1. Uncountability
2. Complex Numbers

$\mathbb{Z}$: $\mathbb{N} \times \mathbb{N}$ $\mathbb{Q}^+$ $\longrightarrow \mathbb{N}$

Reals are not countable

Prove that $(0,1]$ is not countable.

① Represent a real number in $(0,1]$ by

$$0.a_1 a_2 a_3 \ldots \quad \text{where } a_i \in \{0,1,\ldots,9\}$$

This rep. is not unique.

$$0.50000\ldots = 0.49999\ldots$$

For any reals that terminates at some point, we pick the rep. with 9's at the end. Then the rep. is unique.

② If S is infinite and countable, then there exists a bijection $f: S \to \mathbb{N}$.

Proof of the uncountability of $(0,1]$: Suppose $(0,1]$ is countable. Then there exists a bijection $f: (0,1] \to \mathbb{N}$

$f(x)$	x
1	$0.a_{11}a_{12}a_{13}\ldots$
2	$0.a_{21}a_{22}a_{23}\ldots$
3	$0.a_{31}a_{32}a_{33}\ldots$
4	$\vdots$
5	
6	
7	
$\vdots$	

This list includes all real numbers in $(0,1]$. We will construct a real number in $(0,1)$ that is not on the list.

Consider the real number $0.b_1 b_2 b_3 b_4 \ldots$ where

$$b_i = \begin{cases} 7 & \text{if } a_{ii} = 3 \\ 3 & \text{if } a_{ii} \neq 3 \end{cases}$$

b	$0.7373\ldots$
1	$0.\boxed{3}1415\ldots$
2	$0.0\boxed{2}345\ldots$
3	$0.12\boxed{3}45\ldots$
4	$0.222\boxed{2}\ldots$
5	$\vdots$
6	
7	

Since this number differs with the n-th real number on the list at the n-th digit for all n, our number does not appear on the list. Contradiction. So $(0,1]$ is not countable.

<u>Complex Numbers</u>

Fundamental Theorem of Algebra: The polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

with $a_n \neq 0$, $n \geq 1$ has a solution in $\mathbb{C}$.

$$\mathbb{C} = \{ \underbrace{a+bi}_{\text{standard form}} \mid a, b \in \mathbb{R} \}$$

i "imaginary"

Equality: $a+bi = c+di$ iff $a=c$ and $b=d$

Addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$

Multiplication: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$

For $z = a+bi$, a is the real part $\mathrm{Re}(z)$,
b is the imaginary part $\mathrm{Im}(z)$.

$$i^2 = (0+i)(0+i) = (0\cdot 0 - 1\cdot 1) + (0\cdot 1 + 1\cdot 0)i = -1$$

$$-1 = i \cdot i = \sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

We can't say $i = \sqrt{-1}$. We can only say $i^2 = -1$