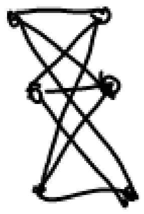
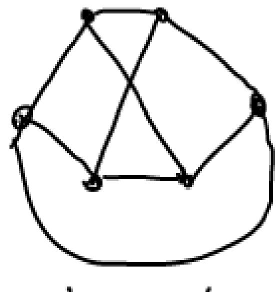
Prop: If G is connected and not a tree, then the boundary of every face in a planar embedding of G contains a cycle.

pf: see 7.5.1 from notes.

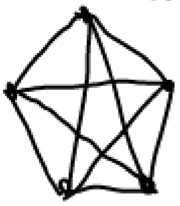


$K_{3,3}$



non-planar?

Prop. If G is a connected planar graph with $p \geq 3$ vertices and q edges, then $q \leq 3p-6$. If G is a tree, then the stmt holds because $q = p-1$.



$p = 5$

$q = \binom{5}{2} = 10$

pf: Consider a planar embedding of G with p vertices, q edges, & r faces. By handshake for faces,

$$2q = \sum_{f \text{ face}} \deg(f) \geq 3r$$

because each face has degree ≥ 3.

So $2q \geq 3r \Rightarrow r \leq \frac{2}{3} q$

By Euler's Formula $2 = p - q + r \leq p - q + \frac{2}{3} q = p - \frac{1}{3} q$

So $q \leq 3p - 6$

Prop. If G is a connected planar graph that is not a tree with p vertices, q edges, and every cycle has length $\geq d$, then $q \leq \frac{d}{d-2}(p-2)$.

pf: $\because$ every face boundary contains a cycle. $\deg(f) \geq d$ for each face f. By handshaking, $2q = \sum \deg(f) \geq dr$. so $r \leq \frac{2}{d} q$ (r = # faces).

Corollary: If G is a (non-tree) connected bipartite graph, with p vertices, q edges, then $q \leq 2p-4$.

pf: Since the graph is bip., every cycle has length ≤ 4.

Since $K_{3,3}$ is bipartite, connected and $p=6$, $q=9$, this implies $K_{3,3}$ is nonplanar.