Phi_S(x) = x^1 + x^2$$

By product Lemma

consider $[x^n]\Phi_S(x)^k = $ # of $k$-tuples $(a_1, a_2, \dots, a_k) \in S^k$ with $a_1 + a_2 + \dots + a_k = n$

$=$ # compositions of $n$ into $k$ (parts of size 1 or 2)

$$[x^n]\Phi_S(x)^k = [x^n](x+x^2)^k$$
$$= [x^n]x^k(1+x)^k$$
$$= [x^{n-k}](1+x)^k$$
$$= \binom{k}{n-k} \quad \text{by binomial theorem}$$

\# compositions of $n$ into $k$ parts of size 1 or 2 is $\binom{k}{n-k}$

$\Rightarrow$ \# comp of $n$ into any \# of parts of size 1 or 2 is $\sum_{k\geq 0}\binom{k}{n-k}$

Alternatively, \# compositions of $n$ into any \# of parts of size 1 or 2 is:

$$\sum_{k\geq 0}[x^n](x+x^2)^k = [x^n]\sum_{k\geq 0}(x+x^2)^k$$
$$= [x^n]\frac{1}{1-x-x^2}$$
$$= n^{th} \text{ Fibonacci Number}$$

Odd parts:

How many compositions of $n$ have each part odd? (any \# of parts)

Let $S=\{1,3,5,7,\dots\}$ $w(\sigma)=\sigma$ for each $\sigma\in S$

$$\Phi_S(x)=x^1+x^3+\dots = x(1+x^2+x^4+\dots)=\frac{x}{1-x^2}$$

\# compositions of $n$ into $k$ odd parts

$=\#\ (a_1,a_2,\dots,a_k)\in S^k$ with $a_1+a_2+\dots+a_k=n$

$=[x^n]\,\Phi_S(x)^k$ by Product Lemma

$\Rightarrow$ \# compositions of $n$ into any \# of odd parts is:

$$\sum_{k\geq 0}[x^n]\,\Phi_S(x)^k=[x^n]\sum_{k\geq 0}\Phi_S(x)^k$$

$$=[x^n]\,\frac{1}{1-\Phi_S(x)}$$

$$=[x^n]\,\frac{1}{1-\left(\frac{x}{1-x^2}\right)}$$

$$=[x^n]\,\frac{1-x^2}{1-x-x^2}$$

Let $\dfrac{1-x^2}{1-x-x^2}=a_0+a_1x+a_2x^2+\dots$

Solving $(1-x-x^2)(a_0+a_1x+a_2x^2+\dots)=1-x^2$

we got
$$\begin{aligned} a_0&=1\\ a_1-a_0&=0\\ a_2-a_1-a_0&=-1\\ &\ \vdots\\ a_k-a_{k-1}-a_{k-2}&=0 \quad k\geq 3\end{aligned}$$

$a_0=1,\ a_1=1,\ a_2=1$

$a_k=a_{k-1}+a_{k-2},\quad k\geq 3$

So $a_n = (n-1)^{th}$ Fibonacci number, $n \geq 1$

$= \#$ compositions of $n$ into odd parts

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Let $A_n = \{$compositions of $n$ into parts of size $1$ or $2\}$

We need $|A_n| = |A_{n-1}| + |A_{n-2}|$ (because $|A_0| = |A_1| = 1$)

$A_0$ $()$

$A_1$ $(1)$

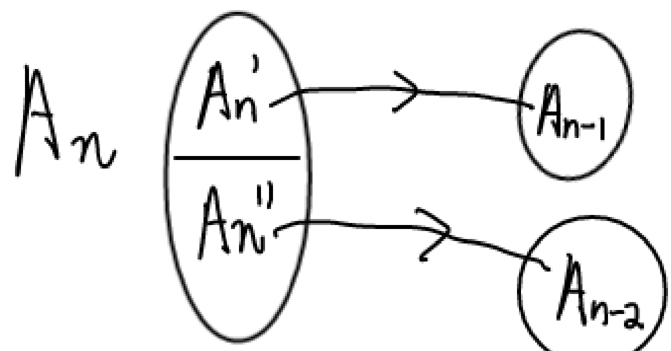
$A_2$ $(1,1)$, $(2)$

$A_3$ $(1,1,1)$, $(1,2)$, $(2,1)$

$A_4$ $(1,1,1,1)$, $(1,2,1)$, $(2,1,1)$, $(1,1,2)$, $(2,2)$

Let $A_n' = \{$comp. of $n$ into parts of size $1$ or $2$ with last part $1\}$

Let $A_n'' = \{$comp of $n$ into parts of size $1$ or $2$ with last part $2\}$



let $f_1 : A_n' \to A_{n-1}$ be defined by

$$f_1(a_1, a_2, \ldots, a_k) = (a_1, a_2, \ldots, a_{k-1})$$

let $g_1 : A_{n-1} \to A_n'$ be defined by

$$g_1(b_1, b_2, \ldots, b_k) = (b_1, b_2, \ldots, b_k, 1)$$

let $f_2 : A_n'' \to A_{n-2}$ be defined by

$$f_2(a_1, \ldots, a_k) = (a_1, \ldots, a_{k-1})$$

$g_2 : A_{n-2} \to A_n''$ be defined by

$$g_2(b_1, \ldots, b_k) = (b_1, \ldots, b_k, 2)$$

$f_1$ & $g_1$ are inverses

$f_2$ & $g_2$ are inverses

So $|A_n'| = |A_{n-1}|$
$|A_n''| = |A_{n-2}|$ $\Rightarrow |A_n| = |A_n'| + |A_n''|$

$$= |A_{n-1}| + |A_{n-2}|$$

$(1, 1, 2), (2, 2)$

| $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|---|---|---|---|---|
| $()$ | $(1)$ | $(1,1)$ | $(1,1,1)$ | $(1,1,1,1)$ |
| | | $(2)$ | $(2,1)$ | $(2,1,1)$ |
| | | | $(1,2)$ | $(1,2,1)$ |
| | | | | $(1,1,2)$ |
| | | | | $(2,2)$ |