Optimization on Closed Bounded Domains

Extreme Value Thm: If f is cts on a closed, bounded set $S \subset \mathbb{R}^2$, then f has an absolute max & min on S.

How to find?

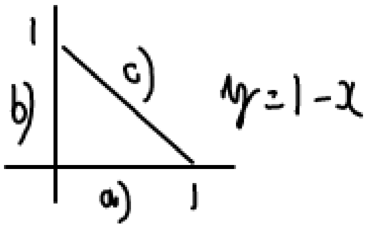
Recall single variable case:

1) find local max/min
2) evaluate f at endpts 0 & 1

For $f: \mathbb{R}^2 \to \mathbb{R}$, same thing:

1) crit pts.
2) evaluate f on the boundary

Ex/ Find the max/min of $f(x,y) = xy$ on the triangle with vertices $(0,0), (1,0), (0,1)$.



1) crit pts inside

$f_x = y$, $f_y = x$

$\nabla f = (0,0) \Rightarrow (x,y) = (0,0)$.

2) On the boundary.

a) $y = 0$, $x \in [0,1]$

$f(x,0) = 0$

b) $x = 0$, $y \in [0,1]$.

$f(0,y) = 0$.

c) $y = 1-x$, $x \in [0,1]$.

$f(x, 1-x) = x(1-x) = x - x^2 = g(x)$ } Find abs max/min of g on $[0,1]$.

1) crit pts. $g'(x) = 1 - 2x = 0$ when $x = \frac{1}{2}$

$g(\frac{1}{2}) = \frac{1}{4}$.

2) end pts. $g(0) = 0 = g(1)$

} All together: The abs max is $1/4$ which occurs at $(1/2, 1/2)$, & the abs min is 0, occurring along the coord. axes.

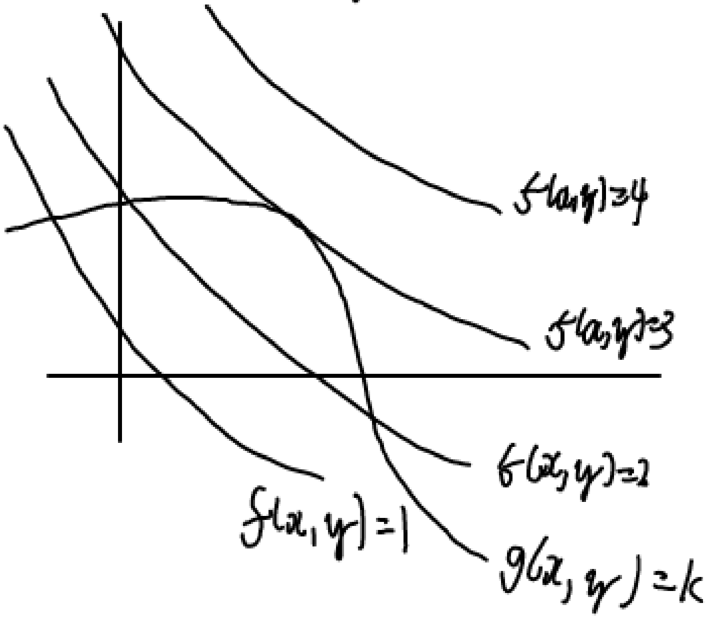
Lagrange Multipliers

- used for optimizing a function given a constraint.

e.g. Find max/min of $f(x,y)$ on the curve $g(x,y)=k$

Idea: draw level curves of f along with $g(x,y)=k$.

Max of f is largest value of c s.t. the level curve $f(x,y)=c$ intersects $g(x,y)=k$.



Intersection happens at a single point.

At this point, the curves share a common tangent.

$\Rightarrow$ The gradient vectors $\nabla f, \nabla g$ are parallel.

(since they are orthog. to level curves)

Therefore, $\nabla f = \lambda \nabla g$, where λ is a const called the Lagrange multiplier.

2 eqns + const eqn = 3 eqn, 3 unkns. x, y, t.

Summary: To find min/max of f on $g(x,y)=k$, find (a,b) that satisfy

1) $\nabla f = \lambda \nabla g$ & $g(x,y)=k$.
2) $\nabla g = (0,0)$ & $g(x,y)=k$
3) (a,b) is an endpt. of $g(x,y)=k$

Evaluate f at all such points & pick out the largest/smallest.

Ex./ Find the max & min of $f(x,y)=x^2+2y^2$ on the unit circle.

Sol^n. $g(x,y) = x^2+y^2 = 1$

$\nabla f = (2x, 4y)$, $\nabla g = (2x, 2y)$

system: $2x = \lambda 2x$ ①

$4y = \lambda 2y$ ②

$x^2+y^2 = 1$ ③

①: $\underbrace{\lambda = 1}$ or $x = 0$

$\Downarrow$

②: $4y = 2y \Longrightarrow y = 0$. put $y=0$ in ③: $x^2 = 1 \Rightarrow x = \pm 1$

$(1,0)$ & $(-1,0)$

If $x = 0$

③: $y^2 = 1 \Longrightarrow y = \pm 1$

②: $\lambda = 2$ or $y = 0$

If $y=0$, then does not satisfy ③, therefore $y \neq 0$.

$\Longrightarrow (0,1), (0,-1)$ from ③

- $\nabla g = (0,0) \Longrightarrow (x,y) = (0,0) \Rightarrow$ not on curve.
- no end pts since closed curve.

Evaluate $f(1,0) = 1$, $f(-1,0) = 1$ $\longleftarrow$ minima

$f(0,1) = 2$, $f(0,-1) = 2$ $\longleftarrow$ maxima

Draw the constraint curve along with the level curves

$f(x,y) = 1$, $f(x,y) = 2$ $(x^2+2y^2 = 1,\ x^2+2y^2 = 2)$