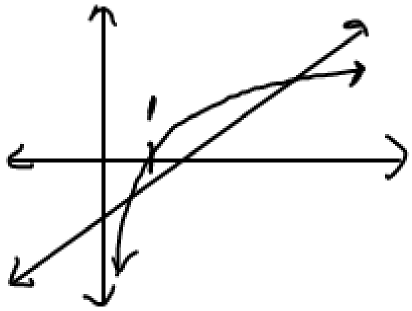
Last Time: Fixed Point Iteration

Solve $f(x)=0$ by writing $x=g(x)$ and using the recurrence relation $x_{n+1}=g(x_n)$.

e.g. Solve $\ln x = 2x-3$

we showed that there were 2 solutions.



2 ways of writing $x=g(x)$:

① $x=e^{2x-3}$

② $x=\frac{1}{2}(\ln x+3)$

start with initial guess $x_0=1$, and see what happens in each scheme.

① $x_0=1$; $x_1=e^{2(1)-3}=e^{-1}\approx 0.3679$;

$x_2\approx 0.1089$; $x_3\approx 0.0613$; $x_4\approx 0.0563$;

$x_5\approx 0.0557$; $x_6\approx 0.0557$; $x_7\approx 0.0556$; $x_8\approx 0.0556$

② $x_0=1$; $x_1=\frac{1}{2}(\ln 1+3)=\frac{3}{2}=1.5$;

$x_2\approx 1.7027$ .... $x_8=1.7915$; $x_9=1.7915$

One root with each method — convenient!
Could we change our intitial guess to find the opposite root with each scheme?

In ①, try $x_0=2$.

$x_0=2$; $x_1=e^{4-3}=e\approx 2.718$;

$x_2\approx 11.4$; $x_3\approx 424900763$; $x_4\rightarrow$ math error.

$\rightarrow$ scheme diverges.

In ②, try $x_0=0.1$

$x_0=0.1$; $x_1\approx 0.3487$; $x_2\approx 0.9732$; $x_3\approx 1.4864$ ....

$\rightarrow$ still approaches the root near 1.79.

Try $x_0 = 0.05$
$x_1 \approx 0.00213$; $x_2 = -1.576$; x_3 doesn't exist (ln of -ve #)
$\rightarrow$ diverges

<u>Condition to guarantee convergence</u>

For the equation $x = g(x)$, we need $|g'(x)| < 1$ for all values of x within an interval that contains the fixed point to guarantee convergence.

· g must be differentiable with a banded derivative.

In ①, $g(x) = e^{2x-3} \rightarrow g'(x) = 2e^{2x-3}$

$$|g'(x)| < 1 \Leftrightarrow 2e^{2x-3} < 1 \Leftrightarrow e^{2x-3} < \tfrac{1}{2} \Leftrightarrow$$
$$2x - 3 < \underbrace{\ln(\tfrac{1}{2})}_{\ln 1 - \ln 2} \Leftrightarrow 2x - 3 < -\ln(2) \Leftrightarrow x < \tfrac{1}{2}(3 - \ln(2)) \approx 1.15$$

$$\boxed{\therefore x < 1.15}$$

For $x_0 = 1$, the method converges
For $x_0 = 2$, the method diverges.

In ②, $g(x) = \frac{1}{2}(3 + \ln x) \rightarrow g'(x) = \frac{1}{2x}$

$$|g'(x)| < 1 \Leftrightarrow \frac{1}{2x} < 1 \Leftrightarrow x > \frac{1}{2}.$$

Showed that with $x_0 = 1$, the method converges.
Also, with $x_0 = 0.1$, the method converges.
(Not guaranteed)
At $x_0 = 0.05$, diverges.
why $|g'(x)| < 1$? see lecture notes.

Polynomial Interpolation

Suppose we have $n+1$ points, and want to draw a smooth curve thru all of them.

The simplest curve is a polynomial of degree n.

(line joins 2 pts, parabola any 3 pts, etc)

Let's say we have the points $(0,y_0),(1,y_1),(2,y_2),(3,y_3)$.

The general eqn for a cubic is.

$$(*) \qquad y = a + bx + cx^2 + dx^3, \text{ where } a,b,c,d \text{ are constants.}$$

plug each point into $(*)$:

$$\left.\begin{aligned}
(0,y_0) &: y_0 = a \\
(1,y_1) &: y_1 = a+b+c+d \\
(2,y_2) &: y_2 = a+2b+4c+8d \\
(3,y_3) &: y_3 = a+3b+9c+27d
\end{aligned}\right\} \begin{array}{l}\text{4 eqns}\\ \text{4 unknowns}\end{array}$$

Newton's idea: <u>finite differences</u>

Define $\Delta y_n = y_{n+1} - y_n$ as the first finite diff.

$$\left.\begin{aligned}
\Delta y_0 &= y_1 - y_0 = b+c+d \\
\Delta y_1 &= y_2 - y_1 = b+3c+7d \\
\Delta y_2 &= y_3 - y_2 = b+5c+19d
\end{aligned}\right\} \begin{array}{l}\text{3 eqns}\\ \text{3 unknowns}\end{array}$$

Define the second finite difference:

$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$$

$$\Rightarrow \left.\begin{aligned}
\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = 2c+6d \\
\Delta^2 y_1 &= \Delta y_2 - y_1 = 2c+12d
\end{aligned}\right\} \begin{array}{l}\text{2 eqns}\\ \text{2 unknowns}\end{array}$$

Third Finite Difference

$$\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n$$

$$\Rightarrow \Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 6d \Rightarrow d = \frac{1}{6}\Delta^3 y_0$$

write a, b, c, d in terms of $y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0$.

$$2c = \Delta^2 y_0 - 6d \Rightarrow c = \frac{1}{2}(\Delta^2 y_0 - \Delta^3 y_0)$$

$$b = \cdots = \Delta y_0 - \frac{1}{2}\Delta^2 y_0 - \frac{1}{6}\Delta^3 y_0$$

$$a = y_0$$

(*): $y = a + bx + cx^2 + dx^3$

$$\Rightarrow y = y_0 + x \cdot \Delta y_0 + \frac{1}{2}x(x-1)\Delta^2 y_0 + \frac{1}{6}x(x-1)(x-2)\Delta^3 y_0$$

write the <u>diff</u> y-values as:

y_0			
y_1	Δy_0	$\Delta^2 y_0$	
y_2	Δy_1		$\Delta^3 y_0$
y_3	Δy_2	$\Delta^2 y_1$	

Formula only uses these