Countability Recall: S is countable if there exists $f: S \to \mathbb{N}$ that is 1-1.

Integers $\mathbb{Z}$: $f: \mathbb{Z} \to \mathbb{N}$

$\mathbb{Z}$ ... $-3, -2, -1, 0, 1, 2, 3, 4$ ...

$\mathbb{N}$ ... $1, 2, 3, 4, 5, 6, 7, 8$ ...

$$f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 1-2x & \text{if } x \le 0 \end{cases}$$

This is a bijection.

Proof (outline): 1-1: suppose $f(x) = f(y)$ 3 cases:

① $x, y > 0$

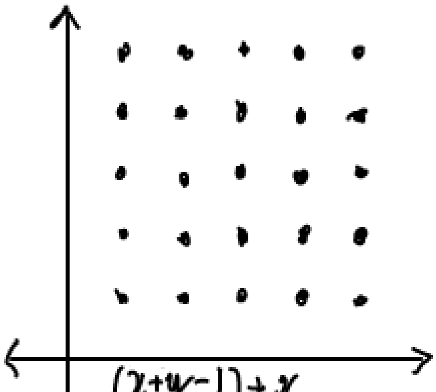
② $x, y \le 0$ $\quad \Rightarrow x = y$

③ $x > 0, y \le 0$

Onto: Suppose $y \in \mathbb{N}$, 2 cases: ① y is odd ② y is even.

This implies that $\mathbb{Z}$ is countable. $|\mathbb{Z}| = |\mathbb{N}|$

Cartesian Product $\mathbb{N} \times \mathbb{N} = \{(x, y) \mid x, y \in \mathbb{N}\}$. Find $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$



$\binom{x+y-1}{2} + x$

Preposition: Every positive integer n can be uniquely written as $n = 2^a b$, where a is non-negative, and b is a positive odd integer.

$36 = 2^2 \cdot 9 \qquad 42 = 2 \cdot 21 \qquad 41 = 2^0 41$

non-neg. $\quad$ positive odd int.

Define $f(x, y) = 2^{x-1}(2y-1)$

1-1: Suppose $f(x, y) = f(x_2, y_2)$. Then $2^{x_1 - 1}(2y_1 - 1) = 2^{x_2 - 1}(2y_2 - 1)$

By prop., they rep. the same int., so $x_1 - 1 = x_2 - 1$, $2y_1 - 1 = 2y_2 - 1$.

So $(x_1, y_1) = (x_2 + y_2)$

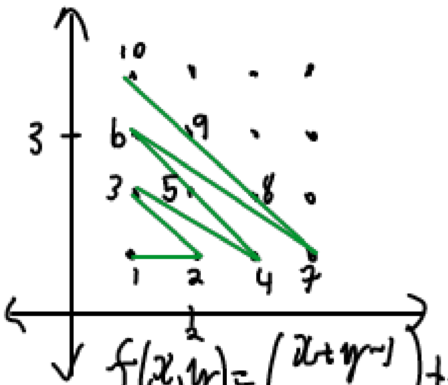
Onto: Let $z \in \mathbb{N}$. By prop, $z = 2^a \cdot b$ for some $a \geq 0$, b is odd.

$f(a+1, \frac{b+1}{2}) = z \Longrightarrow f$ is a bijection.

$f(3,2) = 2^2 \cdot 3 = 12$. $f(1,10) = 2^0 \cdot (19) = 19$

So $\mathbb{N} \times \mathbb{N}$ is countable. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Diff way of defining f: (by diagram)



$f(2,3) = 9$

$f(4,1) = 7$

$f(x,y) = \binom{x+y-1}{2} + y$

maybe ↑

positive rationals: $\mathbb{Q}^+ = \{\frac{p}{q} \mid \gcd(p,q)=1,\ p,q \in \mathbb{N}\}$

$\frac{3}{17} \to (3,17) \to 2^3(33)$ Composition of functions.

$f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ from above where $\gcd(p,q)=1$

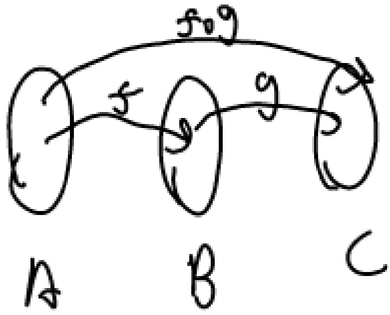
$g: \mathbb{Q}^+ \to \mathbb{N}\times\mathbb{N}$ by $\underbrace{g(\frac{p}{q}) = (p,q)}_{1-1}$

not $\underline{\text{onto}}$

$g(?) = (2,2)$

$\frac{1}{1} \to (1,1)$

Prop: If $f: A \to B$ and $g: B \to C$ are 1-1, then $f \circ g : A \to C$ is also 1-1.



proof: Suppose $f\circ g(x) = f \circ g(y)$ for some $x, y \in A$. Then $f(g(x)) = f(g(y))$.
Since f is 1-1, $g(x) = g(y)$
since g is 1-1, $x = y$.

Back to $\mathbb{Q}^+$: $g: \mathbb{Q}^+ \to \mathbb{N}\times\mathbb{N}$, $f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$.
Both are 1-1, so $g \circ f: \mathbb{Q}^+ \to \mathbb{N}$ is also 1-1.
So $\mathbb{Q}^+$ is countable. $|\mathbb{Q}^+| \leq |\mathbb{N}|$ $|\mathbb{N}| \leq |\mathbb{Q}^+|$

$h: \mathbb{N} \to \mathbb{Q}^+$

$h(x) = x$

So $|\mathbb{Q}^+| = |\mathbb{N}|$